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On the Identification of Sparse plus Low-rank Graphical Models

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Abstract

The improvement of the technology during the recent years is such that the amount of data available for a given application is countinuously increasing. The larger the amount of data, the more important is to provide a graphical description of these data, which gives an intuitive representation of the phenomenon of interest. As a conseguence, the importance of these so-called Graphical models has rapidly grown in the recent years in different areas of science and engeneering.
This thesis proposes an identification procedure for periodic, Gaussian, stationary reciprocal processes, under the assumption that the conditional dependence relations among the manifest (or observed) variables are mainly due to a limited number of latent (or hidden) variables. The identification procedure combines the sparse plus low-rank decomposition of the inverse covariance matrix of the process and the maximum entropy solution for the block-circulant band extension problem recently proposed in the literature.

Contents

1 Introduction

In this Chapter we will introduce the main topics addressed by this thesis. We will briefly review the problems and the motivations that have led to this work and we give an overview of the current state-of-the-art. Finally, we will present the contributions and the organization of the thesis.

The great improvement of the technology in the recent years has led to the frenetic increasing of the number of available data in many important domains such as system biology, social networks, econometrics etc. If this huge amount of data can be modeled in terms of realizations of a Gaussian stochastic process then, under suitable assumptions, a graphical model explaining these data can be provided. This graphical representation allows a better understanding of the phenomenon but it could be still very complicated because of the extremely high number of data. In order to reduce the "graphical complexity" of the model, it is assumed that the observed variables are mainly correlated through a relatively small number of latent (or hidden) variables, see [7], [8], [9]. The corresponding graphical model is composed by a two-layer graphical structure: the upper layer contains the hidden variables and the bottom layer the observed variables. Such models re called *latent-variable graphical models* or *sparse plus low-rank graphical models*. The hope is that the presence of the few (with respect to the observed variables) extra nodes in the hidden layer allows a drastic reduction of edges in the observed layer (sparse layer). As a consequence, the interpretation of this large amount of data is drastically simplified. Moreover, this structure may improve scalability and efficiency of the identification procedure.
When the signal of interest is defined in a finite time interval (or space interval) it may be naturally modeled with a reciprocal process. Reciprocal processes have been introduced at the beginning of the last century and they are actually a generalization of Markov processes since their dynamics is completely specified by their values at the boundary [11]. Under suitable assumptions, these processes can be extended by periodicity on the whole integer line and it turns out that their maximum likelihood estimate leads to a covariance extension problem for circulant covariance matrices, as showed in [12] (see also [13] where the circulant covariance extension problem is linked with the Burg's spectral estimation method [14]). In the same paper, it has been shown that the maximum entropy principle leads to a complete solution of this problem which is actually the acausal counterpart of the autoregressive process. The latter is the solution of the classical covariance extension problem for positive-definite Toeplitz matrices, widely studied in literature.

Contributions and outline of the thesis

In this thesis we develop an identification procedure for latent-variable graphical models associated to reciprocal, Gaussian periodic stationary processes. This procedure combines

the essential steps of the solution of the block-circulant band extension problem presented in [12] and the autoregressive system identification procedure for latent-variable graphical models presented in [7]. In this case the peculiarity is that, in addition to our reciprocal process, we have also the (static) latent components thus, the sparse plus low-rank decomposition of the inverse covariance matrix of the process has to be embedded in the optimization problem that follows from the maximum entropy principle.

