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Holomorphic functional calculus in non-Archimedean geometry

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Introduction

In classical terms (see, for example, [Bou67, §I.3.5]), the joint spectrum $\Sigma_{\boldsymbol{f}}$ of a tuple $\boldsymbol{f}$ of elements $(f_1, \dots, f_n)$ of a unital commutative Banach $\mathbb{C}$-algebra $\mathscr{D}$ can be defined as the image of the map

$$^*\boldsymbol{f} \colon X(\mathscr{D}) \to \mathbb{C}^n, \quad \chi \mapsto (\chi(f_1), \dots, \chi(f_n)), \tag{1}$$

where $X(\mathscr{D})$ is the space of characters of $\mathscr{D}$ (i.e. the space of unital $\mathbb{C}$-algebra homomorphisms $\mathscr{D} \to \mathbb{C}$). Moreover, if we identify $\mathbb{C}^n$ with the space of characters of the $\mathbb{C}$-algebra of polynomials in n variables $\mathbb{C}[T_1, \dots, T_n]$, then we can see $^*\boldsymbol{f}$ as the map (between the corresponding spaces of characters) induced by the homomorphism

$$\varphi \colon \mathbb{C}[T_1, \dots, T_n] \to \mathscr{D}, \quad T_i \mapsto f_i \quad (\forall\, i = 1, \dots, n). \tag{2}$$

As recounted in [Ber07], the problem from which the idea of Berkovich spaces originated was that of defining an analogue of spectra for elements of a unital commutative Banach K-algebra $\mathscr{D}$, where K is a non-Archimedean field. The idea of Berkovich was to allow characters to take values on Banach field extensions of K (something that is superfluous in the complex case because of Gelfand - Mazur theorem, [Bou67, Corollaire I.2.5/2]). This led him (in view of Proposition 1.2.8.iii) to define the following analogues for $X(\mathscr{D})$ and $\mathbb{C}^n$: the Berkovich spectrum $\mathscr{M}(\mathscr{D})$ of $\mathscr{D}$ is the topological space of all bounded multiplicative seminorms on $\mathscr{D}$ (with the weakest topology); the n-dimensional analytic affine space $\mathbb{A}^n_K$ is the topological space of all the multiplicative seminorms on $K[T_1, \dots, T_n]$ which extend the absolute value on K (with the weakest topology). Thus, for any $\boldsymbol{f} \coloneqq (f_1, \dots, f_n) \in \mathscr{D}^n$, he could define an analogue of the map $^*\boldsymbol{f}$ in (1) as the map induced (see Definition/Proposition 1.2.4) by the analogue of φ with K instead of $\mathbb{C}$, and could define the joint spectrum of $\boldsymbol{f}$ as the image of this map. To allow more generality, he further defined the spectrum Σ_φ of a bounded homomorphism $\varphi \colon \mathscr{A} \to \mathscr{D}$ from a K-affinoid algebra $\mathscr{A}$ (see Definition 1.5.4) to a unital commutative Banach K-algebra $\mathscr{D}$ as the image of the induced map $\mathscr{M}(\varphi) \colon \mathscr{M}(\mathscr{D}) \to \mathscr{M}(\mathscr{A})$. One can relate it to the definition of joint spectrum of a tuple of elements by considering $\mathbb{A}_K^n$ as the union of the Berkovich spectra relative to the K-affinoid algebras $K\{\boldsymbol{r}^{-1}\boldsymbol{T}\}$ (see Definition 1.5.2) with $|\boldsymbol{r}| \to \infty$.

Now, for any unital commutative Banach $\mathbb{C}$-algebra $\mathscr{D}$, the (classical) holomorphic functional calculus theorem (see, for example, [Bou67, §I.4]) says that there exists one and only one map which associates to any tuple of elements $\boldsymbol{f} := (f_1, \ldots, f_n) \in \mathscr{D}^n$ a homomorphism $\theta_{\boldsymbol{f}} \colon \Gamma(\Sigma_{\boldsymbol{f}}, \mathscr{O}_{\mathbb{C}^n}) \to \mathscr{D}$ such that, if $z_1, \ldots, z_n$ are the germs of the coordinate functions on $\mathbb{C}^n$, then $\theta_{\boldsymbol{f}}(z_i) = f_i$ for each $i = 1, \ldots, n$ (plus some further properties). Here, $\Gamma(\Sigma_{\boldsymbol{f}}, \mathscr{O}_{\mathbb{C}})$ is the algebra of holomorphic functions on a neighborhood of $\Sigma_{\boldsymbol{f}}$ in $\mathbb{C}^n$. The purpose of this thesis is to provide a complete proof of the analogue theorem for unital commutative Banach algebras over a non-Archimedean field K (Theorem 4.2.1). Omitting some further properties that are proven, it says that there is a way to extend any bounded homomorphism $\varphi \colon \mathscr{A} \to \mathscr{D}$ from a K-affinoid algebra $\mathscr{A}$ to a unital commutative Banach K-algebra $\mathscr{D}$ to a homomorphism $\theta_\varphi \colon \Gamma(\Sigma_\varphi, \mathscr{O}_{\mathscr{M}(\mathscr{A})}) \to \mathscr{D}$, where $\Gamma(\Sigma_\varphi, \mathscr{O}_{\mathscr{M}(\mathscr{A})})$ is the analogue of the algebra of holomorphic functions on a neighborhood of the spectrum.

We start, in Chapter 1, with a brief introduction to Berkovich's setting for non-Archimedean geometry, which ends with a construction of the analogue of the algebra of holomorphic functions on a closed subset (§1.8) and a discussion of the morphisms to be considered (§1.9). Then, in Chapter 2, we discuss relative interiors, which are an analytical analogue of topological interiors for Berkovich spaces. In fact, they are useful in order to prove Proposition 4.1.2, which is our first real step towards the proof of the holomorphic functional calculus theorem. It says that every bounded homomorphism $\psi \colon \mathscr{A} \to \mathscr{D}$ from a K-affinoid algebra $\mathscr{A}$ to a unital commutative Banach K-algebra $\mathscr{D}$ can be extended in one and only one way to a bounded homomorphism $\theta_{\psi, \Sigma_\psi^h} \colon \Gamma(\Sigma_\psi^h, \mathscr{O}_{\mathscr{M}(\mathscr{A})}) \to \mathscr{D}$. Here, Σ_ψ^h is the holomorphically convex envelope of the spectrum Σ_ψ of ψ, and holomorphically convex envelopes and spectra of homomorphisms are the subject of Chapter 3. Finally, in Chapter 4, we state and prove the holomorphically functional calculus theorem (Theorem 4.2.1), after (the analogue of) Arens - Calderon lemma (Lemma 4.1.3).

Given a unital commutative Banach $\mathbb{C}$-algebra $\mathscr{D}$ and a tuple of elements $(f_1, \ldots, f_n)$, the (classical) Arens - Calderon lemma (see, for example, [Gam69, Lemma III.5.2]) says that for any open neighborhood U of the joint spectrum $\Sigma_{(f_1,\ldots,f_n)}$ in $\mathbb{C}^n$ there exist some elements $f_{n+1}, \ldots, f_{n+m}$ in $\mathscr{D}$ such that $\Pi(\Sigma^p_{(f_1,\ldots,f_{n+m})}) \subseteq U$, where $\Pi \colon \mathbb{C}^{n+m} \to \mathbb{C}^n$ is the canonical projection and $\Sigma^p_{(f_1,\ldots,f_{n+m})}$ is the polynomially convex envelope of the joint spectrum $\Sigma_{(f_1,\dots,f_{n+m})}$, i.e. the set

$$\left\{ z \in \mathbb{C}^{n+m} \;\middle|\; |P(z)| \le \max_{w \in \Sigma_{(f_1,\dots,f_{n+m})}} |P(w)| \quad \forall P \in \mathbb{C}[T_1,\dots,T_{n+m}] \right\}.$$

The analogous lemma for a bounded homomorphism $\varphi\colon \mathscr{A} \to \mathscr{D}$ from a K-affinoid algebra $\mathscr{A}$ to a unital commutative Banach K-algebra $\mathscr{D}$ says that for any open neighborhood U of the spectrum Σ_φ in $\mathscr{M}(\mathscr{A})$ there exist $r_1,\dots,r_n \in \mathbb{R}_{>0}$ and a bounded homomorphism $\varphi'\colon \mathscr{A}\{\boldsymbol{r^{-1}T}\} \to \mathscr{D}$ which extends φ and is such that $\Pi(\Sigma^h_{\varphi'}) \subseteq U$, where Π is the continuous map $\mathscr{M}(\mathscr{A}\{\boldsymbol{r^{-1}T}\}) \to \mathscr{M}(\mathscr{A})$ induced by the inclusion of $\mathscr{A}$ into $\mathscr{A}\{\boldsymbol{r^{-1}T}\}$. Thus, for any open neighborhood U of Σ_φ, we can construct a homomorphism $\Gamma(U, \mathscr{O}_{\mathscr{M}(\mathscr{A})}) \to \mathscr{D}$ extending φ by composing the pullback homomorphism (see Definition 1.9.3 and Remark 1.9.6) $\Pi^*_{U,\Sigma^h_{\varphi'}} : \Gamma(U, \mathscr{O}_{\mathscr{M}(\mathscr{A})}) \to \Gamma(\Sigma^h_{\varphi'}, \mathscr{O}_{\mathscr{M}(\mathscr{A}\{\boldsymbol{r^{-1}T}\})})$ with the homomorphism $\theta_{\varphi',\Sigma^h_{\varphi'}} : \Gamma(\Sigma^h_{\varphi'}, \mathscr{O}_{\mathscr{M}(\mathscr{A}\{\boldsymbol{r^{-1}T}\})}) \to \mathscr{D}$ previously described. This is the way in which we will construct the homomorphism θ_φ.

Chapter 1

Conventions and preliminaries

Here we introduce the most important notions and propositions which are needed in the following chapters, together with the notations and conventions used. This is meant to be a quick introduction to non-Archimedean geometry as developed by V. G. Berkovich in the first two chapters of [Ber90]. In particular, we introduce *non-Archimedean fields* and *Banach algebras* (§1.1), *Berkovich spectra* (§1.2), *spectral radii* and *residue rings* (§1.3), *completed tensor products* (§1.4), *affinoid algebras* and *affinoid spaces* (§1.5), *affinoid domains* (§1.6), *special subsets* (§1.7), *sheaves of affinoid functions* (§1.8) and *morphisms of quasiaffinoid spaces* (§1.9). We skip most of the proofs, while referring to [Ber90] or to [BGR84]. We put a bit more attention than [Ber90] to the rings of affinoid functions on closed subsets of affinoid spaces and to pullback homomorphisms (§1.8 and §1.9), since they play an important role in the following chapters.

Convention 1.0.1. We use almost the same notations of [Ber90]. The only difference is that we preferred a more coherent way of assigning names to mathematical objects; here are our choices:

- M, N, i, j, k, l, m, n, s, t, u, and v represent (usually positive) integer numbers;
- ε, δ, C, p, q, r represent (usually positive) real numbers;
- S and T represent indeterminates;
- P and Q represent polynomials;
- K and L represent non-Archimedean fields;
- $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ represent K-affinoid algebras;
- $\mathscr{D}$ and $\mathscr{E}$ represent seminormed (usually Banach) K-algebras;

- $\mathfrak{b}$ and $\mathfrak{p}$ represent ideals of a K-algebra;
- a, b, c, d, e, f, g and h represent elements of a K-algebra;
- X, Y and Z represent Berkovich spectra;
- x, y and z represent elements of a Berkovich spectrum;
- η, θ, ι, ξ, π, ρ, σ, τ, φ and ψ represent homomorphisms of K-algebras;
- χ represents characters of Banach K-algebras;
- Ξ, Π, Φ and Ψ represent (induced) continuous functions between Berkovich spectra and morphisms of K-quasiaffinoid spaces;
- Λ represents subsets of a K-affinoid space which are either open or closed;
- Σ represents closed subsets of a K-affinoid space;
- U represents open subsets of a K-affinoid space;
- V represents special subsets of a K-affinoid space;
- W represents K-affinoid domains of a K-affinoid space.

Convention 1.0.2. Unless otherwise stated, all rings are supposed to be commutative and with identity $1 \neq 0$, and all homomorphisms send the identity to the identity.

Convention 1.0.3. We use the term "canonical" to indicate the maps that are derived or implied by the definitions or the universal properties of the objects involved. For example, $\tau_{\mathscr{D}'}$, $\tau_{\mathscr{D}''}$ and η in Definition 1.4.4 and σ_W in Definition 1.6.1 are all canonical homomorphisms.

1.1 Banach K-algebras

Definition 1.1.1. An *absolute value* on a field K is a map $|\cdot| : K \to \mathbb{R}_{\geq 0}$ such that, for any $c, c' \in K$, we have:

(i) $|c| = 0 \iff c = 0$;

(ii) $|cc'| = |c|\,|c'|$;

(iii) $|c + c'| \leq |c| + |c'|$.

It is said to be *non-Archimedean* if $|c + c'| \leq \max\{|c|, |c'|\}$ for all $c, c' \in K$, and *non-trivial* if there exists an element $c \in K^\times = K \setminus \{0\}$ such that $|c| \neq 1$.

Definition 1.1.2. A field K is called a *non-Archimedean field* if it is endowed with a non-Archimedean absolute value $|\cdot|$ such that the map $K \times K \to \mathbb{R}_{\geq 0}$, $(c, c') \mapsto |c - c'|$ defines a complete metric on K.

Convention 1.1.3. Throughout this thesis we let K be a fixed arbitrary non-Archimedean field. We denote its absolute value by $|\cdot|$.

Definition 1.1.4. A *K-algebra seminorm* on a K-algebra $\mathscr{D}$ is a map $||\cdot|| : \mathscr{D} \to \mathbb{R}_{\geq 0}$ such that, for any $f, g \in \mathscr{D}$, we have:

(i) $f \in K \implies ||f|| = |f|$;

(ii) $||fg|| \leq ||f||\, ||g||$;

(iii) $||f - g|| \leq ||f|| + ||g||$.

It is said to be *non-Archimedean* if $||f - g|| \leq \max\{||f||, ||g||\}$ for all $f, g \in \mathscr{D}$, *power-multiplicative* if $||f^n|| = ||f||^n$ for all $f \in \mathscr{D}$ and $n \in \mathbb{N}$, and *multiplicative* if $||fg|| = ||f||\, ||g||$ for all $f, g \in \mathscr{D}$. Furthermore, it is called a *norm* if $||f|| = 0$ only when $f = 0$.

Definition 1.1.5. A K-algebra $\mathscr{D}$ is called *seminormed* (resp. *normed*) if it is endowed with a K-algebra seminorm (resp. norm). A K-algebra $\mathscr{D}$ is called *Banach* if it is endowed with a K-algebra norm $||\cdot||$ such that the map $\mathscr{D} \times \mathscr{D} \to \mathbb{R}_{\geq 0}$, $(f, g) \mapsto ||f - g||$ defines a complete metric on $\mathscr{D}$.

Convention 1.1.6.

(i) For simplicity, we always suppose the K-algebra seminorm of any seminormed K-algebra to be non-Archimedean.

(ii) If not stated otherwise, we denote the K-algebra seminorm of any seminormed K-algebra by $||\cdot||$. If it is important to point out the K-algebra seminorm, we write $(\mathscr{D}, ||\cdot||)$ instead of just $\mathscr{D}$.

(iii) For simplicity, we will write just "seminorm" instead of "K-algebra seminorm".[1]

(iv) Whenever we write that a map φ between two K-algebras is a homomorphism, it is intended that φ is a K-algebra homomorphism.

Definition 1.1.7. A homomorphism $\varphi\colon \mathscr{D} \to \mathscr{D}'$ between two seminormed K-algebras is said to be *bounded* if there exists a *bound* $C \in \mathbb{R}_{>0}$ such that $||\varphi(f)|| \leq C\, ||f||$ for all $f \in \mathscr{D}$.

[1] The actual definition of seminorm on a ring is like Definition 1.1.4, but with just $||0|| = 0$ and $||1|| = 1$ as condition (i).

It is said to be *contractive* if we can choose $C = 1$.

It is said to be *isometric* (or an *isometry*) if $||\varphi(f)|| = ||f||$ for all $f \in \mathscr{D}$, and it is called an *isometric isomorphism* (resp. an *embedding*) if it is, moreover, bijective (resp. injective).

It is said to be an *admissible epimorphism* if it is bounded and there exists $C' \in \mathbb{R}_{>0}$ such that any $f' \in \mathscr{D}'$ admits a preimage $f \in \mathscr{D}$ with $||f|| \leq C'\,||f'||$; it is called an *admissible isomorphism* if it is, moreover, injective.

Proposition 1.1.8. *Any bounded homomorphism $\varphi\colon \mathscr{D} \to \mathscr{D}'$ between two seminormed K-algebras is continuous. Moreover, the composition of two bounded homomorphisms (resp. two contractive homomorphisms, two isometries, two admissible epimorphisms, ...) is a bounded homomorphism (resp. a contractive homomorphism, an isometry, an admissible epimorphism, ...).*

Definition 1.1.9. Two seminorms $||\cdot||$ and $||\cdot||'$ on a K-algebra $\mathscr{D}$ are said to be *equivalent* if the identity homomorphism $\iota\colon (\mathscr{D}, ||\cdot||) \to (\mathscr{D}, ||\cdot||')$ is an admissible isomorphism.

Definition 1.1.10. Let $\mathfrak{b}$ be an ideal of a seminormed K-algebra $\mathscr{D}$. The *quotient seminorm* on $\mathscr{D}/\mathfrak{b}$ is defined by the formula

$$||f|| \coloneqq \inf\{||h|| \mid h \in \pi^{-1}(\{f\})\} \quad \forall\, f \in \mathscr{D}/\mathfrak{b},$$

where π is the canonical projection of $\mathscr{D}$ into $\mathscr{D}/\mathfrak{b}$.

Convention 1.1.11. If not stated otherwise, we assume any quotient of a seminormed K-algebra to be endowed with the quotient seminorm.

Proposition 1.1.12. *Let $\mathscr{D}$ be a Banach K-algebra.*

(i) The group of units $\mathscr{D}^\times$ is open and any maximal ideal of $\mathscr{D}$ is closed.

(ii) If $\mathfrak{b}$ is a closed ideal of $\mathscr{D}$, then $\mathscr{D}/\mathfrak{b}$ is complete (i.e. it is a Banach K-algebra).

Definition/Proposition 1.1.13. A *completion* of a seminormed K-algebra $\mathscr{D}$ is a Banach K-algebra $\widehat{\mathscr{D}}$ with a homomorphism $\iota\colon \mathscr{D} \to \widehat{\mathscr{D}}$ which is an isometry with dense image. Any two completions are the same up to isometric isomorphisms, and one can be defined as the quotient of the K-algebra of Cauchy sequences in $\mathscr{D}$ modulo the ideal made of the sequences that converge to zero (with $||(f_i)_{i\in\mathbb{N}}|| \coloneqq \lim_{i\to\infty} ||f_i||$, and the isometry $\iota\colon \mathscr{D} \to \widehat{\mathscr{D}}$ sending each element $f \in \mathscr{D}$ to the constant sequence $(f, f, \dots)$).

1.2 Berkovich spectra

Definition 1.2.1. A seminorm $|\cdot|$ on a seminormed K-algebra $(\mathscr{D}, ||\cdot||)$ is said to be *bounded* if there exits a bound $C \in \mathbb{R}_{>0}$ such that $|f| \leq C\,||f||$ for all $f \in \mathscr{D}$.

Proposition 1.2.2.

(i) If $|\cdot|$ is a power-multiplicative bounded seminorm on a seminormed K-algebra $(\mathscr{D}, ||\cdot||)$, then $|f| \leq ||f||$ for all $f \in \mathscr{D}$.

(ii) Any multiplicative norm on a K-algebra $\mathscr{D}$ which is also an integral domain can be extended, in one and only one way, to a multiplicative K-algebra norm on the field of fractions $\operatorname{Frac}(\mathscr{D})$.

(iii) Any bounded seminorm on a seminormed K-algebra $\mathscr{D}$ can be extended, in one and only one way, to seminorm on a completion $\widehat{\mathscr{D}}$. Moreover, the extension is bounded by the same bounds of the original seminorm.

(iv) A non-Archimedean field admits a unique bounded multiplicative seminorm (which is the absolute value).

Definition 1.2.3. The *Berkovich spectrum*, denoted by $\mathscr{M}(\mathscr{D})$, of a Banach K-algebra $\mathscr{D}$ is the set of all bounded multiplicative seminorms[2] on $\mathscr{D}$, with the weakest topology which makes all the maps $|f| : \mathscr{M}(\mathscr{D}) \to \mathbb{R}_{\geq 0}$, $|\cdot| \mapsto |f|$ (for each $f \in \mathscr{D}$) continuous.

Definition/Proposition 1.2.4. Any bounded homomorphism $\varphi\colon \mathscr{D} \to \mathscr{D}'$ between two Banach K-algebras induces a continuous map $\mathscr{M}(\varphi)\colon \mathscr{M}(\mathscr{D}') \to \mathscr{M}(\mathscr{D})$ sending any bounded multiplicative seminorm $|\cdot|$ to the composition $|\cdot| \circ \varphi$.

Remark 1.2.5. If $\varphi\colon \mathscr{D} \to \mathscr{D}'$ is a bounded homomorphism between two Banach K-algebras and it has dense image, then $\mathscr{M}(\varphi)$ is injective.

Definition 1.2.6. Let $\mathscr{D}$ be a Banach K-algebra. A *character* of $\mathscr{D}$ is a bounded homomorphism $\chi\colon \mathscr{D} \to L$ to some non-Archimedean field L extending K.

Two characters $\chi'\colon \mathscr{D} \to L'$ and $\chi''\colon \mathscr{D} \to L''$ are said to be *equivalent* if there exists a non-Archimedean field L and two embeddings $\iota'\colon L' \hookrightarrow L$ and $\iota''\colon L'' \hookrightarrow L$ such that $\iota' \circ \chi' = \iota'' \circ \chi''$. (This is clearly an equivalence relation.)

Definition 1.2.7. Let $\mathscr{D}$ be a Banach K-algebra and let $x \in \mathscr{M}(\mathscr{D})$ be given by a bounded multiplicative seminorm $|\cdot|_x$. It is clear that $\mathfrak{p}_x \coloneqq \{f \in \mathscr{D} \mid |f|_x = 0\}$ is a closed prime

[2]It is easy to see that any bounded multiplicative ring seminorm on a K-algebra is automatically a K-algebra seminorm. Hence, the definition is not affected by the ambiguity given by Convention 1.1.6.iii.

ideal of $\mathscr{D}$. We denote by $\mathscr{H}(x)$ a completion of the field of fractions of the K-algebra $\mathscr{D}/\mathfrak{p}_x$ with the quotient seminorm induced by $|\cdot|_x$. The canonical contractive homomorphism $\mathscr{D} \to \mathscr{H}(x)$ is called the *character associated to* x and it is denoted by χ_x. Nonetheless, we denote the image in $\mathscr{H}(x)$ of any $f \in \mathscr{D}$ by $f(x)$, and we write $|f(x)|$ for its absolute value.

Proposition 1.2.8. *Let $\mathscr{D}$ be a Banach K-algebra.*

(i) If $x \in \mathscr{M}(\mathscr{D})$ is given by a bounded multiplicative seminorm $|\cdot|_x$, then $|f(x)| = |f|_x$ for all $f \in \mathscr{D}$.

(ii) For any $x \in \mathscr{M}(\mathscr{D})$ the image of the continuous map $\mathscr{M}(\chi_x)\colon \mathscr{M}(\mathscr{H}(x)) \to \mathscr{M}(\mathscr{D})$ is precisely $\{x\}$.

(iii) The assignment $x \mapsto \chi_x$ induces a bijection between $\mathscr{M}(\mathscr{D})$ and the set of equivalence classes of characters of $\mathscr{D}$. The inverse map sends the equivalence class of a character $\chi\colon \mathscr{D} \to L$ to the multiplicative bounded seminorm $|\cdot|_L \circ \chi$, where $|\cdot|_L$ is the absolute value on L.

Remark 1.2.9. A basis of open neighborhoods of a point $x \in \mathscr{M}(\mathscr{D})$ is given by the sets $\{y \in \mathscr{M}(\mathscr{D}) \mid |f_i(y)| < |f_i(x)| + \varepsilon_i \wedge |g_j(y)| > |g_j(x)| - \delta_j \;\; \forall\, i = 1, \dots, n, \;\forall\, j = 1, \dots, m\}$, with $n, m \in \mathbb{N}$, $\varepsilon_i, \delta_j \in \mathbb{R}_{>0}$ and $f_i, g_j \in \mathscr{D}$ for each i and j.

Theorem 1.2.10 ([Ber90, Theorem 1.2.1])**.** *The Berkovich spectrum $\mathscr{M}(\mathscr{D})$ of any Banach K-algebra $\mathscr{D}$ (with $0 \neq 1$) is non-empty, compact and Hausdorff.*

1.3 Spectral radii and residue rings

Definition/Proposition 1.3.1. The *spectral radius* of a seminormed K-algebra $\mathscr{D}$ is the map $\rho\colon \mathscr{D} \to \mathbb{R}_{\geq 0}$ defined by the formula

$$\rho(f) \coloneqq \lim_{i \to \infty} \sqrt[i]{||f^i||} \quad \forall\, f \in \mathscr{D}.$$

The limit in the formula is indeed well defined, and it is bounded by $||f||$.

Remark 1.3.2. If the seminorm $||\cdot||$ on $\mathscr{D}$ is power-multiplicative, then it clearly coincide with the spectral radius.

