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Multiparticle Scattering Amplitudes at Two-Loop

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Abstract

In this thesis we present modern techniques needed for the evaluation of one and multi loop amplitudes, and apply some of them in a complete chain that allows the evaluation of a Feynman amplitude. In particular the automated evaluation of a 5 point 2 loop Feynman diagram contributing to the process $e^-e^+ \to \mu^-\mu^+\gamma$ here is presented for the first time. Furthermore we investigate the properties of the integration domain of Feynman integrals in Baikov representation, presenting a new and general formula for their calculation, highlighting an interesting iterative structure beneath the Feynman Integrals. Given this key information in such representation, we found a new parameterization for the Feynman integrals, which needs further studies in order to be better understood. In this thesis, we firstly review the Unitarity based methods, which stems from the Unitarity of the S matrix. Such methods uses *cuts* (i.e. put internal lines on shell) in order to project the amplitude on to its component. For example, in the Cutkosky rule the amplitude is projected in to its imaginary part by means of cuts. Another techniques that relies on cut is the Feynman tree theorem, which by means of complex analysis connect loop level amplitude to tree level one. The most successful approach in such field was the Generalized Unitarity one. Applying the same idea as in the Cutkosky rule, it lead to major automation of one loop calculation. Afterwards we present the issues and the tools that one faces when tackling the calculation of a multiloop Feynman integral, arriving to the analyze the generalized cut and the IBP reduction on the Baikov representation. Lastly, present the *Adaptive integrand decomposition* and an algorithm for the complete automated evaluation of an amplitude. A complete software chain needed to complete such task is then presented, highlighting our contribution to such software.

Contents

Chapter 1

Introduction

In the beginning, the Universe was in a hot dense state in which the particles composing it were highly interacting, scattering one against the other at unbelievable high energies. In order to uncover its great mysteries and knows its laws, such scattering are reproduced, at lower energies, in many laboratories throughout the world nowadays. Studying the data from such experiments, it was found that matter and forces at microscopic scale behaves differently from the macroscopic scale. Quantum Field Theory (QFT), which unifies Special Relativity and Quantum Mechanics, represents the ideal framework to investigate Nature at microscopic level. Nowadays, its exploration has led to the formulation of the *Standard Model of Particle Physics* (SM), the best QFT model which describes matter and forces as interacting elementary particles.

The CERN Large Hadron Collider (LHC) was built to explore its validity at energy scales ranging from the electroweak (EW) scale 100GeV up to energies of some TeV and to search for new phenomena and new particles in this energy domain. The discovery of a Higgs particle at LHC Run 1 in 2012 [1] was a first big achievement in this enterprise. Since first studies of the properties of this Higgs particle show good agreement between measurements and SM predictions, the SM is in better shape than ever to describe all known particle phenomena. Nonetheless there are still many questions at which the Standard Model cannot answer: for examples, it doesn't involves gravity, nor explain what causes the *Dark energy*. Another open problem concerns the particle composition of the *Dark Matter* which, despite having some good candidate being already theorized, is far from being explained. These open problems suggest that the Standard Model of particles cannot describe the whole nature, in fact there should be some physics beyond such model, hidden inside the fundamental particles.

Nonetheless, in view of the absence of spectacular new-physics signals in LHC data, this means that any deviation from the SM hides in small and subtle effects. To extract those differences from data, both experimental analyses and theoretical predictions have to be performed with the highest possible accuracy, i.e. precision can be the key to new discoveries.

Within the theoretical framework of the QFT, such predictions stem from the so called scattering amplitude, an analytical function of the momenta of the involved particles. Such objects derive frome the S-matrix, a *Unitary* matrix which encloses informations about all the possible scattering that could happens. Such amplitude is very difficult to evaluate, indeed apart from special cases, its exact expression cannot be found. A more suitable approach is given by the *perturbative* Quantum

Field Theory: in fact it is possible to expand such amplitude in terms of a small perturbative parameter and then compute it through successive approximation until one achieve the required precision of the result.

It is important to remark that the complexity of the calculation needed to evaluate perturbatively such amplitude grows exponentially as one addresses processes with higher number of particles involved or at higher order in the expansion, in the latter case aiming for results at higher precision. This is due to the fact that many different but quantummechanically indistinguishable processes contribute to each scattering amplitude, and at each order of the perturbative expansion many new such processes appear involving not only real particles, but also virtual one, appearing as particles exchanged between the physical objects involved in the scattering. Such indistinguishable processes that contributes to Scattering Amplitude can be intuitively portrayed by means of the so called Feynman diagrams.

Feynman diagrams are pictorial representations of specific scattering processes, in which occur external and internal legs (the former corresponding to physical particles, while the latter to virtual one which do not satisfy the on-shell condition). The full contribution to an n-point amplitude is the sum of all the n-external legs Feynman diagrams which can be built from the Feynman rules. Contributions to scattering amplitudes, can be classified in to two different kinds, depending on the diagrams with which they can be represented:

- tree-level Feynman diagrams, related to the leading order to the total amplitude. This kind of diagrams can be split in two connected subdiagrams by cutting an internal line;
- l-loop Feynman diagrams, related to l quantum correction of the total amplitudes. Every l-loop Feynman diagrams is an integral in l internal momenta, addressed as loop momenta.

1-loop Feynman amplitudes calculations received a tremendous improvements in the last twenty years [2, 3, 4, 5]: such efficient automation has its roots in the aforementioned *Unitarity* of the scattering matrix: $S^\dagger S = 1$. This property constraint the shape of the interaction, leading to a relation between an amplitude and its complex conjugate as stated by the *Generalised Optical Theorem*. In the perturbative approach such theorem links together amplitude at different perturbative stages: for example one loop amplitude are bound to their tree level counterparts. Such theorem could equivalently be applyed by putting on shell a certan number of virtual particles inside the diagram, through the *Cutkosky rule* [6]. Such operation is defined as *cutting* such feynman diagrams, leading to its projection on the imaginary part.

What happens at one loop is that through the unitarity based methods, one can write the amplitude in terms of a set of scalar integrals which are linearly independent and universal, i.e. every scattering amplitude can be written in terms of these integrals. Such set of integrals is called *Master Integrals*. From these result, genaralizing the idea of cut as the process of putting on shell an internal line, it is possible to reconstruct the full analytical dependence of the amplitude from the master integrals, which are then evaluated.

As a result, the implementation of such unitarity-based methods in automation algorithms for 1-loop amplitudes calculation had a great impact on collider phenomenology, allowing the study of processes involving an high number of particles.

Nonetheless, thanks to the huge amount of data acquired by LHC, in order to keep the pace with the experimental predction, for many processes the 1 loop level of precision isn't enough anymore and multiloop level precision in the theoretical predictions are mandatory to understand clearly the information given by the experimental data.

Moving in to the multiloop case, the complexity of the problem enhance greatly: there the basis of Master Integrals isn't universal anymore: it depends on the process taken in to account and they couldn't be determined in any other way than direct calculation. Moreover the generalised Unitarity approach cannot be extended straightforwardly to this case.

Because of that in this last few years various approaches to face 2-loop corrections were developed, but an efficient evaluation and its automation are still an open problem: limitations occur even at $2 \to 2$ 2-loop non-planar amplitudes with either massive external or internal lines, and $2 \to 3$ amplitudes represents a cap for automatic calculations.

Within this work, techniques that allows to overcome the complexity of the calculation appearing at multiloop level are studyed, arriving to a final application at a $2 \to 3$ particles at 2 loop calculation that draws an original result.

Calculating a loop-correction of a scattering amplitude through a direct integration is a prohibitive task, even with the standard techniques of Feynman parameters: the number of contribution and the complexity of the integral that one has to evaluate make the problem too hard to handle.

For these reasons, a different strategy to evaluate this functions is mandatory. The modern approach to evaluate single Feynman amplitudes is divided in three steps: firstly one performs the tensor reduction [9] or equivalently the adaptive integrand decomposition [17] in order to decompose a single Feynman amplitude in combination of scalar Feynman integrals; at this point it is possible to decompose scalar Feynman integrals in a basis of Master Integrals; lastly, once reduced to such minimal basis, one has to evaluate each integral.

The second step is performed thanks to a set of non-trivial relations, within the dimensional regularization scheme, that derives from the integration-by-parts identities [12]. These identities, known as IBPs, come from the d-dimensional Gauss's divergence theorem and can be exploited to form a linear system of equations that, once solved, allows to write every integral in function of a set of linearly independent one.

In this general picture, interesting results are given by the so called *Baikov-Lee* representation. In that representation the integrand depends on the volume of the parallelotope spanned by the momenta taking part in the interaction. Moreover, in this representation the integration boundaries are determined by the zeroes of such volume, expressed as the *Gram deterinant*. This geometrical object hence encodes crucial information, hidden inside a determinant in a vay that is hard to rea.

A general expression for such boundaries was missing. Studying them during the work of this thesis allowed to find a deeper, iterative structure that needs further studies in order to be fully understood. As an original contribution, a general expression for the boundaries of such integrals were found. Lastly, through such information it is possible to link the Baikov representation to the Hyperspherical coordinates one.

Moreover another application of the cuts draw aattention in the evaluation of Feynman integrals. Cuts are versatile and appears in many application: hteir application within the *Feynman Tree theorem*, in which they relate amplitudes at different perturbative level, gained some attention [11]. There, using the cuts it is possible to write a multiloop amplitude as a sum of tree level one.

In parallel to such though analytical studies, many computational tool was developed in order to apply such methods to the Feynman integrals. Despite the large number of individual tool already existing, at multiloop level a software that performs the whole calculation is still missing. We joined an ongoing project that aims to fill such missing milestone in the research for fenomenological prediction. In such project, we contributed to the development of key component of such software, testing it on a process and bringing a result never obtained before.

This thesis is organised as follows. First of all a brief review of cut-related techniques for scattering amplitudes is given in chapter 1. In particular the so called *Generalized Optical theorem* is introduced which allows to link together amplitude at different perturbative level. Such technique is intuitively portrayed by means of the *Cutkosky rule*: the same result of the Generalized Optical theorem can in fact be obtained by putting on shell a subset of the amplitude's propagators (such operation is defined as *cutting* the amplitude taken in to account). Then another techniqes that link together amplitude at different perturbation level is reviewed: the *Feynman Tree Theorem* that, through the use of *cut* allows to write the amplitude in terms of tree level ones. Such techniques culminate in to the *Generalized Unitarity* approach which, generalizing the use of cuts as projectors, gave a major contribution to the automation of the evaluation for one loop scattering amplitudes. In the last section an application of the Feynman Tree theorem to the QED vertex at one loop is outlined.

In chapter 2 we will review the basic definitions and properties of Feynman integrals in dimensional regularization, with particular attention to the *geometrical* aspects. In fact, after reviewing the notorius *Integration By Parts* identities, thanks to the *Baikov-Lee* a closer look to the generic structure of such integrals is given. Focusing on the integration boundaries in such parametrization, a general formula for its determination is derived *for the first time*, highlighting a nested, iterative structure. Thanks to that a parametrization resembling the Hyperspherical coordinates is found. Further studies are needed in order to understand clearly the possibility given by this approach to the multiloop Feynman integrals. We close the chapter by briefly extends the generalized unitarity approach and the IBPs to the Baikov representation.

After reviewing the notorious techniques for Feynman amplitude evaluation in chapter 1 and 2, in chapter 3 we show how it is possible to apply such methods in order to evaluate a multiloop amplitude. In fact, after reviewing the recently found *Adaptive Integrand Decomposition* approach which sobstitute the traditional tensor decomposition calculation, a complete algorithm for the evaluation of multiloop amplitudes is outlined. Of particular interest is its software implementation, under devolepment. The contribution given during the work of this thesis to such software is then outlined. Lastly, we present such algorithm applyed to a 5 point 2 loop scattering amplitude *for the first time.* There, we briefly outline the results at each intermediate step, arriving finally to the amplitude written in terms of its expansion in the dimensional parameter ϵ, with $D = 4 - 2\epsilon$.

Chapter 2

Unitarity-based methods

In this chapter, we analyze Unitarity based techniques for the evaluation of Feynman diagrams. The first important relation due to Unitarity that ease the calculation of a component of a Feynman amplitude is given by the *Generalized Optical theorem*. Such relation inspired a shortcut to its application: the *Cutkosky rule*, where for the first time *cuts* make their appearance. In fact, by imposing the on-shellness to a set of internal line (i.e. cutting them) one retrieve the same result given by the generalized optical theorem: the imaginary part of the amplitude. Cuts is found to be very versatile and appear in many other techniques. Among them, recently the *Feynman tree theorem*, outlined in this chapter, has drawn renewed attention. This theorem states that it is always possible to write a multiloop amplitude as a sum over tree level diagram integrated over the phase space of the cut propagators. In this thesis, we present its application at the one loop case. Then we move to the *Generalized Unitarity approach*, which brought great results in the evaluation of the one loop amplitudes, used together with the Passarino-Veltman Tensor decomposition. Lastly, we showcase an application of the Feynman tree theorem, evaluating the QED vertex function at one loop.

2.1 Optical Theorem and Cutkosky-Veltman rule

Unitarity arises for the first time in Quantum Mechanics. It is a restriction on the allowed evolution of quantum systems that ensures the sum of probabilities of all possible outcomes of any event always equals 1, thus is conserved through time. Conservation of probability in a quantum theory implies that, in the Schrodinger picture, the norm of a state $|\Psi; t\rangle$ is the same at any time t. For example

$$\langle\Psi; 0|\Psi; 0\rangle = \langle\Psi; t|\Psi; t\rangle. \tag{2.1}$$

Since the operator $S(t)$ regulate the evolution in time of such state through the relation

$$|\Psi; t\rangle = S(t)|\Psi; 0\rangle, \tag{2.2}$$

equation (2.1) can be satisfied only imposing that the operator $S(t)$ is unitary:

$$\langle\Psi; t|\Psi; t\rangle = \langle\Psi; 0|\overbrace{S(t)^\dagger S(t)}^{1}|\Psi; 0\rangle. \tag{2.3}$$

This key property is shared also with its counterpart in Quantum Field Theory the *S-matrix*, or scattering matrix. In this way, from the very general and natural principle that the probability of any possible outcome for a process has to add up to one, it is possible to infer the following property of the scattering matrix S:

$$S^\dagger S = 1. \tag{2.4}$$

This put a constraint on the interaction that could take place as will be shown soon.

The non-trivial part of the S-matrix lies in the *Transfer* matrix T. In order to highlight this component of the scattering matrix, it is possible to write it in the following way:

$$S = 1 - iT. \tag{2.5}$$

Each element of T can be evaluated thanks to the Feynman rules, as they satisfy the following relation

$$\langle f|T|i\rangle = (2\pi)^4\delta^{(4)}(p_i - p_f)\mathcal{M}(i \to f) \tag{2.6}$$

where $\mathcal{M}$ is the Feynman amplitude.

Now substituting the S-matrix written in terms of its non-trivial component inside the Unitarity constraint one obtains that $1 = S^\dagger S = (1 + iT)(1 - iT)$, and hence

$$i(T^\dagger - T) = T^\dagger T. \tag{2.7}$$

Starting from this relation it is possible to draw an important property on the Feynman amplitude $\mathcal{M}$, thanks to equation (2.6). Indeed enclosing the l.h.s. between the bra $\langle f|$ and the ket $|i\rangle$ one obtains the difference between two Feynman amplitude while for what concern the r.h.s. more steps are needed in order to rewrite it in term of $\mathcal{M}$.

After enclosing the left hand side between the final state and the initial one it becomes

$$\begin{aligned} i(\langle f|T^\dagger|i\rangle - \langle f|T|i\rangle) &= i(\langle i|T|f\rangle^* - \langle f|T|i\rangle) \\ &= i(2\pi)^4\delta^{(4)}(p_f - p_i)\left[\mathcal{M}^*(f \to i) - \mathcal{M}(i \to f)\right]. \end{aligned} \tag{2.8}$$

In order to apply equation (2.7) on the right hand side it would be convenient to have two state vectors between the T matrices, since after inserting $\langle f|$ and $|i\rangle$ it becomes

$$\langle f|T^\dagger T|i\rangle. \tag{2.9}$$

Using the completeness relation satisfied by the *complete orthogonal basis* of the Hilbert space of the theory this can be done. This relation states that

$$1 = \sum_x \int d\Pi_x |x\rangle\langle x| \tag{2.10}$$

where x labels all possible single and multi particles states of the theory. The integration measure $d\Pi_x$ is equal to the phase space of the particles in state $|x\rangle$, up to an overall delta:

$$d\Pi_x = \prod_{j\in x} \frac{d^3p_j}{(2\pi)^3}\frac{1}{2E_j}. \tag{2.11}$$

Inserting this completeness relation between the two Transfer matrix in equation (2.9) one finds that

$$\langle f|T^{\dagger}T|i\rangle = \sum_x \int d\Pi_x (2\pi)^4 \delta^{(4)}(p_x - p_f)\mathcal{M}^*(f \to x)(2\pi)^4\delta^{(4)}(p_x - p_i)\mathcal{M}(i \to x). \tag{2.12}$$

where p_x is the sum of all the momenta of the particles appearing in state x, or explicitly $p_x = \sum_{j\in x} p_j$.

After these manipulation equation (2.7) becomes what is called the *generalized Optical Theorem*:

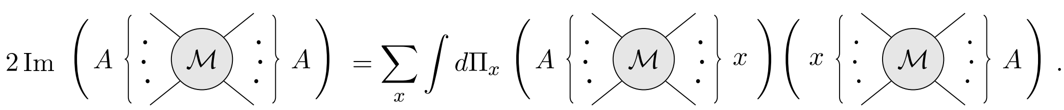
$$i\left[\mathcal{M}^*(f \to i) - \mathcal{M}(i \to f)\right] = \sum_x \int d\Pi_x (2\pi)^4\delta^{(4)}(p_x - p_i)\mathcal{M}^*(f \to x)\mathcal{M}(i \to x) \tag{2.13}$$

where in this case $\sum_x$ means that the r.h.s. is summed over all possible intermediate states.

Moreover, let us stress that, in the case in which $|f\rangle = |i\rangle = |A\rangle$, one obtains the relation

$$2i\mathrm{Im}\mathcal{M}^*(A \to A) = \sum_x \int d\Pi_x (2\pi)^4\delta^{(4)}(p_x - p_i)\mathcal{M}^*(A \to x)\mathcal{M}(A \to x) \tag{2.14}$$

that can be portrayed in the following way:



(2.15)

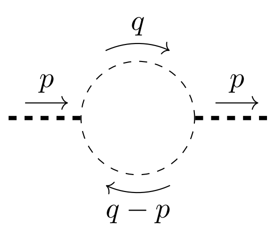
One great feature of this relation is that it links amplitudes at different levels in perturbation theory. Suppose that the amplitude appearing in the l.h.s. is of order λ^2 (with λ being the coupling constant of the theory), the amplitudes on the r.h.s. must be of order λ to match it. Hence *Unitarity* implies that the imaginary part of a one loop amplitude is linked to tree level amplitudes. A closer look to this key concept is given in the example below.

Example

Consider a simple theory which has the following Lagrangian:

$$\mathcal{L} = -\frac{1}{2}\phi(\Box + m^2)\phi - \frac{1}{2}\pi\Box\pi + \frac{\lambda}{2}\phi\pi^2 \tag{2.16}$$

Pictorically, the amplitude $\mathcal{M}(\phi \to \phi)$ at one loop can be represented as

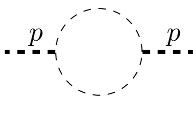


where to distinguish between ϕ and π the former, massive one is represented with a thick line, while the latter with a thinner one. In this process the only possible intermediate state, at one loop, is $|\pi\pi\rangle$. Thus applying the generalized Optical theorem one finds that

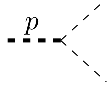
$$2\mathrm{Im}\mathcal{M}(\phi \to \phi) = \int \frac{d^3k_1}{(2\pi)^3(2E_1)} \frac{d^3k_2}{(2\pi)^3(2E_2)} (2\pi)^4 \delta^{(4)}(k_1 + k_2 - p) |\mathcal{M}(\phi \to \pi\pi)|^2 \tag{2.17}$$

.

Diagrammatically it results in

2Im 

$$= \int \frac{d^3k_1}{(2\pi)^3(2E_1)} \frac{d^3k_2}{(2\pi)^3(2E_2)} (2\pi)^4 \delta^{(4)}(k_1 + k_2 - p) \left| \quad \right|^2$$

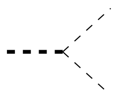


(2.18)

Here the connection between the two different level of the scattering is strikingly clear: a component of the one loop amplitude can be evaluated starting from tree level ones, which are often way simpler to evaluate.

In order to evaluate that integral, firstly it is possible to integrate over d^3k_2, to get rid of 3 out of the 4 delta functions. After this step, acknowledging that the tree level amplitude is

$$i\mathcal{M}(\phi \to \pi\pi) =$$



$$= i\lambda \tag{2.19}$$

the integral appears to be independent from the angular coordinate. In this case the integration over the angular variable can be factorized from the rest and performed easily. Lastly, moving to the center of mass reference frame one arrives to

$$2\mathrm{Im}\mathcal{M}(\phi \to \phi) = \frac{\lambda^2}{16\pi^2} \int d|k| d\Omega \delta(2|k| - M). \tag{2.20}$$

In the end, performing the integration one arrives to the final result

$$\mathrm{Im}\mathcal{M}(\phi \to \phi) = \frac{\lambda^2}{16\pi}. \tag{2.21}$$

■

Even though equation (2.14) is very useful, its application may seem not so intuitive. A simpler approach to evaluate the imaginary part of a diagram is presented by Cutkosky in [6]. He noticed that the term $d\Pi_x \delta^{(4)}(p_x - p_i)$ in the r.h.s. of the generalized optical theorem could be rewritten in a different way. Without loss of generality consider the case in which the state $|x\rangle$ contains l particles. In this case the term $d\Pi_x \delta^{(4)}(p_x - p_i)$ becomes

$$\frac{dk_1^3 \cdots dk_l^3}{(2\pi)^{3l} 2E_1 \cdots 2E_l} \delta^{(4)} \left(\sum_j^l k_j - p_i \right) = \frac{dk_1^3 \cdots dk_{l-1}^3}{(2\pi)^{3l} 2E_1 \cdots 2E_l} \delta \left(\sum_j^l E_j - E \right), \tag{2.22}$$

where E is the total energy of the incoming particles, while the integration over the momentum k_l has been used to simplify the tridimensional component of the delta function. Now it is possible to introduce $l-1$ terms, $dk_{0,j}\,\delta(k_{0,j}-E_j)$, without changing the result of the integral, since the momenta considered belong to particles that are already on shell. This leads to

$$d\Pi_x\,\delta^{(4)}(p_x-p_i)=\frac{dk_1^4\cdots dk_{l-1}^4}{(2\pi)^{3l}}\frac{\delta\left(k_{0,1}-E_1\right)\cdots\delta\left(\sum_j^l k_{0,j}-E\right)}{2E_1\cdots 2E_l}. \tag{2.23}$$

Now the measure of integration contains the ratio between many simple deltas and its zero. This expression recall a remarkable property of the delta function:

$$\sum_i \frac{\delta(x-x_i)}{f'(x_i)}=\delta(f(x)), \tag{2.24}$$

where the sum is over all the x_i such that $f(x_i)=0$. Using it in equation (2.23) allows to write the phase space measure in the following way:

$$d\Pi_x\,\delta^{(4)}(p_x-p_i)=\frac{dk_1^4\cdots dk_{l-1}^4}{(2\pi)^{3l}}\,\theta(k_{0,1})\delta(k_1^2-m_1^2)\cdots\theta(k_{0,l})\delta(k_l^2-m_l^2) \tag{2.25}$$

with $k_l=p_i-p_x$ as a consequence of momentum conservation.

The measure of integration written in this way is exactly the measure of a loop amplitude like the one appearing on the l.h.s. of the generalized Optical theorem, moreover the arguments of the deltas are the denominators of propagators that becomes intermediate states in the r.h.s. of (2.14). Such results can be generalized and applied to *any* amplitude.

This shows a different way to evaluate the imaginary part of a diagram called the *Cutkosky rule* equivalent to the generalized Optical theorem, which can be performed in a three steps fashion:

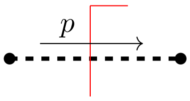
- cut through the diagram $\mathcal{M}$ in all the possible way that allows to put all the cut propagators on shell without violating momentum conservation .
- For each cut propagator, substitute $\frac{i}{p^2-m^2+i\varepsilon}\to(-2\pi i)\theta(p_0)\delta(p^2-m^2)$
- sum over all possible cut, thus obtaining its imaginary part:

$$2\mathrm{Im}\mathcal{M}=\sum_{cuts}\mathcal{M}_{cut} \tag{2.26}$$

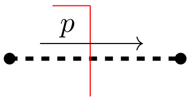
Diagrammatically, a cut propagator will be represented as a propagator cut by a dashed line. Nonetheless cut are directional: due to the presence of $\theta(p_0)$ they depend on the direction of the momentum flow. The direction of the second line on top of the cut represent the argument of the theta function, without it the representation of the cut would be ambiguous. When it points in the same direction of the flow of the momenta, the argument of the theta function will be the energy of the particle flowing in the propagator, otherwise it will be minus the energy of such particle.

Here there are different example of cut propagators, to get used to them.

- scalar

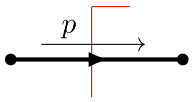


$$= (-2\pi i)\theta(p_0)\delta(p^2 - m^2)$$



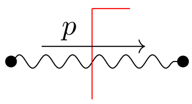
$$= (-2\pi i)\theta(-p_0)\delta(p^2 - m^2) \tag{2.27}$$

- fermion



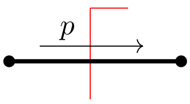
$$= (-2\pi i)(\not{p} + m)\theta(p_0)\delta(p^2 - m^2) \tag{2.28}$$

- photon

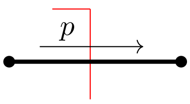


$$= (-2\pi i)(-g^{\mu\nu})\theta(p_0)\delta(p^2) \tag{2.29}$$

- in general



$$= (-2\pi i)\left(\sum_{pol} w.f.\right)\theta(p_0)\delta(p^2 - m^2)$$

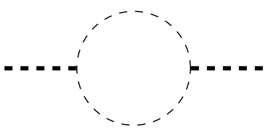
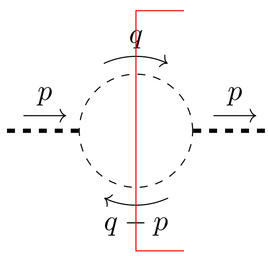


$$= (-2\pi i)\left(\sum_{pol} w.f.\right)\theta(-p_0)\delta(p^2 - m^2) \tag{2.30}$$

Using the new concept of *cuts*, the calculation of the imaginary part becomes more intuitive than before. In order to showcase such improvement, below is presented the example of the self energy of ϕ used also in the example of application for the generalized Optical theorem. Instead now the imaginary part of a diagram is evaluated through the *Cutkosky* rule.

Example

Applying equation (2.26) to the self energy of the massive scalar ϕ at one loop, diagrammatically one obtains:

2 Im  = 

(2.31)

which can be written as

$$2\,\text{Im}\mathcal{M}(\phi \to \pi) = \lambda^2 \int \frac{d^4q}{(2\pi)^2}\theta(q_0)\delta(q^2)\theta(p_0 - q_0)\delta((q-p)^2) \tag{2.32}$$

Applying the property of delta functions outlined in (2.24) and following the same steps performed in the example of application for the Optical theorem one finds

$$\begin{aligned} 2\,\text{Im}\mathcal{M}(\phi \to \pi) &= \lambda^2 \int \frac{d^4q}{(2\pi)^2 2|q|2|q-p|}\delta(q0 - |q|)\delta(p_0 - q_0 - |q-p|) \\ &= \frac{\lambda^2}{16\pi^2}\int \frac{d^3q}{|q|^2}\delta(M - 2|q|) \\ &= \frac{\lambda^2}{16\pi^2}\int d|q|d\Omega\delta(M - 2|q|) \end{aligned} \tag{2.33}$$

which is the result obtained in (2.20) using the generalized Optical theorem, thus confirming the validity of he Cutkosky rule. ■

2.2 Feynman Tree Theorem

In the previous section, thanks to the *Unitarity* of the S-matrix it was possible to draw a relation between amplitudes at different level in perturbation theory: for example, the imaginary part of a one loop diagram has been calculated starting from tree level ones. Nonetheless, with little more effort it is possible to bound more tightly the amplitude at any level to tree level ones, as stated in [7] and [8]. Such relation is exploited through studying the concept of *prescription*, the $+i\varepsilon$ appearing in the Feynman propagators.

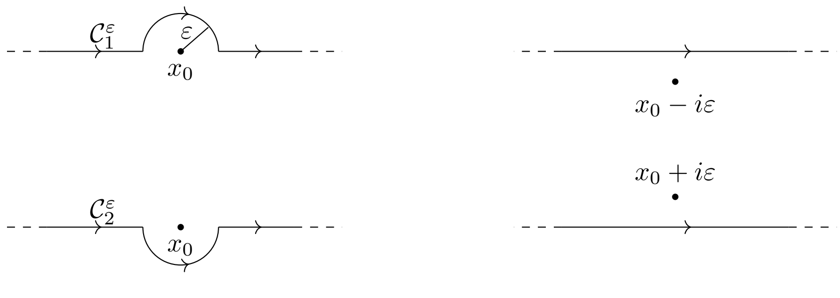
In order to draw such relation, let's consider the following integral:

$$\int dx \frac{f(x)}{x - x_0} \tag{2.34}$$

It is ill defined: it has a singularity on the path of integration, thus it's value isn't uniquely determined. Nonetheless it is possible to regulate it, by analytical continuation of the variable x in the complex plane. In such way it is possible to deform the path of integration so that it turns around the singularity avoiding it, or alternatively the singularity can be moved away from the path of integration. In the latter case it would be moved by a distance of ε with $\varepsilon \to 0^+$. These regularization scheme can be seen below: on the r.h.s. there is the latter, on the l.h.s. the former.

$$\int_{\mathcal{C}^\varepsilon_{1,2}} dx \frac{f(x)}{x - x_0} = \int dx \frac{f(x)}{x - x_0 \pm \varepsilon} \tag{2.35}$$

In this two cases the path of integration in relation to the position of the singularity is:



From here on, in order to regulate an integral which has a singularity along the path of integration, the pole will be shifted away from it.

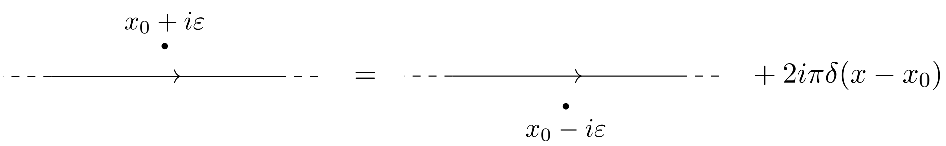
A remarkable fact is that moving the singularity above the integration path leads to a different results from moving it below of it. The difference is a complete loop around the pole, hence the value of its residue. This is clear if one compares the two prescription in integrals along the real axis, obtaining the relation

$$\int \frac{f(x)}{x - x_0 - i\varepsilon} = \int \frac{f(x)}{x - x_0 + i\varepsilon} + 2i\pi f(x_0) \tag{2.36}$$

which can be written as

$$\frac{1}{x - x_0 - i\varepsilon} = \frac{1}{x - x_0 + i\varepsilon} + 2i\pi\delta(x - x_0) \tag{2.37}$$

where the fractions appearing in it are considered as distributions. Portraying each prescription by the position of its pole and its path of integration, intuitively the previous equation becomes



(2.38)

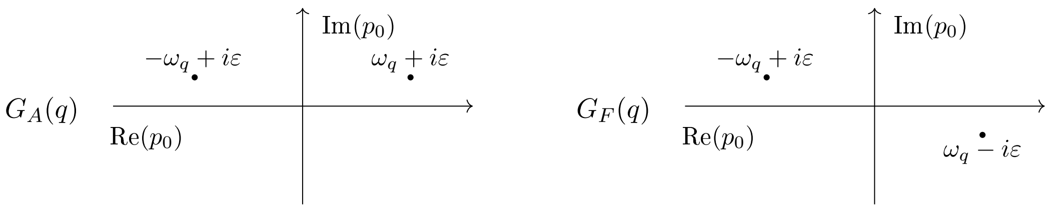
As it is clear by now it is possible to change the prescription with which an integral over a *simple* pole is regulated thanks to this relation. Also the Feynman propagator and the Advenced propagator have simple poles. In particular it is possible to write the Feynman propagator

$$G_F(q) = \frac{i}{q^2 + i\varepsilon} = \frac{i}{(q_0 + \omega_q - i\varepsilon)(q_0 - \omega_q + i\varepsilon)} \tag{2.39}$$

and Advanced propagator

$$G_A(q) = \frac{i}{q^2 - i\varepsilon\, sgn(q_0)} = \frac{i}{(q_0 - \omega_q - i\varepsilon)(q_0 + \omega_q - i\varepsilon)} \tag{2.40}$$

in this way, highlighting its singularity structure which can be represented on the complex plane as



(2.41)

The analogy with (2.38) is striking: two regulated integrals, with a pole in a different position. It should be possible to build a relation between the two propagators like the one previously seen. Using equation (2.38) on G_A, as it is written in (2.40), one obtains:

$$\begin{aligned} G_A(q) &= \frac{i}{(q_0+\omega_q-i\varepsilon)(q_0-\omega_q-i\varepsilon)} \\ &= \frac{i}{(q_0+\omega_q-i\varepsilon)}\left[\frac{1}{(q_0-\omega_q+i\varepsilon)}+2i\pi\delta(q_0-\omega_q)\right] \\ &= G_F(q)-\frac{2\pi\delta(q_0-\omega_q)}{2\omega_q} = G_F(q)-2\pi\theta(q_0)\delta(q^2) \end{aligned} \tag{2.42}$$

Arriving to the final relation

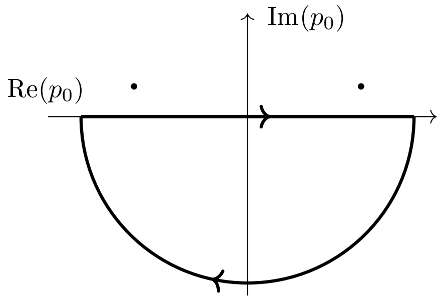
$$G_A(q) = G_F(q) - 2\pi\delta^{(+)}(q^2-m^2). \tag{2.43}$$

Where $\delta^{(+)}(q^2-m^2)$ is $\theta(q_0)\delta(q^2-m^2)$, hence the case with the positive loop momenta inside the theta function.

