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Generalised Geometry in Supergravity theories

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To my parents.

Ai miei genitori.

Abstract

I study how ten-dimensional Type II Supergravity theories can be reformulated using an extension of conventional differential geometry known as "Generalised geometry". I review the dynamics and symmetries of these theories, define the key elements of generalised geometry, including the notion of torsion-free generalised connections, and show how this geometry can be used to give a unified description of the supergravity fields, exhibiting an enlarged local symmetry group. This part will be end showing that the equations of motion for the NSNS sector of Type II Supergravity theories in the framework of Generalised geometry can be reformulated in a similar way of Einstein's equations of motion for gravity in ordinary geometry. In the second part I investigate the notion of "Leibniz generalised parallelisations", the analogue of a local group manifold structure in generalised geometry, aiming to characterise completely such geometries, which play a central role in the study of consistent truncations of supergravity. One of the original results we obtained is the solution of the misterious case of consistent truncation on S^7 showing that in Generalised geometry all spheres S^d are Leibniz generalised parallelisable. I work out also some explicit examples of manifold that are Leibniz generalised parallelisable ($S^2 \times S^1$, H^3, dS_3, AdS_3) and in particular connecting this results with the consistent truncations of supergravity.

Contents

Introduction

String theory is 21st century physics which fell accidentally into the 20th century.
- Edward Witten -

In a physical description of nature, around the seventies, physicists attempted to describe the fundamental interactions of nature using a *Quantum Field Theory* approach, getting the notable *Standard Model*. As far as we know, in nature there are four fundamental forces, which are the gravitational, electromagnetic, strong nuclear and the weak nuclear. The quantum field description was a successful approach, but only for the last three interactions, i.e. for the electromagnetic, nuclear strong and nuclear weak. In particular what makes this approach successful is the possibility to *renormalise* the theory, i.e. the possibility to remove, within a clever procedure, all the infinite quantities that appear in the computations of the n-point correlation functions which, according to the LSZ formula, are related to the observable quantities of the system, among which the cross section [52].

Therefore Physicists tried to find another way to solve the problem. String Theory, which at the beginning was born in order to give an explanation of the strong nuclear interaction, now is recognized to be a theory that explain all the four fundamental interactions in a unified way, which represent the "Einstein's dream". Problems about ultraviolet divergences now do not appear anymore because of three new ingredients inside String Theory: *Supersymmetry*, *Extra dimensions* and the replacement of *point particles* with *extended objects*[1].

Supersymmetry is an appealing idea that consists in trying to enlarge the symmetries of spacetime, which are represented by the Poincarè group. This idea consists in introducing a new set of "fermionic" generators with anticommuting relations between each other, and commuting relations between them and the "bosonic" generators of the Poincarè group. When a fermionic generator is applied to a bosonic state, the result is a fermionic state.

Therefore Supersymmetry can also be seen as a symmetry that maps a *boson* into a *fermion* and vice versa. In particular to each particle existing in nature is associated a *superpartner* with opposite statistics, e.g. photon/ photino, graviton/gravitino, electron/selectron, lepton/slepton, quarks/squarks, W/Wino, gluon/gluino, Higgs/Higgsino, and so on. Now, with the introduction of these new superpartners, naively what happens is that for each bosonic ultraviolet divergent loop diagram, there is a corresponding fermionic diagram, with the same properties, but with opposite sign, and so the sum is zero. This is what is called "miraculous cancellation" [41].

Extra dimensions, the other important ingredient inside String Theory, at the beginning consists in replacing the point particle with a string embedded in a spacetime with arbitrary dimension. The remarkable result of this procedure regards decay processes of one particle into two particles. Replacing point particles by closed strings, the above process is described by a closed string that turns into two closed strings, i.e. by a "pants diagram", and in this way short-distance singularities are avoided. Now the world sheet is a smooth manifold and this has a remarkable consequence: loop amplitudes have no ultraviolet divergences [64].

[1] Actually extra dimensions are not important for removing UV divergences, just only supersymmetry and extended object are needed. However extra dimensions are indispensable to assure *Weyl anomalies* cancellation, which occurs only for the "critical dimension" $d = 26$ for bosonic string theory and $d = 10$ for superstring theories.

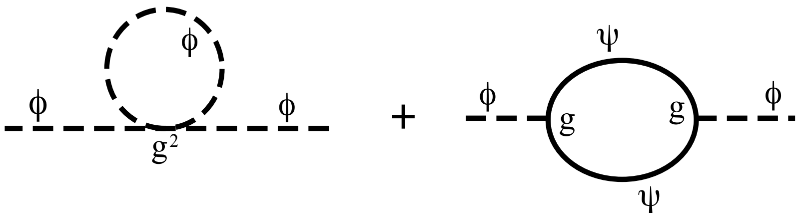

Figure 1: Miraculous cancellation between loop diagrams of supersymmetric partners. In the picture ϕ represents a scalar, ψ a fermion and g is the Yukawa coupling constant. (Courtesy of [55]).

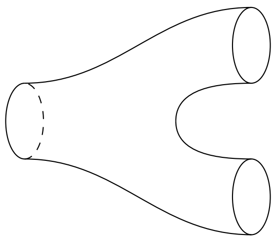

Figure 2: Feynmann diagram of a string interaction vertex.

String Theory at the beginning was born as a bosonic theory in 26 dimensions ($d = 26$ represents the *critical dimension* in order to cancel Weyl anomalies). Only after the introduction of the Supersymmetry we had got what is called "Superstring Theory" which finally lives in 10 dimensions. Superstring Theory is not unique, but it exists in five different versions: Type I, Type IIA, Type IIB, $SO(32)$ Heterotic, $E_8 \times E_8$ Heterotic. These five different theories are regarded as different limits of a single theory, tentatively called M-theory, which lives in 11 dimensions.

Supergravity is a low energy limit of Superstring Theory. Therefore there are five different types of Supergravity theories (plus another one from a different low energy limit of M-theory, which is the 11-dimensional SUGRA). Low energy limit means that the energy must be very less than $1/\sqrt{\alpha'}$, where α' is the fundamental scale in String Theory. However our real world is four-dimensional and superpartners - if they exist - are much heavier then the known observed particles. In order to obtain a more realist theory, we have to break Supersymmetry to provide larger masses for the superpartners. On the other side, in order to remove extra dimensions, firstly we have to perform a *compactification*. This consist in replacing the 10 dimensional Minkowski spacetime with the product of a 4 dimensional Minkowski spacetime and a 6 dimensional compact manifold with characteristic length L.

After a compactification, we have to do a *dimensional reduction*, which consists in squeezing to zero the characteristic length $L \to 0$. If we start with a massless particle in the higher dimensional theory, after compactification we obtain an infinite tower of *massive* particles in the lower dimensional theory [25]. One keeps only the lightest states (usually massless). This is generally not consistent in that these modes will source the heavier modes. However, physically we do not care because at low energies (i.e. below the mass of the first modes we ignore) we will not excite these degrees of freedom, and therefore they will not affect the low-energy (dimensionally reduced) theory.

Otherwise one can take a *consistent truncation* of the full set of modes which consists in identify a finite subset of modes where the omitted modes are not sourced by the kept modes. This means that the two sets of modes, the kept and the neglected, are completely decoupled between each other - the consistent modes do not have to be the lightest one and the space does not even have to be compact. The reader may wonder why we are interested in consistent truncations. The answer is because any solution of the low dimensional theory involving modes from consistent truncation, when uplift to the high dimensional theory, gives an exact solution. This is a requirement of consistency that we want about our theory. Consistent truncations will be explained in details in chapter 1.

A very important question regards which compact manifolds give consistent truncations. Do all compact manifolds provide consistent truncations? The answer unfortunately is no. There is a known result [58], [59] which tells us that consistent truncations appear in manifolds that are *local group manifolds*, which means the manifold M can be written as the quotient of a Lie group G and a discrete group Γ, i.e. $M = G/\Gamma$. For instance consistent truncations are provided by the spheres S^1, S^3 and by the n-dimensional torus T^n, which correspond respectively to the groups $U(1)$, $SU(2)$ and $[U(1)]^n$.

The aim of this thesis is to study the "Generalised geometry", which in one sketch it consists in replacing the tangent bundle of a manifold with the direct sum of the tangent bundle and the cotangent bundle. Actually this represents just one way to define generalised geometry, because it is not unique. Generalised geometry will be defined in chapter 2 and will be applied to Type II Supergravity theories in chapter 3 and 4. In particular the chapter 4 concerns the study of consistent truncations and has original components. The questions we would like to answer with this approach are given by the following motivations

(I) Using the framework of generalised geometry we prove that the equations of motion for the NSNS sector of Type II supergravity theories can be written in a very similar way of Einstein's equations for gravity in ordinary geometry using a metric approach [15]. There is nothing physical new, at least in this moment, about this result. Needless to say, it represents a beautiful formal result.

(II) In the study of consistent truncations of supergravity there are some mysterious cases that are not well understood within the framework of ordinary geometry. What we are talking about are consistent truncations on S^7 [20], on S^4 [47], [48] for eleven-dimensional supergravity and on S^5 [18] for Type IIB supergravity, which are *not* local group manifold. The remarkable fact is that all spheres S^d can give a consistent truncation because they are "Leibniz generalised parallelisable", which in generalised geometry replaced the concept of local group manifold.

(III) Double Field Theory and Generalised Geometry are two approaches which can be shown to be locally equivalent to each other. Double Field Theory [1], which was developed principally by Waren Siegel [62], [63], Chris Hull and Barton Zwiebach [42],[71], consists in defining a field theory in a double configuration space $(x, \tilde{x})$. In this approach the tangent bundle is $2d$-dimensional because of the "double" coordinates. In the approach of generalised geometry the target space remains as usual with the normal d coordinates, but what is double now is the "tangent bundle" (that now is called "generalised tangent bundle") which consists in "$TM \oplus T^*M$". At the end of the day both approaches have a tangent bundle which is $2d$-dimensional and a natural $O(d,d)$-structure.

(IV) Generalised geometry is also a proposal to incorporate "T-duality" [26], in a sense we are going to explain. We mentioned before that generalised geometry has a natural $O(d,d)$-structure. If we try to look now at Type IIA and Type IIB Superstring theories compactified on an n-torus (thus the spacetime is $\mathcal{M}_{10-n} \times T^n$) then we know there is a symmetry between the two theories, called "T-duality", which is represented by the action of the $O(n,n;\mathbb{Z})$ group. An heuristic suggestion, perhaps quite speculative, for investigating in generalised geometry, is that T-duality group $O(n,n;\mathbb{Z})$ is enclosed inside, as a subgroup, to the structure group $O(d,d;\mathbb{R})$ of the generalised tangent bundle. However, although it is often said that Double Field Theory (and hence also generalised geometry) incorporates T-duality or make T-duality manifest, this is not really true. T-duality is a *discrete* symmetry $O(n,n;\mathbb{Z})$ of a particular background geometry (tori T^n), while $O(d,d)$ is the *continuous* structure group symmetry which is present for all backgrounds. They are not the same.

In this thesis the logic is the following. In the chapter 1 will be presented the basic concepts of String Theory. We will start writing down the string action and by variational principle we will be able to write the equations of motion for the string coordinates. Then we will expand the solution and quantised normal modes. From an analysis of the spectrum, which we will not treat but just show the result, will appear three types of massless fields: a graviton $g_{\mu\nu}(x)$, a 2-form antisymmetric tensor gauge field $B_{\mu\nu}(x)$ and a "dilatonic" scalar $\phi(x)$. We will show the effective action and how the *critical dimension* $d = 26$ appears in bosonic string theory. We will also give a connection with superstring theories.

Then we will talk about compactifications and dimensional reductions, explaining the so-called Kaluza-Klein theory. Here, related to this topic, we will talk about consistent truncations. Then we will explain T-duality, which is a symmetry of the bosonic string embedded in a spacetime compactified on a circle and a symmetry between Type IIA and Type IIB superstring theories compactified on a circle. The chapter will end extending T-duality symmetry to toroidal compactification, in particular analysing by which group it is represented.

In chapter 2 we will talk about generalised geometry, focusing more on mathematical aspects. Generalised geometry represents an appealing idea coming from the mathematician Nigel Hitchin and consists in an "extension" of ordinary geometry. In ordinary geometry, one takes a d-dimensional manifold M with its associated tangent bundle TM. In generalised geometry what we do, roughly speaking, is replacing the tangent bundle TM by $TM \oplus T^*M$. Pictorially, in generalised geometry instead of considering only vector we consider "vector + 1-form", which are called "generalised vector".

Then we will define "patching rules", which are relations between fibres on the overlap of the open sets $\{U_i\}$ which cover the manifold. Patching rules are a fundamental ingredient in the theory of *fibre bundle*. The way to define patching rules should reproduce *diffeomorphisms* and *gauge transformations* of $B_{\mu\nu}$.

We will see in the framework of generalised geometry that the B field plays a central role. It establishes the isomorphism between the generalised tangent bundle E and $TM \oplus T^*M$ by the *Splitting Lemma*.

Then we will move to extend in generalised geometry the concept of *diffeomorphism*. Lie derivative and Lie bracket will be generalised, respectively, by the so-called *Dorfman derivative* and *Courant bracket*. This argument will lead us to introduce $O(d, d)$-structure in generalised geometry.

In chapter 3 we will generalise the fundamental tools of General Relativity, which are metric, connection, torsion, Riemann curvature tensor, Ricci tensor and scalar of curvature. This chapter is quite technical. The logic we will follow to perform the generalisation will be close as much as possible to the logic one use for defining the above objects in ordinary geometry.

At the end of the chapter 3 we will be able to write Type II Supergravity equations of motion for the bosonic sector as generalised version of Einstein's equations in vacuum.

In chapter 4, which has original components, we will investigate about *consistent truncations* in the framework of the generalised geometry. We will introduce a conjecture which states that manifolds that are *Leibniz generalised parallelisable* give consistent truncations.

Then we will show that all round spheres S^d are *Leibniz generalised parallelisable*, hence by the conjecture above, they give consistent truncations. This fact gives us an answer about the mysterious cases of consistent truncations on S^4, S^5 and S^7. The proper generalised geometry to study the round spheres is not the Hitchin's one which consists in replacing TM by $TM \oplus T^*M$, rather it is the one which replaces TM by $TM \oplus \bigwedge^{d-2} T^*M$. The case $d = 3$ is special, because the generalised geometry for spheres coincides with Hitchin's generalised geometry.

Then we will study 3-dimensional manifolds, which are described by the same embedding equation of the sphere, i.e. $\eta_{\mu\nu} X^\mu X^\nu = 1$, but with different signature of $\eta_{\mu\nu}$, such that they can be approached with the same generalised geometry of the 3-sphere, which is the Hitchin's one. The manifolds which we choose are S^3, $S^2 \times S^1$, H^3, dS_3, AdS_3. We will show explicitly that all these manifolds are Leibniz generalised parallelisable, and analyse the algebra generated by the generalised vector frame (called simply frame). The reason why we are interested in the algebra generated by the frame is due to the consistent truncations. If we perform a dimensional reduction over a manifold whose frame generates the algebra associated to the Lie group G, than in the dimensional reduced theory the group G plays the role of gauge group.

Related to this argument, we will motivate the conjecture [17] using generalised geometry. The conjecture states that for a dimensional reduction over a Lie group manifold $M = G$, instead of taking the gauge group G provided by the frame, let us pick up $G \times G$. The idea is that generalised geometry, with its property to be "double", gives us a frame which encodes $G \times G$ algebra.

1

Preliminaries in String Theory and Supergravity

I do feel strongly that string theory is our best hope
for making progress at unifying gravity and quantum mechanics.
- Brian Greene -

String theory is an ambitious project. Even thought string theory is not yet fully formulated, and we cannot yet give a detailed description of how the standard model of elementary particles should emerge at low energies, or how the Universe originated, there are some general features of the theory that have been well understood. The first general feature of string theory, and perhaps the most important, is that general relativity is naturally incorporated in the theory. At hight energies/short distances the theory gets modified, but at ordinary energies and distances it is present in exactly the form as proposed by Einstein. This is a remarkable fact because gravity is unified with the other forces of Nature inside a single quantum mechanical framework.

The premise of string theory is that, at the fundamental level, matter does not consist of point-particles but rather of tiny loops of string. It assumes that all particles are different harmonics of small vibrating strings, much in the way the different harmonics of a guitar string correspond to different musical notes. From this slightly absurd beginning, the laws of physics emerge. General relativity, electromagnetism and Yang-Mills gauge theories all appear in a surprising fashion. However, they come with baggage. String theory gives rise to a host of other ingredients, most strikingly extra spatial dimensions of the universe beyond the three that we have observed.

Hence our journey inside string theory will start naturally recalling the physics of particle moving in spacetime, and then generalising to extended objects, called p-branes.

1.1 Relativistic free point particle

The motion of a relativistic particle of mass m in a curved d dimensional spacetime can be formulated as a variational problem. In order to reproduce the non-relativistic equations of motion, the only possible choice for a relativistic action is to be proportional to the length of world line

$$S_0 = -m \int ds \ , \tag{1.1}$$

where the mass m appears because of the dimensionless requirement of the action, and ds is defined as follow

$$ds^2 = -g_{\mu\nu}(X)dX^\mu dX^\nu \ . \tag{1.2}$$

Here $g_{\mu\nu}(X)$, with $\mu, \nu = 0, ..., d-1$, describes the background geometry, which is chosen to have Minkowski signature $(- + \cdots +)$.

The action (1.1) therefore takes the form

$$S_0 = -m \int \sqrt{-g_{\mu\nu}(X)\dot{X}^\mu \dot{X}^\nu}\, d\tau \ , \qquad \text{where} \qquad \dot{X}^\mu = \frac{dX^\mu}{d\tau} \ . \tag{1.3}$$

The action S_0 has a couple of disadvantages. First if does not work for massless particles with $m = 0$. Second S_0 contains a square root, so that it is difficult to quantize in a path integral framework. These problems can be circumvented by introducing an action equivalent to the previous one at the classical level, which is formulated in terms of an auxiliary field $e(\tau)$, called *einbein*

$$\tilde{S}_0 = \frac{1}{2}\int d\tau \left(e^{-1}\dot{X}^2 - m^2 e\right) \ , \qquad \text{where} \qquad \dot{X}^2 = g_{\mu\nu}(X)\dot{X}^\mu \dot{X}^\nu \ . \tag{1.4}$$

We prove now the equivalence between the two actions S_0 and $\tilde{S}_0$. The equation of motion of $e(\tau)$, by variational principle, is

$$\frac{\delta \tilde{S}_0}{\delta e} = 0 \qquad \Longleftrightarrow \qquad m^2 e^2 + \dot{X}^2 = 0 \ , \tag{1.5}$$

Then we can rewrite

$$\dot{X}^2 = g_{\mu\nu}\dot{X}^\mu \dot{X}^\nu = -\sqrt{-g_{\mu\nu}\dot{X}^\mu \dot{X}^\nu}\sqrt{-g_{\mu\nu}\dot{X}^\mu \dot{X}^\nu} \ , \tag{1.6}$$

and then the solution for $e(\tau)$ and its inverse can be written as

$$e = \frac{1}{m}\sqrt{-g_{\mu\nu}\dot{X}^\mu \dot{X}^\nu} \ , \qquad e^{-1} = \frac{m}{\sqrt{-g_{\mu\nu}\dot{X}^\mu \dot{X}^\nu}} \ . \tag{1.7}$$

Substituting (1.7) back into (1.4) we get

$$\begin{aligned} \tilde{S}_0 &= \frac{1}{2}\int d\tau \left(-\frac{m}{\sqrt{-g_{\mu\nu}\dot{X}^\mu \dot{X}^\nu}}\sqrt{-g_{\mu\nu}\dot{X}^\mu \dot{X}^\nu}\sqrt{-g_{\mu\nu}\dot{X}^\mu \dot{X}^\nu} - m^2\frac{1}{m}\sqrt{-g_{\mu\nu}\dot{X}^\mu \dot{X}^\nu}\right) \\ &= -m\int d\tau \sqrt{-g_{\mu\nu}\dot{X}^\mu \dot{X}^\nu} = S_0 \ . \end{aligned} \tag{1.8}$$

1.2 Generalisation to the p-brane action

Let us now generalise the action (1.1) to *extended object*, called p-brane. The 0-brane is represented by the *particle* and it sweeps out a world line in spacetime, the 1-brane is the *string* and it sweeps out a two-dimensional world sheet in spacetime. In general, the p-brane is an object which is extended in p spatial dimensions and it sweeps out a $(p+1)$-dimensional world volume in a d-dimensional spacetime with, of course, $p < d$. The figure 1.1 represents this concept for the string.

The generalisation of the action (1.1) to a p-brane naturally takes the form

$$S_p = -T_p \int d\mu_p \ . \tag{1.9}$$

Here T_p is called the "p-brane tension" and replaces the role which mass plays in (1.1), while $d\mu_p$ is the $(p+1)$-dimensional volume element and replaces the role which the line element ds plays in (1.1). The explicit formula for $d\mu_p$ analogous to (1.2) is

$$d\mu_p = \sqrt{-\det(G_{ab})}\, d^{p+1}\sigma \ , \tag{1.10}$$

where G_{ab} is the *induced metric* on world volume, which is the pull-back of the metric $g_{\mu\nu}$

$$G_{ab} = g_{\mu\nu}(X)\partial_a X^\mu \partial_b X^\nu \qquad\qquad a, b = 0, ..., p \ . \tag{1.11}$$

The indices a and b have a precise meaning. The $(p+1)$-dimensional world volume can be parametrised by $p+1$ parameters, which we call in a compact notation $\sigma^a = (\sigma^0, \sigma^1, ..., \sigma^p)$. The parameter σ^0 is *time-like*, so it can be chosen equal to τ, and the parameters σ^i, with $i = 1, ..., p$, are *p-space-like* coordinates. For us $\partial_a X^\mu \equiv \partial X^\mu / \partial\sigma^a$.

Since $d\mu_p$ has units of $(\text{length})^{p+1}$, the dimension of the p-brane tension is

$$[T_p] = (\text{length})^{-p-1} = \frac{\text{mass}}{(\text{length})^p} \ , \tag{1.12}$$

hence T_p is an energy per unit of (spatial) p-volume.

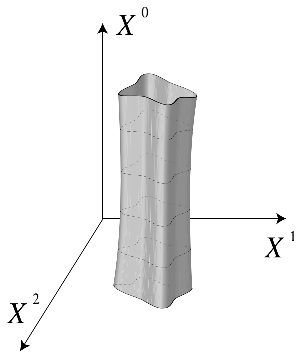

Figure 1.1: The string moving in a tree-dimensional spacetime. The classical trajectory of a string minimizes the area of the world sheet. (Courtesy of [3]).

1.3 The string action

In this section we want to study the particular case of string (or one-brane) propagating in d-dimensional flat Minkowski spacetime ($g_{\mu\nu} = \eta_{\mu\nu}$). The points on the world sheet, sweeped out by the string, are parametrised by two coordinates, $\sigma^0 = \tau$ which is time-like, and $\sigma^1 = \sigma$, which is space-like. If σ is periodic, e.g. $\sigma \in [0, 2\pi]$ with $X(\tau, 0) \cong X(\tau, 2\pi)$, we are describing a *closed string* (right object in figure 1.2). If σ covers a finite interval, but without periodicity condition, we are describing an *open string* (left object in figure 1.2).

The action describing a string propagating in a flat background geometry can be obtained as a special case of (1.9). This action, called the *Nambu-Goto action*, takes the form

$$S_{NG} = -T \int d\sigma d\tau \sqrt{(\dot{X} \cdot X')^2 - \dot{X}^2 X'^2} \ , \tag{1.13}$$

where

$$\dot{X}^\mu = \frac{\partial X^\mu}{\partial \tau} \qquad \text{and} \qquad X'^\mu = \frac{\partial X^\mu}{\partial \sigma} \ , \tag{1.14}$$

and the scalar products are defined by the flat metric, i.e. $A \cdot B = \eta_{\mu\nu} A^\mu B^\nu$. Since the string action has dimension of *area*, the classical string motion must extremizes the world sheet area,

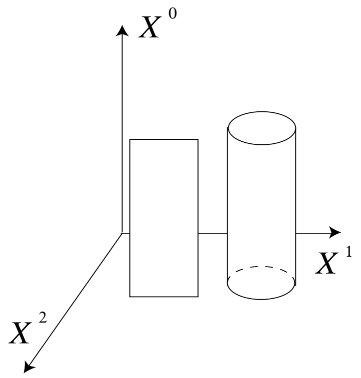

Figure 1.2: Open and closed strings propagating on spacetime. (Courtesy of [3]).

just as classical particle motion makes the length of the world line extremal by moving along a geodesic.

Even thought Nambu-Goto action has a nice physical interpretation as the area of the string world sheet, its quantization in a path integral framework is quite tricky due to the presence of the square root. Also Nambu-Goto action does not work for tensionless strings with $T = 0$. An action that is equivalent to the Nambu-Goto action at the classical level[1] is the *string sigma model action*, called also *Polyakov action.*

The logic followed here is similar to the one presented for the point particle. The string-sigma model action is expressed in terms of an auxiliary world sheet metric $h_{ab}(\tau, \sigma)$, which plays a role analogous to the auxiliary field $e(\tau)$ introduced for the point particle. The string sigma model action is

$$S_\sigma = -\frac{T}{2} \int d^2\sigma \sqrt{-h} h^{ab} \partial_a X \cdot \partial_b X \ , \tag{1.15}$$

where $h = \det(h_{ab})$ and h^{ab} is the inverse of h_{ab}, i.e. $h_{ac}h^{cb} = \delta_a{}^b$, as customary in relativity. Again h_{ab} can be eliminated solving its equations of motion. Also, since the auxiliary field h_{ab} has no kinetic term in the action, its equation of motion implies the vanishing of the world-sheet energy-momentum tensor T_{ab}, i.e.

$$\frac{\delta S_\sigma}{\delta h_{ab}} = 0 \qquad \Longleftrightarrow \qquad T_{ab} = -\frac{2}{T} \frac{1}{\sqrt{-h}} \frac{\delta S_\sigma}{\delta h^{ab}} = 0 \ . \tag{1.16}$$

Hence $T_{ab} = 0$ is a constraint which must be imposted together the equations of motion for X^μ, because it represents the equations of motion for the auxiliary field.

Now we show here the equivalence between the Nambu-Goto action and the Polyakov action. The variation of the Polyakov action respect the auxiliary field h_{ab} reads

$$\delta S_\sigma = -\frac{T}{2} \int d^2\sigma \left(-\frac{1}{2} \sqrt{-h}\, h_{cd} h^{ab} \partial_a X \cdot \partial_b X + \sqrt{-h}\, \partial_c X \cdot \partial_d X \right) \delta h^{cd} \ , \tag{1.17}$$

where we used the fact

$$\delta \sqrt{-h} = -\frac{1}{2} \sqrt{-h}\, h_{ab} \delta h^{ab} \tag{1.18}$$

The equations of motion for h_{ab} are

$$\partial_c X \cdot \partial_d X = \frac{1}{2} h_{cd} h^{ab} \partial_a X \cdot \partial_b X \ . \tag{1.19}$$

[1]Actions equal at the classical level means they give rise to the same equations of motion.

Let us take the square root of minus the determinant of both sides of the above equation. In doing that remind for the string the indices a, b, c, d run from 1 to 2. Hence computing the determinant, the coefficient $\frac{1}{2}h^{ab}\partial_a X \cdot \partial_b X$ on the RHS appears two times. Then we have

$$\sqrt{-\det\left(\partial_c X \cdot \partial_d\right)} = \frac{1}{2}\sqrt{-h}h^{ab}\partial_a X \cdot \partial_b X \ . \tag{1.20}$$

We recognise on the LHS exactly the integrand which appears in the Nambu-Goto action and on the RHS the integrand of the Polyakov action, hence the equivalence is proved.

1.4 Symmetries of the Polyakov action

The Polyakov action for the bosonic string in Minkowski spacetime has three symmetries under the following transformations

- *Poincaré transformations.* These are a global symmetries under which the world sheet fields transform as

$$\delta X^\mu = \omega^\mu{}_\nu X^\nu + a^\mu \qquad \text{and} \qquad \delta h^{ab} = 0 \ , \tag{1.21}$$

  where the constants $\omega^\mu{}_\nu$, with $\omega_{\mu\nu} = -\omega_{\nu\mu}$, describe infinitesimal Lorentz transformations and a^μ describe spacetime translations.

- *Reparametrisation*, also known as *diffeomorphisms.* This is a local symmetry on the world sheet. The string world sheet is parametrized by two coordinates τ and σ, but a change in the parametrisation does not change the action. The fields X^μ transform as world sheet scalars, while h_{ab} transforms as a type $(0, 2)$ tensor

$$\sigma^a \to \sigma'^a = f^a(\sigma) \ , \qquad \Longrightarrow \qquad \begin{cases} X^\mu(\sigma) \to X'^\mu(\sigma') = X^\mu(\sigma) \\ h_{ab}(\sigma) \to h'_{ab}(\sigma') = \frac{\partial\sigma^c}{\partial f^a}\frac{\partial\sigma^d}{\partial f^b}h_{cd}(\sigma) \end{cases} . \tag{1.22}$$

  These local symmetries are called also diffeomorphisms because they correspond to maps of the world sheet into itself. Of course, mathematically f and its inverse f^{-1} are class $\mathcal{C}^\infty$.

- *Weyl transformations.* The action is invariant under the rescaling

$$h_{ab} \to \mathrm{e}^{\phi(\tau,\sigma)}h_{ab} \qquad \text{and} \qquad \delta X^\mu = 0 \ , \tag{1.23}$$

  since $\sqrt{-h} \to \mathrm{e}^{\phi}\sqrt{-h}$ and $h^{ab} \to \mathrm{e}^{-\phi}h^{ab}$ give cancelling factors. This local symmetry is responsible to the fact that the energy-momentum tensor is traceless.

How can we actually think of Weyl invariance? It is not a coordinate change. It represents the invariance of the theory under local change of scale which preserves the angles between all lines, before and after the transformation. Hence a Weyl transformation is a *conformal* transformation.

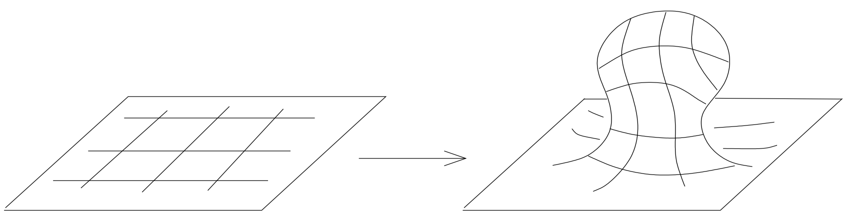

Figure 1.3: An example of Weyl transformation acting on the string world sheet. Angles between lines are preserved (conformal transformation). (Courtesy of [68]).

The two world sheets metric shown in the figure 1.3 are viewed by the Polyakov string as equivalent. This is quite surprising, because theories with these property are extremely rare. Weyl invariance is special to two dimensions, because only in this case the scaling factor coming from $\sqrt{-h}$ cancel that coming from the inverse metric h^{ab}. Theories with Weyl symmetry constrain deeply the kind of interactions that can be added to the action. For example, a potential term like the following one is forbidden

$$\int d^2\sigma\sqrt{-h}V(X) \ . \tag{1.24}$$

Hence a world sheet cosmological constant term on the form

$$\Lambda\int d^2\sigma\sqrt{-h} \ , \tag{1.25}$$

is forbidden.

1.4.1 Gauge fixing

Since the theory possess local symmetries, we can use them to choose a gauge. One possible choice is the *static gauge*, which fixes the longitudinal directions $X^0 = \tau$, $X^1 = \sigma$, while leaving transverse directions X^i, $i = 2, ..., d-1$ free functions of τ and σ.

Another choice of gauge regards the fixing of the auxiliary field components h_{ab}. The auxiliary field has three independent components

$$h_{ab} = \begin{pmatrix} h_{00} & h_{01} \\ h_{10} & h_{11} \end{pmatrix}_{ab} \ , \tag{1.26}$$

where $h_{10} = h_{01}$. Reparametrisation invariance gives us the possibility to choose two functions, f^0 and f^1, in order to fix two of the components of h_{ab}. Again by Weyl invariance we can fix another component of h_{ab} choosing the scaling factor e^{ϕ}. So in the case of the string there is sufficient symmetry to gauge fix h_{ab} completely. Since the signature of h_{ab} is $(-+)$, the auxiliary field can be chosen to be diagonal

$$h_{ab} = \eta_{ab} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}_{ab} \ . \tag{1.27}$$

Actually such a flat world sheet metric is only possible if there is no topological obstruction, which is the case when the world sheet has vanishing Euler characteristic[2] (e.g. cylinder and torus). When a flat world sheet metric is an allowed gauge choice, the string action takes the simple form

$$S = -\frac{T}{2}\int d^2\sigma\, \partial_a X \cdot \partial^a X = \frac{T}{2}\int d^2\sigma(\dot{X}^2 - X'^2) \ , \tag{1.28}$$

which tells us the coordinates X^{μ} can be think as a set of d massless scalar fields which live in the two-dimensional world sheet. In this gauge, which is called *conformal gauge*, the Polyakov action in flat spacetime reduces to a free theory.

The fact we can use Weyl invariance to make any two-dimensional metric flat is an important result. Let us see how this can be done concretely. Consider two metrics related by Weyl transformation, namely $h'_{ab} = e^{2\phi}h_{ab}$. One can check that the Ricci scalars of the two metrics are related by

$$\sqrt{-h'}R' = \sqrt{-h}(R - 2\nabla^2\phi) \ . \tag{1.29}$$

[2]The Euler charateristic is a topological invariant represented by a number that describe the shape of a topological space, regardless of the way it is bent. For example the Euler characteristic of a closed orientable surface can be calculated from its genus g (intuitively, the number of "handles") as $\chi = 2 - 2g$. For a complete definition see [46].

We can pick up a ϕ such that the new metric has vanishing Ricci scalars, $R' = 0$, simply by solving this differential equation for ϕ. However, in two dimensions (but not in higher dimensions) a vanishing Ricci scalar implies a flat metric. In fact, only in two dimensions, symmetry properties of the Riemann curvature tensor and Bianchi identity means it must take the form (see [46])

$$R_{abcd} = \frac{R}{2}\left(h_{ac}h_{bd} - h_{ad}h_{bc}\right) \ . \tag{1.30}$$

Therefore if Ricci scalars vanishes, also Riemann curvature tensors vanishes, which means the manifold is flat.

1.5 Equations of motion and boundary conditions

The equation of motion of X^μ coming from variation of the action (1.28) is the wave equation

$$\partial_a\partial^a X^\mu = 0 \qquad \text{or} \qquad \left(\frac{\partial^2}{\partial\sigma^2} - \frac{\partial^2}{\partial\tau^2}\right)X^\mu = 0 \ . \tag{1.31}$$

Since the metric on the world sheet has been gauge fixed, the vanishing of the energy-momentum tensor, i.e. $T_{ab} = 0$, originating from the equations of motion of the world sheet metric, must now be imposed as an additional constraint condition. In the gauge $h_{ab} = \eta_{ab}$, one find the components of this tensor, which are

$$T_{01} = T_{10} = \dot{X}\cdot X' \qquad \text{and} \qquad T_{00} = T_{11} = \frac{1}{2}(\dot{X}^2 + X'^2) \ . \tag{1.32}$$

Since $T_{00} = T_{11}$ we have the vanishing of the trace of the energy-momentum tensor $Tr(T) = \eta^{ab}T_{ab} = T_{11} - T_{00} = 0$. This is a consequence of Weyl invariance.

In order to give a fully defined variational problem, boundary conditions need to be specified. A string can be either closed or open. For convenience, we will choose the coordinate σ to have the range $0 \leq \sigma \leq 2\pi$. The equations of motion of X^μ are given by the stationary points of the action, which are determined by demanding invariance of the action under the variation

$$X^\mu \to X^\mu + \delta X^\mu \ . \tag{1.33}$$

In addition to the equation of motion (1.31), appears the boundary term

$$T\int d\sigma \left[\dot{X}_\mu\delta X^\mu|_{\tau=\tau_f} - \dot{X}_\mu\delta X^\mu|_{\tau=\tau_i}\right] - T\int d\tau \left[X'_\mu\delta X^\mu|_{\sigma=2\pi} - X'_\mu\delta X^\mu|_{\sigma=0}\right] \ , \tag{1.34}$$

which must be vanish. The first term always appear when using principle of least action. The equations of motion are derived requiring that $\delta X^\mu = 0$ at $\tau = \tau_i$ and $\tau = \tau_f$. However the second term is novel. There are many ways to make it vanishes, which consist in

- *Closed string.* The boundary condition consist in the periodicity of X^μ

$$X^\mu(\tau,\sigma) = X^\mu(\tau,\sigma + 2\pi) \ . \tag{1.35}$$

- *Open string with Neumann boundary conditions.* The component of the momentum normal to the boundary of the world sheet vanishes, i.e.

$$X'_\mu = 0 \qquad \text{at} \qquad \sigma = 0, 2\pi \ . \tag{1.36}$$

  If this choice is made for all μ, Poincaré invariance is not broken. Physically, they mean that no momentum is flowing through the ends of the string.

- *Open string with Dirichlet boundary conditions.* The positions of the two string ends are fixed so that $\delta X^\mu = 0$, and

$$X^\mu|_{\sigma=0} = X_0^\mu \qquad \text{and} \qquad X^\mu|_{\sigma=2\pi} = X_{2\pi}^\mu \ , \tag{1.37}$$

where X_0^μ and $X_{2\pi}^\mu$ are constants and $\mu = 1, ..., d-p-1$ and Neumann boundary conditions are imposed for the other $p+1$ coordinates. Dirichlet boundary conditions break Poincaré invariance.

Concretely, Dirichlet boundary conditions mean

$$X'^a = 0 \qquad \text{for} \quad a = 0, ..., p$$
$$X^I = c^I \qquad \text{for} \quad I = p+1, ..., d-1 \tag{1.38}$$
$$\tag{1.39}$$

with c^I constant. Therefore the end-points of the string must lie in a $(p+1)$-dimensional hypersurface in spacetime such that the $SO(1, d-1)$ Lorents group is broken to

$$SO(1, d-1) \to SO(1, p) \times SO(d-p-1) \ . \tag{1.40}$$

This hypersurface is called a *D-brane*, or better Dp-brane, where D standes Dirichlet and p is the number of spatial dimensions of the brane. The modern interpretation of String Theory is that spacetime is filled of D-branes and strings end on them with Dirichlet boundary conditions.

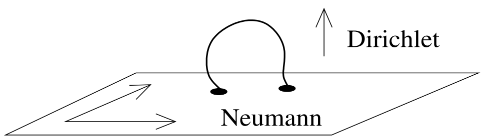

Figure 1.4: Dirichlet boundary condition. The string end-points can move on the Dp-brane with Neumann boundary conditions. (Courtesy of [68]).

The Dp-brane always has Neumann boundary conditions in the X^0 direction. A natural question which may arise is what about to have Dirichlet conditions for X^0. This requirement looks quite wired since the object is now localised at a fixed point in time. But there is a physical interpretation of such an object: it is an *instanton*. This D-instanton is usually think as a D(-1)-brane and it is related to tunneling effects in the quantum theory.