Convention 1.3.3. We denote the spectral radius of any seminormed K-algebra by $\rho(\,\cdot\,)$. If we want to point out that it is associated to a seminormed K-algebra $\mathscr{D}$, we could also use the more precise notation $\rho_{\mathscr{D}}(\,\cdot\,)$.

Proposition 1.3.4. *Let $\mathscr{D}$ be a Banach K-algebra.*

(i) ([Ber90, Theorem 1.3.1]). For any $f \in \mathscr{D}$ we have $\rho(f) = \max_{x \in \mathscr{M}(\mathscr{D})} |f(x)|$.

(ii) ([Ber90, Corollary 1.3.3, 1.3.4.ii]). The spectral radius $\rho(\,\cdot\,)$ is a non-Archimedean power-multiplicative seminorm.

(iii) For any bounded homomorphism $\varphi\colon \mathscr{D} \to \mathscr{D}'$ between two Banach K-algebras, we have $\rho(\varphi(f)) \leq \rho(f)$ for all $f \in \mathscr{D}$.

Definition 1.3.5. For any Banach K-algebra $\mathscr{D}$, we define the ring

$$\mathscr{D}^\circ := \{f \in \mathscr{D} \mid \rho(f) \leq 1\}$$

and its ideal

$$\mathscr{D}^{\circ\circ} := \{f \in \mathscr{D}^\circ \mid \rho(f) < 1\}.$$

The quotient $\mathscr{D}^\circ/\mathscr{D}^{\circ\circ}$ is called the *residue ring* of $\mathscr{D}$, and it is denoted by $\widetilde{\mathscr{D}}$.

For any bounded homomorphism $\varphi\colon \mathscr{D} \to \mathscr{D}'$ between two Banach K-algebras, we denote the induced homomorphism between the residue rings (well defined because of Proposition 1.3.4.iii) by $\widetilde{\varphi}\colon \widetilde{\mathscr{D}} \to \widetilde{\mathscr{D}'}$.

1.4 Completed tensor products

Definition 1.4.1. Let $\mathscr{D}$ be a Banach K-algebra. A *Banach $\mathscr{D}$-algebra* is a Banach K-algebra $\mathscr{D}'$ together with a contractive homomorphism $\mathscr{D} \to \mathscr{D}'$.

Convention 1.4.2. If we say that a map $\varphi\colon \mathscr{D}' \to \mathscr{D}''$ between two Banach $\mathscr{D}$-algebras is a homomorphism, it is intended that φ is a $\mathscr{D}$-algebra homomorphism.

Proposition 1.4.3. *Let $\mathscr{D}$ and $(\mathscr{D}', ||\cdot||)$ be two Banach K-algebras, and let $\varphi\colon \mathscr{D} \to (\mathscr{D}', ||\cdot||)$ be a bounded homomorphism. Then, there exists a norm $||\cdot||'$ on $\mathscr{D}'$ which is equivalent to $||\cdot||$ and is such that $\varphi\colon \mathscr{D} \to (\mathscr{D}', ||\cdot||')$ is contractive, thus making $(\mathscr{D}', ||\cdot||')$ into a Banach $\mathscr{D}$-algebra.*

Proof. We define

$$||f||' := \frac{1}{||\varphi||}\left(\sup_{h \in \mathscr{D}\setminus\{0\}} ||\varphi(h)f||\,||h||^{-1}\right) \quad \forall\, f \in \mathscr{D}',$$

where $||\varphi|| := \sup_{h \in \mathscr{D}\setminus\{0\}} ||\varphi(h)||\,||h||^{-1}$. The verification that $||\cdot||'$ (such defined) is a norm equivalent to $||\cdot||$ and that $\varphi\colon \mathscr{D} \to (\mathscr{D}', ||\cdot||')$ is contractive is analogous to the proof of [BGR84, Proposition 1.2.1/2]. □

Definition/Proposition 1.4.4 ([BGR84, Proposition 3.1.1/2])**.** Let $\mathscr{D}$ be a Banach K-algebra and let $\mathscr{D}'$ and $\mathscr{D}''$ be two Banach $\mathscr{D}$-algebras. Then, there exists a Banach $\mathscr{D}$-algebra $\mathscr{D}' \widehat{\otimes}_{\mathscr{D}} \mathscr{D}''$ together with two contractive homomorphisms $\tau_{\mathscr{D}'} \colon \mathscr{D}' \to \mathscr{D}' \widehat{\otimes}_{\mathscr{D}} \mathscr{D}''$ and $\tau_{\mathscr{D}''} \colon \mathscr{D}'' \to \mathscr{D}' \widehat{\otimes}_{\mathscr{D}} \mathscr{D}''$ satisfying the following universal property: for any two bounded homomorphisms $\eta' \colon \mathscr{D}' \to \mathscr{E}$, bounded by C', and $\eta'' \colon \mathscr{D}'' \to \mathscr{E}$, bounded by C'', there exists a unique homomorphism $\eta \colon \mathscr{D}' \widehat{\otimes}_{\mathscr{D}} \mathscr{D}'' \to \mathscr{E}$, bounded by $C'C''$, such that $\eta \circ \tau_{\mathscr{D}'} = \eta'$ and $\eta \circ \tau_{\mathscr{D}''} = \eta''$. Such a Banach $\mathscr{D}$-algebra $\mathscr{D}' \widehat{\otimes}_{\mathscr{D}} \mathscr{D}''$ is called *completed tensor product* of $\mathscr{D}'$ and $\mathscr{D}''$ over $\mathscr{D}$, and it is unique up to isometric isomorphisms.

Given $d' \in \mathscr{D}'$ and $d'' \in \mathscr{D}''$, we denote by $d' \widehat{\otimes} d''$ the element of $\mathscr{D}' \widehat{\otimes}_{\mathscr{D}} \mathscr{D}''$ given by $\tau_{\mathscr{D}'}(d')\tau_{\mathscr{D}''}(d'')$.

Convention 1.4.5. Given two bounded homomorphisms between Banach K-algebras $\mathscr{D} \to \mathscr{D}'$ and $\mathscr{D} \to \mathscr{D}''$, we will consider the completed tensor product $\mathscr{D}' \widehat{\otimes}_{\mathscr{D}} \mathscr{D}''$ even if they are not contractive. In fact, whenever we do so, we will not make use of the precise norm of any element, but only of Berkovich spectra and the boundedness (or admissibility) of some homomorphisms. Therefore, it is intended that we are considering $\mathscr{D}'$ and $\mathscr{D}''$ to be endowed with the equivalent norms of Proposition 1.4.3.

Definition/Proposition 1.4.6. Let $\mathscr{D}$ be a Banach K-algebra and let $\mathscr{D}'$, $\mathscr{D}''$, $\mathscr{E}'$ and $\mathscr{E}''$ be four Banach $\mathscr{D}$-algebras. Any two bounded homomorphisms $\varphi' \colon \mathscr{D}' \to \mathscr{E}'$, bounded by C', and $\varphi'' \colon \mathscr{D}'' \to \mathscr{E}''$, bounded by C'', induce a unique homomorphism

$$\varphi' \widehat{\otimes}_{\mathscr{D}} \varphi'' \colon \mathscr{D}' \widehat{\otimes}_{\mathscr{D}} \mathscr{D}'' \to \mathscr{E}' \widehat{\otimes}_{\mathscr{D}} \mathscr{E}'',$$

bounded by $C'C''$, such that $(\varphi' \widehat{\otimes}_{\mathscr{D}} \varphi'') \circ \tau_{\mathscr{D}'} = \tau_{\mathscr{E}'} \circ \varphi'$ and $(\varphi' \widehat{\otimes}_{\mathscr{D}} \varphi'') \circ \tau_{\mathscr{D}''} = \tau_{\mathscr{E}''} \circ \varphi''$ (where $\tau_{\mathscr{D}'}$, $\tau_{\mathscr{D}''}$, $\tau_{\mathscr{E}'}$ and $\tau_{\mathscr{E}''}$ are as in Definition 1.4.4).

Proposition 1.4.7 ([BGR84, Proposition 2.1.8/6])**.** *Let $\mathscr{D}$ be a Banach K-algebra. If $\varphi' \colon \mathscr{D}' \to \mathscr{E}'$ and $\varphi'' \colon \mathscr{D}'' \to \mathscr{E}''$ are two admissible epimorphisms of Banach $\mathscr{D}$-algebras, then also the induced homomorphism $\varphi' \widehat{\otimes}_{\mathscr{D}} \varphi'' \colon \mathscr{D}' \widehat{\otimes}_{\mathscr{D}} \mathscr{D}'' \to \mathscr{E}' \widehat{\otimes}_{\mathscr{D}} \mathscr{E}''$ is an admissible epimorphism.*

1.5 K-affinoid algebras and K-affinoid spaces

Convention 1.5.1. We use bold symbols for multi-index notations. In particular, given a K-algebra $\mathscr{D}$ and some positive real numbers $r_1 \ldots, r_n$, we abbreviate $\mathscr{D}\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\}$ as $\mathscr{D}\{\boldsymbol{r}^{-1}\boldsymbol{T}\}$ (see the definition below). Furthermore, given a multi-index $\boldsymbol{u} = (u_1, \ldots, u_n)$

(belonging either to $\mathbb{N}^n$ or to $\mathbb{Z}^n$), then $|\boldsymbol{u}| := \sum_{i=1}^n |u_i|$, $\boldsymbol{T}^{\boldsymbol{u}} := \prod_{i=1}^n T_i^{u_i}$ and $\boldsymbol{r}^{\boldsymbol{u}} := \prod_{i=1}^n r_i^{u_i}$.

Definition 1.5.2. Let $\mathscr{D}$ be a Banach K-algebra. Given $n \in \mathbb{N}$ and $r_1, \ldots, r_n \in \mathbb{R}_{>0}$, we define the following $\mathscr{D}$-subalgebra of the $\mathscr{D}$-algebra of formal power series over $\mathscr{D}$ in the indeterminates $T_1, \ldots, T_n$:

$$\mathscr{D}\{\boldsymbol{r}^{-1}\boldsymbol{T}\} := \left\{ \sum_{\boldsymbol{u}\in\mathbb{N}^n} a_{\boldsymbol{u}}\boldsymbol{T}^{\boldsymbol{u}} \in \mathscr{D}[[T_1, \ldots, T_n]] \mid a_{\boldsymbol{u}} \in \mathscr{D}\ \forall \boldsymbol{u} \in \mathbb{N}^n \ \wedge \lim_{|\boldsymbol{u}|\to\infty} |a_{\boldsymbol{u}}|\,\boldsymbol{r}^{\boldsymbol{u}} = 0 \right\}.$$

We endow it with the *(relative) Gauss norm*, which is given by

$$\left\| \sum_{\boldsymbol{u}\in\mathbb{N}^n} a_{\boldsymbol{u}}\boldsymbol{T}^{\boldsymbol{u}} \right\| := \max_{\boldsymbol{u}\in\mathbb{N}^n} ||a_{\boldsymbol{u}}||\,\boldsymbol{r}^{\boldsymbol{u}}.$$

Proposition 1.5.3. *Let $\mathscr{D}$ be a Banach K-algebra and $\mathscr{D}'$ a Banach $\mathscr{D}$-algebra. Moreover, let us be given some positive real numbers $r_1, \ldots, r_n$ and $q_1, \ldots, q_m$.*

(i) The normed $\mathscr{D}$-algebra $\mathscr{D}\{\boldsymbol{r}^{-1}\boldsymbol{T}\}$ is complete (i.e. Banach).

(ii) If the norm on $\mathscr{D}$ is multiplicative, then so is the Gauss norm on $\mathscr{D}\{\boldsymbol{r}^{-1}\boldsymbol{T}\}$.

(iii) $\rho\big(\sum_{\boldsymbol{u}\in\mathbb{N}^n} a_{\boldsymbol{u}}\boldsymbol{T}^{\boldsymbol{u}}\big) = \max_{\boldsymbol{u}\in\mathbb{N}^n} \rho(a_{\boldsymbol{u}})\boldsymbol{r}^{\boldsymbol{u}}$ for all $\sum_{\boldsymbol{u}\in\mathbb{N}^n} a_{\boldsymbol{u}}T^{\boldsymbol{u}} \in \mathscr{D}\{\boldsymbol{r}^{-1}\boldsymbol{T}\}$.

(iv) $\mathscr{D}' \widehat{\otimes}_{\mathscr{D}} \mathscr{D}\{\boldsymbol{r}^{-1}\boldsymbol{T}\} = \mathscr{D}'\{\boldsymbol{r}^{-1}\boldsymbol{T}\}$.

(v) $\mathscr{D}\{\boldsymbol{r}^{-1}\boldsymbol{T}\} \widehat{\otimes}_{\mathscr{D}} \mathscr{D}\{\boldsymbol{q}^{-1}\boldsymbol{S}\} = \mathscr{D}\{\boldsymbol{r}^{-1}\boldsymbol{T}, \boldsymbol{q}^{-1}\boldsymbol{S}\}$.

Definition 1.5.4. A Banach K-algebra $\mathscr{A}$ is said to be *K-affinoid* if there exits an admissible epimorphism $K\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\} \to \mathscr{A}$ for some $n \in \mathbb{N}$ and $r_1, \ldots, r_n \in \mathbb{R}_{>0}$. If we can choose $r_1 = \cdots = r_n = 1$, then $\mathscr{A}$ is said to be *strictly K-affinoid.*

Definition 1.5.5. Let $\mathscr{A}$ be a K-affinoid algebra and let $\mathscr{B}$ be a Banach $\mathscr{A}$-algebra. $\mathscr{B}$ is said to be *$\mathscr{A}$-affinoid* if there exits an admissible epimorphism $\mathscr{A}\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\} \to \mathscr{B}$ for some $n \in \mathbb{N}$ and $r_1, \ldots, r_n \in \mathbb{R}_{>0}$. If we can choose $r_1 = \cdots = r_n = 1$, then $\mathscr{B}$ is said to be *strictly $\mathscr{A}$-affinoid.*

Proposition 1.5.6. *Let $\mathscr{A}$ be a K-affinoid algebra and let $\mathscr{B}$ and $\mathscr{C}$ be two Banach $\mathscr{A}$-algebras.*

(i) If $\mathscr{B}$ is $\mathscr{A}$-affinoid, then it is also K-affinoid. Moreover, if $\mathscr{B}$ is strictly $\mathscr{A}$-affinoid and $\mathscr{A}$ is strictly K-affinoid, then $\mathscr{B}$ is strictly K-affinoid.

(ii) If $\mathscr{B}$ and $\mathscr{C}$ are (strictly) $\mathscr{A}$-affinoid, then so is $\mathscr{B} \widehat{\otimes}_{\mathscr{A}} \mathscr{C}$.

Definition 1.5.7. *K-affinoid spaces* are the Berkovich spectra of K-affinoid algebras. Moreover, the *morphisms* between two K-affinoid spaces are the continuous maps which are induced by bounded homomorphisms between the underlying K-affinoid algebras.

Remark 1.5.8. Later on, we will endow any K-affinoid space with a sheaf of functions.

Remark 1.5.9. The category of K-affinoid spaces is by construction the opposite of that of K-affinoid algebras. In particular, since the category of K-affinoid algebras admits amalgamated sums in the form of completed tensor products (by Proposition 1.5.6.ii and the universal property of completed tensor products), the category of K-affinoid spaces admits fibered products: $\mathscr{M}(\mathscr{B}) \times_{\mathscr{M}(\mathscr{A})} \mathscr{M}(\mathscr{C}) = \mathscr{M}(\mathscr{B} \widehat{\otimes}_{\mathscr{A}} \mathscr{C})$ for any two bounded homomorphisms of K-affinoid algebras $\mathscr{A} \to \mathscr{B}$ and $\mathscr{A} \to \mathscr{C}$.

Definition 1.5.10. We let $\sqrt{|K^\times|} \coloneqq \{r \in \mathbb{R}_{>0} \mid \exists\, n \in \mathbb{N},\ r^n \in |K^\times|\}$, and we notice that the multiplicative group $\mathbb{R}_{>0}/\sqrt{|K^\times|}$ can be naturally endowed with the structure of a $\mathbb{Q}$-vector space. Now, positive real numbers $r_1, \dots, r_n$ are called *K-free* if their projections to $\mathbb{R}_{>0}/\sqrt{|K^\times|}$ are $\mathbb{Q}$-linearly independent.

Definition 1.5.11. Let $r_1, \dots, r_n$ be K-free positive real numbers. We define the following K-subalgebra of the K-algebra of Laurent series over K in the indeterminates $T_1, \dots, T_n$:

$$K_{\boldsymbol{r}} \coloneqq \left\{ \sum_{\boldsymbol{u} \in \mathbb{Z}^n} a_{\boldsymbol{u}} \boldsymbol{T}^{\boldsymbol{u}} \mid a_{\boldsymbol{u}} \in K \ \forall \boldsymbol{u} \in \mathbb{Z}^n \ \wedge \lim_{|\boldsymbol{u}| \to \infty} |a_{\boldsymbol{u}}|\, \boldsymbol{r}^{\boldsymbol{u}} = 0 \right\}.$$

We endow it with the norm given by $\left|\left|\sum_{\boldsymbol{u} \in \mathbb{Z}^n} a_{\boldsymbol{u}} \boldsymbol{T}^{\boldsymbol{u}}\right|\right| \coloneqq \max_{\boldsymbol{u} \in \mathbb{Z}^n} ||a_{\boldsymbol{u}}||\, \boldsymbol{r}^{\boldsymbol{u}}$.

Proposition 1.5.12. *Let $r_1, \dots, r_n$ be K-free real numbers. Then, $K_{\boldsymbol{r}}$ is a K-affinoid algebra and a non-Archimedean field with non-trivial absolute value. Moreover, $K_{\boldsymbol{r}} = K_{r_1} \widehat{\otimes}_K \dots \widehat{\otimes}_K K_{r_n}$.*

Proposition 1.5.13. *Let $r_1, \dots, r_n$ be K-free real numbers, and let $\mathscr{D}$ be a Banach K-algebra. Then,*

$$\mathscr{D} \widehat{\otimes}_K K_{\boldsymbol{r}} = \left\{ \sum_{\boldsymbol{u} \in \mathbb{Z}^n} a_{\boldsymbol{u}} \boldsymbol{T}^{\boldsymbol{u}} \mid a_{\boldsymbol{u}} \in \mathscr{D} \ \forall \boldsymbol{u} \in \mathbb{Z}^n \ \wedge \lim_{|\boldsymbol{u}| \to \infty} |a_{\boldsymbol{u}}|\, \boldsymbol{r}^{\boldsymbol{u}} = 0 \right\},$$

where the right-hand side has norm given by $\left|\left|\sum_{\boldsymbol{u} \in \mathbb{Z}^n} a_{\boldsymbol{u}} \boldsymbol{T}^{\boldsymbol{u}}\right|\right| \coloneqq \max_{\boldsymbol{u} \in \mathbb{Z}^n} ||a_{\boldsymbol{u}}||\, \boldsymbol{r}^{\boldsymbol{u}}$ and spectral radius $\rho\left(\sum_{\boldsymbol{u} \in \mathbb{Z}^n} a_{\boldsymbol{u}} \boldsymbol{T}^{\boldsymbol{u}}\right) = \max_{\boldsymbol{u} \in \mathbb{Z}^n} \rho(a_{\boldsymbol{u}}) \boldsymbol{r}^{\boldsymbol{u}}$.

Moreover, the map $\mathscr{M}(\mathscr{D} \widehat{\otimes}_K K_{\boldsymbol{r}}) \to \mathscr{M}(\mathscr{D})$ induced by the embedding of $\mathscr{D}$ into $\mathscr{D} \widehat{\otimes}_K K_{\boldsymbol{r}}$ is surjective.

Proposition 1.5.14. *For any K-affinoid algebra $\mathscr{A}$ there exist some K-free real numbers $r_1, \dots, r_n$ such that $\mathscr{A} \widehat{\otimes}_K K_{\boldsymbol{r}}$ is a strictly $K_{\boldsymbol{r}}$-affinoid algebra.*

Remark 1.5.15. We will sometimes need to extend results proven in the strictly affinoid case (and supposing non-trivial absolute value) to the general one. By the previous propositions, to do so it is enough to show that the property passes from $\mathscr{A} \widehat{\otimes}_K K_r$ to $\mathscr{A}$, where $\mathscr{A}$ is any K-affinoid algebra and $r \in \mathbb{R}_{>0} \setminus \sqrt{|K^\times|}$.

Proposition 1.5.16. *Let $\mathscr{A}$ and $\mathscr{B}$ be two K-affinoid algebras. Moreover, let us be given some positive real numbers $r_1, \ldots, r_n$ and $q_1, \ldots, q_n$.*

(i) ([Ber90, Proposition 2.1.4]). For any $f \in \mathscr{A}$ there exist $C \in \mathbb{R}_{>0}$ and $N \in \mathbb{N}$ such that $||f^n|| \leq C\rho(f)^n$ for all $n \geq N$. Moreover, one can take $N = 0$ if f is not nilpotent.

(ii) ([Ber90, Corollary 2.1.5]). Given a bounded homomorphism $\varphi\colon \mathscr{A} \to \mathscr{B}$ and given $f_1, \ldots, f_n \in \mathscr{B}$, then $\rho(f_i) \leq r_i$ for all $i = 1, \ldots, n$ if and only if there exists a (necessarily unique) bounded homomorphism $\varphi'\colon \mathscr{A}\{\boldsymbol{r^{-1}T}\} \to \mathscr{B}$ extending φ and sending T_i to f_i for all $i = 1, \ldots, n$.

(iii) ([Ber90, Corollary 2.1.6]). $\mathscr{A}$ is strictly K-affinoid if and only if $\rho(f) \in \sqrt{|K^\times|} \cup \{0\}$ for all $f \in \mathscr{A}$.

(iv) ([Ber90, Proposition 2.1.7]). Let $\varphi\colon \mathscr{A}\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\} \to \mathscr{B}$ be an admissible epimorphism and let us denote by f_i the image of T_i (for each $i = 1, \ldots, n$). Then, there exists $\varepsilon \in \mathbb{R}_{>0}$ such that, for any choice of elements $f'_1, \ldots, f'_n \in \mathscr{B}$ with $||f_i - f'_i|| \leq \varepsilon$, the homomorphism $\varphi'\colon \mathscr{A}\{\boldsymbol{r^{-1}T}\} \to \mathscr{B}$ sending T_i to f'_i for all $i = 1, \ldots, n$ is an admissible epimorphism.

(v) Clearly, $\mathscr{A}\{\boldsymbol{r^{-1}T}\} \subseteq \mathscr{A}\{\boldsymbol{q^{-1}T}\}$ if $q_i \leq r_i$ for all $i = 1, \ldots, n$. Now, if the absolute value on K is non-trivial and if $\mathscr{B}$ is strictly K-affinoid, then any bounded homomorphism $\varphi\colon \mathscr{A}\{\boldsymbol{r^{-1}T}\} \to \mathscr{B}$ admits a bounded extension $\varphi'\colon \mathscr{A}\{\boldsymbol{q^{-1}T}\} \to \mathscr{B}$ with $q_i \leq r_i$ and $q_i \in \sqrt{|K^\times|}$ for all $i = 1, \ldots, n$.

Proof. Let us prove (v). If $r_i \notin \sqrt{|K^\times|}$, it means, by (iii), that $\rho(\varphi(T_i)) < r_i$. Therefore, we can find $q_i \in \sqrt{|K^\times|}$ such that $\rho(\varphi(T_i)) < q_i < r_i$, because $\sqrt{|K^\times|}$ is dense in $\mathbb{R}_{>0}$ (which follows easily from the fact that the absolute value on K is assumed to be non-trivial). By (ii), we can extend φ to a bounded homomorphism $\mathscr{A}\{r_1^{-1}T_1, \ldots, q_i^{-1}T_i, \ldots, r_n^{-1}T_n\} \to \mathscr{B}$. It suffices to iterate this procedure for each i such that $r_i \notin \sqrt{|K^\times|}$. □

Theorem 1.5.17 ([BGR84, Theorem 6.3.5/1])**.** *Let the absolute value on K be non-trivial and let $\varphi\colon \mathscr{A} \to \mathscr{B}$ be a bounded homomorphism between two strictly K-affinoid algebras. Then, the following are equivalent:*

(a) φ *is finite;*

(b) φ *is integral;*

(c) $\widetilde{\varphi}$ *is finite;*

(d) $\widetilde{\varphi}$ *is integral.*

1.6 K-affinoid domains

Definition 1.6.1. Let $\mathscr{A}$ be a K-affinoid algebra and let $X := \mathscr{M}(\mathscr{A})$. A closed subset $W \subseteq X$ is called a *K-affinoid domain* in X if there exists a K-affinoid algebra $\mathscr{A}_W$ and a bounded homomorphism $\sigma_W \colon \mathscr{A} \to \mathscr{A}_W$ such that:

(i) the induced map $\mathscr{M}(\sigma_V) \colon \mathscr{M}(\mathscr{A}_W) \to X$ has image W;

(ii) for any bounded homomorphism of K-affinoid algebras $\varphi \colon \mathscr{A} \to \mathscr{B}$ such that $\mathscr{M}(\varphi)$ has image inside W there is a unique bounded homomorphism $\varphi_W \colon \mathscr{A}_W \to \mathscr{B}$ such that $\varphi = \varphi_W \circ \sigma_W$.

In such a situation, we say that W is *represented* by the homomorphism $\sigma_W \colon \mathscr{A} \to \mathscr{A}_W$.