This relation acquire great relevance anytime there is an integration over the momentum appearing in the denominator. Indeed, in this case

$$\int_\Gamma dq_0\, G_A(q) = 0 \tag{2.44}$$

because it is possible to close the contour integration in the lower half plane, obtaining a closed circuit Γ which has no singularity within it and hence the integral turns out to be 0. Here there is the path of integration:

$\Gamma =$ 

(2.45)

Generalizing this idea to integrals with more denominators, the result doesn't change since all poles lies on the upper complex half plane, exactly as in the case of only one propagator. Hence one can write

$$\begin{aligned} 0 &= \int \frac{d^Dq}{(2\pi)^D}\mathcal{N}(q)\prod_i G_A^{(i)}(q-p_1-\cdots-p_i) \\ &= \int \frac{d^Dq}{(2\pi)^D}\mathcal{N}(q)\prod_i \left\{G_F^{(i)}(q-p_1-\cdots-p_i) - 2\pi\delta^{(+)}((q-p_1-\cdots-p_i)^2-m^2)\right\} \end{aligned} \tag{2.46}$$

thanks to (2.43), as stated in [8]. $\mathcal{N}(q)$ stands for the numerator of the Feynman amplitude, which can depends on the loop momenta.

Expanding the product in the right hand side and isolating the term that contains only Feynman propagators (which is the amplitude), one obtains

$$\begin{aligned}\mathcal{M} &= \mathcal{M}_{1-cut} - \mathcal{M}_{2-cut} + \cdots + (-1)^{n-1}\mathcal{M}_{n-cut} \\ &= \sum_{j}^{n} (-1)^{j-1}\mathcal{M}_{j-cut}\end{aligned} \tag{2.47}$$

in the case of an amplitude with n internal lines. In order to better understand the relation found here, it is useful to recall the fact that a cut loop amplitude can be seen as a tree level amplitude, integrated over the phase space of the particles flowing in the cut propagator, as one can deduce comparing the example of application of the Optical theorem and the Cutkosky rule. Hence applying this procedure to an l-loop diagram allows to write it as a sum of at least $l-1$ loop diagrams. It is possible then to apply the procedure to the $l-1$ diagram and so on and so on, until one is able to express any loop diagram as the sum of tree level diagram integrated over phase space of the particle flowing in the cut propagators. This procedure is called the *Feynman Tree Theorem*, getting its name from this interpretation of its result.

Example

In order to showcase the application of this theorem, it will be used to evaluate the self energy of the scalar ϕ, presented in the examples outlined before. To perform the calculation it will be used the dimensional regularization scheme, setting $D = 4 - 2\epsilon$. In this particular scheme the amplitude of the self energy becomes

$$A = \lambda^2 \int \frac{d^D q}{(2\pi)^D} \frac{1}{(q^2 + i\varepsilon)[(q-p)^2 + i\varepsilon]} \tag{2.48}$$

Firstly, this diagram will be evaluated using the Feynman parameterization, in order to have a value that allows to check whether the result obtained through the Feynman tree theorem is correct or no.

Feynman parametrization method

Let's recall this useful parameterization briefly. It allows to express rational function in terms of an integral:

$$\frac{1}{A_1 A_2 \cdots A_n} = \Gamma(n) \int_0^1 dx_1 \cdots \int_0^{1-x_1-\cdots-x_{n-2}} dx_{n-1} \frac{1}{[A_1 x_1 + \cdots + A_{n-1}x_{n-1} + A_n(1 - x_1 - \cdots - x_n)]^n} \tag{2.49}$$

Applying it in our case leads to

$$A = \lambda^2 \int_0^1 dx \int \frac{d^D q}{(2\pi)^D} \frac{1}{[(q-px)^2 + M^2 x(1-x)]^2} \tag{2.50}$$

which, shifting the loop momentum, becomes

$$A = \lambda^2 \int \frac{d^D q}{(2\pi)^D} \frac{1}{[q^2 + M^2 x(1-x)]^2} \tag{2.51}$$

It is possible to evaluate this integral thanks to the Wick rotation and by extending the solid angle to D dimensions in the following way:

$$\Omega_{D-1} = \frac{2\pi^{D/2}}{\Gamma(D/2)} \tag{2.52}$$

Knowing that, one obtain the general relation:

$$I_{r,m} = \int \frac{d^D q}{(2\pi^D)} \frac{(q^2)^r}{(q^2 - C)^m} = iC^{r-m+\frac{D}{2}} \frac{(-1)^{r-m}}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(r+\frac{D}{2})}{\Gamma(\frac{D}{2})} \frac{\Gamma(m-r-\frac{D}{2})}{\Gamma(m)} \tag{2.53}$$

that allows to rewrite the amplitude as

$$A = \lambda^2 I_{0,2} \tag{2.54}$$

with

$$C = -M^2 x(1-x) \tag{2.55}$$

Lastly, integrating over the loop momentum gives

$$\begin{aligned} A &= i\lambda^2 \frac{(-M^2)^{-\epsilon}}{(4\pi)^{2-\epsilon}} \Gamma(\epsilon) \int_0^1 dx (x(1-x))^{-\epsilon} \\ &= i \frac{\lambda^2 (-M^2)^{-\epsilon}}{(4\pi)^{2-\epsilon}} \frac{\Gamma(\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(2-2\epsilon)} \end{aligned} \tag{2.56}$$

Feynman Tree theorem method

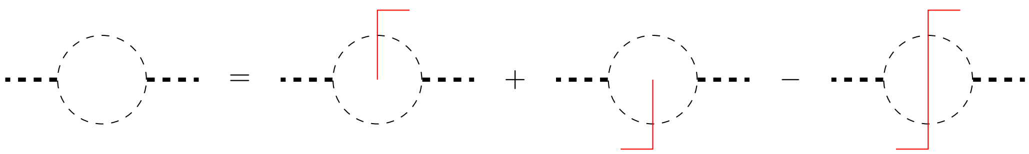
Since the amplitude in this case has two denominators, the application of the Feynman tree theorem will generate the following relation:

$$A = A_1 + A_2 - A_{12} \tag{2.57}$$

with

$$\begin{aligned} A_1 &= \lambda^2 \int \frac{d^D q}{(2\pi)^{D-1}} \theta(q_0)\delta(q^2) G_F(q-p) = \lambda^2 \int \frac{d^D q}{(2\pi)^{D-1}} \delta^{(+)}(q^2) G_F(q-p) \\ A_2 &= \lambda^2 \int \frac{d^D q}{(2\pi)^{D-1}} G_F(q)\theta(q_0 - p_0)\delta((q-p)^2) = \lambda^2 \int \frac{d^D q}{(2\pi)^{D-1}} G_F(q)\delta^{(+)}((q-p)^2) \\ A_{12} &= \lambda^2 \int \frac{d^D q}{(2\pi)^{D-2}} \theta(q_0)\theta(q_0-p_0)\delta(q^2)\delta((q-p)^2) = \lambda^2 \int \frac{d^D q}{(2\pi)^{D-2}} \delta^{(+)}(q^2)\delta^{(+)}((q-p)^2) \end{aligned} \tag{2.58}$$

To clarify the notation used, the amplitude A_i has a cut over the propagator of momentum q_i, with $q_1 = q$ and $q_2 = q - p$, while A_{12} represent the double-cut amplitude. The relation written in equation (2.57) graphically is

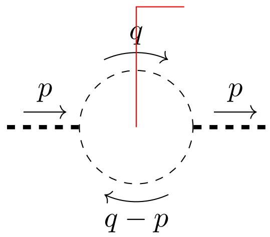


(2.59)

Cutting over $\boldsymbol{q}$. In this case the amplitude is

$$A_1 = i\lambda^2 \int \frac{d^D q}{(2\pi)^{D-1}} \frac{\delta(q_0 - |\boldsymbol{q}|)}{2|\boldsymbol{q}|[(q_0 - p_0)^2 - (\boldsymbol{q} - \boldsymbol{p})^2 + i\varepsilon]} \tag{2.60}$$

where $\boldsymbol{q}$ and $\boldsymbol{p}$ are the $D-1$ dimensional euclidean component of the respective D dimensional Minkowskian vector. It can be represented as



Performing some manipulation, it is possible to write the integral as

$$\begin{aligned} A_1 &= i\lambda^2 \int \frac{d^D q}{2(2\pi)^{D-1}} \frac{\delta(q_0 - q)}{q[(q_0 - M)^2 - p^2 + i\varepsilon]} \\ &= i\lambda^2 \int \frac{d^{D-1} q}{2(2\pi)^{D-1}} \frac{1}{q[(q - M)^2 - q^2 + i\varepsilon]} \\ &= i\lambda^2 \frac{\Omega_{D-2}}{2(2\pi)^{D-1}} \int_0^{+\infty} dq \, \frac{q^{D-3}}{M^2 - 2Mq + i\varepsilon} \end{aligned} \tag{2.61}$$

Setting $D = 4-2\epsilon$, using the explicit formula for the multidimensional angular factor presented in (2.52), and an identity of the gamma function,

$$\Gamma(z + 1/2) = 2^{1-2z}\sqrt{\pi}\frac{\Gamma(2z)}{\Gamma(z)} \tag{2.62}$$

, it is possible to express the prefactor in the following way:

$$\frac{\Omega_{D-2}}{2(2\pi)^{D-1}} = \frac{1}{(4\pi)^{3/2-\epsilon}\Gamma(3/2-\epsilon)} = \frac{1}{4\pi^{1-\epsilon}} \frac{\Gamma(1-\epsilon)}{\Gamma(2-2\epsilon)} \tag{2.63}$$

Then, substituting $|\boldsymbol{q}| = \frac{M}{2} q'$ leads to

$$A_1 = i\lambda^2 \frac{(M^2)^{-\epsilon}}{4\pi^{1-\epsilon}} \frac{\Gamma(1-\epsilon)}{\Gamma(2-2\epsilon)} \int_0^{+\infty} dq' \, \frac{q'^{1-2\epsilon}}{1 - q' + i\varepsilon} \tag{2.64}$$

Notice that

$$\int_0^{+\infty} dq' \, \frac{q'^{1-2\epsilon}}{1 - q' + i\varepsilon} = -\int_{-\infty}^{0} dq' \, \frac{q'^{1-2\epsilon}}{1 - q'} \tag{2.65}$$

where on the r.h.s. the $i\varepsilon$ prescription have been dropped since the pole isn't on the path of integration anymore.

Performing now a change of variables, defining $q' = \frac{u-1}{u}$, the amplitude takes the following shape

$$A_1 = i\lambda^2 \frac{(-1)^{-2\epsilon}(M^2)^{-\epsilon}}{4\pi^{1-\epsilon}} \frac{\Gamma(1-\epsilon)}{\Gamma(2-2\epsilon)} \int_0^1 du \, u^{2\epsilon-2}(1-u)^{1-2\epsilon} \tag{2.66}$$

The integral appearing in it is the notorious Euler's beta function. This particular class of functions are defined as:

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt \tag{2.67}$$

Nonetheless the Euler beta function is connected to another, more common, class of functions (the Gamma functions) by the identity

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{2.68}$$

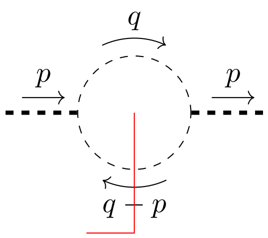
turning A_1 to

$$\begin{aligned} A_1 &= i\lambda^2 \frac{(-1)^{-2\epsilon}(M^2)^{-\epsilon}}{4\pi^{1-\epsilon}} \frac{\Gamma(1-\epsilon)}{\Gamma(2-2\epsilon)} B(2\epsilon, 1-2\epsilon) \\ &= i\lambda^2 \frac{(-1)^{-2\epsilon}(M^2)^{-\epsilon}}{4\pi^{1-\epsilon}} \frac{\Gamma(1-2\epsilon)\Gamma(2\epsilon)\Gamma(1-\epsilon)}{\Gamma(2-2\epsilon)} \end{aligned} \tag{2.69}$$

Cutting over $\boldsymbol{q-p}$. The second cut amplitude is

$$A_2 = i\lambda^2 \int \frac{d^D q}{(2\pi)^{D-1}} \frac{\delta(q_0 - p_0 - \omega_{q-p})}{2\omega_{q-p}(q^2 + i\varepsilon)} \tag{2.70}$$

and could be represented as



Following the same step done for A_1 one arrives to

$$A_2 = i\lambda^2 \frac{(M)^{-2\epsilon}}{(4\pi)^{1-\epsilon}} \frac{\Gamma(1-\epsilon)}{\Gamma(2-2\epsilon)} \int_0^{+\infty} dq' \, \frac{q'^{1-2\epsilon}}{1+q'} \tag{2.71}$$

This time, in order to link the integral appearing in the cut amplitude to the definition of beta function given in (2.68), ones need to define $q' = \frac{u}{1-u}$ obtaining

$$A_2 = i\lambda^2 \frac{(M)^{-2\epsilon}}{4\pi^{1-\epsilon}} \frac{\Gamma(1-\epsilon)}{\Gamma(2-2\epsilon)} \int_0^1 du \, u^{1-2\epsilon}(1-u)^{2\epsilon-2} \tag{2.72}$$

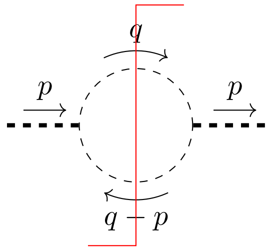
finding that

$$A_2 = (-1)^{2\epsilon} A_1 \tag{2.73}$$

Doublecut. Lastly, the double cut is

$$A_{12} = \lambda^2 \int \frac{d^D q}{(2\pi)^{D-2}} \frac{\delta(q_0 - |\boldsymbol{q}|)\delta(q_0 - p_0 - |\boldsymbol{q} - \boldsymbol{p}|)}{4|\boldsymbol{q}||\boldsymbol{q} - \boldsymbol{p}|} \tag{2.74}$$

and is represented as



specializing again to the center of mass frame one finds

$$\begin{aligned} A_{12} &= \lambda^2 \int \frac{d^D q}{(2\pi)^{D-2}} \theta(q_0)\theta(q_0 - p_0)\delta(q^2)\delta((q-p)^2) \\ &= \lambda^2 \int \frac{d^D q}{(2\pi)^{D-2}} \frac{\delta(q_0 - q)\delta(q_0 - M - q)}{4q^2} = 0 \end{aligned} \tag{2.75}$$

which is 0 because the two delta function can't be fulfilled at once in the case of $M \neq 0$, while for $M = 0$ it is a scaleless integral, hence is zero in dimensional regularization.

A remarkable fact is that this double cut is different from the one obtained via the Cutkosky rule, written in (2.32), having different argument inside the θ function. This is due to the fact that this two techniques stem from different first principle: the Cutkosky rule comes from the *unitarity* of the S-matrix, while the Feynman Tree theorem is obtained via *prescription* manipulation.

In the end, the total amplitude is obtained by the contribution of the two single cut:

$$\begin{aligned} A_{tot} &= A_1 + A_2 = A_1(1 + (-1)^{2\epsilon}) = iA_1(-1)^\epsilon(e^{i\pi\epsilon} + e^{-i\pi\epsilon}) = -2iA_1(-1)^\epsilon \cos(\pi\epsilon) \\ &= i\lambda^2 \frac{(-M^2)^{-\epsilon}}{(4\pi)^{2-\epsilon}} \frac{\Gamma(1-2\epsilon)\Gamma(2\epsilon)\Gamma(1-\epsilon)}{\Gamma(2-2\epsilon)} 2\cos(\pi\epsilon) \end{aligned} \tag{2.76}$$

At this point in order to simplify the expression of the total amplitude it is useful to recall the Euler reflection formula for the Gamma functions. It states that

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \tag{2.77}$$

Using it in the expression of the total amplitude one obtains

$$\begin{aligned} A_{tot} &= i\lambda^2 \frac{(-M^2)^{-\epsilon}}{(4\pi)^{2-\epsilon}} \frac{\Gamma(1-\epsilon)}{\Gamma(2-2\epsilon)} 2 \frac{\pi\cos(\pi\epsilon)}{\sin(2\pi\epsilon)} \\ &= i\lambda^2 \frac{(-M^2)^{-\epsilon}}{(4\pi)^{2-\epsilon}} \frac{\Gamma(1-\epsilon)}{\Gamma(2-2\epsilon)} \frac{\pi}{\sin(\pi\epsilon)} \\ &= i\lambda^2 \frac{(-M^2)^{-\epsilon}}{(4\pi)^{2-\epsilon}} \frac{\Gamma(\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(2-2\epsilon)} \end{aligned} \tag{2.78}$$

, matching the results given by the calculation with the Feynman parameters, reported in (2.56). ■

2.3 Generalized Unitarity

The problem of a systematic approach to the solution of Feynman integrals at one loop will be addressed in this section.

Due to the great number of theories that one can build within the framework of QFT, the number of possible Feynman integrals is countless. Finding relation among them would simplify greatly the quest of the calculation of the loop correction to a given process.

2.3.1 Tensor decomposition

A great tool for starting this calculation is given by the Passarino-Veltman tensor reduction scheme [9]. Through this technique it is possible to separate the Feynman amplitude from its Dirac and Lorentz component, decomposing it into a set of scalar integrals. In order to understand how it works, let's write a generic amplitude with the polarization vector explicit:

$$\mathcal{M} = \epsilon_{1,\alpha_1} \cdots \epsilon_{m,\alpha_m} \mathcal{M}^{\alpha_1 \cdots \alpha_m} \tag{2.79}$$

The Lorentz index appearing on $\mathcal{M}^{\alpha_1 \cdots \alpha_m}$ can come only from the loop momenta. Hence, defining the tensor $Q^{\alpha_1 \cdots \alpha_m} = q^{\alpha_1} \cdots q^{\alpha_m}$, without loss of generality we can write

$$\mathcal{M}^{\alpha_1 \cdots \alpha_m} = I\left[Q^{\alpha_1 \cdots \alpha_m}\right] \equiv \int d^D q \frac{Q^{\alpha_1 \cdots \alpha_m}}{D_1 \cdots D_n} \tag{2.80}$$

where D_i are the denominators of the amplitude. Due to the Lorentz structure of the amplitude, it is possible to write an ansatz for $I\left[Q^{\alpha_1 \cdots \alpha_m}\right]$ in terms of the independent Lorentz vectors and tensors and of the possible Dirac structures:

$$I\left[Q^{\alpha_1 \cdots \alpha_m}\right] = \sum_i c_i T_i^{\alpha_1 \cdots \alpha_m} \tag{2.81}$$

The type of tensors entering in T_i depends on the number of index. For example

- in the case of 1 Lorentz index, the only possible vectors available are the external momenta p_i^α.
- in the case of 2 Lorentz index, T_i can be a couple of external momenta, $p_i^{\alpha_1} p_j^{\alpha_2}$, or the metric tensor $g^{\alpha_1\alpha_2}$.
- in the case of a tensor with 3 Lorentz index, T_i can be made by Lorentz vectors only, $p_i^{\alpha_1} p_j^{\alpha_2} p_k^{\alpha_3}$, or by a combination of a Lorentz vector with the metric tensor $g^{\alpha_1\alpha_2} p^{\alpha_3}$ for example.
- in the case of 4 Lorentz index, the number of possibilities to build T_i start to grows. Other than a combination of Lorentz vector, $p_i^{\alpha_1} p_j^{\alpha_2} p_k^{\alpha_3} p_l^{\alpha_4}$ and a combination of Lorentz vectors and metric tensor, $g^{\alpha_1\alpha_2} p_k^{\alpha_3} p_l^{\alpha_4}$,it is possible to build a tensor with four Lorentz index also by combining metric tensors together, obtaining for example $g^{\alpha_1\alpha_2} g^{\alpha_3\alpha_4}$.

In order to know the shape of the amplitude with its Lorentz structure factorized, it is necessary to evaluate all the different c_i. To do so, it is useful to perform a *projection* by contracting all the indices appearing in both sides of equation (2.81) with different tensors T_j. In this way it is possible to build a system of equation of the form

$$I\left[Q \cdot T_j\right] = \sum_i c_i T_i \cdot T_j \tag{2.82}$$

running on j that labels all the possible tensors that can be used to contract the indices appearing in (2.81). Inverting such system it is possible to determine all the scalar constants, finding that

$$c_i = \sum_j b_{ij} I\left[Q \cdot T_j\right] \tag{2.83}$$

Thus, substituting it back in the formula of the amplitude, defining $\mathcal{E}_{\alpha_1 \cdots \alpha_m} = \epsilon_{1,\alpha_1} \cdots \epsilon_{m,\alpha_m}$, one finds that

$$\mathcal{M} = \sum_{i,j} \mathcal{E} \cdot T_i \, b_{ij} \, I\left[Q \cdot T_j\right]. \tag{2.84}$$

$Q \cdot T_j$ contains all the possible contraction between the loop momentum and the tensors T_j. As seen in the examples made before, T_j can contain at most combination of external momenta and metric tensors, hence it is possible to write the numerator of the integral in a general form:

$$Q \cdot T_j = \left(q^2\right)^a \prod_i (q \cdot p_i)_i^b. \tag{2.85}$$

As shown, through the Passarino-Veltman tensor decomposition it is possible to write a generic amplitude in function of a set of scalar integrals that has, as numerator, scalar products between the loop momentum and the external momenta or powers of the loop momentum.

In order to further simplify the general possible expression of a given Feynman amplitude, it is mandatory to better understand its structure in terms of scalar products and denominators. First of all, in the case of an n particle process at one loop, the scalar Feynman integral can be written as

$$\int \frac{d^4 q}{(2\pi)^4} \frac{N}{D_1^{x_1} \cdots D_n^{x_n}} \tag{2.86}$$

, hence has n denominators. These denominators can be written, in the Euclidean space, as

$$D_i = (q + V_i)^2 + m_i^2 \qquad V_i = \sum_j^i p_j \qquad V_n = 0 \tag{2.87}$$

Substituting the explicit form of V_i inside the expression of the denominator and performing the square of the momenta flowing in the corresponding propagator, it is clear that the denominator can always be written as a combination of scalar products, since the outcome of such procedure is:

$$D_i = q^2 + 2 \sum_i^j q \cdot p_j + \sum_{j,k}^i p_j \cdot p_k + m_i^2. \tag{2.88}$$

Noticing that in an amplitude with n external leg there are $n - 1$ independent momenta due to the momentum conservation condition ($\sum_i^n p_i = 0$), the possible scalar product that involves the loop momentum in an n point function are

$$q^2, q \cdot p_i \qquad i = 1, \ldots, n-1 \tag{2.89}$$

for a total of n independent scalar product, the same as the number of denominators. At one loop hence all scalar product are *reducible*, that means that all scalar products can be expressed in terms of denominators and external variables (such as $p_i^2, p_i \cdot p_j, m_i^2$). For example, using the definition of the denominators given in (2.87), it is possible to write

$$q \cdot p_i = \frac{1}{2}(D_i - D_{i-1} + a) \tag{2.90}$$

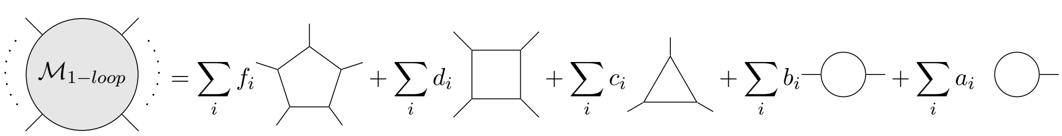
with a made of external variables. Going back to equation (2.84), one now notice that it is possible to write a Feynman amplitude in function of integral with 1 at the numerator, with their value depending only on their denominators. In particular the amplitude will be a rational function of scalar products, and it will be a function of at most 5 independent variable. This is due to the fact that not all the scalar product are independent. Since the external momenta are 4-dimensional vectors, there will be at most 4 independent scalar product like $q \cdot p_i$. On top of that, we can write the D dimensional loop momenta as

$$q^\alpha = q^\alpha_{[4]} + \mu^\alpha \tag{2.91}$$

decomposing it in two orthogonal components: $q^\alpha_{[4]}$, the four dimensional one, and μ^α, the $D-4$ dimensional component. Let us notice that

$$q^2 = q^2_{[4]} + \mu^2 \tag{2.92}$$

with $q^\alpha_{[4]}$ being dependent on the other 4 dimensional scalar product. In this way, any Feynman amplitude at one loop, in dimensional regularization, will at most depend on 5 scalar product (4 that are 4 dimensional, plus one extra dimensional, μ^2), and hence it will depends on at most 5 independent denominators. This property can be portrayed in a fancy way:



(2.93)

2.3.2 Fit on the cut

In order to know the whole Feynman amplitude, it is mandatory to evaluate both the integral in which we can reduce it (the integral with 1 at the numerator and at most 5 independent denominators portrayed in equation (2.93), the *master integrals*) *and* determine the coefficients in front of them. This thesis won't cover how those integrals are evaluated. Instead an efficient way to determine all the coefficients will be shown.

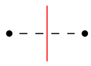
The fit-on-the-cut approach was proposed in Ref. [10] for one-loop amplitudes. As the name suggests, this technique uses the idea of *cut* in order to determine the coefficient of the reduction outlined in (2.93). This technique ,in which one put on-shell certain propagators, was firstly encountered working with *Unitarity*. There, we found that in order to apply the constraint due to the *Unitarity* of the S-matrix, it was necessary to *cut* the diagram taken in to account. As shown in equation

(2.31), through the Cutkosky rule, the *cuts* were used to *project* the amplitude on its imaginary part.

Here in the same way, the attempt is to use *cuts* to *project* the reduced amplitude on a single contribution in order to extract its coefficient, in what is called the *Generalize Unitarity approach*, since it takes inspiration from the effects of *Unitarity* of the scattering matrix.

The idea is to use the *cut* to put propagators on shell, hence applying the following substitution:

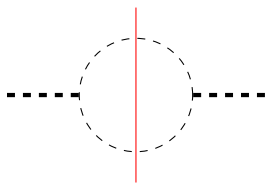
$$\frac{1}{D} \quad \to 2\pi \quad \delta(D)$$

 $\to 2\pi$  (2.94)

where the *generalized cut* is portrayed as a straight red line since it hasn't the θ function in it, differently from the cut appearing in the Cutkosky-Veltman rule.

Example

Here, as an example, the double cut of the 1 loop bubble function written in (2.48) will be calculated. Graphically it is



and its value is

$$A = \lambda^2 \int \frac{d^D q}{(2\pi)^{D-2}} \delta(q^2)\delta((q-p)^2) \tag{2.95}$$

Through the property of the delta function outlined in (2.24) and moving into the center of mass reference frame one arrives to

$$A = \lambda^2 \int \frac{d^D q}{(2\pi)^{D-2}} \frac{\delta(q_0 - |\boldsymbol{q}|) + \delta(q_0 + |\boldsymbol{q}|)}{2|\boldsymbol{q}|} \frac{\delta(q_0 - M - |\boldsymbol{q}|) + \delta(q_0 - M + |\boldsymbol{q}|)}{2|\boldsymbol{q}|} \tag{2.96}$$

Integrating over dq_0 and factorizing the integration measure leads to

$$\begin{aligned} A &= \lambda^2 \int \frac{d|\boldsymbol{q}||\boldsymbol{q}|^{D-4} d\Omega_{D-2}}{4(2\pi)^{D-2}} \left[\delta(-M) + \delta(-M - 2|\boldsymbol{q}|) + \delta(-M) + \delta(2|\boldsymbol{q}| - M)\right] \\ &= \frac{\lambda^2}{4\pi^{1-\epsilon}} \frac{\Gamma(1-\epsilon)}{\Gamma(2-2\epsilon)} \int d|\boldsymbol{q}||\boldsymbol{q}|^{D-4} \delta(2|\boldsymbol{q}| - M) \\ &= \frac{\lambda^2 (M^2)^{-\epsilon}}{2^{3-2\epsilon}\pi^{1-\epsilon}} \frac{\Gamma(1-\epsilon)}{\Gamma(2-2\epsilon)} \end{aligned} \tag{2.97}$$

which, in this particular case, is the D dimensional result for the Cutkosky-Veltman rule, as one can find comparing this result to the 4 dimensional one portrayed in

(2.33). Let's remark that in general, the result of a *generalized* cut is different from the Cutkosky-Veltman cut. ■

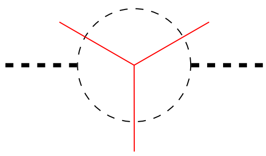
When a cut over a denominator appearing in the integral is performed, it's meaning is clear. But what happens when the amplitude has a cut over a denominator that doesn't appear in it? Take for instance a bubble (two denominators function) that has 3 cuts over 3 independent denominators. Suppose that our amplitude has $D_1 = q^2$ and $D_2 = (q - p_1)^2$ as denominators, and one want to cut it over D_1, D_2 and $D_3 = (q + p_2)^2$, with p_2 linearly independent from p_1. To perform the cut one has to "create" D_3. In order to do so, one multiply and divide by it:

$$A = \int \frac{d^D q}{(2\pi)^{D-3}} \frac{1}{D_1 D_2} = \int \frac{d^D q}{(2\pi)^{D-3}} \frac{D_3}{D_1 D_2 D_3}. \tag{2.98}$$

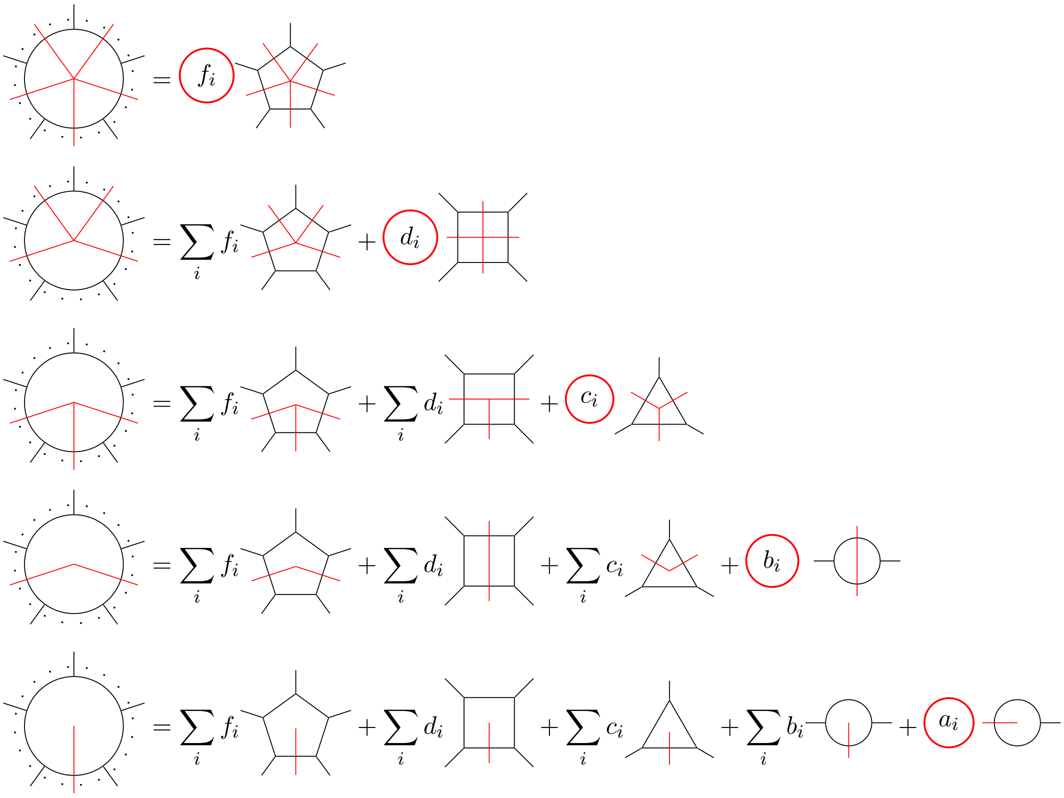
Performing the triple cut now one obtains

$$A_{cut} = \int \frac{d^D q}{(2\pi)^{D-3}} \delta(D_1)\delta(D_2)\delta(D_3) D_3 = 0 \tag{2.99}$$

that is clearly null. Pictorially, it could be represented as

 $= 0$ (2.100)

where the red lines stand for the number of cuts that one has to apply to the amplitude, they are not meant in the traditional convention used in this thesis. In this sense the *generalized* cut acts as a projector. It *project* an amplitude on to the diagram that contains denominators that appear in the cut, or more. For instance, performing a 5 cut on an amplitude, will kill all the integrals with 4 or less denominators that composes it . Cutting 4 time a one loop amplitude will kill the 3 or less denominators function, and all the 4 denominators function that doesn't have one of the cut propagators. In this way, starting from the 5 cut and going down until the single cut, one can build a triangular system, isolating and then determining all the coefficients. Applying this algorithm to (2.93) one obtains, pictorially:



(2.101)

, in this way one can determines all the coefficients appearing in the decomposition in Master Integrals at one loop.