The thesis is organized as follows. In Chapter 2 we set the notation that will be used throughout the thesis. Due to their importance in this work, particular attention will be dedicated to (symmetric) block-circulant matrices for which we will recall some basics results. In Chapter 3 we will review some mathematical notions and results about the Gaussian distribution, Hilbert spaces of random variables with finite second-moment and convex optimization. Chapter 4 will be dedicated to a review of the fundamental ingredients involved in our problem: maximum entropy problems and (Gaussian) graphical models. More precisely, we will briefly mention classical maximum entropy problems and we will focus with more detail on those that are particularly interesting for our case, in order to give an introduction of what we will do in the next chapters. In Chapter 5, after a presentation of periodic, stationary processes, we will derive a model for periodic, stationary reciprocal processes. In Chapter 6 we will construct a latent-variable graphical model for a periodic, Gaussian, stationary reciprocal process assuming that the latent component is static and we will derive and solve the corresponding identification paradigm. Chapter 7 basically specializes what has been presented in Chapter 6 for a particular choice of the structure of the matrices involved in the sparse plus low-rank decomposition of the concentration matrix of the process. Finally, in Chapter 8 we draw some final considerations and we describe possible future extension of this work.

2 Notation

Here we introduce the general notation and terminology used throughout the thesis. Additional nomenclature will be introduced, if needed, at the beginning or within the following Chapters. The table below reports the basic symbols in order to fix the notation that we will use from now on.

$\emptyset$	empty set
$\mathbb{N}$	set of natural numbers $\{0, 1, 2, 3, \dots\}$
$\mathbb{Z}$	ring of integers $\{\dots, -2, -1, 0, 1, 2, \dots\}$
$\mathbb{R}$	field of real numbers
$\mathbb{R}^{p\times q}$	set of $p \times q$ matrices
$\mathbb{S}_n$	set of $n \times n$ symmetric matrices
$A \otimes B$	Kronecker product between matrix A and matrix B
$\langle \cdot\,, \cdot \rangle_{\mathcal{Q}}$	inner product on the inner product space $\mathcal{Q}$
$\|x\|_1$	ℓ^1-norm of vector x defined as $\|x\|_1 := \sum_i \|x_i\|$
$\|x\|_\infty$	ℓ^∞-norm of vector x defined as $\|x\|_\infty := \sup_i \|x_i\|$
$\|A\|$	Frobenious norm of matrix A defined as $\|A\| = \mathrm{tr}(A^\top A)$
$\mathcal{Q}^c$	complement of set $\mathcal{Q}$
$\mathcal{Q}^+$	set of all positive definite matrices contained in $\mathcal{Q}$
I_n	$n \times n$ identity matrix
I_{mN}	$m \times m$ block identity matrix of dimension $mN \times mN$
$(A)_{ij}$	element in position (i, j) in matrix A
$A^\top$	transpose of matrix A
A^{-1}	inverse of matrix A
$\mathrm{tr}(A)$	trace of matrix A
$\ker(A)$	kernel of matrix A
$\det(A)$	determinant of matrix A
$\mathbb{E}[\cdot]$	mathematical expectation

Note that, the space $\mathbb{R}^{p\times q}$ is equipped with the usual inner product $\langle P, Q\rangle = \mathrm{tr}(P^\top Q)$.

2.1 Circulant matrices

Here we introduce the notation and give some preliminary results concerning block-circulant matrices. We start by the definition of block-circulant matrix.

Definition 1. A *block-circulant matrix* $\mathbf{C}$ with N bloks of dimension $m \times m$ is a block-Toeplitz matrix whose columns block (or equivalently rows block) are shifted cyclically, i.e.

$$\mathbf{C} = \begin{bmatrix} C_0 & C_{N-1} & \cdots & C_1 \\ C_1 & C_0 & \cdots & C_2 \\ \vdots & \ddots & \ddots & \vdots \\ C_{N-1} & \cdots & C_1 & C_0 \end{bmatrix}$$

where $C_k \in \mathbb{R}^{m\times m}$, $k = 0, \ldots, N-1$.