Remark 1.6.2. This definition is equivalent to that in [Ber90, §2.2], as proven in [Tem05, Corollary 3.2]

Convention 1.6.3. Let $\mathscr{A}$ be a K-affinoid algebra and let W be a K-affinoid domain in $\mathscr{M}(\mathscr{A})$.

(i) Whenever writing $\mathscr{A}_W$, we imply that W is represented by a bounded homomorphism $\sigma_W \colon \mathscr{A} \to \mathscr{A}_W$.

(ii) By means of Proposition 1.4.3, we assume $\mathscr{A}_W$ to be a Banach $\mathscr{A}$-algebra (i.e. σ_W to be contractive).

(iii) If $W' \subseteq W$ is another K-affinoid domain in $\mathscr{M}(\mathscr{A})$, then the universal property of K-affinoid domains gives a (canonical) bounded homomorphism $\sigma_{W,W'} \colon \mathscr{A}_W \to \mathscr{A}_{W'}$. We denote the image of any element $f \in \mathscr{A}_W$ by $f_{|W'}$.

Proposition 1.6.4. *Let $\mathscr{A}$ be a K-affinoid algebra and let W be a K-affinoid domain in $\mathscr{M}(\mathscr{A})$.*

(i) The bounded homomorphism $\sigma_W \colon \mathscr{A} \to \mathscr{A}_W$ representing W is unique up to the composition with admissible isomorphisms.

(ii) ([Ber90, Proposition 2.2.4.i]). *The map $\mathscr{M}(\sigma_W)\colon \mathscr{M}(\mathscr{A}_W) \to W$ is an homeomorphism.*

(iii) ([Ber90, Remark 2.2.2.iii]). *If $\varphi\colon \mathscr{A} \to \mathscr{B}$ is a bounded homomorphism of K-affinoid algebras and $\Phi\colon \mathscr{M}(\mathscr{B}) \to \mathscr{M}(\mathscr{A})$ is the corresponding morphism of K-affinoid spaces, then $\Phi^{-1}(W)$ is a K-affinoid domain in $\mathscr{M}(\mathscr{B})$, and it is represented by the canonical homomorphism $\mathscr{B} \to \mathscr{B} \widehat{\otimes}_{\mathscr{A}} \mathscr{A}_W$.*

(iv) ([Ber90, Remark 2.2.2.iv]). *If W' is another K-affinoid domain in $\mathscr{M}(\mathscr{A})$, then also $W \cap W'$ is a K-affinoid domain in $\mathscr{M}(\mathscr{A})$, and it is represented by the canonical homomorphism $\mathscr{A} \to \mathscr{A}_W \widehat{\otimes}_{\mathscr{A}} \mathscr{A}_{W'}$.*

Definition/Proposition 1.6.5. Let $\mathscr{A}$ be a K-affinoid algebra and let $X \coloneqq \mathscr{M}(\mathscr{A})$. Given some elements $f_1, \dots, f_m, g_1, \dots, g_n \in \mathscr{A}$ and some positive real numbers $p_1, \dots, p_m$ and $q_1, \dots, q_n$, the closed subset

$$X(\boldsymbol{p^{-1}f}, \boldsymbol{qg^{-1}}) \coloneqq \{x \in X \mid |f_i(x)| \le p_i \ \forall i = 1, \dots, m \ \wedge \ |g_j(x)| \ge q_j \ \forall j = 1, \dots, n\}$$

is said to be a *Laurent domain* in X, and it is a K-affinoid domain represented by the canonical homomorphism

$$\sigma_{X(\boldsymbol{p^{-1}f}, \boldsymbol{qg^{-1}})}\colon \mathscr{A} \to \mathscr{A}\{\boldsymbol{p^{-1}T}, \boldsymbol{qS}\}/\mathfrak{b},$$

where $\mathfrak{b}$ is the ideal generated by the elements $T_i - f_i$ and $g_j S_j - 1$ (for all $i = 1, \dots, m$ and $j = 1, \dots, n$).

If $n = 0$ (i.e. if there are no g_j's nor q_j's), then $X(\boldsymbol{p^{-1}f})$ is said to be a *Weierstrass domain.*

Remark 1.6.6. Let us be given a K-affinoid algebra $\mathscr{A}$ and a point $x \in \mathscr{M}(\mathscr{A})$. By Remark 1.2.9, the neighborhoods of x which are Laurent domains form a basis of neighborhoods of x in $\mathscr{M}(\mathscr{A})$.

Definition/Proposition 1.6.7. Let $\mathscr{A}$ be a K-affinoid algebra and let $X \coloneqq \mathscr{M}(\mathscr{A})$. Given some elements $f_0, f_1, \dots, f_m \in \mathscr{A}$ generating $\mathscr{A}$ (as an ideal) and given some positive real numbers $p_1, \dots, p_m$, the closed subset

$$X(\boldsymbol{p^{-1}f}/f_0) \coloneqq \{x \in X \mid |f_i(x)| \le p_i\,|f_0(x)| \ \ \forall i = 1, \dots, m\}$$

is said to be a *rational domain* in X, and it is a K-affinoid domain represented by the canonical homomorphism

$$\sigma_{X(\boldsymbol{p}^{-1}\boldsymbol{f}/f_0)} \colon \mathscr{A} \to \mathscr{A}\{\boldsymbol{p}^{-1}\boldsymbol{T}\}/\mathfrak{b},$$

where $\mathfrak{b}$ is the ideal generated by the elements $f_0 T_i - f_i$ (for all $i = 1, \dots, m$).

Proposition 1.6.8 ([BGR84, Proposition 7.2.3/7])**.** *The intersection of two rational (resp. Laurent, resp. Weierstrass) domains is a rational (resp. Laurent, resp. Weierstrass) domain.*

Proposition 1.6.9 ([Ber90, Corollary 2.2.10, 2.2.11])**.** *Let $\mathscr{A}$ be a K-affinoid algebra and let W be a K-affinoid domain in $\mathscr{M}(\mathscr{A})$ represented by a bounded homomorphism $\sigma_W \colon \mathscr{A} \to \mathscr{A}_W$.*

(i) If W is a Weierstrass domain, then σ_W has dense image.

(ii) If W is a rational domain in $\mathscr{M}(\mathscr{A})$, then the localization of $\mathscr{A}$ with respect to the elements not vanishing on W has dense image in $\mathscr{A}_W$.

(iii) If W is a rational (resp. Weierstrass) domain in X and W' is a rational (resp. Weierstrass) domain in W, then W' is also a rational (resp. Weierstrass) domain in X.

Theorem 1.6.10 (Gerritzen - Grauert, [BGR84, Corollary 7.3.5/3], [Tem05, Theorem 3.1])**.** *Let $\mathscr{A}$ be a K-affinoid algebra and let W be a K-affinoid domain in $\mathscr{M}(\mathscr{A})$. Then, there exists a finite cover of $\mathscr{M}(\mathscr{A})$ by rational domains $W_1, \dots, W_m$ such that $W \cap W_i$ is a Weierstrass domain in W_i for each $i = 1, \dots, m$.*

Remark 1.6.11. In view of Proposition 1.6.9.iii, the previous theorem tells us that any K-affinoid domain is a finite union of rational domains.

Proposition 1.6.12 ([Ber90, Corollary 2.2.7])**.** *Let $\mathscr{A}$ be a K-affinoid algebra and let $X \coloneqq \mathscr{M}(\mathscr{A})$. If W is a K-affinoid domain in X which is represented by an admissible epimorphism, then $X \setminus W$ is a K-affinoid domain in X and there is an admissible isomorphism between $\mathscr{A}$ and $\mathscr{A}_W \times \mathscr{A}_{X \setminus W}$ (where the norm on the Cartesian product is given by taking the maximum of the norms of the components).*

1.7 Special subsets

Definition 1.7.1. Let $\mathscr{A}$ be a K-affinoid algebra and let $X \coloneqq \mathscr{M}(\mathscr{A})$. We say that V is a *special subset* of X if it is a finite union of K-affinoid domains in X.

Definition 1.7.2. Let $\mathscr{A}$ be a K-affinoid algebra and let $\Sigma \subseteq \mathscr{M}(\mathscr{A})$ be a closed subset. The *special neighborhoods* (resp. *Laurent neighborhoods*, resp. *Weierstrass neighborhoods*) of Σ are the (closed) neighborhoods of Σ that are special subsets (resp. Laurent domains, resp. Weierstrass domains) in $\mathscr{M}(\mathscr{A})$.

Proposition 1.7.3. *Let $\mathscr{A}$ be a K-affinoid algebra and let $\Sigma \subseteq \mathscr{M}(\mathscr{A})$ be a closed subset. Then, the special neighborhoods of Σ form a basis of neighborhoods of Σ.*

Proof. Let U be an arbitrary open neighborhood of Σ. By Remark 1.6.6, there exists, for any point $x \in \Sigma$, a Laurent domain V_x which is a neighborhood of x contained in U. Since Σ is compact (by Theorem 1.2.10), it can be covered using only a finite number of interiors of some V_x's. The finite union of these V_x's is clearly a special neighborhood of Σ contained in U. This concludes the proof. □

Definition/Proposition 1.7.4 ([Ber90, Corollary 2.2.6])**.** Let $\mathscr{A}$ be a K-affinoid algebra and let $X \coloneqq \mathscr{M}(\mathscr{A})$. Let V be a special subset which is the union of a finite family $\{W_i\}_{i\in I}$ of K-affinoid domains in X. We define

$$\mathscr{A}_V \coloneqq \ker\left(\bigoplus_{i\in I} \mathscr{A}_{W_i} \to \bigoplus_{(i,j)\in I^2} \mathscr{A}_{W_i\cap W_j}\right), \tag{1.1}$$

where the map is the homomorphism of $\mathscr{A}$-modules which sends

$$(a_i)_{i\in I} \mapsto (a_{i|W_{i\cap j}} - a_{j|W_{i\cap j}})_{(i,j)\in I^2}.$$

We endow $\mathscr{A}_V$ with the norm given by $||(a_i)_{i\in I}|| \coloneqq \max_{i\in I} ||a_i||$.

It turns out that $\mathscr{A}_V$ is a Banach $\mathscr{A}$-algebra, and (as a consequence of Tate acyclicity theorem, [Ber90, Proposition 2.2.5]) it does not depend, up to admissible isomorphisms, on the choice of the cover $\{W_i\}_{i\in I}$ (and of the representations $\mathscr{A} \to \mathscr{A}_{W_i}$). Thus, we can consider $\mathscr{A}_V$ as if it were determined directly by V. We denote the homomorphism $\mathscr{A} \to \mathscr{A}_V$ making $\mathscr{A}_V$ into an $\mathscr{A}$-algebra by σ_V.

Definition 1.7.5. If $V \subseteq V'$ are two special subsets of a K-affinoid space $X \coloneqq \mathscr{M}(\mathscr{A})$, then there is a (canonical) contractive homomorphism $\sigma_{V',V} \colon \mathscr{A}_{V'} \to \mathscr{A}_V$, obtained by defining $\mathscr{A}_{V'}$ using a cover (made of K-affinoid domains) which extends that used for $\mathscr{A}_V$. We call it *restriction homomorphism* and denote the image of an element $f \in \mathscr{A}_{V'}$ by $f_{|V}$.

Remark 1.7.6. Restriction homomorphisms are compatible with each other, meaning that the composition of two (composable) restriction homomorphisms $\mathscr{A}_{V''} \to \mathscr{A}_{V'}$ and $\mathscr{A}_{V'} \to \mathscr{A}_V$ is the restriction homomorphism between $\mathscr{A}_{V''}$ and $\mathscr{A}_V$.

Proposition 1.7.7. *Let $\mathscr{A}$ be a K-affinoid algebra and let V be a special subset of $\mathscr{M}(\mathscr{A})$ with associated canonical homomorphism $\sigma_V \colon \mathscr{A} \to \mathscr{A}_V$.*

(i) There is a canonical continuous map $V \to \mathscr{M}(\mathscr{A}_V)$ whose composition with the induced map $\mathscr{M}(\sigma_V) \colon \mathscr{M}(\mathscr{A}_V) \to \mathscr{M}(\mathscr{A})$ is the inclusion of V into $\mathscr{M}(\mathscr{A})$.

(ii) V is a K-affinoid domain if and only if $\mathscr{A}_V$ is a K-affinoid algebra and the image of $\mathscr{M}(\sigma_V) \colon \mathscr{M}(\mathscr{A}_V) \to \mathscr{M}(\mathscr{A})$ coincides with V.

Remark 1.7.8. The map of the first point is given by gluing the maps $W \to \mathscr{M}(\mathscr{A}_W)$ of Proposition 1.6.4.ii, for all W in a finite cover of V made of K-affinoid domains.

The second point is a correction of [Ber90, Corollary 2.2.6], which is wrong as it is stated. For a counterexample and a proof of the correct statement, see [Jon19].

Remark 1.7.9. Let $\mathscr{A}$ be a K-affinoid algebra and let V be a special subset of $\mathscr{M}(\mathscr{A})$. By means of the continuous map which is given in the first point of the previous proposition, we can consider V also as a subset of $\mathscr{M}(\mathscr{A}_V)$.

Definition/Proposition 1.7.10. Let $\varphi \colon \mathscr{A} \to \mathscr{B}$ be a bounded homomorphism of K-affinoid algebras and let $\Phi \colon \mathscr{M}(\mathscr{B}) \to \mathscr{M}(\mathscr{A})$ be the corresponding morphism of K-affinoid spaces. It follows from Proposition 1.6.4.iii that $\Phi^{-1}(V)$ is a special subset of $\mathscr{M}(\mathscr{B})$ for any special subset V of $\mathscr{M}(\mathscr{A})$. Moreover, there is a canonical homomorphism $\Phi_V^* \colon \mathscr{A}_V \to \mathscr{B}_{\Phi^{-1}(V)}$ for any special subset V of $\mathscr{M}(\mathscr{A})$. It is induced by the canonical homomorphisms $\mathscr{A}_W \to \mathscr{B}_{\Phi^{-1}(W)} = \mathscr{B} \widehat{\otimes}_{\mathscr{A}} \mathscr{A}_W$ for any K-affinoid domain W in a finite cover of V.

Furthermore, let $\psi \colon \mathscr{A} \to \mathscr{B}_{\widetilde{V}}$ be a bounded homomorphism, where $\mathscr{A}$ and $\mathscr{B}$ are K-affinoid algebras and $\widetilde{V}$ is a special subset in $\mathscr{M}(\mathscr{B})$, and let $\Psi \colon \mathscr{M}(\mathscr{B}_{\widetilde{V}}) \to \mathscr{M}(\mathscr{A})$ be the induced continuous map. For any K-affinoid domain W in $\mathscr{M}(\mathscr{A})$ and any K-affinoid domain $W' \subseteq \Psi^{-1}(W)$ there exists a canonical homomorphism $\Psi^*_{W,W'} \colon \mathscr{A}_W \to \mathscr{B}_{W'}$ by the universal property of $\mathscr{A}_W$. Then, recalling Definition/Proposition 1.7.4 and the universal property of kernels, we get a canonical homomorphism $\Psi^*_{V,V'} \colon \mathscr{A}_V \to \mathscr{B}_{V'}$ for all special subsets V in $\mathscr{M}(\mathscr{A})$ and V' in $\widetilde{V}$ such that $V' \subseteq \Psi^{-1}(V)$.

These homomorphisms are called *pullback homomorphisms* and they are compatible with respect to restriction homomorphisms and compositions, in the sense of Proposition 1.9.4 below.

Remark 1.7.11. Let $\mathscr{A}$ be a K-affinoid algebra and let $\Pi \colon \mathscr{M}(\mathscr{A}\{\boldsymbol{r^{-1}T}\}) \to \mathscr{M}(\mathscr{A})$ be the morphism of K-affinoid spaces induced by the inclusion of $\mathscr{A}$ into $\mathscr{A}\{\boldsymbol{r^{-1}T}\}$ for some $r_1, \dots, r_n \in \mathbb{R}_{>0}$. We notice that for any special subset V of $\mathscr{M}(\mathscr{A})$ we have

$$\mathscr{A}_V\{\boldsymbol{r^{-1}T}\} = \mathscr{A}\{\boldsymbol{r^{-1}T}\}_{\Pi^{-1}(V)}.$$

Indeed, it is true if V is a K-affinoid domain (because of Proposition 1.6.4.iii or by a direct verification of the universal properties), and then it is clear that

$$\mathscr{A}_V\{\boldsymbol{r^{-1}T}\} = \ker\left(\bigoplus_{i\in I}\mathscr{A}_{W_i}\{\boldsymbol{r^{-1}T}\} \to \bigoplus_{(i,j)\in I^2}\mathscr{A}_{W_i\cap W_j}\{\boldsymbol{r^{-1}T}\}\right)$$

when (1.1) holds.

1.8 Sheaves of K-affinoid functions

Definition 1.8.1. Let $X \coloneqq \mathscr{M}(\mathscr{A})$ be a K-affinoid space. For any open subset U of X, we define the *K-algebra of K-affinoid functions on U* as the projective limit $\varprojlim_{V\subseteq U}\mathscr{A}_V$ in the category of K-algebras, where V runs through the special subsets contained in U. It is denoted by $\mathscr{O}_X(U)$ or $\Gamma(U,\mathscr{O}_X)$.

If $U \subseteq U'$ is an inclusion of open subsets in X, then the universal property of projective limits gives a canonical homomorphism $\rho_{U',U}\colon \mathscr{O}_X(U') \to \mathscr{O}_X(U)$, called *restriction homomorphism*.

Remark 1.8.2. It is easy to see that given three open subsets $U \subseteq U' \subseteq U''$, then the restriction homomorphism from $\mathscr{O}_X(U'')$ to $\mathscr{O}_X(U)$ is the composition of the other two. This means that $\mathscr{O}_X$ is a presheaf of K-algebras on X.

Proposition 1.8.3 ([Ber90, §2.3])**.** *Let $X \coloneqq \mathscr{M}(\mathscr{A})$ be a K-affinoid space.*

(i) $\mathscr{O}_X(X) = \mathscr{A}$*;*

(ii) $\mathscr{O}_X$ *is actually a sheaf;*

(iii) $(X,\mathscr{O}_X)$ *is a locally ringed space.*

Definition 1.8.4. Let $X \coloneqq \mathscr{M}(\mathscr{A})$ be a K-affinoid space. For any closed subset Σ of X, we define the *K-algebra of K-affinoid functions on* Σ as the injective limit $\varinjlim_{U\supseteq\Sigma}\Gamma(U,\mathscr{O}_X)$ in the category of K-algebras, where U runs through the open subsets of X containing Σ. It is denoted by $\Gamma(\Sigma,\mathscr{O}_X)$.

Definition/Proposition 1.8.5. It follows easily from the definitions that there is a canonical homomorphism $\rho_{\Lambda',\Lambda}\colon \Gamma(\Lambda',\mathscr{O}_X) \to \Gamma(\Lambda,\mathscr{O}_X)$ for all subsets $\Lambda \subseteq \Lambda'$ of a K-affinoid space X which are either open or closed (even one open and the other closed). We denote the restriction to Λ of any $f \in \Gamma(\Lambda',\mathscr{O}_X)$ by $f_{|\Lambda}$.

If $U \subseteq U'$ are two open subsets of a K-affinoid space X and V is a special subset such that $U \subseteq V \subseteq U'$, then there is a canonical homomorphism $\sigma_{U',V}\colon \Gamma(U',\mathscr{O}_X) \to \mathscr{A}_V$

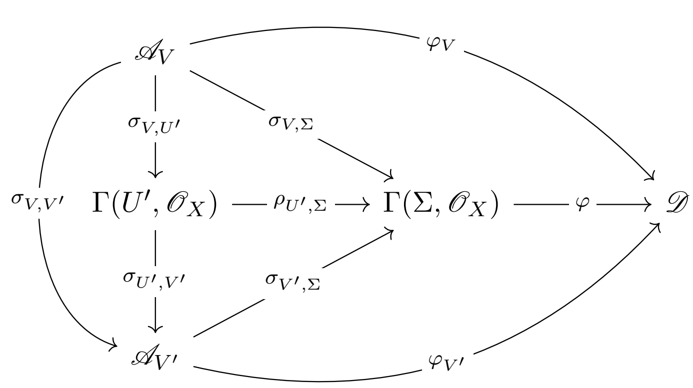
and a canonical homomorphism $\sigma_{V,U} \colon \mathscr{A}_V \to \Gamma(U, \mathscr{O}_X)$. Moreover, if Σ is a closed subset of X and V is a special neighborhood of Σ, then there exists a canonical homomorphism $\sigma_{V,\Sigma} \colon \mathscr{A}_V \to \Gamma(\Sigma, \mathscr{O}_X)$ (which is the composition $\rho_{U,\Sigma} \circ \sigma_{V,U}$, where U is any open neighborhood of Σ inside V).

All these homomorphisms are called *restriction homomorphisms*, and they all are compatible with each other (i.e. the composition of two composable ones is the restriction homomorphism between the corresponding K-algebras).

Convention 1.8.6. Let $X := \mathscr{M}(\mathscr{A})$ be a K-affinoid space and let $\Lambda \subseteq \Lambda'$ be two open or closed subsets of X. Unless otherwise specified, whenever we write an arrow $\Gamma(\Lambda', \mathscr{O}_X) \to \Gamma(\Lambda, \mathscr{O}_X)$ we mean the restriction homomorphism. Anyway, we tend to retain the name ("σ_V", "$\sigma_{U',V}$", "$\sigma_{V,\Sigma}$" and so on) in the case of special subsets.

Proposition 1.8.7. *If Σ is a closed subset of a K-affinoid space $X := \mathscr{M}(\mathscr{A})$, then $\Gamma(\Sigma, \mathscr{O}_X)$ can be calculated as the injective limit $\varinjlim_{V^\circ \supseteq \Sigma} \mathscr{A}_V$ (again in the category of K-algebras) for V running through the special neighborhoods of Σ.*

Proof. It is enough to prove that $\Gamma(\Sigma, \mathscr{O}_X)$, together with the restriction homomorphisms $\sigma_{V,\Sigma} \colon \mathscr{A}_V \to \Gamma(\Sigma, \mathscr{O}_X)$, satisfies the universal property for $\varinjlim_{V^\circ \supseteq \Sigma} A_V$. Suppose that we are given a set of compatible homomorphisms $\varphi_V \colon \mathscr{A}_V \to \mathscr{D}$, where V runs through all the special neighborhoods of Σ. For any open neighborhood U of Σ, we have seen in Proposition 1.7.3 that there exists a special neighborhood V of Σ such that $V \subseteq U$. Therefore, we get a set of compatible homomorphisms $\Gamma(U, \mathscr{O}_X) \xrightarrow{\sigma_{U,V}} \mathscr{A}_V \xrightarrow{\varphi_V} \mathscr{D}$, where U runs through all the open neighborhoods of Σ (and V is a special subset inside U). By the universal property of inductive limits, there exists a unique homomorphism φ from $\Gamma(\Sigma, \mathscr{O}_X) := \varinjlim_{U \supseteq \Sigma} \Gamma(U, \mathscr{O}_X)$ to $\mathscr{D}$ such that $\varphi_V \circ \sigma_{U,V} = \varphi \circ \rho_{U,\Sigma}$ for any special neighborhood V of Σ and any open subset U such that $U \supseteq V$. What we want is a unique homomorphism $\varphi \colon \Gamma(\Sigma, \mathscr{O}_X) \to \mathscr{D}$ such that $\varphi_V = \varphi \circ \sigma_{V,\Sigma}$ for any special neighborhood V of Σ. Therefore, we notice that for any such V there exists an open neighborhood U' of Σ inside V and a special neighborhood V' of Σ inside U', so we get the following diagram:



Now, by the compatibility of the restriction homomorphisms and the fact that (by hypothesis) $\varphi_{V'} \circ \sigma_{V,V'} = \varphi_V$, we obtain that

$$\varphi \circ \sigma_{V,\Sigma} = \varphi \circ \rho_{U',\Sigma} \circ \sigma_{V,U'} = \varphi_{V'} \circ \sigma_{U',V'} \circ \sigma_{V,U'} = \varphi_{V'} \circ \sigma_{V,V'} = \varphi_V,$$

as we wanted. □

1.9 Morphisms of K-quasiaffinoid spaces

Definition 1.9.1. Let U_Y be an open subset of a K-affinoid space $Y := \mathscr{M}(\mathscr{B})$ and let U_X be an open subset of a K-affinoid space $X := \mathscr{M}(\mathscr{A})$. A *morphism of K-quasiaffinoid spaces* $\Xi\colon U_Y \to U_X$ consists of a continuous function $\Xi\colon U_Y \to U_X$ and of a *pullback homomorphism* $\Xi_U^*\colon \Gamma(U, \mathscr{O}_X) \to \Gamma(\Xi^{-1}(U), \mathscr{O}_Y)$ for any open subset U of U_X. The continuous function and the pullback homomorphisms must then satisfy the following properties (with the first two just saying that Ξ is a morphism of locally ringed spaces):

(i) For any two open subsets U and U' of U_X, with $U \subseteq U'$, the following diagram commutes:

$$\begin{array}{ccc} \Gamma(U', \mathscr{O}_X) & \xrightarrow{\Xi_{U'}^*} & \Gamma(\Xi^{-1}(U'), \mathscr{O}_Y) \\ \downarrow & & \downarrow \\ \Gamma(U, \mathscr{O}_X) & \xrightarrow[\Xi_U^*]{} & \Gamma(\Xi^{-1}(U), \mathscr{O}_Y) \end{array}$$

(ii) Recalling Proposition 1.8.3.iii, the homomorphism induced by the pullback homomorphisms sends the maximal ideal of the stalk of the point $\Xi(y)$ to the maximal ideal of the stalk of y, for all $y \in U_Y$.