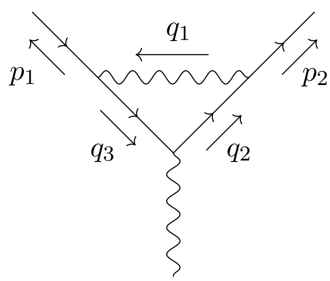
2.4 Application: 1-Loop 3-point amplitudes QED

In this chapter an application of the Feynman Tree theorem will be presented. The amplitude considered is the QED three point function at one loop in the massless spinors limit. The calculation of such amplitude will be computed in two ways: the first time via the well known Feynman parameterization, the second time using the Feynman tree theorem so that it is possible to check the result obtained in the latter framework with the one resulting from the former.

The amplitude considered is:

$$A(p_1, s_1, p_2, s_2, \lambda) = e^3 \mu^{4-D} \int \frac{d^D q_1}{(2\pi)^D} \frac{\bar{u}(p_2, s_2)\gamma_\alpha \not{q}_2 \not{\epsilon}(\lambda) \not{q}_3 \gamma^\alpha u(p_2, s_2)}{(q_1^2 + i\varepsilon)(q_2^2 + i\varepsilon)(q_3^2 + i\varepsilon)} \tag{2.102}$$

and can be represented as the following Feynman diagram:



2.4.1 Feynman parametrization method

In this example, we will adopt the same notation used in [11], in order to compare easily the result drawn by the two calculations.

In order to integrate over q_1 it is necessary to write q_2 and q_3 in function of the former. The relation between the momenta given by the momentum conservation are

$$\begin{aligned} q_2 &= q_1 + p_2 \\ q_3 &= q_1 - p_1 \end{aligned} \tag{2.103}$$

Substituting them in (2.102) one obtains

$$A(p_1, s_1, p_2, s_2, \lambda) = e^3\mu^{4-D} \int \frac{d^D q_1}{(2\pi)^D} \frac{\bar{u}(p_2)\gamma_\alpha(\not{q}_1 + \not{p}_2)\not{\epsilon}(\lambda)(\not{q}_1 - \not{p}_1)\gamma^\alpha u(p_1)}{(q_1^2 + i\varepsilon)((q_1 + p_2)^2 + i\varepsilon)((q_1 - p_1)^2 + i\varepsilon)} \tag{2.104}$$

In order to simplify this expression an identity that contains gamma matrices is needed:

$$\gamma_\alpha\gamma_\rho\gamma_\beta\gamma_\sigma\gamma^\alpha = (4 - D)\gamma_\rho\gamma_\beta\gamma_\sigma - 2\gamma_\sigma\gamma_\beta\gamma_\rho \tag{2.105}$$

in D dimensions. On top of that, another useful relation is given by the Dirac equation that in the massless limits gives $\bar{u}(p_2)\not{p}_2 = 0$ and $\not{p}_1 u(p_1) = 0$. Using them in the total amplitude leads to

$$A(p_1, s_1, p_2, s_2, \lambda) = e^3\mu^{4-D} \int \frac{d^D q_1}{(2\pi)^D} \frac{\bar{u}(p_2)[(2 - D)\not{q}_1\not{\epsilon}\not{q}_1 + 2\not{p}_1\not{\epsilon}\not{q}_1 - 2\not{q}_1\not{\epsilon}\not{p}_2 + 2\not{p}_1\not{\epsilon}\not{p}_2]u(p_1)}{(q_1^2 + i\varepsilon)((q_1 + p_2)^2 + i\varepsilon)((q_1 - p_1)^2 + i\varepsilon)} \tag{2.106}$$

Using the Feynman parameterization in this case, with $n = 3$, gets to:

$$\begin{aligned} A_{tot} &= 2e^3\mu^{4-D} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int \frac{d^D q_1}{(2\pi)^D} \frac{\bar{u}(p_2)[(2 - D)\not{q}_1\not{\epsilon}\not{q}_1 + 2\not{p}_1\not{\epsilon}\not{q}_1 - 2\not{q}_1\not{\epsilon}\not{p}_2 + 2\not{p}_1\not{\epsilon}\not{p}_2]u(p_1)}{((q_1^2 - 2q_1p_1)x_1 + (q_1^2 + 2q_1p_2)x_2 + q_1^2(1 - x_1 - x_2))^3} \\ &= 2e^3\mu^{4-D} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int \frac{d^D q_1}{(2\pi)^D} \frac{\bar{u}(p_2)[(2 - D)\not{q}_1\not{\epsilon}\not{q}_1 + 2\not{p}_1\not{\epsilon}\not{q}_1 - 2\not{q}_1\not{\epsilon}\not{p}_2 + 2\not{p}_1\not{\epsilon}\not{p}_2]u(p_1)}{((q_1 - p_1x_1 + p_2x_2)^2 + s_{12}x_1x_2)^3} \end{aligned} \tag{2.107}$$

Then defining $q_1 = q + p_1x_1 - p_2x_2$ leads to:

$$A_{tot} = 2e^3\mu^{4-D} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int \frac{d^D q}{(2\pi)^D} \frac{N(q, p_1, p_2)}{(q^2 + s_{12}x_1x_2)^3} \tag{2.108}$$

$$\tag{2.109}$$

After some algebra the numerator becomes

$$\begin{aligned} N(q, p_1, p_2) =& \bar{u}(p_2)\{(2 - D)\not{q}\not{\epsilon}\not{q} + [(2 - D)x_1 + 2]\not{p}_1\not{\epsilon}\not{q} - [(2 - D)x_2 + 2]\not{q}\not{\epsilon}\not{p}_2 + [2 - 2x_1 - 2x_2 + \\ &+ (D - 2)x_1x_2]\not{p}_1\not{\epsilon}\not{p}_2\}u(p_1) \end{aligned} \tag{2.110}$$

The total amplitude hence is

$$\begin{aligned} A_{tot} =& e^3\mu^{4-D} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \bar{u}(p_2)\{(2 - D)I^{\rho\sigma}\gamma_\rho\not{\epsilon}\gamma_\sigma + [(2 - D)x_1 + 2]I^\rho\not{p}_1\not{\epsilon}\gamma_\rho \\ & - [(2 - D)x_2 + 2]I^\rho\gamma_\rho\not{\epsilon}\not{p}_2 + [2 - 2x_1 - 2x_2 + (D - 2)x_1x_2]I\not{p}_1\not{\epsilon}\not{p}_2\}u(p_1) \end{aligned} \tag{2.111}$$

with

$$I^{\rho\sigma} = 2\int \frac{d^D q}{(2\pi)^D}\frac{q^\rho q^\sigma}{(q^2+s_{12}x_1x_2)^3} \tag{2.112}$$

$$I^{\rho} = 2\int \frac{d^D q}{(2\pi)^D}\frac{q^\rho}{(q^2+s_{12}x_1x_2)^3} \tag{2.113}$$

$$I = 2\int \frac{d^D q}{(2\pi)^D}\frac{1}{(q^2+s_{12}x_1x_2)^3} \tag{2.114}$$

Due to symmetry argument one has that

$$I^\rho = 0 \tag{2.115}$$

$$I^{\rho\sigma} = 2\frac{g^{\rho\sigma}}{D}\int_0^{x_1} dx_1 \int_0^{1-x_1} dx_2 \int \frac{d^D q}{(2\pi)^D}\frac{q^2}{(q^2+s_{12}x_1x_2)^3} = \frac{g^{\rho\sigma}}{D}\int_0^{x_1} dx_1 \int_0^{1-x_1} dx_2 I_g \tag{2.116}$$

this simplifies A_{tot} greatly, leading to:

$$\begin{aligned} A_{tot} =& e^3\mu^{4-D}\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \bar{u}(p_2)\left\{\frac{(2-D)}{D}I_g\gamma_\rho\not{\epsilon}\gamma^\rho + \left[2-2x_1-2x_2+(D-2)\,x_1x_2\right]I\not{p}_1\not{\epsilon}\not{p}_2\right\}u(p_1) \\ =& e^3\mu^{4-D}\bar{u}(p_2)\not{\epsilon}u(p_1)\int_0^1 dx_1\int_0^{1-x_1} dx_2\left\{\frac{(2-D)^2}{D}I_g - s_{12}[2-2x_1-2x_2+(D-2)x_1x_2]I\right\} \end{aligned} \tag{2.117}$$

where the following identities has been used:

$$\gamma_\rho\gamma_\beta\gamma^\rho = (2-D)\gamma_\beta \tag{2.118}$$

$$\bar{u}(p_2)\not{p}_1\gamma_\beta\not{p}_2 u(p_1) = -s_{12}\bar{u}(p_2)\gamma_\beta u(p_1) \tag{2.119}$$

Applying (2.53) to I and I_g and substituting $D = 4-2\epsilon$ gives

$$\begin{aligned} I &= 2I_{0,3} = i\frac{-(-s_{12}x_1x_2)^{-1-\epsilon}}{(4\pi)^{2-\epsilon}}\Gamma(\epsilon)\epsilon \\ I_g &= 2I_{1,3} = i\frac{(-s_{12}x_1x_2)^{-\epsilon}}{(4\pi)^{2-\epsilon}}\Gamma(\epsilon)(2-\epsilon) \end{aligned} \tag{2.120}$$

The last step needed to evaluate A_{tot} is to compute the integrals over x_1 and x_2. These integrals already appeared in another example: they're all Beta function, as defined in (2.68), thus they can be written as:

$$\begin{aligned}
\int_0^1 dx_1 \int_0^{1-x_1} (x_1x_2)^{-\epsilon} &= \frac{1}{1-\epsilon}\int_0^1 dx_1 x_1^{-\epsilon}(1-x_1)^{1-\epsilon} = \frac{1}{1-\epsilon}B(1-\epsilon, 2-\epsilon) \\
&= \frac{\Gamma(-\epsilon)\Gamma(2-\epsilon)}{\Gamma(3-2\epsilon)}(-\epsilon-\epsilon^2) \\
\int_0^1 dx_1 \int_0^{1-x_1} x_2(x_1x_2)^{-1-\epsilon} &= \frac{1}{1-\epsilon}\int_0^1 dx_1 x_1^{-1-\epsilon}(1-x_1)^{1-\epsilon} = \frac{1}{1-\epsilon}B(-\epsilon, 2-\epsilon) \\
&= \frac{\Gamma(-\epsilon)\Gamma(2-\epsilon)}{\Gamma(3-2\epsilon)}2 \\
\int_0^1 dx_1 \int_0^{1-x_1} x_1(x_1x_2)^{-1-\epsilon} &= -\frac{1}{\epsilon}\int_0^1 dx_1 (x_1(1-x_1))^{-\epsilon} = -\frac{1}{\epsilon}B(1-\epsilon, 1-\epsilon) \\
&= \frac{\Gamma(-\epsilon)\Gamma(2-\epsilon)}{\Gamma(3-2\epsilon)}2 \\
\int_0^1 dx_1 \int_0^{1-x_1} (x_1x_2)^{-1-\epsilon} &= -\frac{1}{\epsilon}\int_0^1 dx_1 x_1^{-1-\epsilon}(1-x_1)^{-\epsilon} = -\frac{1}{\epsilon}B(-\epsilon, 1-\epsilon) \\
&= \frac{\Gamma(-\epsilon)\Gamma(2-\epsilon)}{\Gamma(3-2\epsilon)}\frac{4\epsilon-2}{\epsilon}
\end{aligned} \tag{2.121}$$

The total amplitude then turns out to be

$$A_{tot} = ie^3\bar{u}(p_2)\not{\epsilon}u(p_1)\left(\frac{-s_{12}}{\mu^2}\right)^{-\epsilon}\frac{\Omega_{D-3}}{2(2\pi)^{3-2\epsilon}}\frac{\Gamma(\epsilon)\Gamma(-\epsilon)\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\Gamma(3-2\epsilon)}\{2-\epsilon+2\epsilon^2\} \tag{2.122}$$

Expanding it around $\epsilon = 0$, one obtains the final result

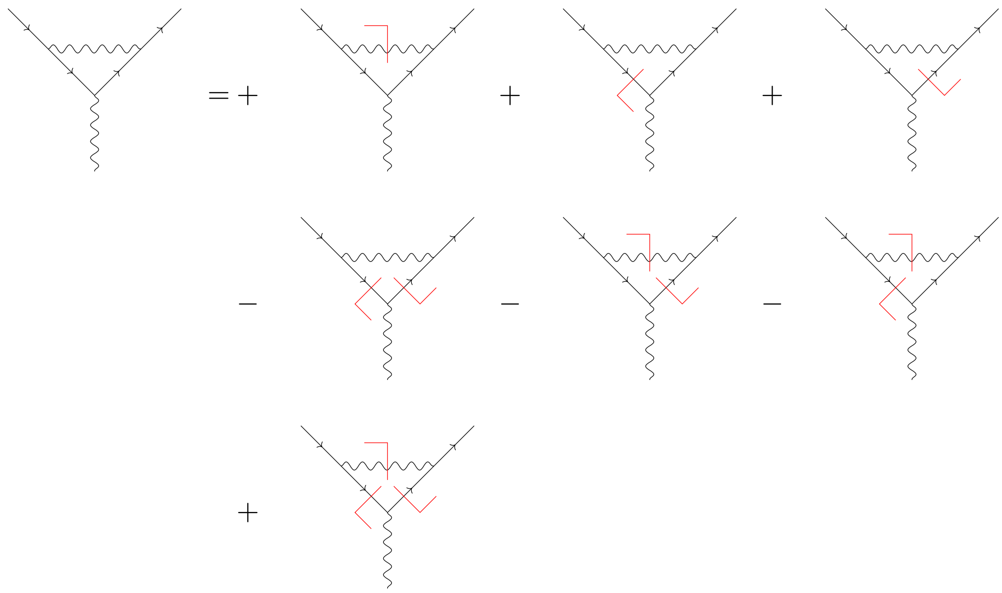
$$\begin{aligned}
A_{tot} =& \frac{e^3}{4}\bar{u}(p_2, s_2)\not{\epsilon}(\lambda)u(p_1, s_1)\frac{\Omega_{D-3}}{(2\pi)^{3-2\epsilon}} \\
&\left\{-\frac{2}{\epsilon^2} + \frac{1}{\epsilon}\left[-3 + 2\log\left(\frac{-s_{12}}{\mu^2}\right)\right] - 8 + 3\log\left(\frac{-s_{12}}{\mu^2}\right) - \log^2\left(\frac{-s_{12}}{\mu^2}\right) + O(\epsilon)\right\}
\end{aligned} \tag{2.123}$$

2.4.2 Feynman tree theorem method

Applying the Feynman tree theorem in this case gives the amplitude written in term of seven cut amplitudes, namely

$$A = A_1 + A_3 + A_2 - A_{23} - A_{13} - A_{12} + A_{123} \tag{2.124}$$

where the convention adopted here is the same as in (2.57). This equation can be represented as



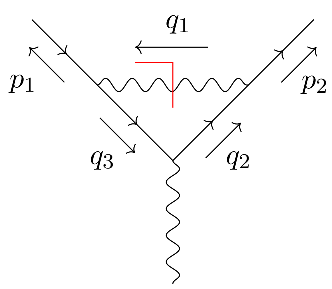
(2.125)

Single cut amplitudes

Cutting over q_1. The first one, with the photon propagator brought on shell by the delta that comes from the Feynman tree theorem, is:

$$A_1(p_1, s_1, p_2, s_2, \lambda) = -e^3 \mu^{4-D} \int \frac{d^D q_1}{(2\pi)^D} 2\pi \delta^{(+)}\left(q_1^2\right) \frac{\bar{u}(p_2)\gamma_\alpha \not{q}_2 \not{\epsilon}(\lambda) \not{q}_3 \gamma^\alpha u(p_1)}{\left(q_2^2 + i\varepsilon\right)\left(q_3^2 + i\varepsilon\right)} \tag{2.126}$$

which can be represented as



Writing all the loop momenta in function of q_1 thanks to equation (2.103) one obtains

$$A_1(p_1, s_1, p_2, s_2, \lambda) = -e^3 \mu^{4-D} \int \frac{d^D q_1}{(2\pi)^D} 2\pi \delta^{(+)}\left(q_1^2\right) \frac{\bar{u}(p_2)\gamma_\alpha(\not{q}_1 + \not{p}_2)\not{\epsilon}\,(\lambda)\,(\not{q}_1 - \not{p}_1)\gamma^\alpha u(p_1)}{\left[(q_1 + p_2)^2 + i\varepsilon\right]\left[(q_1 - p_1)^2 + i\varepsilon\right]} \tag{2.127}$$

Using (2.105) and the Dirac equation for massless spinors simplify the numerator, having it written in the following form

$$\begin{aligned} N(q_1,p_1,p_2) &= \bar{u}(p_2)\left[(4-D)(\not{q}_1+\not{p}_2)\not{\epsilon}(\lambda)(\not{q}_1-\not{p}_1)-2(\not{q}_1-\not{p}_1)\not{\epsilon}(\lambda)(\not{q}_1+\not{p}_2)\right]u(p_1) \\ &= \bar{u}(p_2)\left[(2-D)\not{q}_1\not{\epsilon}(\lambda)\not{q}_1-2\not{q}_1\not{\epsilon}(\lambda)\not{p}_2+2\not{p}_1\not{\epsilon}(\lambda)\not{q}_1+2\not{p}_1\not{\epsilon}(\lambda)\not{p}_2\right]u(p_1) \end{aligned} \tag{2.128}$$

The amplitude then turns out to be a combination of tensor integrals:

$$\begin{aligned} A_1(p_1,s_1,p_2,s_2,\lambda) = e^3\bar{u}(p_2)\Big[&(2-D)I^{(1)}_{\rho\sigma}\gamma^\rho\not{\epsilon}(\lambda)\gamma^\sigma-2I^{(1)}_\rho\gamma^\rho\not{\epsilon}(\lambda)\not{p}_2+2I^{(1)}_\rho\not{p}_1\not{\epsilon}(\lambda)\gamma^\rho+ \\ &+2I^{(1)}\not{p}_1\not{\epsilon}(\lambda)\not{p}_2\Big]u(p_1) \end{aligned} \tag{2.129}$$

with

$$\begin{aligned} I^{(1)} &= -\mu^{4-D}\int\frac{d^Dq_1}{(2\pi)^D}2\pi\delta^{(+)}\left(q_1^2\right)\frac{1}{(2q_1p_2+i\varepsilon)(-2q_1p_1+i\varepsilon)} \\ I^{(1)}_\rho &= -\mu^{4-D}\int\frac{d^Dq_1}{(2\pi)^D}2\pi\delta^{(+)}\left(q_1^2\right)\frac{q_{1,\rho}}{(2q_1p_2+i\varepsilon)(-2q_1p_1+i\varepsilon)} \\ I^{(1)}_{\rho\sigma} &= -\mu^{4-D}\int\frac{d^Dq_1}{(2\pi)^D}2\pi\delta^{(+)}\left(q_1^2\right)\frac{q_{1,\rho}q_{1,\sigma}}{(2q_1p_2+i\varepsilon)(-2q_1p_1+i\varepsilon)} \end{aligned} \tag{2.130}$$

In order to deal with these integrals it is necessary to perform a covariant decomposition using the available momenta p_1 and p_2:

$$I^{(k)}_\rho = p_{1,\rho}C^{(k)}_1+p_{2,\rho}C^{(k)}_2 \qquad I^{(k)}_{\rho\sigma} = C^{(k)}_{00}g_{\rho\sigma}+\sum_{i,j}^{2}p_{i,\rho}p_{j,\sigma}C^{(k)}_{ij} \tag{2.131}$$

Substituting them in A_1 and using (2.118) and (2.119) one gets:

$$A_1 = e^3\bar{u}(p_2)\not{\epsilon}u(p_1)\left\{(2-D)^2C^{(1)}_{00}-s_{12}\left[(2-D)C^{(1)}_{12}-2C^{(1)}_1+2C^{(1)}_2+2I^{(1)}\right]\right\} \tag{2.132}$$

Contracting $I^{(1)}_{\rho\sigma}$and $I^{(1)}_\rho$ with the external momenta or with the metric tensor, one can evaluate all the different constants:

$$\begin{aligned} p_1^\rho I^{(1)}_\rho &= I^{(1)}_1 = \frac{\mu^{4-D}}{2}\int\frac{d^Dq_1}{(2\pi)^D}2\pi\delta^{(+)}\left(q_1^2\right)\frac{1}{(2q_1p_2+i\varepsilon)} \\ p_2^\rho I^{(1)}_\rho &= I^{(1)}_2 = -\frac{\mu^{4-D}}{2}\int\frac{d^Dq_1}{(2\pi)^D}2\pi\delta^{(+)}\left(q_1^2\right)\frac{1}{(-2q_1p_1+i\varepsilon)} \\ g^{\rho\sigma}I^{(1)}_{\rho\sigma} &= I^{(1)}_g = -\mu^{4-D}\int\frac{d^Dq_1}{(2\pi)^D}2\pi\delta^{(+)}\left(q_1^2\right)\frac{q_1^2}{(2q_1p_2+i\varepsilon)(-2q_1p_1+i\varepsilon)} = 0 \\ p_1^\rho p_2^\sigma I^{(1)}_{\rho\sigma} &= p_2^\rho p_1^\sigma I^{(1)}_{\rho\sigma} = I^{(1)}_{12} = \frac{\mu^{4-D}}{4}\int\frac{d^Dq_1}{(2\pi)^D}2\pi\delta^{(+)}\left(q_1^2\right) \end{aligned} \tag{2.133}$$

Using these informations together with (2.131) leads to

$$\begin{aligned} I_1 &= \frac{s_{12}}{2}C^{(1)}_2 \qquad I_2 = \frac{s_{12}}{2}C^{(1)}_1 \\ 0 &= I^{(1)}_g = DC^{(1)}_{00}+s_{12}C^{(1)}_{12} \\ I^{(1)}_{12} &= \frac{s_{12}}{2}C^{(1)}_{00}+\frac{s_{12}^2}{4}C^{(1)}_{12}, \end{aligned} \tag{2.134}$$

knowing that $C^{(1)}_{12} = C^{(1)}_{21}$.
Inverting these relations leads to:

$$\begin{aligned} C^{(1)}_{1} &= \frac{2}{s_{12}} I^{(1)}_{2} & C^{(1)}_{2} &= \frac{2}{s_{12}} I^{(1)}_{1} \\ C^{(1)}_{00} &= \frac{1}{s_{12}} \frac{4}{2-D} I^{(1)}_{12} & C^{(1)}_{12} &= -\frac{4}{s_{12}^2} \frac{D}{2-D} I^{(1)}_{12} \end{aligned} \tag{2.135}$$

Substituting them inside the amplitude gives

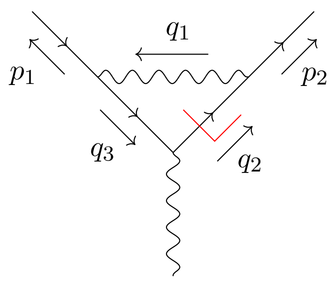
$$A_1 = e^3 \bar{u}(p_2) \not{\epsilon} u(p_1) \left[\frac{8}{s_{12}} I^{(1)}_{12} + 4 I^{(1)}_{2} - 4 I^{(1)}_{1} - 2 s_{12} I^{(1)} \right] \tag{2.136}$$

All the integral reported in (2.133) have a common property: performing a change of variable, defining $q'_\mu = \alpha q_\mu$ leads to $I = \alpha^\beta I$, were β is a generic exponent.
This kind of integral are called scaleless and in dimensional regularization they're equal to zero.
In the end this cut amplitude depends only on integral of this kind so it turns out to be null.

Cutting over q_2. The next amplitude is the one with the cut over the fermionic propagator of momentum q_2. This amplitude is:

$$A_2(p_1, s_1, p_2, s_2, \lambda) = -e^3 \mu^{4-D} \int \frac{d^D q_1}{(2\pi)^D} 2\pi \delta^{(+)}\left(q_2^2\right) \frac{\bar{u}(p_2, s_2) \gamma_\alpha \not{q}_2 \not{\epsilon}(\lambda) \not{q}_3 \gamma^\alpha u(p_1, s_1)}{\left(q_1^2 + i\varepsilon\right)\left(q_3^2 + i\varepsilon\right)} \tag{2.137}$$

and can be represented as



The step needed for the evaluation of A_2, up to the tensor decomposition, are almost identical to those shown in the calculation of A_1, hence they will be omitted. After the tensor decomposition the amplitude becomes

$$A_2(p_1, s_1, p_2, s_2, \lambda) = e^3 \bar{u}(p_2, s_2) \not{\epsilon}(\lambda) u(p_1, s_1) \left\{ \frac{8}{s_{12}} I^{(2)}_{12} + 2(4-D) I^{(2)}_{2} - 4 I^{(2)}_{1} \right\} \tag{2.138}$$

with

$$\begin{aligned}
I_1^{(2)} &= -\mu^{4-D} \int \frac{d^D q_2}{(2\pi)^D} 2\pi\delta^{(+)}\left(q_2^2\right) \frac{q_2 p_1}{\left[(q_2-p_2)^2+i\varepsilon\right]\left[(q_2-p_1-p_2)^2+i\varepsilon\right]} \\
I_2^{(2)} &= -\mu^{4-D} \int \frac{d^D q_2}{(2\pi)^D} 2\pi\delta^{(+)}\left(q_2^2\right) \frac{q_2 p_2}{\left[(q_2-p_2)^2+i\varepsilon\right]\left[(q_2-p_1-p_2)^2+i\varepsilon\right]} \\
I_{12}^{(2)} &= -\mu^{4-D} \int \frac{d^D q_2}{(2\pi)^D} 2\pi\delta^{(+)}\left(q_2^2\right) \frac{(q_2 p_1)(q_2 p_2)}{\left[(q_2-p_2)^2+i\varepsilon\right]\left[(q_2-p_1-p_2)^2+i\varepsilon\right]}
\end{aligned} \tag{2.139}$$

In particular $I_{12}^{(2)} = I_{21}^{(2)}$ and $I_g^{(2)} = 0$ here too so the relation between the constants from the tensor decomposition and the integrals are the same as in (2.135).

A particular parameterization makes the integration easier to be made, hence it will be briefly introduced now.

Firstly notice that for every cut amplitude there is a momentum brought on-shell by the delta function. Hence, knowing that the calculation are performed in the massless limit, it is possible to write the momentum in the following form:

$$q_i = |\boldsymbol{q}_i| \left(1, \sin(\theta_i)\boldsymbol{e}_T, \cos(\theta_i)\right) \tag{2.140}$$

with θ_i being the angle between the three momentum $\boldsymbol{q}_i$ and the z axis. Moreover the modulus of the momentum can be written in function of a dimensionless parameter ξ_i multiplied for $\frac{\sqrt{s_{12}}}{2}$.
Then, changing the parameterization of the angle, thus setting:

$$\cos(\theta_i) = 1 - 2v_i \tag{2.141}$$

allows to express the on-shell momentum as follows:

$$q_i = \frac{\sqrt{s_{12}}}{2}\xi_i \left(1, 2\sqrt{v_i(1-v_i)}\boldsymbol{e}_T, 1-2v_i\right)^T \tag{2.142}$$

where $\xi_i \in [0, \infty[$, $v_i \in [0,1]$ and $\boldsymbol{e}_T$ is a unit vector in the transverse direction.
As expect $q_i^2 = 0$ and

$$|\boldsymbol{q}_i| = q_{i,0} = \frac{\sqrt{s_{12}}}{2}\xi_i \tag{2.143}$$

With this new parameterization the integration over the solid angle changes. Remembering that

$$d\Omega_{D-2} = (\sin(\theta_i))^{D-3} d\theta_i \Omega_{D-3} \tag{2.144}$$

one can write

$$d\Omega_{D-2} = \Omega_{D-3} 2^{D-3} (v_i(1-v_i))^{\frac{D-4}{2}} dv_i. \tag{2.145}$$

With $\boldsymbol{p}_1$ pointing in the positive z direction and $\boldsymbol{p}_2 = -\boldsymbol{p}_1$, the scalar product between the loop momenta and the external momenta becomes:

$$2q_i p_1 = s_{12}\xi_i v_i \qquad\qquad 2q_i p_2 = s_{12}\xi_i(1-v_i) \tag{2.146}$$

Starting from $I_1^{(2)}$ one integrates over dq_0, finding:

$$\begin{aligned} I_1^{(2)} &= -\mu^{4-D}\int \frac{d|\boldsymbol{q}_2||\boldsymbol{q}_2|^{D-3}d\Omega_{D-2}}{2(2\pi)^{D-1}}\frac{q_2p_1}{(-2q_2p_2+i\varepsilon)(s_{12}-2q_2p_1-2q_2p_2+i\varepsilon)} \\ &= \mu^{4-D}\frac{s_{12}^{\frac{D}{2}-2}}{2^D(2\pi)^{D-1}}\int d\xi_2\frac{\xi_2^{D-3}}{1-\xi_2+i\varepsilon}d\Omega_{D-2}\frac{v_2}{1-v_2} \\ &= \left(\frac{s_{12}}{\mu^2}\right)^{-\epsilon}\frac{\Omega_{D-3}}{8(2\pi)^{3-2\epsilon}}\int d\xi_2\frac{\xi_2^{1-2\epsilon}}{1-\xi_2+i\varepsilon}dv_2(v_2(1-v_2))^{-\epsilon}\frac{v_2}{1-v_2} \end{aligned} \tag{2.147}$$

Now there are two integrals that must be evaluated:

$$\int_0^\infty d\xi_2\frac{\xi_2^{1-2\epsilon}}{1-\xi_2+i\varepsilon} \tag{2.148}$$

$$\int_0^1 dv_2(v_2(1-v_2))^{-\epsilon}\frac{v_2}{1-v_2} \tag{2.149}$$

Again, those integrals belong to the family of the Euler's Beta function defined in (2.68), even if they're quite different from each others.