From the definition it is apparent that a block-circulant matrix is completely specified by its first block-column, thus it can be denoted as

$$\mathbf{C} = \mathrm{circ}\{C_0, C_1, \ldots, C_{N-1}\}.$$

It is useful to introduce the *circulant shift matrix*

$$U := \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 1 & 0 & 0 & \ldots & 0 \end{bmatrix} \in \mathbb{R}^{N\times N}, \tag{2.1}$$

such that every block-circulant matrix $\mathbf{C}$ can be written as

$$\mathbf{C} = \sum_{k=0}^{N-1} U^{-k} \otimes C_k.$$

A very important role will be played by the *symbol* of the block-circulant matrix $\mathbf{C}$ namely, the $m \times m$ pseudo-polynomial defined as

$$C(\zeta) = \sum_{k=0}^{N-1} C_k\, \zeta^{-k}. \tag{2.2}$$

Harmonic analysis on $\mathbb{Z}_N$

Let $\zeta^k = e^{j\frac{2\pi}{N}k}$ be the k-th root of unity, $k = 0, \ldots, N-1$. Consider the sequence of $m \times m$ matrices $G_0, \ldots, G_{N-1}$, and assume to extend it by periodicity over the whole integer line $\mathbb{Z}$. Accordingly $G_k = G_{k+N}$, and there is no loss of generality in thinking that

this sequence is defined on the discrete circle group $\mathbb{Z}_N := \{1, 2, \dots, N\}$ with arithmetics mod N[1]. It is well known that the discrete Fourier transform (DFT) of the sequence

$$\hat{G}(\zeta) = \sum_{k=0}^{N-1} G_k \, \zeta^{-k} \tag{2.3}$$

has period N namely. In other words, it is naturally defined on the (discrete) unit circle $\mathbb{T}_N := \{\zeta^0, \zeta^1, \dots, \zeta^{N-1}\}$. Then, the sequence $\{G_k\}$ can be recovered through the inverse DFT

$$G_k = \frac{1}{N} \sum_{j=0}^{N-1} \hat{G}(\zeta^j) \, \zeta^{jk}, \qquad k = 0, \dots, N-1. \tag{2.4}$$

We recall the Plancherel's Theorem for the discrete time Fourier transform, a crucial result in Fourier analysis.

Theorem 1. Let $\{F_k\}$, $\{G_k\}$ be two finite sequences of $m \times m$ matrices and let $\hat{F}(\zeta)$ and $\hat{G}(\zeta)$ be the corresponding discrete time Fourier transforms defined for $\zeta \in \mathbb{T}_N$. Then

$$\langle F, G \rangle := \sum_{k=0}^{N-1} F_k G_k^\top = \frac{1}{N} \sum_{k=0}^{N-1} F(\zeta^k) G^*(\zeta^k) =: \left\langle \hat{F}, \hat{G} \right\rangle$$

namely, the DFT is an isometry between the two spaces involved.

Defining the vectors

$$\hat{\mathbf{G}} := \frac{1}{\sqrt{N}} \begin{bmatrix} \hat{G}(\zeta^0) \\ \hat{G}(\zeta^1) \\ \vdots \\ \hat{G}(\zeta^{N-1}) \end{bmatrix}, \qquad \tilde{G} := \begin{bmatrix} G_0 \\ G_1 \\ \vdots \\ G_{N-1} \end{bmatrix},$$

relations (2.3) and (2.4) can be rewritten in matrix form as

$$\hat{\mathbf{G}} = \mathbf{F} \, \tilde{G}, \qquad \tilde{G} = \mathbf{F}^* \, \hat{\mathbf{G}},$$

respectively, where $\mathbf{F}$ is the Fourier block-matrix

$$\mathbf{F} = \frac{1}{\sqrt{N}} \begin{bmatrix} \zeta^{-0\cdot 0} I_m & \zeta^{-0\cdot 1} I_m & \dots & \zeta^{-0\cdot(N-1)} I_m \\ \zeta^{-1\cdot 0} I_m & \zeta^{-1\cdot 1} I_m & \dots & \zeta^{-1\cdot(N-1)} I_m \\ \vdots & \vdots & \ddots & \vdots \\ \zeta^{-(N-1)\cdot 0} I_m & \zeta^{-(N-1)\cdot 1} I_m & \dots & \zeta^{-(N-1)\cdot(N-1)} I_m \end{bmatrix}. \tag{2.5}$$