(iii) For any K-affinoid domain W in U_X and any K-affinoid domain $W' \subseteq \Xi^{-1}(W^\circ)$[3], the homomorphism

$$\mathscr{A}_W \xrightarrow{\sigma_{W,W^\circ}} \Gamma(W^\circ, \mathscr{O}_X) \xrightarrow{\Xi_{W^\circ}^*} \Gamma(\Xi^{-1}(W^\circ), \mathscr{O}_Y) \xrightarrow{\sigma_{\Xi^{-1}(W^\circ),W'}} \mathscr{B}_{W'}$$

is bounded.

The composition of two (composable) morphisms of K-quasiaffinoid spaces is given by composing in the obvious ways both the continuous functions and the pullback homomorphisms.

[3] W° denotes the topological interior of W in X. It is proven in Proposition 2.2.8 that it coincides with the relative interior $\mathrm{Int}(W/X)$ defined in Definition 2.2.1.

Convention 1.9.2. We will continue using this slight abuse of notation consisting in denoting the underlying continuous function with the same symbol of the morphism of K-quasiaffinoid spaces. Moreover, the asterisk will always indicate pullback homomorphisms (in the generalized sense of Definition 1.9.3 below).

Definition 1.9.3. Let $\Xi\colon U_Y \to U_X$ be a morphism of K-quasiaffinoid spaces, where U_Y is an open subset of a K-affinoid space $Y := \mathscr{M}(\mathscr{B})$ and U_X is an open subset of a K-affinoid space $X := \mathscr{M}(\mathscr{A})$. For any closed subset Σ of U_X there is a canonical homomorphism $\Xi^*_\Sigma\colon \Gamma(\Sigma, \mathscr{O}_X) \to \Gamma(\Xi^{-1}(\Sigma), \mathscr{O}_Y)$ induced by the pullback homomorphisms relative to the open neighborhoods of Σ.

Given an open or closed subset Λ of U_X and an open or closed subset Λ' of $\Xi^{-1}(\Lambda)$, we denote by $\Xi^*_{\Lambda,\Lambda'}$ the composition of Ξ^*_Λ with the restriction homomorphism $\Gamma(\Xi^{-1}(\Lambda), \mathscr{O}_Y) \to \Gamma(\Lambda', \mathscr{O}_Y)$. However, if Λ is declared to be a special subset and $\Lambda' \subseteq \Xi^{-1}(\Lambda^\circ)$[4], then we define $\Xi^*_{\Lambda,\Lambda'}$ as the composition

$$\mathscr{A}_\Lambda \xrightarrow{\sigma_{\Lambda,\Lambda^\circ}} \Gamma(\Lambda^\circ, \mathscr{O}_X) \xrightarrow{\Xi^*_{\Lambda^\circ,\Lambda'}} \Gamma(\Lambda', \mathscr{O}_Y)$$

if Λ' is declared to be an open or closed subset, and as the composition

$$\mathscr{A}_\Lambda \xrightarrow{\sigma_{\Lambda,\Lambda^\circ}} \Gamma(\Lambda^\circ, \mathscr{O}_X) \xrightarrow{\Xi^*_{\Lambda^\circ}} \Gamma(\Xi^{-1}(\Lambda^\circ), \mathscr{O}_Y) \xrightarrow{\sigma_{\Xi^{-1}(\Lambda^\circ),\Lambda'}} \mathscr{B}_{\Lambda'}$$

if Λ' is declared to be a special subset.

We call all these maps *pullback homomorphisms.*

Proposition 1.9.4. *The pullback homomorphisms $\Xi^*_{\Lambda,\Lambda'}$ of the previous definition are compatible with respect to restrictions and compositions, that is:*

(i) For any open or closed subsets $\tilde\Lambda \subseteq \Lambda$ and $\tilde\Lambda' \subseteq \Lambda'$ such that $\tilde\Lambda' \subseteq \Xi^{-1}(\tilde\Lambda)$, the following diagram commutes:

$$\begin{array}{ccc} \Gamma(\Lambda, \mathscr{O}_X) & \xrightarrow{\Xi^*_{\Lambda,\Lambda'}} & \Gamma(\Lambda', \mathscr{O}_Y) \\ \downarrow & & \downarrow \\ \Gamma(\tilde\Lambda, \mathscr{O}_X) & \xrightarrow[\Xi^*_{\tilde\Lambda,\tilde\Lambda'}]{} & \Gamma(\tilde\Lambda', \mathscr{O}_Y) \end{array}$$

(Of course, in case e.g. Λ is declared to be a special subset, then we should substitute $\Gamma(\Lambda, \mathscr{O}_X)$ with $\mathscr{A}_\Lambda$.)

[4]But recall Definition/Proposition 1.7.10, for when $\Lambda' \not\subseteq \Xi^{-1}(\Lambda^\circ)$ but Ξ is a morphism of K-affinoid spaces, $\Lambda' \subseteq \Xi^{-1}(\Lambda)$, and both Λ and Λ' are special subsets.

(ii) *For any other morphism of K-quasiaffinoid spaces $\Psi\colon U_Z \to U_Y$ and for any open or closed subset $\Lambda'' \subseteq \Psi^{-1}(\Lambda')$, we have $(\Xi \circ \Psi)^*_{\Lambda,\Lambda''} = \Psi^*_{\Lambda',\Lambda''} \circ \Xi^*_{\Lambda,\Lambda'}$.*

Proof. It is immediate, if one unravels all the (various) definitions and recall Definition 1.9.1.i and the compatibility of restriction homomorphisms (Definition/Proposition 1.8.5). □

Remark 1.9.5. The homomorphism $\Xi^*_{\Lambda,\Lambda'}\colon \mathscr{A}_\Lambda \to \mathscr{B}_{\Lambda'}$ in the last case of Definition 1.9.3 is bounded. Indeed, if $\{W_i\}_{i\in I}$ is a finite cover of Λ made of K-affinoid domains, then the homomorphisms $\Xi^*_{W_i,\Xi^{-1}(W_i)\cap\Lambda'} \circ \sigma_{\Lambda,W}\colon \mathscr{A}_\Lambda \to \mathscr{B}_{\Xi^{-1}(W_i)\cap\Lambda'}$ are bounded, as a consequence of Definition 1.9.1.iii and the definition of the norm on $\mathscr{B}_{\Xi^{-1}(W_i)\cap\Lambda'}$. Then, since those are the homomorphisms $\sigma_{\Lambda',\Xi^{-1}(W_i)\cap\Lambda'} \circ \Xi^*_{\Lambda,\Lambda'}$, it follows from the definition of the norm on $\mathscr{A}_\Lambda$ that $\Xi^*_{\Lambda,\Lambda'}$ is bounded by the maximum of the bounds of those homomorphisms.

Remark 1.9.6. Let $\psi\colon \mathscr{A} \to \mathscr{B}_{\widetilde{V}}$ be a bounded homomorphism where $\mathscr{A}$ and $\mathscr{B}$ are K-affinoid algebras and $\widetilde{V}$ is a special subset of $\mathscr{M}(\mathscr{B})$. We let $\Psi\colon \mathscr{M}(\mathscr{B}_{\widetilde{V}}) \to \mathscr{M}(\mathscr{A})$ be the induced continuous map, and we denote $\mathscr{M}(\mathscr{B})$ by Y and $\mathscr{M}(\mathscr{A})$ by X. By their compatibility with restriction homomorphisms, the pullback homomorphisms from Definition/Proposition 1.7.10 induce a pullback homomorphism

$$\Gamma(U, \mathscr{O}_X) \to \Gamma(\Psi^{-1}(U) \cap \widetilde{V}^\circ, \mathscr{O}_Y)$$

for any open subset U of $\mathscr{M}(\mathscr{A})$. It is easy to see that $\Psi_{|\widetilde{V}^\circ}\colon \widetilde{V}^\circ \to \mathscr{M}(\mathscr{A})$ together with these pullback homomorphisms is a morphism of K-quasiaffinoid spaces.

Considering the particular case in which $\widetilde{V} = \mathscr{M}(\mathscr{B})$, we obtain that the morphisms of K-affinoid spaces are also morphisms of K-quasiaffinoid spaces.

Proposition 1.9.7. *Let U be an open subset of a K-affinoid space $\mathscr{M}(\mathscr{B})$ and let $X \coloneqq \mathscr{M}(\mathscr{A})$ be another K-affinoid space. Let $\Xi\colon U \to X$ be a morphism of K-quasiaffinoid spaces and let $\widetilde{V} \subseteq U$ be a special subset. Then, the morphism of K-quasiaffinoid spaces $\mathscr{M}(\Xi^*_{X,\widetilde{V}})_{|\widetilde{V}^\circ}\colon \widetilde{V}^\circ \to X$ induced by $\Xi^*_{X,\widetilde{V}}\colon \mathscr{A} \to \mathscr{B}_{\widetilde{V}}$ (in the way explained in the previous remark) coincides with the restriction $\Xi_{|\widetilde{V}^\circ}$.*

Proof. Let us fix a point $y \in \widetilde{V}^\circ \subseteq \mathscr{M}(\mathscr{B}_{\widetilde{V}})$ and let us prove that $\mathscr{M}(\Xi^*_{X,\widetilde{V}})(y) = \Xi_{|\widetilde{V}^\circ}(y)$. We let W be a Laurent neighborhood of $\Xi_{|\widetilde{V}^\circ}(y)$ in X, and we let $W' \subseteq \Xi^{-1}_{|\widetilde{V}^\circ}(W^\circ)$ be a Laurent neighborhood of y in $\widetilde{V}^\circ$ (which exists by Remark 1.6.6). In particular, y is in the image of the map $\mathscr{M}(\sigma_{\widetilde{V},W'})\colon W' = \mathscr{M}(\mathscr{B}_{W'}) \to \mathscr{M}(\mathscr{B}_{\widetilde{V}})$ by Remark 1.7.8. By Proposition 1.9.4, the bounded homomorphism $\Xi^*_{W,W'}\colon \mathscr{A}_W \to \mathscr{B}_{W'}$ makes the following diagram to commute:

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{\Xi^*_{X,\widetilde{V}}} & \mathscr{B}_{\widetilde{V}} \\ {\scriptstyle \sigma_W}\downarrow & & \downarrow{\scriptstyle \sigma_{\widetilde{V},W'}} \\ \mathscr{A}_W & \longrightarrow & \mathscr{B}_{W'} \end{array} \tag{1.2}$$

It follows that $\mathscr{M}(\Xi^*_{X,\widetilde{V}})(y) \subseteq \mathscr{M}(\sigma_W)(\mathscr{A}_W) = W$. Now, since $\mathscr{M}(\mathscr{A})$ is Hausdorff and the Laurent neighborhoods W of $\Xi_{|\widetilde{V}^\circ}(y)$ form a basis of neighborhoods, it follows that their intersection is just $\{\Xi_{|\widetilde{V}^\circ}(y)\}$. Therefore, $\mathscr{M}(\Xi^*_{X,\widetilde{V}})(y) = \Xi_{|\widetilde{V}^\circ}(y)$, as we wanted.

Let us now consider the pullback homomorphisms. If W is a K-affinoid domain in X and W' is a K-affinoid domain in $\Xi^{-1}(W^\circ) \cap \widetilde{V}^\circ = \Xi^{-1}_{|\widetilde{V}^\circ}(W^\circ)$, then the pullback homomorphism $\mathscr{A}_W \to \mathscr{B}_{W'}$ relative to $\mathscr{M}(\Xi^*_{X,\widetilde{V}})_{|\widetilde{V}^\circ}$ and the one relative to $\Xi_{|\widetilde{V}^\circ}$ must coincide, because (by the universal property of $\mathscr{A}_W$) there is a unique bounded homomorphism $\mathscr{A}_W \to \mathscr{B}_{W'}$ making the diagram in (1.2) to commute. Then, recalling Definition/Proposition 1.7.4 and the universal property of kernels, also the pullback homomorphisms $\mathscr{A}_V \to \mathscr{B}_{V'}$ with V a special subset in X and V' a special subset in $\Xi^{-1}(V^\circ) \cap \widetilde{V}^\circ = \Xi^{-1}_{|\widetilde{V}^\circ}(V^\circ)$ must coincide, if they are to satisfy Proposition 1.9.4.i. Finally, by passing to the projective limits, also the pullback homomorphisms $\Gamma(U, \mathscr{O}_X) \to \Gamma(\Xi^{-1}(U) \cap \widetilde{V}^\circ, \mathscr{O}_Y)$ must coincide, for any open subset U in X. This concludes the proof. □

Chapter 2

Inner homomorphisms and relative interiors

Here we introduce *inner homomorphisms* (§2.1), proving the equivalence of 3+1 possible definitions, and *relative interiors* (§2.2), proving some of their properties and in particular their relation with inner homomorphisms (Proposition 2.2.7). Moreover, we prove that the relative interior coincides with the topological one in the case of affinoid domains (Proposition 2.2.8) and we apply this fact in order to study the Weierstrass neighborhoods of a closed subset (Proposition 2.2.10). We are following [Ber90, §2.5] but trying to give more detailed proofs.

Throughout this chapter, we let $\mathscr{A}$ be a K-affinoid algebra, $\mathscr{B}$ an $\mathscr{A}$-affinoid algebra and $\mathscr{D}$ a Banach $\mathscr{A}$-algebra. Moreover, we denote $\mathscr{M}(\mathscr{A})$ by X and $\mathscr{M}(\mathscr{B})$ by Y.

2.1 Inner homomorphisms

Definition/Proposition 2.1.1. A bounded homomorphism $\varphi\colon \mathscr{B} \to \mathscr{D}$ is said to be *inner with respect to* $\mathscr{A}$ if it has one of the following equivalent properties:

(a) There exist $r_1, \dots, r_n \in \mathbb{R}_{>0}$ (for some $n \in \mathbb{N}$) and an admissible epimorphism $\pi\colon \mathscr{A}\{\boldsymbol{r^{-1}T}\} \to \mathscr{B}$ such that $\rho(\varphi(\pi(T_i))) < r_i$ for all $i = 1, \dots, n$.

(b) For any bounded homomorphism $\psi\colon \mathscr{A}\{r^{-1}S\} \to \mathscr{B}$ there exists a polynomial $P = S^m + a_1 S^{n-1} + \cdots + a_m \in \mathscr{A}[S]$ such that $\rho(a_i) \leq r^i$ for all $i = 1, \dots, m$, and $\rho(\varphi(\psi(P))) < r^m$.

(c) For any $\varepsilon \in\,]0,1[$ there exist $r_1, \dots, r_n \in \mathbb{R}_{>0}$ (for some $n \in \mathbb{N}$) and an admissible epimorphism $\pi\colon \mathscr{A}\{\boldsymbol{r^{-1}T}\} \to \mathscr{B}$ such that $||\varphi(\pi(T_i))|| < \varepsilon r_i$ for all $i = 1, \dots, n$.

If, moreover, the absolute value on K is non-trivial and $\mathscr{A}$ and $\mathscr{B}$ are strictly K-affinoid, then the properties above are also equivalent to:

(d) the ring $\widetilde{\varphi}(\widetilde{\mathscr{B}})$ is integral over $\widetilde{\varphi}(\widetilde{\mathscr{A}})$.

Convention 2.1.2. We will refer to these equivalent ways of defining inner homomorphisms as "property (a)", "property (b)", "property (c)" and "property (d)".

Proof. (a) $\Longrightarrow$ (d). We assume that the absolute value on K is non-trivial and that $\mathscr{A}$ and $\mathscr{B}$ are strictly K-affinoid. Moreover, we let $\pi\colon \mathscr{A}\{r_1^{-1}T_1,\dots,r_n^{-1}T_n\} \to \mathscr{B}$ be an admissible epimorphism such that $\rho(\varphi(\pi(T_i))) < r_i$ for all $i = 1,\dots,n$. By Proposition 1.5.16.v, we may assume that $r_i \in \sqrt{|K^\times|}$ for all $i = 1,\dots,n$ (clearly, the extension of an admissible epimorphism is still an admissible epimorphism). It means that we can pick $u_i \in \mathbb{N}_{>0}$ and $c_i \in K^\times$ such that $r_i^{u_i} = \left|c_i^{-1}\right|$ (for all $i = 1,\dots,n$), and consider the bounded homomorphism $\eta\colon \mathscr{A}\{\boldsymbol{T}\} \to \mathscr{A}\{\boldsymbol{r^{-1}T}\}$ sending each T_i to $c_iT^{u_i}$.

We let $\pi' := \pi \circ \eta$, and we notice that, by construction, $\rho(\varphi(\pi'(T_i))) < 1$ for all $i = 1,\dots,n$. Moreover, η is finite because every element of $\mathscr{A}\{\boldsymbol{r^{-1}T}\}$ can be written in the form

$$\sum_{j_1=0}^{u_1-1}\cdots\sum_{j_n=0}^{u_n-1}\left[\left(\sum_{\boldsymbol{k}\in\mathbb{N}^n} a_{\boldsymbol{ku+j}}\boldsymbol{c^{-k}}(\boldsymbol{cT^u})^{\boldsymbol{k}}\right)\boldsymbol{T^j}\right] \quad \text{with} \quad \lim_{|\boldsymbol{k}|\to\infty}|a_{\boldsymbol{ku+j}}|\,\boldsymbol{r^{ku+j}} = 0$$

and, for any fixed $\boldsymbol{j} = (j_1,\dots,j_n)$, the element between the round brackets is the image through η of the element $\sum_{\boldsymbol{k}\in\mathbb{N}^n}(a_{\boldsymbol{ku+j}}\boldsymbol{c^{-k}})\boldsymbol{T^k} \in \mathscr{A}\{\boldsymbol{T}\}$, which is well defined because

$$\lim_{|\boldsymbol{k}|\to\infty}\left|a_{\boldsymbol{ku+j}}\boldsymbol{c^{-k}}\right| = \lim_{|\boldsymbol{k}|\to\infty}|a_{\boldsymbol{ku+j}}|\,\boldsymbol{r^{ku}} = \boldsymbol{r^{-j}}\lim_{|\boldsymbol{k}|\to\infty}|a_{\boldsymbol{ku+j}}|\,\boldsymbol{r^{ku+j}} = 0.$$

Since η is finite and π is surjective, then also the composition $\pi' := \pi\circ\eta$ must be finite. This is equivalent, by Theorem 1.5.17, to the fact that $\widetilde{\pi}'\colon \widetilde{\mathscr{A}\{\boldsymbol{T}\}} \to \widetilde{\mathscr{B}}$ is integral, which clearly implies that $\widetilde{\varphi}(\widetilde{\mathscr{B}})$ is integral over $\widetilde{\varphi}(\widetilde{\pi}'(\widetilde{\mathscr{A}\{\boldsymbol{T}\}}))$. Finally, we have that $\widetilde{\varphi}(\widetilde{\pi}'(\widetilde{\mathscr{A}\{\boldsymbol{T}\}})) = \widetilde{\varphi}(\widetilde{\mathscr{A}})$ because $\rho(\varphi(\pi'(T_i))) < 1$ implies $\widetilde{\varphi}(\widetilde{\pi}'(T_i)) = 0$ for all $i = 1,\dots,n$.

(d) $\Longrightarrow$ (b). We assume that the valuation on K is non-trivial, that $\mathscr{A}$ and $\mathscr{B}$ are strictly K-affinoid and that $\widetilde{\varphi}(\widetilde{\mathscr{B}})$ is integral over $\widetilde{\varphi}(\widetilde{\mathscr{A}})$. We consider a bounded homomorphism $\psi\colon \mathscr{A}\{r^{-1}S\} \to \mathscr{B}$ and we have to produce a polynomial P as in (b). By Proposition 1.5.16.v, we may assume that $r \in \sqrt{|K^\times|}$. Then, we can further assume $r = 1$. Indeed, we can pick $u \in \mathbb{N}_{>0}$ and $c \in K^\times$ such that $r^u = \left|c^{-1}\right|$, and consider the bounded homomorphism $\eta\colon \mathscr{A}\{S\} \to \mathscr{A}\{r^{-1}S\}$ sending S to cS^u: if we can find a polynomial $Q = S^m + a_1S^{m-1} + \cdots + a_m$ such that $\rho(a_i) \leq 1$ for all $i = 1,\dots,m$ and $\rho(\varphi(\psi(\eta(Q)))) < 1$, then the polynomial $P \coloneqq c^{-m}\eta(Q) = S^{um} + c^{-1}a_1 S^{u(m-1)} + \cdots + c^{-m}a_m$ is such that $\rho(c^{-i}a_i) \le r^{ui}$ for all $i = 1, \dots, m$ and $\rho(\varphi(\psi(P))) < r^{um}$, as we wanted.

Thus, it remains only to prove the case in which $r = 1$. In this case, by hypothesis, $\widetilde{\varphi}(\widetilde{\psi}(S)) \in \widetilde{\varphi}(\widetilde{\mathscr{B}})$ is integral over $\widetilde{\varphi}(\widetilde{\mathscr{A}})$. It means that there exists a polynomial $T^m + \widetilde{\varphi}(\widetilde{a}_1)T^{m-1} + \cdots + \widetilde{\varphi}(\widetilde{a}_m) \in \widetilde{\varphi}(\widetilde{\mathscr{A}})[T]$ having $\widetilde{\varphi}(\widetilde{\psi}(S))$ as a root. Then, for any choice of $a_1, \dots, a_m \in \mathscr{A}^\circ$ lifting $\widetilde{a}_1, \dots, \widetilde{a}_m$, the polynomial $P = S^m + a_1 S^{m-1} + \cdots + a_m$ is such that $\rho(a_i) \le 1$ for all $i = 1, \dots, m$, and $\widetilde{\varphi}(\widetilde{\psi}(\widetilde{P})) = 0$, that is $\rho(\varphi(\psi(P))) < 1$.

(a) $\Longrightarrow$ (b). Recalling Proposition 1.4.7, we notice that if property (a) holds for φ, then it holds also for $\varphi \mathbin{\widehat{\otimes}}_K \mathrm{id}_{K_p} \colon \mathscr{B} \mathbin{\widehat{\otimes}}_K K_p \to \mathscr{D} \mathbin{\widehat{\otimes}}_K K_p$ (relatively to $\mathscr{A} \mathbin{\widehat{\otimes}}_K K_p$) for any $p \notin \sqrt{|K^\times|}$. Having already proven the implications (a) $\Longrightarrow$ (d) and (d) $\Longrightarrow$ (b) in the strictly affinoid case, it is enough (by Remark 1.5.15) to show that if property (b) holds for $\varphi \mathbin{\widehat{\otimes}}_K \mathrm{id}_{K_p}$ (relatively to $\mathscr{A} \mathbin{\widehat{\otimes}}_K K_p$) with $p \notin \sqrt{|K^\times|}$, then it holds also for φ (relatively to $\mathscr{A}$). So, let us consider a bounded homomorphism $\psi \colon \mathscr{A}\{r^{-1}S\} \to \mathscr{B}$ and assume the existence of a polynomial $Q = S^m + a_1 S^{m-1} + \cdots + a_m \in (\mathscr{A} \mathbin{\widehat{\otimes}}_K K_p)[S]$ such that $\rho(a_i) \le r^i$ for all $i = 1, \dots, m$, and $\rho(\varphi \mathbin{\widehat{\otimes}}_K \mathrm{id}_{K_p}(\psi \mathbin{\widehat{\otimes}}_K \mathrm{id}_{K_p}(Q))) < r^m$. With reference to Proposition 1.5.13, we let $a_{i,j} \in \mathscr{A}$ be elements such that $a_i = \sum_{j\in\mathbb{Z}} a_{i,j}T^j$, and we consider the polynomial $P \coloneqq S^m + a_{1,0}S^{m-1} + \dots, a_{m,0} \in \mathscr{A}[S]$. Since $\rho(a_i) = \rho(\sum_{j\in\mathbb{Z}} a_{i,j}T^j) = \max_{j\in\mathbb{Z}} \rho(a_{i,j})r^j \ge \rho(a_{i,0})$, we have that $\rho(a_{i,0}) \le \rho(a_i) \le r^i$ for all $i = 1, \dots, m$, and also $\rho(\varphi(\psi(P))) \le \rho(\varphi \mathbin{\widehat{\otimes}}_K \mathrm{id}_{K_p}(\psi \mathbin{\widehat{\otimes}}_K \mathrm{id}_{K_p}(Q))) < r^m$, as we wanted.

(b) $\Longrightarrow$ (c). Since $\mathscr{B}$ is assumed to be $\mathscr{A}$-affinoid, there exists an admissible epimorphism $\eta \colon \mathscr{A}\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} \to \mathscr{B}$. For all $i = 1, \dots, n$, let us denote by η_i the restriction of η to $\mathscr{A}\{r_i^{-1}T_i\}$. Assuming that φ satisfies property (b), then also each $\varphi\circ\eta_i$ satisfies the same property. Indeed, for any bounded homomorphism $\psi \colon \mathscr{A}\{r^{-1}S\} \to \mathscr{A}\{r_i^{-1}T_i\}$, the polynomial P given by property (b) of φ (with respect to the bounded homomorphism $\eta_i\circ\psi$) works also for the property (b) of $\varphi\circ\eta_i$ (with respect to the bounded homomorphism ψ).