As one can notice, the integral over v is:

$$\int_0^1 dv_2v_2^{1-\epsilon}(1-v)^{-1-\epsilon} = B(2-\epsilon,-\epsilon) \tag{2.150}$$

while the integral over ξ has already been faced, in the application of the Feynman Tree theorem on the two point function, in equation (2.65), and its value is

$$\int_0^\infty d\xi_2\frac{\xi_2^{1-2\epsilon}}{1-\xi_2+i\varepsilon} = -(-1)^{-2\epsilon}B(1-2\epsilon,2\epsilon) \tag{2.151}$$

After this calculation $I_1^{(2)}$ becomes

$$I_1^{(2)} = -\left(\frac{s_{12}}{\mu^2}\right)^{-\epsilon}\frac{\Omega_{D-3}}{8(2\pi)^{3-2\epsilon}}(-1)^{-2\epsilon}B(1-2\epsilon,2\epsilon)B(2-\epsilon,-\epsilon) \tag{2.152}$$

Using (2.77) together with (2.68) leads to:

$$I_1^{(2)} = -\left(\frac{s_{12}}{\mu^2}\right)^{-\epsilon}\frac{\Omega_{D-3}}{8(2\pi)^{3-2\epsilon}}(-1)^{-2\epsilon}\frac{\pi}{\sin(2\pi\epsilon)}\frac{\Gamma(2-\epsilon)\Gamma(-\epsilon)}{\Gamma(2-2\epsilon)} \tag{2.153}$$

As it will be shown, also the other cut amplitudes depend on a combination of Gamma functions, so for a simpler evaluation of (2.102) it is helpful to write them in function of a common term:

$$I_1^{(2)} = \left(\frac{s_{12}}{\mu^2}\right)^{-\epsilon}\frac{\Omega_{D-3}}{8(2\pi)^{3-2\epsilon}}(-1)^{-2\epsilon}\frac{\pi}{\sin(2\pi\epsilon)}\frac{\Gamma(2-\epsilon)\Gamma(-\epsilon)}{\Gamma(3-2\epsilon)}(2\epsilon-2) \tag{2.154}$$

Analogously one can calculate $I_2^{(2)}$:

$$\begin{aligned} I_2^{(2)} &= -\mu^{4-D} \int \frac{d|\boldsymbol{q}_2||\boldsymbol{q}_2|^{D-3} d\Omega_{D-2}}{2(2\pi)^{D-1}} \frac{q_2 p_2}{(-q_2 p_2 + i\varepsilon)(s_{12} - 2q_2 p_1 - 2q_2 p_2 + i\varepsilon)} \\ &= \left(\frac{s_{12}}{\mu^2}\right)^{-\epsilon} \frac{\Omega_{D-3}}{8(2\pi)^{3-2\epsilon}} \int d\xi_2 \frac{\xi_2^{1-2\epsilon}}{1-\xi_2+i\varepsilon} dv_2 (v_2(1-v_2))^{-\epsilon} \\ &= -\left(\frac{s_{12}}{\mu^2}\right)^{-\epsilon} \frac{\Omega_{D-3}}{8(2\pi)^{3-2\epsilon}} (-1)^{-2\epsilon} B(1-2\epsilon, 2\epsilon) B(1-\epsilon, 1-\epsilon) \\ &= \left(\frac{s_{12}}{\mu^2}\right)^{-\epsilon} \frac{\Omega_{D-3}}{8(2\pi)^{3-2\epsilon}} (-1)^{-2\epsilon} \frac{\pi}{\sin(2\pi\epsilon)} \frac{\Gamma(2-\epsilon)\Gamma(-\epsilon)}{\Gamma(3-2\epsilon)} (2\epsilon) \end{aligned} \tag{2.155}$$

and $I_{12}^{(2)}$:

$$\begin{aligned} I_{12}^{(2)} &= -\mu^{4-D} \int \frac{d|\boldsymbol{q}_2||\boldsymbol{q}_2|^{D-3} d\Omega_{D-2}}{2(2\pi)^{D-1}} \frac{(q_2 p_1)(q_2 p_2)}{(-2q_2 p_2 + i\varepsilon)(s_{12} - 2q_2 p_1 - 2q_2 p_2 + i\varepsilon)} \\ &= \left(\frac{s_{12}}{\mu^2}\right)^{-\epsilon} \frac{s_{12}\Omega_{D-3}}{16(2\pi)^{3-2\epsilon}} \int d\xi_2 \frac{\xi_2^{2-2\epsilon}}{1-\xi_2+i\varepsilon} dv_2 (v_2(1-v_2))^{-\epsilon} v_2 \\ &= -\left(\frac{s_{12}}{\mu^2}\right)^{-\epsilon} \frac{s_{12}\Omega_{D-3}}{16(2\pi)^{3-2\epsilon}} (-1)^{-2\epsilon} B(1-2\epsilon, 2\epsilon) B(2-\epsilon, 1-\epsilon) \\ &= \left(\frac{s_{12}}{\mu^2}\right)^{-\epsilon} \frac{s_{12}\Omega_{D-3}}{16(2\pi)^{3-2\epsilon}} (-1)^{-2\epsilon} \frac{\pi}{\sin(2\pi\epsilon)} \frac{\Gamma(2-\epsilon)\Gamma(-\epsilon)}{\Gamma(3-2\epsilon)} (\epsilon) \end{aligned} \tag{2.156}$$

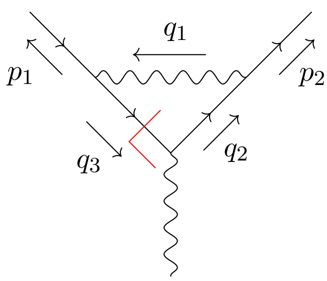
using the same change of variable performed in (2.151) to evaluate the integral over ξ_2.
Substituting these results back in (2.138) leads to:

$$\begin{aligned} A_2(p_1, s_1, p_2, s_2, \lambda) = & e^3 \bar{u}(p_2, s_2) \not{\epsilon}(\lambda) u(p_1, s_1) \left(\frac{s_{12}}{\mu^2}\right)^{-\epsilon} \frac{\Omega_{D-3}}{2(2\pi)^{3-2\epsilon}} \frac{\pi(-1)^{-2\epsilon}}{\sin(2\pi\epsilon)} \frac{\Gamma(2-\epsilon)\Gamma(-\epsilon)}{\Gamma(3-2\epsilon)} \\ & \left\{2 - \epsilon + 2\epsilon^2\right\} \end{aligned} \tag{2.157}$$

Cutting over q_3. The last single cut amplitude missing is A_3:

$$A_3(p_1, s_1, p_2, s_2, \lambda) = -e^3 \mu^{4-D} \int \frac{d^D q_1}{(2\pi)^D} 2\pi\delta^{(+)}(q_3^2) \frac{\bar{u}(p_2, s_2)\gamma_\alpha \not{q}_2 \not{\epsilon}(\lambda) \not{q}_3 \gamma^\alpha u(p_1, s_1)}{(q_1^2 + i\varepsilon)(q_2^2 + i\varepsilon)} \tag{2.158}$$

and can be represented as



Repeating the same steps done for A_2 (the relations written in (2.135) are valid here too) one obtains:

$$A_3(p_1, s_1, p_2, s_2, \lambda) = e^3 \bar{u}(p_2, s_2) \not{\epsilon}(\lambda) u(p_1, s_1) \left\{ \frac{8}{s_{12}} I_{12}^{(3)} - 2(4-D) I_1^{(3)} + 4 I_2^{(3)} \right\} \tag{2.159}$$

$$\tag{2.160}$$

which is pretty similar to (2.138). Here the integrals are:

$$\begin{aligned} I_1^{(3)} &= -\mu^{4-D} \int \frac{d^D q_3}{(2\pi)^D} 2\pi\delta^{(+)}(q_3^2) \frac{(q_3 p_1)}{\left[(q_3+p_1)^2 + i\varepsilon\right] \left[(q_3+p_1+p_2)^2 + i\varepsilon\right]} \\ I_2^{(3)} &= -\mu^{4-D} \int \frac{d^D q_3}{(2\pi)^D} 2\pi\delta^{(+)}(q_3^2) \frac{(q_3 p_2)}{\left[(q_3+p_1)^2 + i\varepsilon\right] \left[(q_3+p_1+p_2)^2 + i\varepsilon\right]} \\ I_{12}^{(3)} &= -\mu^{4-D} \int \frac{d^D q_3}{(2\pi)^D} 2\pi\delta^{(+)}(q_3^2) \frac{(q_3 p_1)(q_3 p_2)}{\left[(q_3+p_1)^2 + i\varepsilon\right] \left[(q_3+p_1+p_2)^2 + i\varepsilon\right]} \end{aligned} \tag{2.161}$$

Applying the parameterization outlined in (2.142) one is able to evaluate the integral written above. Starting from the first, one has that

$$I_1^{(3)} = -\left(\frac{s_{12}}{\mu}\right)^{-\epsilon} \frac{\Omega_{D-3}}{8(2\pi)^{3-2\epsilon}} \int d\xi_3 \frac{\xi_3^{1-2\epsilon}}{1+\xi_3} dv_3 (v_3(1-v_3))^{-\epsilon} \tag{2.162}$$

The integral over v_3 is the same as in (2.155), while the integral over ξ_3 is linked to the one appearing in A_2: defining $\xi_3 = -\xi_2$ it becomes

$$\int_0^{+\infty} d\xi_3 \frac{\xi_3^{1-2\epsilon}}{1+\xi_3} = (-1)^{1-2\epsilon} \int_{-\infty}^0 d\xi_2 \frac{\xi_2^{1-2\epsilon}}{1-\xi_2} = (-1)^{-2\epsilon} \int_0^{+\infty} d\xi_2 \frac{\xi_2^{1-2\epsilon}}{1-\xi_2} \tag{2.163}$$

which is the same integral appearing in (2.148) multiplied by a factor $(-1)^{-2\epsilon}$, so

$$I_1^{(3)} = -(-1)^{-2\epsilon} I_2^{(2)} \tag{2.164}$$

The next one is:

$$I_2^{(3)} = -\left(\frac{s_{12}}{\mu}\right)^{-\epsilon} \frac{\Omega_{D-3}}{8(2\pi)^{3-2\epsilon}} \int d\xi_3 \frac{\xi_3^{1-2\epsilon}}{1+\xi_3} dv_3 (v_3(1-v_3))^{-\epsilon} \frac{1-v}{v} \tag{2.165}$$

Observing that, by substituting $1 - v_3 = v_2$, one gets

$$\int_0^1 dv_3 (v_3(1-v_3))^{-\epsilon} \frac{1-v_3}{v_3} = \int_0^1 dv_2 (v_2(1-v_2))^{-\epsilon} \frac{v_2}{1-v_2} \tag{2.166}$$

it is possible draw a relation between integrals:

$$I_2^{(3)} = -(-1)^{-2\epsilon} I_1^{(2)} \tag{2.167}$$

The next one reads

$$I^{(3)} = -\left(\frac{s_{12}}{\mu}\right)^{-\epsilon} \frac{s_{12}\Omega_{D-3}}{16(2\pi)^{3-2\epsilon}} \int d\xi_3 \frac{\xi_3^{2-2\epsilon}}{1+\xi_3} dv_3 (v_3(1-v_3))^{-\epsilon} (1-v_3) \tag{2.168}$$

By performing the same substitution over v and following the same procedure done in (2.163), one finds that

$$I^{(3)} = (-1)^{-2\epsilon} I^{(2)} \tag{2.169}$$

Substituting every I back into the cut amplitude A_3 leads to

$$\begin{aligned} A_3(p_1, s_1, p_2, s_2, \lambda) =& (-1)^{-2\epsilon} A_2(p_1, s_1, p_2, s_2, \lambda) \\ =& e^3 \bar{u}(p_2, s_2) \not{\epsilon}(\lambda) u(p_1, s_1) \left(\frac{s_{12}}{\mu^2}\right)^{-\epsilon} \frac{\Omega_{D-3}}{2(2\pi)^{3-2\epsilon}} \frac{\pi}{\sin(2\pi\epsilon)} \frac{\Gamma(2-\epsilon)\Gamma(-\epsilon)}{\Gamma(3-2\epsilon)} \\ & \{2 - \epsilon + 2\epsilon^2\} \end{aligned} \tag{2.170}$$

Using (2.62) to match the result outlined in [11] one gets that

$$\frac{\Gamma(2-\epsilon)\Gamma(-\epsilon)}{\Gamma(3-2\epsilon)} = \frac{1}{2} \frac{\Gamma(1-\epsilon)\Gamma(-\epsilon)}{\Gamma(2-2\epsilon)} = \frac{\sqrt{\pi}}{4} 4^{-\epsilon} \frac{\Gamma(-\epsilon)}{\Gamma(3/2-\epsilon)} \tag{2.171}$$

and hence

$$\begin{aligned} A_3(p_1, s_1, p_2, s_2, \lambda) =& e^3 \bar{u}(p_2, s_2) \not{\epsilon}(\lambda) u(p_1, s_1) \left(\frac{s_{12}}{\mu^2}\right)^{-\epsilon} \frac{\Omega_{D-3}}{2(2\pi)^{3-2\epsilon}} \frac{\pi}{\sin(2\pi\epsilon)} \frac{\sqrt{\pi}}{4} 4^{-\epsilon} \frac{\Gamma(-\epsilon)}{\Gamma(3/2-\epsilon)} \\ & \{2 - \epsilon + 2\epsilon^2\} \end{aligned} \tag{2.172}$$

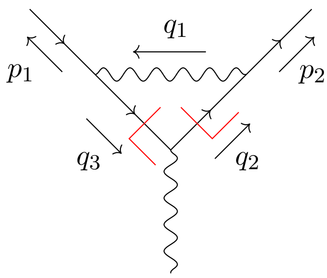
Double cut Amplitudes

The next amplitudes are the double cut.

Cutting over q_2 and q_3. The first that will be evaluated is:

$$A_{23}(p_1, s_1, p_2, s_2, \lambda) = e^3 \mu^{4-D} \int \frac{d^D q_1}{(2\pi)^D} 2\pi \delta^{(+)}(q_2^2) 2\pi \delta^{(+)}(q_3^2) \frac{\bar{u}(p_2, s_2) \gamma_\alpha \not{q}_2 \not{\epsilon}(\lambda) \not{q}_3 \gamma^\alpha u(p_1, s_1)}{(q_1^2 + i\varepsilon)} \tag{2.173}$$

and can be represented as



Shifting the integration variable to q_3, integrating over $dq_{3,0}$ and using (2.142) leads to:

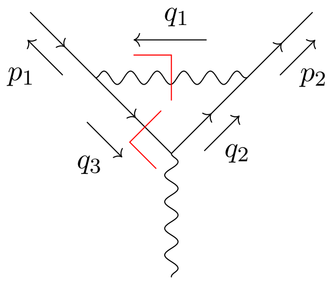
$$\begin{aligned} A_{23}(p_1, s_1, p_2, s_2, \lambda) &= e^3 \mu^{4-D} \int \frac{dq_{3,0} d|\boldsymbol{q}_3| |\boldsymbol{q}_3|^{D-2}}{(2\pi)^{D-2}} \delta^{(+)}\left((q_3 + p_1 + p_2)^2\right) \frac{\delta(q_{3,0} - |\boldsymbol{q}_3|)}{2q_{3,0}} \frac{N(q_3)}{[(q_3 + p_1)^2 + i\varepsilon]} \\ &= \frac{e^3}{2^{(D-1)}(2\pi)^{D-2}} \left(\frac{s_{12}}{\mu^2}\right)^{-\epsilon} \int d\xi_3 d\Omega_{D-2} \delta^{(+)}(s_{12}(1+\xi_3)) N(\xi_3, v_3) \frac{\xi_3^{D-4}}{v_3} = 0 \end{aligned} \tag{2.174}$$

since ξ_3 is integrated over $\xi_3 \in [0, +\infty[$ and in this range the argument of the delta function is different from zero.

Cutting over q_1 and q_3. Proceeding with the calculation, the next amplitude is:

$$A_{13}(p_1,s_1,p_2,s_2,\lambda)=e^3\mu^{4-D}\int\frac{d^Dq_1}{(2\pi)^D}2\pi\delta^{(+)}(q_1^2)2\pi\delta^{(+)}(q_3^2)\frac{\bar{u}(p_2,s_2)\gamma_\alpha\not{q}_2\not{\epsilon}(\lambda)\not{q}_3\gamma^\alpha u(p_1,s_1)}{(q_2^2+i\varepsilon)} \tag{2.175}$$

and can be represented as



Shifting the integration to q_3, this integral decomposes to

$$A_{13}(p_1,s_1,p_2,s_2,\lambda)=e^3\bar{u}(p_2,s_2)\not{\epsilon}(\lambda)u(p_1,s_1)\left\{\frac{8}{s_{12}}I_{12}^{(13)}-2(4-D)I_1^{(13)}+4I_2^{(13)}\right\} \tag{2.176}$$

with

$$\begin{aligned}
I_1^{(13)}=&\mu^{4-D}\int\frac{d^Dq_1}{(2\pi)^D}2\pi\delta^{(+)}(q_1^2)2\pi\delta^{(+)}(q_3^2)\frac{q_3p_1}{(q_2^2+i\varepsilon)}\\
I_2^{(13)}=&\mu^{4-D}\int\frac{d^Dq_1}{(2\pi)^D}2\pi\delta^{(+)}(q_1^2)2\pi\delta^{(+)}(q_3^2)\frac{q_3p_2}{(q_2^2+i\varepsilon)}\\
I_{12}^{(13)}=&\mu^{4-D}\int\frac{d^Dq_1}{(2\pi)^D}2\pi\delta^{(+)}(q_1^2)2\pi\delta^{(+)}(q_3^2)\frac{(q_3p_1)(q_3p_2)}{(q_2^2+i\varepsilon)}
\end{aligned} \tag{2.177}$$

Calculating the first one gets:

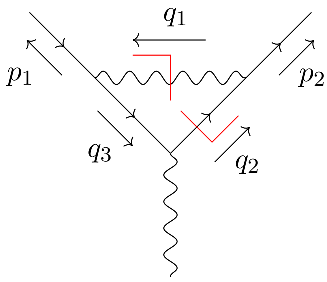
$$\begin{aligned}
I_1^{(13)}(p_1,s_1,p_2,s_2,\lambda)&=e^3\mu^{4-D}\int\frac{d^Dq_1}{(2\pi)^D}2\pi\delta^{(+)}(q_1^2)2\pi\delta^{(+)}(q_3^2)\frac{q_3p_1}{(q_2^2+i\varepsilon)}\\
&=e^3\mu^{4-D}\frac{s_{12}^{D/2-3}}{4(2\pi)^{D-2}}\Omega_{D-3}\int d\xi_3dv_3\theta(\xi_3+1)\delta(\xi_3)\frac{\xi_3^{D-2}}{1+\xi_3}(v_3(1-v_3))^{\frac{D-6}{2}}v_3=0
\end{aligned} \tag{2.178}$$

Because of the $\delta(\xi)$ together with the the fact that the integrand depends on ξ_3^{D-2}. $I_2^{(13)}$ and $I_{12}^{(13)}$ are null for the same reason, so $A_{13}=0$.

Cutting over q_1 and q_2. The last double cut amplitude is:

$$\begin{aligned}
A_{12}(p_1,s_1,p_2,s_2,\lambda)&=e^3\mu^{4-D}\int\frac{d^Dq_1}{(2\pi)^D}2\pi\delta^{(+)}(q_1^2)2\pi\delta^{(+)}(q_2^2)\frac{\bar{u}(p_2,s_2)\gamma_\alpha\not{q}_2\not{\epsilon}(\lambda)\not{q}_3\gamma^\alpha u(p_1,s_1)}{(q_3^2+i\varepsilon)}\\
&=e^3\mu^{4-D}\frac{s_{12}^{D/2-3}}{4(2\pi)^{D-2}}\Omega_{D-3}\int d\xi_2dv_2\theta(\xi_2+1)\delta(\xi_2)\xi_2^{D-4}(v_2(1-v_2))^{\frac{D-6}{2}}N(\xi_2,v_2)
\end{aligned} \tag{2.179}$$

,represented as



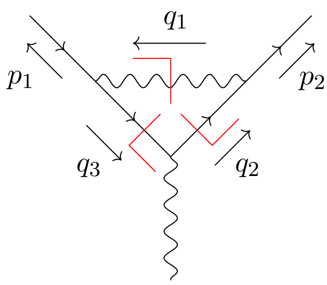
and for the same argument of A_{13}, turns out to be null.

Triple cut Amplitude

Lastly, the triple cut:

$$A_{123}(p_1, s_1, p_2, s_2, \lambda) = -e^3 \mu^{4-D} \int \frac{d^D q_1}{(2\pi)^D} 2\pi\delta^{(+)}\left(q_1^2\right) 2\pi\delta^{(+)}\left(q_2^2\right) 2\pi\delta^{(+)}\left(q_3^2\right) \bar{u}(p_2, s_2)\gamma_\alpha \not{q}_2 \not{\epsilon}(\lambda) \not{q}_3 \gamma^\alpha u(p_1, s_1) \tag{2.180}$$

and can be represented as



which turns out to be 0. In fact shifting the integration over q_3 and using the on-shell parameterization one gets a combination of $\theta(\xi - 1)\delta(\xi v)\delta(\xi(1 - v))$ in the integrand, which is always null.

Conclusions

Now that all terms appearing in equation (2.124) are known, it is finally possible to evaluate the total amplitude. Since the only non vanishing contribution come from

A_2 and A_3, the total amplitude is

$$\begin{aligned}
A(p_1,s_1,p_2,s_2,\lambda) =& A_2 + A_3 = A_2 + (-1)^{-2\epsilon} A_2 \\
&= e^3 \bar{u}(p_2,s_2)\not\epsilon(\lambda) u(p_1,s_1) \left(\frac{s_{12}}{\mu^2}\right)^{-\epsilon} \frac{\Omega_{D-3}}{2(2\pi)^{3-2\epsilon}} \frac{\Gamma(2-\epsilon)\Gamma(-\epsilon)}{\Gamma(3-2\epsilon)} \\
&\left\{2-\epsilon+2\epsilon^2\right\} \frac{\pi(-1)^{-2\epsilon}}{\sin(2\pi\epsilon)} [1+(-1)^{-2\epsilon}] \\
&= e^3 \bar{u}(p_2,s_2)\not\epsilon(\lambda) u(p_1,s_1) \left(\frac{s_{12}}{\mu^2}\right)^{-\epsilon} \frac{\Omega_{D-3}}{2(2\pi)^{3-2\epsilon}} \frac{\Gamma(2-\epsilon)\Gamma(-\epsilon)}{\Gamma(3-2\epsilon)} \\
&\left\{2-\epsilon+2\epsilon^2\right\} \frac{\pi(-1)^{-\epsilon}}{\sin(\pi\epsilon)} \\
&= e^3 \bar{u}(p_2,s_2)\not\epsilon(\lambda) u(p_1,s_1) \left(\frac{-s_{12}}{\mu^2}\right)^{-\epsilon} \frac{\Omega_{D-3}}{2(2\pi)^{3-2\epsilon}} \frac{\Gamma(\epsilon)\Gamma(1-\epsilon)\Gamma(2-\epsilon)\Gamma(-\epsilon)}{\Gamma(3-2\epsilon)} \\
&\left\{2-\epsilon+2\epsilon^2\right\} \\
&= \frac{e^3}{4} \bar{u}(p_2,s_2)\not\epsilon(\lambda) u(p_1,s_1) \frac{\Omega_{D-3}}{(2\pi)^{3-2\epsilon}}
\end{aligned} \tag{2.181}$$

Lastly, expanding in $\epsilon = 0$, one obtains

$$A(p_1,s_1,p_2,s_2,\lambda) = \left\{-\frac{2}{\epsilon^2} + \frac{1}{\epsilon}\left[-3+2\log\left(\frac{-s_{12}}{\mu^2}\right)\right] - 8 + 3\log\left(\frac{-s_{12}}{\mu^2}\right) - \log^2\left(\frac{-s_{12}}{\mu^2}\right) + O(\epsilon)\right\} \tag{2.182}$$

wich is the same result obtained using the Feynman parametrization, outlined in (2.123), as obtained in [11].

Chapter 3

Multiloop Feynman Integrals

In this chapter, after defining Feynman Integrals at multiloop level, we outline some powerful relation that allows major simplification in the evaluation of a scattering amplitude. Among such relation, the *integration by parts identities* are the most powerful. After this introduction, a general parameterization for the Feynman integrals and its properties are presented. Such parameterization is called the *Baikov* representation. Its peculiarity, is how key information of the amplitude is encoded within its integrand: the *Gram determinant*, which has a beautiful and intuitive geometric interpretation: it is proportional to the volume spanned by the momenta appearing in the scattering amplitude. An underestimate aspects of such parameterization, are its integration boundaries. Nonetheless they're of crucial importance for example in the IBPs derivation. After a deep look at such component of the amplitude we analyze a parameterization deriving from the Baikov representation, which is connected to the Hyperspherical coordinates. At last, we review the notorious technique of generalized Unitarity and IBPs in the Baikov representation with denominators as variables, outlining its advantage.

3.1 Definition

In dimensional regularization, hence in a d-dimensional space-time, we define a l-loop Feynman integral with n external legs and t internal propagators as the integral

$$I^{d(l,n)}_{x_1,\ldots,x_t}[\mathcal{N}] = \int \prod_{i=1}^{l} \frac{d^D q_i}{(2\pi)^D} \frac{\mathcal{N}(q_j)}{D_1^{x_1} \cdots D_t^{x_t}} \tag{3.1}$$

with $\mathcal{N}(q_j)$ a generic tensor numerator that may depend on the loop momenta, while the denominators D_i are, extending to multiloop the one loop definition given in (2.87),

$$D_i = l_i^2 + m_i^2 \tag{3.2}$$

where

$$l_i^\alpha = \sum_j \alpha_{ij} q_j^\alpha + \sum_j \beta_{ij} p_j^\alpha \tag{3.3}$$

with p_j^α being the external momenta while α and β are incidence matrices which entries take values in $(0, \pm 1)$. From here on the normalization factor of $(2\pi)^D$ will be omitted. After applying the tensor the composition outlined in 2.3.1 in this multiloop case, one obtains that the starting integral can be written as a rational function of the scalar product as in the single loop case. Nonetheless, the relation between scalar product and loop momenta here are quite different. The number of scalar product now is much greater than n: for each loop momenta we have $n-1$ scalar product like $q_i \cdot p_j$, on top of that we have to add the scalar product between loop momenta. In the end the total number is

$$n_{SP} \equiv r = l(n-1) + \frac{l(l+1)}{2} = \frac{l(2n+l-1)}{2} \tag{3.4}$$

Consider for instance a theory that has only 3 legged vertices. In this case a tree level diagram will have in total $I = n-3$ internal lines, and hence the same number of propagators (n is the number of external legs). A vertex has $n=3$ and $I=0$ as expected. Adding a loop (i. e. connecting 2 points on some lines) increases I by 3. So, I for n-legged l-loop diagrams is

$$I = 3l + n - 3 \tag{3.5}$$

leading to

$$r - I = \frac{(l-1)(l+2n-6)}{2} \tag{3.6}$$

As expected, in the one loop case $n_{SP} - I = 0$, hence there are as many scalar products as denominators. But what happens if one goes to 2 loop? The difference between scalar products and denominators become

$$r - I|_{l=2} = n - 2 \tag{3.7}$$

Thanks to this example one sees that at multi loop level the correspondence between denominators and scalar products no longer exist and there could be Feynman integrals that, after the tensor reduction and further simplification cannot be cast in to the form presented in equation (2.93). This is due to the presence of scalar products that cannot be reduced, so called *irreducible scalar products*(i.e. ISP's). In general a multiloop scalar integral can be cast into

$$I^{d(l,n)}(x_1,\ldots,x_t,y_1,\ldots,y_a) = \int \prod_{i=1}^{l} \frac{d^D q_i}{(2\pi)^D} \frac{S_1^{y_1}\cdots S_a^{y_a}}{D_1^{x_1}\cdots D_t^{x_t}} \tag{3.8}$$

where S_i are the ISP. Differently for what happen at one loop, in this case not all the scalar product can be written in terms of the denominators of the diagram. Because of this, equation (3.8) can be considered as the general formula for the scalar integrals composing the amplitude after the tensor decomposition *and* the integrand reduction. An alternative way to represent a generic integral at multi loop level is to define a set of irreducible numerators (linear functions of the scalar products) $D_m,\ldots,D_r$ to add to the set of denominators such that all the scalar products appearing in the Feynman amplitude can be reduced into a combination of $\{D_1,\ldots,D_r\}$. In the amplitude becomes

$$I^{d(l,n)}(x_1,\ldots,x_r) = \int \prod_{i=1}^{l} \frac{d^D q_i}{(2\pi)^D} \frac{1}{D_1^{x_1}\cdots D_t^{x_t} D_{t+1}^{x_{t+1}}\cdots D_r^{x_r}} \tag{3.9}$$

where this time the powers $x_{t+1}, \ldots, x_r$ are negative.

The object defined at (3.9) is usually referred to as integral families. In principle, we can allow the powers x_i to assume any integer value. It useful to introduce some additional terminology:

- We define a topology (or sector) as an integral of the type (3.9) which corresponds to a graph, i.e. which has the ISPs raised to a negative power.

- Given a topology, its subtopologies correspond to integrals where some denominators are raised to zero power. The graph of a subtopology can be obtained from the one of the âĂIJparent âĂİ topology by pinching (i.e. removing) the corresponding loop propagators.

Any integral family contains, obviously, infinitely many different integrals, each one corresponding to a particular integer tuple $\{x_1, \ldots, x_r\}$. However, only a finite number of such integrals is actually independent, due to the existence of linear relations between Feynman integrals which are a direct consequence of the invariance of eq. (3.9) under Lorentz transformations and re-parametrization of the loop momenta.

3.2 Lorentz invariance identities

It is clear that the first and the last step can be performed at higher loop too, but what kind of relation can be built between integrals at this perturbative level? A key role in this step is played by *symmetries*. For example, we know that the integrals defined in (3.9) are Lorentz scalar, i.e. they are invariant under rotation of the external momenta

$$p_i^\alpha \to p_i^\alpha + \delta\omega^{\alpha\beta} p_{i\beta}, \tag{3.10}$$

with $\delta\omega^{\alpha\beta}$ an antisymmetric tensor ($\delta\omega^{\alpha\beta} = -\delta\omega^{\beta\alpha}$). Imposing the invariance of $I^{d(l,n)}(x_1, \ldots, x_{n_{SP}})$ under this transformation one obtains the identity

$$\begin{aligned} 0 &= \sum_{i=1}^{n} \delta\omega^{\alpha\beta} \left(p_i^\beta \frac{\partial}{\partial p_i^\alpha} \right) I^{d(l,n)}(x_1, \ldots, x_{n_{SP}}) \\ &= \sum_{i=1}^{n} \left(p_i^{[\beta} \frac{\partial}{\partial p_i^{\alpha]}} \right) I^{d(l,n)}(x_1, \ldots, x_{n_{SP}}) \end{aligned} \tag{3.11}$$

where the antisymmetry and arbitrariness of $\delta\omega$ where used in order to draw such relation.

If one contracts eq. (3.11) with all possible antisymmetric tensors built from external momenta, like $p_i^{[\alpha} p_j^{\beta]}$ it is possible to get $(n-1)(n-2)/2$ *Lorentz invariance identities* between Feynman integrals. This is because applying the derivation over the external momenta, eq. (3.11) turns in to a vanishing linear combination of integrals that, after the simplification of the reducible scalar product, becomes a linear identity between different integrals of the type (3.9).

3.3 Integration-by-parts identities

Another fundamental symmetry of the integrals $I^{d(l,n)}(x_1, \ldots, x_{n_r})$, is the invariance under shift of loop momenta of the type

$$q_i^\alpha \to q_i^\alpha + \delta b_{ij} k_j^\alpha \tag{3.12}$$

where $k_j \in \{q_1, \cdots, q_l, p1, \cdots, p_n\}$ and δb_{ij} is again an infinitesimal parameter. This symmetry generate a great number of identities that go under the name of *Integration-by-Part-Identities* or IBP, as shown in [12]. An alternative, intuitive way to derive those relation is given by the D dimensional Gauss theorem. In fact, since our integrals are dimensionally regularized and D is treated as a continuous parameter, one can assume that the integral (3.9) is well defined and hence convergent. In order to have such properties, the integrand must vanish rapidly enough a the boundary of the manifold spanned by the loop momenta. A consequence to this fact is that when integrating by parts $I^{d(l,n)}(x_1, \ldots, x_{n_{SP}})$, no boundary terms is generate. Alternatively the integral of the total derivative of *any* Feynman integrand, in D dimensions, must vanish,

$$\int \prod_{i=1}^{l} \frac{d^D q_i}{(2\pi)^D} \frac{\partial}{\partial q_j^\alpha} \left(\frac{v^\alpha}{D_1^{x_1} \cdots D_r^{x_r}} \right) = 0 \tag{3.13}$$

where v^α is a vector such that $v \in \{q_1, \cdots, q_l, p_1 1, \cdots, p_n\}$. Choosing to differentiate over any loop momenta and by changing the vector v^α over all its possible values it is possible to produce $l(l + n - 1)$ IBPs for each integral

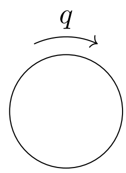
Example

In order to better understand the power of such tool, and to gain confidence in using it, we will apply it in a couple of case.

IBPs for the Tadpole topology

Let's start with the simplest integral of the kind (3.9), the one with only one denominator: the tadpole

$$\int \frac{d^D q}{(2\pi)^D} \frac{1}{k^2 + m^2} = \int \frac{d^D q}{(2\pi)^D} \frac{1}{D_0} = \tag{3.14}$$



As a convention, the integral will be portrayed as a graph which has momentum conservation applied at each vertex, and its internal line will correspond to its denominators. If an internal line has dots over it, it means that the denominator corresponding to that line is raised to the power $a - 1$, with a being the number of dots appearing over it.

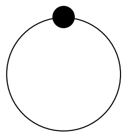
In this case, deriving with respect to the loop momenta, one obtains $\partial_\mu D_0 = 2q_\mu$. Hence choosing $v^\mu = q^\mu$, in the IBP equation (3.13) the following IBP:

$$0 = \int \frac{d^D q}{(2\pi)^D} \partial_\mu \left(\frac{q^\mu}{D_0} \right) = \int \frac{d^D q}{(2\pi)^D} \left(\frac{D}{D_0} - 2\frac{q^2}{D_0^2} \right) = \int \frac{d^D q}{(2\pi)^D} \left(\frac{D-2}{D_0} + \frac{2m^2}{D_0^2} \right) \tag{3.15}$$

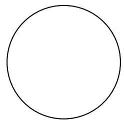
In the end the relation found is

$$\int \frac{d^D q}{(2\pi)^D} \frac{1}{D_0^2} = -\frac{D-2}{2m^2} \int \frac{d^D q}{(2\pi)^D} \frac{1}{D_0} \tag{3.16}$$

and can be portrayed as



$$= -\frac{D-2}{2m^2}$$



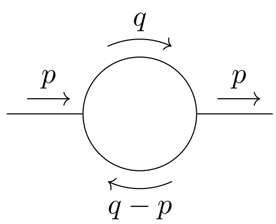
(3.17)

IBPs for the Bubble topology

Now let's derive an IBP in a more interesting case. Let's take now an integral with two denominators, we have:

$$\int \frac{d^D q}{(2\pi)^D} \frac{1}{D_1 D_2} \qquad \text{with} \qquad D_1 = q^2, D_2 = (q-p)^2 + m^2 = q^2 - 2q \cdot p \tag{3.18}$$

where we used the fact that $p^2 = -m^2$ in the Euclidean space. This integral can be represented in this new convention as



In this case, deriving with respect to the integration momenta, one obtains $\partial_\mu D_1 = 2q_\mu, \partial_\mu D_2 = q_m u^2 - 2p_m u$. Choosing $v^\mu = q^\mu$ in this case too, leads to the following IBP:

$$\begin{aligned} 0 &= \int \frac{d^D q}{(2\pi)^D} \partial_\mu \left(\frac{q^\mu}{D_1 D_2} \right) = \int \frac{d^D q}{(2\pi)^D} \left(\frac{D}{D_1 D_2} - \frac{2q^2}{D_1^2 D_2} - \frac{2q^2 - 2q \cdot p}{D_1 D_2^2} \right) \\ &= \int \frac{d^D q}{(2\pi)^D} \left(\frac{D}{D_1 D_2} - 2\frac{D_1}{D_1^2 D_2} - \frac{D_1 + D_2}{D_1 D_2^2} \right) = \int \frac{d^D q}{(2\pi)^D} \frac{D-3}{D_1 D_2} - \int \frac{d^D q}{(2\pi)^D} \frac{1}{D_2} \end{aligned} \tag{3.19}$$

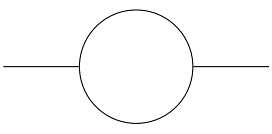
Through a shift in the loop momentum one can write

$$\int \frac{d^D q}{(2\pi)^D} \frac{1}{(q-p)^2 + m^2} = \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2 + m^2} = \int \frac{d^D q}{(2\pi)^D} \frac{1}{D_0} \tag{3.20}$$

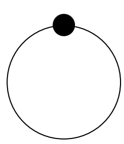
And hence arrives to the final relation

$$\int \frac{d^D q}{(2\pi)^D} \frac{1}{D_1 D_2} = \frac{1}{D-3} \int \frac{d^D q}{(2\pi)^D} \frac{1}{D_0^2} \tag{3.21}$$

that can be drawn as



$$= \frac{1}{D-3}$$

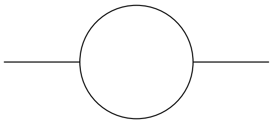


(3.22)

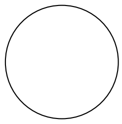
Now, substituting the IBP (3.17) found for the tadpole example, one arrives to the final relation

$$\int \frac{d^D q}{(2\pi)^D} \frac{1}{D_1 D_2} = -\frac{D-2}{D-3} \frac{1}{2m^2} \int \frac{d^D q}{(2\pi)^D} \frac{1}{D_0} \tag{3.23}$$

that can be portrayed in a very intuitive way:



$$= -\frac{D-2}{D-3}\frac{1}{2m^2}$$



(3.24)

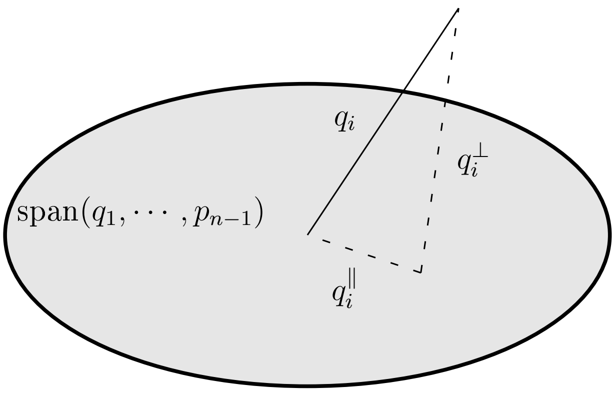
Thanks to the IBP we were able to write an integral that has 2 denominators in function of one with 1 denominator, simplifying greatly the calculation. This simple, but clear example, show the power of such a tool with which is possible to find relation between different Feynman integrals, in order to reduce the problem of the evaluation of an amplitude into the evaluation of a small set of *master integrals*. ■

3.4 Baikov representation

It is possible to write the Feynman integral that appears in eq. (3.9) as an integral in the scalar products that contain the loop momenta. Recalling that l the number of loop and n the number of external legs with $n-1$ independent momenta and defining $m = l + n - 1$ as the total number of independent momenta, it is possible to cast the integration measure in the following form:

$$d^D q_1 \cdots d^D q_l = d^{m-1} q_{1||} d^{D-m+1} q_{1\perp} \cdots d^{m-l} q_{l||} d^{D-m+l} q_{l\perp} \tag{3.25}$$

where $q_{i||}$ lies in the space spanned by $\{q_{i+1} \dots q_l,\, p_1 \dots p_{n-1}\}$, while $q_{i\perp}$ lies in the orthogonal space. To better explain this factorization of the integration measure, it is represented in the Figure below



Introducing $k = (q_1 \dots q_l, p_1 \dots p_{n-1})$, where for $i \le l$ we have that $k_i = q_i$ while for $i > l$, we have $k_i = p_{i-l}$ we can define

$$G(q_1, \dots, q_l, p_1, \dots, p_{n-1}) = \det(k_i \cdot k_j) = \det(s_{ij}) \tag{3.26}$$

which is the determinant of the Gram matrix $\boldsymbol{G}(q_1, \dots, q_l, p_1, \dots, p_{n-1})$, the matrix that takes as entries the scalar product between all the momenta appearing in the amplitude:

$$\boldsymbol{G} = \begin{pmatrix} s_{11} & \cdots & s_{1m} \\ \vdots & \ddots & \vdots \\ s_{m1} & \cdots & s_{mm} \end{pmatrix} \tag{3.27}$$

An intuitive way to understand what the Gram determinant is is given by the fact that $G^{1/2}$ corresponds to the volume of the parallelotope spanned by the vectors $q_1, \dots, q_l,\ p_1, \dots, p_{n-1}$.