Let us solve the equations of motion for the string (1.31). It is convenient to introduce world sheet *light-cone coordinates*, defined as

$$\sigma^\pm = \tau \pm \sigma \ . \tag{1.41}$$

In these coordinates the equations of motion becomes

$$\partial_+ \partial_- X^\mu = 0 \ . \tag{1.42}$$

In light-cone coordinates the energy-momentum tensor becomes

$$T_{++} = \partial_+ X^\mu \partial_+ X_\mu \qquad T_{+-} = 0 \tag{1.43}$$
$$T_{-+} = 0 \qquad T_{--} = \partial_- X^\mu \partial_- X_\mu \ . \tag{1.44}$$

The most general solution is

$$X^\mu(\tau, \sigma) = X_L^\mu(\sigma^+) + X_R^\mu(\sigma^-) \ , \tag{1.45}$$

where X_L^μ and X_R^μ are arbitrary functions which describe *left*-moving and *right*-moving waves respectively. There are two other conditions we must add. One is the reality of X^μ, the other is the vanishing of energy-momentum tensor, which is

$$(\partial_- X_R)^2 = (\partial_+ X_L)^2 = 0 \ . \tag{1.46}$$

For the *closed string*, which satisfies periodicity boundary conditions $X^\mu(\tau,\sigma) = X^\mu(\tau,\sigma+2\pi)$, the most general solution can be expanded in Fourier modes

$$\begin{aligned} X_R^\mu(\tau-\sigma) &= \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p^\mu(\tau-\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu \mathrm{e}^{-in(\tau-\sigma)} \ , \\ X_L^\mu(\tau+\sigma) &= \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p^\mu(\tau+\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^\mu \mathrm{e}^{-in(\tau+\sigma)} \ , \end{aligned} \tag{1.47}$$

where x^μ is a *center-of-mass* position and p^μ is the total string momentum, describing the free motion of the string center of mass. This can be checked by studying the Nöther currents arising from spacetime translation symmetry $X^\mu \to X^\mu + a^\mu$. The parameter α', called *Regge-slope parameter*, is the fundamental string constant and is related to the string tension T and the string length scale l_s by

$$T = \frac{1}{2\pi\alpha'} \qquad \text{and} \qquad l_s^2 = \alpha' \ . \tag{1.48}$$

The exponential terms in (1.47) represent the string excitation modes. The requirement that X_R^μ and X_L^μ are real implies that x^μ and p^μ are real, while positive and negative modes are conjugate to each other

$$\alpha_{-n}^\mu = (\alpha_n^\mu)^* \qquad \text{and} \qquad \tilde{\alpha}_{-n}^\mu = (\tilde{\alpha}_n^\mu)^* \ . \tag{1.49}$$

The terms linear in σ cancel from the sum $X_R^\mu + X_L^\mu$, therefore closed-string boundary conditions are satisfied.

In order to quantise the theory, one can introduce the canonical momentum conjugate to X^μ, given by

$$P^\mu(\tau,\sigma) = \frac{\partial\mathcal{L}}{\partial \dot{X}_\mu} = T\dot{X}^\mu \ . \tag{1.50}$$

The classical Poisson brackets, at equal time, are

$$\{X^\mu(\tau,\sigma), X^\nu(\tau,\sigma')\} = \{P^\mu(\tau,\sigma), P^\nu(\tau,\sigma')\} = 0 \ , \tag{1.51}$$
$$\{P^\mu(\tau,\sigma), X^\nu(\tau,\sigma')\} = \eta^{\mu\nu}\delta(\sigma-\sigma') \ . \tag{1.52}$$

Inserting on (1.51) the mode expansion for X^μ and $\dot{X}^\mu$ gives the Poisson brackets satisfied by the modes

$$\{\alpha_m^\mu, \alpha_n^\nu\} = \{\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu\} = im\,\eta^{\mu\nu}\delta_{m+n,0} \ , \tag{1.53}$$
$$\{\alpha_m^\mu, \tilde{\alpha}_n^\nu\} = 0 \ . \tag{1.54}$$

1.6 The quantum string

The world sheet theory developed up to now can be quantised. The *canonical quantisation* consists in replacing Poisson brackets by commutators

$$\{\cdot,\cdot\} \to \frac{i}{\hbar}[\cdot,\cdot] \ , \tag{1.55}$$

and in promoting the modes α_n^μ and $\tilde{\alpha}_n^\mu$ as operator. The commutation relations of α_n^μ and $\tilde{\alpha}_n^\mu$ are those of harmonic oscillator creation and annihilation operator. In fact if we define

$$a_n^\mu \equiv \frac{\alpha_n^\mu}{\sqrt{n}} \ , \qquad a_n^{\mu\dagger} \equiv \frac{\alpha_{-n}^\mu}{\sqrt{n}} \qquad \text{with} \qquad n > 0 \ , \tag{1.56}$$

and with the same definition for the modes $\tilde{\alpha}_n^\mu$, then we have the familiar commutation relations

$$[a_m^\mu, a_n^{\nu\dagger}] = \delta_{mn}\eta^{\mu\nu} \qquad [\tilde{a}_m^\mu, \tilde{a}_n^{\nu\dagger}] = \delta_{mn}\eta^{\mu\nu} \qquad [a_m^\mu, \tilde{a}_n^\nu] = ... = 0 \ . \tag{1.57}$$

Therefore each scalar field X^μ, for fixed μ, gives rise to two towers of creation and annihilation, decoupled to each other. There are two towers because we have right-moving modes α_n^μ, and left-moving modes $\tilde{\alpha}_n^\mu$.

After defined commutation relations, one can starts building the Fock space of our theory. One can introduce *vacuum state* of the string $|0\rangle$, defined as

$$a_n^\mu |0\rangle = \tilde{a}_n^\mu |0\rangle = 0 \ , \tag{1.58}$$

which represent the ground state for the harmonic oscillators. Actually this is not the ground state for the whole system, because we have to keep in consideration also the operator x^μ and p^μ, which they give informations about the motion of the center-of-mass. The true ground state is $|0\rangle \otimes \Psi(x)$, where $\Psi(x)$ is a spatial wavefunction, which carries, if we work in momentum space, a quantum number given by p^μ, which is eigenvalue of the momentum operator $\hat{p}^\mu$. Therefore we can write the vacuum state as $|0;p\rangle$, which still obeys (1.58), but now also

$$\hat{p}^\mu |0;p\rangle = p^\mu |0;p\rangle \ . \tag{1.59}$$

A generic state living in the Fock space can be generated acting with any number of creation operators $a_n^{\mu\dagger}$ and $\tilde{a}_n^{\mu\dagger}$

$$(a_1^{\mu_1\dagger})^{n_{\mu_1}}(a_2^{\mu_2\dagger})^{n_{\mu_2}}\cdots(\tilde{a}_1^{\nu_1\dagger})^{n_{\nu_1}}(\tilde{a}_2^{\nu_2\dagger})^{n_{\nu_2}}\cdots|0;p\rangle \ . \tag{1.60}$$

Each state in the Fock space represent a different excited state of the string. Since the excited states of the string have interpretation of different species of particles, and since there are infinite number of way to excite the string ($n \in \mathbb{N}$), the conclusion is that in our particle spectra there are an infinite number of different species of particles. This is different from the point of view of the Standard Model, where the number of species of particles is finite.

It should be emphasized that this is *first quantisation*, and all of these states (including the ground state) are one-particle states. Second quantisation requires *string field theory* [61].

Already at this stage emerge some crick of the theory. For example the states build from the time component operators a_m^0 and $\tilde{a}_m^0$ have negative norm

$$a_m^{0\dagger}|0\rangle \qquad \text{with norm} \qquad \langle 0|\, a_m^0 a_m^0 \,|0\rangle = -1 \ , \tag{1.61}$$

where the ground state is normalised as $\langle 0\,|\,0\rangle = 1$. The reason is due to the minus sign in the commutation relations which is coming from $\eta_{00} = -1$.

Therefore now we have to go deeper inside the quantisation of the string, removing non-physical states. There are two ways available, both compatible with the canonical formalism shown above. These two choices reflect the fact the string action enjoys a gauge symmetries (i.e. reparametrisation and Weyl invariance) and whenever we wish to quantise a gauge theory we are presented with a number of different ways in which we can proceed.

- *Covariant quantisation.* It consists first in quantising the system (as we did before) and then subsequently imposing the constraints that arise from gauge fixing, intended as operator equations acting on the physical states of the system and modding out by residual symmetry. For instance, in QED, this is provided by the Gupta-Bleuler method of quantisation which comes from fixing the Lorentz gauge. In string theory, covariant quatisation consists of treating all fields X^μ as operators and imposing the constraint of vanishing energy-momentum tensor $T_{ab} = 0$ on the states and modding out by residual conformal symmetry.

- *Lightcone quatisation.* This alternative method consists first in solving all of the constraints of the system in order to determine the space of physically distinct classical solutions. Second, one can quantise these physical solutions. Again, in QED this is the way one proceed in Coulomb gauge.

At the end of the day the two methods must agree. Each of them presents different challenges and offers a different viewpoint.

This work is not intended to show in details this part, which can be found in a classical String Theory's book [30], [3], [53].

1.6.1 The string spectrum

In this section we present the outcomes from the analysis of the spectrum of a single, free string. Before do that, let us define what is the *mass* of a string. From the relativistic mass-shell condition, *mass* is defined as

$$M^2 = -p_\mu p^\mu \ , \tag{1.62}$$

where p_μ is the total momentum of the string, which is given by

$$p^\mu = T \int_0^\pi d\sigma \, \dot{X}^\mu(\sigma) \ . \tag{1.63}$$

Writing $\dot{X}^\mu$ in modes expansion, one have for the closed string the *mass formula*

$$M^2 = \frac{2}{\alpha'} \sum_{n=1}^{\infty} (\alpha_{-n} \cdot \alpha_n + \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n) \ . \tag{1.64}$$

The tachyon.
Let us start with the ground state $|0;p\rangle$. If one applies the mass operator (1.64) to the ground state, the result is [3]

$$M^2 = -\frac{1}{\alpha'} \frac{D-2}{6} \ . \tag{1.65}$$

There is an evident problem: this particle has a negative mass-squared. Such particles are called *tachyons*. These objects, in the context of special relativity, are interpreted as particles moving faster then the speed of light. But in the language of quantum field theory there is a better interpretation. Suppose to have a field in spacetime - let us call it $\Phi(x)$ - whose quantisation will give rise to a tachyonic particle. Suppose $V(\Phi)$ is the potential term which enters in the lagrangian density $\mathcal{L}$. The mass-squared of the particle is simply the quadratic term in the lagrangian, i.e.

$$M^2 = \left.\frac{\partial^2 V(\Phi)}{\partial \Phi^2}\right|_{\Phi=0} . \tag{1.66}$$

Therefore the negative squared-mass is telling us we are expanding the system around a vacuum which is situated in a maximum of the potential $V(\Phi)$. Since this vacuum is unstable, one should shift and expand around a minimum of the potential, which is stable. This shift is made explicit in string field theory. The natural question is whether the potential has a good minimum elsewhere. Unfortunately, no one know the answer to this.

The tachyon is a problem just for the bosonic string. If we introduce fermions on the world sheet and study superstring, then tachyons will disappear.

The first excited states.

Once imposed contraint conditions in the quantisation process, one can find the first excited state. It contains $(d-2)^2$ particle states, because the constraint conditions restrict the index μ to runs from 1 to $d-2$ (i.e. $i = 1, ..., d-2$). Therefore the states are given by[3]

$$\left|\Omega^{ij}\right\rangle = \tilde{a}_1^{i\dagger} a_1^{j\dagger} \left|0;p\right\rangle . \tag{1.67}$$

The indices i and j of the operators $\tilde{a}_1^{i\dagger}$ and $a_1^{j\dagger}$ transform in the vector representation of the group $SO(d-2)$, or equivalently, in the *massless* vector representation of the full Lorentz group $SO(1, d-1)$. Therefore the state $\left|\Omega^{ij}\right\rangle$ lies in the tensor product of two massless vectors representations.

One can compute the *mass* of these particle states applying the mass operator (1.64) on (1.67). The result is

$$M^2 = \frac{4}{\alpha'}\left(1 - \frac{d-2}{24}\right) , \tag{1.68}$$

but since the states $\left|\Omega^{ij}\right\rangle$ must be massless, for consistency reason the spacetime dimension must be

$$d = 26 . \tag{1.69}$$

The dimension $d = 26$ is also called *critical dimension* of the bosonic string theory.

Hence the states $\left|\Omega^{ij}\right\rangle$ must transform in the $\mathbf{24} \otimes \mathbf{24}$ representation of $SO(24)$. Obviously this representation is *reducible* because decompose into three *irreducible* representations

$$\mathbf{24} \otimes \mathbf{24} = \mathbf{299} \oplus \mathbf{276} \oplus \mathbf{1} , \tag{1.70}$$

where $\mathbf{299}$ is the *traceless symmetric* second rank tensor, $\mathbf{276}$ is the *antisymmetric* second rank tensor and $\mathbf{1}$ is the *singlet* (i.e. trace), and are described by the fields

$$g_{\mu\nu}(X) , \qquad B_{\mu\nu}(X) , \qquad \phi(X) . \tag{1.71}$$

The quantum of these fields give rise to particles[4]. The particle in the symmetric tracless representation of $SO(24)$ is particularly interesting. It is a massless spin 2 particle. There are general arguments, due to originally to Feynman and Weinberg, that *any* theory of interacting massless spin 2 particles must be equivalent to general relativity. We should therefore identify the field $g_{\mu\nu}(X)$ with the metric of spacetime. The antisymmetric representation correspond to the so called "Kalb-Ramond field" or, in the language of differential geometry, the "2-form". The real scalar field is call the *dilaton*. The remarkable fact is that these three massless fields are

[3] In the lightcone quantisation one finds the so called *level matching* condition which for the first excited state tells us if we act with a creation operator, then we have to act also with an annihilation operator.

[4] By Wigner's theorem, particles are in correspondence 1 to 1 with irreducible representations of the little group $SO(d-1)$ in the massive case, and $SO(d-2)$ in the massless case.

common to all string theories. The set $(g_{\mu\nu}(X), B_{\mu\nu}(X), \phi(X))$ is also called (massless) *background*. The flat spacetime we have considered so far is represented by the background $(\eta_{\mu\nu}, 0, 0)$.

Higher excited states.
Let us treat first the second excited states, then the higher excited states have similar behaviour. In the right-moving sector, appear two different states $a_1^i a_1^j |0\rangle$ and $a_2^i |0\rangle$. The same is true for the left-moving sector. Therefore the second excited states can be written as[5]

$$(a_1^i a_1^j \oplus a_2^i) \otimes (\tilde{a}_1^i \tilde{a}_1^j \oplus \tilde{a}_2^i) |0; p\rangle \ . \tag{1.72}$$

Again, one can compute the mass of these states and finds

$$M^2 = \frac{4}{\alpha'} \ . \tag{1.73}$$

The general feature is that all these excited states will have mass close to the Planck scale, so there is a low probability to found them in a particle physics experiment. But they play an important role when one come to discuss scattering amplitudes. It is thanks to this infinite tower of states that ultra-violet behaviour of gravity is suppressed.

1.7 Low energy effective action

So far, we have only discussed strings propagating in flat spacetime. In this section we want to study strings propagating in different backgrounds $(g_{\mu\nu}(X), B_{\mu\nu}(X), \phi(X))$.

There is a natural generalisation of the Polyakov action. In order to preserve symmetries and renormalisability, it is [3]

$$S_{gb} = S_g + S_B + S_\phi \ , \tag{1.74}$$

where

$$\begin{aligned}
S_g &= -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-h}\, h^{ab} g_{\mu\nu}(X) \partial_a X^\mu \partial_b X^\nu \\
S_B &= -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-h}\, \epsilon^{ab} B_{\mu\nu}(X) \partial_a X^\mu \partial_b X^\nu \\
S_\phi &= -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-h}\, \alpha' R(h) \phi(X)
\end{aligned}$$

The first term involving $g_{\mu\nu}$ is the natural generalisation of the Polyakov action for flat Minkowski spacetime to curved spacetime. The second term which include $B_{\mu\nu}$ represent how strings couple with the antisymmetric field $B_{\mu\nu}$. The tensor ϵ^{ab} is the Levi-Civita tensor in 2-dimensions normalized such that $\sqrt{-h}\epsilon^{12} = +1$. The third term represent how strings couple with the dilaton field $\phi(X)$. In this term appears $R(h)$, which is the 2-dimensional Ricci scalar of the world sheet (in some sense the string contributions enter here). The Regge-slope parameter α' appears for dimensional reason.

Let us look the properties which possess the action (1.74)

- *Spacetime diffeomorphisms invariant.* Since all spacetime indices are contracted together the action S_{gb} is invariant under spacetime general coordinate transformations $X^\mu \to X'^\mu$, provided $g_{\mu\nu}$ and $B_{\mu\nu}$ transform as tensors and ϕ as scalar, i.e.

$$g'_{\mu\nu}(X') = \frac{\partial X^\alpha}{\partial X'^\mu} \frac{\partial X^\beta}{\partial X'^\nu} g_{\alpha\beta}(X) \ , \qquad B'_{\mu\nu}(X') = \frac{\partial X^\alpha}{\partial X'^\mu} \frac{\partial X^\beta}{\partial X'^\nu} B_{\alpha\beta}(X) \ , \qquad \phi'(X') = \phi(X) \ . \tag{1.75}$$

[5] Again, by level matching conditions.

- *Gauge invariance.* The $B_{\mu\nu}(X)$ field is invariant under gauge transformation

$$B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu \ . \tag{1.76}$$

  This transformation causes the integrand of S_{gn} to vary by a total derivative. In electromagnetism, one can construct the gauge invariant electric and magnetic fields which are packaged in the 2-form field strength $F = dA$. Similarly, for $B_{\mu\nu}$, the gauge invariant field strength $H = dB$ is a 3-form

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu} \ . \tag{1.77}$$

  This 3-form h is sometimes known as *torsion*, because it plays the same role of torsion in general relativity, providing the antisymmetric component to the affine connection.

- *Reparametrisation invariance.* This symmetry enjoyed by the Polyakov action, is still preserved by S_{gb}.

- *Weyl transformation.* Under the rescaling

$$h_{ab} \to \mathrm{e}^{\phi(\tau,\sigma)}\, h_{ab} \ , \tag{1.78}$$

  the term which contains $B_{\mu\nu}$ is invariant because ϵ^{ab} is a *tensor*, while $\sqrt{-h}\epsilon^{ab} \equiv \varepsilon^{ab}$ is a *density tensor* (of weight $+1$), such that $\varepsilon^{12} = +1$. Since the density tensor ε^{ab} is h_{ab} independent, the B-term is Weyl invariant. However the dilaton coupling $R(h)\phi$ is not Weyl invariant. This is intentional: the dilaton coupling is introduced precisely to complete the cancellation of Weyl-symmetry anomalies in the perturbative α' expansion.

Let us analyse better what does α' really mean and what does perturbation expansion mean.

1.7.1 The meaning of α'

Recall that in the conformal gauge, the Polyakov action in flat spacetime reduces to free theory. This fact was extremely useful because allows us to compute exactly the spectrum of the theory. But in a curved background, it is no longer the case. Here we want to analyse these solutions. For convenience we consider as a curved background $(g_{\mu\nu}, 0, 0)$, i.e. fixing to zero the 2-form B and the dilaton ϕ. In conformal gauge, the action S_{gb} for the background specified above, is

$$S_{gb} = \frac{1}{4\pi\alpha'} \int d^2\sigma g_{\mu\nu}(X) \partial_a X^\mu \partial^a X^\nu \ . \tag{1.79}$$

Let us now expand the string coordinate $X^\mu(\tau, \sigma)$ around a classical solution achieved in flat background. We choose to pick up the trivial solution, i.e. the string sitting at the constant point $\bar{x}^\mu$,

$$X^\mu(\tau,\sigma) = \bar{x}^\mu + \sqrt{\alpha'} Y^\mu(\tau,\sigma) \ . \tag{1.80}$$

Here Y^μ are the dynamical fluctuations about the point which we assume to be small. The factor $\sqrt{\alpha'}$ is there for dimensional reasons. Since $[\alpha'] = -2$ and $[X] = -1$, we have $[Y] = 0$. Hence Y is small if $Y \ll 1$. Expanding the Lagrangian which gives the action (1.79)

$$g_{\mu\nu}(X)\partial X^\mu \partial X^\nu = \alpha' \left[g_{\mu\nu}(\bar{x}) + \sqrt{\alpha'} \partial_\rho g_{\mu\nu}(\bar{x}) Y^\rho + \frac{\alpha'}{2} \partial_\rho \partial_\sigma g_{\mu\nu}(\bar{x}) Y^\rho Y^\sigma + \mathcal{O}(\alpha'^{3/2}) \right] \partial Y^\mu \partial Y^\nu \ . \tag{1.81}$$

What we get is the Lagrangian of d fields interacting between each other. Each of the coefficients $\partial\partial ... g_{\mu\nu}$ in the Taylor expansion are coupling constants for the interactions of the fluctuations

Y^μ. The theory has an infinite number of coupling constants and they are nicely encodes by the function $g_{\mu\nu}(X)$.

Now, a natural question one may ask, is when this field theory is weakly coupled. Obviously this requires the whole infinite set of coupling constants to be small. Let us proceed now heuristically. Suppose that the target space has characteristic radius of curvature r_c, meaning that

$$\frac{\partial g}{\partial X} \sim \frac{1}{r_c} \ . \tag{1.82}$$

Since the radius of curvature is a length scale, $[r_c] = -1$, if we look to the expansion of the Lagrangian (1.81), we see that the effective dimensionless coupling is given by

$$\frac{\sqrt{\alpha'}}{r_c} \ . \tag{1.83}$$

Now we have a physical interpretation when the perturbation theory is valid. One can approach perturbatively the action (1.79) if and only if the spacetime metric vary only on scales much greater than $\sqrt{\alpha'}$. If there are regions of spacetime where the radius of curvature becomes comparable to the string length scale, $r_c \sim \sqrt{\alpha'}$, then the fields Y^μ are strongly coupled and one has to develop new methods to solve it.

If one study the Feynman graphs coming from the interacting theory, one can discover that the parameter α' plays also the role of the *loop-expansion* parameter.

1.7.2 Beta function

Classically, the theory defined by (1.79) is conformally invariant. But no one assures us this is necessarily true in the quantum theory. In fact. in order to regulate divergences which appears after quantisation, one introduces a UV cut-off and, typically, after renormalisation, physical quantities depend on the scale of a given process, μ. After this, the theory is no longer conformally invariant. There are plenty of theories which classically possess scale invariance which is broken quantum mechanically. The most famous of these is Yang-Mills.

In string theory conformal invariance is a gauge symmetry, which we want to preserve jealously. The goal now is to understand in which circumstance the action (1.79) retains conformal invariance at quantum level.

In quantum field theory, the object which describe how coupling depend on a scale μ is called β-function. Since our couplings involves functions, we should really talk about a β-functional. Heuristically

$$\beta_{\mu\nu}(g) \sim \mu \frac{\partial g_{\mu\nu}(X;\mu)}{\partial \mu} \ . \tag{1.84}$$

The quantum theory will be conformally invariant only if

$$\beta_{\mu\nu}(g) = 0 \ , \tag{1.85}$$

which means that the coupling constants expressed in terms of g do not depend by the energy scale μ.

Reintroduce now the fields B and ϕ in the action. Again, one can expand around classical solution and now we have three types of coupling constants, which are packaged into the functions $g_{\mu\nu}(X)$, $B_{\mu\nu}(X)$ and $\phi(X)$. Therefore the beta functions now are

$$\beta_{\mu\nu}(g) \sim \mu \frac{\partial g_{\mu\nu}(X;\mu)}{\partial \mu} \ , \qquad \beta_{\mu\nu}(B) \sim \mu \frac{\partial B_{\mu\nu}(X;\mu)}{\partial \mu} \ , \qquad \beta_{\mu\nu}(\phi) \sim \mu \frac{\partial \phi_{\mu\nu}(X;\mu)}{\partial \mu} \ . \tag{1.86}$$

Our goal now is to explain how the classical lack of Weyl invariance in the dilaton coupling can be compensated by a one-loop contribution arising from the coupling to $g_{\mu\nu}$ and $B_{\mu\nu}$.

To see this explicitly, the best way is to look at the breakdown of Weyl invariance as seen as traceless condition on energy-momentum tensor T_{ab}. If one compute the beta functions for the two-dimensional theory (1.74), one can notice that the energy-momentum tensor receives three different kinds of contribution from the three different spacetime fields. the trace of energy-momentum tensor is

$$T^a{}_a = -\frac{1}{2\alpha'}\beta_{\mu\nu}(g)h^{ab}\partial_a X^\mu \partial_b X^\nu + \frac{1}{2\alpha'}\beta_{\mu\nu}(B)\epsilon^{ab}\partial_a X^\mu \partial_b X^\nu - \frac{1}{2}\beta(\phi)R(h)\ . \tag{1.87}$$

The one-loop beta functions can be computed [13], and the result is

$$\begin{aligned}
\beta_{\mu\nu}(g) &= \alpha'\mathcal{R}_{\mu\nu} + 2\alpha'\nabla_\mu\nabla_\nu\phi - \frac{\alpha'}{4}H_{\mu\lambda\kappa}H_\nu{}^{\lambda\kappa} \\
\beta_{\mu\nu}(B) &= -\frac{\alpha'}{2}\nabla^\lambda H_{\lambda\mu\nu} + \alpha'\nabla^\lambda\phi H_{\lambda\mu\nu} \\
\beta(\phi) &= -\frac{\alpha'}{2}\nabla^2\phi + \alpha'\nabla_\mu\phi\nabla^\mu\phi - \frac{\alpha'}{24}H_{\mu\nu\lambda}H^{\mu\nu\lambda}
\end{aligned}$$

A consistent background of string theory must preserve Weyl invariance, which requires

$$\beta_{\mu\nu}(g) = \beta_{\mu\nu}(B) = \beta(\phi) = 0\ . \tag{1.88}$$

Since $\beta_{\mu\nu}(g)$ is symmetric in (μ,ν) and $\beta_{\mu\nu}(B)$ is antisymmetric in (μ,ν), then the conditions (1.88) represent d^2 equations, which match the d.o.f. coming from the background $(g_{\mu\nu}, B_{\mu\nu}, \phi)$. This means if we did not insert the dilaton coupling in the action S_{gb}, we would not be able to assure Weyl invariance at one-loop level.

1.7.3 Effective field equations

The equations (1.88) can be viewed as the equations of motion for the background in which the string propagates. Now we can invert the perspective: we look for a $d = 26$ dimensional spacetime action which reproduces these beta functions equations of motion. This is the *low-energy effective action* of the bosonic string

$$S_{eff} = \frac{1}{2\kappa^2}\int d^{26}X\,\sqrt{-g}\,\mathrm{e}^{-2\phi}\left(\mathcal{R} - \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda} + 4\partial_\mu\phi\partial^\mu\phi\right)\ , \tag{1.89}$$

where κ is the gravitational coupling constant.

Varying the action (1.89) with respect to the three fields, can be shown to yield the beta functions,

$$\begin{aligned}
\delta S_{eff} = \frac{1}{2\kappa^2\alpha'}\int d^{26}X\,\sqrt{-g}\,\mathrm{e}^{-2\phi}[&\delta g_{\mu\nu}\beta^{\mu\nu}(g) - \delta B_{\mu\nu}\beta^{\mu\nu}(B) \\
&- (2\delta\phi + \frac{1}{2}g^{\mu\nu}\delta g_{\mu\nu})(\beta^\lambda{}_\lambda(g) - 4\beta(\phi))]\ .
\end{aligned} \tag{1.90}$$

Action (1.89) governs low-energy dynamic of the spacetime fields. The appellative "low-energy" refers to the fact that we have truncated the beta functions to one-loop expansion.

Something remarkable has happened here. Remind we started looking how a single string moves in flat spacetime. Then we introduced a generic background and we tried to figure out how the string moves in this new spacetime configuration. Now, at the end of this procedure, we found that the background fields fluctuate. This represents how the tiny string governs the way the whole universe moves.

The action (1.89) actually does not looks like the familiar Einstein-Hilbert action, because of the strange factor $e^{-2\phi}$ sitting out front. However it can be eliminated by a field redefinition. In d dimensions, we define a new metric $\tilde{g}_{\mu\nu}$, called Einstein-frame metric, which is related to the previous string-frame metric $g_{\mu\nu}$ by

$$\tilde{g}_{\mu\nu}(X) = e^{-4\phi/(d-2)} g_{\mu\nu}(X) \ . \tag{1.91}$$

Note that this is not a changed of metric due to a changed of coordinates. It is merely a redefinition of the fields, which one can always make in any field theory. At the and of the day, restricting to $d = 26$, the action (1.89) becomes

$$S_{eff} = \frac{1}{2\kappa^2} \int d^{26}X \sqrt{-\tilde{g}} \left(\tilde{\mathcal{R}} - \frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda} - \frac{1}{6} \partial_\mu \phi \partial^\mu \phi \right) \ . \tag{1.92}$$

Notice that the kinetic terms for ϕ are now with the right sign. Also for the dilaton there is no potential term, which means there is nothing that dynamically sets its expectation value in the bosonic string. However the backgrounds of superstring develop a potential term for the dilaton, fixing the string coupling constant.

Now we can recognise that the gravitational part of the action (1.92) takes the standard Einstein-Hilbert form. Hence now, we can give a meaning to the gravitational coupling, which must be related to the Newton's constant G in 26-dimensions, by

$$\kappa^2 = 8\pi G \ . \tag{1.93}$$

The possibility of defining two metrics really arises because we have a *massless* scalar field ϕ in the game. Whenever such a field exists, we can always measure distances in different ways by including ϕ in our ruler. In another perspective, massless scalar fields give rise a long range attractive forces which can mix with gravitational forces and violate the principle of equivalence. Therefore, if we want to connect with Nature, we need to find a way to make ϕ massive. Such a mechanisms exist in the context of superstring.

1.8 Superstring at a Glance

In this section we will not provide details about superstring, but rather we will enunciate its general features and what is different from bosonic string theory. Further material can be found in [53], [30].

First of all, the key difference between the bosonic string and the superstring is the addition of fermionic modes on its world sheet. Bosonic modes satisfy commutation relations, while for fermionic modes we have *anti*-commutation relations. The resulting world sheet theory is supersymmetric[6]. In fact the name *superstring* comes from "supersymmetry" plus "(bosonic) string". One can again follow the procedure discussed up to now for the bosonic string. In particular after quantisation, one find the following features of superstring

- The critical dimension of the superstring is $d = 10$.
- There is no tachyon in the spectrum.
- Appears again the massless bosonic fields $g_{\mu\nu}(X)$, $B_{\mu\nu}(X)$ and $\phi(X)$.

These massless bosonic fields are all part of the spectrum of the superstring. In this context, $B_{\mu\nu}$ is also called Neveu-Schwarz 2-form. In superstring appears also massless spacetime fermions, as well as further massless bosonic fields. The exact form of the extra bosonic fields depends on which superstring theory we consider.

[6] At least this happens in the so-called Neveu-Schwarz-Ramond formalism

While the bosonic string is unique, there a a number of choices that one can make when adding fermios to the world sheet. This give rise to a different classes of superstring theories. Later developments reveal that they are actually all part of the same framework, which commonly goes by the name of *M-theory*. What makes the types of superstring theories different between each other is the way one choose to add fermions on the world sheet. We can add fermions in both left-moving and right-moving sectors of the string, or whether we choose the fermions to move only in one direction (which usually taken to be right-moving). This procedure gives rise to two different classes of string theory.

(I) Type II strings have both left and right-moving world sheet fermions. This theory contains 32 supercharges, hence it is maximally supersymmetric. This correspond to a spacetime theory in $d = 10$ dimensions, with $\mathcal{N} = 2$ supersymmetry.

(II) Heterotic strings have just right-moving fermions. The resulting spacetime theory has $\mathcal{N} = 1$ supersymmetry in $d = 10$, which corresponds to 16 supercharges.

Actually each of these two families divides into other two choices. This leaves us with four superstring theories[7]. These four theories are called Type IIA, Type IIB, Heterotic $SO(32)$ and Heterotic $E_8 \times E_8$. Each of them contains the fields $g_{\mu\nu}$, $B_{\mu\nu}$ and ϕ that appear in the bosonic string, together with a number of extra fields. The dynamics of these fields is described by the low-energy effective action, which naturally splits up into three pieces

$$S_{eff}^{superstring} = S_1 + S_2 + S_{fermi} \; . \tag{1.94}$$

Here S_{fermi} is the *fermionic sector* and describes how fermions interact. We will not describe them. $S_1 + S_2$ describes the *bosonic sector*, and now we will talk briefly about it.

S_1 is essentially the same for all the four theories listed above and is given by the action we found for the bosonic string in the string frame (1.89). S_1 represents the *Neveu-Schwarz sector*, NSNS. In this part we will use form notation, and denote $H_{\mu\nu\rho}$ simply as H_3, where the subscript tells us the degree of the form. Then the action reads

$$S_1 = \frac{1}{2\kappa^2}\int d^{10}X\sqrt{-g}\mathrm{e}^{-2\phi}\left(\mathcal{R} - \frac{1}{2}|\tilde{H}_3|^2 + 4\partial_\mu\phi\partial^\mu\phi\right) \; , \tag{1.95}$$

where $\tilde{H}_3$ is not quite the same as the original H_3, but it depends by which type of theory we are choosing.

The second part of the bosonic action, which is S_2, describes the dynamics of the extra bosonic fields which are specific to each different theory. For Type II theories, S_2 represents the *Ramond* sector RR. Now we go through the four different theories and explain the S_2 term.

- **Type IIA**: For this theory $\tilde{H}_3 = H_3$. The extra bosonic fields are: the 1-form C_1 and the 3-form C_3. The action for these extra bosonic fields is

$$S_2 = -\frac{1}{4\kappa^2}\int d^{10}X\left[\sqrt{-g}\left(|F_2|^2 + |\tilde{F}_4|^2\right) + B_2 \wedge F_4 \wedge F_4\right] \; . \tag{1.96}$$

The field strengths are given by

$$F_2 = dC_1 \; , \qquad F_4 = dC_3 \; , \qquad \tilde{F}_4 = F_4 - C_1 \wedge H_3 \; . \tag{1.97}$$

Notice that the final term in the action does not depend by the metric. It is commonly called *Chern-Simons* term.

Type IIA is a *non-chiral* theory because each kind of particle appears within both types of chirality.

[7] Actually there is another one which we will describe later. It is the Type I theory.

- **Type IIB**: Again, $\tilde{H}_3 = H_3$. The extra bosonic fields are: a scalar C_0, a 2-form C_2 and a 4-form C_4. The action is given by

$$S_2 = -\frac{1}{4\kappa^2}\int d^{10}X\left[\sqrt{-g}\left(|F_1|^2 + |\tilde{F}_3|^2 + \frac{1}{2}|\tilde{F}_5|^2\right) + C_4 \wedge H_3 \wedge F_3\right] . \tag{1.98}$$

  The field strengths are given by

$$F_1 = dC_0 \ , \qquad F_3 = dC_2 \ , \qquad F_5 = dC_4 \ , \tag{1.99}$$

$$\tilde{F}_3 = F_3 - C_0 \wedge H_3 \ , \qquad \tilde{F}_5 = F_5 - \frac{1}{2}C_2 \wedge H_3 + \frac{1}{2}B_2 \wedge F_3 \ . \tag{1.100}$$

  The low-energy dynamics described by the action (1.98) is not complete. We have to add the constraint of self-duality of $\tilde{F}_5$, i.e.

$$\tilde{F}_5 = \star\tilde{F}_5 \ . \tag{1.101}$$

  Type IIB is a *chiral* theory because each kind of particle appears within only one type of chirality.

- **Heterotic** $SO(32)$: The heterotic strings do not have Ramond-Ramond fields. Heterotic strings $SO(32)$ has an non-abelian gauge field strength F_2, with gauge group $SO(32)$. The dynamics of this field is governed simply by the Yang-Mills action in ten dimensions

$$S_2 = \frac{\alpha'}{8\kappa^2}\int d^{10}X\,\sqrt{-g}\,Tr|F_2|^2 \ . \tag{1.102}$$

  Here $\tilde{H}_3$ is defined as $\tilde{H}_3 = dB_2 - \alpha'\omega_3/4$, where ω_3 is the Chern-Simons 3-forms constructed from the non-abelian gauge field A_1

$$\omega_3 = Tr\left(A_1 \wedge dA_1 + \frac{2}{3}A_1 \wedge A_1 \wedge A_1\right) \ . \tag{1.103}$$

  This strange looking combination is due to the most intricate aspects of the heterotic string, known as anomaly cancellation.

- **Heterotic** $E_8 \times E_8$: This theory has the same description of the Heterotic $SO(32)$, with the only difference about the gauge group, which now is $E_8 \times E_8$.

The actions that we have written down probably look a little arbitrary. But they have very important properties. In particular the full action $S_{eff}^{superstring}$ for Type II theories is invariant under $\mathcal{N} = 2$ spacetime supersymmetry, and it is unique! This is the only action (module equivalent actions) which enjoys this property. This fact motivates the choice done before.

I want to clarify a bit more about chirality: in both Type IIA and Type IIB theories we have two sets of chiral fermions. But for Type IIA the two sets have different chirality (so overall the theory is chirality symmetric) while for Type IIB both sets have the same chirality. Of the bosons, only the self-dual 5-form $\tilde{F}_5$ in Type IIB is chiral.

Actually there are five superstring theories in ten dimensions, and not only four. The remaining theory is called Type I and includes open strings moving in flat ten dimensional space as well as closed strings.Type I superstring theory can also be understood as arising from projection of Type IIB superstring theory. Type IIB superstrings are oriented, and their world sheets are orientable. The world-sheet parity transformation

$$\Omega : \sigma \to -\sigma \ , \tag{1.104}$$

reverses the orientation of the world sheet. World-sheet parity exchanges the left- and right-modes of the world-sheet fields (which are bosons X^μ and also fermions ψ^μ). This $\mathbb{Z}_2$ transformation is a *symmetry* of the type IIB theory but not of the type IIA theory, because type IIB contains both left- and right- moving fermions carry the same chirality. When we gauge this $\mathbb{Z}_2$ symmetry, the Type I theory emerges[8]. The Type I closed-string spectrum is obtained by keeping the states that are *even* under the world-sheet parity transformation, and eliminating the one that are *odd*. The states in the NSNS sector of Type IIB are given by the tensor product of two vectors. Hence only states which are symmetric in the two vectors survive. These are the dilaton ϕ and the graviton $g_{\mu\nu}$, while the antisymmetric tensor $B_{\mu\nu}$ is eliminated.

Finally, the low-energy effective action of each superstring theory gives rise to a low-energy effective theory, which goes under the name of *Supergravity* theory. Despite Supergravity fails as a theory of Quantum Gravity, it has retained its importance for investigating superstring theories at low energies.

1.9 Kaluza-Klein theory

We have seen that for consistency reasons, bosonic string theory lives in $d = 26$ dimensions, and superstring theories in $d = 10$ (or eleven for M-theory). But we do not. Or better, we *are able* to observe only three macroscopic large dimensions. Hence we should develop a strategy which allows us to restore a connection with the $d = 4$ spacetime, where General Relativity and Standard Model live. The technique we are going to talk about goes under the name of *Kaluza-Klein theory*, and consists in two steps: a *compactification* followed by a *dimensional reduction*.

1.9.1 Compactifications

Since string theory is a theory which includes gravity, and since gravity curves spacetime, there are no obstructions to stop extra dimensions of the universe from curling up. Therefore the idea is to make compact extra dimensions, just replacing the d dimensional spacetime $\mathbb{R}^{1,d-1}$ with the product of four-dimensional spacetime $\mathbb{R}^{1,3}$ and a compact $(d-4)$-dimensional manifold $\mathcal{C}_{d-4}$, i.e.

$$\mathbb{R}^{1,d-1} \quad \longrightarrow \quad \mathbb{R}^{1,3} \times \mathcal{C}_{d-4} \; . \tag{1.105}$$

Consider the compactified spacetime (1.105) and assign coordinates (x, y) to the whole target space, where x are coordinates in $\mathbb{R}^{1,3}$, and y are coordinates in $\mathcal{C}_{d-4}$. The d-dimensional metric can be decomposed according to

$$ds^2(\mathbb{R}^{1,d-1}) = g_{\mu\nu} dx^\mu dx^\nu + ds^2(\mathcal{C}_{d-4}) \; , \tag{1.106}$$

which means the metric is a direct product of metrics on the two subspaces. The form of $ds^2(\mathcal{C}_{d-4})$ can be constrained. Indeed the properties which $\mathcal{C}_{d-4}$ must satisfy are imposed by the dynamics. In fact the spacetime equations of motion coming from the low-energy effective action are

$$\beta_{\mu\nu}(g) = \beta_{\mu\nu}(B) = \beta(\phi) = 0 \; , \tag{1.107}$$

which have many solutions! This is part of the story of vacuum selection in string theory. In fact there is an open question which regards what solution, if any, describes the world we see around us. Does this putative solution have other special properties, or is it a random choice from many

[8] Gauging a *discrete* symmetry, such as $\mathbb{Z}_2$, could sound heretic, but actually is a well defined operation. The curios reader can see [50]

possibility? There is no answer to this question and currently there is no known principle which uniquely selects a solution which looks like our world.

Here suppose to pick up a simple solution of the low-energy effective action which solves (1.107). We set $H_{\mu\nu\rho} = 0$ (i.e. set B to a closed 2-form) and the dilaton ϕ to a constant value. Then our equations of motion consist simply in searching for a Ricci flat backgrounds obeying $\mathcal{R}_{\mu\nu} = 0$. Therefore $\mathcal{C}_{d-4}$ must be a compact $(d-4)$-dimensional Ricci-flat manifold.