Suppose, for the moment, that we are able to prove the implication (b) $\Longrightarrow$ (c) when $\mathscr{B} = \mathscr{A}\{r^{-1}S\}$, for any $r \in \mathbb{R}_{>0}$. Applying this to $\varphi \circ \eta_i$ (for each i), with fixed $\varepsilon \in\,]0,1[$, we get admissible epimorphisms $\pi_i \colon \mathscr{A}\{r_{i,1}^{-1}T_{i,1} \dots, r_{i,m_i}^{-1}T_{i,m_i}\} \to \mathscr{A}\{r_i^{-1}T_i\}$ such that $||\varphi(\eta_i(\pi_i(T_{i,j})))|| < \varepsilon r_{i,j}$ for all $i = 1, \dots, n$ and $j = 1, \dots, m_i$. By Proposition 1.4.7 and Proposition 1.5.3.v, we can put all the π_i's together and form an admissible epimorphism

$$\pi' \colon \mathscr{A}\{r_{1,1}^{-1}T_{1,1}, \dots, r_{n,m_n}^{-1}T_{n,m_n}\} \to \mathscr{A}\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}$$

such that, for all $i = 1, \dots, n$, the restriction of π' to $\mathscr{A}\{r_{i,1}^{-1}T_{i,1} \dots, r_{i,m_i}^{-1}T_{i,m_i}\}$ coincides with π_i (up to the inclusion of $\mathscr{A}\{r_i^{-1}T_i\}$ into $\mathscr{A}\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}$). Therefore, $||\varphi(\eta(\pi'(T_{i,j})))|| = ||\varphi(\eta_i(\pi_i(T_{i,j})))|| < \varepsilon r_{i,j}$ for all $i = 1, \dots, n$ and $j = 1, \dots, m_i$. Hence, $\eta \circ \pi'$ is an admissible epimorphism proving property (c).

Thus, it remains only to prove the implication (b) $\Longrightarrow$ (c) when $\mathscr{B} = \mathscr{A}\{r^{-1}S\}$. We let $P = S^m + a_1 S^{m-1} + \cdots + a_m \in \mathscr{A}[S]$ be such that $\rho(a_i) \leq r^i$ for all $i = 1, \dots, m$ and $\rho(\varphi(P)) < r^m$ (we are considering $\psi = \mathrm{id}_{\mathscr{B}}$ in the text of property (b)). For any $M \in \mathbb{N}_{>0}$, we consider the polynomial $P^M = S^{mM} + b_1^{(M)} S^{mM-1} + \cdots + b_{mM}^{(M)}$. It is such that $\rho(b_i^{(M)}) \leq r^i$ for all $i = 1, \dots, mM$, as it can be seen from the expansion

$$P^M = \sum_{\substack{k_0,\dots,k_m \in \mathbb{N} \\ k_0+\cdots+k_m = M}} \frac{M!}{k_0! \cdots k_m!} (S^m)^{k_0} (a_1 S^{n-1})^{k_1} \cdots (a_m)^{k_m},$$

(in a way analogous to the one used later in the proof of the claim), or from the fact that otherwise we would have

$$\rho(P^M) \geq \max_{i=1,\dots,mM} \rho(b_i^{(M)}) r^{mM-i} > r^{mM} = \rho(P)^M.$$

Moreover, it follows from Proposition 1.5.16.i that there exists a constant $C' \in \mathbb{R}_{>0}$ such that $\left|\left|\varphi(P^M)\right|\right| \leq C' \rho(\varphi(P))^M$ for all M large enough. Then,

$$\lim_{M\to\infty} \frac{\left|\left|\varphi(P^M)\right|\right|}{r^{mM}} = \lim_{M\to\infty} \left(\frac{\rho(\varphi(P))}{r^m}\right)^M = 0.$$

It means that for any $\varepsilon > 0$ we can find $M \in \mathbb{N}$ such that $\left|\left|\varphi(P^M)\right|\right| < C^{-1}\varepsilon r^{mM}$, where C is any fixed bound of φ.

From now on, let us fix $\varepsilon \in]0, 1[$, and let us denote by

$$Q = S^n + a'_1 S^{n-1} + \cdots + a'_n$$

the polynomial P^M (so that $n := mM$) for M large enough, so that $||\varphi(Q)|| < C^{-1}\varepsilon r^n$. Recall that $\rho(a'_i) \leq r^i$ for all $i = 1, \dots, n$ and that this fact implies that $\rho(Q) \leq r^n$.

Let us pick $q \in \mathbb{R}_{>0}$ such that $r \leq q$ and $Cr\varepsilon^{-1} < q$. We define a homomorphism

$$\pi \colon \mathscr{A}\{q^{-1}T_0, r^{-n}T_1, r^{-(n+1)}T_2, \dots, r^{-(2n-1)}T_n\} \to \mathscr{A}\{r^{-1}S\} = \mathscr{B}$$

by sending T_0 to S and T_i to $S^{i-1}Q$ (for all $i = 1, \dots, n$). It is well defined and bounded because of Proposition 1.5.16.ii, since $\rho(S) = r \leq q$ and $\rho(S^{i-1}Q) \leq \rho(S^{i-1})\rho(Q) \leq$

$r^{i-1}r^n = r^{n+i-1}$ for all $i = 1, \ldots, n$. Finally, we notice that

$$||\varphi(\pi(T_0))|| = ||\varphi(S)|| \leq C\,||S|| = Cr < \varepsilon q$$

and

$$||\varphi(\pi(T_i))|| \leq \left|\left|\varphi(S^{i-1})\right|\right| ||\varphi(Q)|| \leq Cr^{i-1}\,||\varphi(Q)|| < \varepsilon r^{n+i-1} \quad \forall\, i = 1, \ldots, n,$$

so it remains only to prove that π is an admissible epimorphism. This will follow from the following claim:

Claim. *For all $j \in \mathbb{N}$ there exist two polynomials $G_j \in \mathscr{A}[T_1, \ldots, T_m]$ and $H_j \in \mathscr{A}[T_0]$ such that $\pi(G_j + H_j) = S^j$ and H_j has degree at most $n-1$. Moreover, there exists a constant $C \in \mathbb{R}_{>0}$ such that $||G_j|| \leq Cr^j$ for all $j \in \mathbb{N}$ and $\left|\left|h_i^{(j)}\right|\right| \leq Cr^j$ for all $j \in \mathbb{N}$ and all $i = 1, \ldots, n$, where $h_1^{(j)}T_0^{n-1} + \cdots + h_{n-1}^{(j)}T_0 + h_n^{(j)} = H_j$.*

Proof of the claim. We construct the polynomials G_j and H_j by induction on j. The base case is simple: for all $j \leq n-1$ we can take $H_j = T_0^j$ and $G_j = 0$. Now, let us assume the validity of first part of the claim for all $j < l$, with $l \geq n$. By euclidean division, let us write $l = tn + s$ with $t, s \in \mathbb{N}$ and $s < n$. We notice that S^sQ^t must be a monic polynomial of degree $tn + s = l$; let us give a name to its coefficients: $S^sQ^t = S^l + b_1^{(l)}S^{l-1} + \cdots + b_l^{(l)}$. Now, we define

$$G_l \coloneqq T_1^{t-1}T_{s+1} - (b_1^{(l)}G_{l-1} + \ldots b_{l-n}^{(l)}G_n)$$

and

$$H_l \coloneqq -(b_1^{(l)}H_{l-1} + \ldots b_{l-n}^{(l)}H_n) - (b_{l-n+1}^{(l)}T_0^{n-1} + b_{l-n+2}^{(l)}T_0^{n-2} + \cdots + b_l^{(l)})$$

(which means that $G_l = T_1$ and $H_l = -(b_1^{(l)}T_0^{n-1} + b_2^{(l)}T_0^{n-2} + \cdots + b_l^{(l)})$ in case $l = n$). These two polynomials satisfy the first part of the claim for $j = l$. Indeed, by inductive hypothesis, $\deg(H_l) \leq n-1$ and

$$\begin{aligned}
\pi(G_l + H_l) &= \pi(T_1)^{t-1}\pi(T_{s+1}) - \sum_{i=1}^{l-n} b_i^{(l)}\pi(G_{l-i} + H_{l-i}) - \pi(b_{l-n+1}^{(l)}T_0^{n-1} + \cdots + b_l^{(l)}) = \\
&= Q^{t-1}(S^sQ) - \sum_{i=1}^{l-n} b_i^{(l)}S^{l-i} - (b_{l-n+1}^{(l)}S^{n-1} + \cdots + b_l^{(l)}) = \\
&= S^sQ^t - (b_1^{(l)}S^{l-1} + \cdots + b_l^{(l)}) = S^l.
\end{aligned}$$

Let us now prove the "moreover" part. For any $l \in \mathbb{N}$, we notice that $b_i^{(l)} = 0$ if $i > l - s$, and otherwise $b_i^{(l)}$ can be seen (formally) as a homogeneous polynomial of degree i in the formal ring of polynomials $\mathbb{Z}[a'_1, \ldots, a'_n]$, when considering each a'_k to have degree exactly k. Indeed, $b_i^{(l)}$ is the coefficient of the monomial of Q^t with degree (in the variable S) $l - i - s = tn - i$, and

$$Q^t = \sum_{\substack{k_0, \ldots, k_n \in \mathbb{N} \\ k_0 + \cdots + k_n = t}} \frac{t!}{k_0! \cdots k_n!} (S^n)^{k_0} (a'_1 S^{n-1})^{k_1} \cdots (a'_n)^{k_n}.$$

Clearly, the terms of degree $tn - i$ (in the variable S) are those such that

$$nk_0 + (n-1)k_1 + \cdots + k_{n-1} = tn - i$$

and hence (considering now $\deg a'_k = k$ for all $k = 1, \ldots, n$) their coefficients have degree equal to

$$\begin{aligned} k_1 + \cdots + nk_n &= k_1 + \cdots + nk_n + tn - (k_0 + \cdots + k_n)n = \\ &= tn - [nk_0 + (n-1)k_1 + \cdots + (n - (n-1))k_{n-1}] = \\ &= i, \end{aligned}$$

as we wanted.

Now, since $\rho(a'_k) \leq r^k$ for all $k = 1, \ldots, n$ (and since ρ is a non-Archimedean seminorm), it follows that $\rho(b_i^{(l)}) \leq r^i$ for all $l \in \mathbb{N}$ and $i = 1, \ldots, l$. By induction, it is easy to see that, for every $l \in \mathbb{N}$, the coefficients $h_1^{(l)}, \ldots, h_n^{(l)}$ of H_l are homogeneous polynomials of degree at most l in the formal variables a_k (always with $\deg a_k = k$, for all $k = 1, \ldots, n$). Therefore, if $C \in \mathbb{R}_{>0}$ is such that $||a'_k|| \leq Cr^k$ for all $k = 1, \ldots, n$, then $\left|\left|h_i^{(l)}\right|\right| \leq Cr^l$ for all $l \in \mathbb{N}$ and all $i = 1, \ldots, n$.

We want now to prove that there exists a constant $C' \in \mathbb{R}_{>0}$ such that $\rho(G_j) \leq C'r^j$ for all $j \in \mathbb{N}$. We use induction again, and show that, for any $l \geq n$, if $C' > 1$ is such that $\rho(G_{l-i}) \leq C'r^{l-i}$ for all $i = 1, \ldots, n$, then the same constant is such that $\rho(G_l) \leq C'r^l$. Indeed, recalling that $\rho(b_i^{(l)}) \leq r^i$ for all i, we obtain that

$$\begin{aligned} \rho(G_l) &= \rho(T_1^{t-1}T_{s+1} - (b_1 G_{l-1} + \cdots + b_{l-n} G_n)) \leq \\ &\leq \max\{\rho(T_1^{t-1}T_{s+1}), \rho(b_1 G_{l-1}), \ldots, \rho(b_{l-n} G_n)\} \leq \\ &\leq \max\{r^{n(t-1)} r^{n+s}, r(C'r^{l-1}), \ldots, r^{l-n}(C'r^n)\} = C'r^l, \end{aligned}$$

as we wanted. Then, we notice that every G_l is not nilpotent, because of its monic monomial $T_1^{q-1}T_{s+1}$. Therefore, by Proposition 1.5.16.i, we can find a constant $C'' \in \mathbb{R}_{>0}$

such that $||G_l|| \leq C''\rho(G_l) \leq C''C'r^l$ for all $l \in \mathbb{N}$.

To conclude, we can just pick $C \in \mathbb{R}_{>0}$ such that $C \geq C''C'$ (where C' is such that $C' > 1$ and $\rho(G_i) \leq C'r^i$ for all $i = 0, \dots, n-1$) and $||a'_k|| \leq Cr^k$ for all $k = 1, \dots, n$. □

Let us use the claim to prove that π is indeed an admissible epimorphism. For any element $f = \sum_{j\in\mathbb{N}} b_j S^j \in \mathscr{A}\{r^{-1}S\}$ (meaning that $\lim_{j\to\infty} ||b_j||\, r^j = 0$), we define $g := \sum_{j\in\mathbb{N}} b_j G_j$ and $h := \sum_{j\in\mathbb{N}} b_j H_j = d_1 T_0^{n-1} + \cdots + d_n$, where $d_i := \sum_{l\in\mathbb{N}} b_l h_i^{(l)}$ for all $i = 1, \dots, n$. They are all well defined because of the upper estimates in the claim, which give also the bounds

$$||g|| \leq C \max_{j\in\mathbb{N}} ||b_j||\, r^j = C\, ||f||$$

and

$$||h|| = \max_{i=1,\dots,n} ||d_i||\, q^{n-i} \leq \max_{i=1,\dots,n} \max_{j\in\mathbb{N}} C\, ||b_j||\, r^j q^{n-i} \leq C\Big(\max_{i=1,\dots,n} q^{n-i}\Big)\, ||f||\,.$$

Finally, since $\pi(G_j + H_j) = S^j$ for all $j \in \mathbb{N}$, it is clear that $\pi(g + h) = f$.

(c) $\Longrightarrow$ (a). This implication is trivial. □

Corollary 2.1.3. *Let $r \notin \sqrt{|K^\times|}$. If $\varphi\colon \mathscr{B} \to \mathscr{D}$ is a bounded homomorphism and $\varphi \widehat{\otimes}_K \mathrm{id}_{K_r}\colon \mathscr{B} \widehat{\otimes}_K K_r \to \mathscr{D} \widehat{\otimes}_K K_r$ is inner with respect to $\mathscr{A} \widehat{\otimes}_K K_r$, then φ is inner with respect to $\mathscr{A}$.*

Proof. Already proven, inside the proof of the implication (a) $\Longrightarrow$ (b). □

Corollary 2.1.4. *Let $\varphi\colon \mathscr{B} \to \mathscr{D}$ be a bounded homomorphism which is inner with respect to $\mathscr{A}$. Then, for any bounded homomorphism of $\mathscr{A}$-affinoid algebras $\psi\colon \mathscr{B}' \to \mathscr{B}$ and any bounded homomorphism of Banach $\mathscr{A}$-algebras $\xi\colon \mathscr{D} \to \mathscr{D}'$, the composition $\xi\circ\varphi\circ\psi\colon \mathscr{B}' \to \mathscr{D}'$ is inner with respect to $\mathscr{A}$.*

Proof. By Proposition 1.3.4.iii, it is clear that the composition $\xi \circ \varphi \circ \psi\colon \mathscr{B} \to \mathscr{D}'$ satisfies property (a) if $\varphi \circ \psi\colon \mathscr{B}' \to \mathscr{D}$ does. Then, it is enough to show that the composition $\varphi \circ \psi$ satisfies property (b). This follows because for any bounded homomorphism $\eta\colon \mathscr{A}\{r^{-1}S\} \to \mathscr{B}'$ there exists a polynomial $P = S^m + a_1 S^{n-1} + \cdots + a_m \in \mathscr{A}[S]$ with $\rho(a_i) \leq r^i$ for all $i = 1, \dots, m$, such that $\rho((\varphi \circ \psi)(\eta(P))) = \rho(\varphi((\psi \circ \eta)(P))) < r^m$ (by property (b) of the inner morphism φ, with respect to the bounded homomorphism $\psi \circ \eta$), and this polynomial (depending on η, which is arbitrary) proves property (b) for $\varphi \circ \psi$. Note that we have already used this fact at the beginning of the proof of the implication (b) $\Longrightarrow$ (c). □

2.2 Relative interiors

Recall that $\mathscr{A}$ is a K-affinoid algebra, that $\mathscr{B}$ is an $\mathscr{A}$-affinoid algebra and that $X \coloneqq \mathscr{M}(\mathscr{A})$ and $Y \coloneqq \mathscr{M}(\mathscr{B})$.

Definition 2.2.1. The *relative interior* of a morphism of K-affinoid spaces $\Xi\colon Y \to X$ is the set $\mathrm{Int}(Y/X)$ of points $y \in Y$ such that the associated characters $\mathscr{B} \to \mathscr{H}(y)$ are inner with respect to $\mathscr{A}$.

Moreover, the complement of $\mathrm{Int}(Y/X)$ in Y is called the *relative boundary* of Ξ and is denoted by $\partial(Y/X)$.

Remark 2.2.2. More explicitly, using property (a), $y \in \mathrm{Int}(Y/X)$ if and only if $y \in Y$ and there exist $r_1, \dots, r_n \in \mathbb{R}_{>0}$ (for some $n \in \mathbb{N}$) and an admissible epimorphism $\pi\colon \mathscr{A}\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} \to \mathscr{B}$ such that $|\pi(T_i)(y)| < r_i$ for all $i = 1, \dots, n$.

It follows that $\mathrm{Int}(Y/X)$ is always an open subset of Y, since it is a union of open subsets (one for any admissible epimorphisms like π).

Example 2.2.3. Let us consider the particular case $\mathscr{A} = K$. The existence of an admissible epimorphism $\pi\colon K\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} \to \mathscr{B}$ such that $|\pi(T_i)(y)| < r_i$ for all $i = 1, \dots, n$ (property (a)) means the existence of an embedding of Y into the closed polydisc $E(\mathbf{0}, \boldsymbol{r}) \coloneqq \mathscr{M}(K\{\boldsymbol{r^{-1}T}\})$ such that y lies in the open polydisc $D(\mathbf{0}, \boldsymbol{r}) \coloneqq \{x \in E(\mathbf{0}, \boldsymbol{r}) \mid |T_i(x)| < r_i \ \forall\, i = 1, \dots, n\}$. On the other hand, the existence, for any bounded homomorphism $\psi\colon K\{r^{-1}S\} \to \mathscr{B}$, of a polynomial $P = S^m + a_1 S^{n-1} + \cdots + a_m \in K[S]$ such that $|\psi(P)(y)| < r^m$ while $|a_i| \leq r^i$ for all $i = 1, \dots, m$ (property (b)) implies that the image of y in $E(0, r) \coloneqq \mathscr{M}(K\{r^{-1}S\})$ cannot be the Gauss point (i.e. the point corresponding to the Gauss norm).

In particular, considering the extension K_r of K when $r \notin \sqrt{|K^\times|}$, we see that $\mathrm{Int}(K_r/K) = \emptyset$. In fact, the norm on K_r extends the Gauss norm on $K\{r^{-1}S\}$ after the canonical inclusion of $K\{r^{-1}S\}$ into K_r; it follows that the only point in $\mathscr{M}(K_r)$ (recall Proposition 1.2.2.iv) is sent by the induced map $\mathscr{M}(K_r) \to \mathscr{M}(K\{r^{-1}S\})$ precisely to the Gauss point.

Proposition 2.2.4.

(i) In view of Remark 1.5.9, any two morphisms of K-affinoid spaces $\Phi\colon Y \to X$ and $\Psi\colon X' \to X$ induce a morphism $\Psi'\colon Y' \to Y$, where $Y' \coloneqq Y \times_X X'$. Then, $\Psi'^{-1}(\mathrm{Int}(Y/X)) \subseteq \mathrm{Int}(Y'/X')$.

(ii) For any two morphisms of K-affinoid spaces $\Xi\colon Z \to Y$ and $\Psi\colon Y \to X$, we have $\mathrm{Int}(Z/X) = \mathrm{Int}(Z/Y) \cap \Xi^{-1}(\mathrm{Int}(Y/X))$.

(iii) If $Z \subseteq Y \subseteq X$ are K-affinoid spaces, then $\mathrm{Int}(Z/X) \subseteq \mathrm{Int}(Y/X)$.

Proof. (i). Let $y' \in Y'$ be such that $y := \Psi'(y') \in \mathrm{Int}(Y/X)$; we need to prove that $y' \in \mathrm{Int}(Y'/X')$. We let $X' = \mathscr{M}(\mathscr{A}')$ and $Y' = \mathscr{M}(\mathscr{B}')$, where $\mathscr{B}' := \mathscr{B} \widehat{\otimes}_{\mathscr{A}} \mathscr{A}'$. By Remark 2.2.2, there is an admissible epimorphism $\pi\colon \mathscr{A}\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} \to \mathscr{B}$ with $|\pi(T_i)(y)| < r_i$ for all $i = 1, \dots, n$. By Proposition 1.4.7, the induced homomorphism

$$\pi' := \pi \mathbin{\widehat{\otimes}_{\mathscr{A}}} \mathrm{id}_{\mathscr{A}'}\colon \underbrace{\mathscr{A}'\{\boldsymbol{r^{-1}T}\}}_{\mathscr{A}\{\boldsymbol{r^{-1}T}\} \widehat{\otimes}_{\mathscr{A}} \mathscr{A}'} \to \underbrace{\mathscr{B}'}_{\mathscr{B} \widehat{\otimes}_{\mathscr{A}} \mathscr{A}'}$$

is an admissible epimorphism too. Moreover, if we denote the underlying bounded homomorphisms by $\psi\colon \mathscr{A}\{\boldsymbol{r^{-1}T}\} \to \mathscr{A}'\{\boldsymbol{r^{-1}T}\}$ and $\psi'\colon \mathscr{B} \to \mathscr{B}'$, then we have a commutative diagram

$$\begin{array}{ccc} \mathscr{A}\{\boldsymbol{r^{-1}T}\} & \xrightarrow{\pi} & \mathscr{B} \\ \psi\big\downarrow & & \big\downarrow\psi' \\ \mathscr{A}'\{\boldsymbol{r^{-1}T}\} & \xrightarrow{\pi'} & \mathscr{B}' \end{array}$$

and

$$\left|\pi'(T_i)(y')\right| = \left|\pi'(\psi(T_i))(y')\right| = \left|\psi'(\pi(T_i))(y')\right| = \left|\pi(T_i)(\Psi'(y'))\right| = |\pi(T_i)(y)| < r_i.$$

This shows that $y' \in \mathrm{Int}(Y'/X')$, as we wanted.

(ii). We let $X = \mathscr{M}(\mathscr{A})$, $Y = \mathscr{M}(\mathscr{B})$ and $Z = \mathscr{M}(\mathscr{C})$, and we let $\Phi := \Psi \circ \Xi$. For any $z \in Z$, we define $y := \Xi(z)$ and $x := \Psi(y) = \Phi(z)$. We have to prove three clauses:

$z \in \mathrm{Int}(Z/X) \implies z \in \mathrm{Int}(Z/Y)$. This follows by the previous point (using the same names for functions, even if not for spaces), noticing that $Z = Z \times_X Y$, with Ψ' being nothing but the identity of Z (since Z, with the identity and Ξ, clearly satisfy the universal property for the Cartesian product).

$z \in \mathrm{Int}(Z/X) \implies y \in \mathrm{Int}(Y/X)$. Considering property (b), let $\eta\colon \mathscr{A}\{r^{-1}S\} \to \mathscr{B}$ be any bounded homomorphism. Since $z \in \mathrm{Int}(Z/X)$, there exists a polynomial $P = S^m + a_1S^{m-1} + \dots + a_m \in \mathscr{A}[S]$ such that $\rho(a_i) \le r^i$ for all $i = 1, \dots, m$ and $|(\xi \circ \eta(P))(z)| < r^m$, where $\xi\colon \mathscr{B} \to \mathscr{C}$ is the bounded homomorphism underlying Ξ. We conclude that $y \in \mathrm{Int}(Y/X)$ because $|\eta(P)(y)| = |\eta(P)(\Xi(z))| = |(\xi \circ \eta(P))(z)| < r^m$.

$z \in \mathrm{Int}(Z/Y) \wedge y \in \mathrm{Int}(Y/X) \implies z \in \mathrm{Int}(Z/X)$. By hypothesis we have two admissible epimorphisms $\pi\colon \mathscr{A}\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} \to \mathscr{B}$ and $\eta\colon \mathscr{B}\{q_1^{-1}S_1, \dots, q_m^{-1}S_m\} \to \mathscr{C}$ such that $|\pi(T_i)(y)| < r_i$ for all $i = 1, \dots, n$ and $|\eta(S_j)(z)| < q_j$ for all $j = 1, \dots, m$. We extend π to an admissible epimorphism $\mathscr{A}\{\boldsymbol{r^{-1}T}, \boldsymbol{q^{-1}S}\} \to \mathscr{B}\{\boldsymbol{q^{-1}S}\}$ in the obvious way (i.e. sending S_j to S_j for each j). Its composition with η then gives an admissible epimorphism $\sigma\colon \mathscr{A}\{\boldsymbol{r^{-1}T}, \boldsymbol{q^{-1}S}\} \to \mathscr{C}$ such that $|\sigma(T_i)(z)| < r_i$ for all $i = 1, \dots, n$ and $|\sigma(S_j)(z)| < q_j$ for all $j = 1, \dots, m$, as we wanted.