Then, by geometric considerations, the volume elements $d^{m-i}q_{i||}$ are

$$d^{m-i}q_{i||} = \frac{ds_{i,i+1}ds_{i,i+2}\cdots ds_{i,m}}{G^{1/2}(q_i, q_{i+1}, \dots, q_l, p_1, \dots, p_n - 1)} \tag{3.28}$$

in which the denominators are Gram determinants, hence they corresponds to the volumes of the parallelogram formed by the momenta $(q_i, q_{i+1}, \dots, q_l, p_1, \dots, p_{n-1})$.

For the orthogonal component instead the measure becomes

$$d^{D-m+i}q_{i\perp} = \Omega_{D-m+i-1}|q_{i\perp}|^{D-m+i-1}d|q_{i\perp}| = \frac{1}{2}\Omega_{D-r+i-1}(|q_{i\perp}|)^{D-m+i-2}dq_{i\perp}^2 \tag{3.29}$$

where $|q_{i\perp}|$ is the height of the parallelogram with the base formed by $q_i, q_{i+1}, \dots, q_l, p_1, \dots, p_{n-1}$. $|q_{i\perp}|$ can also be written as the volume of the whole parallelogram divided by the area of its base, thus replacing $dq_{i\perp}^2$ with ds_{ii} we arrive to

$$d^{D-m+i}q_{i\perp} = \frac{1}{2}\Omega_{D-m+i-1}\left(\frac{G(q_i + 1, q_{i+2}, \dots, q_l, p_1, \dots, p_{n-1})}{G(q_{i+1}, q_{i+2}, \dots, q_l, p_1, \dots, p_{n-1})}\right)^{\frac{D-m+i-2}{2}} ds_{ii} \tag{3.30}$$

After all these considerations, putting everything back together leads to

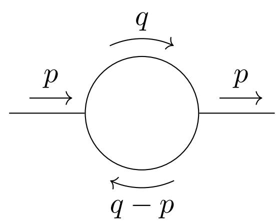
$$\int_{\mathcal{D}}\prod_1^l \frac{d^Dq_i}{(2\pi)^D}\frac{1}{D_1^{x_1}\cdots D_r^{x_r}} = \frac{1}{(2\pi)^D}\frac{\pi^{Dl/2-l(l-1)/4-l(n-1)/2}}{\prod_{i=1}^l \Gamma(\frac{D-m+i}{2})}G(p_1, \dots, p_{n-1})^{\frac{-D+n}{2}}$$
$$\int_{\mathcal{D}}\prod_{i=1}^l\prod_{j\geq i}^r ds_{ij}G(q_1, q_2, \dots, q_l, p_1, \dots, p_{n-1})^{\frac{D-m-1}{2}}\frac{1}{D_1^{x_1}\cdots D_r^{x_r}} \tag{3.31}$$

with $\mathcal{D}$ beeing the region of integration. $\mathcal{D}$ can be rather complicate, nevertheless as thoroughly proven in [13] it's boundaries are determined by the brunch cat of the integrand. Due to the fact that it has the Gram determinant raised to a non integer power, this condition can be translated as the Gram determinant vanishes at its boundaries.

Examples

Bubble amplitude

To showcase the validity of this parameterization, let's apply it to the case of the bubble amplitude with massless propagators. Namely to the graph



which represents the integral

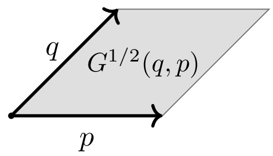
$$\int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2(q-p)^2} \tag{3.32}$$

The bubble integral corresponds to the case in which $l = 1, n = 2, m = 2$ in (3.31). Applying that formula to this case leads to

$$I = \frac{1}{(2\pi)^D} \frac{\pi^{3/2-\epsilon}}{\Gamma(3/2-\epsilon)} \left[G(p)\right]^{-1+\epsilon} \int \frac{ds_{11}}{s_{11}} \int \frac{ds_{12}}{s_{11}+p^2-2s_{12}} \left[G(q,p)\right]^{1/2-\epsilon} \tag{3.33}$$

The variables appearing in such integral are $s_{11} = q^2$ and $s_{12} = q \cdot p = |q||p|\cos\theta$ with $\theta \in [0,\pi]$. This makes the integration variable range between $s_{11} \in [0, +\infty]$ and $s_{12} \in [-\sqrt{s_{11}p^2}, \sqrt{s_{11}p^2}]$, that correspond to the condition of obvious positiveness of s_{11} and to the fact that G vanishes at the boundaries.

Being $G^{1/2}$ the volume spanned by the momenta p and q, let's remark the fact that the condition that determines the integration boundaries is equivalent to imposing that such volume collapses into a lower dimensional volume. This is due to the fact that G is zero when at least a couple of the momenta that composes its scalar product are linearly dependent. In this case, graphically:



$G^{1/2}(q,p) = 0 \quad \Leftrightarrow$

(3.34)

As one notice, the Gram determinant approach zero as the vectors that composes it approach one another, becoming linearly dependent. This, as the Figure above show, happens when $\theta = 0, \pi$.

The integral then becomes

$$I = \frac{1}{(2\pi)^D} \frac{\pi^{3/2-\epsilon}}{\Gamma(3/2-\epsilon)} (p^2)^{-1+\epsilon} \int_0^{+\infty} \frac{ds_{11}}{s_{11}} \int_{-\sqrt{s_{11}p^2}}^{\sqrt{s_{11}p^2}} \frac{ds_{12}}{s_{11}+p^2-2s_{12}} \left(s_{11}p^2 - s_{12}^2\right)^{1/2-\epsilon} \tag{3.35}$$

It is useful to turn the integration boundaries on ds_{12} to 0 and 1, hence we perform some change of variables, inverting the condition that $s_{12} = \sqrt{s_{11}p^2}(2u-1)$ and then rescaling ds_{11}

$$
\begin{aligned}
I &= \frac{1}{(2\pi)^D}\frac{2^{2-2\epsilon}\pi^{3/2-\epsilon}}{\Gamma(3/2-\epsilon)}\int_0^{+\infty} dx_1 \frac{s_{11}^{-\epsilon}}{(\sqrt{s_{11}}+\sqrt{p^2})^2}\int_0^1 du\left[1-\frac{4\sqrt{s_{11}p^2}}{(\sqrt{s_{11}}+\sqrt{p^2})^2}u\right]^{-1}(u(1-u))^{1/2-\epsilon}\\
&= \frac{1}{(2\pi)^D}\frac{2^{2-2\epsilon}\pi^{3/2-\epsilon}}{\Gamma(3/2-\epsilon)}(p^2)^{-\epsilon}\int_0^{+\infty} dv\frac{v^{-\epsilon}}{(1+\sqrt{v})^2}\int_0^1 du\left[1-\frac{4\sqrt{v}}{(1+\sqrt{v})^2}u\right]^{-1}(u(1-u))^{1/2-\epsilon}.
\end{aligned}
\tag{3.36}
$$

It could resemble quite a difficult integration to perform, however the integral over du can be written in terms of the integral representation of the Hypergeometric function, the analytic continuation of the Gaussian Hypergeometric series:

$$
{}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 u^{b-1}(1-u)^{c-a-1}(1-zu)^{-a} \tag{3.37}
$$

valid for $\Re(c) > \Re(b) > 0$.
Our integral then becomes

$$
\int_0^1 du(u(1-u))^{1/2-\epsilon}\left[1-\frac{4\sqrt{v}}{(1+\sqrt{v})^2}u\right]^{-1} = \frac{\Gamma^2(3/2-\epsilon)}{\Gamma(3-2\epsilon)}\,{}_2F_1\left(1,\frac{3}{2}-\epsilon;3-2\epsilon;\frac{4\sqrt{v}}{(1+\sqrt{v})^2}\right) \tag{3.38}
$$

obtaining:

$$
\begin{aligned}
I &= \frac{2^{2-2\epsilon}\pi^{3/2-\epsilon}}{(2\pi)^D}\frac{\Gamma(3/2-\epsilon)}{\Gamma(3-2\epsilon)}(p^2)^{-\epsilon}\int_0^{+\infty} dv\frac{v^{-\epsilon}}{(1+\sqrt{v})^2}\,{}_2F_1\left(1,\frac{3}{2}-\epsilon;3-2\epsilon;\frac{4\sqrt{v}}{(1+\sqrt{v})^2}\right)\\
&= \frac{1}{(2\pi)^D}\frac{\pi^{2-\epsilon}}{\Gamma(2-2\epsilon)}(p^2)^{-\epsilon}\int_0^{+\infty} dv\frac{v^{-\epsilon}}{(1+\sqrt{v})^2}\,{}_2F_1\left(1,\frac{3}{2}-\epsilon;3-2\epsilon;\frac{4\sqrt{v}}{(1+\sqrt{v})^2}\right)
\end{aligned}
\tag{3.39}
$$

where the property of the Gamma function presented in (2.62) has been used.
Luckily, Hypergeometric functions satisfy a plethora of useful identities that can greatly simplify the calculation, as for example, the following:

$$
{}_2F_1\left(a,b;2b;\frac{4z}{(1+z)^2}\right) = (1+z)^{2a}\,{}_2F_1\left(a,a-b+\frac{1}{2};b+\frac{1}{2};z^2\right) \tag{3.40}
$$

valid for $|z| < 1$.

The Hypergeometric function in (3.39) would have $z = \sqrt{v}$, but $v \in [0,+\infty[$ so this property isn't suitable for our integral as it is right now.In order to proceed with the calculation it is useful to split our integral in two:

$$
I = \frac{1}{(2\pi)^D}\frac{\pi^{2-\epsilon}}{\Gamma(2-2\epsilon)}(p^2)^{-\epsilon}\left[\int_0^1+\int_1^{+\infty}\right]\left[dv\frac{v^{-\epsilon}}{(1+\sqrt{v})^2}\,{}_2F_1\left(1,\frac{3}{2}-\epsilon;3-2\epsilon;\frac{4\sqrt{v}}{(1+\sqrt{v})^2}\right)\right] = I_1+I_2 \tag{3.41}
$$

where I_1 is the integral between 0 and 1, while I_2 cover the remaining value of v. Now it is possible to apply the (3.40), starting from the first one gets:

$$
\begin{aligned}
I_1 &= \frac{(p^2)^{-\epsilon}}{(4\pi)^{2-\epsilon}\Gamma(2-2\epsilon)}\int_0^1 dv\frac{v^{-\epsilon}}{(1+\sqrt{v})^2}\,{}_2F_1\left(1,\frac{3}{2}-\epsilon;3-2\epsilon;\frac{4\sqrt{v}}{(1+\sqrt{v})^2}\right)\\
&= \frac{(p^2)^{-\epsilon}}{(4\pi)^{2-\epsilon}\Gamma(2-2\epsilon)}\int_0^1 dv v^{-\epsilon}\,{}_2F_1\left(1,\epsilon;2-\epsilon;;v\right).
\end{aligned}
\tag{3.42}
$$

For the second integral, it is mandatory to set $v' = 1/v$ in order to apply (3.40), obtaining

$$\begin{aligned} I_2 &= \frac{(p^2)^{-\epsilon}}{(4\pi)^{2-\epsilon}\Gamma(2-2\epsilon)} \int_1^{+\infty} dv \frac{v^{-\epsilon}}{(1+\sqrt{v})^2} {}_2F_1\left(1, \frac{3}{2}-\epsilon; 3-2\epsilon; \frac{4\sqrt{v}}{(1+\sqrt{v})^2}\right) \\ &= \frac{(p^2)^{-\epsilon}}{(4\pi)^{2-\epsilon}\Gamma(2-2\epsilon)} \int_0^1 \frac{dv'}{v'^2} \frac{1}{v'^{-\epsilon}} \frac{v'}{(1+\sqrt{v'})^2} {}_2F_1\left(1, \frac{3}{2}-\epsilon; 3-2\epsilon; \frac{4\sqrt{v'}}{(1+\sqrt{v'})^2}\right) \\ &= \frac{(p^2)^{-\epsilon}}{(4\pi)^{2-\epsilon}\Gamma(2-2\epsilon)} \int_0^1 dv' \, v'^{-1+\epsilon} {}_2F_1\left(1, \epsilon; 2-\epsilon; v'\right). \end{aligned} \tag{3.43}$$

Summing everything back and setting $v' = v$ in I_2 leads to

$$I = I_1 + I_2 = \frac{(p^2)^{-\epsilon}}{(4\pi)^{2-\epsilon}\Gamma(2-2\epsilon)} \int_0^1 dv \left(v^{-\epsilon} + v^{-1+\epsilon}\right) {}_2F_1\left(1, \epsilon; 2-\epsilon; v\right) \tag{3.44}$$

This is what is called the generalized Hypergeometric function, which is defined in his integral representation as:

$${}_3F_2(a_1, a_2, a_3; b_1, b_2; z) = \frac{\Gamma(b_2)}{\Gamma(a_3)\Gamma(b_2 - a_3)} \int_0^1 dv v^{a_3-1}(1-v)^{b_2-a_3-1} {}_2F_1(a_1, a_2; b_1, vz). \tag{3.45}$$

So in the end our integral reduces to

$$\begin{aligned} I &= \frac{(p^2)^{-\epsilon}}{(4\pi)^{2-\epsilon}\Gamma(2-2\epsilon)} \left[\frac{\Gamma(1-\epsilon)}{\Gamma(2-\epsilon)} {}_3F_2(1,\epsilon,1-\epsilon;2-\epsilon,2-\epsilon;1) + \frac{\Gamma(\epsilon)}{\Gamma(1+\epsilon)} {}_3F_2(1,\epsilon,\epsilon;2-\epsilon,1+\epsilon;1)\right] \\ &= \frac{(p^2)^{-\epsilon}}{(4\pi)^{2-\epsilon}\Gamma(2-2\epsilon)} \left[\frac{1}{1-\epsilon} {}_3F_2(1,\epsilon,1-\epsilon;2-\epsilon,2-\epsilon;1) + \frac{1}{\epsilon} {}_3F_2(1,\epsilon,\epsilon;2-\epsilon,1+\epsilon;1)\right]. \end{aligned} \tag{3.46}$$

Using the fact that

$$\begin{aligned} {}_3F_2(a_1,a_2,a_3;b_1,1+a_3;1) =& \frac{\Gamma(b_1)\Gamma(1-a_2)\Gamma(1+a_3)\Gamma(a_1-a_3)}{\Gamma(a_1)\Gamma(1-a_2+a_3)\Gamma(b_1-a_3)} \\ &+ \frac{\Gamma(b_1)\Gamma(1-a_2)\Gamma(1+a_3)\Gamma(a_3-a_1)}{\Gamma(a_3)\Gamma(1+a_1-a_2)\Gamma(1-a_1+a_3)\Gamma(b_1-a_1)} \\ &{}_3F_2(a_1, a_1-a_3, 1+a_1-b_1; 1+a_1-a_2, 1+a_1-a_3; 1), \end{aligned} \tag{3.47}$$

applied on the first generalized Hypergeometric function, one gets

$$\begin{aligned} \frac{1}{1-\epsilon} {}_3F_2(1,\epsilon,1-\epsilon;2-\epsilon,2-\epsilon;1) =& \frac{1}{1-\epsilon} \frac{\Gamma(2-\epsilon)\Gamma(1-\epsilon)\Gamma(2-\epsilon)\Gamma(\epsilon)}{\Gamma(1)\Gamma(2-2\epsilon)\Gamma(1)} \\ &+ \frac{1}{1-\epsilon} \frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)\Gamma(-\epsilon)\Gamma(2-\epsilon)}{\Gamma^3(1-\epsilon)\Gamma(2-\epsilon)} {}_3F_2(1,\epsilon,\epsilon;2-\epsilon,1+\epsilon;1) \\ =& \frac{\Gamma(2-\epsilon)\Gamma^2(1-\epsilon)\Gamma(\epsilon)}{\Gamma(2-2\epsilon)} - \frac{1}{\epsilon} {}_3F_2(1,\epsilon,\epsilon;2-\epsilon,1+\epsilon;1) \end{aligned} \tag{3.48}$$

Given this result, this integral finally becomes

$$\begin{aligned} I &= \frac{(p^2)^{-\epsilon}}{(4\pi)^{2-\epsilon}} \frac{\Gamma(\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(2-2\epsilon)} \\ &= \frac{(-M^2)^{-\epsilon}}{(4\pi)^{2-\epsilon}} \frac{\Gamma(\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(2-2\epsilon)} \end{aligned} \tag{3.49}$$

which is exactly the amplitude one obtains using the Feynman parametrization outlined in (2.56) up to a phase factor as expected. This difference is due to the fact that we're operating in Euclidean space as stated in [13]

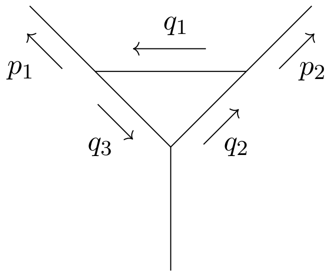
■

Integration domain in the 3 point function

In this example we will take a closer look to the integration domain in the Baikov representation. The integral taken in to account is

$$I = \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2(q-p_1)^2(q+p_2)^2} \tag{3.50}$$

which can be represented as



where $q_2 = q + p_2$ while $q_3 = q - p_1$.
Applying (3.31) in this case leads to

$$I = \frac{\pi^{1-\epsilon}}{\Gamma(1-\epsilon)} \frac{[G(p_1,p_2)]^{-1/2+\epsilon}}{(2\pi)^D} \int \frac{ds_{11}}{s_{11}} \int \frac{ds_{12}}{s_{11} - 2s_{12} + s_{22}} \int ds_{13} \frac{[G(q,p_1,p_2)]^{-\epsilon}}{s_{11} + 2s_{13} + s_{33}} \tag{3.51}$$

As stated before, *any* couple of vectors contained in the Gram determinant that becomes linearly dependent make the Gram determinant null. This property gives a powerful insight on how it is possible to fix the integration boundaries of *all* the scalar products variables.

s_{11} is bounded by its positivity condition as in the case of the bubble. s_{12} by imposing that the vectors that builds it becomes parallel at the integration boundaries ($s_{12} \in [-\sqrt{s_{11}p_1^2}, \sqrt{s_{11}p_1^2}]$, hence through $\det G(q,p_1) = 0$, setting the minor of $G(q,p_1,p_2)$ containing q and p_1 equal to 0.

At this point one has to find the integration boundaries for s_{13} at fixed s_{11} and s_{12}, or at fixed modulus of q and at fixed angle θ_{12}, between q and p_1.

We know that

$$s_{13} = \sqrt{s_{11}s_{33}} \cos(\theta_{13}) \tag{3.52}$$

nonetheless it's difficult to determine the dependence of s_{13} in function of s_{12}, it is in fact a lot easier to do it graphically. At fixed θ_{12}, q can only rotate around p_1 hence it lies on a cone around the external momentum as shown in Figure 3.1 on the following page.

In this figure the external momenta are depicted with a thicker line and the value of q corresponding to the maximum and minimum value of θ_{13} (and hence of s_{13}) are drawn in red.

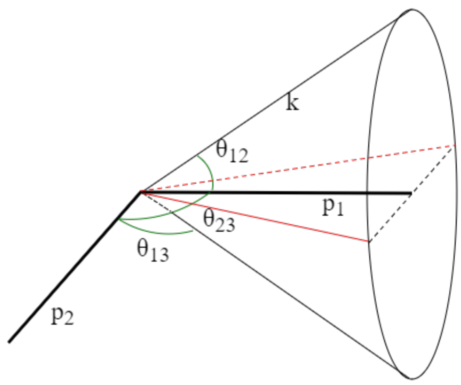


Figure 3.1: Possible value of θ_{23}

As one can notice, those values took place when the three momenta lies in a common plane, hence again when they are linearly dependent (satisfying the condition $\det G(q, p_1, p_2) = 0$ as stated before). Solving the latter gives

$$s_{13} = \frac{s_{12}s_{23} \pm \sqrt{G(q,p_1)G(p_1,p_2)}}{s_{22}} \tag{3.53}$$

,the integration boundaries for s_{13}. In the end the amplitude can be written as

$$I = \frac{\pi^{1-\epsilon}}{\Gamma(1-\epsilon)} \frac{[G(p_1,p_2)]^{-1/2+\epsilon}}{(2\pi)^D} \int_0^{+\infty} ds_{11} \int_{G(q,p_1)\geq 0} ds_{12} \int_{G(q,p_1,p_2)\geq 0} ds_{13} \, [G(q,p_1,p_2)]^{-\epsilon} \, F\left(s_{11}, s_{12}, s_{13}\right) \tag{3.54}$$

where F is, in this case

$$F^{-1} = s_{11}\left(s_{11} + s_{22} - 2s_{12}\right)\left(s_{11} + s_{33} + 2s_{13}\right) \tag{3.55}$$

■

3.4.1 Integration Boundaries

Looking back at the previous example, one finds that an iterative structure starts to arise. In fact, one can start by fixing all the boundaries of s_{ii} by their positivity condition, and then fixes the boundaries of s_{ij} with $i \neq j$ by $G(q, \cdots, p_{n-1}) \geq 0$, proceeds with the remaining scalar products, setting greater and greater minors of the Gram determinant to 0 until one arrives to impose this condition on the Gram determinant built with all the momenta appearing in the amplitude to , in order to determine the boundaries for the last scalar product. In this way one can determine the whole integration domain. This idea leads to the following expression, with the integration domain written explicitly:

$$
\begin{aligned}
\int d^D q_1 \dots d^D q_l F =& \frac{\pi^{Dl/2-l(l-1)/4-l(n-1)/2}}{\prod_{i=1}^{l}\Gamma(\frac{D-m+i}{2})} G(p_1,\dots,p_{n-1})^{\frac{-D+n}{2}} \int_0^{+\infty} ds_{11} \cdots \int_0^{+\infty} ds_{ll} \\
& \int_{G(q_1,p_1)\geq 0} ds_{1,l+1} \cdots \int_{G(q_1,p_1,\dots,p_{n-1})\geq 0} ds_{1m} \cdots \int_{G(q_l,p_1,\dots,p_{n-1})\geq 0} ds_{lm} \\
& \int_{G(q_1,q_2,p_1,\dots,p_{n-1})\geq 0} ds_{12} \cdots \int_{G(q_1,q_l,p_1,\dots,p_{n-1})\geq 0} ds_{1l} \\
& \int_{G(q_1,q_2,q_3,\dots,p_{n-1})\geq 0} ds_{23} \cdots \int_{G(q_1,\dots,q_l,p_1,\dots,p_{n-1})\geq 0} ds_{l-1,l}
\end{aligned} \tag{3.56}
$$

alternatively we can write it as a multiplication of productoria

$$
\begin{aligned}
\int d^D q_1 \dots d^D q_l F =& \frac{\pi^{Dl/2-l(l-1)/4-l(n-1)/2}}{\prod_{i=1}^{l}\Gamma(\frac{D-m+i}{2})} G(p_1,\dots,p_{n-1})^{\frac{-D+n}{2}} \\
& \prod_{i=1}^{l}\left[\int_0^{+\infty} ds_{ii}\right] \prod_{i=1}^{l}\prod_{j=1}^{n-1}\left[\int_{G(q_i,p_1,\dots,p_j)\geq 0} ds_{i,l+j}\right] \\
& \prod_{i=1}^{L}\left[\int_{G(q_1,q_i,p_1,\dots,p_{n-1})\geq 0} ds_{1i}\right] \prod_{i=1}^{L}\prod_{j=i+1}^{l}\left[\int_{G(q_1,\dots,q_i,q_j,p_1,\dots,p_{n-1})\geq 0} ds_{ij}\right] F
\end{aligned} \tag{3.57}
$$

The advantage of using this formula is that the 0 of a generic Gram determinant is known. Looking at the boundaries determined in (3.53) one notice that they have a rather lucky shape. This is not due to luck alone. In fact using Laplace's formula to expand the determinant one finds

$$
G(q_1, q_2, \dots, q_L, p_1, \dots, p_E) = (s_{ij})^2 a + s_{ij} b_{ij} + c_{ij} \tag{3.58}
$$

whith

$$
\begin{aligned}
a_{ij} =& \frac{1}{2}\frac{\partial^2 G}{\partial s_{ij}^2} \\
b_{ij} =& \left.\frac{\partial G}{\partial s_{ij}}\right|_{s_{ij}=0} \\
c_{ij} =& \left. G\right|_{s_{ij}=0}
\end{aligned} \tag{3.59}
$$

hence to evaluate explicitly the integration boundaries one needs to solve a second order equation, obtaining[1]

$$
G = 0 \;\Rightarrow\; s_{ij} = \frac{b_{ij} \pm \sqrt{G_i^i G_j^j}}{G_{ij}^{ij}} \tag{3.60}
$$

[1]The structure of the zeroes in (3.60), used in (3.57), was originally conjectured by us, and late proven together with R. Sameshima (PhD student at New York City College of Technology)

with

$$G_i^j = \det \begin{pmatrix} s_{11} & \cdots & s_{1,j-1} & s_{1,j+1} & \cdots & s_{1M} \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ s_{i-1,1} & \cdots & s_{i-1,j-1} & s_{i-1,j+1} & \cdots & s_{i-1,M} \\ s_{i+1,1} & \cdots & s_{i+1,j-1} & s_{i+1,j+1} & \cdots & s_{i+1,M} \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ s_{M1} & \cdots & s_{M,j-1} & s_{M,j+1} & \cdots & s_{MM} \end{pmatrix} \tag{3.61}$$

the determinant of the Gram matrix without the i-th row and j-th column. This result generalize the one obtained in (3.53). By the direct calculation in some case one finds for example:

$$\begin{aligned} G(q,p_1,p_2) = 0 \;\Rightarrow\; s_{13} &= \frac{b_{13} \pm \sqrt{G(q,p_1)G(p_1,p_2)}}{G(p_2)} \\ G(q,p_1,p_2,p_3) = 0 \;\Rightarrow\; s_{14} &= \frac{b_{14} \pm \sqrt{G(q,p_1,p_2)G(p_1,p_2,p_3)}}{G(p_1,p_2)} \\ G((q,p_1,p_2,p_3,p_4) = 0 \;\Rightarrow\; s_{15} &= \frac{b_{15} \pm \sqrt{G(q,p_1,p_2,p_3)G(p_1,p_2,p_3,p_4)}}{G(p_1,p_2,p_3)} \end{aligned} \tag{3.62}$$

where

$$\begin{aligned} b_{13} &= s_{12}s_{23} \\ b_{14} &= s_{12}s_{24}s_{33} + s_{13}s_{22}s_{34} - s_{13}s_{23}s_{24} - s_{12}s_{23}s_{34} \\ b_{15} &= -s_{13}s_{24}s_{25}s_{34} + s_{12}s_{25}s_{34}^2 + s_{13}s_{24}^2s_{35} - s_{12}s_{24}s_{34}s_{35} + s_{13}s_{23}s_{25}s_{44} \\ &- s_{12}s_{25}s_{33}s_{44} - s_{13}s_{22}s_{35}s_{44} + s_{12}s_{23}s_{35}s_{44} - s_{13}s_{23}s_{44}s_{45} + s_{12}s_{24}s_{33}s_{45} \\ &+ s_{13}s_{22}s_{34}s_{45} - s_{12}s_{23}s_{34}s_{45} + s_{14}s_{24}s_{25}s_{33} - s_{14}s_{23}s_{25}s_{34} - s_{14}s_{23}s_{24}s_{35} \\ &+ s_{14}s_{22}s_{34}s_{35} + s_{14}s_{23}^2s_{45} - s_{14}s_{22}s_{33}s_{45} \end{aligned}$$

which are consistent to (3.60).

A proof to (3.60) will be given in the next section.

Proof

The strategy adopted to demonstrate the relation (3.60) will be to prove it first for s_{12}, and then to generalize this result to *any* scalar product. That being said, let's prove that

$$G = (s_{12})^2 a + s_{12}b_{12} + c_{12} = 0 \;\Rightarrow\; s_{12} = \frac{b_{12} \pm \sqrt{G_1^1 G_2^2}}{G_{12}^{12}} \tag{3.63}$$

Using the formula for the zeroes of a second grade polynomial on G one get

$$s_{12} = \frac{b \pm \sqrt{\Delta_{12}}}{G_{12}^{12}} \tag{3.64}$$

with

$$\Delta = b^2 - 4ac \tag{3.65}$$

where the subscription is omitted in order to do the calculation with a lighter notation.

As we know from (3.59), the parameters can be written as derivatives of G. In light of the fact that the derivative of a matrix can be calculated in the following way:

$$\frac{d}{dt}A(t) = \sum_{i=1}^{m} \begin{pmatrix} A_1 & \cdots & \frac{d}{dt}A_i & \cdots & A_m \end{pmatrix} \tag{3.66}$$

$$\tag{3.67}$$

with A an $m \times m$ matrix while A_i are the different column of the latter, one can write a, b, c as:

$$a = \det \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & s_{33} & \cdots & s_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & s_{m3} & & s_{mm} \end{pmatrix} = \det \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & \boldsymbol{G}_{12}^{12} & \\ 0 & 0 & & & \end{pmatrix} \tag{3.68}$$

$$b = \det \begin{pmatrix} 0 & 0 & s_{13} & \cdots & s_{1m} \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & s_{23} & s_{33} & \cdots & s_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & s_{m2} & s_{m3} & & s_{mm} \end{pmatrix} = \det \begin{pmatrix} 0 & 0 & s_{13} & \cdots & s_{1m} \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & s_{23} & & & \\ \vdots & \vdots & & \boldsymbol{G}_{12}^{12} & \\ 0 & s_{m2} & & & \end{pmatrix} \tag{3.69}$$

$$\tag{3.70}$$

$$c = \det \begin{pmatrix} s_{11} & 0 & s_{13} & \cdots & s_{1m} \\ 0 & s_{22} & s_{23} & \cdots & s_{2m} \\ s_{13} & s_{23} & s_{33} & \cdots & s_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{m1} & s_{m2} & s_{m3} & & s_{mm} \end{pmatrix} = \det \begin{pmatrix} s_{11} & 0 & s_{13} & \cdots & s_{1m} \\ 0 & s_{22} & s_{23} & \cdots & s_{2m} \\ s_{13} & s_{23} & & & \\ \vdots & \vdots & & \boldsymbol{G}_{12}^{12} & \\ s_{m1} & s_{m2} & & & \end{pmatrix} \tag{3.71}$$

where the bold character is used to distinguish the matrix itself (example: $\boldsymbol{G}_{ij}^{ij}$ is the gram matrix without a set of rows and columns) from its determinant (exemple: G_{ij}^{ij} is the determinant of a minor of the gram matrix).