For example, in bosonic string theory in $d = 26$ dimensions, the simplest such a manifold is just $\mathcal{C}_{22} = T^{22}$, the torus endowed with a flat metric. But there are also a whole host of other possibilities [27]. Compact, complex manifolds that admit such Ricci-flat metrics are called *Calabi-Yau* manifolds[9].

The idea consists in the fact that the compact manifold $\mathcal{C}_{d-4}$ can be enclosed inside a $(d-4)$-dimensional sphere of radius L, which goes under the name of *characteristic length scale.* If the characteristic length scale L is small enough then the presence of these extra dimensions would not have been observed in experiment. The Standard Model has been accurately tested to energies of a TeV, or so. Hence if we suppose the validity of the Standard Model *also* on $\mathcal{C}_{(d-4)}$, then, by uncertainty principle of Quantum Mechanics, the characteristic length scale must be

$$L \lesssim (\mathrm{TeV})^{-1} \sim 10^{-16}\mathrm{cm}\ . \tag{1.108}$$

1.9.2 Dimensional reduction

After we have compactified the theory, we have to squeeze to zero the compact manifold $\mathcal{C}_{d-4}$ by the limit $L \to 0$. This procedure is called *dimensional reduction.*

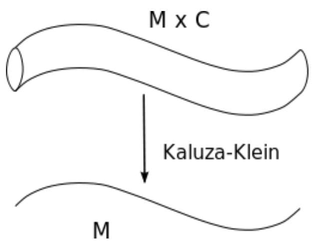

Figure 1.5: Compactification on a circle followed by a dimensional reduction. (Courtesy of Wikipedia).

Compactification of the theory gives rise to a remarkable fact. The fields defined in the non-compactified theory behave different after a compactification. A massless field defined in the original theory, after a compactification gives rise to an infinite tower of *massive* fields [66]. Consider a massless complex scalar field $\Phi(x)$ living in a d-dimensional Minkowski spacetime. In the Kaluza-Klein theory, compactification is performed on a circle of radius L,

$$M_d \quad \longrightarrow \quad M_{d-1} \times S^1\ . \tag{1.109}$$

Assign coordinates $x^0, x^1, ..., x^{d-2}$ on the $(d-1)$ Minkowski spacetime M_{d-1}, and y on the circle S^1, with range $0 \leq y \leq 2\pi L$. The complex scalar field $\Phi(x)$ must obey the Klein-Gordon equation

$$\Box_d\, \Phi(x^0, ..., x^{d-2}, y) = 0\ . \tag{1.110}$$

Expand now the complex scalar field in Fourier transform only for the variable y

$$\Phi(x^0, ..., x^{d-2}, y) = \int_{-\infty}^{+\infty} \frac{dp}{2\pi}\, \tilde{\Phi}(x^0, ..., x^{d-2}, p)\, \mathrm{e}^{ipy}\ . \tag{1.111}$$

[9]Strictly speaking, Calabi-Yau manifolds are complex manifolds with vanishing first Chern class. There is a theorem, due to Yau, which guaranteed the existence of a *unique* Ricci-flat metric on these spaces.

The field $\Phi(x)$ must satisfies periodicity condition on the circle

$$\Phi(x^0, ..., x^{d-2}, y) = \Phi(x^0, ..., x^{d-2}, y + 2\pi L) \ , \tag{1.112}$$

and this equation, if insert back into (1.111), gives us the "quantisation" of the momentum along y direction, which now is not a continue variable anymore, but is discrete and labelled by an integer k

$$p_n = \frac{k}{L} \qquad\qquad k \in \mathbb{Z} \ . \tag{1.113}$$

Since now the momentum is a discrete variable, the Fourier transform becomes a Fourier expansion replacing the integral by the sum

$$\int_{-\infty}^{+\infty} \frac{dp}{2\pi} \quad \longrightarrow \quad \sum_{k\in\mathbb{Z}} \ . \tag{1.114}$$

This gives us the discrete expansion of the field $\Phi(x)$

$$\Phi(x^0, ..., x^{d-2}, y) = \sum_{k\in\mathbb{Z}} \Phi_k(x^0, ..., x^{d-2})\, \mathrm{e}^{i\frac{ky}{L}} \ . \tag{1.115}$$

Insert now the discrete expansion (1.115) back into (1.110) and we get

$$\sum_{k\in\mathbb{Z}} \left[\Box_{d-1} - \left(\frac{k}{L}\right)^2\right] \Phi_k(x^0, ..., x^{d-2}) = 0 \ . \tag{1.116}$$

This equation is satisfies if the spacetime function associated to the k-th Fourier mode, namely Φ_k, satisfies

$$\left[\Box_{d-1} - \left(\frac{k}{L}\right)^2\right] \Phi_k(x^0, ..., x^{d-2}) = 0 \ , \tag{1.117}$$

which represents the equation of motion of a scalar (complex) field of mass $m_k^2 = (k/L)^2$. Therefore the spectrum of the theory, as viewed in d-dimensional Minkowski spacetime, contains an infinite tower of massive scalars.

A similar procedure can be repeated in the cases of massless spinor field and vector field in M_d. In the case of massless vector field the procedure is almost the same of what we have seen here for the massless scalar, while in the case of massless spinor filed there is one ingredient more. The spinor filed Ψ is not observable, rather bilinear quantities, such as the energy density $T^{00} = -\bar{\Psi}\gamma^0\partial^0\Psi$ are observable, and they *must* be periodic. This implies that spinor field can be *periodic* or *anti*-periodic. This has a repercussion on the mode number k which is an *integer* in the periodic case or *half*-integer in the anti-periodic case. Therefore at the end of the day we would observe an infinite tower of massive spinor particles with distinct spectra for the periodic and anti-periodic cases.

The dimensional reduction contemplates after compactification the limit $L \to 0$ and in this procedure all masses m_k go to infinity. We have to keep in mind that we are getting involve in a "physical limit" and not in a merely "mathematical limit". This means the limit $L \to 0$ in Physics does not read simply as sending L to zero, as one do in Mathematics. We have to ask what actually does it mean send L to zero. One reasonable answer could be "L is almost zero when we are not able to detect it anymore with an experiment". An famous physical length which cannot be detected by an experiment is the *Planck length* l_p

$$l_p = \sqrt{\frac{\hbar G}{c^3}} \approx 1.616199(97) \times 10^{-35} \ \mathrm{m} \ . \tag{1.118}$$

Therefore the "mathematical" limit $L \to 0$ can be thought as sending $L < l_p$. This has a consequence in terms of value of masses m_n. All of them, except the zero mode which remains massless, become of the order of Planck mass m_P

$$m_P = \sqrt{\frac{\hbar c}{G}} \approx 1.2209 \times 10^{19} \text{ GeV}/c^2 \ . \tag{1.119}$$

1.10 Consistent truncations

At the end of the Kaluza-Klein dimensional reduction one keeps only the lightest states, usually massless, of the entire infinite set of harmonic modes. However this process is not consistent because in general these modes will source the heavier modes [54]. In order to avoid this inconsistency one can rather take a *consistent truncation* of the full set of modes, which by definition is

Definition (Consistent truncation). *Let be $\{\Phi_k\}$ the whole set of harmonic modes. Let be $\mathcal{A} \subset \{\Phi_k\}$ a subset. Then $\mathcal{A}$ is a consistent truncation of the full set if the field equations of the omitted modes $\{\Phi_k\} \setminus \mathcal{A}$ are not sourced by the modes that are kept $\mathcal{A}$. Thus setting the omitted modes to zero is consistent with the field equations.*

Explain here with an example what concretely a consistent truncation means. Let be φ_1 and φ_2 real scalar fields. Suppose the dynamic is governed by the Lagrangian

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\varphi_1\partial^\mu\varphi_1 - \frac{1}{2}m_1^2\varphi_1^2 - \frac{1}{2}\partial_\mu\varphi_2\partial^\mu\varphi_2 - \frac{1}{2}m_2^2\varphi_2^2 + \lambda\varphi_1\varphi_2^2 \ , \tag{1.120}$$

where $\lambda\varphi_1\varphi_2^2$ is the interaction term. The equations of motion for the fields ϕ_1 and ϕ_2 are

$$\begin{aligned} \left(\Box - m_1^2\right)\varphi_1 &= \lambda\varphi_2^2 \ , \\ \left(\Box - m_2^2\right)\varphi_2 &= 2\lambda\varphi_1\varphi_2 \ . \end{aligned} \tag{1.121}$$

Instead of considering the whole set of fields (φ_1, φ_2), let us consider a subset, which can be $(\varphi_1, 0)$ or $(0, \varphi_2)$.

- $(\varphi_1, 0)$ is a consistent truncation of (φ_1, φ_2). If we set $\varphi_2 = 0$ inside the equations of motion (1.121) we have that φ_1 satisfies the Klein-Gordon equation and the field φ_2 does not appear anymore as a source.

  $$\left(\Box - m_1^2\right)\varphi_1 = 0 \qquad\qquad \varphi_2 = 0 \ , \tag{1.122}$$

  and this solution is consistent with the chosen subset $(\varphi_1, 0)$.

- $(0, \varphi_2)$ is *not* a consistent truncation of (φ_1, φ_2). In fact if we try to set $\varphi_1 = 0$ in the second equation of (1.121) we have the Klein-Gordon equation for φ_2, but the first equation implies also $\varphi_2 = 0$, therefore the equations of motion for this subset are

  $$\varphi_1 = 0 \qquad\qquad \varphi_2 = 0 \ , \tag{1.123}$$

  and this solution is, of course, *not* consistent with the chosen subset $(0, \varphi_2)$.

A consistent truncation, thanks to its property to decouple from the omitted modes, has a remarkable consequence, which is notable from the above example. Any solution of the lower dimensional theory (provided by the limit $L \to 0$) involving modes which are a consistent truncation, when up-lifts, it represents also an exact solution of the equations of motion in the higher dimensional theory.

The intriguing aspect of this topic is that exists a few consistent truncations, and they depend by which *manifold* one chooses to compactify the theory and by the number of *supersymmetries* which are put inside the theory.

Poincarè invariance and renormalisability constrain the Lagrangian to assume some particular forms. Supersymmetry invariance imposes more strictly conditions to the form of the Lagrangian. Therefore if we increase the number of supersymmetries some interaction terms are not allowed inside the Lagrangian and this assures less coupling between the fields which we are playing with.

About the delicate aspect how the choice of manifold can determine consistent truncations, there is the entire chapter 4 dedicated to it.

1.11 T-duality

One way of motivating the necessity of D-branes is bases on T-duality, so now in this section we will explain T-duality of the bosonic string theory. Under T-duality transformations, closed bosonic strings transform into closed strings of the same type (i.e. with same boundary conditions) in the T-dual geometry. The situation is different for open strings, however. The key is to focus on the type of boundary conditions imposed at the end of the open strings. Even though we start with Neumann boundary conditions, that are compatible with Poincaré invariance, after T-duality transformation we end with Dirichlet boundary conditions and Poincaré invariance is broken. Open strings with Dirichlet boundary conditions *must* end in a specified positions, which means in a specified hypersurfaces, called Dp-branes [14].

Physically, much of the importance of Dp-branes resides in the fact that they provides a remarkable way of introducing non-abelian gauge symmetries in string theory. Indeed non-abelian gauge fields naturally appear confined to the world volume of multiple coincident Dp-branes. In this Dp-branes approach to non-abelian gauge theories in string theory is also possible to define a Higgs mechanism in order to get symmetry breaking.

Let us start considering bosonic string theory in $d = 26$ with a background of the form

$$M_{25} \times S^1 \ . \tag{1.124}$$

The circle is taken to have radius R, so the coordinate on S^1 has periodicity

$$X^{25} \sim X^{25} + 2\pi R \ . \tag{1.125}$$

This periodicity condition on the *spacetime* has a consequence on the boundary condition of the string. Consider the *closed string*. As we move around the string, we no longer need for the compact direction X^{25} the condition $X^{25}(\tau, \sigma + 2\pi) = X^{25}(\tau, \sigma)$, but can be relaxed to

$$X^{25}(\tau, \sigma + 2\pi) = X^{25}(\tau, \sigma) + 2\pi R W \qquad\qquad W \in \mathbb{Z} \ , \tag{1.126}$$

where W is called *winding number* and it indicates the number of times the string winds around the circle and its sign encodes the direction, as shown in Fig. 1.6.

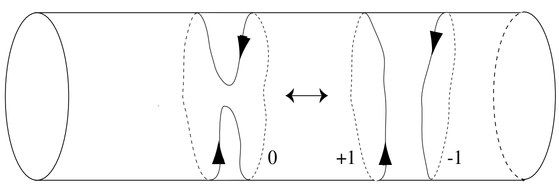

Figure 1.6: Winding modes with $W = -1, 0, +1$. (Courtesy of [3]).

Let us now consider the mode expansion for a closed string with winding number W. The expansion for the coordinates X^μ with $\mu = 0, ..., 24$ remains unchanged from the expansion in flat 26-dimensional Minkowski spacetime. What change here is only the expansion of $X^{25}(\tau,\sigma)$ which must take into account the new boundary condition (1.126). This is given by

$$X^{25}(\tau,\sigma) = x^{25} + \alpha' p^{25}\tau + WR\sigma + \text{oscillator modes} . \tag{1.127}$$

Since X^{25} is a compact dimension, the momentum eigenvalue along this direction, p^{25} must be quantised. Remember that the quantum mechanical wave function contains the factor $exp(ip^{25}x^{25})$. As a result, if x^{25} is increased by $2\pi R$, corresponding to going once around the circle, the wave function should return to its original value, or in other words, it should be single-valued on the circle. This implies that momentum in the 25-th direction is of the form

$$p^{25} = \frac{K}{R} \qquad\qquad K \in \mathbb{Z} , \tag{1.128}$$

where K is called *Kaluza-Klein excitation number*. Before splitting $X^{25}(\tau,\sigma)$ into right- and left-moving parts, it is useful to introduce the quantities

$$p_L = \frac{K}{R} + \frac{WR}{\alpha'} , \qquad\qquad p_R = \frac{K}{R} - \frac{WR}{\alpha'} . \tag{1.129}$$

The right- and left- modes appearing in $X^{25}(\tau,\sigma) = X_L^{25}(\sigma^+) + X_R^{25}(\sigma^-)$ read

$$\begin{aligned} X_R(\tau-\sigma) &= \frac{1}{2}x^{25} + \frac{1}{2}\alpha' p_R(\tau-\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^{25}\mathrm{e}^{-in(\tau-\sigma)} , \\ X_L(\tau+\sigma) &= \frac{1}{2}x^{25} + \frac{1}{2}\alpha' p_L(\tau+\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^{25}\mathrm{e}^{-in(\tau+\sigma)} . \end{aligned} \tag{1.130}$$

How the spectrum of this theory looks like to an observer living in $d = 25$ non-compact directions? Each particle state will be described by a momentum p^μ with $\mu = 0, ..., 24$ and by a Kaluza-Klein excitation (labelled by K). The 25-dimensional mass squared is given by

$$M^2 = -\sum_{\mu=0}^{24} p_\mu p^\mu . \tag{1.131}$$

Again, the mass of these particles if fixed in terms of oscillator modes. The formula, which we do not give here a proof, reads

$$M^2 = \left(\frac{K}{R}\right)^2 + \left(\frac{WR}{\alpha'}\right)^2 + 2\left(N_L + N_R - 2\right) , \tag{1.132}$$

where N_L and N_R are the number operators of the left- and right- harmonic modes, often called *levels*, defined as

$$N_R = \sum_{i=1}^{d-2}\sum_{n>0}\alpha_{-n}^i\alpha_n^i , \qquad\qquad N_L = \sum_{i=1}^{d-2}\sum_{n>0}\tilde{\alpha}_{-n}^i\tilde{\alpha}_n^i . \tag{1.133}$$

The first and the second terms in the mass formula (1.132) are novel. The first term tells us that a string with $K > 0$ units of momentum around the circle gains a contribution to its mass of K/R. And this result match with what we have discovered in the Kaluza-Klein theory. The second term tells us that a string which winds $W > 0$ times around the circle picks up a contribution $2\pi WRT$ to its mass, where $T = 1/2\pi\alpha'$ is the tension of the string.

The formula (1.132) suggests a striking symmetry of string theory that is not present for particle theories. Interchanging $W \leftrightarrow K$ and simultaneously inverting the compactification radius, i.e.

$$R \quad \longrightarrow \quad \tilde{R} = \alpha'/R \ , \tag{1.134}$$

leaves the spectrum invariant. This symmetry of the bosonic string is called *T-duality*. It suggests that compactification on a circle of radius R is physically equivalent to compactification on a circle of radius $\tilde{R}$. This equivalence is a clear indication that ordinary geometric concepts and intuition can break down in string theory at the string scale. As a symmetry at this level, T-duality is a $\mathbb{Z}_2$ discrete symmetry. It can be enlarged, as we will see later, with toroidal compactification of the theory.

1.11.1 T-duality for Superstrings

Let us nod another time towards the superstring. It turns out the ten-dimensional superstring theories are not invariant under T-duality. Instead, they map into each other. More precisely, Type IIA and IIB transform into each other under T-duality, which means that Type IIA string theory on a circle of radius R is equivalent to Type IIB string theory on a circle of radius α'/R.

$$\begin{matrix} \text{Type IIA} & & \text{Type IIB} \\ & \overset{T}{\longleftrightarrow} & \\ R & & \alpha'/R \end{matrix} \ .$$

This joint with the transformation of D-branes, since Type IIA has Dp-branes with p even, while IIB has p odd. Similarly, the two heterotic strings transform into each other under T-duality, and Type I into Type I′[10].

1.11.2 Toroidal compactification

So far we discussed T-duality symmetry for theories compactified on a circle. However this argument can be extended to theories compactified on a n-dimensional torus T^n, which adds additional interesting structure. References are [69], [3], [66].

Let us consider *closed* bosonic string on a toroidal compactified spacetime. Specifically the spacetime manifold is described by the metric

$$ds^2 = \sum_{\mu,\nu=0}^{d-1} \eta_{\mu\nu}\, dX^\mu dX^\nu + \sum_{I,J=1}^{n} G_{IJ}\, dY^I dY^J \ , \tag{1.135}$$

where $d + n = 26$. Here the first term describes flat d-dimensional Minkowski spacetime parametrised by the coordinates X^μ and the second term describes the "internal" torus T^n with dimensionless coordinates Y^I, each of which has period 2π. The physical sizes and angles that characterise the T^n can be encoded into the constant internal metric G_{IJ}. For example, in the special case of a rectangular torus, the n internal circle are all perpendicular and the internal metric is diagonal

$$G_{IJ} = R_I^2 \delta_{IJ} \ , \tag{1.136}$$

where R_I is the radius of the Y^I circle. The coordinate Y^I have periodicity

$$Y^I \sim Y^I + 2\pi \qquad\qquad I = 1, ..., n \ . \tag{1.137}$$

With these conditions, the closed string boundary conditions can relax to

[10]Since Type I theory can be seen as an orientifold projection of the Type IIB theory, by analogy Type I′ theory is an orientifold projection of the Type IIA theory.

$$
\begin{aligned}
&X^\mu(\tau,\sigma+2\pi) = X^\mu(\tau,\sigma)\ , \\
&Y^I(\tau,\sigma+2\pi) = Y^I(\tau.\sigma) + 2\pi W^I \qquad\qquad \text{with} \qquad W^I \in \mathbb{Z}\ .
\end{aligned}
\tag{1.138}
$$

Here W^I are the winding number which give the number of times that the string winds around the Y^I cycle.

Analyse now the mode expansion. As before, the expansion for the coordinates X^μ with $\mu = 0, ..., d-1$ remains unchanged. For the compact coordinates $Y^I(\tau,\sigma)$ we have

$$
\begin{aligned}
Y^I(\tau,\sigma) \quad &= Y^I_L(\sigma^+) + Y^I(\sigma^-)\ , \\
Y^I_L(\tau+\sigma) &= \frac{1}{2}y^I + p^I_L(\tau+\sigma) + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}^I_n \mathrm{e}^{-in(\tau+\sigma)} \\
Y^I_R(\tau-\sigma) &= \frac{1}{2}y^I + p^I_R(\tau-\sigma) + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\alpha^I_n \mathrm{e}^{-in(\tau-\sigma)}\ .
\end{aligned}
\tag{1.139}
$$

Because of the winding modes W^I and the Kaluza-Klein excitations K_I coming from the compact geometry of spacetime, we have

$$
p^I_L = W^I + \frac{1}{2}K_I\ , \qquad\qquad p^I_R = -W^I + \frac{1}{2}K_I\ , \qquad\qquad W^I, K_I \in \mathbb{Z}\ . \tag{1.140}
$$

All the above results hold for the case of no background B fields and a diagonal internal metric G_{IJ}. Now consider turning on constant background values for the antisymmetric two-form B_{IJ} and the internal metric G_{IJ}. One can show that in this case p^I_L and p^I_R get an enhancement from the fields B_{IJ} and G_{IJ}, which is

$$
\begin{aligned}
p^I_L &= W^I + G_{IJ}\left(\frac{1}{2}K_J - B_{JK}W^K\right)\ , \\
p^I_R &= -W^I + G_{IJ}\left(\frac{1}{2}K_J - B_{JK}W^K\right)\ ,
\end{aligned}
\tag{1.141}
$$

where, as usual, G^{IJ} denotes the inverse metric.

Again, one can compute the mass spectrum from the point of view of an observer living in d non-compact directions. Without prove it, the result is

$$
M^2 = M_0^2 + (N_R + N_L - 2)\ , \tag{1.142}
$$

where the second term is the common one which always appear from harmonic oscillators, while the first one is

$$
M_0^2 = \begin{pmatrix} W & K \end{pmatrix} \mathcal{G} \begin{pmatrix} W \\ K \end{pmatrix}\ , \tag{1.143}
$$

where

$$
\mathcal{G} = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix}\ , \tag{1.144}
$$

or the inverse

$$
\mathcal{G}^{-1} = \begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{pmatrix}\ . \tag{1.145}
$$

These are $2n \times 2n$ matrices written in terms of $n \times n$ blocks.

We can recognise some remarkable symmetries of the mass formula. The first consist in the *inversion* symmetry

$$W^I \longleftrightarrow K_I \ , \qquad\qquad \mathcal{G} \longleftrightarrow \mathcal{G}^{-1} \ , \tag{1.146}$$

and this discrete symmetry represent the extended version of T-duality symmetry of circle compactification. The check is the follow

$$M_0^2 = \begin{pmatrix} W & K \end{pmatrix} \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix} \begin{pmatrix} W \\ K \end{pmatrix} = W(G - BG^{-1}B)W + 2WBG^{-1}K + KG^{-1}K \ ,$$

and after the transformation (1.146), again we have

$$M_0'^2 = \begin{pmatrix} K & W \end{pmatrix} \begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{pmatrix} \begin{pmatrix} K \\ W \end{pmatrix} = W(G - BG^{-1}B)W + 2WBG^{-1}K + KG^{-1}K \ ,$$

so, transformation (1.146), which is a "generalised T-duality" transformation, is actually a *symmetry* of the mass formula, since $M_0^2 = M_0'^2$.

But now, in toroidal compactifications, there are additional discrete *shift* symmetries given by

$$B_{IJ} \to B_{IJ} + N_{IJ} \qquad \text{and} \qquad W^I \to W^I \ , \ K_I \to K_I + N_{IJ}W^J \ , \tag{1.147}$$

where N_{IJ} is an antisymmetric matrix of integers. Explicitly, after the transformation (1.147), we have

$$\begin{aligned}
\tilde{M}_0^2 &= \begin{pmatrix} W & K + NW \end{pmatrix} \begin{pmatrix} G - (B+N)G^{-1}(B+N) & (B+N)G^{-1} \\ -G^{-1}(B+N) & G^{-1} \end{pmatrix} \begin{pmatrix} W \\ K + NW \end{pmatrix} \\
&= W(G - BG^{-1}B)W + 2WBG^{-1}K + KG^{-1}K + \\
&- WBG^{-1}NW + WBG^{-1}NW - WNG^{-1}NW + WNG^{-1}NW + \\
&- KG^{-1}NW + KG^{-1}NW - NWG^{-1}NW + NWG^{-1}NW + \\
&+ (WN + NW)G^{-1}K - (WN + NW)G^{-1}BW \\
&= W(G - BG^{-1}B)W + 2WBG^{-1}K + KG^{-1}K = M_0^2 \ .
\end{aligned}$$

the terms in the second and third lines cancel in couples, while the terms in the fourth line cancels because N_{IJ} is antisymmetric.

We can recast the problem more geometrically reproducing the above symmetries with the action of a matrix $\mathcal{A}$ on $\mathcal{G}$ and $(W \ Z)$ as

$$\mathcal{G} \to \mathcal{A}\mathcal{G}\mathcal{A}^{-1} \qquad \text{and} \qquad \begin{pmatrix} W \\ K \end{pmatrix} \to \begin{pmatrix} W' \\ K' \end{pmatrix} = \mathcal{A} \begin{pmatrix} W \\ K \end{pmatrix} \ . \tag{1.148}$$

Then the two symmetries above becomes

- *Inversion*:
$$\mathcal{A}_I = \begin{pmatrix} 0 & \mathbb{1}_n \\ \mathbb{1}_n & 0 \end{pmatrix} \ . \tag{1.149}$$

- *Shift*:
$$\mathcal{A}_S = \begin{pmatrix} \mathbb{1}_n & 0 \\ N_{IJ} & \mathbb{1}_n \end{pmatrix} \ . \tag{1.150}$$

These two matrices are element of the $O(n,n;\mathbb{Z})$ group since they preserve a metric which is similar to the $O(n,n;\mathbb{Z})$ metric, i.e.

$$\mathcal{A}_{I,S}^T \begin{pmatrix} 0 & \mathbb{1}_n \\ \mathbb{1}_n & 0 \end{pmatrix} \mathcal{A}_{I,S} = \begin{pmatrix} 0 & \mathbb{1}_n \\ \mathbb{1}_n & 0 \end{pmatrix} . \tag{1.151}$$

Combinations of $\mathcal{A}_I$ and$\mathcal{A}_S$ are still symmetries of the mass formula and they recover the whole group $O(n,n;\mathbb{Z})$. We prove here this result. A generic element of the group $O(n,n;\mathbb{Z})$ is of the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} , \tag{1.152}$$

with A, B, C, D $n \times n$ matrices with integer entries, constraint by

$$\begin{pmatrix} A^T & C^T \\ B^T & D^T \end{pmatrix} \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} , \tag{1.153}$$

which reads

$$\begin{cases} A^T C = -C^T A \\ A^T D + C^T B = \mathbb{1} \\ B^T C + D^T A = \mathbb{1} \\ B^T D = -D^T B \end{cases} . \tag{1.154}$$

Then consider the following compositions of the transformations $\mathcal{A}_I$ and$\mathcal{A}_S(N)$, where by $\mathcal{A}_S(N)$ we intend the shift given by the antisymmetric matrix N,

$$\mathcal{A}_I \mathcal{A}_S(N) = \begin{pmatrix} N & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} , \tag{1.155}$$

$$\mathcal{A}_S(N) \left(\mathcal{A}_I \mathcal{A}_S(M)\right) = \begin{pmatrix} M & \mathbb{1} \\ NM & N \end{pmatrix} , \tag{1.156}$$

$$\left[\mathcal{A}_S(I) \left(\mathcal{A}_I \mathcal{A}_S(L)\right)\right] \left[\mathcal{A}_S(N) \left(\mathcal{A}_I \mathcal{A}_S(M)\right)\right] = \begin{pmatrix} LM + NM & L + N \\ ILM + INM & IL + IN \end{pmatrix} . \tag{1.157}$$

Since the composed transformation (1.157) satisfies the $O(n,n;\mathbb{Z})$ conditions (1.154) and since it is a very general $O(n,n;\mathbb{Z})$ transformation parametrised by the antisymmetric matrices N, M, I, L, then the group reached by compositions of inversion and shift transformations is the $O(n,n;\mathbb{Z})$ discrete group.

Then we have found an important result:

"The T-duality symmetry of toroidal compactifications T^n
is represented by the group $O(n,n;\mathbb{Z})$."

In fact the T-duality symmetry on circle, which means $T^1 = S^1$, is represented by the group $O(1,1;\mathbb{Z})$, which is the discrete group $\mathbb{Z}_2$.

2

Generalised Geometry

If you can't explain it simply, you don't understand it well enough.
- Albert Einstein -

Einstein's General Relativity is a theory whose natural formulation is inside the framework of differential geometry, called "ordinary geometry". By "natural formulation" I mean that the proper way to define all the objects inside the theory, and in particular the equations of motion, makes use of the language of *Riemannian Geometry.* At least this is the best result that we are able to achive nowadays. This approach consists in taking a 4-dimensional manifold M, equipped with a metric g, and we define a vector field over M as a section of the fibre bundle TM. Point by point of M a *vector* is an element of the tangent space T_pM, and a *covector*, that is its dual, is an element of the cotangent space T_p^*M. Using the technique of *tensor product* between vectorial spaces, one defines a generic tensor field of type (q, r) as a section of $\bigotimes^q TM \bigotimes^r T^*M$. The "*general covariance principle*", whose statement tells us that the laws of physics must be the same in every reference frame, has direct realization in the language of tensors. What this tells us is that if an equation is written covariantly and is true in a certain reference frame, than it is true for each reference frame. Hence the tangent bundle is a very important key concept if one wants to define in a covariant way General Relativity.

Generalised Geometry is an appealing idea given by the mathematician Nigel Hitchin [38] that he had when he was spending great time in Spain, as he tells in many of his talks. The idea consists in an extension of ordinary geometry, in particular it consists in "replacing" the tangent bundle TM by $TM \oplus T^*M$ in a sense. In this fashion one has to provide the generalisation of all tools of ordinary geometry. What happens in particular is that one has to give a generalisation of the definition of Lie derivative, connections, torsions and Riemann tensor. The way one can do this generalisation is not unique. For this reason it does not exist only one type of generalised geometry. The type of generalised geometry to choose is suggested by the physical problem one has to describe. The generalised geometry that we mostly study in this work is the "Hitchin and Gualtieri" generalised geometry, that is the "$TM \oplus T^*M$" generalised geometry which is equipped "for free" with the $O(d, d)$-structure group. But in the other side, in order to investigate some geometrical properties, such as "Leibniz generalised parallelisability" of round spheres S^d, that is what we are going to do later, the most natural generalised geometry to choose is "$TM \oplus \Lambda^{d-2}T^*M$". Another example of generalised geometry that we are not going to study in this work, is the generalised geometry with $E_{d(d)} \times \mathbb{R}^+$-structure group, which is usefull to reformulate eleven-dimensional supegravity, as presented in [16]. What we want to do in this work is "geometrising" the bosonic structure that appears in Type II Supergravity. What will happen is that the graviton g, the 2-form B and the dilaton ϕ are enclosed in a new object that appears in generalised geometry: the *generalised metric*.

2.1 The beginning of the generalisation

What we want to do now is to provide a *generalisation* of the whole set of geometric objects that we can define in ordinary geometry. The hope is that in this framework we would be able to understand more "physics" than what we can do up to now. In order to get inside that, the first step consist to define what is the new space where the new objects live. In the old-fashioned ordinary geometry this space is the *tangent bundle* (or tensor products of it). Now it is represented by the *generalised tangent bundle* (or tensor products of it). Any generalisation we are going to do should be motivated by *analogy* with ordinary geometry, which will represent our guide, and by preservation of symmetries, such as the invariance under gauge transformations of B-field or the invariance under diffeomorphisms.

2.1.1 The generalised tangent bundle

Suppose to take a d-dimensional manifold M equipped with an atlas $\mathcal{A} = \{U_i, \phi_i\}$. Point by point of the manifold remain defined the tangent space T_pM and its dual, the cotangent space T_p^*M. Recall now how the *tangent bundle* is defined in ordinary geometry. It is a *fibre bundle* where the fibre attached at point p is the tangent space in p. This means we are performing a union point by point of all the tangent space of the manifold, hence the tangent bundle is

$$TM \equiv \bigcup_{p\in M} T_pM \ . \tag{2.1}$$

Now let's try to do a generalisation in the way conjectured by Hitchin. Rather than considering only the tangent space T_pM and making a union of them point by point in order to obtain the tangent bundle, let's try to replace T_pM by $T_pM \oplus T_p^*M$. Therefore the *generalised tangent bundle* is defined as

$$E \equiv \bigcup_{p\in M} T_pM \oplus T_p^*M \ . \tag{2.2}$$

Seems clear that a generic element of the generalised tangent bundle (that in a mathematical language is called "section of E", or in a physical language "generalised vector fields") is represented by a "double component vector field", in particular the first d entries are the "vector entries" and, point by point, they live in T_pM, while the second d entries are the "one-form entries" and they live in T_p^*M. Formally we can recast all these components into a single generalised vector field V with components

$$V^M = \begin{pmatrix} v^\mu \\ \lambda_\mu \end{pmatrix} \ , \tag{2.3}$$

where the capital index M runs from 1 to $2d$, while the greek index μ runs from 1 to d.

There is an intriguing aspect concerning the fact that E is not writable trivially as $TM\oplus T^*M$ because the fibres are intertwined together. However the two bundles, E and $TM \oplus T^*M$, are isomorphic to each other, and what we will see further is that the isomorphism map between the two spaces is given by the B-field, which has got the important role in generalised geometry to "untie" the fibres.

There is another way, more mathematical, to define the generalised tangent bundle, and consists in defining it via "exact sequence". For more details see appendix B.

At this point we are not done about the definition of the generalised tangent bundle. We remind that a complementary and indispensable information inside its definition is given by the *patching rules*. Each definition of fibre bundle must keep into account the patching rules, and the generalised tangent bundle does not give an exception. So what we are going to do now is defining the "patching rules" for the generalised tangent space[1].

2.1.2 Patching rules

Before merely talking about patching rules, that seems a purely mathematical topic, I would like to provide some physical reasons why patching rules are important, in order to make happy the physical sense of the reader (and the writer as well).

Let's consider *electromagnetism.*. The electromagnetic theory is a *gauge theory* whose geometrical interpretation is best described by a $U(1)$-principle bundle (further informations about principle bundle are available in appendix A). Suppose to take a manifold M which is equipped of a maximal atlas $\mathcal{A} = \{U_i, \phi_i\}$ (for concreteness the reader can think to a 4-dimensional Minkowski space). To each point p in the manifold remains associated a fibre that is an element of $U(1)$ group. Since $U(1)$ is one-dimensional Lie group, the adjoint representation of $U(1)$ elements is also one-dimensional. Therefore the fibre at point p is of the form $e^{i\alpha}$. Let us pick up a generic point p of the manifold which is covered by two charts U_i and U_j. Let be $e^{i\alpha_i}$ and $e^{i\alpha_j}$ the two fibres in p associated to the two charts respectively. The patching rule consist in providing a mathematical formula that expresses how the fibres "match together" in the overlap of the open sets U_i and U_j. This rule is *local* because it has validity only in the neighborhood of the point p where the overlap of the two charts is not the empty set. Now we have to figure out how to define this patching rule. We should start by the fact that $U(1)$ is a compact Lie group. Therefore if we have two generic elements A and B, then exists a third element C such that $A = B \circ C$. This is just only by the closure property of groups. Hence we can guess that there is a real function Λ_{ij}, that depends by the chosen point p, such that

$$e^{i\alpha_i} = e^{i\Lambda_{ij}} e^{i\alpha_j} \ , \tag{2.4}$$

which means that the two fibres $e^{i\alpha_i}$ and $e^{i\alpha_j}$ are equal to each other up to a rotation in the complex plane.

But if we want to make striking the physical meaning of the formula (2.4) we have to look at the connection on the $U(1)$-principle bundle, which is represented by the one-form $A = A_\mu \, dx^\mu$. In physics the $U(1)$-principle bundle connection is called "*electromagnetic gauge vector field*" and it represents the field associated to the photon. Let be A_i and A_j the connections defined respectively in the charts U_i and U_j. This two connections are related between each other, and the way they are related is imposed by the patching rule (2.4). This particular relation, that we are not going to prove, but that the curious reader can find in [46], is the following

$$A_i = A_j + d\Lambda_{ij} \ , \tag{2.5}$$

and in components reads

$$A_{i\mu} = A_{j\mu} + \partial_\mu \Lambda_{ij} \ . \tag{2.6}$$

This is nothing else than the well-known gauge transformation rule of the electromagnetic potential A_μ.

The real function Λ_{ij}, which is manifestly antisymmetric in i and j, must satisfy a cocycle relation in order to patch correctly in the threefold intersections. In fact, given a point p and the covering charts $U_i \cap U_j \cap U_k \neq \emptyset$, by consistency we have

$$A_i = A_j + d\Lambda_{ij} = A_k + d\Lambda_{ij} + d\Lambda_{jk} = A_i + d\Lambda_{ij} + d\Lambda_{jk} + d\Lambda_{ki} \ , \tag{2.7}$$

that implies

[1]The patching information is actually *the definition* of the extension given in appendix B. They are not distinct.

$$d(\Lambda_{ij} + \Lambda_{jk} + \Lambda_{ki}) = 0 \qquad \Longleftrightarrow \qquad \Lambda_{ij} + \Lambda_{jk} + \Lambda_{ki} = const \ . \tag{2.8}$$

The constant can be determined using a physical condition. Let's consider the wave function that describes our physical state in a Hilbert space. After a gauge transformation we know that the two wave functions are related each other by an element of the $U(1)$ gauge group. Suppose φ_i is the initial wave function and φ_j is the final wave function after a gauge transformation, then they are related by

$$\varphi_i = e^{i\Lambda_{ij}} \varphi_j \ . \tag{2.9}$$

After three gauge transformations in the triple patching, the wave function must return to its initial value

$$\varphi_i = e^{i\left(\Lambda_{ij}+\Lambda_{jk}+\Lambda_{ki}\right)} \varphi_i \qquad \Longleftrightarrow \qquad e^{i\left(\Lambda_{ij}+\Lambda_{jk}+\Lambda_{ki}\right)} = 1 \ , \tag{2.10}$$

whose solution is

$$\Lambda_{ij} + \Lambda_{jk} + \Lambda_{ki} = 2\pi n \qquad\qquad n \in \mathbb{Z} \ . \tag{2.11}$$

This argument, if extended, leads to the existence of the *magnetic monopole*, but we won't to pursue this topic here.

Now that we are motivated enough by some physical reasons in order to study the patching rules, let's start to review how the patching rules are defined for the tangent bundle in ordinary geometry.

Again, let's take a manifold M which is equipped of a maximal atlas $\mathcal{A} = \{U_i, \phi_i\}$, pick up a generic point p and look at the non trivial overlap of two open sets $p \in U_i \cap U_j \neq \emptyset$ that contain the point p. Let's take coordinates x in U_i and y in U_j. Let be v a generic vector field. Requiring the coordinate-free property of v one can write

$$v = v^\nu \frac{\partial}{\partial x^\nu}\bigg|_p = v'^\mu \frac{\partial}{\partial y^\mu}\bigg|_p \ , \tag{2.12}$$

and using the chain rule, one obtain the patching rule between the vector components

$$v'^\mu = \frac{\partial y^\mu}{\partial x^\nu}\bigg|_p v^\nu \ . \tag{2.13}$$

Now we should figure out the patching rules for the generalised tangent bundle starting from some analogy with the patching rules for the tangent bundle. What we can say is that

(I) when the "form components" of a generalised vector field are switched off, then the generalised vector field must transform as an ordinary vector field.

(II) the "form components" must transform in a way that recover the gauge transformations of the B-field.

In order to satisfy the previous two conditions, the patching rules for a generalised vector field must be of the form

$$\begin{aligned} v_{(i)} &= A_{ij}\, v_{(j)} \ , \\ \lambda_{(i)} &= A_{ij}^{-T}\, \lambda_{(j)} + i_{v_{(j)}} d\Lambda_{ij} \ , \end{aligned} \tag{2.14}$$

where A_{ij} is an element of $GL(d, \mathbb{R})$ and it represents the jacobian matrix $\frac{\partial y^\mu}{\partial x^\nu}$ that appear in the patching rules for the tangent bundle. Acting on vectors it is of the form A_{ij}, while acting on one-forms, in order to preserve the inner product between vectors and one-forms, it is on the form A^{-T}_{ij}. Λ_{ij} is a one-form that provides the gauge transformations of the B-field

$$B_{(i)} = B_{(j)} - d\Lambda_{ij} \ . \tag{2.15}$$

I want to make a clarification about the notation used: $i_{(\cdot)}(\cdot)$ is the inner product between vectors and one-forms using the contraction of the indices, i.e. $i_v\lambda = v^\mu\lambda_\mu$. The subscripts i and j are *not* indices, but are labels referred to the two open sets U_i and U_j in which the objects are defined.