(iii). This is just a particular case of the implication $z \in \operatorname{Int}(Z/X) \implies y \in \operatorname{Int}(Y/X)$ given in the proof of the previous point. □

Proposition 2.2.5. *Let W be a K-affinoid domain in Y and let V be a special subset of W with a finite cover $\{W_i\}_{i=1,\dots,n}$ by K-affinoid domains. Then, the canonical homomorphism $\sigma_{W,V}\colon \mathscr{B}_W \to \mathscr{B}_V$ is inner with respect to $\mathscr{A}$ if and only if all the canonical homomorphisms $\sigma_{W,W_i}\colon \mathscr{B}_W \to \mathscr{B}_{W_i}$ are inner with respect to $\mathscr{A}$.*

Proof. The "only if" part follows immediately from Corollary 2.1.4. Let us prove that $\sigma_{W,V}$ satisfies property (b) if all the homomorphisms σ_{W,W_i} do. Let $\psi\colon \mathscr{A}\{r^{-1}S\} \to \mathscr{B}_W$ be a bounded homomorphism. By hypothesis, we assume to have polynomials $P_i = S^{m_i} + a_1^{(i)}S^{m_i-1} + \dots + a_{m_i}^{(i)}$, for all $i = 1, \dots, n$, such that $\rho(a_j^{(i)}) \leq r^j$ and $\rho(\sigma_{W,W_i}(\psi(P_i))) < r^{m_i}$ for all $i = 1, \dots, n$ and $j = 1, \dots, m_i$. Let

$$P \coloneqq \prod_{i=1}^{n} P_i = S^m + a_1 S^{m-1} + \dots + a_m.$$

From $\rho(a_j^{(i)}) \leq r^j$ for all i and j, we obtain that $\rho(a_i) \leq r^i$, and also that $\rho(\sigma_{W,W_i}(\psi(P_k))) \leq \rho(P_k) \leq r^{m_k}$ for all $i, k = 1, \dots, n$. From these second inequalities and the strict one $\rho(\sigma_{W,W_i}(\psi(P_i))) < r^{m_i}$, it follows that $\rho(\sigma_{W,W_i}(\psi(P))) < r^m$ for all $i = 1, \dots, n$. Since the norm on $\mathscr{B}_V$ is by definition the maximum of the norms on the $\mathscr{B}_{W_i}$'s (after the restriction homomorphisms), it follows that $\rho(\sigma_{W,V}(\psi(P))) = \max_{i=1,\dots,n} \rho(\sigma_{W,W_i}(\psi(P))) < r^m$, and this finishes the proof. □

Proposition 2.2.6. *Let Σ be a closed subset of Y. Then, Σ is contained in $\operatorname{Int}(Y/X)$ if and only if for any $\varepsilon \in \,]0,1[$ there exist $r_1, \dots, r_n \in \mathbb{R}_{>0}$ (for some $n \in \mathbb{N}$) and an admissible epimorphism $\pi\colon \mathscr{A}\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} \to \mathscr{B}$ such that $\Sigma \subseteq Y(\boldsymbol{(\varepsilon r)^{-1} f})$, where $f_i \coloneqq \pi(T_i)$ for all $i = 1, \dots, n$.*

Proof. The "if" part is trivial. For the other direction, let us fix $\Sigma \subseteq \operatorname{Int}(Y/X)$ and $\varepsilon \in \,]0,1[$. For any $y \in \Sigma$ there exists an admissible epimorphism $\pi_y\colon \mathscr{A}\{r_{y,1}^{-1}T_1, \dots, r_{y,n_y}^{-1}T_{n_y}\} \to \mathscr{B}$ such that $|\pi_y(T_i)(y)| < r_{y,i}$ for all $i = 1, \dots, n_y$. For all such i, we set $f_{y,i} \coloneqq \pi_y(T_i)$ and pick $q_{y,i} \in \mathbb{R}_{>0}$ such that $|\pi_y(T_i)(y)| < q_{y,i} < r_{y,i}$. Then, the Weierstrass domain $W_y \coloneqq Y(\boldsymbol{q_y^{-1} f_y})$ is a neighborhood of y in Y, and the canonical homomorphism $\sigma_{W_y}\colon \mathscr{B} \to \mathscr{B}_{W_y}$ is inner with respect to $\mathscr{A}$ by property (a) (since $\rho(\sigma_{W_y}(\pi_y(T_i))) = \rho(\sigma_{W_y}(f_{y,i})) \leq q_{y,i} < r_{y,i}$ for all $i = 1, \dots, n_y$).

Because Σ is compact, it lies in a finite union $V \coloneqq \bigcup_{i=1}^m W_{y_i}$, for some points y_i in Σ. By the previous proposition, the canonical homomorphism $\sigma_V : \mathscr{B} \to \mathscr{B}_V$ is inner with respect to $\mathscr{A}$. This means that we can find an admissible epimorphism $\pi : \mathscr{A}\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} \to \mathscr{B}$ such that $||\sigma_V(\pi(T_i))|| < \varepsilon r_i$ for all $i = 1, \dots, n$ (property (c)). Denoting $f_i \coloneqq \pi(T_i)$ and recalling that the image of $\mathscr{M}(\mathscr{B}_V)$ in Y contains V, we have that

$$\max_{y \in V} |f_i(y)| \leq \max_{y' \in \mathscr{M}(\mathscr{B}_V)} \left|\sigma_V(f_i)(y')\right| \leq ||\sigma_V(f_i)|| < \varepsilon r_i \quad \forall\, i = 1, \dots, n.$$

Hence, $\Sigma \subseteq V \subseteq Y((\boldsymbol{\varepsilon r})^{-1}\boldsymbol{f})$, as we wanted. □

Proposition 2.2.7. *Any bounded homomorphism $\varphi : \mathscr{B} \to \mathscr{D}$ is inner with respect to $\mathscr{A}$ if and only if the induced map $\mathscr{M}(\varphi) : \mathscr{M}(\mathscr{D}) \to Y$ has image inside* $\mathrm{Int}(Y/X)$.

Proof. First, let us denote the image of $\mathscr{M}(\varphi)$ by Σ, and let us notice that

$$\rho(\varphi(f)) = \max_{z \in \mathscr{M}(\mathscr{D})} |\varphi(f)(z)| = \max_{y \in \Sigma} |f(y)| \quad \forall\, f \in \mathscr{B}. \tag{2.1}$$

Now, if $\varphi : \mathscr{B} \to \mathscr{D}$ is inner with respect to $\mathscr{A}$, it means that there exists an admissible epimorphism $\pi : \mathscr{A}\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} \to \mathscr{B}$ such that $\rho(\varphi(\pi(T_i))) < r_i$ for all $i = 1, \dots, n$. By the previous formula (with $f = \pi(T_i)$), it follows that $|\pi(T_i)(y)| < r_i$ for all $y \in \Sigma$ and all $i = 1, \dots, n$. This shows (by Remark 2.2.2) that $\Sigma \subseteq \mathrm{Int}(Y/X)$ if φ is inner with respect to $\mathscr{A}$.

For the other direction, let us suppose that $\Sigma \subseteq \mathrm{Int}(Y/X)$. By Proposition 2.2.6, fixed $\varepsilon \in]0, 1[$, there exists an admissible epimorphism $\pi : \mathscr{A}\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} \to \mathscr{B}$ such that $\Sigma \subseteq Y((\boldsymbol{\varepsilon r})^{-1}\boldsymbol{f})$, where $f_i \coloneqq \pi(T_i)$ for all $i = 1, \dots, n$. Using formula (2.1) again, it follows that $\rho(\varphi(\pi(T_i))) = \max_{y \in \Sigma} |\pi(T_i)(y)| \leq \varepsilon r_i$ for all $i = 1, \dots, n$. This shows that φ is inner with respect to $\mathscr{A}$ if $\Sigma \subseteq \mathrm{Int}(Y/X)$. □

Proposition 2.2.8. *If Y is an affinoid domain in X, then* $\mathrm{Int}(Y/X)$ *coincides with the topological interior Y° of Y in X.*

Proof. Let us prove the inclusion $Y^\circ \subseteq \mathrm{Int}(Y/X)$ first. By Remark 1.2.9, given any point y in the topological interior of Y, there must exist an open neighborhood of y inside Y of the form $\{x \in X \mid |f_i(x)| < r'_i \wedge |g_j(x)| > q'_j \;\; \forall\, i = 1, \dots, n, \; \forall\, j = 1, \dots, m\}$, for some f_i's and g_j's in $\mathscr{A}$. For each i and j, we pick $r_i < r'_i$ and $q_j > q'_j$ in such a way that

$$y \in U \coloneqq \{x \in X \mid |f_i(x)| < r_i \wedge |g_j(x)| > q_j \;\; \forall\, i = 1, \dots, n, \; \forall\, j = 1, \dots, m\}.$$

Then, by construction, $y \in W \subseteq Y$, where W is the Laurent domain

$$X(\boldsymbol{r^{-1}f}, \boldsymbol{qg^{-1}}) \coloneqq \{x \in X \mid |f_i(x)| \le r_i \wedge |g_j(x)| \ge q_j \ \ \forall\, i = 1, \dots, n,\ \forall\, j = 1, \dots, m\}.$$

It follows directly from the construction of $\mathscr{A}_W$ in Definition/Proposition 1.6.5 that there exists an admissible epimorphism $\pi\colon \mathscr{A}\{\boldsymbol{r^{-1}T}, \boldsymbol{qS}\} \to \mathscr{A}_W$ which sends T_i to f_i (for all i) and S_j to g_j^{-1} (for all j). In particular, $|\pi(T_i)(y)| < r_i$ for all $i = 1, \dots, n$ and $|\pi(S_j)(y)| < q_j^{-1}$ for all $j = 1, \dots, m$. This shows that $y \in \operatorname{Int}(W/X) \subseteq \operatorname{Int}(Y/X)$, where the last inclusion follows from Proposition 2.2.4.iii.

Now, let us prove the inclusion $\operatorname{Int}(Y/X) \subseteq Y^\circ$. By Theorem 1.6.10, there exists a finite cover of X by K-affinoid domains $W_1, \dots, W_m$ such that $Y_j \coloneqq Y \cap W_j$ is a Weierstrass domain in W_j for each $j = 1, \dots, m$. Let us be given an arbitrary point $y \in \operatorname{Int}(Y/X)$ and let $\overline{j} \in \{1, \dots, m\}$ be such that $y \in Y_{\overline{j}}$. We apply Proposition 2.2.4.i with $X' \coloneqq W_{\overline{j}}$, with $Y' \coloneqq Y \times_X X' = Y \cap W_{\overline{j}} = Y_{\overline{j}}$ and with Ψ and Ψ' being respectively the inclusions of $W_{\overline{j}}$ in X and of $Y_{\overline{j}}$ in Y. Since $\Psi'^{-1}(\operatorname{Int}(Y/X)) = Y_{\overline{j}} \cap \operatorname{Int}(Y/X) \ni y$, we obtain that $y \in \operatorname{Int}(Y_{\overline{j}}/X_{\overline{j}})$. If $\operatorname{Int}(Y_{\overline{j}}/X_{\overline{j}}) \subseteq Y_{\overline{j}}^\circ$, then we would get that $y \in Y_{\overline{j}}^\circ \subseteq Y^\circ$. Therefore, we can restrict ourself to case of a Weierstrass domain: for simplicity, we assume Y to be a Weierstrass domain in X (forgetting $Y_{\overline{j}}$ and $X_{\overline{j}}$) and prove the inclusion $\operatorname{Int}(Y/X) \subseteq Y^\circ$ in this case.

Let y be any point in $\operatorname{Int}(Y/X)$; it means that there exists an admissible epimorphism $\pi\colon \mathscr{A}\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} \to \mathscr{A}_Y$ such that

$$y \in U \coloneqq \{y' \in Y \mid \left|f_i(y')\right| < r_i \ \ \forall\, i = 1, \dots, n\},$$

where $f_i \coloneqq \pi(T_i)$ for each $i = 1, \dots, n$. We know that $\mathscr{A}$ is dense in $\mathscr{A}_Y$ because we assumed Y to be a Weierstrass domain. Hence, we can find $f_1', \dots, f_n' \in \mathscr{A}$ such that $||f_i' - f_i|| < r_i$ for all $i = 1, \dots, n$, and such that (by Proposition 1.5.16.iv) the homomorphism $\pi'\colon \mathscr{A}\{\boldsymbol{r^{-1}T}\} \to \mathscr{A}_Y$ sending T_i to f_i' for all $i = 1, \dots, n$ is an admissible epimorphism.

Let us set

$$U' \coloneqq \{x \in X \mid \left|f_i'(x)\right| < r_i \ \ \forall\, i = 1, \dots, n\}$$

and

$$W \coloneqq X(\boldsymbol{r^{-1}f'}) = \{x \in X \mid \left|f_i'(x)\right| \le r_i \ \ \forall\, i = 1, \dots, n\}.$$

We notice that $Y \subseteq W$ because $|f_i'(y')| \le \max\{|f_i(y')|, ||f_i' - f_i||\} \le r_i$ for all $y' \in Y$

and $i = 1, \dots, n$. For an analogous reason $U' \cap Y = U$. Moreover, π' clearly induces an admissible epimorphism from $\mathscr{A}_W = \mathscr{A}\{\boldsymbol{r^{-1}T}\}/(T_1 - f_1', \dots, T_n - f_n')$ to $\mathscr{A}_Y$. It coincides with the restriction homomorphism $\mathscr{A}_W \to \mathscr{A}_Y$ by the universal property of $\mathscr{A}_W$; therefore, by Proposition 1.6.12, W must be the disjoint union of Y and of another K-affinoid domain W'. We notice that $U = U' \cap Y = U' \setminus W'$, and since U' is open (in X) and W' closed (in X), then U must be open (in X). This concludes the proof because, by construction, $y \in U \subseteq Y$. □

Corollary 2.2.9. *Let $W \subseteq W'$ be two K-affinoid domains in X. The restriction homomorphism $\sigma_{W',W} \colon \mathscr{A}_{W'} \to \mathscr{A}_W$ is inner with respect to $\mathscr{A}$ if and only if W lies in the topological interior of W'.*

Proof. Since W and W' are the images in $X = \mathscr{M}(\mathscr{A})$ of $\mathscr{M}(\mathscr{A}_W)$ and $\mathscr{M}(\mathscr{A}_{W'})$ (respectively), it follows immediately from Proposition 2.2.7 that $\sigma_{W',W}$ is inner with respect to $\mathscr{A}$ if and only if $W \subseteq \mathrm{Int}(W'/X)$. But $\mathrm{Int}(W'/X)$ coincides with the topological interior of W' by the previous proposition. □

Proposition 2.2.10. *A closed subset $\Sigma \subseteq X$ lies in the topological interior of a Weierstrass domain W if and only if for any $\varepsilon \in \,]0,1[$ there exists a representation $W = X(\boldsymbol{r^{-1}f})$ such that $\Sigma \subseteq X((\boldsymbol{\varepsilon r})^{-1}\boldsymbol{f})$ for some $f_1, \dots, f_n \in \mathscr{A}$ and $r_1, \dots, r_n \in \mathbb{R}_{>0}$.*

Proof. The "if" part is trivial. For the other direction, the hypothesis becomes that Σ is contained in $\mathrm{Int}(W/X)$, by Proposition 2.2.8. Then, by Proposition 2.2.6, there exists an admissible epimorphism $\pi \colon \mathscr{A}\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} \to \mathscr{A}_W$ such that $\Sigma \subseteq W((\boldsymbol{\varepsilon r})^{-1}\boldsymbol{f})$, where $f_i := \pi(T_i)$ for all $i = 1, \dots, n$. We know that $\mathscr{A}$ is dense in $\mathscr{A}_W$; hence, by Proposition 1.5.16.iv, we can assume that $f_i \in \mathscr{A}$ for all $i = 1, \dots, n$. We let $W' := X(\boldsymbol{r^{-1}f})$, and we notice that $\ker(\pi)$ contains the ideal generated by all $T_i - f_i$, so that π induces an admissible epimorphism

$$\mathscr{A}_{W'} = \mathscr{A}\{\boldsymbol{r^{-1}T}\}/(T_1 - f_1, \dots, T_n - f_n) \to \mathscr{A}_W,$$

which must coincide with the restriction homomorphism. By Proposition 1.6.12, the complement $W' \setminus W$ is a K-affinoid domain in X with $\mathscr{A}_{W'} \cong \mathscr{A}_W \times \mathscr{A}_{W' \setminus W}$. We pick a real number $0 < r_{n+1} < 1$, and we let e be the element of $\mathscr{A}_{W'}$ corresponding to the pair $(0,1)$ under the previous identification. By the density of $\mathscr{A}$ in $\mathscr{A}_{W'}$, we can pick $f_{n+1} \in \mathscr{A}$ such that $||e - f_{n+1}|| < \varepsilon r_{n+1}$. Then, it is easy to see that

$$W = \{x \in W' \mid |f_{n+1}(x)| \le r_{n+1}\} = X(\boldsymbol{r^{-1}f})$$

and $\Sigma \subseteq X((\boldsymbol{\varepsilon r})^{-\mathbf{1}}\boldsymbol{f})$, where, now, $\boldsymbol{r}$ and $\boldsymbol{f}$ are tuples of $n+1$ elements. $\square$

Chapter 3

Holomorphically convex envelopes and spectra of homomorphisms

Here we introduce *holomorphically convex envelopes* of closed subsets (§3.1), proving their relations with Weierstrass neighborhoods (Proposition 3.1.2 and Corollary 3.1.3), and *spectra of homomorphisms* (§3.2), proving some of their properties, especially with respect to holomorphically convex envelopes. We are following [Ber90, §2.6, §7.3] but trying to give more detailed proofs.

3.1 Holomorphically convex envelopes

Throughout this section, we let $X := \mathscr{M}(\mathscr{A})$ be a K-affinoid space and we let Σ be any closed subset of X.

We recall that if $f \in \Gamma(U, \mathscr{O}_X)$, where U is an open neighborhood of Σ, then we denote its restriction to Σ by $f_{|\Sigma}$. In particular, this applies to all $f \in \mathscr{A} = \Gamma(X, \mathscr{O}_X)$.

Definition/Proposition 3.1.1.

(i) For any $x \in \Sigma$ and any special neighborhood V of Σ there is a canonical homomorphism $\mathscr{A}_V \to \mathscr{H}(x)$. Hence, by Proposition 1.8.7 and the universal property of inductive limits, we get a canonical homomorphism $\Gamma(\Sigma, \mathscr{O}_X) \to \mathscr{H}(x)$. We denote the image in $\mathscr{H}(x)$ of any $f \in \Gamma(\Sigma, \mathscr{O}_X)$ by $f(x)$ (and the absolute value of $f(x)$ by $|f(x)|$).

(ii) We notice that $\left|f_{|\Sigma}(x)\right| = |f(x)|$ for any $f \in \mathscr{A}$ and $x \in \Sigma$.

(iii) The assignment $||f||_\Sigma := \max_{x\in\Sigma} |f(x)|$ for $f \in \Gamma(\Sigma, \mathscr{O}_X)$ defines a non-Archimedean seminorm $||\cdot||_\Sigma$ on $\Gamma(\Sigma, \mathscr{O}_X)$.

(iv) We denote the completion of $(\Gamma(\Sigma, \mathscr{O}_X), ||\cdot||_\Sigma)$ by $\mathscr{H}(\Sigma)$, and the closure of the image of $\mathscr{A} = \Gamma(X, \mathscr{O}_X)$ in $\mathscr{H}(\Sigma)$ by $\mathscr{P}(\Sigma)$.

(v) The *holomorphically convex envelope* of Σ in X is the topological space

$$\Sigma^h := \{x \in X \mid |f(x)| \leq \left|\left|f_{|\Sigma}\right|\right|_\Sigma \ \forall f \in \mathscr{A}\}$$

(with the topology induced form that of X).

(vi) We say that Σ is *holomorphically convex* in X if $\Sigma = \Sigma^h$.

Proof. (i). The canonical homomorphism $\mathscr{A}_V \to \mathscr{H}(x)$ is given by the composition of the restriction homomorphism $\mathscr{A}_V \to \mathscr{A}_W$, where W is any K-affinoid domain inside V and containing x, and the homomorphism $\mathscr{A}_W \to \mathscr{H}(x)$ obtained by the universal property of K-affinoid domains. Indeed, given two affinoid domains W and W' inside V and containing x, then the two homomorphisms $\mathscr{A}_V \to \mathscr{A}_W \to \mathscr{H}(x)$ and $\mathscr{A}_V \to \mathscr{A}_{W'} \to \mathscr{H}(x)$ are the same, since they both coincide with the homomorphism $\mathscr{A}_V \to \mathscr{A}_{W\cap W'} \to \mathscr{H}(x)$ (by the compatibility of restriction homomorphisms and the universal property of K-affinoid domains).

(ii). It follows directly from the construction in the previous point (considering $V = X$) that $f_{|\Sigma}(x)$ and $f(x)$ are the same point of $\mathscr{H}(x)$.

(iii). Clearly, $||c||_\Sigma = |c|$ for all $c \in K$, and for all $f, g \in \Gamma(\Sigma, \mathscr{O}_X)$ we have

$$||f+g||_\Sigma = \left|(f+g)(x')\right| \leq \left|f(x')\right| + \left|g(x')\right| \leq ||f||_\Sigma + ||g||_\Sigma$$

and

$$||fg||_\Sigma = \left|(fg)(x'')\right| \leq \left|f(x'')\right| \left|g(x'')\right| \leq ||f||_\Sigma \, ||g||_\Sigma \,,$$

where x' and x'' are points of Σ (which exist by the compactness of Σ) realizing the maxima. □

Proposition 3.1.2.

(i) Any Weierstrass neighborhood W of Σ is also a neighborhood of Σ^h.

(ii) The intersection of all Weierstrass neighborhoods of Σ coincides with Σ^h.

(iii) There is an homeomorphism between $\mathscr{M}(\mathscr{P}(\Sigma))$ and Σ^h.

Proof. (i). By proposition 2.2.10, we can write $W = X(\boldsymbol{r}^{-1}\boldsymbol{f})$ and $\Sigma \subseteq X((\boldsymbol{\varepsilon r})^{-1}\boldsymbol{f})$ for some $f_1, \ldots, f_n \in \mathscr{A}$, $r_1, \ldots, r_n \in \mathbb{R}_{>0}$ and $\varepsilon \in]0,1[$, i.e. $W = \{x \in X \mid |f_i(x)| \leq r_i \ \forall i =$

$1,\dots,n\}$ and $\Sigma \subseteq \{x \in X \mid |f_i(x)| \leq \varepsilon r_i \ \ \forall i = 1,\dots,n\}$. This immediately implies that $\left|\left|f_{i|\Sigma}\right|\right|_{\Sigma} \leq \varepsilon r_i$ for all $i = 1,\dots,n$, and therefore $\Sigma^h \subseteq \{x \in X \mid |f_i(x)| \leq \varepsilon r_i \ \ \forall i = 1,\dots,n\}$. This set is clearly contained in the interior of $W = X(\boldsymbol{r^{-1}f})$, so we are done.

(ii). By the previous point, it is clear that Σ^h is contained in the intersection of all the Weierstrass neighborhoods of Σ. Let us prove the converse: we suppose that $x \notin \Sigma^h$ and find a Weierstrass neighborhood W of Σ such that $x \notin W$. By our assumption, there exists $f \in \mathscr{A}$ such that $\left|\left|f_{|\Sigma}\right|\right|_{\Sigma} < |f(x)|$. Then we take $r \in \mathbb{R}_{>0}$ such that $\left|\left|f_{|\Sigma}\right|\right|_{\Sigma} < r < |f(x)|$ and define the Weierstrass domain $W := X(r^{-1}f) = \{y \in X \mid |f(y)| \leq r\}$. It is clear that it does not contain x and that Σ lies in its interior.

(iii). With a slight abuse of notation, let us denote by $||\cdot||_{\Sigma}$ also the norm on $\mathscr{P}(\Sigma)$. Moreover, let us denote by φ the canonical homomorphism $\mathscr{A} \to \mathscr{P}(\Sigma)$. It is easy to see that $||\varphi(f)||_{\Sigma} = \left|\left|f_{|\Sigma}\right|\right|_{\Sigma} \leq ||f||$ for all $f \in \mathscr{A}$. Therefore, φ induces a continuous function $\mathscr{M}(\varphi)\colon \mathscr{M}(\mathscr{P}(\Sigma)) \to X$. Since $\mathscr{M}(\mathscr{P}(\Sigma))$ is compact and X is Hausdorff, it is enough to show that $\mathscr{M}(\varphi)$ is injective and has image Σ^h. We notice that $\mathscr{M}(\varphi)$ is indeed injective, because the image of $\mathscr{A}$ in $\mathscr{P}(\Sigma)$ is dense by construction. Now, an element x is in the image of $\mathscr{M}(\varphi)$ if and only if there exists $y \in \mathscr{M}(\mathscr{P}(\Sigma))$ such that $|f(x)| = |\varphi(f)(y)|$ for all $f \in \mathscr{A}$. But $|\varphi(f)(y)| \leq ||\varphi(f)|| = \left|\left|f_{|\Sigma}\right|\right|_{\Sigma}$. Thus, the inequality $|f(x)| = |\varphi(f)(y)| \leq \left|\left|f_{|\Sigma}\right|\right|_{\Sigma}$ (for all $f \in \mathscr{A}$) shows that $x \in \Sigma^h$ if x is in the image of $\mathscr{M}(\varphi)$. As for the converse implication, we notice that for any $x \in \Sigma^h$ we have $|f(x)| \leq \left|\left|f_{|\Sigma}\right|\right|_{\Sigma} = ||\varphi(f)||_{\Sigma}$ for all $f \in \mathscr{A}$. Hence, x extends in a unique way to a seminorm on the completion of $(\mathscr{A}, ||\cdot||_{\Sigma} \circ \varphi)$. It remains only to notice that $\mathscr{P}(\Sigma)$ is such a completion: $\mathscr{P}(\Sigma)$ is complete (since it is a closed subspace of a complete metric space) and $\varphi\colon (\mathscr{A}, ||\cdot||_{\Sigma} \circ \varphi) \to \mathscr{P}(\Sigma)$ is an isometry with dense image almost by definition. □

Corollary 3.1.3. *If Σ is holomorphically convex in X, then $\Gamma(\Sigma, \mathscr{O}_X)$ can be calculated as the inductive limit $\varinjlim_{W^\circ \supset \Sigma} A_W$ (in the category of K-algebras) for W running through the Weierstrass neighborhoods of Σ.*

Proof. We start with an easy topological lemma: let U be an open subset of a compact set X and let $\{W_i\}_{i \in I}$ be compact subsets of X such that their intersection is inside U; then there exists a finite number of them whose intersection is inside U. In fact, $X \setminus U$ is compact, $\{X \setminus W_i\}_{i \in I}$ is an open cover of $X \setminus U$, and the complements of any finite subcover give a finite number of W_i's whose intersection is inside U.