In order to proceed with the calculation the following relation

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A - B\,D^{-1}C)\det D \tag{3.72}$$

is mandatory. In order to derive it, some small manipulation must be made. Let us define

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{3.73}$$

and

$$Y = \begin{pmatrix} 1 & -B\,D^{-1} \\ 0 & 1 \end{pmatrix} \tag{3.74}$$

In this case Z becomes

$$Z = Y\,X = \begin{pmatrix} A - B\,D^{-1}C & 0 \\ C & D \end{pmatrix} \tag{3.75}$$

Knowing that

$$\det \begin{pmatrix} A & 0 \\ C & D \end{pmatrix} = \det A \det D \tag{3.76}$$

it is possible to evaluate $\det Z$, obtaining

$$\det Z = \det(A - B\,D^{-1}C)\det D \tag{3.77}$$

thus

$$\det X = \det Z = \det(A - B\,D^{-1}C)\det D \tag{3.78}$$

Now using (3.72) to evaluate the different determinant appearing in (3.68) one gets

$$\begin{aligned}
a =& G^{12}_{12} \det \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right] = -G^{12}_{12} \\
b =& G^{12}_{12} \det \left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} s_{13} & \cdots & s_{1M} \\ 0 & \cdots & 0 \end{pmatrix} (\boldsymbol{G}^{12}_{12})^{-1} \begin{pmatrix} 0 & s_{23} \\ \vdots & \vdots \\ 0 & s_{2m} \end{pmatrix} \right] \\
=& \, G^{12}_{12} \det \left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & t_{12} \\ 0 & 0 \end{pmatrix} \right] = G^{12}_{12} t_{12} \\
c =& G^{12}_{12} \det \left[\begin{pmatrix} s_{11} & 0 \\ 0 & s_{22} \end{pmatrix} - \begin{pmatrix} s_{13} & \cdots & s_{1m} \\ s_{23} & \cdots & s_{2m} \end{pmatrix} (\boldsymbol{G}^{12}_{12})^{-1} \begin{pmatrix} s_{13} & s_{23} \\ \vdots & \vdots \\ s_{1M} & s_{2m} \end{pmatrix} \right] \\
=& \, G^{12}_{12} \det \left[\begin{pmatrix} s_{11} & 0 \\ 0 & s_{22} \end{pmatrix} - \begin{pmatrix} t_{11} & t_{12} \\ t_{12} & t_{22} \end{pmatrix} \right] = G^{12}_{12} \det \begin{pmatrix} s_{11} - t_{11} & -t_{12} \\ -t_{12} & s_{22} - t_{22} \end{pmatrix} \\
=& \, G^{12}_{12} [(s_{11} - t_{11})(s_{22} - t_{22}) - t^2_{12}]
\end{aligned} \tag{3.79}$$

where

$$t_{ji} = t_{ij} = \begin{pmatrix} s_{i3} & \cdots & s_{iM} \end{pmatrix} (\boldsymbol{G}^{12}_{12})^{-1} \begin{pmatrix} s_{j3} \\ \vdots \\ s_{jm} \end{pmatrix} \tag{3.80}$$

Substituting these explicit relation in Δ ones obtain

$$\begin{aligned}
\Delta &= \left(G^{12}_{12}\right)^2 t^2_{12} - \left(G^{12}_{12}\right)^2 [(s_{11} - t_{11})(s_{22} - t_{22}) - t^2_{12}] \\
&= \left(G^{12}_{12}\right)^2 (s_{11} - t_{11})(s_{22} - t_{22}) \\
&= G^{12}_{12} \left[s_{22} - \begin{pmatrix} s_{23} & \cdots & s_{2m} \end{pmatrix} (\boldsymbol{G}^{12}_{12})^{-1} \begin{pmatrix} s_{23} \\ \vdots \\ s_{2M} \end{pmatrix} \right] G^{12}_{12} \left[s_{11} - \begin{pmatrix} s_{13} & \cdots & s_{1m} \end{pmatrix} (\boldsymbol{G}^{12}_{12})^{-1} \begin{pmatrix} s_{13} \\ \vdots \\ s_{1M} \end{pmatrix} \right] \\
&= G^1_1 G^2_2
\end{aligned} \tag{3.81}$$

where in the second last step we used (3.72) backward to rebuild our block matrix. This proves the validity of (3.59) using s_{12} as variable, because substituting the values of Δ found in (3.81) one finds that

$$G = 0 \;\Rightarrow\; s_{12} = \frac{b_{12} \pm \sqrt{G_1^1 G_2^2}}{G_{12}^{12}} \tag{3.82}$$

Let's recall the definition of the Gram determinant appearing in the result:

$$\begin{aligned} G_1^1 &= \det \begin{pmatrix} s_{22} & s_{23} & \cdots & s_{2m} \\ s_{23} & & & \\ \vdots & & \boldsymbol{G}_{12}^{12} & \\ s_{2M} & & & \end{pmatrix} = \det \begin{pmatrix} s_{22} & \cdots & s_{2m} \\ \vdots & \ddots & \vdots \\ s_{2M} & \cdots & s_{mm} \end{pmatrix} \\ G_2^2 &= \det \begin{pmatrix} s_{11} & s_{13} & \cdots & s_{2m} \\ s_{13} & & & \\ \vdots & & \boldsymbol{G}_{12}^{12} & \\ s_{2M} & & & \end{pmatrix} = \det \begin{pmatrix} s_{11} & s_{13} & \cdots & s_{2m} \\ s_{13} & s_{33} & \cdots & s_{3m} \\ \vdots & \vdots & \ddots & \vdots \\ s_{2m} & s_{3m} & \cdots & s_m \end{pmatrix} \end{aligned} \tag{3.83}$$

Generalization. It is possible to extend this result to the case with s_{ij} a general off diagonal term of the Gram matrix.

In that case the parameter of the equation obtained by imposing $G = 0$ are

$$a_{ij} = \det \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & \boldsymbol{G}_{ij}^{ij} & \\ 0 & 0 & & & \end{pmatrix} \tag{3.84}$$

$$b_{ij} = \det \begin{pmatrix} 0 & 0 & v_{[i,j]}^i & \\ 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & (v_{[ij]}^j)^T & \boldsymbol{G}_{ij}^{ij} & \\ 0 & & & \end{pmatrix} \tag{3.85}$$

$$c_{ij} = \det \begin{pmatrix} s_{ii} & 0 & v_{[i,j]}^i \\ 0 & s_{jj} & v_{[i,j]}^j \\ & & \\ (v_{[i,j]}^i)^T & (v_{[ij]}^j)^T & \boldsymbol{G}_{ij}^{ij} \end{pmatrix} \tag{3.86}$$

$$\tag{3.87}$$

where

$$v_{[i]}^k = \det \begin{pmatrix} s_{k1} & \cdots & s_{k,i-1} & s_{k,i+1} & \cdots & s_{k,m} \end{pmatrix} \tag{3.88}$$

$$\tag{3.89}$$

Then, following the procedure outlined for s_{12} one can extend the previous result to any s_{ij} with $i \neq j$, thus proving that in

$$G = 0 \;\Rightarrow\; s_{ij} = \frac{b_{ij} \pm \sqrt{\Delta_{ij}}}{G_{ij}^{ij}} \tag{3.90}$$

Δ_{ij} is always of the form

$$\Delta_{ij} = G_i^i G_j^j \tag{3.91}$$

as stated in eq. (3.59)

■

3.4.2 Hypersferical parametrization

From the form of the integral appearing in (3.57) it is possible to modify the integration boundaries so that the integration will be performed over an Hypersphere in the one loop case, that can be extended to the multiloop case. In fact, shifting the integration boundaries the Gram determinants changes shape. This is due to the fact that the integration boundaries and the zero of the determinant are deeply connect. Hence changing the boundaries make the Gram determinant become a different polynomial that still is zero on the boundaries.

With that in mind, a transformation that shift the domain of integration to $[-1, +1]$ let the Gram determinant becomes a polynomial that is zero at $+1$ and -1, for example $1-u^2$. This is in fact the case: it is possible to rewrite the Baikov representation outlined in (3.31) in a parameterization such that one get the integration measure to be

$$C\int_0^{+\infty} ds_{11} s_{11}^{1-\epsilon} \prod_{i=1}^{t} \left(\int_{-1}^{1} da_i (1-a_i^2)^{(2-i)/2-\epsilon}\right) = C\int_0^{+\infty} ds_{11} s_{11}^{\frac{D-2}{2}} \prod_{i=1}^{t} \left(\int_{-1}^{1} da_i (1-a_i^2)^{\frac{D-2-i}{2}}\right) \tag{3.92}$$

with t the number of denominators appearing in the integral. This parameterization recall the integration in polar coordinates in d dimensions, which is

$$\int_0^{+\infty} dq q^{D-1} \prod_{i=1}^{d-2} \left(\int_0^{\pi} d\theta_i \sin^{d-1-i}(\theta_i)\right) \int_0^{2\pi} d\theta_{d-1} \tag{3.93}$$

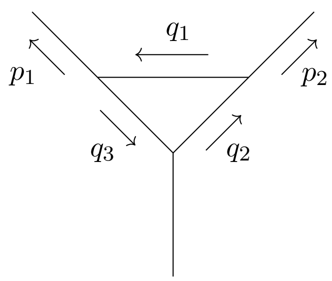
or, provided that $cos(\theta_i) = a_i$,

$$\int_0^{+\infty} dq^{D-1} \prod_{i=1}^{d-2} \left(\int_{-1}^{+1} da_i (1-a_i^2)^{(d-2-i)/2}\right) \int_0^{2\pi} d\theta_{d-1}. \tag{3.94}$$

Below, various examples are provided.

1-loop Examples

3 point function. We intend to take a deeper glance to the function treated in the example 3.4, wich can be represented as



Given the integral in the Baikov representation with explicit boundaries outlined in (3.54), in order to turn the integration domain to $[-1,+1]$ some manipulation is needed. In that case, starting from the amplitude written above and changing the integration variable s_{13} to u through the relation

$$s_{13} = \frac{s_{12}s_{23}}{s_{22}} + u\frac{\sqrt{G(q,p_1)G(p_1,p_2)}}{s_{22}} \tag{3.95}$$

one finds

$$I = \frac{\pi^{1-\epsilon}}{\Gamma(1-\epsilon)}\frac{[G(p_1)]^{-1+\epsilon}}{(2\pi)^D}\int_0^{+\infty} ds_{11}\int_{-\sqrt{s_{11}s_{22}}}^{\sqrt{s_{11}s_{22}}} ds_{12}\,[G(q,p_1)]^{1/2-\epsilon}\int_{-1}^{1} du(1-u^2)^{-\epsilon}F\left(s_{11},s_{12},u\right) \tag{3.96}$$

Defining now

$$s_{12} = v\sqrt{s_{11}s_{22}} \tag{3.97}$$

the integration finally turns to

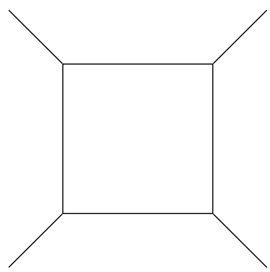
$$I = \frac{\pi^{1-\epsilon}}{\Gamma(1-\epsilon)}\frac{1}{(2\pi)^D}\int_0^{+\infty} ds_{11}s_{11}^{1-\epsilon}\int_{-1}^{1} dv(1-v^2)^{1/2-\epsilon}\int_{-1}^{1} du(1-u^2)^{-\epsilon}F\left(s_{11},v,u\right) \tag{3.98}$$

where now F turned in a much more complicated expression:

$$F^{-1} = \frac{s_{11}}{s_{22}}\left(s_{11}+s_{22}-2v\sqrt{s_{11}s_{22}}\right)\left(s_{11}s_{22}+s_{22}s_{33}+2v\,s_{23}\sqrt{s_{11}s_{22}}+2u\sqrt{1-v^2}\sqrt{s_{11}s_{22}G(p_1,p_2)}\right) \tag{3.99}$$

■

4 point function. In this example the diagram with 4 external legs will be considered. It can be represented as



which corresponds to the integral

$$I = \int\frac{d^Dq}{(2\pi)^D}\frac{1}{q^2(q+p_1)^2(q-p_2)^2(q+p_1+p_3)^2} \tag{3.100}$$

Applying the Baikov parameterization in this case gives

$$\begin{gathered} I = \frac{\pi^{1/2-\epsilon}}{\Gamma(1/2-\epsilon)}[G(p_1,p_2,p_3)]^{\epsilon}\int\frac{ds_{11}}{s_{11}}\int\frac{ds_{12}}{s_{11}+s_{22}+2s_{12}} \\ \int\frac{ds_{13}}{s_{11}+s_{33}-2s_{13}}\int ds_{14}\frac{[G(q,p_1,p_2,p_3)]^{-1/2-\epsilon}}{s_{11}+s_{22}+s_{44}+2s_{24}+2s_{12}+2s_{14}} \end{gathered} \tag{3.101}$$

Following the same framework applied to the Triangle case, one can determine the integration boundaries for s_{11}, s_{12} and s_{13}, while for s_{14}it is mandatory to find the zeros of $G(q, p_1, p_2)$. Using the formula (3.59) one finds

$$s_{14} = \frac{b_{14} \pm \sqrt{G(q,p_1,p_2)G(p_1,p_2,p_3)}}{G(p_1,p_2)} \tag{3.102}$$

The integral, with the integration boundaries specified, hence becomes

$$I = \frac{\pi^{1/2-\epsilon}}{\Gamma(1/2-\epsilon)}[G(p_1,p_2,p_3)]^{\epsilon}\int_0^{+\infty}\frac{ds_{11}}{s_{11}}\int_{G(q,p_1)\geq 0}\frac{ds_{12}}{s_{11}+s_{22}+2s_{12}}$$
$$\int_{G(q,p_1,p_2)\geq 0}\frac{ds_{13}}{s_{11}+s_{33}-2s_{13}}\int_{G(q_1,p_1,p_2,p_3)\geq 0}ds_{14}\frac{[G(q,p_1,p_2,p_3)]^{-1/2-\epsilon}}{s_{11}+s_{22}+s_{44}+s_{24}+2s_{12}+2s_{14}} \tag{3.103}$$

In order to change the integration domain of s_{14} to $[-1,+1]$ it is important to define s_{14} in function of t,

$$s_{14} = \frac{s_{12}s_{24}s_{33}+s_{13}s_{22}s_{34}-s_{13}s_{23}s_{24}-s_{12}s_{23}s_{34}}{G(p_1,p_2)} + t\frac{\sqrt{G(q,p_1,p_2)G(p_1,p_2,p_3)}}{G(p_1,p_2)} \tag{3.104}$$

In that way the integral becomes

$$\begin{aligned} I = \frac{\pi^{1/2-\epsilon}}{\Gamma(1/2-\epsilon)}[G(p_1p_2)]^{-1/2+\epsilon}\int_0^{+\infty}ds_{11}\int_{G(q,p_1)\geq 0}ds_{12} \\ \int_{G(q,p_1,p_2)\geq 0}ds_{13}G(q,p_1,p_2)^{-\epsilon}\int_{-1}^{1}dt(1-t^2)^{-1/2-\epsilon}F(s_{11},s_{12},s_{13},t) \end{aligned} \tag{3.105}$$

Performing now the change of variables outlined in (3.97) and in (3.95), we arrive to the final form of the integral:

$$I = \frac{\pi^{1/2-\epsilon}}{\Gamma(1/2-\epsilon)}\int_0^{+\infty}ds_{11}s_{11}^{1-\epsilon}\int_{-1}^{1}dv(1-v^2)^{1/2-\epsilon}\int_{-1}^{1}du(1-u^2)^{-\epsilon}\int_{-1}^{1}dt(1-t^2)^{-1/2-\epsilon}F(s_{11},v,u,t) \tag{3.106}$$

with F that contains all the information about the denominators appearing in the integrals.

■

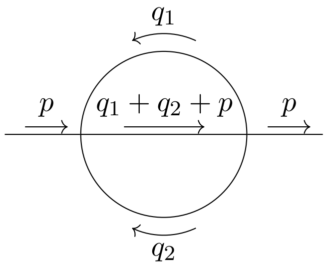
As shown in these example, the iterative structure of this integral arises clearly, since to obtain the box in this new parameterization it was necessary to use the same transformation used for the triangle. This parameterization can be extended easily into the multiloop case too. Below, an example of it in the case of a 2-loop 2 point function.

2-loop Examples

Sunrise. The amplitude in which we're interested is

$$I = \int\frac{d^Dq_2}{(2\pi)^D}\int\frac{d^Dq_1}{(2\pi)^D}\frac{1}{q_1^2(q_1+q_2+p)^2q_2^2}. \tag{3.107}$$

Such amplitude can be portrayed as the following Feynman diagram:



Using (3.31), it is possible to rewrite it in function of the scalar products, obtaining:

$$I = \frac{1}{(2\pi)^{3D}} \frac{\pi^{D-3/2}}{\Gamma(1-\epsilon)\Gamma(3/2-\epsilon)} [G(p)]^{-1+\epsilon} \int_0^{+\infty} ds_{11} \int_0^{+\infty} ds_{22} \int_{G(q_1,p)\geq 0} ds_{13} \int_{G(q_2,p)\geq 0} ds_{23} \int_{G(q_1,q_2,p)\geq 0} ds_{12} \, [G(q_1,q_2,p)]^{-\epsilon} \, F(s_{11}, s_{12}, s_{13}) \tag{3.108}$$

with

$$F^{-1} = s_{11}\, s_{22}\, (s_{11} + s_{22} + s_{33} + 2s_{12} + 2s_{13} + 2s_{23}) \tag{3.109}$$

where we choose to fix the integration boundaries starting from the scalar products between the loop momenta and the external one.

In order to change the integration domain to $[-1, +1]$ we need to define

$$s_{12} = \frac{s_{12}s_{23}}{s_{33}} + t\frac{\sqrt{G(q_1,p)G(q_2,p)}}{s_{33}} \tag{3.110}$$

turning the integral to

$$I = \frac{1}{(2\pi)^{3D}} \frac{\pi^{D-3/2}}{\Gamma(1-\epsilon)\Gamma(3/2-\epsilon)} [G(p)]^{-2+\epsilon} \int_0^{+\infty} ds_{11} \int_0^{+\infty} ds_{22} \int_{G(q_1,p)\geq 0} ds_{13} \, [G(q_1,p)]^{1/2-\epsilon} \int_{G(q_2,p)\geq 0} ds_{23} \, [G(q_2,p)]^{1/2-\epsilon} \int_{-1}^{1} dt (1-t^2)^{-\epsilon} F(s_{11}, s_{22}, t, s_{13}, s_{23}) \tag{3.111}$$

Lastly, performing some other change of variables over s_{13} and s_{23} one arrives to

$$I = \frac{1}{(2\pi)^{3D}} \frac{\pi^{D-3/2}}{\Gamma(1-\epsilon)\Gamma(3/2-\epsilon)} \int_0^{+\infty} ds_{11} s_{11}^{1-\epsilon} \int_0^{+\infty} ds_{22} s_{22}^{1-\epsilon} \int_{-1}^{1} dv (1-v^2)^{1/2-\epsilon} \int_{-1}^{1} du (1-u^2)^{1/2-\epsilon} \int_{-1}^{1} dt (1-t^2)^{-\epsilon} F(s_{11}, s_{22}, v, u, t) \tag{3.112}$$

with

$$F^{-1} = \frac{s_{11}s_{22}}{s_{33}} \Big(s_{11}s_{33} + s_{22}s_{33} + s_{33}^2 + 2v\, s_{33}\sqrt{s_{11}s_{33}} + 2u\, s_{33}\sqrt{s_{22}s_{33}} + 2v\, u\, s_{33}\sqrt{s_{11}s_{22}} + 2t\, \sqrt{1-v^2}\, \sqrt{1-u^2}\, \sqrt{s_{11}s_{22}} s_{33} \Big) \tag{3.113}$$

■

This parameterization can be applied in a generic case, hence it is very versatile. Further studies will be performed on it, in order to understand its application together with the generalized cut and/or the IBP approach.

3.4.3 Denominators as variables

Thanks to the auxiliary numerators defined in (3.9), there is the same numbers of denominators as there is of scalar products. Moreover, naming the formers as x_a, it exist a a linear transformation A_a^{ij} such that we can write

$$x_a = \sum_{i=1}^{l} \sum_{j=i}^{m} A_a^{ij} s_{ij} + m_a^2 \tag{3.114}$$

Note that we are working in the euclidean prescription.
It happens that A_a^{ij} isn't always invertible: at one loop the number of scalar products and denominators are the same, but at more loop it is not granted because we can have irreducible scalar products. In that case we should introduce new variables x_a, or new denominators, that depend on them and write them with a positive exponent. In the end, A_a^{ij} can always become invertible (considering ij as a single index spanning from 1 to r, as done by the index a). Hence we can write:

$$s_{ij} = \sum_{a=1}^{r} A_{ij}^a (x_a - m_a^2) \tag{3.115}$$

Given the above relation, we can perform a change of variable integrating now over the denominator, not only the scalar product.
In the end , considering the general case where

$$F(q_1, \dots, q_l, p_1, \dots, p_{n-1}) = \frac{1}{x_1^{\alpha_1} \cdots x_r^{\alpha_r}} \tag{3.116}$$

our amplitude becomes

$$I = \frac{\pi^{Dl/2 - l(l-1)/4 - l(n-1)/2}}{\prod_{i=1}^{l} \Gamma(\frac{D-m+i}{2})} G(p_1, \dots, p_{n-1})^{\frac{-D+n}{2}} det A_{ij}^a \int \frac{\prod_{a=1}^{N} dx_a}{x_1^{\alpha_1} \cdots x_r^{\alpha_r}} P(x_1 - m_1^2, \dots, x_r - m_r^2)^{\frac{D-m-1}{2}} \tag{3.117}$$

with $P(x_1 - m_1^2, \dots, x_r - m_r^2)$ being the Gram determinant written in function of the denominators and the couple (ij) in the $det A_{ij}^a$ considered as a single index.

3.4.4 Cuts in Baikov representation

Settin the denominators as integration variables becomes pretty useful when one is trying to evaluate a generalized cut diagram. In fact in this case the delta has a simple zero, while in the representation with momenta as variables things are a little more complicate as shown in the example 2.3.2.

Formally, cutting a propagator and hence putting it on shell, is performed by deforming the contour of integration around the pole appearing for $x_j = 0$. Explicitly:

$$\begin{aligned} Cut_j\,[I] =& \frac{\pi^{Dl/2 - l(l-1)/4 - l(n-1)/2}}{\prod_{i=1}^{l} \Gamma(\frac{D-m+i}{2})} G(p_1, \dots, p_{n-1})^{\frac{-D+n}{2}} det A_{ij}^a \\ & \int \prod_{\substack{a=1 \\ i \neq j}}^{N} dx_a \oint_{x_j=0} dx_j \frac{P(x_1 - m_1^2, \dots, x_r - m_r^2)^{\frac{D-m-1}{2}}}{x_1^{\alpha_1} \cdots x_r^{\alpha_r}} \end{aligned} \tag{3.118}$$

From this equation one can see that, if the denominator is raised to power 1, the action of cutting the amplitude turns out to be

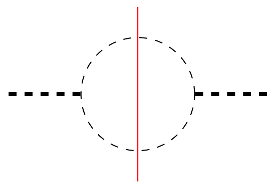
$$Cut_j\,[I] = \frac{\pi^{Dl/2-l(l-1)/4-l(n-1)/2}}{\prod_{i=1}^{l}\Gamma(\frac{D-m+i}{2})} G(p_1,\dots,p_{n-1})^{\frac{-D+n}{2}} detA^a_{ij} \int \prod_{\substack{a=1\\ i\neq j}}^{N} dx_a \frac{P(x_1-m_1^2,\dots,x_r-m_r^2)^{\frac{D-m-1}{2}}}{x_1^{\alpha_1}\cdots x_{j-1}x_{j+1}\cdots x_r^{\alpha_r}}\Bigg|_{x_j=0} \tag{3.119}$$

and hence reconstruct the classic generalized Unitarity substitution rule:

$$\frac{1}{x_j} \to 2\pi i\delta(x_j) \tag{3.120}$$

Example: double cut of a Bubble

Here this new parameterization will be applied to the case of the bubble amplitude already discussed thoroughly in the previous chapter, precisely in example 2.3.2. Namely, we want to evaluate



First of all, let's apply the parameterization outlined in (3.117) to the integral with two denominators. From (3.35), setting

$$\begin{aligned} x_1 &= q^2 \\ x_2 &= (q-p)^2 \end{aligned} \tag{3.121}$$

one can evaluate the Gram determinant in these new variables, and also determine the matrix of change of variables:

$$\begin{aligned} det(A^a_{ij}) &= det\begin{pmatrix}1 & 1/2\\ 0 & -1/2\end{pmatrix} = \frac{1}{2} \\ P(x_1-m_1^2,\dots,x_N-m_N^2) &= det\begin{pmatrix} x_1 & \frac{x_1-x_2+p^2}{2} \\ \frac{x_1-x_2+p^2}{2} & p^2\end{pmatrix} = x_1p^2 - \frac{(x_1-x_2+p^2)^2}{4} \end{aligned} \tag{3.122}$$

leading to the final form of our integral:

$$I = \frac{\lambda^2}{(2\pi)^D}\frac{\pi^{3/2-\epsilon}}{\Gamma(3/2-\epsilon)}\frac{1}{2(p^2)^{1-\epsilon}}\int_0^{+\infty}\frac{dx_1}{x_1}\int_{x_1+p^2-2\sqrt{x_1p^2}}^{x_1+p^2+2\sqrt{x_1p^2}}\frac{dx_2}{x_2}\left(x_1p^2-\frac{(x_1-x_2+p^2)^2}{4}\right)^{1/2-\epsilon} \tag{3.123}$$

Now applying the cut one obtains

$$I = -\frac{\lambda^2}{(2\pi)^{D-2}}\frac{\pi^{3/2-\epsilon}}{\Gamma(3/2-\epsilon)}\frac{1}{2(p^2)^{1-\epsilon}}\int dx_1dx_2\delta(x_1)\delta(x_2)\left(x_1p^2-\frac{(x_1-x_2+p^2)^2}{4}\right)^{1/2-\epsilon} \tag{3.124}$$

which is easily evaluated:

$$
\begin{aligned}
I &= -\frac{\lambda^2}{(2\pi)^{D-2}} \frac{\pi^{3/2-\epsilon}}{\Gamma(3/2-\epsilon)} \frac{(-p^2)^{-\epsilon}}{2^{2-2\epsilon}} \\
&= -\frac{\lambda^2(M^2)^{-\epsilon}}{2^{3-2\epsilon}} \frac{\Gamma(1-\epsilon)}{\Gamma(2-2\epsilon)}
\end{aligned} \tag{3.125}
$$

which is consistent with the result obtained in (2.97) up to a phase, which is expected since we're working in the Euclidean. ■

3.4.5 Integration by parts identities in Baikov representation

As seen before, the condition for the validity of the integration by parts identities, outlined in (3.13), that the integrand vanishes at the boundaries of the domain of integration. This is exactly the property of the integrand in the Baikov representation, since the Gram determinant vanishes at the boundaries of integration. This led to the formulation of such identities in this parametrization too, as formulated in [14]. Analogously to the original formulation of the IBP, these identities revolves around

$$
\int_{\mathcal{D}} dx_1 \cdots dx_r \sum_i^N \frac{d}{dx_i} \frac{c_i(x) P(x)^{\frac{D-h}{2}}}{x_1^{\alpha_1} \cdots x_r^{\alpha_r}} = 0 \tag{3.126}
$$

where $P(x)$ is the Baikov polynomial in the denominators variables, $c_i(x)$ are arbitrary polynomials of the denominators $x_1, \cdots, x_r$, $h = m-1$ and $\mathcal{D}$ is the domain of integration. From here on we will omit such domain, nonetheless it's role is crucial in such identities since they're verified because the shape of $\mathcal{D}$ grant that $P(x)|_{\partial\mathcal{D}} = 0$.

Applying the derivative to the integrand one obtains the relation

$$
\int dx_1 \cdots dx_r \sum_i^r \left(\frac{\partial c_i(x)}{\partial x_i} + \frac{D-h}{2} \frac{c_i(x)}{P(x)} \frac{\partial P(x)}{\partial x_i} - \alpha_i \frac{c_i(x)}{x_i} \right) \frac{P(x)^{\frac{D-h}{2}}}{x_1^{\alpha_1} \cdots x_r^{\alpha_r}} = 0 \tag{3.127}
$$

Still, it's not the IBP we're searching of. This identity relates among the other terms the integral

$$
\frac{D-2-h}{2} \int dx_1 \cdots dx_r \sum_i^N \left(\frac{\partial P(x)}{\partial x_i} c_i(x) \right) \frac{P(x)^{\frac{D-2-h}{2}}}{x_1^{\alpha_1} \cdots x_r^{\alpha_r}} \tag{3.128}
$$

which can be interpreted as an amplitude in $D-2$ dimensions. Such relation, that links amplitude in different dimensions, is not what we're interested in. In order to overcome this problem it is necessary to put a constraint to the c_i used in the IBP. Such polynomials must satisfy what is called a *syzygy* equation:

$$
\sum_i^r c_i(x) \frac{\partial P(x)}{\partial x_i} = bP(x) \tag{3.129}
$$

Substituting it in (3.127) one obtain useful identities:

$$
\int dx_1 \cdots dx_r \sum_i^r \left(\frac{\partial c_i(x)}{\partial x_i} + \frac{D-h}{2} b - \alpha_i \frac{c_i(x)}{x_i} \right) \frac{P(x)^{\frac{D-h}{2}}}{x_1^{\alpha_1} \cdots x_r^{\alpha_r}} = 0 \tag{3.130}
$$

Moreover sometimes it would be convenient to have IBPs that relate integrals with the same total power at the denominators. This can be done imposing the second *syzygy* equation

$$c_i(x) = b_i x_i \tag{3.131}$$

Thanks to this condition the third terms in the identity doesn't add any power to the denominators, hence obtaining the result we were looking for. Such particular type of IBP algorithm has recentrly received a software adaptation, found in [15, 16]

Chapter 4

Automatic evaluation of 2-loop Amplitudes

Great tools for the automatic evaluation of the scattering amplitudes were presented in previous chapters. Among them, remarkable ones are the Generalized Unitarity approach, that together with the tensor and the integrand decomposition gave a huge boost on the calculation of one loop amplitude, and the IBPs reduction technique, which is the best way to reduce the number of integrals appearing during the calculation of the amplitude. In this chapter we firstly outline the *Adaptive integrand approach*, where adaptive means that such techniques depends on which integral is used, it adapts to the problem. Such powerful techniques, recalling the integrand reduction already encountered in the one loop case, is a promising alternative to the Tensor decomposition, as shown by the example at which it's been applied. After introducing this new token, the puzzle is finally complete: it is possible automatize the calculation, starting from the process and arriving directly at the Laurent expansion. How it can be done is firstly described from an algorithm based point of view, afterwards a possible implementation is used, outlining the contribution given during the work of this thesis, in order to carry out the evaluation of a $2 \to 3$ process in QED.

4.1 Adaptive integrand decomposition

A viable approach to the calculation of multiloop scattering amplitudes, is given by the decomposition of the integrand in independent contribution. In the one loop case this type of decomposition is performed almost effortlessly thanks to the generalized Unitarity approach. Nonetheless, as stated before, this approach can't be trasposed immediately on the multiloop level due to the presence of *Irreducible Scalar Products*. The higher complexity of the problem takled needs a new approach to the subject. The *Adaptive Integrand Decomposition* [17, 18] sheds new light on the problem, tackling it with a new approach.

4.1.1 Parallel and orthogonal space

As outlined in the chapter 3.4, the Baikov representation is very versatile and in some cases its uses simplify the calculation as seen for example in the generalized cut approach. Using it together with the parameterization in which the loop momenta is split in two orthogonal component, as seen in (2.91), leads to some major simplification

In general one can write a multiloop Feynman amplitude as

$$I^{d(l,n)}[\mathcal{N}] = \int \prod_i^l \frac{d^D q_i}{(2\pi)^D} \frac{\mathcal{N}(q_i)}{\prod_j D_j} \tag{4.1}$$

with

$$D_j = l_j^2 + m_j^2 \quad \text{with} \quad l_i^\alpha = \sum_j \alpha_{ij} q_j^\alpha + \sum_j \beta_{ij} p_j^\alpha \tag{4.2}$$

where p_j^α are the external momenta while α and β are incidence matrices which entries take values in $(0, \pm 1)$, recalling the definition of the denominators given in (2.87).

Generalizing the idea of splitting the loop momenta in to a 4-dimensional component and extra dimensional component outlined in (2.91) to the multiloop case, one has that

$$q_i^\alpha = q_{[4]i}^\alpha + \mu_i^\alpha \quad q_i \cdot q_j = q_{[4]i} \cdot q_{[4]j} + \mu_{ij} \quad (\mu_{ij} = \mu_i \cdot \mu_j). \tag{4.3}$$

Substituting such decomposition in the expression for a multiloop denominator (4.2), one obtains that

$$D_i = l^2 + m_i^2 = l_{[4]i}^2 + \sum_{j,k} \alpha_{ij}\alpha_{ik}\mu_{jk} + m_i^2 \tag{4.4}$$

where

$$l_{[4]i}^\alpha = \sum_j \alpha_{ij} q_{[4]j}^\alpha + \sum_j \beta_{ij} p_j^\alpha \tag{4.5}$$

With this choice of parameterization for the loop momenta both the numerator and the denominators will depend on the scalar products μ_{ij} and the components of $q_{[4]i}$ with respect to a four-dimensional basis of vectors $\{e_i^\alpha\}$, such that $q_{[4]i}^\alpha = \sum_j x_{ij} e_j^\alpha$. It is then possible to apply the Baikov parameterization seen in section 3.4 to the extra dimensional component of the integration measure, writing a generic multiloop integral as

$$I^{d(l,n)}[\mathcal{N}] = \Omega_d^{(l)} \int \prod_{i=1}^l d^4 q_{[4]i} \int \prod_{1 \le i \le j \le l} d\mu_{ij} G(\mu_{ij})^{\frac{d-5-l}{2}} \frac{\mathcal{N}(q_{[4]i}, \mu_{ij})}{\prod_j D_j(q_{[4]i}, \mu_{ij})} \tag{4.6}$$

with $G(\mu_{ij}$ the Gram determinant as defined in (3.26), hence $G(\mu_{ij}) = \det(\mu_i \cdot \mu_j$. The prefactor $\Omega_d^{(l)}$ is the result of the angular integration over the angular directions.