In a more familiar notation using indices, the equations (2.14) and (2.15) becomes

$$\begin{aligned} v^\mu_{(i)} &= (A_{ij})^\mu{}_\nu \, v^\nu_{(j)} \ , \\ \lambda_{(i)\,\mu} &= (A_{ij})_\mu{}^\nu \, \lambda_{(j)\,\nu} + v^\nu_{(j)}\partial_{[\nu}\Lambda_{|ij|\mu]} \ , \end{aligned} \tag{2.16}$$

and

$$B_{(i)\,\mu\nu} = B_{(j)\,\mu\nu} - \partial_{[\mu}\Lambda_{|ij|\nu]} \ . \tag{2.17}$$

We can see that talking about the patching rules the B-field makes its first appearance in generalised geometry. In fact the patching rules must taking into account the gauge transformations of the B-field in the way showed above. But a natural question that should arise at this point is: "does the B-field play a particular role in generalised geometry?". The answer is *yes, it does* and in the paragraph 2.3.4 we are going to explain its role. But before do that we have to introduce the $O(d,d)$ structure in generalised geometry.

2.2 The extension of the concept of "diffeomorphism"

The concept of diffeomorphism is very important in physics, especially in General Relativity that is a (geometrical) theory of the *gravitation interaction*. In nature there are four fundamental interactions. They are the *gravitational*, the *electromagnetic*, the *strong nuclear* and the *weak nuclear*. What makes each interaction different from the others is the *gauge group*. When you define a *gauge group*, you define an *interaction*. In a gauge group, that is a Lie group, in order to choose a particular element, we have to specify the value of a set of parameters that are in number equal to the dimension of the group. For each parameter remains associated a *gauge boson* and its mass determines the range of the interaction. The *electromagnetic* interaction is governed by the gauge group $U(1)$ with one massless gauge boson that is called "photon", therefore it is a long range interaction. The *nuclear strong* interaction has the gauge group $SU(3)$ with eight massless gauge bosons, called "gluons". Again, it is a long range interaction (with property of confinement at low energies and asymptotic freedom at hight energies). The *nuclear weak* interaction is characterized by the $SU(2)$ gauge group with three massive gauge bosons, called W^+, W^- and Z^0 bosons. It is a short range interaction. Actually these are not gauge bosons inside the only $SU(2)$ gauge group, but they are a *combination* of the four gauge bosons inside the group $SU(2)\times U(1)$ that appears in the *electro-weak* unification in the Standard Model.

The gauge group for the gravitational interaction is the *diffeomorphism* group, indicated by $\mathrm{Diff}(M)$[2]. The generators of the algebra associated to the diffeomorphism group are the vector fields defined over the manifold. Each algebra is equipped by a *brackets* which represent the "multiplication rule", and the one associated to the algebra of the diffeomorphism group are the Lie brackets. Given $v = v^\mu\partial_\mu$ and $w = w^\mu\partial_\mu$ vector fields, they are defined as

$$[v,w] \equiv (v^\mu\partial_\mu v^\nu - w^\mu\partial_\mu v^\nu)\,\partial_\nu \tag{2.18}$$

[2]If the manifold M is simply $\mathbb{R}^d$, then the diffeomorphism group becomes the general linear group $GL(d,\mathbb{R})$, which consist of all the possible invertible transformation one can think in d dimensions. To be honest, $GL(d,\mathbb{R})$ is not exactly the diffeomorphism group of $\mathbb{R}^d$, but it is its *deformation retract*, which means that $\mathrm{Diff}(\mathbb{R}^d)$ can be continuously deformed into $GL(d,\mathbb{R})$ in the sense of homotopy. I want to stress the fact the reader should keep in mind that only on $\mathbb{R}^d$ the retract of $\mathrm{Diff}(\mathbb{R}^d)$ is the group $GL(d)$, but for a general manifold is not.

Inside the abstract definition of a Lie algebra $\mathfrak{g}$ on a field $\mathbb{F}$, the bracket $[\cdot,\cdot]:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ must satisfies the properties

1. (**Bilinear**) $\forall\ X,Y,Z\in\mathfrak{g}$ and $a,b\in\mathbb{F}$

$$\begin{aligned}[aX+bY,Z]&=a[X,Z]+b[Y,Z]\ ,\\ [Z,aX+bY]&=a[Z,X]+b[Z,Y]\ ;\end{aligned}$$

2. (**Skew-symmetric**) $\forall\ X,Y\in\mathfrak{g}$

$$[X,Y]=-[Y,X]\ ;$$

3. (**Leibniz rule**) $\forall\ X,Y,Z\in\mathfrak{g}$

$$[X,[Y,Z]]=[[X,Y],Z]+[Y,[X,Z]]\ .$$

It is quite trivial to show that the Lie bracket defined as in the equation (2.18) satisfies the above properties of bilinearity, skew-symmetry and the Leibniz rule. In particular, thanks to the skew-symmetry property, the Leibniz rule can be rewritten as the *Jacoby identity*:

$$[X,[Y,Z]]+[Y,[Z,X]+[Z,[X,Y]]=0\ . \tag{2.19}$$

The physical interpretation of the Lie bracket is given by the fact that if we move in the direction of v and then in the direction of w we will not end at the same point that we would be if we move first in the direction of w and then in the direction of v. But the difference between the two end points is (a vector) given by the Lie bracket $[v,w]$.

Now consider a *flow* generated by a vector field v. If we follow point by point the "arrows" given by the vector field we can draw a trajectory which can be parametrised by a real parameter τ that can be think to be, for instance, the proper time of the particle. If at the beginning the particle is at the point x, after the proper time τ it will be at point y that is given by the flow associated to the vector field v, in formula we write $y=\phi_v(\tau,x)$.

The flow $\phi_v(\tau,x)$ can be think as a "one-parameter group of diffeomorphism". For each vector field v remains defined a one-parameter group of diffeomorphism $\phi_v(\tau,x)$, and vice-versa.

Now suppose to take another vector field w, or to be more general, a tensor field with components $T^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_q}$. Now we wonder what happens to the tensor component if we perform an infinitesimal diffeomorphism $y^\mu=x^\mu+\tau v^\mu+\mathcal{O}(\tau^2)$. The answer is given by the *Lie derivative* along the vector field v. In particular the Lie derivative tells us how a generic vector field changes along the direction of v. In formula

$$\delta T^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_q}(x)=T'^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_q}(x)-T^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_q}(x)=\mathcal{L}_vT^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_q}(x)\ . \tag{2.20}$$

Therefore the Lie derivative plays an important role in encoding the infinitesimal diffeomorphism. Concretely the Lie derivative respect v for a vector field w coincides with the Lie bracket

$$\mathcal{L}_vw=[v,w]\ , \tag{2.21}$$

hence also the Lie derivative satisfies the properties of bilinearity , skew-symmetry and the Jacoby identity.

Summarizing we understood that there are two important objects to consider when we talk about the diffeomorphism group, i.e. the gauge group for the gravitational interaction. They are the Lie brackets and the Lie derivative which *a priori* they are two different objects, with different physical meaning. Only at the end one can show that their expressions coincide.

Keeping in our mind this facts of ordinary geometry, let's generalise Lie derivative and Lie brackets inside the framework of generalised geometry.

2.2.1 Dorfman derivative

In trying to generalise the Lie derivative, the best natural way to do that is requiring that the properties of the Lie derivative in ordinary geometry are preserved. This means that for each real scalar function $\alpha \in \mathcal{F}(M)$, generalised vector fields $V, W \in E$ and generalised tensor fields $A \in E^{\otimes n}$, $B \in E^{\otimes m}$, the Dorfman derivative is a map

$$L_{(\cdot)}(\cdot) : \Gamma(E) \times \Gamma(E^{\otimes n}) \to \Gamma(E^{\otimes n}) \tag{2.22}$$

with the properties

(1) $L_V(\alpha) = \pi(V)[\alpha] = v[\alpha]$,

(2) $L_V(\alpha A) = \alpha L_V A + V[\alpha] L_V A$,

(3) $L_V(A \otimes B) = (L_V A) \otimes B + A \otimes (L_V B)$,

(4) $L_V(W[\alpha]) = V[W[\alpha]] = (L_V W)[\alpha] + W[V[\alpha]]$,

(5) $\pi(L_V W) = \mathcal{L}_{\pi(V)}\pi(W) = \mathcal{L}_v w$.

where π is the projection map $\pi : E \to TM$ defined in the appendix B, i.e. $\pi(V) = v$, where v is the vector component of V, and $V[\alpha]$ is the action of the generalised vector fields V on the function α, i.e. $V[\alpha] = V^M \partial_M \alpha$, where ∂_M can be chosen as discussed in section 2.3.1.

The request (1) to (4) are the natural generalisation of the property of the ordinary Lie derivative. The really new one is the request (5) which tells us that the projection of the generalised Lie derivative of a generalised vector (that is a generalised vector) is simply the ordinary Lie derivative.

Keeping in mind the above properties, a way to define the "generalised Lie derivative", due to Irene Dorfmann (1987), is the following

Definition (The Dorfman derivative). *Let be $V = \begin{pmatrix} v^\mu & \lambda_\mu \end{pmatrix}^T$ and $W = \begin{pmatrix} w^\mu & \zeta_\mu \end{pmatrix}^T$ generalised vector fields, then the Dorfman derivative of a vector field is defined as*

$$L_V W \equiv \begin{pmatrix} \mathcal{L}_v w \\ \mathcal{L}_v \zeta - i_w d\lambda \end{pmatrix} . \tag{2.23}$$

This definition can be extended to a generic generalised tensor field of rank n,

$$A = \begin{pmatrix} a^{\mu_1 \dots \mu_n} \\ b_{\mu_1 \dots \mu_n} \end{pmatrix} \in \Gamma(E^{\otimes n}) .$$

but it is more easy to do that using the formalism that we will treat in the paragraph 2.3.

Notice that inside the definition 2.23 appear only Lie derivatives respect the vector field v, i.e. $\mathcal{L}_v(\cdot)$. This is related the fact that the Lie derivative respect a one-form doesn't exist as a mathematical object. Another question concern the extra term "$-i_w d\lambda$". The reason because we introduce it is due the fact that the Dorfman derivative must encode the infinitesimal transformations of the *whole* symmetry group $\Omega^2_{cl}(M) \rtimes \mathrm{Diff}(M)$, which consist in the diffeomorphism and in the gauge transformations of the B-field. Therefore since the gauge transformation of the B-field is $\delta B = d\lambda$, the extra term $-i_w d\lambda$ is a consequence of this gauge transformation on the 1-form. The Dorfman derivative enjoys the following results

Proposition 2.1. *The Dorfman derivative, defined as in (2.23), satisfies the property of bilinearity and the Leibniz rule, but it does not satisfy the property of skew-symmetry.*

These properties are easy to check and in particular it is immediate to see from the definition (2.23), that Dorfman derivative loses the skew-symmetry property. For this reason Dorfman derivative does not satisfy the Jacoby identity, because Jacoby identity is a consequence of two properties: skew-symmetry and Leibniz rule. But if Dorfman derivative was also skew-symmetric, than it would be bought as "Lie bracket" of an algebra. Actually this is not the case, thus we have to define the "Courant bracket".

We will see later in the paragraph 2.3 how one can rewrite the Dorfman derivative in a more familiar way, which will represent the formal link between Generalised Geometry and Double Field Theory.

2.2.2 Courant bracket

After have generalised the Lie derivative, one should generalise the Lie bracket as well. Remind that the Lie bracket must be skew-symmetric because are bracket over a Lie algebra. Therefore we would like to preserve this property also in generalised geometry. Firs we can not define the generalised Lie bracket as the same of the Dorfman derivative because, as we noticed in the previous paragraph, the last one is not skew-symmetric. Therefore one define the *Courant bracket*

Definition (Courant bracket)**.** *Let be* $V = \begin{pmatrix} v^\mu & \lambda_\mu \end{pmatrix}^T$ *and* $W = \begin{pmatrix} w^\mu & \zeta_\mu \end{pmatrix}^T$ *generalised vector fields, then the Courant bracket is defined as as the antisymmetric part of the Dorfman derivative, i.e.*

$$[\![V, W]\!] \equiv \frac{1}{2}\left(L_V W - L_W V\right) \ , \tag{2.24}$$

which can be written more explicitly

$$[\![V, W]\!] = \begin{pmatrix} [v, w] \\ \mathcal{L}_v\zeta - \mathcal{L}_w\lambda - \frac{1}{2}d\left(i_v\zeta - i_w\lambda\right) \end{pmatrix} . \tag{2.25}$$

The Courant bracket satisfies the following properties

Proposition 2.2 (Properties of Courant bracket)**.** *The Courant bracket* $[\![\ ,\ ]\!]$ *satisfies the properties*

(I) skew-symmetry

$$[\![V, W]\!] = -[\![W, V]\!]$$

(II) Failure of the Jacoby identity

$$[\![[\![V, W]\!], Z]\!] + [\![[\![W, Z]\!], V]\!] + [\![[\![Z, V]\!], W]\!] = \frac{1}{3}d(\langle[\![V, W]\!], Z\rangle + \langle[\![W, Z]\!], V\rangle + \langle[\![Z, V]\!], W\rangle$$

The property of skew-symmetry is immediate from the definition of Courant bracket, while the proof of the second property is available in [38].

2.3 The $O(d, d)$ structure

I want to advise the reader that in this paragraph we are going to follow an heuristic procedure. This procedure will allow us to define "for free" an $O(d, d)$ structure on the generalised tangent bundle. It is perfectly acceptable the axiomatic approach, followed by [38], with consist to assume from the beginning the existence of a generalised tangent bundle equipped with an $O(d, d)$ structure.

Let us consider the Dorfman derivative and the Courant bracket of the generalised vector fields $V = \begin{pmatrix} v^\mu & \lambda_\mu \end{pmatrix}^T$ and $W = \begin{pmatrix} w^\mu & \zeta_\mu \end{pmatrix}^T$. From their definitions it is clear the two objects don't coincide. Thus let's try to compute the difference between them.

$$\llbracket V,W\rrbracket \equiv \frac{1}{2}\left(L_V W - L_W V\right) = \frac{1}{2}\begin{pmatrix} \mathcal{L}_v w - \mathcal{L}_w v \\ \mathcal{L}_v\mu - i_w d\lambda - \mathcal{L}_w\lambda + i_v d\mu \end{pmatrix} \tag{2.26}$$
$$= \begin{pmatrix} [v,w] \\ \frac{1}{2}\left(\mathcal{L}_v\mu - i_w d\lambda - \mathcal{L}_w\lambda + i_v d\mu\right) \end{pmatrix} = \begin{pmatrix} [v,w] \\ \mathcal{L}_v\mu - \mathcal{L}_w\lambda - \frac{1}{2}d\left(i_v\mu - i_w\lambda\right) \end{pmatrix}$$
$$= \begin{pmatrix} [v,w] \\ \mathcal{L}_v\mu - i_w d\lambda \end{pmatrix} - \begin{pmatrix} 0 \\ \frac{1}{2}d\left(i_v\mu + i_w\lambda\right) \end{pmatrix} = L_V W - \begin{pmatrix} 0 \\ \frac{1}{2}d\left(i_v\mu + i_w\lambda\right) \end{pmatrix} \ ,$$

where we used the *Cartan formula* for the Lie derivative of a 1-form μ respect the vector field v

$$\mathcal{L}_v\mu = i_v d\mu + d i_v\mu \ . \tag{2.27}$$

Hence we have

$$L_V W - \llbracket V,W\rrbracket = \begin{pmatrix} 0 \\ \frac{1}{2}d(i_v\zeta + i_w\lambda) \end{pmatrix} \ . \tag{2.28}$$

The form component of this difference is the differential of a scalar which is *bilinear* and *symmetric* in the V and W components. Therefore the question now is the following, why not buy this expression as a scalar product between V and W defined by a certain metric? If we are happy with that, we can figure out the metric that induce the scalar product $\langle\cdot,\cdot\rangle$

$$\langle V,W\rangle = \frac{1}{2}(i_v\zeta + i_w\lambda) = \begin{pmatrix} v & \lambda \end{pmatrix} \frac{1}{2}\begin{pmatrix} \mathbb{0}_{d\times d} & \mathbb{1}_{d\times d} \\ \mathbb{1}_{d\times d} & \mathbb{0}_{d\times d} \end{pmatrix}\begin{pmatrix} w \\ \lambda \end{pmatrix} \ . \tag{2.29}$$

The metric which defines the scalar product has eigenvalues

$$\det\left[\lambda\mathbb{1}_{2d\times 2d} - \frac{1}{2}\begin{pmatrix} \mathbb{0}_{d\times d} & \mathbb{1}_{d\times d} \\ \mathbb{1}_{d\times d} & \mathbb{0}_{d\times d} \end{pmatrix}\right] = \lambda^{2d} - \frac{1}{4} = 0 \qquad \Longrightarrow \ \lambda = \pm\frac{1}{2} \ , \tag{2.30}$$

and therefore it is similar, up to a constant factor 1/2, to the metric preserved by the elements of the group[3] $O(d,d)$,

$$\eta_{MN} = \frac{1}{2}\begin{pmatrix} \mathbb{0}_{d\times d} & \mathbb{1}_{d\times d} \\ \mathbb{1}_{d\times d} & \mathbb{0}_{d\times d} \end{pmatrix}_{MN} \simeq \frac{1}{2}\begin{pmatrix} \mathbb{1}_{d\times d} & \mathbb{0}_{d\times d} \\ \mathbb{0}_{d\times d} & -\mathbb{1}_{d\times d} \end{pmatrix}_{MN} \ . \tag{2.31}$$

Since the two metrics are *similar*, then the two groups are *isomorphic*. Hence we are allowed to consider the group $O(d,d)$ up to an isomorphism.

The idea now consists in promoting $O(d,d)$ group as a structure group on the generalised tangent bundle. This is what I mean when we say "generalised geometry has an $O(d,d)$ structure for free". In fact in ordinary geometry there is not a structure over the tangent bundle, or alternatively, there is only the trivial structure given by the group $GL(d,\mathbb{R})$. Imposing a G-structure, geometrically, means that we are allowed to choose on the tangent bundle only the reference frames related to each other by a transformation induced by an element of the group G. For instance, an $O(d)$ structure in ordinary geometry restrict the choice to the orthonormal frames on the tangent bundle.

2.3.1 Connection with Double Field Theory

The Dorfman derivative, defined by the equation (2.23), can be rewritten in a way more close to the definition of Lie derivative of a generic tensor field.

Recall the definition of Lie derivative of a vector field w in a coordinate basis $\{\partial_\mu\}$

$$\mathcal{L}_v w^\mu = v^\nu\partial_\nu w^\mu - w^\nu\partial_\nu v^\mu \ , \tag{2.32}$$

[3] I just remind that almost every group of matrices is defined by a quadratic constraint, which usually consists in preserving a particular type of matrix. The choice of the matrix which is preserved defined the group. For instance, the group $O(p,q)$ is defined as $\{M \in GL(d,\mathbb{R}) | M\eta M^T = \eta, \ p+q=d\}$, with $\eta = \begin{pmatrix} \mathbb{1}_{p\times p} & \mathbb{0} \\ \mathbb{0} & -\mathbb{1}_{q\times q} \end{pmatrix}$.

recall also the definition of Lie derivative of a 1-form ζ

$$\mathcal{L}_v \zeta_\mu = v^\nu \partial_\nu \zeta_\mu + (\partial_\mu v^\nu)\zeta_\nu \ . \tag{2.33}$$

The most general definition of Lie derivative of a tensor field $T^{\mu_1\dots\mu_p}_{\nu_1\dots\nu_q} \in \Gamma(TM^{\otimes p}\otimes T^*M^{\otimes q})$ is

$$\begin{aligned}\mathcal{L}_v T^{\mu_1\dots\mu_p}_{\nu_1\dots\nu_q} = v^\mu \partial_\mu T^{\mu_1\dots\mu_p}_{\nu_1\dots\nu_q} \\ + (\partial_\mu v^{\mu_1}) T^{\mu\mu_2\dots\mu_p}_{\nu_1\dots\nu_q} + \dots + (\partial_\mu v^{\mu_p}) T^{\mu_1\dots\mu_{p-1}\mu}_{\nu_1\dots\nu_q} \\ - (\partial_{\nu_1} v^\mu) T^{\mu_1\dots\mu_p}_{\mu\nu_2\dots\nu_q} - \dots - (\partial_{\nu_q} v^\mu) T^{\mu_1\dots\mu_p}_{\nu_1\dots\nu_{q-1}\mu} \ ,\end{aligned} \tag{2.34}$$

where the terms on the second and third lines can be viewed as the adjoint action of the $\mathfrak{gl}(d,\mathbb{R})$ matrix $a^\mu{}_\nu = \partial_\nu v^\mu$ on the particular tensor field T. A sketch of proof of the formula (2.34) can be found in [46].

All the above definitions can be extended also in generalised geometry, taking care to define opportunely the coordinate basis $\{\partial_M\}$. Let's try to compute more explicitly the Dorfman derivative of a generalised vector field W using the index notation

$$\begin{aligned}L_V W &= \begin{pmatrix} [v^\mu \partial_\mu w^\nu - (\partial_\mu v^\nu) w^\mu]\, \partial_\mu \\ [v^\nu \partial_\nu \zeta_\mu + (\partial_\mu v^\nu)\zeta_\nu - w^\nu \partial_\nu \lambda_\mu + w^\nu \partial_\mu \lambda_\nu]\, dx^\mu \end{pmatrix} \\ &= \begin{pmatrix} v^\nu \partial_\nu w^\mu \partial_\mu \\ v^\nu \partial_\nu \zeta_\mu dx^\mu \end{pmatrix} + \begin{pmatrix} 0 \\ [(\partial_\mu v^\nu)\zeta_\nu + (\partial_\mu \lambda_\nu) w^\nu]\, dx^\mu \end{pmatrix} - \begin{pmatrix} (\partial_\nu v^\mu) w^\nu \partial_\mu \\ (\partial_\nu \lambda_\mu) w^\nu dx^\mu \end{pmatrix}\end{aligned} \tag{2.35}$$

First notice that one can embed the action of the partial derivative operator into generalised geometry using the map[4] $T^*M \xrightarrow{\iota} E$. Now we perform the following choice of ∂, viewed as a map to a section of E^*

$$\partial_M = \begin{cases} \partial_\mu & \text{for} \quad M = \mu \\ 0 & \text{for} \quad M = \mu + d \end{cases} \ , \tag{2.36}$$

and one can rewrite the equation (2.35) in terms of generalised objects

$$L_V W = \left[V^N \partial_N W^M + \left(\partial^M V^N - \partial^N V^M \right) W_N \right] \partial_M \ . \tag{2.37}$$

Raising and lowering indices is done by the $O(d,d)$ metric η_{MN}. I want to stress a subtle point, related to the definition (2.36) of the partial derivative operator. This definition is not covariant and is intended to work when the operator ∂ is applied to a generalised tensor field. In fact the operator ∂ is a map into section of E^* and has no sense define it to be zero in the form components when it is not applied to something. Therefore what we intend here is

$$\partial_{\mu+d}[\,\text{everything}\,] = 0 \ . \tag{2.38}$$

The formula (2.36) is very important for another reason: it represents the link with the Double Field Theory! In fact in Double Field Theory one makes double the target space introducing extra coordinates $\tilde{x}$, defines a generalisation of the objects and at the end of the day one imposes a constraint in order to get rid of the extra degrees of freedom. One choice of constraint in Double Field Theory is

$$(x, \tilde{x} + a) \sim (x, \tilde{x}) \qquad\qquad \forall\ a \in \mathbb{R}^d \ . \tag{2.39}$$

Following the procedure of making double the set of coordinates has a direct consequence in the tangent bundle, which now it is a $2d$ dimensional vector space. The new thing which appears is a *structure*. In particular one can show (see [1], [42], [71]) that the group structure is exactly $O(d,d)$, the same which appears in generalised geometry. A brief review about Double

[4]Further informations about formal aspect of the generalised geometry are available in the appendix B.

Field Theory is available in appendix C. The constrain (2.39) actually represent the analogous of the constraint we use in generalised geometry, given by the equation (2.36). Since the partial differential operator in Generalised Geometry is locally defined in a chart, and since the constraint equation for Double Field Theory (2.39) is a local relation valid point by point x on the manifold, the two approaches match at least locally.

I want to mention that some Physicist use the constraint

$$\tilde{x} = 0 \ , \tag{2.40}$$

instead the less strictly one given by the equation (2.39). The difference between the two choices is subtle and it manifests in terms of the $O(d,d)$ metric. In fact in the "non covariant" constraint (2.40) we can require to define the $O(d,d)$ metric η just only on a submanifold of the double target space, i.e. only on the subspace defined by $\tilde{x} = 0$. On the other side, the constraint (2.39) requires the metric η is globally defined in the *whole* double target space. It is clear that in the last case the constraint in less strictly but we have require more conditions in terms of the metric η, while in the first case we have the opposite situation.

The constraint (2.36) for the generalised geometry is not covariant and it is more similar to the constraint (2.40) for the Double Field Theory. However there is a covariant form of the *same* constraint given by

$$\eta^{MN}\partial_M f \ \partial_N g = 0 \qquad\qquad \forall \quad f, g \quad \text{real functions} \ , \tag{2.41}$$

which has solution for the operator ∂_M exactly the equation (2.39). However I want to remark that the above equation does not add new informations, but it is just only a *covariant* way to express the same information.

Come back now to the equation (2.35) of the Dorfman derivative expresses in terms of generalised indices. For the Lie derivative of a tensor field (2.34), we showed that appears the adjoint action of the $\mathfrak{gl}(d,\mathbb{R})$ matrix, i.e. an element of the Lie algebra of the trivial structure group $GL(d,\mathbb{R})$ which appears in ordinary geometry. For the Dorfman derivative happens something similar. Remember that in generalised geometry there is a non trivial $O(d,d)$ structure group. In fact what happens is that the second and the third terms which appear in the equation (2.35) represent the adjoint action of the $\mathfrak{o}(d,d)$ matrix $A^{MN} = \partial^M V^N - \partial^N V^M$.

After this consideration it is easy to guess the form of the Dorfman derivative for a type (p,q) generalised tensor field

$$L_{(\cdot)}(\cdot) : \Gamma(E) \times \Gamma(E^{\otimes p}) \otimes \Gamma(E^{*\otimes q}) \to \Gamma(E^{\otimes p}) \otimes \Gamma(E^{*\otimes q})$$

$$\begin{aligned} L_V T^{M_1\dots M_p}_{N_1\dots N_q} &= V^N \partial_N T^{M_1\dots M_p}_{N_1\dots N_q} \\ &+ \left(\partial^{M_1} V^N - \partial^N V^{M_1}\right) T_{N N_1 \dots N_q}{}^{M_2\dots M_p} + \dots + \left(\partial^{M_p} V^N - \partial^N V^{M_p}\right) T^{M_1\dots M_{p-1}}_{N_1\dots N_q}{}_N \\ &- \left(\partial_M V_{N_1} - \partial_{N_1} V_M\right) T^{M M_1\dots M_p}_{N_1\dots N_q} - \dots - \left(\partial^M V^{N_q} - \partial^{N_q} V^M\right) T^{M_1\dots M_p}_{N_1\dots N_{q-1}}{}^M \end{aligned} \tag{2.42}$$

In a similar way, one can rewrite also the Courant bracket in the indices notation

$$[\![V, W]\!]^M = V^N \partial_N W^M - W^N \partial_N V^M - \frac{1}{2}\left(V_N \partial^M W^N - W_N \partial^M V^N\right) \tag{2.43}$$

2.3.2 What is a G-structure?

The formal definition in Differential Geometry is the following

Definition (G-structure)**.** *Given a d-manifold M, for a given structure group G, a G-structure is a G-principal subbundle of the tangent frame bundle FM.*

Definition (Tangent frame bundle)**.** *The tangent frame bundle of a smooth manifold M is the frame bundle associated to the tangent bundle of M.*

Definition (Frame bundle)**.** *The frame bundle is a principal fibre bundle associated to any vector bundle V. Its fibre over a point $p \in M$ is the set of all ordered bases, or frames, for $V|_p$.*

Definition (Principal fibre bundle)**.** *A principal fibre bundle (or simply, principal bundle) is a special case of fibre bundle where the fibre is a Lie group G.*

Definition (Vector bundle)**.** *A vector bundle is a fibre bundle whose fibre attached point by point on the manifold is a vector space.*

Concretely, a G-structure represents a way to chose the reference frame in the tangent bundle. In more details, one chooses a point $p \in M$, take the tangent space T_pM. and pick up a basis for this vector space. For instance, if one wants to choose the coordinate basis, then the basis for the tangent space is induced by chart chosen.

Suppose to define the group G as the set of matrices which preserve a certain metric tensor $\mathcal{A}$, which we call $O_{\mathcal{A}}$

$$O_{\mathcal{A}} \equiv \{M \in GL(d, \mathbb{R}) | M\mathcal{A}M^T = \mathcal{A}\} , \tag{2.44}$$

e.g. if $\mathcal{A} = \delta$ then we have the group $O(d)$, if $\mathcal{A} = \eta^{(p,q)}$ then we have the group $O(p, q)$, if $\mathcal{A} = \Omega$, i.e. the symplectic matrix, then we have the symplectic group[5] $Sp(d, \mathbb{R})$.

Imposing a G-structure means that for the basis $\{\hat{e}_a|_p\}$ the scalar products between the couples of vectors must satisfy the following condition

$$\langle \hat{e}_a, \hat{e}_b \rangle = \mathcal{A}_{ab} \ . \tag{2.45}$$

Any other reference frame $\{\hat{e}'_a|_p\}$ related to the above by an $O_{\mathcal{A}}$ transformation, i.e.

$$\hat{e}'_a = A_a{}^b \ \hat{e}_b \qquad\qquad A \in O_{\mathcal{A}} \tag{2.46}$$

is still a legitimate reference frame which satisfies the equation (2.45). This is what we mean by "choosing the reference frames".

For instance, choosing the $O(d)$-structure means that we are restricting our attention only to the orthonormal frames.

2.3.3 Symmetries of E

We showed that generalised geometry is equipped with an $O(d, d)$-structure. This fact has a repercussion on the choice of reference frame in the generalised tangent bundle. In particular, what we mean, is that the choice of reference frames is restricted to a basis of generalised vector fields $\{\hat{E}_M\}$ which satisfies

$$\langle \hat{E}_M, \hat{E}_N \rangle = \frac{1}{2} \begin{pmatrix} \mathbb{0} & \mathbb{1} \\ \mathbb{1} & \mathbb{0} \end{pmatrix} \tag{2.47}$$

The choice of the basis $\{\hat{E}_M\}$ is not unique, but is define up to an $O(d, d)$ transformation. This arbitrariness on the choice of the basis is what we mean by "symmetries of E".

In this paragraph we analyse in detail which are the symmetries of E, studying the group $O(d, d)$. Since $O(d, d)$ is a Lie group, it is more easy to hand out the formulas on its algebra $\mathfrak{o}(d, d)$, and then, exponentiating linear combinations of the generators, we will reconstruct the group $O(d, d)$[6].

[5] With d an even number.

[6] Actually what we are able to reconstruct continuously is only the part connected to the identity $\mathbb{1}$ of the group $O(d, d)$, which is $SO(d, d)$. But then each element of $O(d, d)$ can be reached from an element of $SO(d, d)$ by a multiplication of an element of $\mathbb{Z}_2$.

One can start to linearise around the identity an element of $O(d,d)$,

$$O = \mathbb{1} + \epsilon T + \mathcal{O}(\epsilon^2) \qquad\qquad \epsilon \ll 1 \ , \tag{2.48}$$

and impose the constraint

$$O\eta O^T = \eta \qquad\qquad \text{with} \qquad \eta = \frac{1}{2}\begin{pmatrix} \mathbb{0} & \mathbb{1} \\ \mathbb{1} & \mathbb{0} \end{pmatrix} \ . \tag{2.49}$$

Then we have

$$\left[\mathbb{1} + \epsilon T + \mathcal{O}(\epsilon^2)\right] \eta \left[\mathbb{1} + \epsilon T + \mathcal{O}(\epsilon^2)\right]^T = \eta \ , \tag{2.50}$$

which gives us, neglecting terms $\mathcal{O}(\epsilon^2)$

$$T\eta = -(T\eta)^T \ . \tag{2.51}$$

Start from a generic form of T,

$$T = \begin{pmatrix} A & C \\ B & D \end{pmatrix} \qquad\qquad \text{with} \qquad A, B, C, D \in GL(d, \mathbb{R}) \ , \tag{2.52}$$

and imposing the constraint (2.51), we obtain the following conditions

$$\begin{cases} A = -D^T \\ B = -B^T \\ C = -C^T \end{cases} \ . \tag{2.53}$$

Renaming $C = \beta$, we have found by the conditions above, that any generator of the $O(d,d)$ group, in the representation which acts on sections of E, is fixed by

- *endomorphism* $A : TM \to TM$,
- skew-symmetric map $B : TM \to T^*M$,
- skew-symmetric map $\beta : T^*M \to TM$,

and at the end we have

$$T = \begin{pmatrix} A & \beta \\ B & -A^T \end{pmatrix} \ . \tag{2.54}$$

Exponentiating T we reach a generic element $SO(d,d)$. We analize now the three type of $SO(d,d)$ transformations that we can have. Notice that the exponentiation process is straightforward because of

$$\begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}^2 = 0 \qquad\qquad \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}^2 = 0 \tag{2.55}$$

which is a consequence to be upper/lower triangular matrix.

(I) **(B-transformation)** Set $A = 0$, $\beta = 0$,

$$exp\left[\begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \ . \tag{2.56}$$

It is useful to think of a B-transformation as a *shear* transformation, which fixes projections on TM and acts by shearing in the T^*M direction. For a shorter notation we will adopt e^B instead the above expression.

(II) **(β-transformation)** Set $A = 0$, $B = 0$,

$$exp\left[\begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \ . \tag{2.57}$$

Again, it is useful to think of a β-transformation as a *shear* transformation, which fixes projections on T^*M and acts by shearing in the TM direction.

(III) **($GL(d)$-transformation)** Set $\beta = 0$, $B = 0$,

$$exp\left[\begin{pmatrix} A & 0 \\ 0 & -A^T \end{pmatrix}\right] = \begin{pmatrix} exp(A) & 0 \\ 0 & exp(A^T)^{-1} \end{pmatrix} \ . \tag{2.58}$$

Since $GL(d) \subset O(d,d)$, the above transformation represent a way to embed $GL(d)$ inside $O(d,d)$.

The B-transformation is more fundamental than the β transformation. A mathematical reason is due to the fact the $(0,2)$ tensor fields (B-like tensor fields) can be a connections, while the $(2,0)$ tensor fields (β-like tensor fields does not.

The physical motivation is given by the fact the B field is related to the 2-form B, which represent $d(d-1)/2$ d.o.f. inside the NSNS sector in Type II Supergravity theories. Therefore from here when we will talk about $O(d,d)$ transformations we will always restrict to A and B transformations.

The B-field must satisfy another property more than skew-symmetry. This requirement is due to the fact generalised geometry must be "invariant" under $O(d,d)$ transformations. Generalised geometry actually is defined to be the set $(E, \langle\ ,\ \rangle, [\![\ ,\ ]\!], \pi)$ which has the property to be an *exact Courant algebroid*. Here E is not defined in the heuristic way presented by the formula (2.2), but is defined by an "exact sequence". Further details about these topics are available in the appendix B. What we have to impose is that the $O(d,d)$ transformations let invariant the whole set $(E, \langle\ ,\ \rangle, [\![\ ,\ ]\!], \pi)$. We have the following result

Proposition 2.3. *The exact Courant algebroid* $(E, \langle\ ,\ \rangle, [\![\ ,\ ]\!], \pi)$ *is* $O(d,d)$ *invariant if and only if* B *is closed.*

Proof. We have to prove that the scalar product and the Courant bracket and anchor are invariant. The only non trivial part is related to the B transformation. Let be

$$V = \begin{pmatrix} v \\ \lambda \end{pmatrix} \qquad\qquad W = \begin{pmatrix} w \\ \zeta \end{pmatrix} \ , \tag{2.59}$$

and the transformed by a B-transformation

$$V' = e^B V = \begin{pmatrix} v \\ \lambda + i_v B \end{pmatrix} \qquad\qquad W' = e^B W = \begin{pmatrix} w \\ \zeta + i_w B \end{pmatrix} \ . \tag{2.60}$$

- **Scalar product**

$$\langle V', W' \rangle = \frac{1}{2}\left[i_v(\zeta + i_w B) + i_w(\lambda + i_v B)\right] = \frac{1}{2}\left(i_v\zeta + i_w\lambda\right) + \frac{1}{2}\left(i_v i_w + i_w i_v\right) B = \langle V, W \rangle \ , \tag{2.61}$$

where in the last step we used the skew-symmetry property of B in order to cancel the term $(i_v i_w + i_w i_v)\, B$.

Notice that the condition of preserving the scalar product does not impose something new on B.

- **Courant bracket**

  From the definition of Courant bracket (2.25), first we expand

$$[\![V',W']\!] = [\![V,W]\!] + \begin{pmatrix} 0 \\ \mathcal{L}_v i_w B - \mathcal{L}_w i_v B - \frac{1}{2} d\left(i_v i_w B - i_w i_v B\right) \end{pmatrix} . \tag{2.62}$$

  The last two terms give $d(i_v i_w B) = \mathcal{L}_v i_w B - i_v d(i_w B)$ by the Cartan formula, and so yield

$$\begin{aligned} [\![e^B V, e^B W]\!] &= [\![V,W]\!] + \begin{pmatrix} 0 \\ \mathcal{L}_v i_w B - \mathcal{L}_w i_v B - d\left(i_v i_w B\right) \end{pmatrix} = \\ &= [\![V,W]\!] + \begin{pmatrix} 0 \\ \mathcal{L}_v i_w B - i_w d\left(i_v B\right) \end{pmatrix} = \\ &= [\![V,W]\!] + \begin{pmatrix} 0 \\ i_{[v,w]} B + i_w \mathcal{L}_v B - i_w d\left(i_v B\right) \end{pmatrix} = \\ &= [\![V,W]\!] + \begin{pmatrix} 0 \\ i_{[v,w]} B + i_w i_v dB \end{pmatrix} = \\ &= e^B \left([\![V,\ W]\!]\right) + \begin{pmatrix} 0 \\ i_w i_v dB \end{pmatrix} . \end{aligned}$$

  Therefore we see that e^B is an automorphism of the Courant bracket if and only if $i_v i_w dB = 0$ for all v, w, which happens precisely when $dB = 0$.

- **Anchor**

$$\pi(V') = \pi\left[\begin{pmatrix} v \\ \lambda + i_v B \end{pmatrix}\right] = v = \pi(V) , \tag{2.63}$$

  which is automatically satisfied for every B.

Since the the B field must be a closed 2-form, we decide to choose it as the differential of a 1-form, i.e. $B = d\Lambda$. In this way B is also exact. This B field is related to the 2-form $\mathcal{B}$, d.o.f. inside the NSNS sector. $\mathcal{B}$ is defined modulo a gauge transformation

$$\mathcal{B}_j = \mathcal{B}_i + d\Lambda_{ij} , \tag{2.64}$$

and the physical connection between generalised geometry and physics is given from the choice of B exactly to be the gauge transformation $d\Lambda_{ij}$. In this manner B is defined locally on the intersection of two charts $U_i \cap U_j$, therefore it has got two chart indices $B_{ij} = d\Lambda_{ij}$ and by consistency it must satisfy the cocycle condition on threefold intersection

$$B_{ij} + B_{jk} + B_{ki} = 0 . \tag{2.65}$$

Now we are able to motivate explicitly the 1-form transformation of the patching rule (2.14). In the intersection $U_i \cap U_j$ for a section of E we have

$$\begin{pmatrix} v_j \\ \lambda_j \end{pmatrix} = \begin{pmatrix} v_i \\ \lambda_i - i_{v_i} d\Lambda_{ij} \end{pmatrix} . \tag{2.66}$$

Now we "rotate" the generalised vector $\begin{pmatrix} v_j & \lambda_j \end{pmatrix}^T$ with an element of the structure group $O(d,d)$ in order to make it equal to $\begin{pmatrix} v_i & \lambda_i \end{pmatrix}^T$. The transformation that we pick up is $exp(B_{ij}) = exp(\mathcal{B}_j - \mathcal{B}_i)$

$$exp(B_{ij})\begin{pmatrix} v_j \\ \lambda_j \end{pmatrix} = \begin{pmatrix} v_j \\ \lambda_j + i_{v_j}B_{ij} \end{pmatrix} = \begin{pmatrix} v_i \\ \lambda_i - i_{v_i}d\Lambda_{ij} + i_{v_i}(\mathcal{B}_j - \mathcal{B}_i) \end{pmatrix} \doteq \begin{pmatrix} v_i \\ \lambda_i \end{pmatrix} , \tag{2.67}$$

and imposing the last equality we have exactly the gauge transformation of the $\mathcal{B}$ field

$$\mathcal{B}_j = \mathcal{B}_i + d\Lambda_{ij} . \tag{2.68}$$

The physical observable is not the $\mathcal{B}$ field, which is defined only local in a chart, neither by the B field, which is defined on the intersection of two charts, but is given by the flux $H = d\mathcal{B}$, which is globally defined.