It is very easy to see that $\mathbf{F}$ is a unitary matrix ($\mathbf{F}\mathbf{F}^* = \mathbf{F}^*\mathbf{F} = \mathbf{I}$) and that $\mathbf{F} = F \otimes I_m$ where

$$F = \frac{1}{\sqrt{N}} \begin{bmatrix} \zeta^{-0\cdot 0} & \zeta^{-0\cdot 1} & \dots & \zeta^{-0\cdot(N-1)} \\ \zeta^{-1\cdot 0} & \zeta^{-1\cdot 1} & \dots & \zeta^{-1\cdot(N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \zeta^{-(N-1)\cdot 0} & \zeta^{-(N-1)\cdot 1} & \dots & \zeta^{-(N-1)\cdot(N-1)} \end{bmatrix} \tag{2.6}$$

is the usual $m \times m$ Fourier matrix. The following result is a generalization of the fact that circulant matrices are diagonalized by the Fourier matrix in the case of block-circulant matrices. The proof generalizes what one can prove in the case of scalar-entries matrices (see e.g. [1] p.649).

[1] Hence $N + h = h$ so that N plays the role of the zero element.

Lemma 1. Let $\mathbf{C}$ be a block-circulant matrix with symbol $C(\zeta)$ defined by (2.2). Then

$$\mathbf{C} = \mathbf{F}^* \mathrm{diag}\left\{C(\zeta^0), C(\zeta^1), \ldots, C(\zeta^{N-1})\right\} \mathbf{F}, \tag{2.7}$$

where $\mathbf{F}$ is the $mN \times mN$ unitary matrix (2.5).

Proof. Let us first prove that

$$FU^{-1}F^* = \mathrm{diag}\{1, \zeta^{-1}, \zeta^{-2}, \ldots, \zeta^{-N+1}\} =: D, \tag{2.8}$$

where U is the $m \times m$ circulant-shift matrix. By direct computation

$$(F^*DF)_{hk} = \frac{1}{N} \sum_{\ell=0}^{N-1} \zeta^{\ell(h-1)} \zeta^{-\ell} \zeta^{-\ell(k-1)} = \frac{1}{N} \sum_{\ell=0}^{N-1} \left(\zeta^{h-k-1}\right)^{\ell},$$

hence, $(F^*DF)_{hk} = 1$ if $h - k - 1 = \alpha N$, for some $\alpha \in \mathbb{Z}$, while

$$(F^*DF)_{hk} = \frac{1}{N} \frac{1 - (\zeta^{h-k-1})^N}{1 - \zeta^{h-k-1}} = 0$$

otherwise. We conclude that $F^*DF = U^{-1}$ and (2.8) is proved. The remaining of the proof follows directly by the properties of the Kronecker product.

$$\mathbf{FCF}^* = \mathbf{F} \left(\sum_{k=0}^{N-1} U^{-k} \otimes C_k \right) \mathbf{F}^* = \sum_{k=0}^{N-1} \mathbf{F}(U^{-k} \otimes C_k)\mathbf{F}^*$$

where, for each $k = 0, 1, \ldots N-1$,

$$\begin{aligned} \mathbf{F}(U^{-k} \otimes C_k)\mathbf{F}^* &= (F \otimes I_m)(U^{-k} \otimes C_k)(F^* \otimes I_m) \\ &= (FU^{-k} \otimes C_k)(F^* \otimes I_m) \\ &= FU^{-k}F^* \otimes C_k \\ &= D^k \otimes C_k \end{aligned}$$

by (2.8). Finally

$$\mathbf{FCF}^* = \sum_{k=0}^{N-1} D^k \otimes C_k = \mathrm{diag}\left\{C(\zeta^0), C(\zeta^1), \ldots, C(\zeta^{N-1})\right\},$$