In the same sense, it is always possible to split the loop momenta in to orthogonal components, choosing directions that one prefers in order to perform the splitting. A more convenient choice of directions for this procedure is found to be along the space parallel to the external momenta and the space orthogonal to them. This parameterization in the case of $n \geq 5$ external leg coincides with the previous case, while in the other cases is different, but nonetheless ease the calculation of the integral.

Remarkably, the choice of the splitting depends on the amplitude taken in to account (more precisely, on its number of external legs), so this parameterization changes times to times. It *adapts* to the integral to which is applied, getting its name (*Adaptive Integrand Decomposition*) from this key feature.

Defining $d_{||}$ as the dimension of the space spanned by the external momenta, one finds that it is possible to choose $4d_{||}$ of the vectors that belongs to the basis $\{e_i^\alpha\}$ to lie into the subspace orthogonal to the external kinematics, i.e. such that

$$e_i \cdot p_j = 0 \quad for\ i > d_{||}, \forall j \tag{4.7}$$

and

$$e_i \cdot e_j = \delta_{ij}. \tag{4.8}$$

In this way the loop momenta can be written in its $d = d_{||} + d_{\perp}$ component as

$$q_i^\alpha = q_{||}^\alpha + \lambda_i^\alpha. \tag{4.9}$$

where

$$q_{||}^\alpha = \sum_{j=1}^{d_{||}} x_{ij} e_j^\alpha \qquad \lambda_i^\alpha = \sum_{j=d_{||}+1}^{4} x_{ij} e_j^\alpha + \mu_i^\alpha \tag{4.10}$$

with $q_{||}$ that lies in the $d_{||}$ space, while λ_i belongs in the orthogonal $d_{\perp}$ dimensional one. This parameterization has a very useful feature: all the denominators become independent from the single orthogonal component of the transverse loop momenta, they only depend on the scalar products between them. In fact the denominators appear as

$$D_i = l_{||i}^2 + \sum_{j,k} \alpha_{ij}\alpha_{ik}\lambda_{jk} + m_i^2 \tag{4.11}$$

with

$$l_{||i}^\alpha = \sum_j \alpha_{ij} q_{||j}^\alpha + \sum_j \beta_{ij} p_j^\alpha \tag{4.12}$$

and

$$\lambda_{jk} = \sum_{l=d_{||}+1}^{4} x_{jl} x_{lk} + \mu_{jk} \tag{4.13}$$

In this way the integral becomes

$$I^{d(l,n)}[\mathcal{N}] = \Omega_d^{(l)} \int \prod_{i=1}^{l} d^{n-1} q_{||i} \int \prod_{1 \le i \le j} d\lambda_{ij} G(\lambda_{ij})^{\frac{d_\perp - 1 - l}{2}} \frac{\mathcal{N}(q_{||i}, \lambda_{ij}, \Theta_\perp)}{\prod_j D_j(q_{||i}, \lambda_{ij})} \tag{4.14}$$

where $\Theta_\perp$ parametrizes the integral over the single orthogonal component λ_i that lies in the four dimensional space. Remarkably, it is possible always possible to integrate away the $\Theta_\perp$, obtaining a simplified integral.

4.1.2 Polinomial division

Another powerful tool in the integrand decomposition is the integrand reduction via *multivariate polynomial division.*

As stated in 2.3.1, at one loop it is possible to write the integrand of an amplitude as a rational function of the scalar products. This statement can be extended easyly to the multiloop level too. This gives the idea that the integrand of a multiloop process can be written as a rational function in certain variables that we will call z_i. At this point a suitable approach is to perform the polynomial division until we're left with *irreducible* polynomals in the z_i variables, i.e. polynomials that are no longer divisible for denominators appearing in the integrand.

One can write the tensor integral of a multiloop amplitude as

$$I^{d(l,n)}_{i_1,\dots,i_a}[\mathcal{N}] = \int \frac{d^D q_1}{(2\pi)^D} \cdots \frac{d^D q_l}{(2\pi)^D} \mathcal{I}_{i_1,\dots,i_a}, \qquad \mathcal{I}_{i_1,\dots,i_a} \equiv \frac{\mathcal{N}_{i_1,\dots,i_a}}{D_{i_1} \cdots D_{i_a}}, \tag{4.15}$$

Through the polynomial division it is possible to write the numerator as a quotient $\mathcal{Q}_{i_1,\dots,i_a}$, that depends on the denominators, plus a remainder $\Delta_{i_1,\dots,i_a}$ that is :

$$\begin{aligned} \mathcal{N}_{i_1,\dots,i_a} &= \mathcal{Q}_{i_1,\dots,i_a} + \Delta_{i_1,\dots,i_a} \\ &= \sum_{k=1}^{a} \mathcal{N}_{i_1,\dots,i_{k-1},i_{k+1},\dots,i_a} D_k + \Delta_{i_1,\dots,i_a} \end{aligned} \tag{4.16}$$

In this way it is possible to *reduce* the integrand, in fact substituting (4.16) back into $\mathcal{I}_{i_1,\dots,i_a}$ one finds that

$$\mathcal{I}_{i_1,\dots,i_a} = \sum_{k=1}^{a} \mathcal{I}_{i_1,\dots,i_{k-1},i_{k+1},\dots,i_a} + \frac{\Delta_{i_1,\dots,i_a}}{D_{i_1} \cdots D_{i_a}}. \tag{4.17}$$

Iterating such procedure on the resulting $\mathcal{I}_{i_1,\dots,i_{k-1},i_{k+1},\dots,i_a}$ it is possible to write the integrand as a combination of irreducible remainder, arriving to the final form

$$\mathcal{I}_{i_1,\dots,i_a} \equiv \frac{\mathcal{N}_{i_1,\dots,i_a}}{D_{i_1} \cdots D_{i_a}} = \sum_{k=1}^{a} \sum_{\{j_1 \cdots j_k\}} \frac{\Delta_{j_1 \cdots j_k}}{D_{j_1} \cdots D_{j_k}} \tag{4.18}$$

In order to understand such procedure from a deeper mathematical point of view, let us introduce the concept of *ideal.* Defining $P[z_i]$ as the ring of all polynomials in the z_i variables, every set of indices $i_1, \dots, i_a$ defines the ideal

$$\mathscr{I}_{i_1 \dots i_a} \equiv \langle D_{i_1}, \dots, D_{i_a} \rangle = \left\{ \sum_{k=1}^{a} h_k(z_i) D_{i_k}(z_i) \ : \ h_k(z_i) \in P[z_i] \right\} \tag{4.19}$$

With that definition in mind, the goal of the integrand reduction can be expressed as to write the integrand as a contribution of *irreducible* polynomials $\Delta_{j_1 \cdots j_k}$, i.e. polynomials which contain no contribution belonging in the corresponding ideal $\mathscr{I}_{j_1 \cdots j_k}$.

Depending on the choice of variables z_i the picture presented in this section can significantly simplify. A particular convenient choice of variables turns out to be the one presented in eq. (4.9). In this way using the Adaptive Integrand Decomposition together with the polynomial division method greatly simplify the Feynman amplitude.

This approach has been tested and found trustworthy in many two loop cases, reported in Figure **??** and **??**, making a crucial step forward in the evaluation of two loop amplitudes.

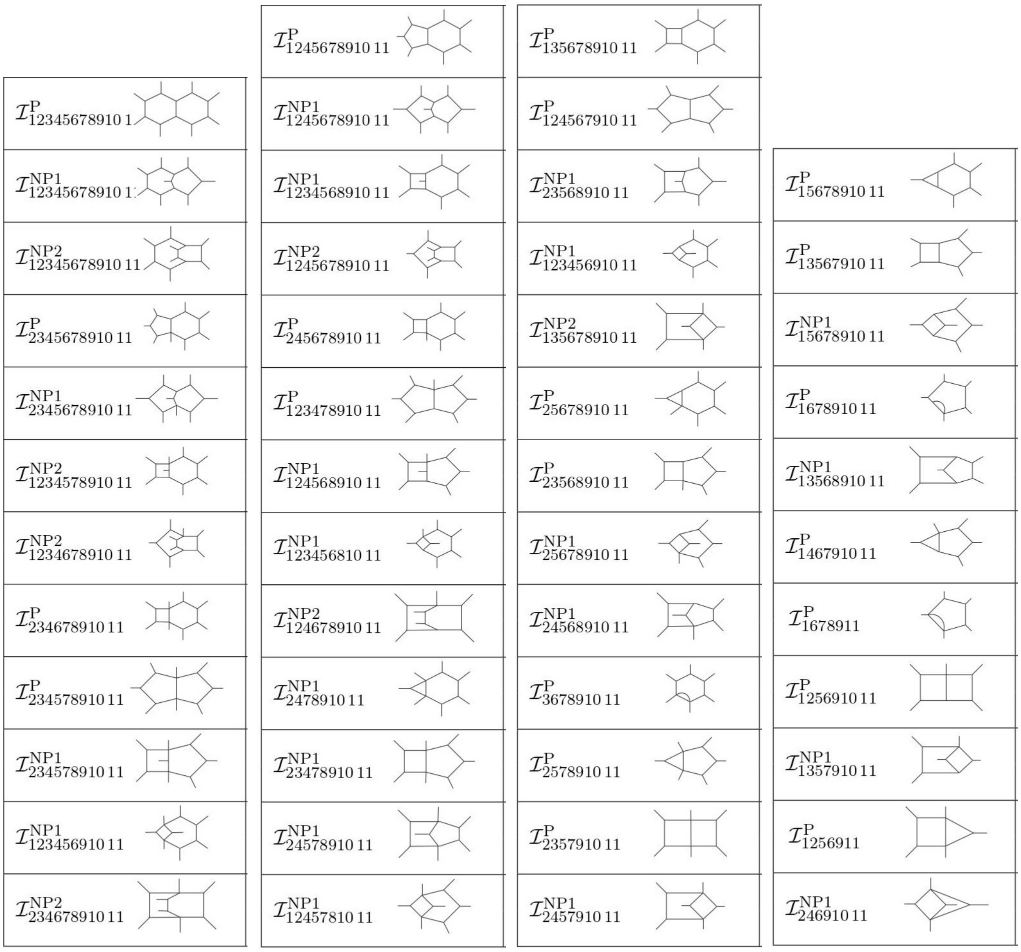


Figure 4.1: Examples in which the Adaptive integrand decomposition has been successfully applyed:8,7,6, and 5 external legs cases

4.2 A complete chain for Amplitude evaluation

In the last years many techniques have been developed in order to tackle the problem of the evaluation of two loop amplitudes. Since at multi loop level a unique basis of Master Integrals doesn't exist due to the presence of the *Irreducible Scalar Products*, the generalized Unitarity approach that brought great improvement in the one loop case cannot be used straightforwardly at this level. Since the huge number of integrals that contribute at a two loop process, the direct evaluation of them is unpracticable. A better philosophy is to reduce *as much as possible* the number of Master Integrals that one has to evaluate. In this mindset, thanks to the tool reported in this thesis, it is possible to outline a general algorithm for the evaluation of *any* amplitude, with particular interest in the two loop case. The algorithm, reported in Figure 4.3, consists in 4 steps.

Firstly one has to generate all the Feynman diagrams that can contribute to the process taken in to account, obtaining the amplitude written as a combination of *tensor* integrals

$$\mathcal{M} = \sum_i I_i. \tag{4.20}$$

In order to apply the IBP algorithm efficiently, it is crucial to write the amplitude

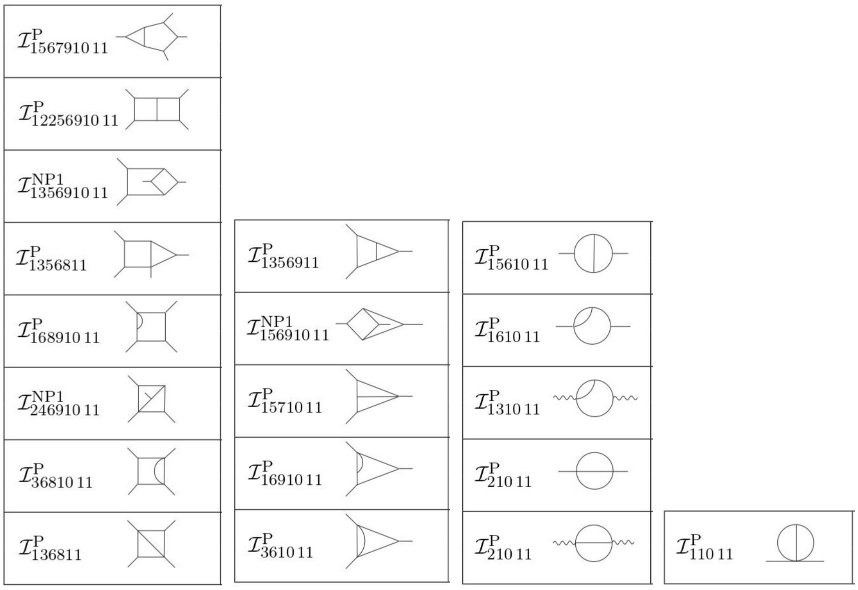


Figure 4.2: Examples in which the Adaptive integrand decomposition ha been successfully applyed:4,3,2 and 1 external legs cases

as a combination of scalar integral reduced as much as possible. This task can be completed thanks to the Adaptive Integrand decomposition outlined in Section 4.1, substituting the classic tensor decomposition. After this step, the integral that contribute to the amplitudes as written in equation (4.20) becomes

$$I_i = \int dq_1 \cdots dq_l \sum_{k=1}^{a} \sum_{\{j_1 \cdots j_k\}} \frac{\Delta_{j_1 \cdots j_k}}{D_{j_1} \cdots D_{j_k}} \tag{4.21}$$

with

$$\int dq_1 \cdots dq_l \frac{\Delta_{j_1 \cdots j_k}}{D_{j_1} \cdots D_{j_k}} = c_{j_1 \cdots j_k} \int dq_1 \cdots dq_l \frac{S_1 \cdots S_m}{D_{j_1} \cdots D_{j_k}} \tag{4.22}$$

with S_r being an *irreducible scalar product*. Since any further algebraic simplification is impossible, in order to reduce the number of Master Integrals it is mandatory to use IBPs to minimize the number of direct calculation. In this way the amplitude can be written as

$$\mathcal{M} = \sum_i c_i I_i^{IM} \tag{4.23}$$

with I_i^{IM} being the minimal basis of Mater Integrals that are needed for that process. After this steps, one can proceed to the direct evaluation of the remaining integrals, or if such integrals are already evaluated, one can take their value from the literature. In this way, starting from the definition of the process needed, it is possible to arrive to the Amplitude written as a Laurent expansion around ϵ, where the dimensional parameter is defined such that the relation $D = 4 - \epsilon$ is satisfied. Thus arriving to

$$\mathcal{M} = \sum_i b_i \epsilon^i \tag{4.24}$$

This could resemble a great achievement, noneheless the complexity of the calculation in the intermediate steps is not to be underestimated. In fact many software that

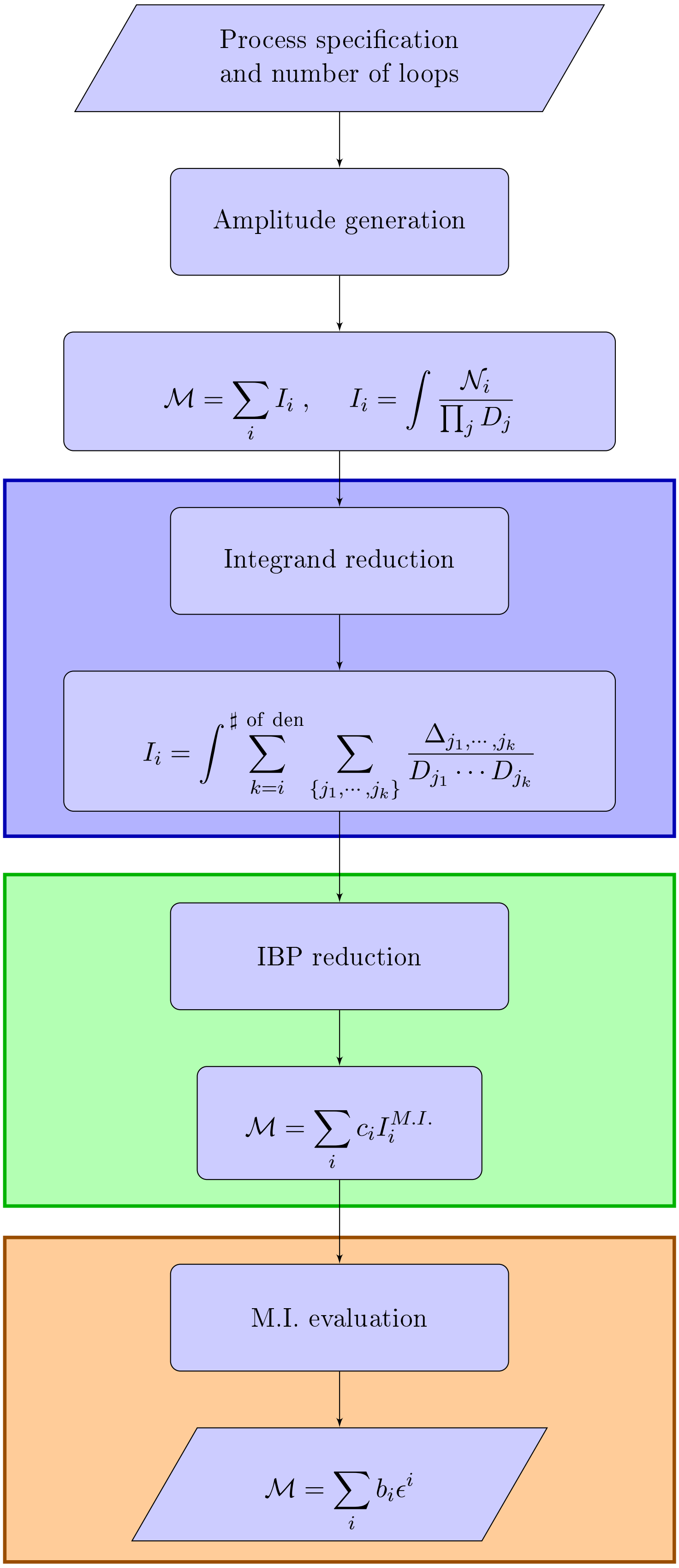


Figure 4.3: Flow chart of the algorithm for the evaluation of scattering amplitude

are able to perform these single steps has been developed. Citing some remarkable example, for the amplitude generation software such as FeynArts [19] together with FeynCalc [20, 21], QGRAF are well known and widely used. For the IBP reduction, suitable tools are Reduze [22], FIRE [23], Kira [24], Azurite [15], while for the direct evaluation one can use software such as SecDec [25] or FIESTA [26]. Lastly, a code that performs the Adaptive Integrand Decomposition is under development [27]. Despite the large number of individual tools already existing, a software that perform the whole calculation is still missing.

During this work of thesis important contribution towards the completion of such goal has been given. Such contribution are part of a wider project already underway. Connecting tools as pictured in Figure 4.5, a *complete chain* for the evaluation of Master Integrals is under development.

Inside such project, during this work of thesis the step of the numerical evaluation of analytical Master Integrals and of interfacing the IBP reduction performed by Reduze[1] and the numerical evaluation done by SecDec has been perfected

4.2.1 Interface between Reduze and SecDec

After performing the integrand reduction using Aida, a great tool to perform the IBP reduction is given by the software Reduze. Using the *Laporta algorithm* [28] it can generate and solve the integration by parts identities. In order to simplify such calculation, it relates different integrals, keeping track of such manipulation with an internal notation.

Even though this step simplify the calculation, it has the drawback of giving a hard to read output. In order to translate such output in to a readable format for the evaluation of the MIs, one has to undergo a couple of steps.

Firstly, it is mandatory to extract the information about the denominators present in the integrals before the IBP reduction. Such information can be extracted from the output of the previous step, namely in the output file of Aid, that we will call "file.output.m". Then, it is necessary to combine this information together with the information that allows to translate the output of Reduze that is stored inside an internal file, called "crossing.yaml", and finally one can translate the Master Integrals list given in output from Reduze in to a format readable by SecDec. This process is depicted into Figure 4.6.

[1]Starting from what was done with Reduze, Kira was added to the complete chain afterwards thanks to the work of R. Sameshima (PhD student at New York City College of Technology)

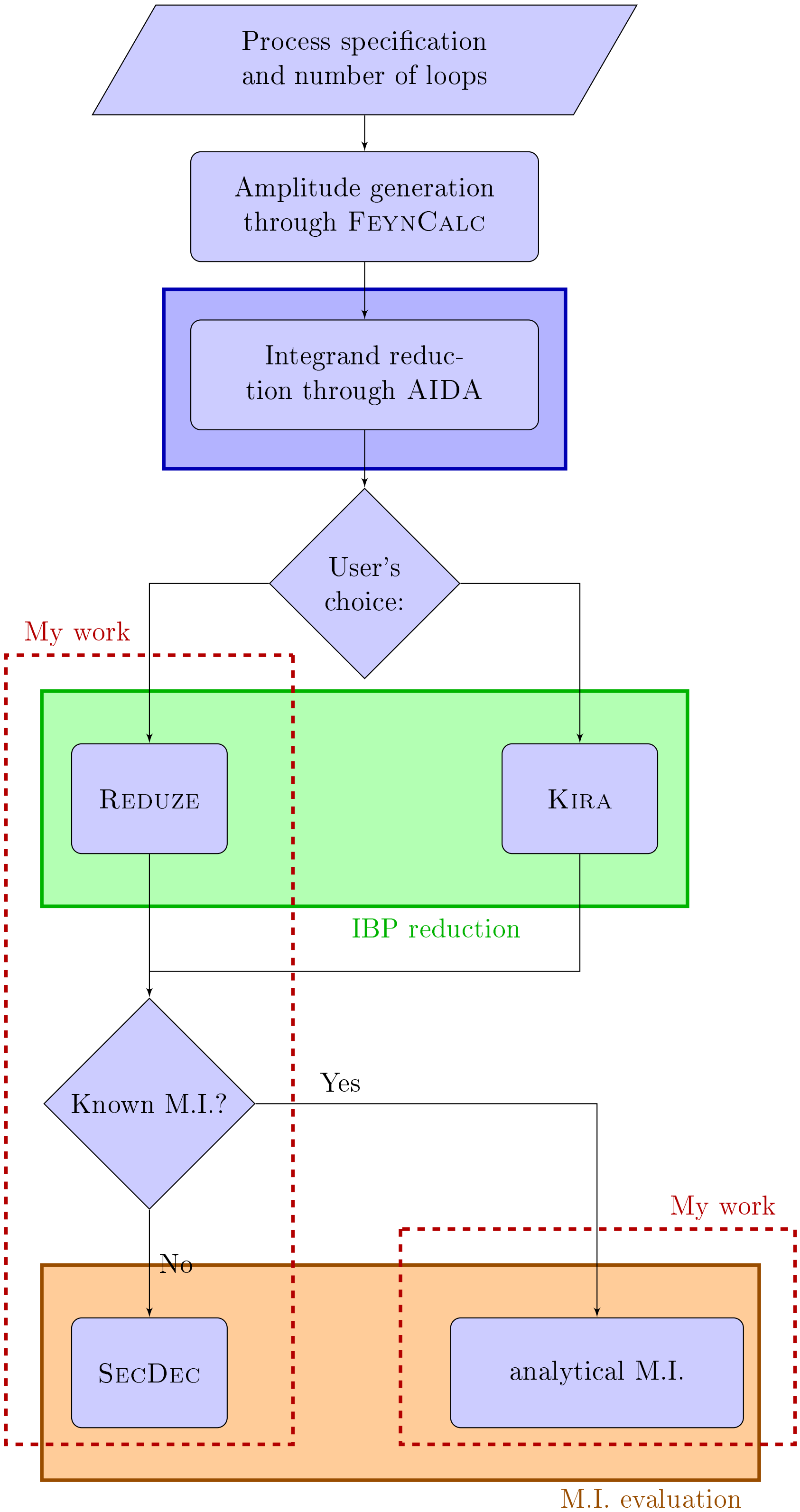


Figure 4.4: Flowchart of...example

Figure 4.5: Possible structure from a software point of view

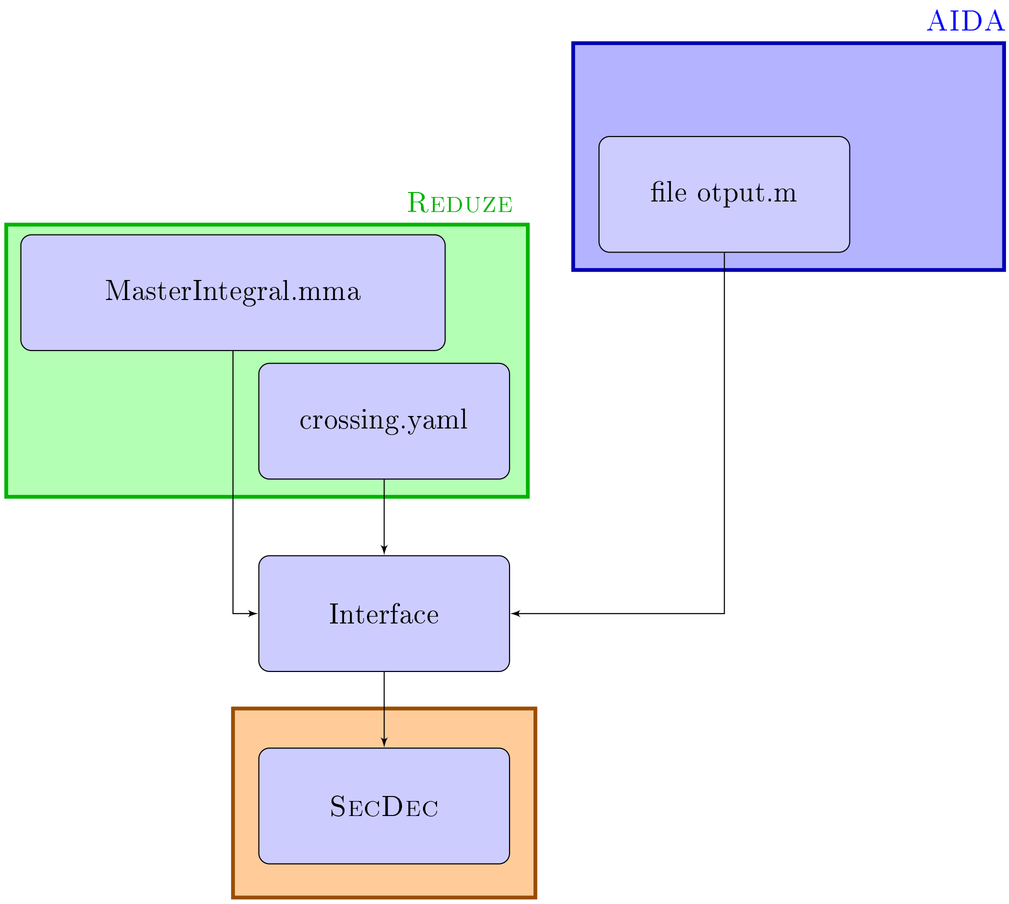


Figure 4.6: Flowchart of the interface between Reduze and Secdec

4.2.2 Evaluation of available Master Integrals

When possible, the use of Master Integral already present in literature would greatly ease the calculation needed for the complete evaluation of an amplitude. Unfortunately, often such integrals are written as a combination of special function, preventing their immediate numerical evaluation. It is mandatory then that an efficient software that performs the complete calculation of a Feynman amplitude for a process has to have the possibility of extract such information from what is already there in the literature.

During this work of thesis a code for the numerical evaluation of already known Master Integral has been developed, with the aim of join such step to the complete chain, with further work in the future.

Such code uses the software GiNaC to evaluate the integrals. We used it on the set of integrals given by Papadopoulos [29]. Its internal operation are depicted in Figure 4.7.

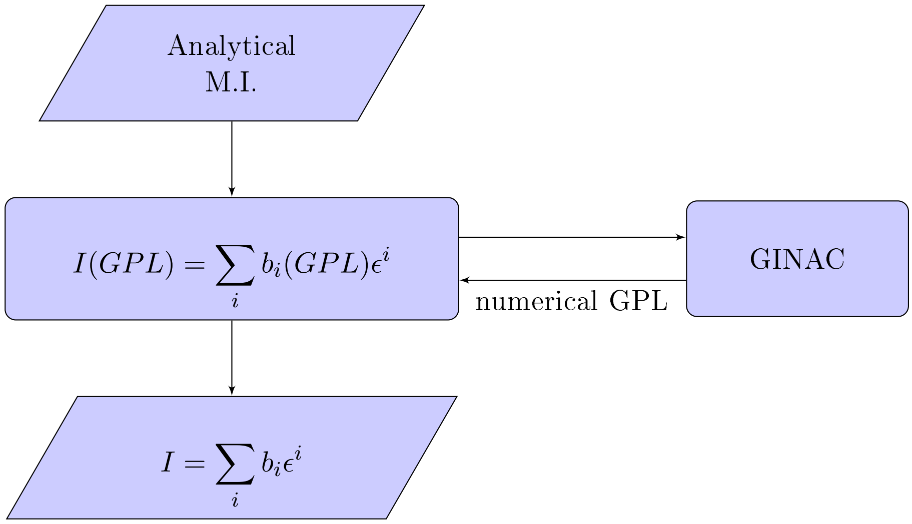


Figure 4.7: Flowchart of the evaluation of analytic MIs

The Mater Integral for the five point function at two loop, given by Papadopoulos are given in terms of *Generalised PolyLogarithm*, or GPLs for short. After reading such expression, a list of all the GPLs appearing in the MI is extracted and passed to GiNac which evaluates it. In return, the list of values of the respective GPLs is given as output by the algebraic manipulation software, so that the GPLs appearing in the expression of the Master Integral can be substituted with their value, as pictured in the example reported in Figure 4.8.

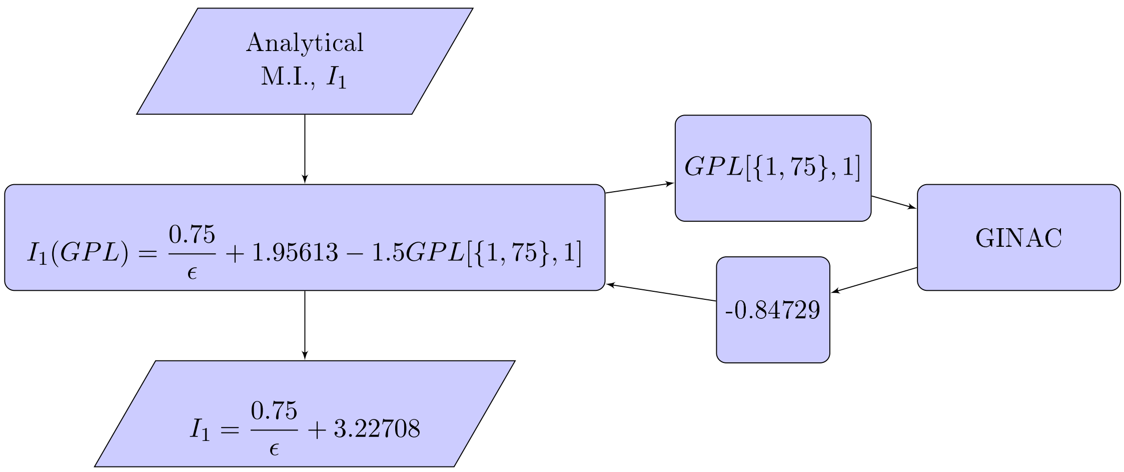
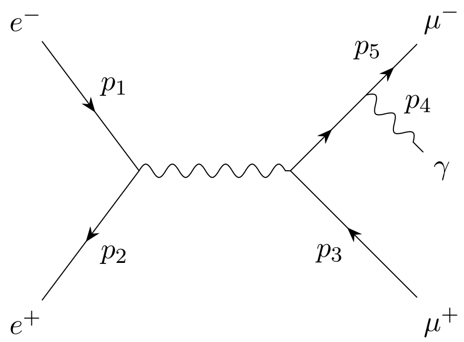


Figure 4.8: Esempio di funzionamento dell'interfaccia

4.3 Application: $e^+e^- \to \mu^+\mu^-\gamma$ scattering at 2-loop

The evaluation of 5 point functions at two loop is a hot topic in the scientific community, as proved by the recent results given in [30, 31, 32]. Hence an interesting test for the validity of the complete chain could be the evaluation of such processes.

The complexity of the problem in this case makes the analytical path towards the result impracticable, hence the application of the complete chain has been done numerically, choosing a phase space in which evaluate the process. A feasible objective in the time given for the work of this thesis, was the QED process $e^-e^+ \to \mu^-\mu^+\gamma$, of which its tree level amplitude is depicted below:



The software setup used for such calculation is depicted in Figure 4.9.

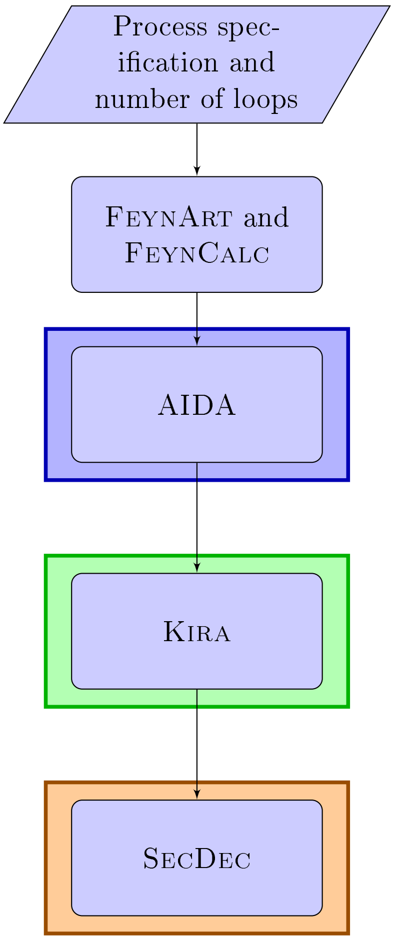


Figure 4.9: Flowchart of the setup used for the evaluation of the 5 point process $e^+e^- \to \mu^+\mu^-\gamma$

4.3.1 Amplitude generation: FeynCalc+FeynArts

The first step encountered in the evaluation of the amplitude for such process was the generation of the integral and its translation in to a readable format. The former was executed thanks to FEYNARTS. In fact, defining the process, the number of loop requested and in which theory one wants to compute it, FEYNARTS is able to generate all the Feynman integrals that contributes to it.