2.3.4 The B-field: a way to "untie" the fibres

In the above paragraph we have seen that the B-field is related to the physical d.o.f. in the NSNS sector. In generalised geometry the B-field plays an important role.

Generalised geometry can be seen formally as an *exact Courant algebroid.* For further details the reader can see the appendix B. The generalised tangent bundle is defined by the exact sequence

$$0 \longrightarrow T^*M \underset{B}{\overset{\iota}{\rightleftarrows}} E \underset{B}{\overset{\pi}{\rightleftarrows}} TM \longrightarrow 0 , \tag{2.69}$$

where we know the map B exists.

For any short exact sequence there is a remarkable theorem which is formulated inside *category theory.* The statement of the theorem, which is called "splitting lemma", is the following[7]

Theorem 2.1 (Splitting lemma)**.** *Given a short sequence with maps f and g*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 , \tag{2.70}$$

one writes the additional arrows ψ and φ for maps that may not exist

$$0 \to A \underset{\psi}{\overset{f}{\rightleftarrows}} B \underset{\varphi}{\overset{g}{\rightleftarrows}} C \to 0 . \tag{2.71}$$

Then the following statements are equivalent

(I) ***Left split***

there exists a map $\psi : B \to A$ such that the composition $\psi \circ f$ is the identity on A,

(II) ***Right split***

there exists a map $\varphi : C \to B$ such that the composition $g \circ \varphi$ is the identity on C,

(III) ***Direct sum***

B is isomorphic to the direct sum of A and C,

$$B \simeq A \oplus C , \tag{2.72}$$

with f corresponding to the natural injection of A and g corresponding to the natural projection onto C.

[7]There is a technical requirement in the hypothesis of the *splitting lemma*, which consists the sets A, B, C must be elements of an abelian category. We do not prove here that the category of tangent bundle is an abelian category.

For the proof of the *splitting lemma* the reader can see [33].

Since in the case of generalised geometry we have the *right* and *left splits* B, then by the equivalence of the three statements, we have that E is isomorphic to the direct sum of TM and T^*M,

$$E \simeq TM \oplus T^*M \ , \tag{2.73}$$

and ι is the natural injection (inclusion map) of T^*M and π is the natural projection (anchor) onto TM. This represent a formal result which prove the isomorphism between E and $TM \oplus T^*M$ thanks to the B map. For this reason we say the B field has the property to "untie" the fibres on E. In fat a generic section of $TM \oplus T^*M$ is on the form "vector $\oplus$ 1-form", i.e.

$$\begin{pmatrix} v \\ \lambda \end{pmatrix} \in TM \oplus T^*M \qquad\qquad v \in TM \ , \ \lambda \in T^*M \ , \tag{2.74}$$

and a generic section of E is given by a B transformation of the (2.74),

$$\begin{pmatrix} v \\ \lambda + i_v B \end{pmatrix} \in E \ . \tag{2.75}$$

Pictorially we say that the fibres in E are "intertwined" together.

2.3.5 Generalised coordinate basis

A question the reader may ask is "how can we define a coordinate basis for generalised vector?". We have seen in the chapter 2.3.1 the definition of the operator ∂_M. I want to stress that this object does not represent a choice of generalised coordinate basis, because inside its definition the symmetry between vectors and 1-forms is broken. In fact it represents just only an operator " ∂ " useful for rewriting Dorfman derivative in the same formal way of Lie derivative. In a sense, since ∂_μ represents how the exterior derivative operator d acts in components, ∂_M represent how the "generalised exterior derivative" works in components.

The definition of generalised coordinate basis is the following. Given a d-manifold, consider the coordinate basis $\{\hat{e}_\mu|_p\} = \{\partial_\mu|_p\}$ for the tangent space T_pM. By duality, we can define $\{e^\mu|_p\} = \{dx^\mu|_p\}$ such that

$$i_{\hat{e}_\mu} e^\nu|_p = \delta_\mu{}^\nu \ , \tag{2.76}$$

which provides a coordinate basis for the cotangent space T^*_pM.

Then we can define the coordinate basis for the space $TM \oplus T^*M$ simply putting in direct sum the two basis

$$\begin{pmatrix} \hat{e}_\mu \\ e^\mu \end{pmatrix} \ . \tag{2.77}$$

Since $E \simeq TM \oplus T^*M$, by a B field transformation of the (2.77) we get the generalised non coordinate basis $\{\hat{E}_M\}$

$$\hat{E}_M = \begin{cases} \begin{pmatrix} \hat{e}_\mu \\ i_{\hat{e}_\mu} B \end{pmatrix} & \qquad M = \mu \\ \\ \begin{pmatrix} 0 \\ \hat{e}^\mu \end{pmatrix} & \qquad M = \mu + d \end{cases} \ . \tag{2.78}$$

Since we have an $O(d,d)$ structure, the above basis must satisfy the condition on the scalar products

$$\langle \hat{E}_M, \hat{E}_N \rangle = \eta_{MN} \ , \qquad \eta_{MN} = \frac{1}{2}\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \ , \tag{2.79}$$

which is satisfied in force to the (2.76). The check is straightforward and it is the following

$$\hat{E}_M = \begin{pmatrix} \hat{e_\mu} \\ \hat{e}^\mu + i_{\hat{e}_\mu B} \end{pmatrix} \ , \qquad \hat{E}_N = \begin{pmatrix} \hat{e_\nu} \\ \hat{e}^\nu + i_{\hat{e}_\nu B} \end{pmatrix}$$

$$\begin{aligned} \langle \hat{E}_M, \hat{E}_N \rangle &= \frac{1}{2}\left[i_{\hat{e}_\mu}(\hat{e}^\nu + i_{\hat{e}_\nu} B) + i_{\hat{e}_\nu}(\hat{e}^\mu + i_\mu B) \right] \\ &= \frac{1}{2}\left[\delta_\mu{}^\nu + \delta_\nu{}^\mu + (i_{\hat{e}_\mu} i_{\hat{e}_\nu} + i_{\hat{e}_\nu} i_{\hat{e}_\mu}) B \right] = \eta_{MN} \ , \end{aligned}$$

By construction, the generalised tangent bundle is equipped with an $O(d,d)$ structure. Therefore one can always make a change of basis

$$V^M \mapsto V'^M = O^M{}_N V^N \ , \qquad \hat{E}_M \mapsto \hat{E}'_M = \hat{E}'_N (O^{-1})^N{}_M \ , \tag{2.80}$$

where $O \in O(d,d)$, and the basis $\{\hat{E}'_M\}$ still satisfies the condition (2.79). Since $GL(d) \subset O(d,d)$, if we choose

$$O = \begin{pmatrix} M & 0 \\ 0 & M^{-T} \end{pmatrix} \ , \tag{2.81}$$

where $M \in GL(d)$, then we have a frame rotation due to the trivial structure group $GL(d)$ acting in ordinary geometry. This argument will lead us to define in the next chapter the *generalised vielbein*.

3

Construction of generalised geometrical objects

Even before string theory, especially as physics developed in the 20th century,
it turned out that the equations that really work in describing nature
with the most generality and the greatest simplicity are very elegant and subtle.
- Edward Witten -

In this chapter we want to reconstruct into the framework of generalised geometry all geometrical objects that one defines in ordinary geometry. The logic we will follow, at least most of the time, consists in considering the definition of the particular object that we want to generalise contextualized in ordinary geometry, and trying to generalised it in the most natural way inside generalised geometry that we developed in the chapter 2.

The geometrical objects which we are going to generalised are the *metric*, *connection*, *torsion*, *Riemann tensor*, *Ricci tensor* and *scalar of curvature*. They represent the fundamental tools which we need in order to construct a geometrical theory of gravity. In fact General Relativity, without the intention to take off the extremely important physical meaning, can be seen as a particular application of Differential Geometry, where the tools involved are the one mentioned above.

Our purpose at the end of this chapter will be to rewrite Type II Supergravity theories as a theory of gravity but in the framework of generalised geometry, getting the so-called "generalised gravity".

3.1 Generalised metric

The beginning of generalisation starts from the construction of a *metric*, which is known to be a fundamental object of every geometrical theory of gravity.[1]

The way we follow for constructing generalised metric use an analogy with ordinary geometry and regards the splitting of the $O(d,d)$ structure.

[1]The main failure of *Loop Quantum Gravity* is given by the fact nowadays it is not understood how to construct a *metric*.

3.1.1 The $O(d) \times O(d)$ substructure

We have seen that Ordinary geometry is equipped by the trivial structure, provided by the $GL(d, \mathbb{R})$ group. Consider now its maximal compact subgroup, which is $O(d)$. If we restrict our attention only to $O(d)$, then we have an $O(d)$ substructure over the tangent bundle. When we restrict the structure group from $GL(d, \mathbb{R})$ to $O(d)$ means that we are restricting the choice of reference frame on the tangent bundle, between the all possible choices of reference frames, only to the orthonormal frames.

Extend now this idea to generalised geometry. The structure group is $O(d, d)$ and its maximal compact subgroup is $O(d) \times O(d)$. To be general, consider a spacetime of generic signature (p, q), with $p + q = d$. Therefore the maximal compact subgroup becomes $O(p, q) \times O(q, p)$.

Geometrically, an $O(p, q) \times O(q, p)$ substructure has an important consequence. The generalised tangent bundle E splits into two d-dimensional sub-bundles

$$E = C_+ \oplus C_- \ , \tag{3.1}$$

such that the $O(d, d)$ metric defined by the scalar product (2.29) , restricts to a separate metric of signature (p, q) on C_+ and a metric of signature (q, p) on C_-. In fact for a space spacetime of signature $(p, q) = (d, 0)$, the basis for C_+ and C_- is the basis that diagonalize the metric (2.29) according to the formula (2.31).

One can define a frame $\{\hat{E}^+_a\} \cup \{\hat{E}^-_{\bar{a}}\}$, such that $\{\hat{E}^+_a\}$ form an orthonormal basis for C_+, and $\{\hat{E}^-_{\bar{a}}\}$ form an orthonormal basis for C_-. The indices a and $\bar{a}$ runs both from 1 to d. The above statement means they satisfy

$$\begin{aligned}
\langle \hat{E}^+_a, \hat{E}^+_b \rangle &= \eta_{ab} \ , \\
\langle \hat{E}^-_{\bar{a}}, \hat{E}^-_{\bar{b}} \rangle &= -\eta_{\bar{a}\bar{b}} \ , \\
\langle \hat{E}^+_a, \hat{E}^-_{\bar{a}} \rangle &= \langle \hat{E}^-_{\bar{a}}, \hat{E}^+_a \rangle = 0 \ ,
\end{aligned} \tag{3.2}$$

where η_{ab} and $\eta_{\bar{a}\bar{b}}$ are flat metrics with signature (p, q). There is thus a manifest $O(p, q) \times O(q, p)$ symmetry with the first factor $O(p, q)$ acting on $\hat{E}^+_a$ and the second factor $O(q, p)$ acting on $\hat{E}^-_{\bar{a}}$.

Conceptually this represent an important point. One can think $O(p, q) \times O(q, p)$ as a “residual symmetry” after breaking the $O(d, d)$ symmetry. And in particular, the fascinating thing, is that in the two sub-bundle C_+ and C_- we get “ordinary geometry” with structure group $O(p, q)$ and $O(q, p)$ respectively.

We can recast the two basis $\{\hat{E}^+_a\}$ and $\{\hat{E}^-_{\bar{a}}\}$ into a single object

$$\hat{E}_A = \begin{cases} \hat{E}^+_a & \text{for} \qquad A = a \ , \\ \hat{E}^-_{\bar{a}} & \text{for} \qquad A = \bar{a} + d \ , \end{cases} \tag{3.3}$$

which satisfies

$$\langle \hat{E}_A, \hat{E}_B \rangle = \eta_{AB} \ , \qquad \text{where} \qquad \eta_{AB} = \begin{pmatrix} \eta_{ab} & 0 \\ 0 & -\eta_{\bar{a}\bar{b}} \end{pmatrix} \ . \tag{3.4}$$

We will adopt the convention that we will always raise and lower the C_+ indices $a, b, c, ...$ with η_{ab}, and the C_- indices $\bar{a}, \bar{b}, \bar{c}, ...$ with $\eta_{\bar{a}\bar{b}}$, while we continue to raise and lower $2d$ dimensional indices $A, B, C, ...$ with the $O(d, d)$ metric η_{AB}. Thus, for example we have

$$\hat{E}^A = \begin{cases} \hat{E}^{+a} & \text{for} \qquad A = a \ , \\ -\hat{E}^{-\bar{a}} & \text{for} \qquad A = \bar{a} + d \ , \end{cases} \tag{3.5}$$

when we raise the A index on the frame.

The two sub-bundle C_+ and C_- are not equal to TM and T^*M, but they are isomorphic. In fact the basis $\{\hat{E}_A\}$ must be a linear combinations of vectors and 1-forms, because the metric

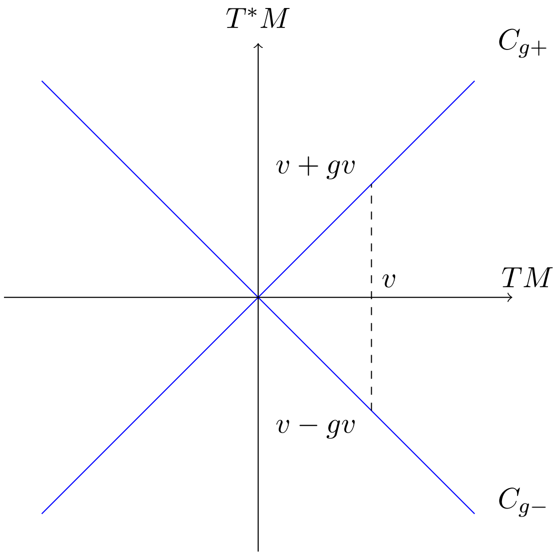

Figure 3.1: Splitting of E in $C_+ \oplus C_-$. In the picture it is represented, for $\mathcal{B} = 0$, how reach elements of C_{g+} and C_{g-} starting from a vector $v \in TM$.

η_{AB} is the diagonalized form of the $O(d,d)$ metric η_{MN} in (2.29), which is defined in the basis that does not mix vectors with forms. A graphic representation of the splitting is given by the picture (3.1).

A generic element $V_+ \in C_+$ can be written as

$$V_+ = \begin{pmatrix} v \\ Mv \end{pmatrix} , \tag{3.6}$$

where $v \in TM$ and, in components, the form part is given by $M_{\mu\nu}v^\nu$, for some general matrix M.This actually describes an isomorphism between TM and C_+. If we write $M_{\mu\nu} = \mathcal{B}_{\mu\nu} + g_{\mu\nu}$, where g is symmetric and $\mathcal{B}$ antisymmetric[2], then, by patching conditions (2.14) and fixing to zero the diffeomorphisms $A = 0$, one can check that g and $\mathcal{B}$ transform on the overlap $U_i \cup U_j$ as

$$g_i = g_j \ , \qquad\qquad \mathcal{B}_i = \mathcal{B}_j + d\Lambda_{(ij)} \ , \tag{3.7}$$

and hence $\mathcal{B}$ can be interpreted as the $\mathcal{B}$-field in the NSNS sector and g as the metric field for the graviton.

Orthogonality between C_+ and C_- implies that a generic element $V_- \in C_-$ can be written as

$$V_- = \begin{pmatrix} v \\ i_v(\mathcal{B} - g) \end{pmatrix} . \tag{3.8}$$

The check is the following

$$\langle V_+, V_- \rangle = \begin{pmatrix} v & i_v(\mathcal{B} + g) \end{pmatrix} \left[\frac{1}{2} \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \right] \begin{pmatrix} v \\ i_v(\mathcal{B} - g) \end{pmatrix} = i_v i_v(\mathcal{B} - g) + i_v i_v(\mathcal{B} + g) = 2 i_v i_v \mathcal{B} = 0 \ .$$

3.1.2 Generalised metric as a "projector"

In the previous paragraph we have shown the splitting of E into C_+ and C_- by the formula (3.1). The idea now is to define *projectors* Π_+ and Π_- which project a generalised vector V inside C_+ and C_- respectively.

[2]Every tensor field of type $(2,0)$ or $(0,2)$ can be decomposed in the symmetric plus the antisymmetric parts.

Decompose a generalised vector V in the basis $\{\hat{E}^+_a\}$ and $\{\hat{E}^-_{\bar{a}}\}$

$$V = V^a_+ \hat{E}^+_a + V^{\bar{a}}_- \hat{E}^-_{\bar{a}} \ . \tag{3.9}$$

Define projectors

$$\Pi_+ : E \to C_+ \ , \qquad \Pi_- : E \to C_- \ , \tag{3.10}$$

such that

$$\Pi_+(V) = V^a_+ \hat{E}^+_a \ , \qquad \Pi_-(V) = V^{\bar{a}}_- \hat{E}^-_{\bar{a}} \ . \tag{3.11}$$

The projectors $\Pi_\pm$ must satisfy the "equations of projectors"

$$\Pi^2_\pm = \Pi_\pm \qquad \Pi_+\Pi_- = \Pi_-\Pi_+ = 0 \ . \tag{3.12}$$

Let us parametrized $\Pi_\pm$ in terms of G in the following way [67]

$$\Pi_\pm = \frac{1}{2}\left(\mathbb{1} \pm G\right) \ , \tag{3.13}$$

and find a solution for G. It is easy to check that the equations (3.12) give us the condition $G^2 = \mathbb{1}$. The other condition we have to impose is

$$\eta(GV, GW) = \eta(V, W) \ , \tag{3.14}$$

which is equivalent to the three conditions (3.2). Therefore at the end of the day, G must satisfy the equations

$$G^2 = \mathbb{1} \ , \qquad G^T \, \eta \, G = \eta \ . \tag{3.15}$$

The solution of G in the basis $\{\hat{E}^+_a\}$ and $\{\hat{E}^-_{\bar{a}}\}$ is the following

$$G = \eta^{ab} \hat{E}^+_a \otimes \hat{E}^+_b + \eta^{\bar{a}\bar{b}} \hat{E}^-_{\bar{a}} \otimes \hat{E}^-_{\bar{a}} \ , \tag{3.16}$$

and the solution in the coordinate frame $\{\hat{E}_M\}$ has the familiar expression

$$G_{MN} = \frac{1}{2} \begin{pmatrix} g - \mathcal{B}g^{-1}\mathcal{B} & \mathcal{B}g^{-1} \\ -g^{-1}\mathcal{B} & g^{-1} \end{pmatrix}_{MN} \ . \tag{3.17}$$

The procedure to obtain G_{MN} in (3.17) from G_{AB} in (3.16) is explained in the next paragraph and make use of the *generalised vielbein.*

I want to remark why the solution obtained is reasonable. Since a projector is a map from E to $C_\pm$, it must get rid of d degrees of freedom. Therefore G must contain inside d degrees of freedom. These d degrees of freedom are parametrized by a second rank symmetric tensor $g_{\mu\nu}$, which has $d(d+1)/2$ d.o.f., and by a 2-form $\mathcal{B}_{\mu\nu}$, which has $d(d-1)/2$ d.o.f.. In total the solution (g, B) has d^2 d.o.f. and it must parametrise the coset

$$\frac{O(d,d)}{O(p,q) \times O(q,p)} \ , \tag{3.18}$$

which represent the "extra d.o.f." that the projectors take out.

We have seen in the previous section the physical meaning of g and $\mathcal{B}$, which *a priori* they represent only a way to parametrize a solution. Since inside G there is the metric of the manifold g, then G is also called *generalised metric.* This represent an heuristic way to make appear the generalised metric. There are other non heuristic way to define generalised metric (see [26]).

3.1.3 Generalised vielbein

Suppose to define a metric over a manifold M, which represent a smooth assignment of an inner product map on each $T_pM \otimes T_pM \to \mathbb{R}$. In coordinates the metric is specified by a covariant second rank symmetric tensor field $g_{\mu\nu}(x)$, and the inner product of two contravariant vectors $V^\mu(x)$ and $U^\mu(x)$ is $g_{\mu\nu}V^\mu(x)U^\nu(x)$, which is a scalar field.

There is a particular metric with which the scalar products are easier. We are talking about the "Minkowski metric" η, which is a diagonal metric and depends only by the signature of the spacetime, and not by the particular point $p \in M$.

The idea of the *veilbein* is to diagonalize locally, point by point on the manifold, the metric $g_{\mu\nu}(x)$, leading it to the Minkowski metric. This procedure is obtain changing the basis for T_pM point by point. Suppose the metric $g_{\mu\nu}(x)|_p$ may be diagonalized by the orthogonal matrix $O \in O(d)$, where $O(d)$ represent also the maximal compact

$$g_{\mu\nu} = O_\mu{}^a \, D_{ab} \, O^b{}_\nu \ , \tag{3.19}$$

where D is a diagonal matrix

$$D = \mathrm{diag}(-\lambda^0, \lambda^1, ..., \lambda^{d-1}) \qquad \text{with} \qquad \lambda^i > 0 \quad \forall\, i = 0, ..., d-1 \ . \tag{3.20}$$

The form of D is determined by the facts that D and g have got the same eigenvalues, g is non-degenerate, and our convention for the spacetime signature is the most employed in Supergravity, i.e. $(- + \cdots +)$.

Define now the new matrix $e_a{}^\mu$ and its inverse $e^a{}_\mu$, which satisfies

$$e^a{}_\mu e_a{}^\nu = \delta_\mu{}^\nu \ , \qquad e^a{}_\mu e_b{}^\mu = \delta^a{}_b \ , \tag{3.21}$$

such that

$$e^a{}_\mu(x) = \sqrt{\lambda^a(x)} O^a{}_\mu(x) \ , \tag{3.22}$$

where in the RHS the index a is not intended to be contracted. In four dimensions this matrix is commonly called the *tetrad* or *vierbein*, while in general dimensions it is called *vielbein*.

With this formalism the metric can be written as

$$g_{\mu\nu}(x) = e^a{}_\mu(x)\, \eta_{ab}\, e^b{}_\nu(x) \ , \tag{3.23}$$

where η is the Minkowski metric in d dimensions.

The new basis $\{\hat{e}_a|_p\}$ for T_pM is related to the old one $\{\hat{e}_\mu|_p\}$ by the vielbein transformation

$$\hat{e}_a|_p = e_a{}^\mu(x)\, \hat{e}_\mu|_p \ . \tag{3.24}$$

Often μ is called "curved index" and a "flat index", for the form which the metric g assumes in the two basis

$$g(\hat{e}_\mu, \hat{e}_\nu) = g_{\mu\nu} \ , \qquad g(\hat{e}_a, \hat{e}_b) = \eta_{ab} \ . \tag{3.25}$$

A similar construction is also available in generalised geometry. The index M, which is referred to the $O(d,d)$ frame, is the analogue of the curved index μ in ordinary geometry, while the index A, which is referred to the $O(d) \times O(d)$ frame, is the analogue of the flat index a.

The *generalised vielbein* is defined as follow

$$\hat{E}_A = E_A{}^M \hat{E}_M \ , \tag{3.26}$$

and it represents the matrix of change of basis. Again, its inverse is $\hat{E}^A{}_M$ such that

$$E^A{}_M E_A{}^N = \delta_M{}^N \ , \qquad E^A{}_M E_B{}^M = \delta^A{}_B \ . \tag{3.27}$$

The expression of the generalised vielbein is given in terms of g and $\mathcal{B}$ and it comes from considering how the vectors V_+ and V_- can be written in terms of vector component and 1-form component.

The solution is the following

$$E^A{}_M = \frac{1}{\sqrt{2}} \begin{pmatrix} e_+ & e_- \\ e_+(g-\mathcal{B}) & e_-(g+\mathcal{B}) \end{pmatrix} , \tag{3.28}$$

where $e_\pm$ are the ordinary vielbeins which satisfies

$$\begin{aligned} g = e_+^T \eta \, e_+ \ , \qquad &\Longrightarrow \qquad g_{\mu\nu} = e^a_{+\,\mu} \, \eta_{ab} \, e^b_{+\,\nu} \ , \\ g = e_-^T \eta \, e_- \ , \qquad &\Longrightarrow \qquad g_{\mu\nu} = e^{\bar a}_{-\,\mu} \, \eta_{\bar a \bar b} \, e^{\bar b}_{+\,\nu} \ , \end{aligned} \tag{3.29}$$

and $1/\sqrt{2}$ is a normalization factor.

Now we can check that starting from generalised vectors in TM and in T^*M, such as

$$V^M = \begin{pmatrix} v \\ 0 \end{pmatrix} , \qquad W^M = \begin{pmatrix} 0 \\ \lambda \end{pmatrix} , \tag{3.30}$$

we can apply on them a vielbein transformation and obtain

$$V_+^A = E^A{}_M V^M = \frac{1}{\sqrt{2}} \begin{pmatrix} e_+ v \\ e_+(g-\mathcal{B})v \end{pmatrix} , \qquad V_-^A = E^A{}_M W^M = \frac{1}{\sqrt{2}} \begin{pmatrix} e_- \lambda \\ e_-(g+\mathcal{B})\lambda \end{pmatrix} , \tag{3.31}$$

which they are the vectors V_+ and V_- in the desired form. The contractions between "μ type" indices between vielbein, $g, \mathcal{B}$ tensors and v, λ are intended to be done with the $\eta_{\mu\nu}$ metric of $O(p,q)$.

With the formalism of vielbain one can rewrite objects written in "curved indices", like the $O(d,d)$ metric η_{MN} and the generalised metric G_{MN} in therms of objects written in "flat indices". In fact we have the following result

$$\eta_{MN} = E^A{}_M \, \eta_{AB} \, E^B{}_N \qquad\qquad G_{MN} = E^A{}_M \, G_{AB} \, E^B{}_N \ , \tag{3.32}$$

and in matrix form

$$\eta = E^T \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} E \ , \qquad G = E^T \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} E \ . \tag{3.33}$$

This is the procedure we adopted in the above paragraph in order to write down G_{MN}. In fact the formulæ (3.32) give rise to (2.31) and to (3.17).

3.1.4 The importance of $O(p,q) \times O(q,p)$-substructure

In general, given a G-structure, the substructure generated by the maximal compact subgroup plays a very important role which is not only related to the definition of generalised metric. In fact, focusing to the $O(p,q) \times O(q,p)$-substructure of $O(d,d)$-structure, one can do the following observations.

In type II String Theory compactified on a 6-dimensional manifold, the sub-bundles $C_\pm$ have a natural interpretation in terms of the world-sheet theory. They are associated to the fermions left and right moving and $e_\pm$ are the corresponding vielbeins. The spinor fields associated to fermions transforms under the *double cover* of the group $O(p,q)$, which is the group $Pin(p,q)$. Usually one considers only the connected to the identity component[3] of $O(p,q)$, which is $SO(p,q)$, thus the double cover is $Spin(p,q)$.

[3]Because it is the only component that can be reached exponentiating the generators of the algebra.

Let us now assume we have a $Spin(p,q) \times Spin(q,p)$-structure. Then we can define $S(C_\pm)$ which are the spinor bundles associated to the sub-bundles $C_\pm$. Then one can define the corresponding gamma matrices γ^a and $\gamma^{\bar{a}}$ and spinor fields associated to left/right moving fermions as sections $\epsilon^\pm \in \Gamma(S(C_\pm))$. These represent the fundamental ingredients for inserting fermions inside generalised geometry. Since this is not the aim of this work, we will not talk about spinors anymore. Further informations about this topic are available in [15].

I just want to mention a result about maximal compact subgroup. There is not a unique way to define generalised geometry, and in each definition the structure group which appears can be different. Therefore one may ask if in other generalised geometry we lose the possibility to find a maximal structure group, and hence lose the possibility to define a generalised metric and construct a theory for fermions. Fortunately this is not the case, because of the following result

Theorem 3.1 (Cartan-Iwasawa-Malcev)**.** *Every connected Lie group G admits maximal compact subgroup K. It is in general not unique, but is unique up to conjugation, meaning that given two maximal compact subgroup, K and H, there is an element $g \in G$ such that*

$$gKg^{-1} = H \ . \tag{3.34}$$

The proof of the theorem can be found in [11] and [34]. Hence this theorem is important because it assures that in whatever generalised geometry one can always follow the procedure developed for $T \oplus T^*$ generalised geometry.

3.2 Generalised connection

Let us to remind briefly the key points to keep in mind in order to define a connection in ordinary geometry. Then by analogy we will move to generalised connection.

3.2.1 Connection in ordinary geometry

A *connection* (called also *covariant derivative*) is a map

$$\nabla_{(\cdot)}(\cdot) : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM) \ , \tag{3.35}$$

which satisfies the following properties

- $\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z$,
- $\nabla_{X+Y} Z = \nabla_X Z + \nabla_Y Z$,
- $\nabla_{fX} Y = f \nabla_X Y$,
- $\nabla_X(fY) = X[f]Y + f\nabla_X Y$,

for each $X, Y, Z \in \Gamma(TM)$ and for each function f from the manifold M to $\mathbb{R}$. In a coordinate basis $\{e_\mu\} = \{\partial/\partial x^\mu\}$, by definition, the connection acts as

$$\nabla_{e_\nu} e_\mu \equiv \nabla_\nu e_\mu = \Gamma_\mu{}^\lambda{}_\nu \, e_\lambda \ . \tag{3.36}$$

$\Gamma_\mu{}^\lambda{}_\nu$ is called *affine connection* and represents how we choose to transport vectors around the manifold M.

So far the affine connection can be arbitrary. If a *metric* is defined over the manifold M, then it is reasonable to put some restriction on Γ demanding that the metric $g_{\mu\nu}$ be *covariantly constant*, which reads

$$(\nabla_\kappa g)_{\mu\nu} = 0 \ . \tag{3.37}$$

If two vectors X and Y are parallel transported along the curve $c(t)$ which has tangent vector $V = [dx^\mu(c(t))/dt]\, e_\mu|_{c(t)}$, i.e.

$$\nabla_V X = 0 \ , \qquad\qquad \nabla_V Y = 0 \ , \tag{3.38}$$

then, using the constraint (3.37) and (3.38), the inner product between X and Y along $c(t)$ is constant, i.e.

$$\nabla_V \left[g(X,Y)\right] = V^\mu \left[(\nabla_\mu g)(X,Y) + g(\nabla_\mu X, Y) + g(X, \nabla_\mu Y)\right] = 0 \ . \tag{3.39}$$

The connection ∇ which satisfies (3.37) is called *metric connection*. The condition (3.37) gives us an expression for Γ (the derivation can be found in [46])

$$\Gamma_\mu{}^\lambda{}_\nu = \tilde{\Gamma}_\mu{}^\lambda{}_\nu + \frac{1}{2}\left(T_\mu{}^\lambda{}_\nu + T_\nu{}^\lambda{}_\mu + T^\lambda{}_{\mu\nu}\right) \ , \tag{3.40}$$

where $\tilde{\Gamma}_\mu{}^\lambda{}_\nu$ is the *Christoffel symbols*, symmetric in μ and ν, defined by

$$\tilde{\Gamma}_\mu{}^\lambda{}_\nu = \frac{1}{2} g^{\lambda\varepsilon}\left(\partial_\mu g_{\varepsilon\nu} + \partial_\nu g_{\mu\varepsilon} - \partial_\lambda g_{\mu\nu}\right) \ , \tag{3.41}$$

and $T^\lambda{}_{\mu\nu}$ is antisymmetric respect to the lower indices μ and ν and is called *torsion tensor*. We will define properly this object in the paragraph 3.3 when we will define the "generalised torsion".

If the torsion tensor vanishes on a manifold M, the metric connection ∇ is called the *Levi-Civita connection*, or *torsion-free metric connection*. Levi-Civita connections are natural generalization of the connection defined in the classical geometry of surfaces and always exists at least for (pseudo)-Riemannian manifolds. This is guaranteed by the following

Theorem 3.2 (The fundamental theorem of Riemannian geometry). *On any Riemannian manifold (or pseudo-Riemannian manifold) (M, g), there is a unique Levi-Civita connection.*

In particular the affine connection for a Levi-Civita connection are equal to the Cristoffel symbols.

Now, instead of considering the coordinate basis $\{e_\mu\}$, we pick up the non coordinate basis $\{e_a\}$ such that $g_{ab} = \eta_{ab}$, obtained acting with a vielbein on the coordinate basis. Let be $\{\theta^a\}$ the dual non coordinate basis. The action of connection on the non coordinate basis is the following

$$\nabla_\mu e_a = \omega_\mu{}^b{}_a\, e_b \ , \tag{3.42}$$

where $\omega_\mu{}^b{}_a$ is called *spin connection*. Therefore when we consider a tensor T which possesses both types of indices (coordinate and non coordinate type), i.e.

$$T = T^{\mu a}{}_{\nu b}\, \partial_\mu \otimes e_a \otimes dx^\nu \otimes \theta^b \ , \tag{3.43}$$

the connection acts on this object as

$$(\nabla_\gamma T)^{\mu a}{}_{\nu b} = \partial_\gamma T^{\mu a}{}_{\nu b} + \Gamma_\gamma{}^\mu{}_\varphi T^{\varphi a}{}_{\nu b} + \omega_\gamma{}^a{}_c T^{\mu c}{}_{\nu b} - \Gamma_\gamma{}^\varphi{}_\nu T^{\mu a}{}_{\varphi b} - \omega_\gamma{}^c{}_b T^{\mu a}{}_{\nu c} \ . \tag{3.44}$$

The metric compatibility condition has an implication on the symmetry of ω. Let us consider the metric expressed in non coordinate basis $g = \eta_{ab}\theta^a \otimes \theta^b$. Then we have

$$\begin{aligned} 0 = \nabla_\mu g &= (\partial_\mu \eta_{ab})\, \theta^a \otimes \theta^b + \eta_{ab}(\nabla_\mu \theta^a) \otimes \theta^b + \eta_{ab}\, \theta^a \otimes (\nabla_\mu \theta^b) \\ &= \eta_{ab}\left(\omega_\mu{}^a{}_c\, \theta^c \otimes \theta^b + \theta^a \otimes \omega_\mu{}^b{}_c\, \theta^c\right) = (\omega_{\mu bc} + \omega_{\mu cb})\, \theta^c \otimes \theta^b \ , \end{aligned} \tag{3.45}$$

which implies

$$\omega_{\mu bc} = -\omega_{\mu cb} \ . \tag{3.46}$$

The equation (3.46) has a geometrical interpretation. When one defines a metric g on a manifold M in ordinary geometry means we are defining an $O(d)$-structure. The Lie algebra of $O(d)$ group is

$$\mathfrak{o}(d) = \mathfrak{so}(d) = \{M \in GL(d, \mathbb{R}) | M = -M^T\} \ . \tag{3.47}$$

Therefore $\omega_{\mu ab}$ is an element of the $\mathfrak{o}(d)$ algebra. This is a general result: affine connections defined on a G-principle bundle are element of the corresponding $\mathfrak{g}$ algebra[4].

3.2.2 Connection in generalised geometry

In this section we will follow closely the procedure treated in the above paragraph in order to extend the concept of connection in generalised geometry.

A *generalised connection* must be a map

$$D_{(\cdot)}(\cdot) : \Gamma(E) \times \Gamma(E) \to \Gamma(E) \tag{3.48}$$

which satisfies the same properties of the ordinary connection presented before, just only replacing sections of TM with sections of E. The definition of generalised connection in a coordinate basis $\{\hat{E}_M\}$ is

$$D_M \hat{E}_N = \Gamma_M{}^P{}_N \hat{E}_P \ , \tag{3.49}$$

where here $\Gamma_M{}^P{}_N$ is called *generalised affine connection* and it is the generalisation of the affine connection $\Gamma_\mu{}^\lambda{}_\nu$. Now we have to take into account the fact in generalised geometry there is an $O(d, d)$-structure. Therefore the first constraint to impose in order to determine $\Gamma_M{}^P{}_N$, is the compatibility with the $O(d, d)$ metric $\eta = \eta_{MN} \hat{E}^M \otimes \hat{E}^N$.

$$\begin{aligned} 0 = D_P \eta &= (\partial_P \eta_{MN}) \, \hat{E}^M \otimes \hat{E}^N + \eta_{MN} (D_P \hat{E}^M) \otimes \hat{E}^N + \eta_{MN} \hat{E}^M \otimes (D_P \hat{E}^N) && (3.50) \\ &= \eta_{MN} \left(\Gamma_P{}^M{}_Q \hat{E}^Q \otimes \hat{E}^N + \hat{E}^M \otimes \Gamma_P{}^N{}_Q \hat{E}^Q \right) = (\Gamma_{PNQ} + \Gamma_{PQN}) \, \hat{E}^Q \otimes \hat{E}^N \ , && (3.51) \end{aligned}$$

again, which implies

$$\Gamma_{PNQ} = -\Gamma_{PQN} \ . \tag{3.52}$$

Therefore Γ_{PMN} must be an element of $\mathfrak{o}(d, d) = \mathfrak{so}(d, d)$ algebra. This symmetry property does not have analogous in ordinary geometry for $\Gamma_\mu{}^\lambda{}_\nu$ and it is due to the $O(d, d)$-structure.

Starting from a conventional connection ∇ in ordinary geometry, there is a procedure to construct the corresponding generalised connection, which is denoted by D^∇ for emphasizing the construction. Let us take a generalised vector V. The idea is to act with connection on the vector component and 1-form component separately in $TM \oplus T^*M$ and then perform a B-shift in order to come back again in E. In details, let be

$$V^M = \begin{pmatrix} v^\mu \\ \lambda_\nu \end{pmatrix} \in \Gamma(TM \oplus T^*M) \ . \tag{3.53}$$

The action of ∇_M on $TM \oplus T^*M$ is

$$\begin{pmatrix} \nabla_\mu v \\ \nabla_\mu \lambda \end{pmatrix} \ , \tag{3.54}$$

[4] Remind that a G-structure can be seen as a G-principal sub-bundle of the tangent frame bundle.

where by definition

$$\nabla_M \equiv \begin{cases} \nabla_\mu & \text{if} \quad M = \mu = 1, ..., d \\ 0 & \text{if} \quad M = \mu + d \end{cases} . \tag{3.55}$$

Now performing a B-shift on (3.54) we end on E with the desired expression for generalised connection

$$D^\nabla_M V = \begin{cases} \begin{pmatrix} \nabla_\mu v \\ \nabla_\mu \lambda + i_{\nabla_\mu v} B \end{pmatrix} & \text{if} \quad M = \mu = 1, ..., d \\ \\ 0 & \text{if} \quad M = \mu + d \end{cases} . \tag{3.56}$$

Another way to define generalised connection starting from ordinary connection consists to impose by definition the null action of D_M on the B field. In this procedure one starts from sections of E

$$V = \begin{pmatrix} v \\ \lambda + i_v B \end{pmatrix} \in \Gamma(E) , \tag{3.57}$$

and then, by definition

$$D^\nabla_M V = \begin{pmatrix} \nabla_M v \\ \nabla_M \lambda + i_{\nabla_M v} B \end{pmatrix} \tag{3.58}$$

where we can see ∇_M did not act on $i_{\nabla_M v} B$. Again, ∇_M is defined as in (3.53).