We apply this lemma to any open subset $U \subseteq X$ containing Σ, with the compact subsets being the Weierstrass neighborhoods of Σ. Their intersection is indeed inside U because it coincides with Σ, by the second point of the previous proposition (under our hypothesis that $\Sigma = \Sigma^h$). A finite intersection of Weierstrass domains is again a Weierstrass domain by Proposition 1.6.8. Hence, since U was arbitrary, the neighborhoods of Σ which are Weierstrass domains form a basis of neighborhoods of Σ. The thesis now follows by a proof completely analogous to the one of Proposition 1.8.7. $\square$

3.2 Spectra of homomorphisms

Throughout this section, we let $\mathscr{A}$ be a K-affinoid algebra and $\mathscr{D}$ a Banach K-algebra.

Definition 3.2.1. The *spectrum* of a bounded homomorphism $\varphi\colon \mathscr{A} \to \mathscr{D}$ is the image of the induced function $\mathscr{M}(\varphi)\colon \mathscr{M}(\mathscr{D}) \to \mathscr{M}(\mathscr{A})$ as a closed subset of $\mathscr{M}(\mathscr{A})$. It is denoted by Σ_φ.

Example 3.2.2. By Proposition 1.2.8.ii, the spectrum of a character $\chi_x\colon \mathscr{A} \to \mathscr{H}(x)$ is the point $\{x\}$ inducing the character.

Remark 3.2.3. Let $\varphi\colon \mathscr{A} \to \mathscr{D}$ be a bounded homomorphism. Unravelling all the definitions, we get

$$\left|\left|f_{|\Sigma_\varphi}\right|\right|_{\Sigma_\varphi} = \max_{x\in\Sigma_\varphi} |f(x)| = \max_{z\in\mathscr{M}(\mathscr{D})} |\varphi(f)(z)| = \rho(\varphi(f)) \quad \forall\, f \in \mathscr{A}. \tag{3.1}$$

Proposition 3.2.4. *Let $\varphi\colon \mathscr{A} \to \mathscr{D}$ be a bounded homomorphism. For any Weierstrass neighborhood W of the spectrum Σ_φ there exists one and only one bounded homomorphism $\varphi_W\colon \mathscr{A}_W \to \mathscr{D}$ which extends φ.*

Proof. For any W as in the statement and for any fixed $\varepsilon \in \,]0,1[$, Proposition 2.2.10 tells us that $W = X(\boldsymbol{r^{-1}f})$ and $\Sigma_\varphi \subseteq X((\boldsymbol{\varepsilon r})^{-1}\boldsymbol{f})$ for some $f_1, \dots, f_n \in \mathscr{A}$ and $r_1, \dots, r_n \in \mathbb{R}_{>0}$. Let us prove that we can construct a (clearly unique) bounded homomorphism $\varphi'\colon \mathscr{A}\{\boldsymbol{r^{-1}T}\} \to \mathscr{D}$ which extends φ and sends T_i to $\varphi(f_i)$ for all $i = 1, \dots, n$. We start by noticing that $\rho(\varphi(f_i)) = \max_{x\in\Sigma_\varphi} |f_i(x)| \le \varepsilon r_i$ for all $i = 1, \dots, n$, where the equality is shown in (3.1) and the inequality is because $\Sigma_\varphi \subseteq X((\boldsymbol{\varepsilon r})^{-1}\boldsymbol{f})$. Now, the fact that

$$\rho(\varphi(f_i)) \coloneqq \lim_{u\to\infty} \sqrt[u]{||\varphi(f_i)^u||} \le \varepsilon r_i$$

implies that (for each $i = 1, \dots, n$) there exists $m_i \in \mathbb{N}$ such that $||\varphi(f_i)^u|| < r_i^u$ for all $u \ge m_i$. Then, we pick a real number C such that $C \ge 1$ and $C \ge \prod_{i=1}^n ||\varphi(f_i)^{u_i}||$ for any possible choice (for each $i = 1, \dots, n$) of $u_i = 1, \dots, m_i$. It follows that $||a_{\boldsymbol{u}}\boldsymbol{\varphi(f)^u}|| \le C\,||a_{\boldsymbol{u}}||\, r^{\boldsymbol{u}}$ for all $a_{\boldsymbol{u}} \in \mathscr{A}$. This shows that φ' is well defined and bounded, as we wanted.

To conclude, it is clear that $\ker(\varphi')$ contains the ideal generated by all $T_i - f_i$, so that φ' factors (in a unique way) as a composition of the canonical projection and of a homomorphism φ_W (extending φ) from $\mathscr{A}_W = \mathscr{A}\{\boldsymbol{r^{-1}T}\}/(T_1 - f_1, \dots, T_n - f_n)$ to $\mathscr{D}$. Moreover, the homomorphism $\varphi_W \colon \mathscr{A}_W \to \mathscr{D}$ extending φ is unique because the composition of any such homomorphism with the canonical projection $\mathscr{A}\{\boldsymbol{r^{-1}T}\} \to \mathscr{A}\{\boldsymbol{r^{-1}T}\}/(T_1 - f_1, \dots, T_n - f_n)$ must coincide with φ'. □

Proposition 3.2.5. *Let $\varphi \colon \mathscr{A} \to \mathscr{B}$ and $\psi \colon \mathscr{B} \to \mathscr{D}$ be two bounded homomorphisms, with $\mathscr{B}$ a K-affinoid algebra. Let $\Psi \colon Z \to Y$ and $\Phi \colon Y \to X$ be the induced continuous functions between Berkovich spectra. Then, $\Phi(\Sigma^h_\psi) \subseteq \Sigma^h_{\psi\circ\varphi} \subseteq \Sigma^h_\varphi$.*

Proof. For any $y \in \Sigma^h_\psi$ we have $|g(y)| \le \max_{z\in Z} |\psi(g)(z)|$ for all $g \in \mathscr{B}$, by formula (3.1). Then, $\Phi(y)$ is such that

$$|f(\Phi(y))| = |\varphi(f)(y)| \le \max_{z\in Z} |\psi(\varphi(f))(z)| = \left|\left| f_{|\Sigma_{\psi\circ\varphi}} \right|\right|_{\Sigma_{\psi\circ\varphi}} \quad \forall f \in \mathscr{A}.$$

This shows the first inclusion. The second one follows from the two simple facts that $\Sigma_{\psi\circ\varphi} = \Phi(\Psi(Z)) \subseteq \Phi(Y) = \Sigma_\varphi$ and that taking the holomorphically convex envelopes preserves inclusions. □

Corollary 3.2.6. *Let $\xi' \colon \mathscr{A}\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} \to \mathscr{D}$ and $\xi'' \colon \mathscr{A}\{q_1^{-1}S_1, \dots, q_m^{-1}S_m\} \to \mathscr{D}$ be two bounded homomorphisms, and let $\xi \colon \mathscr{A}\{\boldsymbol{r^{-1}T}, \boldsymbol{q^{-1}S}\} \to \mathscr{D}$ be the (bounded) homomorphism acting as ξ' on the T_i's and as ξ'' on the S_j's. Furthermore, let Π, Π' and Π'' be the morphisms of K-affinoid spaces induced by the inclusions of $\mathscr{A}$ in $\mathscr{A}\{\boldsymbol{r^{-1}T}, \boldsymbol{q^{-1}S}\}$, in $\mathscr{A}\{\boldsymbol{r^{-1}T}\}$ and in $\mathscr{A}\{\boldsymbol{q^{-1}S}\}$, respectively. Then $\Pi(\Sigma^h_\xi) \subseteq \Pi'(\Sigma^h_{\xi'}) \cap \Pi''(\Sigma^h_{\xi''})$.*

Proof. We apply Proposition 3.2.5 with $\psi = \xi$ and with φ being the inclusion of $\mathscr{A}\{\boldsymbol{r^{-1}T}\}$ into $\mathscr{A}\{\boldsymbol{r^{-1}T}, \boldsymbol{q^{-1}S}\}$. Denoting by Φ the map induced by φ, we obtain that $\Phi(\Sigma^h_\xi) \subseteq \Sigma^h_{\xi'}$. Hence, $\Pi(\Sigma^h_\xi) = \Pi'(\Phi(\Sigma^h_\xi)) \subseteq \Pi'(\Sigma^h_{\xi'})$, where the first equality follows from the fact that $\Pi = \Pi' \circ \Phi$. The other inclusion $\Pi(\Sigma^h_\xi) \subseteq \Pi''(\Sigma^h_{\xi''})$ is completely analogous. □

Proposition 3.2.7. *Let $\varphi \colon \mathscr{A} \to \mathscr{D}$ be any bounded homomorphism, let $\mathscr{D}'$ be the closed K-subalgebra of $\mathscr{D}$ generated by the image of φ, and let φ' denote the induced homomorphism $\mathscr{A} \to \mathscr{D}'$. Then, $\Sigma_{\varphi'} = \Sigma^h_\varphi$.*

Proof. Recalling formula (3.1), we have

$$\left|\left| f_{|\Sigma_\varphi} \right|\right|_{\Sigma_\varphi} = \rho_{\mathscr{D}}(\varphi(f)) = \rho_{\mathscr{D}'}(\varphi'(f)) = \left|\left| f_{|\Sigma_{\varphi'}} \right|\right|_{\Sigma_{\varphi'}} \quad \forall f \in \mathscr{A}.$$

Thus, $\Sigma^h_{\varphi'} = \Sigma^h_{\varphi}$, and it remains to prove that $\Sigma^h_{\varphi'} \subseteq \Sigma_{\varphi'}$. We notice that for any $x \in \Sigma^h_{\varphi'}$ we have

$$|f(x)| \leq \big|\big|f_{|\Sigma_\varphi}\big|\big|_{\Sigma_\varphi} = \rho(\varphi(f)) \leq ||\varphi(f)|| \quad \forall\, f \in \mathscr{A}.$$

It follows that x can be extended to a seminorm on the completion of $(\mathscr{A}, ||\cdot|| \circ \varphi)$. It remains only to notice that $\mathscr{D}'$ is such a completion: $\mathscr{D}'$ is complete (since it is a closed subspace of a complete metric space) and $\varphi\colon (\mathscr{A}, ||\cdot||_\Sigma \circ \varphi) \to \mathscr{D}'$ is an isometry with dense image almost by definition. $\square$

Chapter 4

The holomorphic functional calculus

Here we prove *Arens - Calderon lemma* (§4.1), and we use it to prove the *holomorphic functional calculus theorem* (§4.2). We are following [Ber90, §7.3] but trying to give more detailed proofs.

4.1 Preliminary results

Throughout this section we let $\mathscr{A}$ be a K-affinoid algebra and $\mathscr{D}$ a Banach K-algebra. Moreover, we denote the K-affinoid space $\mathscr{M}(\mathscr{A})$ by X.

Definition 4.1.1. Let Σ be a closed subset of X. A homomorphism $\varphi\colon \Gamma(\Sigma, \mathscr{O}_X) \to \mathscr{D}$ is said to be *bounded* if the compositions $\mathscr{A}_V \xrightarrow{\sigma_{V,\Sigma}} \Gamma(\Sigma, \mathscr{O}_X) \xrightarrow{\varphi} \mathscr{D}$ (with the restriction homomorphisms) are bounded for all special neighborhoods V of Σ.

Proposition 4.1.2. *Every bounded homomorphism $\varphi\colon \mathscr{A} \to \mathscr{D}$ can be extended in one and only one way to a bounded homomorphism $\theta_{\varphi,\Sigma^h}\colon \Gamma(\Sigma^h, \mathscr{O}_X) \to \mathscr{D}$, where Σ^h is any holomorphically convex subset of X containing the spectrum Σ_φ.*

Proof. By Corollary 3.1.3, we have that $\Gamma(\Sigma^h, \mathscr{O}_X) = \varinjlim_{W^\circ \supseteq \Sigma^h} \mathscr{A}_W$, where W runs through the Weierstrass neighborhoods of Σ^h. It now follows from Proposition 3.2.4 (and the universal property of inductive limits) that φ has one and only one bounded extension $\theta_{\varphi,\Sigma^h}\colon \Gamma(\Sigma^h, \mathscr{O}_X) \to \mathscr{D}$. $\square$

The aim of this chapter is to construct an extension of any bounded homomorphism $\varphi\colon \mathscr{A} \to \mathscr{D}$ not just to $\Gamma(\Sigma^h_\varphi, \mathscr{O}_X)$, but to $\Gamma(\Sigma_\varphi, \mathscr{O}_X)$.

Lemma 4.1.3 (Arens - Calderon)**.** *Let $\varphi\colon \mathscr{A} \to \mathscr{D}$ be a bounded homomorphism. For any open neighborhood U of the spectrum Σ_φ there exist $r_1, \dots, r_n \in \mathbb{R}_{>0}$ (for some $n \in \mathbb{N}_{>0}$) and a bounded homomorphism $\varphi'\colon \mathscr{A}\{\boldsymbol{r}^{-1}\boldsymbol{T}\} \to \mathscr{D}$ which extends φ and is such that*

$\Pi(\Sigma^h_{\varphi'}) \subseteq U$, *where* Π *is the continuous map* $\mathscr{M}(\mathscr{A}\{\boldsymbol{r^{-1}T}\}) \to X$ *induced by the inclusion of* $\mathscr{A}$ *into* $\mathscr{A}\{\boldsymbol{r^{-1}T}\}$.

Proof. First, we prove the following claim:

Claim. *Let* x *be a point in* $X \setminus \Sigma_\varphi$. *Then, there exist* $q_1, \ldots, q_m \in \mathbb{R}_{>0}$ *(for some* $m \in \mathbb{N}_{>0}$*) and a bounded homomorphism* $\varphi_x \colon \mathscr{A}\{\boldsymbol{q^{-1}S}\} \to \mathscr{D}$ *which extends* φ *and is such that* $x \notin \Pi_x(\Sigma^h_{\varphi_x})$, *where* Π_x *is the continuous map induced by the inclusion of* $\mathscr{A}$ *into* $\mathscr{A}\{\boldsymbol{q^{-1}S}\}$.

Proof of the claim. Let us denote by $\mathfrak{b}$ the ideal of $\mathscr{D} \mathbin{\widehat{\otimes}}_K \mathscr{H}(x)$ generated by the elements $\varphi(a) \mathbin{\widehat{\otimes}} 1 - 1 \mathbin{\widehat{\otimes}} a(x)$ for all $a \in \mathscr{A}$. Let us suppose, by contradiction, that $\mathfrak{b} \neq \mathscr{D} \mathbin{\widehat{\otimes}}_K \mathscr{H}(x)$. Then, also the closure $\overline{\mathfrak{b}}$ of $\mathfrak{b}$ must be different from $\mathscr{D} \mathbin{\widehat{\otimes}}_K \mathscr{H}(x)$, otherwise $\mathfrak{b}$ would intersect the group of units $(\mathscr{D} \mathbin{\widehat{\otimes}}_K \mathscr{H}(x))^\times$, since this is open by Proposition 1.1.12.i. It follows that we have a commutative diagram

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{\ \varphi\ } & \mathscr{D} \\ {\scriptstyle \chi_x}\downarrow & & \downarrow \\ \mathscr{H}(x) & \longrightarrow & \mathscr{D}'' \end{array}$$

with $\mathscr{D}'' \coloneqq \mathscr{D} \mathbin{\widehat{\otimes}}_K \mathscr{H}(x)/\overline{\mathfrak{b}} \neq \{0\}$ being a Banach K-algebra. The image of the continuous map $\mathscr{M}(\mathscr{D}'') \to X$ induced by the two coinciding compositions $\mathscr{A} \to \mathscr{D}''$ must be the point $\{x\}$ by Example 3.2.2. On the other hand, the image of that map must ly inside Σ_φ. This contradicts our hypothesis that $x \notin \Sigma_\varphi$, and thus $\mathfrak{b} = \mathscr{D} \mathbin{\widehat{\otimes}}_K \mathscr{H}(x)$. This means that we can write

$$1 = \sum_{i=1}^{m} (\varphi(a_i) \mathbin{\widehat{\otimes}} 1 - 1 \mathbin{\widehat{\otimes}} a_i(x)) g_i \mathbin{\widehat{\otimes}} h_i \tag{4.1}$$

for some $a_i \in \mathscr{A}$, $g_i \in \mathscr{D}$ and $h_i \in \mathscr{H}(x)$.

For each $i = 1, \ldots, m$, we let $r_i \in \mathbb{R}_{>0}$ be such that $r_i > ||g_i||$. Then, recalling Proposition 1.5.16.ii, we let $\varphi_x \colon \mathscr{A}\{\boldsymbol{r^{-1}T}\} \to \mathscr{D}$ be the bounded homomorphism which extends φ and sends T_i to g_i for all $i = 1, \ldots, m$. We denote by $\mathscr{D}'$ the closed K-subalgebra of $\mathscr{D}$ generated by the image of φ_x (i.e. generated by $\varphi(\mathscr{A})$ and the g_i's), and we denote by φ'_x (resp. φ') the bounded homomorphism obtained by restricting the codomain of φ_x (resp. φ) to $\mathscr{D}'$. Proposition 3.2.7 tells us that $\Sigma_{\varphi'_x} = \Sigma^h_{\varphi_x}$, so we want to prove that $x \notin \Pi(\Sigma_{\varphi'_x})$.

We suppose, by contradiction, that there exists $y \in \mathscr{M}(\mathscr{D}')$ such that $\mathscr{M}(\varphi')(y) = x$. By Proposition 1.2.8.iii, it means that the two characters $\chi_y \circ \varphi' \colon \mathscr{A} \to \mathscr{H}(y)$ and $\chi_x \colon \mathscr{A} \to \mathscr{H}(x)$ are equivalent, i.e. there exist two embeddings $\iota_y \colon \mathscr{H}(y) \hookrightarrow L$ and $\iota_x \colon \mathscr{H}(x) \hookrightarrow L$ to a non-Archimedean field L extending K such that $\iota_y \circ \chi_y \circ \varphi' = \iota_x \circ \chi_x$. By the universal property of the completed tensor product, $\iota_y \circ \chi_y$ and ι_x induce a bounded homomorphism

$\mathscr{D}' \mathbin{\widehat{\otimes}}_K \mathscr{H}(x) \to L$, and it is easy to see that all the elements $\varphi(a) \mathbin{\widehat{\otimes}} 1 - 1 \mathbin{\widehat{\otimes}} a(x)$ with $a \in \mathscr{A}$ are in its kernel. It follows that $\iota_x \colon \mathscr{H}(x) \to L$ factors as a composition $\mathscr{H}(x) \to \mathscr{D}' \mathbin{\widehat{\otimes}}_K \mathscr{H}(x)/\mathfrak{b}' \to L$, where $\mathfrak{b}'$ is the ideal of $\mathscr{D}' \mathbin{\widehat{\otimes}}_K \mathscr{H}(x)$ generated by the elements $\varphi(a) \mathbin{\widehat{\otimes}} 1 - 1 \mathbin{\widehat{\otimes}} a(x)$ for all $a \in \mathscr{A}$. This is contradictory because $\mathfrak{b}'$ must be the whole ring $\mathscr{D}' \mathbin{\widehat{\otimes}}_K \mathscr{H}(x)$ by formula (4.1). We have thus proven that $x \notin \mathscr{M}(\varphi')(\mathscr{M}(\mathscr{D}'))$. Finally, since $\mathscr{M}(\varphi') = \Pi_x \circ \mathscr{M}(\varphi'_x)$ (because φ'_x extends φ' by construction), this is equivalent to saying that $x \notin \Pi_x(\mathscr{M}(\varphi'_x)(\mathscr{M}(\mathscr{D}'))) = \Pi_x(\Sigma_{\varphi'_x})$. □

Let us fix a neighborhood U of Σ_φ. For all $x \in X \setminus U \subseteq X \setminus \Sigma_\varphi$, we let φ_x and Π_x be as in the statement of the claim, which tells us that the set $U_x \coloneqq X \setminus \Pi_x(\Sigma^h_{\varphi_x})$ is an open neighborhood of x (in fact $\Pi_x(\Sigma^h_{\varphi_x})$ is compact and hence closed in X). Since $X \setminus U$ is compact, it is covered by a finite number of U_x's: let us denote the corresponding points by $x_1, \dots, x_t$. We construct φ' (and Π) by putting together all the φ_{x_i}'s (resp. Π_{x_i}'s) in the same way as ξ is constructed after ξ' and ξ'' (resp. Π after Π' and Π'') in Corollary 3.2.6. That corollary ensures us that

$$\Pi(\Sigma^h_{\varphi'}) \subseteq \bigcap_{i=1}^{t} \Pi_{x_i}(\Sigma^h_{\varphi_{x_i}}) = X \setminus \left(\bigcup_{i=1}^{t} U_{x_i} \right) \subseteq U,$$

as we wanted. (The corollary gives the first inclusion, the subsequent equality is an application of De Morgan laws and the last inclusion follows from the fact that the U_{x_i}'s cover $X \setminus U$ by construction.) □

4.2 The theorem

Theorem 4.2.1 (holomorphic functional calculus)**.** *There exists one and only one way, satisfying the following properties, to extend any bounded homomorphism $\varphi \colon \mathscr{A} \to \mathscr{D}$ from a K-affinoid algebra $\mathscr{A}$ to a Banach K-algebra $\mathscr{D}$ to a bounded homomorphism $\theta_\varphi \colon \Gamma(\Sigma_\varphi, \mathscr{O}_X) \to \mathscr{D}$ (where $X \coloneqq \mathscr{M}(\mathscr{A})$).*

(Composition property). For any bounded homomorphism $\psi \colon \mathscr{D} \to \mathscr{D}'$ between two Banach K-algebras, the diagram

$$\begin{array}{ccc} \Gamma(\Sigma_\varphi, \mathscr{O}_X) & \xrightarrow{\theta_\varphi} & \mathscr{D} \\ \downarrow & & \downarrow \psi \\ \Gamma(\Sigma_{\psi\circ\varphi}, \mathscr{O}_X) & \xrightarrow[\theta_{\psi\circ\varphi}]{} & \mathscr{D}' \end{array} \tag{4.2}$$

commutes.

(Superposition property). Any morphism of K-quasiaffinoid spaces $\Xi \colon U \to X'$ from

an open neighborhood U of Σ_φ to a K-affinoid space $X' := \mathscr{M}(\mathscr{A}')$ gives rise to a bounded homomorphism

$$\psi\colon \mathscr{A}' \xrightarrow{\Xi^*_{X',\Sigma_\varphi}} \Gamma(\Sigma_\varphi, \mathscr{O}_X) \xrightarrow{\theta_\varphi} \mathscr{D}$$

such that $\Sigma_\psi = \Xi(\Sigma_\varphi)$ and such that the diagram

$$\begin{array}{ccc} \Gamma(\Sigma_\psi, \mathscr{O}_{X'}) & \xrightarrow{\Xi^*_{\Sigma_\psi,\Sigma_\varphi}} & \Gamma(\Sigma_\varphi, \mathscr{O}_X) \\ & {\scriptstyle\theta_\psi}\searrow \quad \swarrow{\scriptstyle\theta_\varphi} & \\ & \mathscr{D} & \end{array} \tag{4.3}$$

commutes.

Proof. (Construction of θ_φ). Let V be any special neighborhood of Σ_φ. Making use of Lemma 4.1.3, we let $\varphi'\colon \mathscr{A}\{\boldsymbol{r^{-1}T}\} \to \mathscr{D}$ be an extension of φ such that $\Pi(\Sigma^h_{\varphi'}) \subseteq V^\circ$, where Π is the canonical morphism from $X' := \mathscr{M}(\mathscr{A}\{\boldsymbol{r^{-1}T}\})$ to X. By Proposition 4.1.2, it follows that φ' can be extended in one and only one way to a bounded homomorphism $\theta_{\varphi',\Sigma^h_{\varphi'}}\colon \Gamma(\Sigma^h_{\varphi'}, \mathscr{O}_{X'}) \to \mathscr{D}$. Then, we define the bounded homomorphism $\theta_{\varphi,V}\colon \mathscr{A}_V \to \mathscr{D}$ as the composition $\theta_{\varphi',\Sigma^h_{\varphi'}} \circ \Pi^*_{V,\Sigma^h_{\varphi'}}$, where $\Pi^*_{V,\Sigma^h_{\varphi'}}\colon \mathscr{A}_V \to \Gamma(\Sigma^h_{\varphi'}, \mathscr{O}_{X'})$ is the pullback homomorphism. We can see that it is indeed bounded by decomposing it further as $(\theta_{\varphi',\Sigma^h_{\varphi'}} \circ \sigma_{\Pi^{-1}(V),\Sigma^h_{\varphi'}}) \circ \Pi^*_V$.