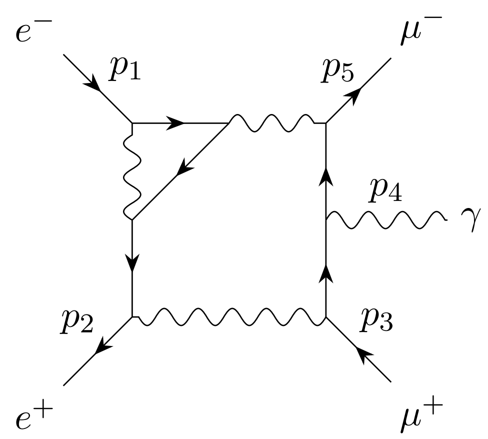
Other than that, the phase space point was also specified. We evaluated the process in the massless limit, hence imposing that $p_i^2 = 0$, while fixing the 5 independent mandelstam variables to

$$\begin{aligned} 2p_1 \cdot p_2 &= s_{12} = -4 \\ 2p_2 \cdot p_3 &= s_{23} = -6 \\ 2p_3 \cdot p_4 &= s_{34} = -10 \\ 2p_4 \cdot p_5 &= s_{45} = -14 \\ 2p_5 \cdot p_1 &= s_{51} = -22 \end{aligned} \tag{4.25}$$

In this step, 120 different diagrams were generated. In principle, one would have

$$\mathcal{M} = \sum_i I_i\ , \qquad I_i = \int \frac{\mathcal{N}_i}{\prod_j D_j} \tag{4.26}$$

, instead from that list, we chose one diagram to apply the complete chain software:

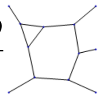
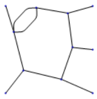


In order to obtain an input readable by AIDA it was contracted with complex conjugate of the tree level one, presented at the beginning of the section, thus evaluating an *interference term* that involves the diagram pictured above. Thanks to FEYNCALC it was finally translated in to an input readable by Aida and printed in a temporary file.

4.3.2 Integrand reduction: AIDA

Aida then took it and reduced such integral by means of the Adaptive integrand decomposition. As an output, one obtains the diagram written in terms of fraction that has at the numerator irreducible polynomials. In this case there was 639 of such integrands, starting from the amplitude pictured above. Here there is a little example of the output, representing integrals (that contains the reduced integrands) by their topology:

$$I = \int \sum_{k=i}^{8} \sum_{\{j_1,\cdots,j_k\}} \frac{\Delta_{j_1,\cdots,j_k}}{D_{j_1}\cdots D_{j_k}}$$

$$= -\frac{124928 + 62464D}{33}$$  $$-\frac{56672 + 33856D - 2760D^2}{33}$$  $$+\dots \qquad (4.27)$$

4.3.3 IBPs reduction: Kira

From the output of Aida, the topology of the problem was extracted, obtaining the following integral:

$$I^{[d]}[\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}\} = \int d^D q_1 d^D q_2 \frac{1}{D_1^{x_1} D_2^{x_2} D_3^{x_3} D_4^{x_4} D_5^{x_5} D_6^{x_6} D_7^{x_7} D_8^{x_8} S_1^{x_9} S_2^{x_{10}} S_3^{x_{11}}} \qquad (4.28)$$

where S_i are the ISPs, in fact x_9, x_{10}, x_{11} are typically negative. In this particular case, the diagram had the following list of denominators and ISPs:

$$\begin{aligned} D_1 &= q_1^2 \\ D_2 &= q_2^2 \\ D_3 &= (q_1 + q_2)^2 \\ D_4 &= (q_2 + p_2)^2 \\ D_5 &= (q_2 + p_2 + p_3)^2 \\ D_6 &= (q_2 + p_2 + p_3 + p_4)^2 \\ D_7 &= (q_2 + p_2 + p_3 + p_4 + p_5)^2 \\ D_8 &= (q_1 - p_2 - p_3 - p_4 - p_5)^2 \\ S_1 &= q_1 \cdot p_3 \\ S_2 &= q_1 \cdot p_4 \\ S_3 &= q_1 \cdot p_5 \end{aligned} \qquad (4.29)$$

In this step, the huge number of 639 integrals were reduced in to 28 Master Integrals, being a crucial simplification if one aim to evaluate an amplitude. After the IBP reduction the amplitude was written in the form

$$\mathcal{M} = \sum_{i=1}^{28} c_i I_i^{MI} \qquad (4.30)$$

The topologies of such Master Integrals are depicted in Figure 4.10.

4.3.4 MIs evaluation: SecDec

Lastly, the output of Kira was collected and translated into a language readable by SecDec, which returns the Master Integrals written as an expansion around $D = 4 - 2\epsilon$, namely

$$\mathcal{M} = \sum_i b_i \epsilon^i. \qquad (4.31)$$

The result of such calculation are outlined in the Tables 4.1 and 4.2.

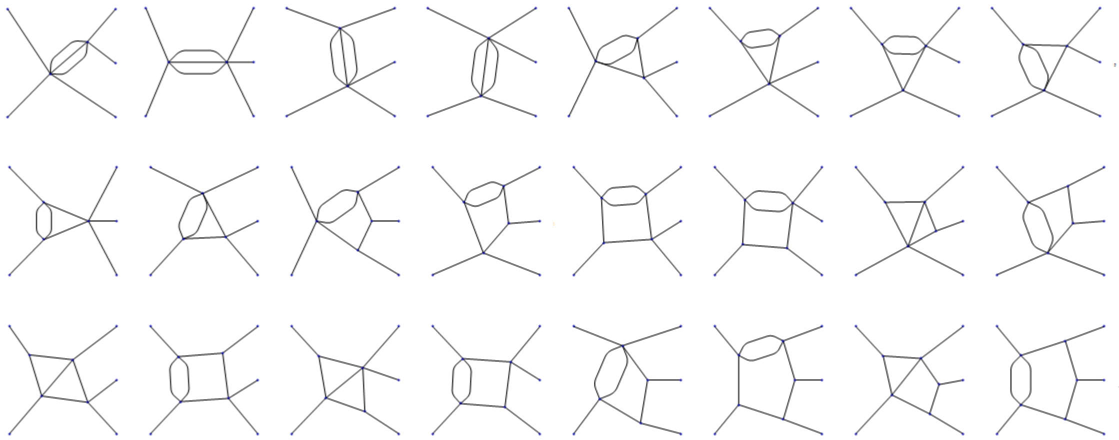


Figure 4.10: Topologies of the master integrals appearing in the decomposition of the 2-loop 5-leg integral family of Section 4.3.3

4.3.5 Result

After performing the steps reported above, all the results were collectedand substituted back into the output of Aida, resulting at last in

$$I = \frac{13.4435}{\epsilon^4} - \frac{75.9369}{\epsilon^3} + \frac{280.556}{\epsilon^2} + \frac{1.9349 \cdot 10^9}{\epsilon} - 3.2168 \cdot 10^8 \tag{4.32}$$

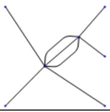
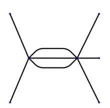
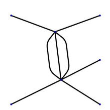
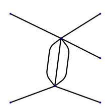
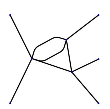
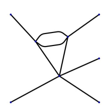
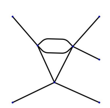
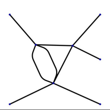
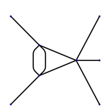
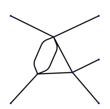
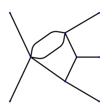
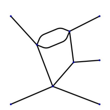
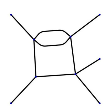
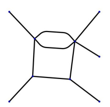
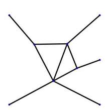
graph	$I^{[d]}$-**integral**	$I^{[d=4-2\epsilon]}_{(s_{12}=-4,s_{23}=-6,s_{34}=-10,s_{45}=-14,s_{51}=-22)}$
	$I^{[d]}[\{0,0,1,0,1,0,0,1,0,0,0\}]$	$\frac{3.5}{\epsilon}+0.23609+20.93819\epsilon$
	$I^{[d]}[\{0,0,1,1,0,0,0,1,0,0,0\}]$	$\frac{1}{\epsilon}+2.57298+9.29018\epsilon$
	$I^{[d]}[\{1,0,1,0,0,1,0,0,0,0,0\}]$	$\frac{5.5}{\epsilon}-4.60084+34.81470\epsilon-28.2164\epsilon^2+158.170\epsilon^3$
	$I^{[d]}[\{1,0,1,0,1,0,0,0,0,0,0\}]$	$\frac{1.5}{\epsilon}+2.64307+11.29871\epsilon$
	$I^{[d]}[\{0,0,1,1,0,1,0,1,00,0\}]$	$\frac{0.5}{\epsilon^2}-\frac{0.37980}{\epsilon}+3.29937-3.36067\epsilon+15.4036\epsilon^2$
	$I^{[d]}[\{0,1,1,0,0,1,0,1,0,0,0\}]$	$\frac{0.5}{\epsilon^2}-\frac{1.16826}{\epsilon}+5.43729-9.95395\epsilon$
	$I^{[d]}[\{0,1,1,0,1,0,0,1,0,0,0\}]$	$\frac{0.5}{\epsilon^2}+\frac{0.13102}{\epsilon}+1.40115+1.42641\epsilon$
	$I^{[d]}[\{1,0,1,0,1,0,1,0,00,0\}]$	$\frac{0.5}{\epsilon^2}-\frac{0.71627}{\epsilon}+3.62518-5.63423\epsilon+18.1484\epsilon^2$
	$I^{[d]}[\{1,0,1,1,0,0,1,0,0,0,0\}]$	$\frac{0.5}{\epsilon^2}+\frac{0.5364}{\epsilon}+4.36029+5.09703\epsilon+25.52848\epsilon^2$
	$I^{[d]}[\{1,0,1,1,0,1,0,0,0,0,0\}]$	$\frac{0.5}{\epsilon^2}-\frac{0.37980}{\epsilon}+1.61438-0.26682\epsilon+3.23000\epsilon^2$
	$I^{[d]}[\{0,0,1,1,1,1,0,1,0,0,0\}]$	$-\frac{0.1}{\epsilon^3}+\frac{0.40961}{\epsilon^2}-\frac{0.61849}{\epsilon}+1.37594-2.4683\epsilon$
	$I^{[d]}[\{0,1,1,0,1,1,0,1,0,0,0\}]$	$\frac{0.081205}{\epsilon^2}-\frac{0.28532}{\epsilon}+0.63034$
	$I^{[d]}[\{0,1,1,1,0,1,0,1,0,0,0\}]$	$\frac{0.06570}{\epsilon^2}-\frac{0.15884}{\epsilon}+0.49844$
	$I^{[d]}[\{0,1,1,1,1,0,0,1,0,0,0\}]$	$-\frac{0.16667}{\epsilon^3}+\frac{0.38875}{\epsilon^2}-\frac{0.52154}{\epsilon}+1.92108$
	$I^{[d]}[\{1,0,1,0,1,1,0,1,0,0,0\}]$	$-\frac{0.1123}{\epsilon^2}-\frac{0.58269}{\epsilon}-1.45504$

Table 4.1: Values of the MIs for the process $e^+e^- \to \mu^+\mu^-\gamma$

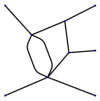
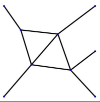
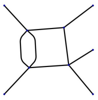
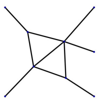
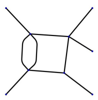
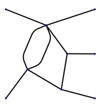
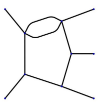
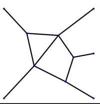
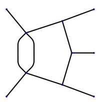
graph	$I^{[d]}$-**integral**	$I^{[d=4-2\epsilon]}_{(s_{12}=-4,s_{23}=-6,s_{34}=-10,s_{45}=-14,s_{51}=-22)}$
	$I^{[d]}[\{1,0,1,0,1,1,1,0,0,0,0\}]$	$-\frac{0.07143}{\epsilon^3}+\frac{0.3489}{\epsilon^2}-\frac{0.69302}{\epsilon}+1.41019-2.63516\epsilon$
	$I^{[d]}[\{1,0,1,1,0,1,0,1,0,0,0\}]$ $I^{[d]}[\{1,-1,1,1,0,1,0,1,0,0,0\}]$	$\frac{0.4169}{\epsilon}-0.83657+0.20589\epsilon$ $-\frac{0.5}{\epsilon^2}+\frac{0.27947}{\epsilon}+0.2491-3.16215\epsilon$
	$I^{[d]}[\{1,0,1,1,0,1,1,0,0,0,0\}]$	$\frac{0.15272}{\epsilon^2}-\frac{0.43336}{\epsilon}+0.84943-2.43561\epsilon$
	$I^{[d]}[\{1,0,1,1,1,0,0,1,0,0,0\}]$	$-\frac{0.25383}{\epsilon^2}+\frac{0.78731}{\epsilon}-1.42448$
	$I^{[d]}[\{1,0,1,1,1,0,1,0,0,0,0\}]$	$\frac{0.12528}{\epsilon^2}-\frac{0.20079}{\epsilon}+0.67472-1.91593\epsilon$
	$I^{[d]}[\{1,0,1,1,1,1,0,0,0,0,0\}]$	$-\frac{0.1}{\epsilon^3}+\frac{0.32488}{\epsilon^2}-\frac{0.57686}{\epsilon}+1.58360-2.1124\epsilon$
	$I^{[d]}[\{0,1,1,1,1,1,0,1,0,0,0\}]$ $I^{[d]}[\{-1,1,1,1,1,1,0,1,0,0,0\}]$	$\frac{0.03333}{\epsilon^3}-\frac{0.07235}{\epsilon^2}+\frac{0.07949}{\epsilon}+0.22502$ $\frac{0.1}{\epsilon^3}-\frac{0.09421}{\epsilon^2}+\frac{0.05847}{\epsilon}-0.14308$
	$I^{[d]}[\{1,0,1,1,1,1,0,1,0,0,0\}]$ $I^{[d]}[\{1,-1,1,1,1,1,0,1,0,0,0\}]$	$-\frac{0.01059}{\epsilon^3}+\frac{0.08118}{\epsilon^2}-\frac{0.17178}{\epsilon}+0.22502$ $\frac{0.06355}{\epsilon^3}-\frac{0.5145}{\epsilon^2}+\frac{1.43464}{\epsilon}-2.76172$
	$I^{[d]}[\{1,0,1,1,1,1,1,0,0,0,0\}]$ $I^{[d]}[\{1,-1,1,1,1,1,1,0,0,0,0\}]$	$\frac{0.01429}{\epsilon^3}-\frac{0.07359}{\epsilon^2}+\frac{0.09566}{\epsilon}-0.16079+0.20589\epsilon$ $-\frac{0.2}{\epsilon^3}+\frac{0.78933}{\epsilon^2}-\frac{1.32655}{\epsilon}+2.34087-3.91084\epsilon$

Table 4.2: Values of the MIs for the process $e^+e^- \to \mu^+\mu^-\gamma$

4.4 Analytic expressions from functional reconstruction

Let us stress the fact that computing the amplitude for such process *numerically* can become a key result for the analytical evaluation of the latter.

Multiple numerical evaluations of a function in fact can provide useful information about its structure. In the case of rational functions, not only some properties but the entire analytic expression can be obtained by sampling repeatedly on different phase-space points. A detailed discussion of this topic, called functional reconstruction, is far beyond the scope of this thesis; we refer the interested reader to [33] and [34]. Scattering amplitudes over Finite Fields and multivariate functional reconstruction). In this section we will first briefly present the general idea behind functional reconstruction, then we will report some results obtained applying this technique to the coefficients multiplying the master integrals in the loop-level expansion, which happen to be rational functions.

4.4.1 A simple example: univariate polynomials

Consider as an example one of the simplest possible rational functions, i.e. a polynomial in only one variable, lets say of degree two:

$$\mathcal{P}(x) = x^2 - 5x + 3 \tag{4.33}$$

We want to guess eq.4.33 starting from something like

$$\begin{cases} \mathcal{P}(3) = -3 \\ \mathcal{P}(5) = 3 \\ \mathcal{P}(7) = 17 \\ \quad \vdots \end{cases} \tag{4.34}$$

If we knew the degree d of $\mathcal{P}$ in advance we could simply build and solve a system of $d+1$ equations of the form

$$\mathcal{P}(x) = n_0 + n_1 x + n_2 x^2 \Big|_{x=x_i} \tag{4.35}$$

with $x_i = 3, 5, 7$. However, since in general one does not know the degree d, another method must be used. The task at hand can be solved rewriting the polynomial in an alternative representation due to Newton:

$$\begin{aligned} \mathcal{P}(x) &= \sum_{i=0}^{R} a_i \prod_{j=0}^{i-1} (x - y_j) \\ &= a_0 + (x - y_0)\Big(a_1 + (x - y_1)\big(a_2 + (x - y_2)(\cdots + (x - y_{R-1})a_R)\big)\Big) \end{aligned} \tag{4.36}$$

where the a_i depend on the y_i, and the latter can be chosen arbitrarily. The values of the coefficients a_i can then be immediately extracted performing a smart sampling on x. Baring in mind the second line of eq.4.36 we have:

$$\text{choose } y_0 = 3 \Rightarrow \begin{cases} \mathcal{P}(3) = a_0 + (3-3)(\cdots) \\ \qquad\;\; = a_0 \\ \qquad\;\; \Downarrow \\ \quad a_0 = -3 \end{cases}$$

$$\text{choose } y_1 = 5 \Rightarrow \begin{cases} \mathcal{P}(5) = -3 + (5-3)(a_1 + (5-5)(\cdots)) \\ \qquad\;\; = -3 + (5-3)a_1 \\ \qquad\quad \Downarrow \\ a_1 = 3 \end{cases}$$

$$\text{choose } y_2 = 7 \Rightarrow \begin{cases} \mathcal{P}(7) = -3 + (7-3)(3 + (7-5)(a_2 + (7-7)(\cdots))) \\ \qquad\;\; = -3 + (7-3)(3 + (7-5)a_2) \\ \qquad\quad \Downarrow \\ a_2 = 1 \end{cases}$$

One can see that $a_i = 0$ for any y_i when $i > 2$, thus the final expression we find is

$$\mathcal{P}(x) = -3 + (x-3)(3 + (x-5)(1)) \tag{4.37}$$

which, upon expansion of all the products, is none other than eq.4.33.

4.4.2 Univariate rational functions

In a similar fashion, for a proper rational function i.e. a ratio of two polynomials

$$\mathcal{R}(x) = \frac{n_0 + n_1 x + n_2 x^2 + \cdots + n_R x^R}{d_0 + d_1 x + d_2 x^2 + \cdots + d_{R'} x^{R'}} \tag{4.38}$$

one can use the so called Thiele interpolation formula, which is an alternative representation of 4.38:

$$\begin{aligned} \mathcal{R}(x) &= a_0 + \cfrac{x - y_0}{a_1 + \cfrac{x - y_1}{a_2 + \cfrac{x - y_2}{\cfrac{\vdots}{a_{N-1} + \cfrac{x - y_{N-1}}{a_N}}}}} \\ &= a_0 + (x - y_0)\left(a_1 + (x - y_1)\left((x - y_2)\left(\cdots + \frac{x - y_{N-1}}{a_N}\right)^{-1}\right)^{-1}\right)^{-1} \end{aligned} \tag{4.39}$$

Again, at the cost of introducing a set of arbitrary constants y_i, the computation of the coefficients a_i is reduced to a systematic evaluation of $\mathcal{R}$ on the chosen y_i. The analytic expression of the a_i is computed recursively and is given by

$$\begin{aligned} a_0 &= \mathcal{R}(y_0) \\ a_1 &= (\mathcal{R}(y_1) - a_0)^{-1}(y_1 - y_0) \\ &\vdots \\ a_r &= \left(\left((\mathcal{R}(y_N) - a_0)^{-1}(y_N - y_0) - a_1\right)^{-1}(y_N - y_1) - \cdots - a_{N-1}\right)^{-1}(y_N - y_{N-1}) \end{aligned} \tag{4.40}$$

In a very similar way, through multiple nested applications of the univariate algorithm [34], it is possible to obtain the analytic expression of the multivariate rational functions. In particular, since the coeficients multiplying the master integrals in the loop expansion happen to be rational functions, we can apply these techniques to them.

4.5 Application to integration by parts identities

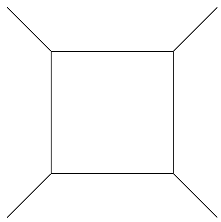
In this section we are going to present some results obtained by applying functional reconstruction techniques to the coefficients multiplying the master integrals in a loop-level expansion. In the tables we are going to list the analytic expressions of the coefficients along with the number of numerical evaluations which where necessary to compute them. Such reconstructions are performed by means of routines developed in [35]

4.5.1 One loop: box topology

The integral's topology taken in to account in this example is the following:

$$I[\{1,1,1,1\}] = \int d^D q \frac{1}{q^2(q+p_1)^2(q+p_1+p_2)^2(q+p_1+p_2+p_3)^2} \tag{4.41}$$

and can be depicted as



Through the use of REDUZE it was possible to reduce the integral $I[\{2,1,1,1\}]$ in terms of easier integrals, thanks to the IBPs. Such reduction lead to an identity of the form

$$I[\{2,1,1,1\}] = b_1 I[\{1,1,1,1\}] + b_2 I[\{0,1,0,1\}] \tag{4.42}$$

In the table, the coefficients are presented together with the number of numerical samples one has to perform in order to reconstruct such relation analytically.

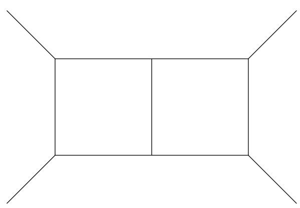
Coefficient	Number of evaluations	Analytic expression
b_1	7	$-\frac{d-5}{s}$
b_2	44	$\frac{4(d-5)(d-3)}{(d-6)s(s+t)^2}$

(4.43)

Table 4.3: Table of the analytic coefficients for the IBPs reported in equation (4.42)

4.5.2 Two loops: ladder topology

Moving to a more interesting case, let's address a two loop case. In particular the ladder topology:



which is characterized by the following set of denominators and ISPs:

$$\begin{aligned}
D_1 &= q_1^2 \\
D_2 &= q_2^2 \\
D_3 &= (q_1 + 1_2)^2 \\
D_4 &= (q_1 + p_1)^2 \\
D_5 &= (q_1 + p_1 + p_2)^2 \\
D_6 &= (q_1 + q_2 + p_1 + p_2)^2 \\
D_7 &= (q_1 + q_2 + p_1 + p_2 + p_3)^2 \\
D_8 &= (q_2 + p_2)^2 \\
D_9 &= (q_2 + p_3)^2
\end{aligned} \tag{4.44}$$

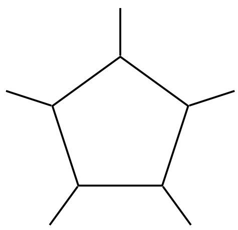
obtaining an IBP of the form

$$\begin{aligned}
I[\{1,1,1,1,2,1,1,-1,-1\}] &= c_1 I[\{1,1,1,1,1,1,1,-1,0\}] + c_2 I[\{1,1,1,1,1,1,1,0,0\}] \\
&+ c_3 I[\{0,1,1,1,1,0,1,0,0\}] + c_4 I[\{1,1,0,1,1,0,1,0,0\}] + c_5 I[\{1,0,1,0,1,1,0,0,0\}] \\
&+ c_6 I[\{0,1,1,1,0,1,0,0,0\}] + c_7 I[\{0,1,1,0,1,0,0,0,0\}] + c_8 I[\{0,1,1,0,1,0,0,0,0\}]
\end{aligned} \tag{4.45}$$

The result of the reconstruction of such IBP is reported in 4.4

4.5.3 One loop: pentagon

Taking in to account an example closer to the amplitude evaluated in Section 4.3 we address the reconstruction of a five point one loop topology. This example also provide information on the difference in the reconstruction of an IBP in an higher loop cases vs. a case with higher external leg number.
The topology taken in to account is



with the following denominators:

$$\begin{aligned}
D_1 &= q^2 \\
D_2 &= (q + p_1)^2 \\
D_3 &= (q + p_1 + p_2)^2 \\
D_4 &= (q + p_1 + p_2 + p_3)^2 \\
D_5 &= (q + p_1 + p_2 + p_3 + P_4)^2
\end{aligned} \tag{4.46}$$

The IBP reconstructed is of the form

$$\begin{aligned}
I[\{1,2,1,1,1\}] =& d_1 I[\{1,1,1,1,1\}] + d_2 I[\{0,1,1,1,1\}] + d_3 I[\{1,0,1,1,1\}] + d_4 I[\{1,1,0,1,1\}] \\
&+ d_5 I[\{1,1,1,0,1\}] + d_6 I[\{1,1,1,1,0\}] + d_7 I[\{0,0,1,0,1\}] + d_8 I[\{0,1,0,0,1\}] \\
&+ d_9 I[\{0,1,0,1,0\}] + d_{10} I[\{1,0,0,1,0\}] + d_{11} I[\{1,0,1,0,0\}]
\end{aligned} \tag{4.47}$$

The results for its reconstruction are outlined in Table 4.5.

Coefficient	Number of evaluations	Analytic expression
c_1	3	$-\frac{3}{2}(d-4)$
c_2	11	$\frac{1}{2}(d-4)(s+t)$
c_3	63	$-\frac{3t\left(3d^3s+3d^3t-40d^2s-38d^2t+177ds+157dt-258s-210t\right)}{(d-6)(d-3)s^2(s+t)}$
c_4	33	$-\frac{3\left(3d^2s+3d^2t-28ds-20dt+60s+28t\right)}{2(d-4)s(s+t)}$
c_5	31	$\frac{2\left(6d^2s+2d^2t-41ds-16dt+71s+30t\right)}{(d-4)s^3}$
c_6	68	$-\frac{3(3d-10)\left(d^3s+d^3t-17d^2s-15d^2t+91ds+71dt-154s-106t\right)}{(d-6)(d-4)^2s^2(s+t)}$
c_7	123	$\frac{3(3d-10)(3d-8)\left(3d^3s+6d^3t-42d^2s-76d^2t+194ds+314dt-292s-420t\right)}{2(d-6)(d-4)^3s^2(s+t)^2}$
c_8	127	$-\frac{3(3d-10)(3d-8)\left(2d^5s+2d^5t-47d^4s-45d^4t+430d^3s+392d^3t-1925d^2s-1661d^2t+4236ds+3444dt-3684s-2820t\right)}{(d-6)^2(d-4)^3(d-3)s^3(s+t)}$

(4.48)

Table 4.4: Table of the analytic coefficients for the IBPs reported in equation (4.45), together with the number of numerical samples provided in order to reconstruct such relations.

Coefficient	Number of evaluations	Analytic expression
d_1	208	$-\frac{(d-6)\left(2s_{12}^2+s_{15}s_{12}+s_{23}s_{12}-s_{34}s_{12}-3s_{45}s_{12}+s_{45}^2-s_{15}s_{34}-s_{23}s_{45}\right)}{s_{12}(s_{12}+s_{23}-s_{45})(s_{12}-s_{34}-s_{45})}$
d_2	72	$\frac{4(d-5)s_{45}}{s_{12}(-s_{12}-s_{23}+s_{45})(-s_{12}+s_{34}+s_{45})}$
d_3	84	$\frac{4(d-5)(s_{12}+s_{15}-s_{45})}{s_{12}(s_{12}+s_{23}-s_{45})(s_{12}-s_{34}-s_{45})}$
d_4	84	$\frac{4(d-5)(s_{15}-s_{23}+s_{45})}{s_{12}(s_{12}+s_{23}-s_{45})(-s_{12}+s_{34}+s_{45})}$
d_5	82	$\frac{4(d-5)(s_{23}+s_{34}-s_{45})}{s_{12}(s_{12}+s_{23}-s_{45})(-s_{12}+s_{34}+s_{45})}$
d_6	74	$-\frac{4(d-5)s_{15}}{s_{12}(s_{12}+s_{23}-s_{45})(s_{12}-s_{34}-s_{45})}$
d_7	173	$\frac{64(d-5)(d-3)}{(d-6)s_{12}s_{34}(s_{12}+s_{23}-s_{45})(s_{12}-s_{34}-s_{45})}$
d_8	149	$-\frac{64(d-5)(d-3)}{(d-6)s_{12}^2(s_{12}+s_{23}-s_{45})(s_{12}-s_{34}-s_{45})}$
d_9	275	$-\frac{64(d-5)(d-3)}{(d-6)s_{12}(s_{12}+s_{23}-s_{45})(s_{12}-s_{34}-s_{45})^2}$
d_{10}	467	$\frac{64(d-5)(d-3)}{(d-6)s_{12}(s_{15}-s_{23}-s_{34})(s_{12}+s_{23}-s_{45})(s_{12}-s_{34}-s_{45})}$
d_{11}	275	$\frac{64(d-5)(d-3)}{(d-6)s_{12}(s_{12}+s_{23}-s_{45})^2(s_{12}-s_{34}-s_{45})}$

(4.49)

Table 4.5: Table of the analytic coefficients for the IBPs reported in equation (4.47), together with the number of numerical samples provided in order to reconstruct such relations.

Conclusions

In this work we presented in detail all the techniques needed to perform the of a scattering amplitude, both at 1 and multiloop level, discussing their application in a 2 loop case.
We started by introducing the Unitarity based approach, starting from their root: the Unitarity of the S matrix and its consequences, from which the *generalized Optical theorem* stems. After describing with an example such interesting relation between amplitudes at different perturbative levels, we described an efficient shortcut that allows to arrive at the same results of the Optical theorem: the *Cutkosky rule.* By means of on shell methods it is in fact possible to evaluate the imaginary part of an amplitude into a straightforward way: putting on shell all the possible subset of the internal lines (i.e. *cutting* them) the amplitude is *projected* on to its imaginary component, thus showing the power of Unitarity based methods. After a brief example to outline the validity of such procedure, we reviewed another technique that revolves around the *cut.* In fact, by means of complex analysis and contour deformation, it is possible to draw a relation between the Feynman propagator and its Advanced counterpart. This relation is of crucial importance when dealing with integration over the momenta appearing in the denominator. In fact in this case the latter turns out to be zero, releasing a powerful identity which allows to write an arbitrary l loop amplitude in function of tree level amplitude, integrated over the phase space of the internal leg putted on shell. Such tree level amplitudes can be seen as cut amplitude, thus showcasing once more the great power that lies inside the Unitarity based techniques.
Such techniques culminated in to the generalized Unitarity approach, applied at one loop level. In such framework, by means of tensor decomposition and integrand reduction the amplitude is cast in to a combination of 5 *universal* Master Integrals. Such result is valid for *any* amplitude at one loop. After that, inspired by the Unitarity based technique of the Cutkosky rule, in which the amplitude is projected in to its imaginary component through the use of cuts, it is possible to evaluate in an easy way the coefficient in front of all Master Integrals, thus determining the expansion. Such technique lead to major breakthrough in the automation of one loop amplitudes calculations.
Unfortunately, such framework cannot be applied straightforwardly to the multiloop level. In fact after defining a Feynman integrals we showcase how, due to the presence of a higher number of internal line, at this perturbation level scalar products which cannot be written in terms of denominators appears, i.e. the *Irreducible Scalar Products.* Such object are the greatest speed bump in the automation of multiloop calculation. We then outline a different strategy in order to evaluate such amplitude. A core step in this framework is the reduction of the number of integral that one has to evaluate to obtain the total value of the amplitude. We presented the *integration by parts identities* which are a set of identities due to the *symmetry* of the integral under

redefinition of the loop momenta. After showcasing an example of application of such reduction, we present a promising parameterization that highlight the geometrical properties of Feynman amplitude: the Baikov representation. Such parametrization is obtained by performing a change of variables, from loop momenta to the scalar products involving them. In such a way, the integrand becomes proportional to the *Gram determinant*, which is deeply related to a volume, in this case the volume spanned by the momenta flowing in the amplitude. We then propose some example in order to reach a deeper understanding of this mathematical object, discovering an unexpected iterative structure that revolves around the *boundaries* of such integrals in this parameterization. Thanks to that, we were able to find a new parameterization of the scattering amplitude of which very little is known. It is our hope that such parameterization hides useful properties which will be studied in the future.
After that we presented the application of the well known IBP reduction and Unitarity cut approach, from a different point of view. They are in fact applied in together with the Baikov representation, showcasing the strength of such a parameterization. Lastly, after presenting the *Adaptive integrand decomposition*, a promising alternative to the tensor reduction method, we focus on the computational aspects of the evaluation of the Feynman amplitude. After developing a discrete knowledge of such techniques in fact it is mandatory to apply them through the usage of a software. With that in mind, we contributed to the development of a *complete chain* for the evaluation of the multiloop amplitude, based on tools that uses techniques studied in this work of thesis.
Lastly, a first test for such chain was completed, evaluating the amplitude of the process $e^-e^+ \to \mu^-\mu^+\gamma$ at two loop in QED. In future, feasible objective could be the evaluation of process like $pp \to HH$ or $pp \to Hjj$ at two loop in QCD, the latter in the heavy top effective theory.
This is meant to be a starting point for a deeper research in scattering amplitudes, both on the analytical and computational point of view.

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