We can also define generalised connection in a non coordinate basis $\{\hat{E}_A\}$ as follow

$$D_M \hat{E}_N = \Omega_M{}^P{}_N \hat{E}_P , \tag{3.59}$$

where $\Omega_M{}^P{}_N$ is called *generalised spin connection*. Again, if we consider a generalised tensor with both coordinate type and non coordinate type indices, i.e.

$$T = T^{MA}{}_{NB} \hat{E}_M \otimes \hat{E}_A \otimes \hat{E}^N \otimes \hat{E}^B , \tag{3.60}$$

then generalised connection acts as

$$\begin{aligned} D_P T^{MA}{}_{NB} = \partial_P T^{MA}{}_{NB} + \Gamma_P{}^M{}_Q T^{MA}{}_{NB} + \Omega_P{}^A{}_C T^{MC}{}_{NB} \\ - \Gamma_P{}^Q{}_N T^{MA}{}_{QB} - \Omega_P{}^C{}_B T^{MA}{}_{NC} . \end{aligned} \tag{3.61}$$

Suppose now to introduce the generalised metric G on the manifold M. This means we are restricting on the $O(p,q) \times O(q,p)$-substructure. A generalised connection D is compatible with the $O(p,q) \times O(q,p)$-substructure if it is compatible with the metric G, which is the analogous requirement of metric connection in ordinary geometry. Choosing the non coordinate basis the metric assumes the form $G = \eta^{ab} \hat{E}^+_a \otimes \hat{E}^+_b + \eta^{\bar{a}\bar{b}} \hat{E}^-_{\bar{a}} \otimes \hat{E}^-_{\bar{b}}$, and we have

$$\begin{aligned} 0 = D_M G &= (\partial_M \eta^{ab})\, \hat{E}^+_a \otimes \hat{E}^+_b + \eta^{ab} (D_M \hat{E}^+_a) \otimes \hat{E}^+_b + \eta^{ab} \hat{E}^+_a \otimes (D_M \hat{E}^+_b) \\ &+ (\partial_M \eta^{\bar{a}\bar{b}})\, \hat{E}^-_{\bar{a}} \otimes \hat{E}^-_{\bar{b}} + \eta^{\bar{a}\bar{b}} (D_M \hat{E}^-_{\bar{a}}) \otimes \hat{E}^-_{\bar{b}} + \eta^{\bar{a}\bar{b}} \hat{E}^-_{\bar{a}} \otimes (D_M \hat{E}^-_{\bar{b}}) \\ &= \eta^{ab} \left(\Omega_M{}^a{}_c \hat{E}^+_c \otimes \hat{E}^+_b + \hat{E}^+_a \otimes \Omega_M{}^b{}_c \hat{E}^+_c \right) + \eta^{\bar{a}\bar{b}} \left(\Omega_M{}^{\bar{a}}{}_{\bar{c}} \hat{E}^-_{\bar{c}} \otimes \hat{E}^-_{\bar{b}} + \hat{E}^-_{\bar{a}} \otimes \Omega_M{}^{\bar{b}}{}_{\bar{c}} \hat{E}^-_{\bar{c}} \right) \\ &= (\Omega_{Mbc} + \Omega_{Mcb})\, \hat{E}^+_c \otimes \hat{E}^+_b + (\Omega_{M\bar{b}\bar{c}} + \Omega_{M\bar{c}\bar{b}})\, \hat{E}^-_{\bar{c}} \otimes \hat{E}^-_{\bar{b}} . \end{aligned} \tag{3.62}$$

and again we have

$$\Omega_{Mab} = -\Omega_{Mba} \ , \qquad\qquad \Omega_{M\bar{a}\bar{b}} = -\Omega_{M\bar{b}\bar{a}} \ , \tag{3.63}$$

which means $\Omega_{Mab} \in \mathfrak{o}(p,q)$ and $\Omega_{M\bar{a}\bar{b}} \in \mathfrak{o}(q,p)$.

Equivalently a generalised connection D is compatible with the $O(p,q)\times O(q,p)$-substructure if the derivative acts separately in the two C_+ and C_- sub-bundle, without mixing the two types of indices. This is actually what we did in (3.62) because terms like $\Omega_M{}^{\bar{c}}{}_a \hat{E}^{\bar{c}}$ when D_M acts on $\bar{E}^a$ do not appear.

Therefore the "generalised metric connection", for a generalised vector written in the basis $\{\hat{E}^+_a\} \cup \{\hat{E}^-_{\bar{a}}\}$

$$V = V^a_+ \hat{E}^+_a + V^{\bar{a}}_- \hat{E}^-_{\bar{a}} \ , \tag{3.64}$$

it acts as

$$D_M V^A = \begin{cases} \partial_M V^a_+ + \Omega_M{}^a{}_b V^b_+ & \text{for} \quad A = a \\ \partial_M V^{\bar{a}}_- + \Omega_M{}^{\bar{a}}{}_{\bar{b}} V^{\bar{b}}_- & \text{for} \quad A = \bar{a} \end{cases} \ , \tag{3.65}$$

Now there is a natural choice for the spin connection $\Omega_M{}^a{}_b$. We mentioned above that if a Levi-Civita connection ∇ is defined on a (pseudo) Riemannian manifold (M,g), then there is a natural construction for the generalised connection D^∇ associated to ∇. This procedure leads us to the following fact. Let be $v \in \Gamma(TM)$. In accord to the picture 3.1 we can parametrize v using the vector components of the basis $\{\hat{E}^+_a\}$ for C_+, which is $\{\hat{e}^+_a\}$, or equivalently, using the vector components of the basis $\{\hat{E}^-_{\bar{a}}\}$ for C_-, which is $\{\hat{e}^-_{\bar{a}}\}$. In each $C_\pm$ sub-bundle there is an $O(d)$-structure which constraint the spin connections $\omega^+_{\mu ab}$ and $\omega^-_{\mu\bar{a}\bar{b}}$, by compatibility with $O(d)$ metric, to be antisymmetric in the two flat indices, i.e.

$$\omega^+_{\mu ab} = -\omega^+_{\mu ba} \ , \qquad\qquad \omega^-_{\mu\bar{a}\bar{b}} = -\omega^-_{\mu\bar{b}\bar{a}} \ . \tag{3.66}$$

The two description of v are equivalent. Given $v = v^a \hat{e}^+_a = v^{\bar{a}} \hat{e}^-_{\bar{a}}$, the connection acts on v as

$$\nabla_\mu v^\nu = \left(\partial_\mu v^a + \omega^+_\mu{}^a{}_b v^b\right) \hat{e}^{+\,\nu}_a = \left(\partial_\mu v^{\bar{a}} + \omega^-_\mu{}^{\bar{a}}{}_{\bar{b}} v^{\bar{b}}\right) \hat{e}^{-\,\nu}_{\bar{a}} \tag{3.67}$$

The natural choice is to take the generalised spin connections Ω_{Mab} and $\Omega_{M\bar{a}\bar{b}}$ to be equal the ordinary spin connections in the $C_\pm$ sub-bundles

$$\Omega_{\mu ab} = \omega^+_{\mu ab} \ , \qquad\qquad \Omega_{\mu\bar{a}\bar{b}} = \omega^-_{\mu\bar{a}\bar{b}} \ , \tag{3.68}$$

and by definition of D_M and ∂_M, we have

$$\Omega_{\mu+d\,ab} = 0 \ , \qquad\qquad \Omega_{\mu+d\,\bar{a}\bar{b}} = 0 \ . \tag{3.69}$$

Therefore at the end of the day, the generalised connection (3.65) acts as follow

$$D^\nabla_M V^a_+ = \begin{cases} \nabla_\mu V^a_+ & \text{for } M = \mu \\ 0 & \text{for } M = \mu + d \end{cases} \ , \qquad D^\nabla_M V^{\bar{a}}_- = \begin{cases} \nabla_\mu V^{\bar{a}}_- & \text{for } M = \mu \\ 0 & \text{for } M = \mu + d \end{cases} \tag{3.70}$$

By construction the above generalised connection is compatible with the $O(p,q) \times O(q,p)$-structure. However it is not *torsion-free*, which means the generalised torsion tensor, computed with the generalised connection D^∇, is not identically zero. In the next paragraph we will define generalised torsion and show that

Proposition 3.1. *Given an $O(p,q) \times O(q,p)$-structure, there always exists a Levi-Civita connection, i.e. torsion-free and metric compatible connection. However it is not unique.*

3.3 Generalised torsion

In ordinary geometry one introduced *torsion tensor* as a map

$$\mathcal{T} : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM) \ , \tag{3.71}$$

defined by

$$\mathcal{T}(v, w) = \nabla_v w - \nabla_w v - [v, w] \qquad\qquad v, w \in \Gamma(TM) \ . \tag{3.72}$$

The torsion tensor enjoys a formal remarkable fact which consists in the possibility to be expressed only in terms of Lie derivatives. Given a coordinates basis $\{\partial_\mu\}$ and an affine connection ∇, the torsion tensor can be expressed as

$$\mathcal{T}(v, w) = \mathcal{L}^{\nabla}_v w - \mathcal{L}_v w = v^\mu \nabla_\mu w - w^\mu \nabla_\mu v - v^\mu \partial_\mu w + w^\mu \partial_\mu v \ , \tag{3.73}$$

where the symbol $\mathcal{L}^{\nabla}_{(\cdot)}(\cdot)$ is a notation for the Lie derivative expresses in coordinate basis with ∂ replaced by ∇.

This time, instead of generalising the torsion tensor starting from the definition, we prefer to perform the generalization of the formal remarkable fact which satisfies and buy it as a *definition* of generalised torsion tensor. Therefore we have

Definition (Generalised torsion tensor)**.** *Given a generalised connection D, "generalised torsion tensor" is a map*

$$T : \Gamma(E) \times \Gamma(E) \to \Gamma(E) \ , \tag{3.74}$$

acting on $V, W \in \Gamma(E)$ as follow

$$T(V, W) = L^D_V W - L_V W \ . \tag{3.75}$$

where $L^D_{(\cdot)}(\cdot)$ is the Dorfman derivative with the operator ∂ replaced by D.

What we are going to do now is provide a sketch of proof of proposition (3.1), giving the references for long computations we will skip. The generalised torsion tensor computed with the generalised connection D^∇ is not zero, but it is proportional to the flux $H = d\mathcal{B}$,

$$T(D^\nabla) = -4H \ . \tag{3.76}$$

The sketch of proof can be found in [15]. However we can redefine generalised connection adding a generic generalised tensor Σ, as following

$$D_M W^A = D^\nabla_M W^A + \Sigma_M{}^A{}_B W^B \ . \tag{3.77}$$

The compatibility of D with $O(p, q) \times O(p, q)$-structure give us the following constraint on Σ

$$\Sigma_{Mab} = \Sigma_{Mba} \ , \qquad \Sigma_{M\bar{a}\bar{b}} = \Sigma_{M\bar{b}\bar{a}} \ , \qquad \Sigma_M{}^a{}_{\bar{b}} = \Sigma_M{}^{\bar{a}}{}_b = 0 \ . \tag{3.78}$$

The generalised torsion tensor, computed with the new generalised connection D, in [15] is shown to be

$$T_{ABC}(D) = -4H_{ABC} - 3\Sigma_{[ABC]} \ , \tag{3.79}$$

where the components H_{ABC} are the components of H in the frame basis (i.e. with "flat indices"). The same rule is valid for the other quantities which appears in the equation (3.79).

Now one requires the generalised torsion tensor $T(D)$ vanishes identically. This will give us the following equation for $\Sigma_{[ABC]}$

$$3\Sigma_{[ABC]} = -4H_{ABC} \ . \tag{3.80}$$

The 3-form H, in coordinate basis is

$$H = \frac{1}{3!} H_{\mu\nu\rho} \, dx^\mu \wedge dx^\nu \wedge dx^\rho \ , \tag{3.81}$$

and we have to express it the frame basis. Using the generalised vielbein, one can derive the expression of dx^μ in terms of $\{\hat{E}_A\}$, which is

$$dx^\mu = \frac{1}{2}\left(e_a^{+\ \mu} \hat{E}^{+a} - e_{\bar{a}}^{-\ \mu} \hat{E}^{-\bar{a}} \right) \ . \tag{3.82}$$

The formula (3.82) is an explicit expression for the embedding $T * M \hookrightarrow E = C_+ \oplus C_-$. There is a similar decomposition of H under the embedding

$$\bigwedge^3 T^* M \hookrightarrow \bigwedge^3 E = \bigwedge^3 C_+ \oplus 3\left(\bigwedge^2 C_+ \otimes C_- \right) \oplus 3\left(C_+ \otimes \bigwedge^2 C_- \right) \oplus \bigwedge^3 C_- \ , \tag{3.83}$$

where we used the binomial expansion (as for polynomials). In fact this coefficient decomposition appears in the frame basis expression of H, obtained inserting (3.82) inside (3.81)

$$\begin{aligned} H = \frac{1}{6} H_{\mu\nu\rho} \, dx^\mu \wedge dx^\nu \wedge dx^\rho = \frac{1}{8} \Bigg(& \frac{1}{6} H^{abc} \hat{E}_a^+ \wedge \hat{E}_b^+ \wedge \hat{E}_c^+ - \frac{3}{6} H^{ab\bar{c}} \hat{E}_a^+ \wedge \hat{E}_b^+ \wedge \hat{E}_{\bar{c}}^- \\ & + \frac{3}{6} H^{a\bar{b}\bar{c}} \hat{E}_a^+ \wedge \hat{E}_{\bar{b}}^- \wedge \hat{E}_{\bar{c}}^- - \frac{1}{6} H^{\bar{a}\bar{b}\bar{c}} \hat{E}_{\bar{a}}^- \wedge \hat{E}_{\bar{b}}^- \wedge \hat{E}_{\bar{c}}^- \Bigg) \ , \end{aligned} \tag{3.84}$$

where the factor 1/8 appears from $(1/2)^3$ using (3.82) on $dx^\mu \wedge dx^\nu \wedge dx^\rho$. The equation (3.80) becomes

$$\begin{aligned} \Sigma_{[abc]} &= -\tfrac{1}{6} H_{abc} \ , \quad \Sigma_{\bar{a}bc} = -\tfrac{1}{2} H_{\bar{a}bc} \ , \\ \Sigma_{[\bar{a}\bar{b}\bar{c}]} &= +\tfrac{1}{6} H_{\bar{a}\bar{b}\bar{c}} \ , \quad \Sigma_{a\bar{b}\bar{c}} = +\tfrac{1}{2} H_{a\bar{b}\bar{c}} \ , \end{aligned} \tag{3.85}$$

there is also another condition, which comes considering a theory with dilaton (see [15])

$$\Sigma_a{}^a{}_b = 0 \ , \qquad \Sigma_{\bar{a}}{}^{\bar{a}}{}_{\bar{b}} = 0 \ . \tag{3.86}$$

The equations (3.85) and (3.86) admit solution for Σ, but it is not *unique*. Thus we can always find a torsion-free and metric compatible connection D which is not uniquely determined. Specifically, one finds

$$\begin{aligned} D_a V_+^b &= \nabla_a V_+^b - \frac{1}{6} H_a{}^b{}_c V_+^c + A_a^{+\ b}{}_c V_+^c \ , \\ D_{\bar{a}} V_+^b &= \nabla_{\bar{a}} V_+^b - \frac{1}{2} H_{\bar{a}}{}^b{}_c V_+^c \ , \\ D_a V_-^{\bar{b}} &= \nabla_a V_-^{\bar{b}} + \frac{1}{2} H_a{}^{\bar{b}}{}_{\bar{c}} V_-^{\bar{c}} \ , \\ D_{\bar{a}} V_-^{\bar{b}} &= \nabla_{\bar{a}} V_-^{\bar{b}} + \frac{1}{6} H_{\bar{a}}{}^{\bar{b}}{}_{\bar{c}} V_-^{\bar{c}} + A_{\bar{a}}^{-\ \bar{b}}{}_{\bar{c}} V_-^{\bar{c}} \ , \end{aligned} \tag{3.87}$$

where the undetermined tensors $A^\pm$ satisfy

$$\begin{aligned} A_{abc}^+ &= -A_{acb}^+ \ , \quad A_{[abc]}^+ = 0 \ , \quad A_a^{+\ a}{}_b = 0 \ , \\ A_{\bar{a}\bar{b}\bar{c}}^- &= -A_{\bar{a}\bar{c}\bar{b}}^- \ , \quad A_{[\bar{a}\bar{b}\bar{c}]}^- = 0 \ , \quad A_{\bar{a}}^{-\ \bar{a}}{}_{\bar{b}} = 0 \ , \end{aligned} \tag{3.88}$$

such that the equations (3.85) and (3.86) are still satisfied.

3.4 Generalised Riemann curvature tensor

Having constructed a generalised connection, in particular a not unique Levi-Civita generalised connection, it is natural to ask if one can also introduce a notion of generalised Riemann curvature tensor.

Before do that, let us remind how *Riemann curvature tensor* is defined in ordinary geometry. Given a connection ∇, it is a map

$$\mathcal{R} : \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM) \ , \tag{3.89}$$

defined by

$$\mathcal{R}(v, w, z) = \nabla_v \nabla_w z - \nabla_w \nabla_v z - \nabla_{[v,w]} z \qquad\qquad v, w, z \in \Gamma(TM) \ . \tag{3.90}$$

By analogy with ordinary geometry, we have

Definition (Generalised Riemann curvature tensor)**.** *Given a generalised connection D, "generalised Riemann curvature tensor" is a map*

$$R : \Gamma(E) \times \Gamma(E) \times \Gamma(E) \to \Gamma(E) \ , \tag{3.91}$$

acting on $U, V, W \in \Gamma(E)$ as follow

$$R(U, V, W) = D_U D_V W - D_V D_U W - D_{[\![U,V]\!]} W \ . \tag{3.92}$$

The definition (3.92) is the natural generalisation of the definition (3.90), where connection ∇ is replaced by generalised connection D and Lie bracket $[\cdot, \cdot]$ is replaced by Courant bracket $[\![\cdot, \cdot]\!]$. Therefore generalised Riemann curvature R and Riemann curvature in ordinary geometry $\mathcal{R}$ have the same properties of symmetry, i.e.

- $R(U, V, W) = -R(V, U, W)$,
- $\langle R(U, V, W), Z\rangle = -\langle R(U, V, Z), W\rangle$,
- $R(U, V, W) + R(V, W, U) + R(W, U, V) = 0$,
- $\langle R(U, V, W), Z\rangle = -\langle R(W, Z, U), V\rangle$.

However, the object defined in (3.92) is non-tensorial [32]. In fact it seems to be a differential operator and, *a priori*, it is not obvious that it is multilinear object. For example, Riemann tensor defined in (3.90) still seems to be differential operator, but one can check that it satisfies multilinearity property, i.e.

$$\mathcal{R}(fv, gw, hz) = fgh\ \mathcal{R}(v, w, z) \ , \qquad\qquad \forall \quad f, g, h : M \to \mathbb{R} \ . \tag{3.93}$$

For generalised Riemann tensor (3.92) one finds

$$\begin{aligned} R(fU, gV, hW) &= D_{fU} D_{gV} hW - D_{gV} D_{fU} hW - D_{[\![fU,gV]\!]} hW \\ &= fgh\ \left(D_U D_V W - D_V D_U W - D_{[\![U,V]\!]} W\right) - \frac{1}{2} h\langle U, V\rangle D_{(fdg-gdf)} W \ . \end{aligned} \tag{3.94}$$

The term which breaks multilinearity is $-\frac{1}{2} h\langle U, V\rangle D_{(fdg-gdf)} W$. Therefore if we restrict to evaluate generalised Riemann tensor only over vectors which are orthogonal respect $O(d, d)$ metric, then the *restricted* generalised Riemann tensor is a proper tensor, i.e. it satisfies multilinearity property. One possible choice of vectors U and V such that $\langle U, V\rangle = 0$ is

$$U \in C_+ \ , \qquad\qquad V \in C_- \ . \tag{3.95}$$

Therefore we can build a *tensorial* $O(p,q)\times O(q,p)$ generalised Riemann curvature, such that the index structure would be

$$\left(R_{a\bar{b}}{}^{c}{}_{d}, R_{a\bar{b}}{}^{\bar{c}}{}_{\bar{d}}\right) \qquad \text{and} \qquad \left(R_{\bar{a}b}{}^{c}{}_{d}, R_{\bar{a}b}{}^{\bar{c}}{}_{\bar{d}}\right) \ , \tag{3.96}$$

where the first and second index tells us in which sub-bundle U and V are chosen, while the third and fourth index are the "representation indices", which tell us if generalised Riemann curvature is acting on C_+ or in C_- sub-bundle. Moreover the representation indices posses a property. Remind the *Cartan's structure equation* in ordinary geometry

$$d\omega^{a}{}_{b} + \omega^{a}{}_{c}\wedge\omega^{c}{}_{b} = \mathcal{R}^{a}{}_{b} \ , \tag{3.97}$$

where the *spin connection 1-form* is

$$\omega^{a}{}_{b} \equiv \Gamma^{a}{}_{cb}\theta^{c} \ , \tag{3.98}$$

and the *curvature 2-form* is

$$\mathcal{R}^{a}{}_{b} \equiv \frac{1}{2}\mathcal{R}^{a}{}_{bcd}\theta^{c}\wedge\theta^{d} \ . \tag{3.99}$$

In generalised geometry Cartan's structure equations are more complicated, but still they put in relation the generalised spin connection Ω and the generalised Riemann tensor R. Because of the generalised version of equations (3.97), representation indices of generalised Riemann curvature has the same behaviour of the spin connection's one. Since $\Omega_{Mab}\in\mathfrak{o}(p,q)$ and $\Omega_{M\bar{a}\bar{b}}\in\mathfrak{o}(q,p)$, then we have also $R_{ab}\in\mathfrak{o}(p,q)$ and $R_{\bar{a}\bar{b}}\in\mathfrak{o}(q,p)$, which means generalised Riemann curvature is an object acting in the adjoint representation.

The generalised Riemann curvature is not a uniquely determined object, because of the two possible choices shown in (3.96). But the surprising fact is that *generalised Ricci tensor*, which is the object that enter in the Einstein's equation, is unique!

Definition (Generalised Ricci tensor)**.** *The generalised Ricci tensor is the trace of the generalised Riemann curvature*

$$R_{AB} = R_{CA}{}^{C}{}_{B} \ . \tag{3.100}$$

The two possible generalised Riemann curvature (3.96) give us the following generalised Ricci tensors

$$R^{(\mathbf{I})}_{BD} = \begin{cases}\eta^{ac}R_{a\bar{b}cd} = R_{\bar{b}d}\\ \eta^{ac}R_{a\bar{b}\bar{c}\bar{d}} = 0\end{cases} \ , \qquad R^{(\mathbf{II})}_{BD} = \begin{cases}\eta^{\bar{a}c}R_{\bar{a}bcd} = 0\\ \eta^{\bar{a}\bar{c}}R_{\bar{a}b\bar{c}\bar{d}} = R_{b\bar{d}}\end{cases} \ . \tag{3.101}$$

Since generalised Riemann curvature R and Riemann curvature in ordinary geometry $\mathcal{R}$ have the same properties of symmetry, then generalised Ricci tensor is symmetric, i.e. $R_{AB}=R_{BA}$ and this ensure that the two generalised Ricci tensor computed, $R_{a\bar{b}}$ and $R_{\bar{a}b}$ are the same quantity. Hence we have a *unique* generalised Ricci tensor, which comes from a *restricted* generalised Riemann tensor. I want to stress that the restriction on the two sub-bundles C_+ and C_- is crucial for the uniqueness of generalised Ricci tensor, which would not be guaranteed without this important condition.

3.5 Type II Supergravity as $O(9,1)\times O(1,9)$ generalised gravity

The fact generalised Ricci tensor is unique plays a central role in our attempt to "geometrise" Type II Supergravity theories. Indeed the object $R_{a\bar{b}}$, which is equal to $R_{\bar{a}b}$, contains d^2 d.o.f., because a and $\bar{b}$ run both from 1 to d. The other components of generalised Ricci tensor, like R_{ab} and $R_{\bar{a}\bar{b}}$ are identically zero, hence they do not contribute to the d.o.f.. This is a good news, because our aim is to capture the NSNS bosonic fields of type II theories by packaging them into a generalised metric G and a conformal factor Φ (see later the paragraph 3.5.1 for dilaton inclusion in generalised geometry). The NSNS sector contains a metric g of signature $(9,1)$, a 2-form $\mathcal{B}$ and a dilaton ϕ, which at each point $p \in M$ they bring the d.o.f shown in table 3.1.

Field	d.o.f. per spacetime point
$g_{\mu\nu}$	$d(d+1)/2$
$\mathcal{B}_{\mu\nu}$	$d(d-1)/2$
ϕ	1
$\{g_{\mu\nu}, B_{\mu\nu}, \phi\}$	d^2+1

Table 3.1: NSNS sector's d.o.f. per spacetime point

Therefore the d.o.f. of NSNS sector, which in total are d^2+1, match with the number of equations for (G,Φ) available, which they comes from generalised Ricci tensor $R_{a\bar{b}}$ and generalised scalar curvature R. Before moving to explain that, let us briefly review how dilaton can be "geometrised".

3.5.1 Dilaton description in generalised geometry

Dilaton cannot be described inside generalised geometry developed in the chapter 2. In order to include dilaton ϕ correctly inside generalised geometry, we have to perform a small different choice of generalised tangent bundle. Doing that, generalised geometry explained before will be slightly modified. I show here, without details which can be found in [15], the key changing to do.

The generalised tangent bundle to consider now is $\tilde{E}$, which is E "weighted" by $\det T^*M$ so that

$$\tilde{E} \equiv \det T^*M \otimes E \ . \tag{3.102}$$

The basis of $\tilde{E}$ now must be a *conformal basis* $\{\hat{E}_M\}$ such that

$$\langle \hat{E}_M, \hat{E}_N \rangle = \Phi^2 \eta_{MN} \qquad \text{where} \qquad \eta = \frac{1}{2}\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \ , \tag{3.103}$$

where $\Phi \in \det T^*M$ is called *conformal factor*. This choice of basis define an $O(d,d)\times\mathbb{R}^+$-structure.

Dorfman derivative and Courant bracket remain defined as before, where the action of Lie derivative now is

$$\begin{aligned} \mathcal{L}_v w^\mu &= v^\nu \partial_\nu w^\mu - w^\nu \partial_\nu v^\mu + p(\partial_\nu v^\nu) w^\mu \ , \\ \mathcal{L}_v \zeta_\mu &= v^\nu \partial_\nu \zeta_\mu + (\partial_\mu v^\nu)\zeta_\nu + p(\partial_\nu v^\nu)\zeta^\mu \ , \end{aligned} \tag{3.104}$$

where $p \in \mathbb{R}^+$ is a *weight*, which tells us in which representation of $\mathbb{R}^+$ the tensors v and w are acting on.

The maximal compact subgroup of $O(d,d)\times\mathbb{R}^+$ is still $O(p,q)\times O(q,p)$. Geometrically, choosing an $O(p,q)\times O(q,p)$-substructure, now does two things. First it fixes a nowhere vanishing section $\Phi \in \Gamma(\det T^*M)$, and this provides an isomorphism between weighted and un-weighted generalised tangent space $\tilde{E}$ and E. Second it defines the familiar splitting of E into C_+ and C_- discussed before. Again one can define projectors $\Pi_\pm$, but now the d.o.f. it must take out when one projects a vector of $\tilde{E}$ into $C_\pm$ are d^2+1. The one more d.o.f. is captured by the conformal factor Φ which can be parametrized by the metric g and the dilaton ϕ as

$$\Phi = \mathrm{e}^{-2\phi}\sqrt{-g} \ , \qquad \text{where} \qquad g \equiv \det g_{\mu\nu} \ . \tag{3.105}$$

Therefore at the end of this procedure, the NSNS sector acts as an element of the coset

$$\{g, \mathcal{B}, \phi\} \in \frac{O(10,10)}{O(9,1)\times O(1,9)} \times \mathbb{R}^+ \ . \tag{3.106}$$

This briefly explanation does not pretend to be completed, but it give just only the key points in order to include dilaton in generalised geometry.

3.5.2 Equations of motion

Generalised Ricci tensor $R_{a\bar{b}}$ can be explicitly worked out. Choosing the two orthonormal frames to be aligned so $e_a^+ = e_a^- = e_a$ we get the following result

$$R_{ab} = \mathcal{R}_{ab} - \frac{1}{4} H_{acd} H_b{}^{cd} + 2\nabla_a \nabla_b \phi + \frac{1}{2} \mathrm{e}^{2\phi} \nabla^c \left(\mathrm{e}^{-2\phi} H_{cab} \right) \ , \tag{3.107}$$

where $\mathcal{R}_{ab}$ is the Ricci tensor in ordinary geometry. Generalised scalar of curvature, obtained contracting generalised Ricci tensor with generalised metric, is

$$R = \mathcal{R} + 4\nabla^2\phi - 4\left(\partial\phi\right)^2 - \frac{1}{12} H^2 \ . \tag{3.108}$$

The NSNS action takes the form

$$S_{NS} = \frac{1}{2\kappa^2} \int d^{10}X \sqrt{-g}\, \mathrm{e}^{-2\phi} \left[\mathcal{R} + 4\left(\partial\phi\right)^2 - \frac{1}{12} H^2 \right] \ . \tag{3.109}$$

where κ is the gravitational coupling constant, related to the Newton's constant G and string length l_s and string coupling constant g_s by

$$\kappa^2 = 8\pi G = \frac{1}{4\pi} (2\pi l_s)^8 g_s^2 \ . \tag{3.110}$$

The following action, which formally seems the Einstein-Hilbert action for General Relativity written in the language of generalised geometry

$$S_{GEH} = \frac{1}{2\kappa^2} \int d^{10}X \ \Phi R \ , \tag{3.111}$$

is equivalent to the NSNS action (3.109), up to integration by parts of the $\nabla^2\phi$ term.

Varying the action S_{GEH} respect to the generalised metric G, we find the equations of motion for g and B, which are captured inside the compact equation

$$R_{a\bar{b}} = 0 \ , \tag{3.112}$$

which looks like Einstein's equations in vacuum.

Again, varying the action S_{GHE} respect to the conformal factor Φ, we find equations of motion for ϕ

$$R = 0 \ . \tag{3.113}$$

What we did here is to rewrite the supergravity equations of motion with local $O(9,1) \times O(1,9)$ covariance. In this description there are two things more one can do. First one should introduce RR fields, which appear in equation (3.112) as a source term. Also the RR sector find a geometrisation inside generalised geometry, which we do not want to pursue here. Second remark concern fermions. The fermionic action in generalised geometry can be written using the ordinary action just replacing connection by generalised connection. The fermionic equations of motion are reproduced correctly and the whole set of equations are locally $Spin(9,1)\times Spin(1,9)$ covariant.

4

Generalised parallelisability and consistent truncations

The scientist does not study nature because it is useful;
he studies it because he delights in it, and he delights in it because it is beautiful.
If nature were not beautiful, it would not be worth knowing,
and if nature were not worth knowing, life would not be worth living.
- Henri Poincaré -

Consistent truncations represent a saddle topic in Physics. Not all compact manifolds provide them. The classic example of consistent truncation comes from compactification on *local group manifold* (LGM), which is shown to work in [58], [59]. However exist some mysterious cases of consistent truncations which do not happen on local group manifold. They are S^7 [20], S^4 [47], [48] and S^5 [18]. All of them are contemplated in the classical supergravity solutions, which are: the S^3 near-horizon NS-fivebrane background, $AdS_7 \times S^4$ in eleven-dimensional supergravity, $AdS_5 \times S^5$ in Type IIB, and $AdS_4 \times S^7$ in eleven-dimensional supergravity. In this chapter we will show how to solve these mysterious cases using the concept of *generalised Leibniz parallelisation* (GLP). By the conjecture [45] due to K. Lee, C. Strickland-Constable and D. Waldram, the GLP condition replaces the LGM condition in the consistent truncations. The important result is that all round spheres are GLP. In the explanation we will follow the "historical path", which gives rise naturally to the concept of GLP.

4.1 Local group manifold and parallelisability

Let us start defining what is a *local group manifold*

Definition (Local group manifold (LGM))**.** *A local group manifold is a manifold M which can be written as*

$$M = G/\Gamma \ , \tag{4.1}$$

where G is a Lie group and Γ is a discrete, freely-acting subgroup of G[1]*.*

[1] The action of a group H on a set X can be *transitive*, *free* or *effective*. Free means that the action of each element $h \in H$ on a point $x \in X$ has no fixed points, i.e. $h(x) \neq x \ \ \forall h \in H, x \in H$.

The fact to be LGM can be read mathematically in terms of the vectors fields defined over the manifold M. First of all we can do the following considerations

- If Γ is *not* a *normal* subgroup of G, which means

$$g\Gamma g^{-1} = \Gamma \ , \qquad\qquad \forall\ g \in G \ , \tag{4.2}$$

  then G/Γ is *not* a group, but simply a manifold. If we are in this particular case, has not sense talk about Lie algebra associated to G/Γ, since it is not a Lie group. However remains defined the definition of tangent space $T_{\mathbb{1}}M$, where $\mathbb{1}$ is the identity element in G, which is represented by a point p in M.

- The set of tangent vectors $\{\hat{e}_\mu|_p\}$, which generates the whole tangent space T_pM, satisfies always the Lie bracket relation

$$[\hat{e}_\mu, \hat{e}_\nu]|_p = f_{\mu\nu}{}^\rho(p)\, \hat{e}_\rho|_p \ , \tag{4.3}$$

  where $f_{\mu\nu}{}^\rho$ are functions which depend by the point $p \in M$ chosen.

- If the manifold M is *also* a group, let us say $M = G$, $\Gamma = \{\mathbb{1}\}$, then exists two special classes of vector fields characterised by an invariance under group action, which are the *left*-invariant vector fields and the *right*-invariant vector fields. They are defined in the appendix A. The choice of one type rather then the other is just a convention, since the two families give rise to equivalent theories. Suppose to pick up the left choice. Then we have the remarkable theorem

  Theorem 4.1 (Lie's theorem)**.** *Let be $\{\hat{e}_a|_g\}$ the set of left-invariant vector fields evaluated at the element $g \in G$. Then holds the following Lie bracket relation*

$$[\hat{e}_a, \hat{e}_b]|_g = f_{ab}{}^c\, \hat{e}_c|_g \ , \qquad\qquad \forall\ g \in G \ , \tag{4.4}$$

  where $f_{ab}{}^c$ are constant, i.e. they do not depend by the element $g \in G$ chosen.

  $f_{ab}{}^c$ are called *structure constants* of the Lie group G. The proof of the theorem can be found in the appendix A.

Keeping in mind these facts, come back to the case of local group manifold $M = G/\Gamma$. Since Γ is discrete, at linearised level $T_{\mathbb{1}}M$ looks like $T_{\mathbb{1}}G$. Therefore if the discrete group Γ acts on the left, then the left-invariant vector fields $\{\hat{e}_a\}$ defined for the group G still continue to satisfy the relation (4.4)[2].

Summarizing, a *local group manifold* is characterised by the following mathematical condition on the left-invariant vector fields $\{\hat{e}_a\}$

$$[\hat{e}_a, \hat{e}_b] = \mathcal{L}_{\hat{e}_a}\hat{e}_b = f_{ab}{}^c\, \hat{e}_c \ , \tag{4.5}$$

where the coefficients $f_{ab}{}^c$ are constant (and are called *structure constant* in the case $\Gamma \lhd G$).

It is well known from [58] and [59] that local group manifolds gives consistent truncations[3]. But is this request too much strong? First we can notice one remarkable fact about local group manifold: they are also *parallelisable*. Let us define what parallelisability means

[2]An explicit example is provided by the two groups $SO(3) \simeq SU(2)/\mathbb{Z}_2$, which have same algebras $\mathfrak{so}(3) \simeq \mathfrak{su}(2)$.

[3]Actually in these papers is shown that one should require in addition the "unimodular" condition $f_{ab}{}^b = 0$. The condition of unimodularity is automatically satisfied for compact Lie groups, which in practice is where our interest lies.

Definition (Parallelisable manifold). *A differentiable manifold M of dimension d is called "parallelisable" if there exist d smooth vector fields globally defined*

$$\{v_1, ..., v_d\} \ , \tag{4.6}$$

on the manifold, such that at any point $p \in M$ the tangent vectors

$$\{v_1(p), ..., v_d(p)\} \ , \tag{4.7}$$

are linearly independent and nowhere vanishing, i.e. they form a basis of the tangent space at p.

The particular choice of vector fields $\{v_1, ..., v_d\}$ which satisfies the definition above is called *parallelisation*, or an *absolute parallelism*, of M. Then we have the result

Theorem 4.2. *A manifold M is parallelisable if, and only if, the tangent bundle TM is trivial, i.e. $TM \simeq M \times \mathbb{R}^d$.*

We have also other necessary conditions for the parallelisability.

Proposition 4.1. *If a manifold M is parallelisable, then*

- *it must have Euler characteristic $\chi = 0$.*
- *there is always a flat connection, i.e. with zero curvature.*

The fact a parallelisable manifold has a flat connection is a consequence of the theorem on the triviality of tangent bundle. In general TM looks trivial only locally, i.e. if U is open subset of M, then there is a diffeomorphism from TU to $U \times \mathbb{R}^d$. But if TM looks trivial globally, this means all tangent spaces of the manifold are isomorphic. This isomorphism establishes that two tangent vectors of the spaces T_pM and T_qM having the same coordinates with respect to the frame $\{e|_p\}$ and $\{e|_q\}$ are identified. But, by definition, this assigns to the manifold a connection ∇ of zero curvature. However, in general, this flat connection has torsion, see for example $SU(3) = S^3$.

Come back now to the main logic we were following. For a manifold M which is LGM, the set of left-invariant vector fields provides a parallelisation of M, because point by point $p \in M$ they provide a basis for T_pM. Hence a LGM is also parallelisable. The parallelisability condition is weaker than LGM condition. Hence if we buy the parallelisability condition as a new condition to give rise consistent truncations, it seems to be a way out for the mysterious case of consistent truncation on S^7 in eleven-dimensional supergravity.

In fact in ordinary geometry, the only *parallelisable spheres* are (S^0), S^1, S^3 and S^7 and this is related to the fact the only normed division algebras over the real[4] (up to isomorphism) are: the real number $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$ and the octonions $\mathbb{O}$. Other spheres S^d with $d \neq 0, 1, 3, 7$ are *not* parallelisable[5].

However there are also other cases of consistent truncations on spheres which are even not parallelisable. We are talking about S^4 and S^5. Hence this argument push us to search another condition that guarantees consistent truncations, which must be weaker than the parallelisability.

4.2 Generalised Leibniz parallelisability

First ideas began to appear in [26] moving in generalised geometry. The idea is to generalise the concept of LGM (4.5) in generalised geometry. This gives rise to the new concept of *generalised Leibniz parallelisability*

[4] Roughly speaking, they are algebras over the real field where is defined a *norm* and *division* operation is possible.

[5] For example, the *Hairy ball theorem* states the 2-dimensional sphere is not parallelisable. The theorem can be generalised to S^{2n}, with $n \geq 1$.

Definition (Generalised Leibniz parallelisability (GLP)). *A differentiable manifold M, equipped with a global frame $\{\hat{E}_A\}$ on E, is "generalised Leibniz parallelisable" if*

$$L_{\hat{E}_A} \hat{E}_B = F_{AB}{}^C \hat{E}_C \ , \tag{4.8}$$

where $L_{(\cdot)}(\cdot)$ is the Dorfman derivative and $F_{AB}{}^C$ are constant.

By definition, a GLP manifold is also parallelisable in the appropriated notion of generalised geometry, which is

Definition (Generalised parallelisable manifold). *A differentiable manifold M of dimension d is called "generalised parallelisable" if there exist $2d$ smooth generalised vector fields globally defined*

$$\{V_1, ..., V_{2d}\} \ , \tag{4.9}$$

on the manifold, such that at any point $p \in M$ the generalised tangent vectors

$$\{V_1(p), ..., V_{2d}(p)\} \ , \tag{4.10}$$

are linearly independent and nowhere vanishing, i.e. they form a basis of the generalised tangent space at p.

By analogy with ordinary geometry, the particular choice of generalised vector fields $\{V_1, ..., V_{2d}\}$ which satisfies the definition above is called *generalised parallelisation*, or an *absolute generalised parallelism*, of M. In the definition of GLP, the "global frame" is intended to be a generalised parallelisation.

Again, we have the result

Theorem 4.3. *A manifold M is generalised parallelisable if, and only if, the generalised tangent bundle E is trivial.*

Triviality of the generalised tangent bundle, in the Hitchin's generalised geometry, means $E \simeq M \times \mathbb{R}^{2d}$, but in general it depends case by case how one defines the generalised tangent bundle.

At this point, one is led to the conjecture [45]

Conjecture 4.1. *A manifold M which is GLP gives consistent truncations.*

This conjecture has a remarkable consequence. In the next section we will consider spheres, and we will show that all round spheres S^d are GLP. Therefore by the conjecture stated before, all round spheres give consistent truncations and this fact sheds light in the last two mysterious cases of consistent truncations on S^4 in eleven-dimensional supergravity and S^5 in Type IIB supergravity.