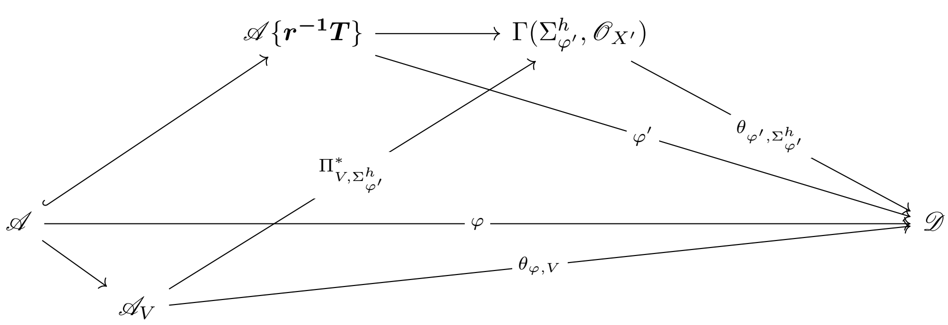


Figure 4.1: Homomorphisms involved in the construction of $\theta_{\varphi,V}$.

Let us now prove that the homomorphism $\theta_{\varphi,V}$ just defined does not depend upon φ'. For this, let us use for a moment the notations of Corollary 3.2.6, that is, let (ξ', Π') and (ξ'', Π'') stand for two possible couples (φ', Π), and let (ξ, Π) be constructed as in the corollary. Now, the corollary tells us that $\Pi(\Sigma^h_\xi) \subseteq V^\circ$, and so it suffices to prove that $\theta_{\xi,\Sigma^h_\xi} \circ \Pi^*_{V,\Sigma^h_\xi} = \theta_{\xi',\Sigma^h_{\xi'}} \circ \Pi'^*_{V,\Sigma^h_{\xi'}}$ (since the other equality with ξ'' and Π'' instead of ξ' and Π' is then completely analogous). Let us denote by Φ the morphism of K-affinoid spaces induced by the inclusion of $\mathscr{A}\{\boldsymbol{r^{-1}T}\}$ into $\mathscr{A}\{\boldsymbol{r^{-1}T}, \boldsymbol{q^{-1}S}\}$ (again with reference to the text of Corollary 3.2.6). By Proposition 3.2.5, we have that $\Phi(\Sigma^h_\xi) \subseteq \Sigma^h_{\xi'}$, so there exists a pullback homomorphism $\Phi^*_{\Sigma^h_{\xi'},\Sigma^h_\xi}\colon \Gamma\big(\Sigma^h_{\xi'}, \mathscr{O}_{\mathscr{M}(\mathscr{A}\{\boldsymbol{r^{-1}T}\})}\big) \to \Gamma\big(\Sigma^h_\xi, \mathscr{O}_{\mathscr{M}(\mathscr{A}\{\boldsymbol{r^{-1}T},\boldsymbol{q^{-1}S}\})}\big)$. By the uniqueness of the extension $\theta_{\xi',\Sigma^h_{\xi'}}\colon \Gamma\big(\Sigma^h_{\xi'}, \mathscr{O}_{\mathscr{M}(\mathscr{A}\{\boldsymbol{r^{-1}T}\})}\big) \to \mathscr{D}$ of ξ', we obtain that $\theta_{\xi,\Sigma^h_\xi} \circ \Phi^*_{\Sigma^h_{\xi'},\Sigma^h_\xi} = \theta_{\xi',\Sigma^h_{\xi'}}$. On the other hand, since $\Pi = \Pi' \circ \Phi$, the compatibility of pullbacks with respect to compositions (Proposition 1.9.4.ii) gives $\Pi^*_{V,\Sigma^h_\xi} = \Phi^*_{\Sigma^h_{\xi'},\Sigma^h_\xi} \circ \Pi'^*_{V,\Sigma^h_{\xi'}}$. Hence,

$$\theta_{\xi,\Sigma^h_\xi} \circ \Pi^*_{V,\Sigma^h_\xi} = \theta_{\xi,\Sigma^h_\xi} \circ \Phi^*_{\Sigma^h_{\xi'},\Sigma^h_\xi} \circ \Pi'^*_{V,\Sigma^h_{\xi'}} = \theta_{\xi',\Sigma^h_{\xi'}} \circ \Pi'^*_{V,\Sigma^h_{\xi'}},$$

as we wanted.

By Proposition 1.8.7, in order to construct $\theta_\varphi\colon \Gamma(\Sigma_\varphi, \mathscr{O}_X) \to \mathscr{D}$, it remains only to show that the bounded homomorphisms $\theta_{\varphi,V}$ are compatible. Hence, we let $V \subseteq V'$ be two special neighborhoods of Σ_φ. Since we have proven that $\theta_{\varphi,V'} := \theta_{\varphi',\Sigma^h_{\varphi'}} \circ \Pi^*_{V',\Sigma^h_{\varphi'}}$ does not depend upon φ', we can take φ' such that $\Pi(\Sigma^h_{\varphi'}) \subseteq V^\circ \subseteq V'^\circ$. Then, $\theta_{\varphi',\Sigma^h_{\varphi'}}$ is (by construction) independent of whether we are considering V or V', while the triangle

$$\begin{array}{ccc} \mathscr{A}_{V'} & \xrightarrow{\sigma_{V',V}} & \mathscr{A}_V \\ & {\scriptstyle \Pi^*_{V',\Sigma^h_{\varphi'}}} \searrow \quad \swarrow {\scriptstyle \Pi^*_{V,\Sigma^h_{\varphi'}}} & \\ & \Gamma(\Sigma^h_{\varphi'}, \mathscr{O}_{X'}) & \end{array}$$

commutes by Proposition 1.9.4.i. This concludes the proof of the compatibility of the $\theta_{\varphi,V}$'s and our construction of θ_φ, which is bounded because the homomorphisms $\theta_{\varphi,V}$ are bounded, and extends φ because $\theta_{\varphi,X} = \theta_\varphi$.

(Composition property). Let V be a special neighborhood of Σ_φ and let φ' be as above, in the construction of θ_φ. By Proposition 3.2.5 we have that $\Sigma^h_{\psi\circ\varphi'} \subseteq \Sigma^h_{\varphi'}$, and in view of Proposition 1.9.4.i there is a commutative diagram

$$\begin{array}{ccc} & \mathscr{A}_V & \\ & {\scriptstyle \Pi^*_{V,\Sigma^h_{\varphi'}}} \swarrow \quad \searrow {\scriptstyle \Pi^*_{V,\Sigma^h_{\psi\circ\varphi'}}} & \\ \Gamma(\Sigma^h_{\varphi'}, \mathscr{O}_{X'}) & \xrightarrow{\rho_{\Sigma^h_{\varphi'},\Sigma^h_{\psi\circ\varphi'}}} & \Gamma(\Sigma^h_{\psi\circ\varphi'}, \mathscr{O}_{X'}) \end{array}$$

Moreover, we notice that both $\psi \circ \theta_{\varphi',\Sigma^h_{\varphi'}}$ and $\theta_{\psi\circ\varphi',\Sigma^h_{\psi\circ\varphi'}} \circ \rho_{\Sigma^h_{\varphi'},\Sigma^h_{\psi\circ\varphi'}}$ extend $\psi \circ \varphi'$. This extension must be unique by Proposition 4.1.2; hence,

$$\begin{aligned} \psi \circ \theta_{\varphi,V} = \psi \circ \theta_{\varphi',\Sigma^h_{\varphi'}} \circ \Pi^*_{V,\Sigma^h_{\varphi'}} &= \\ = \theta_{\psi\circ\varphi',\Sigma^h_{\psi\circ\varphi'}} \circ \rho_{\Sigma^h_{\varphi'},\Sigma^h_{\psi\circ\varphi'}} \circ \Pi^*_{V,\Sigma^h_{\varphi'}} &= \\ = \theta_{\psi\circ\varphi',\Sigma^h_{\psi\circ\varphi'}} \circ \Pi^*_{V,\Sigma^h_{\psi\circ\varphi'}} &= \theta_{\psi\circ\varphi,V}. \end{aligned}$$

The commutativity of the diagram in (4.2) then follows by considering the inductive limit on the special neighborhoods V of Σ_φ.

(Superposition property). By Proposition 1.7.3, we can find a special neighborhood $\widetilde{V}$ of Σ_φ inside U. We fix such a special subset $\widetilde{V}$ and notice that the restriction homomorphism $\Gamma(U, \mathscr{O}_X) \to \Gamma(\Sigma_\varphi, \mathscr{O}_X)$ factors through $\mathscr{A}_{\widetilde{V}}$ by Definition/Proposition 1.8.5. Recalling the construction of θ_φ, it follows that $\psi := \theta_\varphi \circ \Xi^*_{X',\Sigma_\varphi} = \theta_{\varphi,\widetilde{V}} \circ \Xi^*_{X',\widetilde{V}}$. After this new decomposition of ψ and after Remark 1.9.5, we can be sure that ψ is indeed bounded. Furthermore, by Proposition 1.9.7 and the simple fact that $\Sigma_{\theta_{\varphi,\widetilde{V}}} = \Sigma_\varphi$ (as subsets of $\widetilde{V}^\circ$), we obtain that

$$\Sigma_\psi = \mathscr{M}(\Xi^*_{X',\widetilde{V}})(\Sigma_{\theta_{\varphi,\widetilde{V}}}) = \Xi(\Sigma_\varphi),$$

as we wanted.

Now we need to define a lot of homomorphisms: see Figure 4.2 for a representation of all of them. Using Lemma 4.1.3 (as in the construction of θ_φ), we let $\varphi' \colon \mathscr{A}\{\boldsymbol{q}^{-1}\boldsymbol{S}\} \to \mathscr{D}$ be an extension of φ such that $\Pi(\Sigma^h_{\varphi'}) \subseteq \widetilde{V}^\circ$, where Π is the morphism induced by the inclusion of $\mathscr{A}$ into $\mathscr{A}\{\boldsymbol{q}^{-1}\boldsymbol{S}\}$. Let us denote $\widetilde{V}' := \Pi^{-1}(\widetilde{V})$ and let $\theta_{\varphi',\widetilde{V}'} \colon \mathscr{A}\{\boldsymbol{q}^{-1}\boldsymbol{S}\}_{\widetilde{V}'} \to \mathscr{D}$ be the composition $\theta_{\varphi',\Sigma^h_{\varphi'}} \circ \sigma_{\widetilde{V}',\Sigma^h_{\varphi'}}$ (well defined because $\Sigma^h_{\varphi'} \subseteq \widetilde{V}'^\circ$).

Now, let V be any special neighborhood of Σ_ψ. Using Lemma 4.1.3 again, we let $\psi' \colon \mathscr{A}'\{\boldsymbol{r}^{-1}\boldsymbol{T}\} \to \mathscr{D}$ be an extension of ψ such that $\Pi'(\Sigma^h_{\psi'}) \subseteq V^\circ$, where Π' is the morphism induced by the inclusion of $\mathscr{A}'$ into $\mathscr{A}'\{\boldsymbol{r}^{-1}\boldsymbol{T}\}$. If Π'' denotes the morphism induced by the inclusion of $\mathscr{A}$ into $\mathscr{A}\{\boldsymbol{q}^{-1}\boldsymbol{S}, \boldsymbol{r}^{-1}\boldsymbol{T}\}$ and $\widetilde{V}'' := \Pi''^{-1}(\widetilde{V})$, we notice that

$$\mathscr{A}\{\boldsymbol{q}^{-1}\boldsymbol{S}\}_{\widetilde{V}'}\{\boldsymbol{r}^{-1}\boldsymbol{T}\} = \mathscr{A}\{\boldsymbol{q}^{-1}\boldsymbol{S}\}_{\widetilde{V}'} \widehat{\otimes}_{\mathscr{A}'} \mathscr{A}'\{\boldsymbol{r}^{-1}\boldsymbol{T}\} = \mathscr{A}\{\boldsymbol{q}^{-1}\boldsymbol{S}, \boldsymbol{r}^{-1}\boldsymbol{T}\}_{\widetilde{V}''}$$

by Proposition 1.5.3.iv and Remark 1.7.11. Let us denote by $\theta_{\varphi'',\widetilde{V}''} \colon \mathscr{A}\{\boldsymbol{q}^{-1}\boldsymbol{S}, \boldsymbol{r}^{-1}\boldsymbol{T}\}_{\widetilde{V}''} \to \mathscr{D}$ the homomorphism induced (through the universal property of completed tensor products) by $\theta_{\varphi',\widetilde{V}'}$ and ψ'. We let $\varphi'' := \theta_{\varphi'',\widetilde{V}''} \circ \sigma_{\widetilde{V}''}$, and we notice that φ'' extends φ'. In particular, by Proposition 3.2.5, $\Pi''(\Sigma^h_{\varphi''}) \subseteq \Pi'(\Sigma^h_{\varphi'}) \subseteq \widetilde{V}^\circ$, which means that $\Sigma^h_{\varphi''} \subseteq \widetilde{V}''^\circ$. Hence, using Proposition 4.1.2, we can decompose $\theta_{\varphi'',\widetilde{V}''}$ as a the composition $\theta_{\varphi'',\Sigma^h_{\varphi''}} \circ \sigma_{\widetilde{V}'',\Sigma^h_{\varphi''}}$.

Let us denote by $\Xi' \colon \widetilde{V}''^\circ \to \mathscr{M}(\mathscr{A}'\{\boldsymbol{r}^{-1}\boldsymbol{T}\})$ the morphism of K-quasiaffinoid spaces associated (as explained in Remark 1.9.6) to the canonical homomorphism $\mathscr{A}'\{\boldsymbol{r}^{-1}\boldsymbol{T}\} \to \mathscr{A}\{\boldsymbol{q}^{-1}\boldsymbol{S}, \boldsymbol{r}^{-1}\boldsymbol{T}\}_{\widetilde{V}''}$. If we denote the inclusion of $\mathscr{A}'$ into $\mathscr{A}'\{\boldsymbol{r}^{-1}\boldsymbol{T}\}$ by ι and the canonical homomorphism $\mathscr{A}'\{\boldsymbol{r}^{-1}\boldsymbol{T}\} \to \mathscr{A}\{\boldsymbol{q}^{-1}\boldsymbol{S}, \boldsymbol{r}^{-1}\boldsymbol{T}\}_{\widetilde{V}''}$ by τ, then

$$\Pi' \circ \Xi' = \mathscr{M}(\iota) \circ \mathscr{M}(\tau)_{|\widetilde{V}''^\circ} = \mathscr{M}(\Xi^*_{X',\widetilde{V}})_{|\widetilde{V}^\circ} \circ \mathscr{M}(\Pi''^*_{X,\widetilde{V}''})_{|\widetilde{V}''^\circ} = \Xi_{|\widetilde{V}^\circ} \circ \Pi''_{|\widetilde{V}''^\circ} \tag{4.4}$$

by Proposition 1.9.7 and the fact that, by construction, $\tau \circ \iota = \Pi^*_{\widetilde{V},\widetilde{V}''} \circ \Xi^*_{X',\widetilde{V}}$ (and that

$\widetilde{V}'' = \Pi''^{-1}(\widetilde{V})$). Moreover, the fact that by construction

$$\theta_{\varphi'',\Sigma^h_{\varphi''}} \circ \Xi'_{\Sigma^h_{\psi'},\Sigma^h_{\varphi''}} \circ \rho_{\mathscr{M}(\mathscr{A}'\{\boldsymbol{r^{-1}T}\}),\Sigma^h_{\psi'}} = \theta_{\varphi'',\Sigma^h_{\varphi''}} \circ \sigma_{\widetilde{V}'',\Sigma^h_{\varphi''}} \circ \tau = \theta_{\varphi'',\widetilde{V}''} \circ \tau = \psi'$$

implies that $\theta_{\psi',\Sigma^h_{\psi'}} = \theta_{\varphi'',\Sigma^h_{\varphi''}} \circ \Xi'_{\Sigma^h_{\psi'},\Sigma^h_{\varphi''}}$ by the uniqueness of the bounded homomorphism $\Gamma(\Sigma^h_{\psi'}, \mathscr{O}_{\mathscr{M}(\mathscr{A}'\{\boldsymbol{r^{-1}T}\})}) \to \mathscr{D}$ extending ψ' (Proposition 4.1.2).

Now, let V' be a special neighborhood of Σ_φ inside $\Xi^{-1}(V^\circ) \cap \widetilde{V}^\circ$. We recall that, by construction, θ_φ is the homomorphism induced by the homomorphisms $\theta_{\varphi,V'} := \theta_{\varphi'',\Sigma^h_{\varphi''}} \circ \Pi''^*_{V',\Sigma^h_{\varphi''}}$ for such special neighborhoods V' (we can define $\theta_{\varphi,V'}$ in that way because we have shown it to be independent of the choice of the decomposition). By (4.4) and Proposition 1.9.4.ii, we obtain the following commutative diagram:

$$\begin{array}{ccccc}
\mathscr{A}'_V & \xrightarrow{\Pi'^*_{V,\Sigma^h_{\psi'}}} & \Gamma\big(\Sigma^h_{\psi'}, \mathscr{O}_{\mathscr{M}(\mathscr{A}'\{\boldsymbol{r^{-1}T}\})}\big) & \xrightarrow{\theta_{\psi',\Sigma^h_{\psi'}}} & \\
\downarrow{\scriptstyle \Xi^*_{V,V'}} & & \downarrow{\scriptstyle \Xi'^*_{\Sigma^h_{\psi'},\Sigma^h_{\varphi''}}} & & \mathscr{D} \\
\mathscr{A}_{V'} & \xrightarrow[\Pi''^*_{V',\Sigma^h_{\varphi''}}]{} & \Gamma\big(\Sigma^h_{\varphi''}, \mathscr{O}_{\mathscr{M}(\mathscr{A}\{\boldsymbol{q^{-1}S},\boldsymbol{r^{-1}T}\})}\big) & \xrightarrow{\theta_{\varphi'',\Sigma^h_{\varphi''}}} &
\end{array}$$

Hence,

$$\theta_{\varphi,V'} \circ \Xi^*_{V,V'} = \theta_{\varphi'',\Sigma^h_{\varphi''}} \circ \Pi''^*_{V',\Sigma^h_{\varphi''}} \circ \Xi^*_{V,V'} = \theta_{\psi',\Sigma^h_{\psi'}} \circ \Pi'^*_{V,\Sigma^h_{\psi'}} = \theta_{\psi,V}.$$

The commutativity of the diagram in (4.3) then follows by considering the inductive limit on the special neighborhoods V of Σ_ψ (and, consequently, the inductive limit on the special neighborhoods V' of Σ_φ).

(Uniqueness). Let $\tilde{\theta}_\varphi \colon \Gamma(\Sigma_\varphi, \mathscr{O}_X) \to \mathscr{D}$ be a holomorphic functional calculus extension of φ (possibly different from that constructed before). Let V be any special neighborhood of Σ_φ and let $\varphi' \colon \mathscr{A}\{\boldsymbol{r^{-1}T}\} \to \mathscr{D}$ and $\Pi \colon X' \to X$ be as in the construction of θ_φ. By the superposition property (relative to the morphism Π), the triangle

$$\begin{array}{ccc}
\Gamma(\Sigma_\varphi, \mathscr{O}_X) & \xrightarrow{\Pi^*_{\Sigma_\varphi,\Sigma_{\varphi'}}} & \Gamma(\Sigma_{\varphi'}, \mathscr{O}_{X'}) \\
{\scriptstyle \tilde{\theta}_\varphi} \searrow & & \swarrow {\scriptstyle \tilde{\theta}_{\varphi'}} \\
& \mathscr{D} &
\end{array}$$

must commute. Moreover, the diagram

$$\begin{array}{ccc}
\mathscr{A}_V & \xrightarrow{\sigma_{V,\Sigma_\varphi}} & \Gamma(\Sigma_\varphi, \mathscr{O}_X) \\
\downarrow{\scriptstyle \Pi^*_{V,\Sigma^h_{\varphi'}}} & & \downarrow{\scriptstyle \Pi^*_{\Sigma_\varphi,\Sigma_{\varphi'}}} \\
\Gamma(\Sigma^h_{\varphi'}, \mathscr{O}_{X'}) & \xrightarrow[\rho_{\Sigma^h_{\varphi'},\Sigma_{\varphi'}}]{} & \Gamma(\Sigma_{\varphi'}, \mathscr{O}_{X'})
\end{array}$$

commutes by Proposition 1.9.4.i. Recalling the homomorphisms $\theta_{\varphi',\Sigma^h_{\varphi'}}$ and $\theta_{\varphi,V}$ from the construction of θ_φ, we have that

$$\tilde{\theta}_\varphi \circ \sigma_{V,\Sigma_\varphi} = \tilde{\theta}_{\varphi'} \circ \Pi^*_{\Sigma_\varphi,\Sigma_{\varphi'}} \circ \sigma_{V,\Sigma_\varphi} = \tilde{\theta}_{\varphi'} \circ \rho_{\Sigma^h_{\varphi'},\Sigma_{\varphi'}} \circ \Pi^*_{V,\Sigma^h_{\varphi'}} = \theta_{\varphi',\Sigma^h_{\varphi'}} \circ \Pi^*_{V,\Sigma^h_{\varphi'}} = \theta_{\varphi,V} = \theta_\varphi \circ \sigma_{V,\Sigma_\varphi},$$

where $\tilde{\theta}_{\varphi'} \circ \rho_{\Sigma^h_{\varphi'},\Sigma_{\varphi'}} = \theta_{\varphi',\Sigma^h_{\varphi'}}$ by the uniqueness of the homomorphism $\theta_{\varphi',\Sigma^h_{\varphi'}}$ extending φ'. It follows that $\tilde{\theta}_\varphi = \theta_\varphi$, as we wanted, by considering the universal property of the inductive limit $\varinjlim_{V^\circ \supseteq \Sigma} \mathscr{A}_V = \Gamma(\Sigma_\varphi, \mathscr{O}_X)$. □

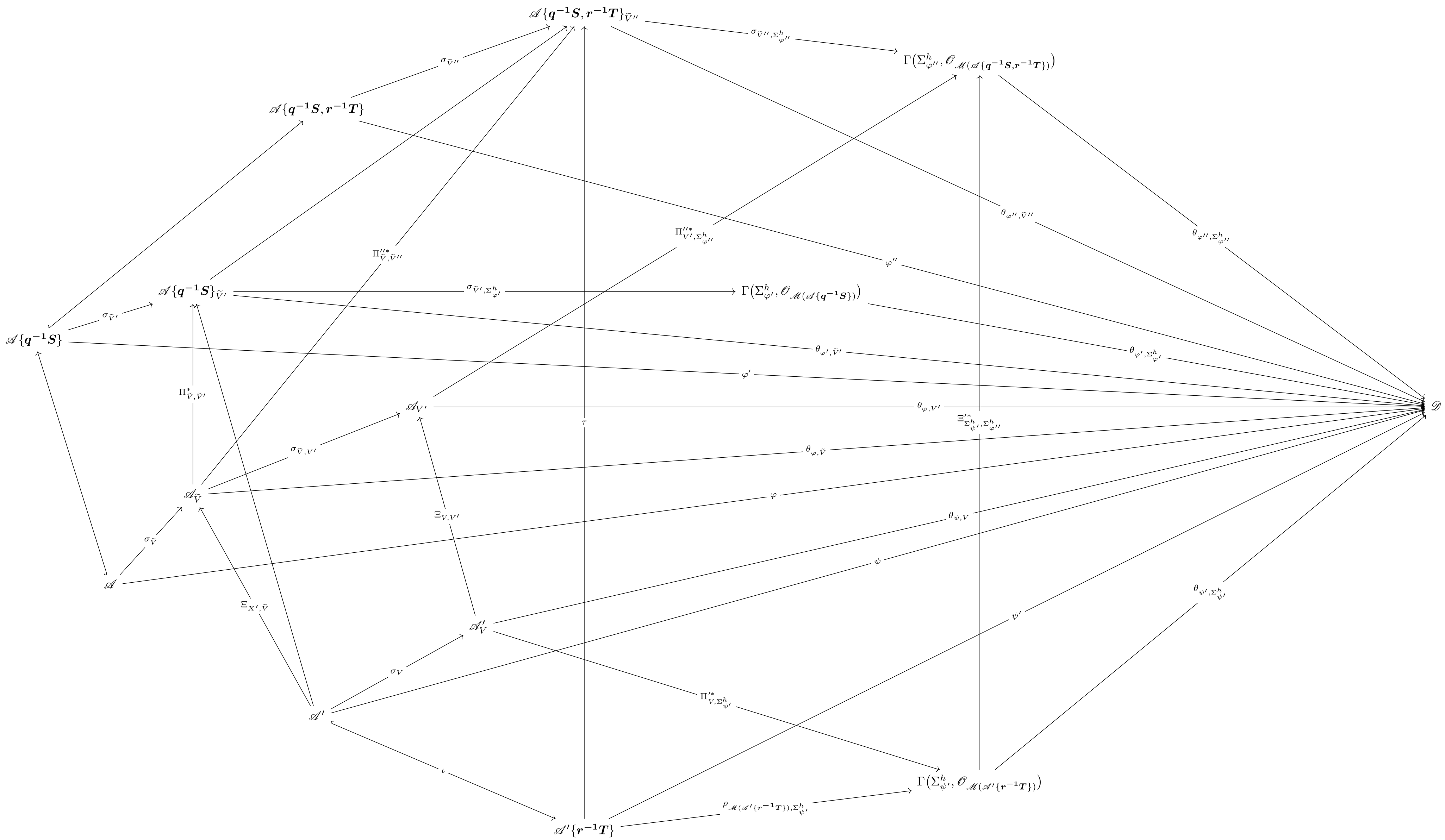


Figure 4.2: Homomorphisms involved in the proof of the superposition property.

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