Just notice that the GLP condition is weaker than the (ordinary) parallelisability, and generalised parallelisability is weaker than GLP condition. The strength of the four concepts we introduced is shown on the Table[6] 4.1.

If a manifold M is parallelisable in ordinary sense, then it is also generalised parallelisable. In fact, let $\{e_a\}$ be a parallelisation of M. Let $\{\theta^a\}$ be its dual. Then a generalised parallelisation of M is given reminding the formula (2.78) which now reads

$$\hat{E}_A = \begin{cases} \begin{pmatrix} e_a \\ i_{e_a} B \end{pmatrix} & \text{for} \quad A = a \\ \\ \begin{pmatrix} 0 \\ \theta^a \end{pmatrix} & \text{for} \quad A = a + d \end{cases} \ . \tag{4.11}$$

[6] The table shows just only the *strength* of the definitions. Which means the class of GLP manifolds is broader than the class of parallelisable manifolds, but it is not true all parallelisable manifolds are also GLP manifolds.

Local group manifold (LGM)

$\downarrow$

Parallelisable manifold

$\downarrow$

Generalised Leibniz parallelisability (GLP)

$\downarrow$

Generalised parallelisable manifold

Table 4.1: Condition's strength. It decrease from top to down.

It is easy to check that the generalised vectors in (4.11), because of the properties of the (ordinary) parallelisation, they are globally defined, linearly independent and nowhere vanishes, hence they provide a generalised parallelisation.

The careful reader will have noticed that we did not define the concept of GLP manifold as a generalisation of the merely definition of LGM (4.1). Conversely we prefer to generalise the equation (4.5) which is a mathematical condition to be LGM. Furthermore the generalisation regarded only the Lie derivative, which was replaced by the Dorfman derivative, but we said anything about Lie bracket, which it should be replaced by Courant bracket. Moreover, despite ordinary geometry, Courant bracket and Dorfman derivative do not coincide. Hence seems we are struck in an inconsistency. Fortunately this is not the case thanks to the following fact

Lemma 4.1. *Let $\{\hat{E}_A\}$ a generalised parallelisation of M, $O(d,d)$-compatible. Then we have*

$$L_{\hat{E}_A}\hat{E}_B = [\![\hat{E}_A, \hat{E}_B]\!] \ . \tag{4.12}$$

Proof. Recall the relation between Courant bracket and Dorfman derivative acting on $\{\hat{E}_A\}$

$$[\![\hat{E}_A, \hat{E}_B]\!] = L_{\hat{E}_A}\hat{E}_B + d\,\langle \hat{E}_A, \hat{E}_B\rangle \ . \tag{4.13}$$

The generalised parallelisation is $O(d,d)$-compatible, which means

$$\langle \hat{E}_A, \hat{E}_B\rangle = \eta_{AB} \ . \tag{4.14}$$

Therefore we have

$$[\![\hat{E}_A, \hat{E}_B]\!] = L_{\hat{E}_A}\hat{E}_B + d\,(\eta_{AB}) = L_{\hat{E}_A}\hat{E}_B \ . \tag{4.15}$$

□

Now we close the circle about GLP, showing a result which tells us how the GLP manifold looks like.

Theorem 4.4. *A manifold M which is GLP is also a homogeneous space, i.e.*

$$M = G/H \ , \tag{4.16}$$

where $G \subset O(d,d)$ is a $2d$-dimensional Lie group and H is a d-dimensional Lie subgroup of G.

Proof. Let us pick a generalised parallelisation $\{\hat{E}_A\}$ $O(d,d)$-compatible, which satisfies

$$[\![\hat{E}_A, \hat{E}_B]\!] = F_{AB}{}^C\hat{E}_C \ , \tag{4.17}$$

where $F_{AB}{}^C$ are constant. The Courant bracket on $\{\hat{E}_A\}$ satisfies the Jacoby identity and hence defines a Lie algebra $\mathfrak{g}$ whose structure constants are $F_{AB}{}^C$. Given Proposition 3.18 of [31], we also have

$$\eta_{CD}F^D{}_{AB} + \eta_{BD}F^D{}_{AC} = 0 \; . \tag{4.18}$$

This implies that the adjoint representation of the algebra $\mathfrak{g}$, where the generators are given by $(T_A)_B{}^C = F_{AB}{}^C$, acts as a sub-algebra $\mathfrak{g} \subset \mathfrak{o}(d,d)$.

Recall now that the Courant bracket, under the projection $\pi : E \to TM$, reduces to the Lie bracket for ordinary vectors

$$\pi\left([\![\hat{E}_A, \hat{E}_B]\!] \right) = [v_A, v_B] = F_{AB}{}^C v_C \; , \tag{4.19}$$

where $v_A \equiv \pi(\hat{E}_A)$, and they are $2d$ vector fields on M. Since the set $\{\hat{E}_A\}$ provide a basis for E, the set $\{v_A\}$ must provide a basis for TM. Therefore some of v_A can be vanishes at some points $p \in M$, or cannot be linearly independent, but surely there must be at least d of them non-vanishing at each point $p \in M$ and linearly independent. Let us fix some point $p \in M$. We can identify vectors $X = X^A\hat{E}_A$, with constant X^A, as elements of the lie algebra $\mathfrak{g}$. Now we look at the set of vectors X with vanishing $\pi(X)$ at a given point $p \in M$

$$\mathfrak{h} = \{X \in \mathfrak{g} | \pi(X) = 0\} \; . \tag{4.20}$$

This must be at least d-dimensional subset of $\mathfrak{g}$ because it must span T^*M. But also its complementary set $\mathfrak{g} \setminus \mathfrak{h}$ must be at least d-dimensional in order to span TM. Hence $\mathfrak{h}$ must be exactly a d-dimensional subset of $\mathfrak{g}$. Since the Lie bracket of two vector fields that vanish at $p \in M$ must itself vanish at $p \in M$, we have that $\mathfrak{h}$ must form a closed sub-algebra. Hence, exponentiating the algebra, the manifold M must be a coset space

$$M = G/H \tag{4.21}$$

with $G \subset O(d,d)$ a $2d$-dimensional Lie group and $H \subset G$ a d-dimensional Lie subgroup. □

4.2.1 Gauge group on the reduced theory

There are many methods to do a dimensional reduction, as one can see in the literature. The most ancient was the Kaluza-Klein compactification performed over a circle S^1. The first generalisation was considered by Pauli in 1953. He starting point was the six-dimensional spacetime $M_4 \times S^2$. The extra dimensions form a 2-sphere S^2, and with an appropriate Ansatz he constructed a non-abelian theory with gauge group $SU(2)$.

In the years after that, further generalisations were proposed. They can be classified into three (plus one) families [17]. The ingredients are the choice of compact manifold $\mathcal{C}_d$ and the gauge group for the ending theory.

(I) **Toroidal reduction** $\mathcal{C}_d = T^d$. This represents a d-dimensional torus reduction and is a generalisation of the Kaluza-Klein circle reduction. The reduced theory gets the gauge group $[U(1)]^d$.

(II) **DeWitt reduction** $\mathcal{C}_d = G$. This is a reduction on a group manifold, where G is a compact Lie group. The group G becomes also the gauge group for the reduced theory.

(III) **Pauli reduction** $\mathcal{C}_d = G/H$. Coset space reduction, where H is the maximal compact subgroup of G. The most common examples are sphere-reductions $S^d = SO(d+1)/SO(d)$. The gauge group appearing in the reduced theory consists in the isometry group of $\mathcal{C}_d$, which is G.

(IV) **Inhomogeneous spaces** or spaces without any isometries.

In most cases coset reductions are preferred above group manifold reductions, since less extra dimensions are needed to obtain a certain gauge group. For example, if we want the gauge group $SO(8)$ we could use the $SO(8)$ group manifold or the coset $SO(8)/SO(7)$, which corresponds to the 7-sphere. In the first case we have $\dim SO(8) = 28$, while in the second case $\dim S^7 = 7$.

In the article [17], the authors M. Cvetic, G.W. Gibbons, H. Lü and C.N. Pope have presented a conjecture for Pauli reduction on group manifold $\mathcal{C}_d = G$. This conjecture leads to a gauge group $G \times G$ on the reduced theory instead the only gauge group G appearing in DeWitt reduction. The conjecture refers only to reductions of string theory (i.e. we must have a metric, B-field and dilaton), not of arbitrary theories. The truncation on this group manifold with $G \times G$ gauge group on the reduced theory is *consistent* [23]. As the authors wrote in the article, "there is no obvious group-theoretical explanation why this truncation can be performed consistently".

Using the framework of generalised geometry we are able to give a theoretical motivation of this conjecture (section 4.6). In generalised geometry the gauge group which appears in the reduced theory is generated by the algebra defined by the generalised parallelisation of $\mathcal{C}_d$, which must satisfy the GLP condition. We will show that the generalised parallelisation of G, which satisfy the GLP condition, generates the $\mathfrak{g} \oplus \mathfrak{g}$ algebra, hence exponentiating the $G \times G$ group.

4.3 Spheres as generalised parallelisable spaces

In this section we reproduce just only the fundamental steps about the result that all round spheres are GLP. The complete explanation can be found in [45].

The d-dimensional sphere can emerge dynamically as solution of equations of motion. Let us consider a theory in d dimensions with metric g and d-form filed strength $F = dA$, satisfying the equations of motion

$$R_{\mu\nu} = \frac{1}{d-1} F^2 g_{\mu\nu} \ , \qquad\qquad F = \frac{d-1}{R} \mathrm{vol}_g \ , \tag{4.22}$$

where $R_{\mu\nu}$ is the Ricci tensor, $R = g^{\mu\nu} R_{\mu\nu}$ is the scalar of curvature, $F^2 = \frac{1}{d!} F^{\mu_1 \dots \mu_d} F_{\mu_1 \dots \mu_d}$ and vol_g is the volume form computed with the metric g. The indices μ and ν run from 1 to d. The equations of motion (4.22) admits a solution with a round sphere S^d metric of radius R.

Since the sphere background has a d-form field strength F, it is natural to consider a generalised geometry where the generalised tangent bundle is $\frac{1}{2}d(d+1)$-dimensional defined as[7]

$$E \equiv \bigcup_{p \in M} T_p M \oplus \bigwedge^{d-2} T_p^* M \simeq TM \oplus \bigwedge^{d-2} T^* M \ . \tag{4.23}$$

We notice immediately that in the case $d = 3$ we get the Hitchin's generalised geometry $E \simeq TM \oplus T^*M$ and the 3-form F becomes the flux 3-form $H = dB$. Generalised vectors V now are

$$V^M = \begin{pmatrix} v^\mu \\ \lambda_{\mu_1 \dots \mu_{d-2}} \end{pmatrix} \ , \tag{4.24}$$

and also here one can define patching rules, Dorfman derivative, Courant bracket and so on, in the same way we did for $TM \oplus T^*M$ generalised geometry.

The structure group which appears here is not $O(d,d)$ anymore, but the positive determinant general group $GL^+(d+1, \mathbb{R})$. The maximal subgroup of $GL^+(d+1, \mathbb{R})$ is the Lie group $O(d+1)$, and the degrees of freedom encoded by the generalised metric G on the d-sphere are given by the coset

$$G \in \frac{GL(d+1)}{O(d+1)} \ . \tag{4.25}$$

[7] Actually E is defined as a short exact sequence, in the same way of the Hitchin's generalised geometry, getting at the end an exact Courant algebroid.

Consider now the sphere S^d resulting as solution of equations of motion (4.22). The sphere S^d can be embedded into $\mathbb{R}^{d+1}$ with embedding equation

$$\delta_{ij} y^i y^j = 1 \ , \tag{4.26}$$

where y^i are constrained dimensionless coordinates with $i = 1, ..., d+1$. The metric is

$$ds^2 = R^2 \delta_{ij} dy^i dy^j \ . \tag{4.27}$$

The generalised tangent bundle is $\frac{1}{2}d(d+1)$-dimensional, therefore a generalised parallelisation must contains $\frac{1}{2}d(d+1)$ generalised vectors. Since the sphere S^d is a *maximally symmetric manifold*, it admits $\frac{1}{2}d(d+1)$ Killing vector and we can use them in order to construct the generalised parallelisation. From the point of view of the embedding space $\mathbb{R}^{d+1}$, the sphere is invariant under rotational $SO(d+1)$ Killing vectors v_{ij},

$$v_{ij} = R^{-1}\left(y_i \frac{\partial}{\partial y^j} - y^j \frac{\partial}{\partial y^i}\right) \ , \tag{4.28}$$

which they gives the $\mathfrak{so}(d+1)$ algebra under Lie bracket

$$[v_{ij}, v_{kl}] = R^{-1}\left(\delta_{ik} v_{lj} - \delta_{il} v_{kj} - \delta_{jk} v_{li} + \delta_{jl} v_{ki}\right) \ . \tag{4.29}$$

The Ansatz for the global frame $\hat{E}_{ij}$ is

$$\hat{E}_{ij} = \begin{pmatrix} v_{ij} \\ \sigma_{ij} + i_{v_{ij}} A \end{pmatrix} \ , \tag{4.30}$$

where the term $i_{v_{ij}}A$ represent the action of the isomorphism $E \simeq TM \oplus \bigwedge^{d-2} T^*_p M$, and σ_{ij} is defined as

$$\sigma_{ij} = \star_d (R^2 dy_i \wedge dy_j) = \frac{R^{d-2}}{(d-2)!} \epsilon_{ijk_1 \ldots k_{d-1}} y^{k_1} dy^{k_2} \wedge \cdots \wedge dy^{k_{d-1}} \ , \tag{4.31}$$

where the Hodge star is computed in d dimensions, see the appendix A for details.

We have to check that the vector fields (4.30) are nowhere vanishing. Analysing separately when vanishes the vector components and the form components, we have

vector components	$v_{ij} = 0$	when $y_i = y_j = 0$
form components	$dy_i \wedge dy_j = 0$	when $y_i^2 + y_j^2 = 1$

The Killing vectors vanishes in the poles of the sphere, while $dy_i \wedge dy_j$ vanishes in the equator, i.e. when y^i and y^j are one functions of the other and the external product vanishes. Since this two conditions can never be satisfied together, the set of vector fields (4.11) gives us a generalised parallelisation of S^d.

One can check also the generalised parallelisation (4.11) is orthogonal respect to the generalised metric on the round sphere, hence Dorfman derivative coincides with the Courant bracket on the generalised parallelisation. The explicit computation give us

$$L_{\hat{E}_{ij}} \hat{E}_{kl} = [\![\hat{E}_{ij} . \hat{E}_{kl}]\!] = R^{-1}\left(\delta_{ik}\hat{E}_{lj} - \delta_{il}\hat{E}_{kj} - \delta_{jk}\hat{E}_{li} + \delta_{jl}\hat{E}_{ki}\right) \ . \tag{4.32}$$

We recognise that the Lie algebra generated by the generalised parallelisation is $\mathfrak{so}(d+1)$. This is in accord to the fact the Pauli reductions over S^2, S^7 give rise respectively to the gauge groups $SO(3)$ and $SO(8)$. Also the formula (4.32) shows that all round spheres S^d are GLP. Hence all of them, by the conjecture 4.1, provide consistent truncations.

4.3.1 The three-dimensional case

The case $d = 3$ is a particular one because it gives rise a connection with the Hitchin's generalised geometry. The indices i and j runs form 1 to 4 and we can defined self-dual and antiself-dual generalised vectors

$$\hat{E}^{\pm}_{ij} = \hat{E}_{ij} \mp \frac{1}{2}\epsilon_{ijkl}\hat{E}_{kl} \ . \tag{4.33}$$

We have six generalised vectors $\hat{E}^{+}_{ij}$ which only three are linearly independent. The same argument works for $\hat{E}^{-}_{ij}$. Therefore we can choose

$$\hat{E}^{+}_{a} \equiv \hat{E}^{+}_{4a} \ , \qquad a = 1,2,3 \ , \tag{4.34}$$

$$\hat{E}^{-}_{\bar{a}} \equiv \hat{E}^{-}_{4\bar{a}} \ , \qquad \bar{a} = 1,2,3 \ . \tag{4.35}$$

and $\hat{E}^{+}_{a}$, $\hat{E}^{-}_{\bar{a}}$ represent two $SO(3)$ triplets which give rise to the algebra

$$\begin{aligned} L_{\hat{E}^{+}_{a}}\hat{E}^{+}_{b} &= [\![\hat{E}^{+}_{a}, \hat{E}^{+}_{b}]\!] = R^{-1}\epsilon_{abc}\hat{E}^{+}_{c} \ , \\ L_{\hat{E}^{-}_{\bar{a}}}\hat{E}^{-}_{\bar{b}} &= [\![\hat{E}^{-}_{\bar{a}}, \hat{E}^{-}_{\bar{b}}]\!] = R^{-1}\epsilon_{\bar{a}\bar{b}\bar{c}}\hat{E}^{-}_{\bar{c}} \ , \\ L_{\hat{E}^{+}_{a}}\hat{E}^{-}_{\bar{b}} &= [\![\hat{E}^{+}_{a}, \hat{E}^{-}_{\bar{b}}]\!] = 0 \ . \end{aligned} \tag{4.36}$$

Hence the self-dual and antiself-dual frames give us two copies of the $\mathfrak{su}(2) = \mathfrak{so}(3)$ algebra. When the complete frame is exponentiated we get the Lie group $SO(3) \times SO(3)$, which is the maximal compact subgroup of $SO(3,3)$. But $SO(3) \times SO(3)$ is also isomorphic to the Lie group $SO(4)$, which is the maximal compact subgroup of the Lie group $SL(4,\mathbb{R})$, and $SO(3,3)$ is also isomorphic to $SL(4,\mathbb{R})$. Hence we have closed the circle proving the theory developed for the spheres leads back in the case $d = 3$ to the Hitchin's geometry, in particular showing explicitly the splitting $E \simeq C_+ \oplus C_-$ provided by self-dual and antiself-dual frames.

$$\begin{array}{cccl} SO(3,3) & \supset & SO(3) \times SO(3) & \text{Hitchin's generalised geometry} \\ \wr & & \wr & \\ SL(4,\mathbb{R}) & \supset & SO(4) & \text{Sphere's generalised geometry} \end{array}$$

4.4 Homogeneous spaces: S^3, H^3, dS_3 and AdS_3

In this section we keep on the case $d = 3$. We analyse maximally symmetric spaces which have the same embedding equation of the 3-sphere embedded in $\mathbb{R}^4$, but with different metric signature with n number of plus and m number of minus, i.e.

$$\eta^{(n,m)}_{ij} y^i y^j = 1 \ . \tag{4.37}$$

We consider the following spaces

Sphere	$S^3 = \frac{SO(4)}{SO(3)} \simeq \frac{SO(3)\times SO(3)}{SO(3)}$	$y_1^2 + y_2^2 + y_3^2 + y_4^2 = 1$
de Sitter	$dS_3 = \frac{SO(3,1)}{SO(2,1)}$	$y_1^2 + y_2^2 + y_3^2 - y_4^2 = 1$
Anti-de Sitter	$AdS_3 = \frac{SO(2,2)}{SO(2,1)} \simeq \frac{SO(2,1)\times SO(1,2)}{SO(2,1)}$	$-y_1^2 - y_2^2 + y_3^2 + y_4^2 = 1$
Hyperbolic space	$H^3 = \frac{SO(3,1)}{SO(3)}$	$-y_1^2 - y_2^2 - y_3^2 + y_4^2 = 1$

All these spaces arise as solution of the equations of motion (4.22) considering different signatures of the metric g in $d=3$. Hence the description of all these spaces is captured by the sphere's generalised geometry defined in (4.23), which in our case leads back to Hitchin's generalised geometry. Hence at the end of the day we would expect to find the splitting given by the $O(p,q)\times O(q,p)$ structure. Actually we will discover that this procedure can be done canonically for the sphere and Anti-de Sitter spaces, but not for the de Sitter and hyperbolic spaces.

The generalised parallelisation for all these spaces is given by

$$L_{\hat{E}_{ij}}\hat{E}_{kl} = [\![\hat{E}_{ij}.\hat{E}_{kl}]\!] = R^{-1}\left(\eta^{(n,m)}_{ik}\hat{E}_{lj} - \eta^{(n,m)}_{il}\hat{E}_{kj} - \eta^{(n,m)}_{jk}\hat{E}_{li} + \eta^{(n,m)}_{jl}\hat{E}_{ki}\right) . \tag{4.38}$$

Let us take self-dual and antiself-dual generalised vectors defined by (4.33). Then the linearly independent $\hat{E}^+_a$ and $\hat{E}^-_{\bar{a}}$ represent two triplets under the groups $SO(3)$ for the sphere, and $SO(2,1)$ for the Anti-de Sitter space. These two sets of $\hat{E}^+_a$ and $\hat{E}^-_{\bar{a}}$ give rise to two copies of $\mathfrak{so}(3)$ algebra for the sphere, and $\mathfrak{so}(2,1)$ algebra for the Anti-de Sitter space. Hence also for the Anti-de Sitter space we have the splitting of the generalised tangent bundle $E\simeq C_+\oplus C_-$ provided by the choice of the $SO(2,1)\times SO(1,2)$-substructure. An important fact to notice is that the group definition in terms of coset of these two spaces admit naturally a "splitting" given by the isomorphisms between $SO(4)\simeq SO(3)\times SO(3)$ and $SO(2,2)\simeq SO(2,1)\times SO(1,2)$. This is not true for the de Sitter and hyperbolic spaces.

In fact, if the frame $\{\hat{E}^+_a,\hat{E}^-_{\bar{a}}\}$ gives rise to a splitting for the sphere and the Anti-de Sitter space, from the other side, it do *not* for the de Sitter and hyperbolic spaces. The reason is due to the fact for the sphere and Anti-de Sitter space there are an *even* number of minus signs in $\eta^{(n,m)}$, while for the de Sitter and hyperbolic spaces an *odd* number.

However we can "reabsorb" the extra minus sign considering the complexification of the self-dual and antiself-dual basis

$$\hat{\mathcal{E}}^{\pm}_{ij} = \hat{E}_{ij} \pm \frac{i}{2}\epsilon_{ijkl}\hat{E}_{kl} . \tag{4.39}$$

Again, the linearly independent $\{\mathcal{E}^+_a,\mathcal{E}^-_{\bar{a}}\}$ fulfil

$$\begin{aligned}
L_{\hat{\mathcal{E}}^+_a}\hat{\mathcal{E}}^+_b &= [\![\hat{\mathcal{E}}^+_a,\hat{\mathcal{E}}^+_b]\!] = R^{-1}\epsilon_{abc}\hat{\mathcal{E}}^+_c ,\\
L_{\hat{\mathcal{E}}^-_{\bar{a}}}\hat{\mathcal{E}}^-_{\bar{b}} &= [\![\hat{\mathcal{E}}^-_{\bar{a}},\hat{\mathcal{E}}^-_{\bar{b}}]\!] = R^{-1}\epsilon_{\bar{a}\bar{b}\bar{c}}\hat{\mathcal{E}}^-_{\bar{c}} ,\\
L_{\hat{\mathcal{E}}^+_a}\hat{\mathcal{E}}^-_{\bar{b}} &= [\![\hat{\mathcal{E}}^+_a,\hat{\mathcal{E}}^-_{\bar{b}}]\!] = 0 .
\end{aligned} \tag{4.40}$$

This show the isomorphism between the complexified algebras

$$\mathfrak{so}(3,1)_{\mathbb{C}} \simeq \mathfrak{su}(2)_{\mathbb{C}}\oplus\mathfrak{su}(2)_{\mathbb{C}} \simeq \mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C}) . \tag{4.41}$$

Therefore we get the splitting also for the group $SO(3,1)$, which leads to a definition of generalised metric. But now we have to remind that the splitting is available only on the complexified basis. Hence we are not using generalised geometry anymore, but *Generalised Complex Geometry*. The generalised metric G in generalised geometry must satisfies the condition

$$G^2 = \mathbb{1} . \tag{4.42}$$

In generalised complex geometry the generalised metric J must fulfils

$$J^2 = -\mathbb{1} , \tag{4.43}$$

but the degrees of freedom encoded by the two generalised metric are the same, g and $\mathcal{B}$.

4.5 A non-trivial case: $S^2 \times S^1$

The manifold $S^2 \times S^1$ is not a coset space, hence one should be tempted to think it is not GLP, since it represents a necessary condition from the theorem (4.4). In ordinary geometry there is a theorem which states "product of parallelisable manifolds is parallelisable". Hence one would like to check if it is also true "product of GLP manifolds is GLP". We do not prove here this statement, but we show that is it true in the case $S^2 \times S^1$. The fact $S^2 \times S^1$ is GLP tells us this space is also an homogeneous space. Since the case $S^2 \times S^1$ is original, all the computations are given in detailed in the appendix D.1 and here we just follow the main logic.

Let us take y^i constrained coordinates, with $i = 1, 2, 3$, which described S^2 embedded in $\mathbb{R}^3$ by the equation $\delta_{ij} y^i y^j = 1$. For the circle S^1 we do not choose constrained coordinates, but just the free angle parameter $\psi \in [0, 2\pi[$.

The generalised frame can be constructed as[8]

$$\hat{E}_{ij} = \begin{pmatrix} v_{ij} \\ -i_{v_{ij}} B + \star_2 (dy_i \wedge dy_j) \wedge d\psi \end{pmatrix} , \qquad \hat{E}'_i = \begin{pmatrix} y_i \partial_\psi \\ -i_{y_i \partial_\psi} B + \epsilon_{ijk}\, y^j dy^k \end{pmatrix} , \tag{4.44}$$

where $\{\hat{E}_i\}$ is the generalised parallelisation on S^2 and $\{\hat{E}'_i\}$ are generalised vectors for the circle reproduced in each direction y^1, y^2, y^3 of the embedded space $\mathbb{R}^3$. We need them in order to have a set of six generalised vector fields, which corresponds to the dimension of the generalised tangent bundle E.

These generalised vector fields are nowhere vanishing. The set $\{\hat{E}_{ij}\}$ was already discussed in the section 4.3, while for $\{\hat{E}'_i\}$ we can see that[9]

vector components	$y_i \partial_\psi = 0$	when $y_i = 0$
form components	$\epsilon_{ijk}\, y^j dy^k = 0$	when $y^i = \pm 1$

Instead of considering the two-indices generalised vectors $\hat{E}_{ij}$, let us consider their Hodge dual one-index generalised vectors

$$\hat{E}_i = \epsilon_{ijk} \hat{E}_{jk} \ . \tag{4.45}$$

The frame $\{\hat{E}_i, \hat{E}_j\}$ is orthogonal respect the $O(d,d)$ metric and fulfils the algebra

$$\begin{aligned} L_{\hat{E}_i} \hat{E}_j &= [\![\hat{E}_i, \hat{E}_j]\!] = -\epsilon_{ijk} \hat{E}_k \ , \\ L_{\hat{E}_i} \hat{E}'_j &= [\![\hat{E}_i, \hat{E}'_j]\!] = -\epsilon_{ijk} \hat{E}'_k \ , \\ L_{\hat{E}'_i} \hat{E}'_j &= [\![\hat{E}'_i, \hat{E}'_j]\!] = 0 \ . \end{aligned} \tag{4.46}$$

From these relations we recognise the algebra recovered here is nothing more than the algebra of the Euclidean group in 3 dimensions, $E(3) \simeq ISO(3) \simeq SO(3) \ltimes \mathbb{R}^3$.

The frame (4.44) is not the only possible generalised parallelisation of $S^2 \times S^1$. Another choice, which is still nowhere vanishes, is given by

$$\hat{\mathcal{E}}_{ij} = \begin{pmatrix} v_{ij} \\ -i_{v_{ij}} B + \star_2 (dy_i \wedge dy_j) \wedge d\psi \end{pmatrix} , \qquad \hat{\mathcal{E}}'_i = \begin{pmatrix} y_i \partial_\psi \\ -i_{y_i \partial_\psi} B + dy_i \end{pmatrix} . \tag{4.47}$$

Despite this frame gives a generalised parallelisation of $S^2 \times S^1$, it is not orthogonal respect the $O(d,d)$ metric. Hence it is not a good candidate in order to provide an algebra.

[8] Here we choose $A = -B$.

[9] It may be useful draw a 2-sphere and see that the form component vanishes only in the intersection of two circles orthogonal to the y^j and y^k directions.

4.6 The conjecture $G \times G$

Consider a group manifold $M = G$ equipped with *left* and *right* invariant vector fields, Let $l_{\bar{a}}$ and r_a, with $a, \bar{a} = 1, ..., d$, be a *basis* for the left and right invariant vector fields respectively. They must fulfil the following Lie bracket relations

$$
\begin{aligned}
[r_a, r_b] &= f_{ab}{}^c r_c \ , \\
[l_{\bar{a}}, l_{\bar{b}}] &= f_{\bar{a}\bar{b}}{}^{\bar{c}} l_{\bar{c}} \ , \\
[r_a, l_{\bar{b}}] &= 0 \ .
\end{aligned}
\tag{4.48}
$$

The dual basis ρ^a and $l_{\bar{a}}$ with respect to r_a and $l_{\bar{a}}$ are defined as

$$
i_{r_a}\rho^b = \delta_a{}^b, \qquad i_{l_{\bar{a}}}\lambda^{\bar{b}} = \delta_{\bar{a}}{}^{\bar{b}}, \tag{4.49}
$$

Raising and lowering indices is made by the *Cartan-Killing form* which remains associated to the Lie algebra $\mathfrak{g}$, i.e.

$$
g = g_{ab}\, \rho^a \otimes \rho^b = g_{\bar{a}\bar{b}}\, \lambda^{\bar{a}} \otimes \lambda^{\bar{b}}. \tag{4.50}
$$

Remember that left and right invariant vector fields give rise to two copies of the same theory. Let us pick up the set of generalised vector fields $\{E_a^+\}$ and $\{E_{\bar{a}}^-\}$ defined as

$$
\hat{E}_a^+ = \begin{pmatrix} r_a \\ \rho_a - i_{r_a}B \end{pmatrix} , \qquad \hat{E}_{\bar{a}}^- = \begin{pmatrix} l_{\bar{a}} \\ -\lambda_{\bar{a}} - i_{l_{\bar{a}}}B \end{pmatrix} . \tag{4.51}
$$

The frame (4.51) is globally defined, nowhere vanishes and all vector fields are linearly independent. Hence it provides a generalised parallelisation of G. We search a generalised parallelisation which is orthogonal with respect to the $O(d,d)$ metric, i.e.

$$
\langle \hat{E}_a^+, \hat{E}_b^- \rangle = \eta_{ab} \ , \tag{4.52}
$$
$$
\langle \hat{E}_{\bar{a}}^+, \bar{E}_{\bar{b}}^- \rangle = \eta_{\bar{a}\bar{b}} \ , \tag{4.53}
$$
$$
\langle \hat{E}_a^+, \hat{E}_{\bar{b}}^- \rangle = 0 \ . \tag{4.54}
$$

The orthogonality conditions (4.52) and (4.53) are automatically satisfied. While the conditions (4.54) is satisfied if and only if holds the following relation

$$
i_{r_a}\lambda_{\bar{a}} = i_{l_{\bar{a}}}\rho_a \ . \tag{4.55}
$$

The generalised parallelisation (4.51) satisfies the GLP condition, and thanks to the (4.55), we have the algebra

$$
\begin{aligned}
L_{\hat{E}_a^+}\hat{E}_b^+ &= [\![\hat{E}_a^+, \hat{E}_a^+]\!] = f_{ab}{}^c \hat{E}_c^+ \ , \\
L_{\hat{E}_{\bar{a}}^-}\hat{E}_{\bar{b}}^- &= [\![\hat{E}_{\bar{a}}^-, \hat{E}_{\bar{b}}^-]\!] = f_{\bar{a}\bar{b}}{}^{\bar{c}} \hat{E}_{\bar{c}}^- \ , \\
L_{\hat{E}_a^+}\hat{E}_{\bar{b}}^- &= [\![\hat{E}_a^+, \hat{E}_{\bar{b}}^-]\!] = 0 \ .
\end{aligned}
\tag{4.56}
$$

The algebra (4.56) reproduced here by (4.51) consists in two copies of the Lie algebra $\mathfrak{g}$ associated to the Lie group G. Hence for a group manifold $M = G$, in the framework of generalised geometry, it is always possible to find a generalised parallelisation of G which gives rise to the algebra $\mathfrak{g} \oplus \mathfrak{g}$, and therefore, if $M = G$ is used for dimensional reductions, to a gauge group $G \times G$ in the reduced theory.

Conclusions

Generalised geometry is a modern topic in between mathematics and physics, which is still nowadays developing and is giving great results, as we have seen in this thesis. Hence I strongly believe that this approach should be pursued further inside Supergravity and M-theory in order to give a better explanation of some mysterious fact, for example about U-duality and the Mirror Symmetry, already approached by the double field theory formalism [10]. Some ideas in exceptional generalised geometry about U-duality have also arisen in [7].

Despite generalised geometry is growing fast, there are many questions which still remain unanswered. One of the most important concern how *supersymmetry* can enter in generalised geometry. In [15] the authors shown how to incorporate all the fermions inside generalised geometry, but still supersymmetry remains left outside. Instead asking how supersymmetry can enter in generalised geometry, the question should be more a conceptual one: why is generalised geometry the appropriate formalism to describe supergravity theories? Surely the fact that the metric and the B field can be enclosed inside a generalised metric and Type II Supergravity equations of motion can be written as $O(9,1) \times O(1,9)$ covariant Einstein's equations in vacuum gives us a good clue, but this is not enough. One might also ask whether the structure group $O(d,d)$ could be extended to include fermionic symmetries. However at the moment we do not know if it is possible to introduce a *superspace* inside generalised geometry framework, or even if generalised geometry does *not* need a superspace in order to encode supersymmetry.

About the topic of consistent truncations of supergravity in generalised geometry framework, there are still open questions. The main question probably regards if the conjecture 4.1 is true or not. If we assume it is true, then we do not know if GLP is the *weakest* requirement for a manifold in order to have a consistent truncation in the reduced theory. If we suppose GLP represents the weakest requirement, then we do not know which topological properties a manifold must hold. Putting in another way, we do not know the *sufficient* conditions which a manifold must have in order to be GLP. When we discover these informations, then we will be able to catalogue which manifolds can give consistent truncations and which not.

Since this way is quite hard to figure out immediately, the tactics consist in pick up other manifolds, which are not discovered to give consistent truncations and try to see if they are GLP. This approach gives a broad vision about properties which GLP manifolds must fulfil.

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A
Differential Geometry

A.1 Some concepts about manifolds

In this section we talk about some concepts quite formal about manifolds, which sometimes are not in the daily language, but that we used in this thesis. The topics we cover here are *induced maps*, *flows* and *Lie derivative*, *left and right invariant vector fields* and *embedded manifolds*. We assume known the general knowledge about manifolds.

A.1.1 Induced maps: pullback and pushforward

Let us consider the smooth map $f : M \to N$ from an m-dimensional manifold M to an n-dimensional manifold N. Smooth means if we take a chart (U, ϕ) on M and (V, ϕ) on N, where $p \in U$ and $f(p) \in V$, then the "coordinate presentation" of f, i.e.

$$\psi \circ f \circ \phi^{-1} : \mathbb{R}^m \to \mathbb{R}^n \ , \tag{A.1}$$

is $\mathcal{C}^\infty$. The *smooth* map f induces naturally a map f_*, called *pushforward*, such as

$$f_* : T_pM \to T_{f(p)}N \ . \tag{A.2}$$

Let us pick up a point $p \in M$ and let be $x = \phi(p)$ and $y = \psi(f(p))$. Let be $V = V^\mu \partial/\partial x^\mu \in T_pM$ and $f_*V = W^\alpha \partial/\partial y^\alpha \in T_{f(p)}N$. Then, in a chart, the pushforward components are defined as

$$W^\alpha = V^\mu \frac{\partial y^\alpha(x)}{\partial x^\mu} \ . \tag{A.3}$$

Note that the matrix $(\partial y^\alpha/\partial x^\mu)$ is nothing but the Jacobian of the map $f : M \to N$.

There smooth map f induces also another map f^*, calles *pullback*, such as

$$f^* : T^*_{f(p)}M \to T^*_pM \ . \tag{A.4}$$

Again, in the same notation as before, consider the covector $\omega = \omega_\alpha dy^\alpha \in T^*_{f(p)}N$ and $f^*\omega = \lambda_\mu dx^\mu \in T^*_pM$. The pullback components are defined as

$$\lambda_\mu = \omega_\alpha \frac{\partial y^\alpha(x)}{\partial x^\mu} \ . \tag{A.5}$$

Note that f_* works between tangent bundles and goes in the same direction as f, while f^* works between cotangent bundles and goes backward, hence the names *pushforward* and *pullback*.

A.1.2 Flows and Lie derivative

Let X be a vector field in M. An integral curve $\mathcal{C}(t)$ of X is a curve

$$\mathcal{C} : \mathbb{R} \to M \ , \tag{A.6}$$

whose tangent vector at $\mathcal{C}(t)$ is $X|_{\mathcal{C}(t)}$. Given a chart (U, ϕ), such that $x(t) = \phi(\mathcal{C}(t))$, than this read

$$\frac{dx^\mu}{dt} = X^\mu(x(t)) \ . \tag{A.7}$$

The problem of finding integral curves is equivalent to solving the autonomous system of ordinary differential equations (ODEs) given by (A.7). The initial condition $x_0^\mu = x^\mu(0)$ must be specified in order to have a well defined Chauchy problem. The existence and uniqueness theorem of ODEs guarantees there is a unique solution of (A.7). Therefore we have a family of integral curves, labelled by the initial data x_0^μ, and it is called *flow*. Formally a flow is a map

$$\sigma : \mathbb{R} \times M \to M \ , \tag{A.8}$$

such that in a chart

$$\frac{d}{dt}\sigma^\mu(t, x_0) = X^\mu(\sigma(t, x_0)) \ , \qquad\qquad \sigma^\mu(0, x_0) = x_0^\mu \ . \tag{A.9}$$

The flow $\sigma(t, x)$ represent a one-parameter group of diffeomorphisms from M to M because for fixed $t \in \mathbb{R}$ the map $\sigma_t : M \to M$ is a diffeomorphism, and it satisfies the "group properties"

- $\sigma_t \circ \sigma_s(x) = \sigma_{t+s}(x) \qquad \forall\ t, s \in \mathbb{R}$,
- $\sigma_0 = \mathbb{1}$,
- $\sigma_{-t} = (\sigma_t)^{-1}$.

Since $\sigma_t \circ \sigma_s(x) = \sigma_s \circ \sigma_t(x)$, it is also a *commutative* group.

Consider now the two flows $\sigma(t, x)$ and $\tau(t, x)$ generated by the vector fields X and Y,

$$\frac{d\sigma^\mu(t, x)}{dt} = X^\mu(\sigma(t, x)) \tag{A.10}$$

$$\frac{d\tau^\mu(t, x)}{dt} = Y^\mu(\tau(t, x)) \ . \tag{A.11}$$

The *Lie derivative* of Y along the flow σ of X evaluates how the vector fields Y changes along the vector fields X. To do this we have to compare the vector Y at a point x with the nearby point $x' = \sigma_\epsilon(x)$, with $\epsilon \ll 1$. However we cannot take simply the difference between the components of Y at the two points since they belong to difference tangent spaces T_pM and $T_{\sigma_\epsilon(x)}M$. However we can "pull back" the vector $Y|_{\sigma_\epsilon(x)} \in T_{\sigma_\epsilon(x)}M$ to T_xM using the pushforward map $(\sigma_{-\epsilon})_* : T_{\sigma_\epsilon(x)}M \to T_xM$. Hence the *Lie derivative* is defined as

$$\mathcal{L}_X Y \equiv \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left[(\sigma_{-\epsilon})_* Y|_{\sigma_\epsilon(x)} - Y|_x\right] \ . \tag{A.12}$$

In a chart (U, ϕ) with coordinates x, $X = X^\mu \partial/\partial x^\mu$ and $Y = Y^\mu \partial/\partial x^\mu$, one can show the Lie derivatives has the expression

$$\mathcal{L}_X Y = (X^\mu \partial_\mu Y^\nu - Y^\nu \partial_\mu X^\mu)\, \partial_\nu \ . \tag{A.13}$$

The above expression holds also in a non coordinate basis $\{\hat{e}_a\}$, just replacing ∂_μ by $\hat{e}_a$.

One can define *Lie bracket* $[X, Y]$ as the commutator of the two vector fields, i.e.

$$[X, Y]f = X[Y[f]] - Y[X[f]] \ , \tag{A.14}$$

where $f \in \mathcal{F}(M)$. Accidentally, Lie derivative and Lie bracket coincide. Lie derivative can be extended to tensor fields.

The properties of Lie derivative are

- Linearity

$$\mathcal{L}_X(T_1 + T_2) = \mathcal{L}_X T_1 + \mathcal{L}_X T_2 \ , \tag{A.15}$$

  with T_1 and T_2 generic tensor fields.

- Leibniz rule

$$\mathcal{L}_X(Y \otimes \omega) = Y \otimes (\mathcal{L}_X \omega) + (\mathcal{L}_X Y) \otimes \omega \ , \tag{A.16}$$

  where $Y \in \Gamma(TM)$ and $\omega \in \Gamma(T^*M)$.

- Given T a generic tensor.

$$\mathcal{L}_{[X,Y]}T = \mathcal{L}_X \mathcal{L}_Y T - \mathcal{L}_Y \mathcal{L}_X T \ . \tag{A.17}$$

- Cartan's formula

$$\mathcal{L}_X \omega = d i_X \omega + i_X d\omega \ , \tag{A.18}$$

  where $\omega \in \Gamma(T^*M)$.

A.1.3 Left and right invariant vector fields

Let a and g be element of a Lie group G. The *right-translation* $R - a : G \to G$ and the l*left-translation* $L_a : G \to G$ of g by a are defined by

$$R_a \, g = g \, a \ , \tag{A.19}$$

$$L_a \, g = a \, g \ . \tag{A.20}$$

Since the group composition in a Lie group is a *smooth* map by definition, R_a and L_a are smooth maps. Then remains defined the induced maps

$$R_{a*} : T_g G \to T_{ga} G \ , \qquad L_{a*} : T_g G \to T_{ag} G \ . \tag{A.21}$$

Let X be a vector field on a Lie group G. X is said to be *left-invariant vector fields* if

$$L_{a*} \, X|_g = X|_{ag} \ . \tag{A.22}$$

The same definition holds for right-invariant vector fields, just replacing L_{a*} by R_{a*}. In components, the relation (A.22) reads

$$L_{a*} \, X|_g = X^\mu(g) \frac{\partial x^\nu(ag)}{\partial x^\mu(g)} \frac{\partial}{\partial x^\nu}\bigg|_{ag} = X^\nu(ag) \frac{\partial}{\partial x^\nu}\bigg|_{ag} \ , \tag{A.23}$$

where $x^\mu(g)$ and $x^\mu(ag)$ are coordinates of g and ag, respectively.

Left and right invariant vector fields are very useful for proving the following fact, which goes under the name of *Lie's theorem*

$$[X_a, X_b]|_g = f_{ab}{}^c X_c|_g \ , \qquad\qquad \forall\ g \in G \ , \tag{A.24}$$

where $\{X_a\}$ is a set of left or right invariant vector fields, and $f_{ab}{}^c$ are constants which do not depend by g. Let us prove this fact. Consider the algebra $T_{\mathbb{1}}G = \mathfrak{g}$ generated by the left invariant vector fields $\{X_a|_{\mathbb{1}}\}$. They must satisfy the algebra relation

$$[X_a, X_b]|_{\mathbb{1}} = f_{ab}{}^c X_c|_{\mathbb{1}} \ , \tag{A.25}$$

where $f_{ab}{}^c$ are the structure constants of the algebra $\mathfrak{g}$. The idea now is that we want to "move" the algebra in another point $g \in G$ different from $\mathbb{1}$. If we apply L_{g*} to both members of (A.25), and reminding the definition of left invariant vector fields, for the LHS we have

$$L_{g*}[X_a, X_b]|_{\mathbb{1}} = [L_{g*}X_a|_{\mathbb{1}}, L_{g*}X_b|_{\mathbb{1}}] = [X_a, X_b]|_g \ , \tag{A.26}$$

for the RHS we have

$$L_{g*}f_{ab}{}^c X_c|_{\mathbb{1}} = f_{ab}{}^c X_c|_g \ . \tag{A.27}$$

Therefore $f_{ab}{}^c$ do not change. Since this holds for each $g \in G$, we have the thesis (A.24).

A.1.4 Embedded manifolds

Let M be a d-dimensional manifold embedded in $\mathbb{R}^{d+n}$. Our aim now is provide a way to compute the *volume form* and the *Hodge dual* in M using constrained coordinates $x^\mu \in \mathbb{R}^{d+n}$. First we have to define the concept of *normal bundle*.

Fix a point $p \in M$ and let N_pM be the vector space which is normal to T_pM in $\mathbb{R}^{d+n}$, which means $\delta_{\mu\nu}U^\mu V^\nu = 0$ where $U \in N_pM$, $V \in T_pM$, $\mu, \nu = 1, ..., d+n$ and δ is the Euclidean metric in $\mathbb{R}^{d+n}$. The vector space N_pM is isomorphic to $\mathbb{R}^n$. The *normal bundle* is defined as

$$NM \equiv \bigcup_{p\in M} N_pM \ . \tag{A.28}$$

Recall the volume form on $\mathbb{R}^{d+n}$

$$\mathrm{vol}_{\mathbb{R}^{d+n}} = dx^1 \wedge \cdots \wedge dx^{d+n} = \frac{1}{(d+n)!}\epsilon_{\mu_1 \ldots \mu_{d+n}} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{d+n}} \ . \tag{A.29}$$

Let $V_1, ..., V_n$ a set of n normal vector fields in NM. Then the volume form on M is given by

$$\mathrm{vol}_M = i_{V_1} \cdots i_{V_n} \mathrm{vol}_{\mathbb{R}^{d+n}} \ . \tag{A.30}$$

Given a p-form ω, with $p \leq d$, i.e.

$$\omega = \frac{1}{p!}\omega_{\mu_1\ldots\mu_p} dx^{\mu_1} \wedge \cdots dx^{\mu_p} \ , \tag{A.31}$$

the Hodge dual of ω in $\mathbb{R}^{d+n}$ is defined as

$$\star_{d+n}\,\omega \equiv \frac{1}{p!(d+n-p)!}\omega_{\nu_1\ldots\nu_p}\epsilon^{\nu_1\ldots\nu_p}{}_{\mu_1\ldots\mu_{d+n-p}} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{d+n-p}} \ . \tag{A.32}$$

The Hodge dual of ω in M is given by

$$\star_d\,\omega = i_{V_1} \cdots i_{V_n} \star_{d+n}\,\omega \ . \tag{A.33}$$

Let us particularise these formulæin the case $M = S^d$ and $n = 1$. Let us take polar coordinates on S^d. Then the normal vector $V \in NS^d$ lies along the radial direction r. Using the chain rule we have the expression

$$V = \partial_r = \frac{x^\mu}{r}\partial_\mu \ . \tag{A.34}$$

If we insert (A.34) inside (A.30) and (A.33) then we get the expressions used in the chapter 4.

A.2 Fibre bundle

Fibre bundle represents a technique in order to put over a manifold a second topological space. This approach is very useful in physics. In fact many theories, such as general relativity and gauge theories, are described naturally in terms of fibre bundles. Heuristically, a fibre bundle consists in taking a manifold and associating to each point an element of a topological space which is called *fibre*. Let us give here the formal definition

Definition (Fibre bundle). *A (differentiable) fibre bundle is a set of elements* (E, π, M, F, G) *which consists in*

- *a d-dimensional differentiable manifold M, called* base space*,*
- *a d-dimensional differential manifold F, called* fibre*,*
- *a $2d$-dimensional differentiable manifold E, called* total space*,*
- *a surjection $\pi : E \to M$, called* projection*,*
- *a Lie group G, called* structure group*, which acts on F on the left,*
- *a* local trivialization*. Given a set of open covering $\{U_i\}$ of M, they are maps $\phi_i : U_i \times F \to \pi^{-1}(U_i)$ such that $\pi \circ \phi_i(p, f) = p$. The inverse ϕ_i^{-1} maps $\pi^{-1}(U_i)$ onto the direct product $U_i \times F$, this is the reason of the name "local trivialization".*
- *on the intersection $U_i \cap U_j \neq \emptyset$, we require that the* transition functions $t_{ij} \equiv \phi_i^{-1} \circ \phi_j$ *are elements of G.*

If the *transition functions* are trivial everywhere, than we have the so-called *trivial bundle*. The tangent bundle is just a particular case of fibre bundle, where the fibre F at each point p is the tangent space T_pM. In this case the structure group G is the general linear transformation $GL(d, \mathbb{R})$, which represents the diffeomorphism group. In generalised geometry G is the $O(d, d)$ group, which is larger than the $GL(d, \mathbb{R})$ group, and it encodes inside also the gauge transformations of the B field.

B

Formal aspects of Generalised Geometry

The way we presented the definition of the generalised tangent bundle in the chapter 2 actually does not represent the canonical construction one should do. But in that moment we preferred to give to the reader a more "friendly" definition, in order to make the exposure less heavy.

Here I want to give the more precise definition of generalised tangent bundle, which make use of some mathematical tools, such as the *exact sequence*.

Definition (Exact sequence)**.** *An exact sequence is a sequence, either finite or infinite, of objects* $\{G_i\}_{i=0,...,n}$ *and morphisms* $\{\varphi_i\}_{i=1,...,n}$ *between them, such that the image of one morphism equals the kernel of the next.*

$$G_0 \xrightarrow{\varphi_1} G_1 \xrightarrow{\varphi_2} G_2 \xrightarrow{\varphi_3} \cdots \xrightarrow{\varphi_n} G_n \ , \tag{B.1}$$

and

$$Im(\varphi_i) = ker(\varphi_{i+1}) \qquad\qquad \forall i = 1, ..., n-1 \ . \tag{B.2}$$

In the above definition, by *sequence* we mean an ordered list. For a sequence, unlike set, order matters, and the same elements can appear multiple times at different positions in the sequence. The *objects* mentioned above can be generic sets, vectorial spaces, topological spaces, groups, etc ... Depending on the chosen objects, the morphisms can be respectively functions, linear transformations, continuous functions, group homomorphisms, etc ...

The exact sequence is called "short exact sequence" when the sequence is finite.

Definition (Generalised Tangent Bundle)**.** *The generalised tangent bundle E is defined by the short exact sequence*

$$0 \longrightarrow T^*M \xrightarrow{\iota} E \xrightarrow{\pi} TM \longrightarrow 0 \ , \tag{B.3}$$

where

- *ι is the inclusion map, which maps elements of T^*M in itself inside E*

$$\iota(\lambda) = \begin{pmatrix} 0 \\ \lambda \end{pmatrix} \ ,$$

- *π is called "anchor", which is a projection function, i.e. a map that takes the TM part of a sections of E.*

$$\pi\left[\begin{pmatrix} v \\ \lambda \end{pmatrix}\right] = v \ .$$

B.1 The exact Courant algebroid

In the chapter 2 we defined the Courant bracket $[\![\ ,\ ]\!]$, which maps sections of E in sections of E, and we showed, in an heuristic way, that the generalised geometry has an $O(d,d)$ structure induced by the scalar product $\langle\ ,\ \rangle$.

In a mathematical point of view, we have that

"Generalised geometry is an *exact Courant algebroid*"

Definition (Courant algebroid). *The set* $(E, \langle\ ,\ \rangle, [\![\ ,\ ]\!], \pi)$ *is called Courant algebroid, where*

- E *is a real vector bundle;*
- $\pi : E \to TM$ *is an anchor;*
- $\langle\ ,\ \rangle : E \times E \to \mathbb{R}$ *is a non degenerate symmetric bilinear form (scalar product);*
- $[\![\ ,\ ]\!] : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ *Courant bracket such that*
  - $[\![\pi(A), \pi(B)]\!] = \pi([\![A, B]\!])$
  - $[\![[\![A, B]\!], C]\!] + [\![[\![B, C]\!], A]\!] + [\![[\![C, A]\!], B]\!] = 0$ *(Jacoby identity)*
  - $[\![A, fB]\!] = f[\![A, B]\!] + \pi(A)[f]B$
  - $\pi(A\langle B, C\rangle) = \langle [\![A, B]\!], C\rangle + \langle B, [\![A, C]\!]\rangle$
  - $[\![A, A]\!] = \iota(d\langle A, A\rangle)$

  $\forall\ A, B, C \in \Gamma(E)$.

This structure of generalised geometry is not only a beautiful mathematical formalism, but it gives us some important results. For instance, in the B-transformation, the fact that B must be closed comes from the preservation of the exact Courant algebroid by the $O(d,d)$ action.

C
Double Field Theory

Double Field Theory is an attempt on making T-duality explicit in *field* theory Lagrangians. Heuristically we can make the following argument. Consider a string living in a $D = (d + n)$-dimensional spacetime with n compactified coordinates on a n-dimensional torus, thus the spacetime looks like $M_d \times T^n$. As we have seen in the section 1.11.2, upon quantisation there will be momentum modes and winding modes for each compact direction. Let us denote the non-compact coordinates by x^α and the compact coordinates by x^μ, in total $x^i = (x^\mu, x^\alpha)$. The compact coordinates x^α give rise to string momentum excitations K_α, but since strings are extended objects, there are also winding quantum number W^a. For these last novel quantum numbers there are not any coordinates associated with them. Hence it is reasonable to *introduce* some new coordinates $\tilde{x}_\alpha$ which take into account the winding numbers.

If one attempts to write down the complete field theory of closed strings in coordinate space it must include the x^α as well as the $\tilde{x}_\alpha$. Thus the argument of all fields in such a theory must be doubled and for this reason this theory is called *Double Field Theory*. The doubled fields $\phi(x^\alpha, \tilde{x}_\alpha, x^\mu)$ are said to be functions of momentum *and* winding. The action must include a suitable integration over the additional coordinates, i.e. must be in the form

$$S = \int dx^\alpha d\tilde{x}_\alpha dx^\mu \mathcal{L}(x^\alpha, \tilde{x}_\alpha, x^\mu) \ . \tag{C.1}$$

Actually we will see that for covariance reasons we have to doubled all the coordinates $(x^i, \tilde{x}^i)$.

Let us write down the sigma-model action for string propagating in a background

$$S = -\frac{1}{4\pi}\int_0^{2\pi} d\sigma \int_{-\infty}^{+\infty} d\tau \left(\eta^{ab}\partial_a x^i \partial_b x^j g_{ij} + \epsilon^{ab}\partial_a x^i \partial_b x^j B_{ij}\right) \ , \tag{C.2}$$

where we choose the background

$$g_{ij} = \begin{pmatrix} g_{\alpha\beta} & 0 \\ 0 & \eta_{\mu\nu} \end{pmatrix} \ , \qquad B_{ij} = \begin{pmatrix} B_{\alpha\beta} & 0 \\ 0 & 0 \end{pmatrix} \ . \tag{C.3}$$

The Hamiltonian H, given as integration of the Hamiltonian density, is given by

$$H = \frac{1}{2} Z^T \mathcal{H}(g, B) Z + N_L + N_R + \ldots \ , \tag{C.4}$$

where the dots indicate terms irrelevant to the discussion and

$$Z^M = \begin{pmatrix} W^i \\ K_i \end{pmatrix} \ , \qquad M = 1, \ldots, 2D \ , \tag{C.5}$$

where

$$W^i = \begin{pmatrix} W^\alpha \\ 0 \end{pmatrix} \ , \qquad\qquad K_i = \begin{pmatrix} K_\alpha \\ 0 \end{pmatrix} \ , \tag{C.6}$$

where winding and momentum modes along non-compact x^μ coordinates are null. $\mathcal{H}(g, B)$ is called *generalised metric* and is defined as

$$\mathcal{H}(g, B) = \begin{pmatrix} g - Bg^{-1}B & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix} \ . \tag{C.7}$$

There is a constraint in closed string theory which matches the level of the right and left moving excitations in any physical state. This condition reads

$$N_L - N_R = W^\alpha K_\alpha \equiv \frac{1}{2} Z^T \eta Z \ , \tag{C.8}$$

where η is defined as

$$\eta_{MN} = \begin{pmatrix} 0 & \mathbb{1}_D \\ \mathbb{1}_D & 0 \end{pmatrix}_{MN} \ , \tag{C.9}$$

which is similar to the $O(D, D)$ metric. Therefore under $O(D, D)$ transformations of the type

$$Z' = hZ \ , \qquad\qquad h^T \eta h = \eta \ , \tag{C.10}$$

the physics should not change. This lead us to introduce *extra coordinates* associated to h transformations,

$$X'^M = h^M{}_N X^N \ , \qquad\qquad X \equiv \begin{pmatrix} \tilde{x} \\ x \end{pmatrix} \ . \tag{C.11}$$

The physical constraint of level matching has a differential expression which reads

$$N_L - N_R = \partial \cdot \tilde{\partial} = \frac{1}{2} \partial^M \partial_M \equiv \Delta \ . \tag{C.12}$$

On massless fields holds $N_L = N_R = 1$. Therefore Δ operator must annihilate on them, i.e.

$$\frac{1}{2} \partial^M \partial_M \phi = 0 \ . \tag{C.13}$$

This is called *strong constraint* and tells us $\tilde{\partial}_i \phi = 0$, i.e. ϕ does not depend by extra coordinates.

If two fields ϕ_i and ϕ_j are annihilated separately by Δ, it is not true the product $\phi_i \phi_j$ is annihilated again by Δ, because

$$\partial \cdot \tilde{\partial}(\phi_1 \phi_2) = (\partial \cdot \tilde{\partial} \phi_1) \phi_2 + (\tilde{\partial} \phi_1) \cdot (\partial \phi_2) + (\partial \phi_1) \cdot (\tilde{\partial} \phi_2) + \phi_1 (\partial \cdot \tilde{\partial} \phi_2) \ . \tag{C.14}$$

Hence there is another constraint, called *strong constraint*, which requires

$$\eta^{MN} \partial_M \phi_i \, \partial_N \phi_j = 0 \ , \qquad\qquad \forall \ i, j \ . \tag{C.15}$$

The strong constraint (C.15) looks very similar to the constraint (2.41) in generalised geometry, just replacing the fields ϕ_i and ϕ_j with functions f and g. At the end we have the following result

Proposition C.1. *For a set of fields $\phi_i(x, \tilde{x})$ that satisfies (C.15) there is a duality frame $(\tilde{x}'_i, x'^i)$ in which the fields do not depend on $\tilde{x}'_i$.*

D Computations on GLP manifolds

In this appendix we show the computations about the original part of the thesis, i.e. the $S^2 \times S^1$ case and the conjecture $G \times G$. During the thesis we used a notation which was clear about the vector components and the form components of generalised vectors. However that notation is quite painful when one get inside computations. Hence, for this reason, here we adopt another notation, mostly used in literature, which consist in writing generalised vectors as a "sum" of vectors and forms, i.e.

$$V = \begin{pmatrix} v \\ \lambda \end{pmatrix} \qquad \longrightarrow \qquad V = v + \lambda \ . \tag{D.1}$$

D.1 Computations on $S^2 \times S^1$

The Ansatz for the generalised parallelisation is

$$\begin{aligned} \hat{E}_i &= v_i - i_{v_i} B + a\, y_i\, d\psi \ , \\ \hat{E}'_i &= b(y_i \partial_\psi - i_{y_i \partial_\psi} B) + c\, \epsilon_{ijk}\, y^j\, dy^k \ , \end{aligned} \tag{D.2}$$

where a, b, c are coefficients to determine by GLP and $O(3,3)$ orthogonality conditions and here, instead considering $\hat{E}_{ij}$ on S^2 embedded on $\mathbb{R}^3$, we consider their dual $\hat{E}_i \equiv \epsilon_{ijk} \hat{E}_{jk}$. Remind that v_i are rotational $SO(3)$ Killing vectors, $v_i = \epsilon_{ijk} v_{jk} = \epsilon_{ijk}\, y_j \partial_k$. We impose now $O(3,3)$ orthogonality condition on the frame (D.2). We can neglect in this computation the term proportional to B since the scalar product $\langle \cdot, \cdot \rangle$ is invariant under B-shift.

$$\begin{aligned} \langle \hat{E}_i, \hat{E}_j \rangle &= \frac{1}{2}[i_{v_i}(a\, y_j d\psi) + i_{v_j}(a\, y_i d\psi)] = 0 && \text{(identically)} \ , \\ \langle \hat{E}'_i, \hat{E}'_j \rangle &= \frac{1}{2}[i_{b\, y_i \partial_\psi}(c\, \epsilon_{jkl} y^k dy^l) + i_{b\, y_j \partial_\psi}(c\, \epsilon_{ikl} y^k dy^l)] = 0 && \text{(identically)} \ , \\ \langle \hat{E}_i, \hat{E}'_j \rangle &= \frac{1}{2}[i_{v_i}(c\, \epsilon_{jkl} y^k dy^l) + i_{b\, y_j \partial_\psi}(a\, y_i d\psi)] \\ &= \frac{1}{2}(c\, \epsilon_{jkl} y^k i_{v_i} dy^l + ab y_j y_i i_{\partial_\psi} d\psi) \\ &= \frac{1}{2}(c\, \epsilon_{ljk} y^k \epsilon_{lis} y^s + ab y_j y_i) \\ &= \frac{1}{2}(c\, (\delta_{ji}\delta_{ks} - \delta_{js}\delta_{ki}) y^k y^s + ab y_j y_i) \\ &= \frac{1}{2}(c\, \delta_{ij} y^s y^s - c\, y_i y_j + ab y_j y_i) = \frac{1}{2} c \delta_{ji} && (ab = c) \ , \end{aligned}$$

where in the last step we used that $y^i y^i = 1$. Hence, imposing the condition $ab = c$ we find that this frame is indeed an $O(d,d)$ frame.

$$\langle \hat{E}_i, \hat{E}_j \rangle = 0 \ , \qquad \langle \hat{E}'_i, \hat{E}'_j \rangle = 0 \ , \qquad \langle \hat{E}_i, \hat{E}'_j \rangle = \frac{1}{2}\delta_{ij} \ . \tag{D.3}$$

Next we impose, by equations of motion, the flux form H to be proportional to the wedge product of the volume forms of S^2 and S^1

$$H = h' \, \mathrm{vol}_{S^2} \wedge \mathrm{vol}_{S^1} \ .$$

To construct it first we need to calculate these volume forms using the technique shown in A.1.4. Suppose the two spheres S^1 and S^2 have radius respectively R_1 and R_2.

$$\begin{aligned}
\mathrm{vol}_{S^1} &= i_{\partial_r} \mathrm{vol}_{\mathbb{R}^2}\big|_{r=R_1} = \frac{y^k}{r} i_{\partial_k} \left(\frac{r^2}{2!} \epsilon_{ij} dy^i \wedge dy^j \right) \Bigg|_{r=R_1} \\
&= \frac{r}{2} \epsilon_{ij} (y^i \, dy^j - y^j \, dy^i)\big|_{r=R_1} = r \epsilon_{ij} y^i \, dy^j \big|_{r=R_1} = R_1 d\psi \\
\mathrm{vol}_{S^2} &= i_{\partial_r} \mathrm{vol}_{\mathbb{R}^3}\big|_{r=R_2} = \frac{y^l}{r} i_{\partial_l} \left(\frac{r^3}{3!} \epsilon_{ijk} dy^i \wedge dy^j \wedge dy^k \right) \Bigg|_{r=R_2} \\
&= \frac{r^2}{3!} \epsilon_{ijk} (y^i \, dy^j \wedge dy^k - y^j \, dy^i \wedge dy^k + y^k \, dy^i \wedge dy^j)\big|_{r=R_2} \\
&= \frac{3r^2}{3!} \epsilon_{ijk} y^i \, dy^j \wedge dy^k \big|_{r=R_2} = \frac{R_2^2}{2} \epsilon_{ijk} y^i \, dy^j \wedge dy^k .
\end{aligned}$$

Putting all together and redefining the proportionality constant we find

$$H = h' \frac{R_1 R_2^2}{2} \epsilon_{ijk} y^i \, dy^j \wedge dy^k \wedge d\psi = \frac{h}{2} \epsilon_{ijk} y^i \, dy^j \wedge dy^k \wedge d\psi . \tag{D.4}$$

Now we want to compute the Dorfman derivatives. Let us start to compute term by term which is inside it, using the Cartan's formula (A.18)

$$\begin{aligned}
[v_i, v_j] &= -\epsilon_{ijk} v_k \\
i_{v_i} dy_j &= dy_j (\epsilon_{ikl} y_k \partial_l) = \epsilon_{ikl} y_k \delta_{jl} = -\epsilon_{ijk} y_k \\
\mathcal{L}_{v_i} y_j &= i_{v_i} dy_j = -\epsilon_{ijk} y_k \\
\mathcal{L}_{v_i} dy_j &= d(i_{v_i} dy_j) = d(-\epsilon_{ijl} y_l) = -\epsilon_{ijl} dy_l \\
\mathcal{L}_{v_i} y_j \partial_\psi &= [v_i, y_j \partial_\psi] = (\mathcal{L}_{v_i} y_j) \partial_\psi = -\epsilon_{ijk} y_k \partial_\psi \\
\mathcal{L}_{v_i} \epsilon_{jkl} y^k dy^l &= \epsilon_{jkl} [(\mathcal{L}_{v_i} y^k) dy^l + (\mathcal{L}_{v_i} dy^l) y^k] = \epsilon_{jkl} (-\epsilon_{ikr} y^r dy^l - \epsilon_{ilr} y^k dy^r) \\
&= -(\epsilon_{jkl} \epsilon_{ikr} + \epsilon_{jrk} \epsilon_{ikl}) y^r dy^l = -\epsilon_{ijk} \epsilon_{krl} y^r dy^l .
\end{aligned}$$

Note that in the last calculation we used the fact that the ϵ symbols are also the structure constants of $\mathfrak{so}(3)$, thus they must satisfy the Jacobi identity

$$\epsilon_{jlk} \epsilon_{kir} + \epsilon_{kjr} \epsilon_{lik} + \epsilon_{ijk} \epsilon_{klr} = 0.$$

We need to calculate the inner products between the flux form H and the vector components of the frame, since these terms also appear in the Dorfman derivative

$$\begin{aligned}
i_{by_i \partial_\psi} H &= \frac{bh}{2} y_i \epsilon_{lmn} y^l i_{\partial_\psi} (dy^n \wedge dy^m \wedge d\psi) \\
&= \frac{bh}{2} y_i \epsilon_{lmn} y^l dy^n \wedge dy^m \\
&= \frac{bh}{2} \epsilon_{imn} y_l y^l dy^n \wedge dy^m \\
&= \frac{bh}{2} \epsilon_{lmn} dy^n \wedge dy^m .
\end{aligned}$$

$$
\begin{aligned}
i_{v_i}H &= \frac{h}{2}\epsilon_{lnm}y^l i_{v_i}(dy^n \wedge dy^m \wedge d\psi)\\
&= \frac{h}{2}\epsilon_{lnm}y^l(dy^n(v_i)dy^m - dy^m(v_i)dy^n) \wedge d\psi\\
&= \frac{h}{2}\epsilon_{lnm}y^l(-\epsilon_{inj}y^j dy^m + \epsilon_{imj}y^j dy^n) \wedge d\psi\\
&= \frac{h}{2}\epsilon_{lmn}y^l(2\epsilon_{inj}y^j dy^m) \wedge d\psi\\
&= h\epsilon_{nlm}\epsilon_{nji}y^l y^j dy^m \wedge d\psi\\
&= h(\delta_{lj}\delta_{mi} - \delta_{li}\delta_{mj})y^l y^j dy^m \wedge d\psi\\
&= h(y^j y^j dy^i - y^i y^j dy^j) \wedge d\psi\\
&= h dy^i \wedge d\psi
\end{aligned}
$$

Where in the first product we used that $y_{[i}\epsilon_{lmn]} = \frac{1}{4!}(y_i\epsilon_{lmn} - y_l\epsilon_{mni} + y_m\epsilon_{nli} - y_n\epsilon_{lm}) = 0$ and in both the fact that $y^i y^i = 1$, which implies $dy^i y^i = 0$. Finally, we calculate the algebra using all this results

$$
\begin{aligned}
L_{\hat{E}_i}\hat{E}_j &= [v_i, v_j] - i_{[v_i,v_j]}B + \mathcal{L}_{v_i}ay_j d\psi + i_{v_j}(i_{v_i}H + d(ay_i d\psi))\\
&= -\epsilon_{ijk}v_k - i_{-\epsilon_{ijk}v_k}B + a(\mathcal{L}_{v_i}y_j)d\psi + i_{v_j}(hdy^i \wedge d\psi + ady_i \wedge d\psi)\\
&= -\epsilon_{ijk}(v_k - i_{v_k}B + a\epsilon_{ijk}y_k d\psi) + (a+h)i_{v_j}dy_i \wedge d\psi\\
&= -\epsilon_{ijk}\hat{E}_k + (a+h)\epsilon_{ijk}y_k d\psi\\
&= -\epsilon_{ijk}\hat{E}_k\\
L_{\hat{E}'_i}\hat{E}'_j &= [by_i\partial_\psi, by_j\partial_\psi] - i_{[by_i\partial_\psi, by_j\partial_\psi]}B + \mathcal{L}_{by_i\partial_\psi}(c\epsilon_{jkl}y^k dy^l) + i_{by_j\partial_\psi}(i_{by_i\partial_\psi}H + d(c\epsilon_{ikl}y^k dy^l))\\
&= 0 - i_0 B + bc\epsilon_{jkl}\mathcal{L}_{y_j\partial_\psi}(y^k dy^l) + i_{by_j\partial_\psi}(\frac{bh}{2}\epsilon_{ikl}dy^k \wedge dy^l + c\epsilon_{ikl}dy^k \wedge dy^l)\\
&= (\frac{bh}{2} + c)\epsilon_{ikl}by_j i_{\partial_\psi}(dy^k \wedge dy^l) = 0\\
L_{\hat{E}_i}\hat{E}'_j &= [v_i, by_j\partial_\psi] - i_{[v_i, by_j\partial_\psi]}B + \mathcal{L}_{v_i}(c\epsilon_{jlm}y^l dy^m) + i_{by_j\partial_\psi}(i_{v_i}H + d(ay_i d\psi))\\
&= -b\epsilon_{ijk}y_k\partial_\psi - i_{-b\epsilon_{ijk}y_k\partial_\psi}B - c\epsilon_{ijk}\epsilon_{klm}(y^l dy^m) + (a+h)i_{by_j\partial_\psi}dy_i \wedge d\psi\\
&= -\epsilon_{ijk}\hat{E}'_k - b(a+h)y_j dy_i = -\epsilon_{ijk}\hat{E}'_k
\end{aligned}
$$

So imposing the condition $a = -h$ we find

$$
\begin{aligned}
L_{\hat{E}_i}\hat{E}_j &= [\![\hat{E}_i, \hat{E}_j]\!] = -\epsilon_{ijk}\hat{E}_k \ , && \text{(D.5)}\\
L_{\hat{E}_i}\hat{E}'_j &= [\![\hat{E}_i, \hat{E}'_j]\!] = -\epsilon_{ijk}\hat{E}'_k \ , && \text{(D.6)}\\
L_{\hat{E}'_i}\hat{E}'_j &= [\![\hat{E}'_i, \hat{E}'_j]\!] = 0 \ . && \text{(D.7)}
\end{aligned}
$$

As a last step, we fix the remaining normalisation constants as $a = b = c = -h = 1$, so that the frame becomes

$$
\begin{aligned}
\hat{E}_i &= v_i - i_{v_i}B + y_i\, d\psi\\
\hat{E}'_i &= y_i\partial_\psi - i_{y_i\partial_\psi}B + \epsilon_{ijk}y^j\, dy^k
\end{aligned}
\tag{D.8}
$$

and the flux three-form

$$
H = -\frac{1}{2}\epsilon_{ijk}y^i\, dy^j \wedge dy^k \wedge d\psi. \tag{D.9}
$$

D.2 Computations on the conjecture $G \times G$

The Ansatz for the generalised parallelisation is

$$\begin{aligned}\hat{E}^+_a &= r_a + \rho_a - i_{r_a} B \ , \\ \hat{E}^-_{\bar{a}} &= l_{\bar{a}} - \lambda_{\bar{a}} - i_{l_{\bar{a}}} B \ .\end{aligned} \tag{D.10}$$

Since we are using a frame for the splitting bundle $E \simeq C_+ \oplus C_-$, we require the $O(d) \times O(d)$ orthogonality of the frame (D.10). This condition reads

$$\begin{aligned}\langle \hat{E}^+_a , \hat{E}^+_b \rangle &= \frac{1}{2}(i_{r_a}\rho_b + i_{r_b}\rho_a) = \delta_{ab} \ , \\ \langle \hat{E}^-_{\bar{a}} , \hat{E}^-_{\bar{b}} \rangle &= \frac{1}{2}(i_{l_{\bar{a}}}\lambda_{\bar{b}} + i_{l_{\bar{b}}}\lambda_{\bar{a}}) = \delta_{\bar{a}\bar{b}} \ , \\ \langle \hat{E}^+_a , \hat{E}^-_{\bar{b}} \rangle &= \frac{1}{2}(i_{r_a}\lambda_{\bar{b}} + i_{l_{\bar{b}}}\rho_a) \ .\end{aligned}$$

Hence if holds the condition

$$i_{r_a}\lambda_{\bar{a}} = i_{l_{\bar{a}}}\rho_a \ , \tag{D.11}$$

then the frame is $O(d) \times O(d)$ orthogonal, i.e.

$$\langle \hat{E}^+_a , \hat{E}^+_b \rangle = \eta_{ab} \ , \qquad \langle \hat{E}^-_{\bar{a}} , \hat{E}^-_{\bar{b}} \rangle = \delta_{\bar{a}\bar{b}} \ , \qquad \langle \hat{E}^+_a , \hat{E}^-_{\bar{b}} \rangle = 0 \ . \tag{D.12}$$

In a group manifold $M = G$ there are no conditions on the flux 3-from H.

Let us compute the terms involved in the Dorfman derivative. For the right-invariant sector

$$\begin{aligned}& d\rho^a = -f^a{}_{bc}\rho^b \wedge \rho^c \ , \\ & \mathcal{L}_{r_a} r_b = f_{ab}{}^c r_c \ , && \text{(algebra transformation)} \\ & \mathcal{L}_{r_a}\rho^b = f_{abc}\rho^c \ , && \text{(adjoint transformation)}\end{aligned}$$

and the same for the left-invariant sector

$$\begin{aligned}& d\lambda^{\bar{a}} = -f^{\bar{a}}{}_{\bar{b}\bar{c}}\lambda^{\bar{b}} \wedge \lambda^{\bar{c}} \ , \\ & \mathcal{L}_{l_{\bar{a}}} l_{\bar{b}} = f_{\bar{a}\bar{b}}{}^{\bar{c}} l_{\bar{c}} \ , && \text{(algebra transformation)} \\ & \mathcal{L}_{l_{\bar{a}}}\lambda^{\bar{b}} = f_{\bar{a}\bar{b}\bar{c}}\lambda^{\bar{c}} \ . && \text{(adjoint transformation)}\end{aligned}$$

The first equality comes from the general formula

$$d\omega\,(X,Y) = X\,[\omega\,(Y)] - Y\,[\omega\,(X)] - \omega\big([X,Y]\big) \ , \tag{D.13}$$

which holds for a 1-form ω and two vector fields X, Y. If we particularise the expression for the basis $\omega = \rho^a$, $X = r_b$ and $Y = r_c$, then we have

$$d\rho^a\,(r_b, r_c) = r_b\,[\delta^a{}_c] - r_c\,[\delta^a{}_b] - \rho^a\big([r_b, r_c]\big) = -f^a{}_{bc} \ , \tag{D.14}$$

hence

$$d\rho^a = -f^a{}_{bc}\rho^b \wedge \rho^c \ . \tag{D.15}$$

The same procedure holds for proving $d\lambda^{\bar{a}} = -f^{\bar{a}}{}_{\bar{b}\bar{c}}\lambda^{\bar{b}} \wedge \lambda^{\bar{c}}$.

The Dorfman derivative becomes

$$\begin{aligned}L_{\hat{E}^+_a}\hat{E}^+_b &= [r_a, r_b] + \mathcal{L}_{r_a}(\rho_b - i_{r_b}B) - i_{r_b}d(\rho_a - i_{r_a}B) \\ &= f_{ab}{}^c r_c + \mathcal{L}_{r_a}\rho_b - i_{[r_a,r_b]}B - i_{r_b}i_{r_a}dB - i_{r_b}d(i_{r_a}B) - i_{r_b}d\rho_a + i_{r_b}d(i_{r_a}B) \\ &= f_{ab}{}^c r_c + f_{ab}{}^c\rho_c - f_{ab}{}^c i_{r_c}B = f_{ab}{}^c\hat{E}^+_c\end{aligned}$$

$$
\begin{aligned}
L_{\hat{E}^-_{\bar{a}}}\hat{E}^-_{\bar{b}} &= [l_{\bar{a}}, l_{\bar{b}}] + \mathcal{L}_{l_{\bar{a}}}(-\lambda_{\bar{b}} - i_{l_{\bar{b}}}B) - i_{l_{\bar{b}}}d(-\lambda_{\bar{a}} - i_{l_{\bar{a}}}B) \\
&= f_{\bar{a}\bar{b}}{}^{\bar{c}} l_{\bar{c}} - \mathcal{L}_{l_{\bar{a}}}\lambda_{\bar{b}} - i_{[l_{\bar{a}}, l_{\bar{b}}]}B - i_{l_{\bar{b}}} i_{l_{\bar{a}}} dB - i_{l_{\bar{b}}} d(i_{l_{\bar{a}}}B) - i_{l_{\bar{b}}} d\lambda_{\bar{a}} + i_{l_{\bar{b}}} d(i_{l_{\bar{a}}}B) \\
&= f_{\bar{a}\bar{b}}{}^{\bar{c}} l_{\bar{c}} - f_{\bar{a}\bar{b}}{}^{\bar{c}}\lambda_{\bar{c}} - f_{\bar{a}\bar{b}}{}^{\bar{c}} i_{l_{\bar{c}}}B = f_{\bar{a}\bar{b}}{}^{\bar{c}}\hat{E}^-_{\bar{c}}
\end{aligned}
$$

$$
\begin{aligned}
L_{\hat{E}^+_{a}}\hat{E}^-_{\bar{b}} &= [r_a, l_{\bar{b}}] + \mathcal{L}_{r_a}(-\lambda_{\bar{b}} - i_{l_{\bar{b}}}B) - i_{l_{\bar{b}}}d(\rho_a - i_{r_a}B) \\
&= -\mathcal{L}_{r_a}\lambda_{\bar{b}} - i_{[r_a, l_{\bar{b}}]}B - i_{l_{\bar{b}}} i_{r_a} dB - i_{l_{\bar{b}}} d(i_{r_a}B) + i_{l_{\bar{b}}} d\rho_a + i_{l_{\bar{b}}} d(i_{r_a}B) = 0 \ .
\end{aligned}
$$

Therefore the following algebra is fulfilled

$$
\begin{aligned}
L_{\hat{E}^+_a}\hat{E}^+_b &= [\![\hat{E}^+_a, \hat{E}^+_b]\!] = f_{ab}{}^{c}\hat{E}^+_c \ , && \text{(D.16)} \\
L_{\hat{E}^-_{\bar{a}}}\hat{E}^-_{\bar{b}} &= [\![\hat{E}^-_{\bar{a}}, \hat{E}^-_{\bar{b}}]\!] = f_{\bar{a}\bar{b}}{}^{\bar{c}}\hat{E}^-_{\bar{c}} \ , && \text{(D.17)} \\
L_{\hat{E}^+_a}\hat{E}^-_{\bar{b}} &= [\![\hat{E}^+_a, \hat{E}^-_{\bar{b}}]\!] = 0 \ , && \text{(D.18)}
\end{aligned}
$$

if and only if

$$
i_{r_a}H = -d\rho_a = f^{abc}\rho^b \wedge \rho^c \ , \qquad i_{l_{\bar{a}}}H = d\lambda_{\bar{a}} = -f^{\bar{a}\bar{b}\bar{c}}\lambda^{\bar{b}} \wedge \lambda^{\bar{c}} \ . \tag{D.19}
$$

These two conditions give us a form for the flux 3-form H, which is

$$
H = \frac{1}{3!} f_{abc}\ \rho^a \wedge \rho^b \wedge \rho^c = -\frac{1}{3!} f_{\bar{a}\bar{b}\bar{c}}\ \lambda^{\bar{a}} \wedge \lambda^{\bar{b}} \wedge \lambda^{\bar{c}}. \tag{D.20}
$$

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