which concludes the proof. □

2.1.1 Symmetric block-circulant matrices

Particularly important in this work will be the vector space $\mathcal{C}$ of the $N \times N$ symmetric, block-circulant matrices with blocks of dimension $m \times m$. As long as we consider symmetric matrices it is useful to distinguish the case of N even from the case of N odd, because the

two cases lead to slightly different results. Accordingly, the general element $\mathbf{C} \in \mathcal{C}$ will have the structure

$$\mathbf{C} = \begin{bmatrix} C_0 & C_1^\top & \cdots & C_{\frac{N}{2}-1}^\top & C_{\frac{N}{2}}^\top & C_{\frac{N}{2}-1} & \cdots & C_1 \\ C_1 & C_0 & \ddots & & \ddots & & \ddots & \vdots \\ \vdots & \ddots & & C_1^\top & & \cdots & & C_{\frac{N}{2}-1} \\ C_{\frac{N}{2}-1} & \cdots & C_1 & \ddots & \ddots & & \ddots & C_{\frac{N}{2}}^\top \\ C_{\frac{N}{2}} & \ddots & & \ddots & C_0 & \ddots & & C_{\frac{N}{2}-1}^\top \\ C_{\frac{N}{2}-1}^\top & \ddots & \cdots & \ddots & C_1 & \ddots & C_1^\top & \vdots \\ \vdots & & \ddots & & \vdots & \ddots & \ddots & C_1^\top \\ C_1^\top & \cdots & C_{\frac{N}{2}-1}^\top & C_{\frac{N}{2}} & C_{\frac{N}{2}-1} & \cdots & C_1 & C_0 \end{bmatrix} \in \mathbb{R}^{mN \times mN}$$

if N is even while it will be

$$\mathbf{C} = \begin{bmatrix} C_0 & C_1^\top & \cdots & C_{\frac{N-1}{2}}^\top & C_{\frac{N-1}{2}} & \cdots & C_1 \\ C_1 & C_0 & \cdots & \cdots & \ddots & \ddots & \vdots \\ \vdots & C_1 & \ddots & \cdots & \cdots & \ddots & C_{\frac{N-1}{2}} \\ C_{\frac{N-1}{2}} & \vdots & \ddots & C_0 & C_1^\top & \cdots & C_{\frac{N-1}{2}}^\top \\ C_{\frac{N-1}{2}}^\top & \cdots & \cdots & C_1 & \ddots & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \ddots & \ddots & C_1^\top \\ C_1^\top & \cdots & C_{\frac{N-1}{2}}^\top & C_{\frac{N-1}{2}} & \cdots & C_1 & C_0 \end{bmatrix} \in \mathbb{R}^{mN \times mN},$$

if N is odd. Using the circulant shift matrix (2.1) each $\mathbf{C} \in \mathcal{C}$ can be written in the form

$$\mathbf{C} = \sum_{k=0}^{N-1} U^{-k} \otimes C_k, \qquad \text{where} \qquad C_k = \begin{cases} C_{N-k}^\top & \text{for} \quad k > \dfrac{N}{2}, & \text{if } N \text{ is even,} \\ \\ C_{N-k}^\top & \text{for} \quad k > \dfrac{N-1}{2}, & \text{if } N \text{ is odd.} \end{cases}$$

We endow the vector space $\mathcal{C}$ with the inner product $\langle \mathbf{C}, \mathbf{D} \rangle_{\mathcal{C}} = \operatorname{tr}(\mathbf{C}^\top \mathbf{D})$. We are also interested in the subspace $\mathcal{B} \subset \mathcal{C}$ of symmetric, banded block-circulant matrices of band-

width n, with $N > 2n$, containing all the matrices $\mathbf{B}$ of the form

$$\mathbf{B} = \begin{bmatrix} B_0 & B_1^\top & \cdots & B_n^\top & 0 & \cdots & 0 & B_n & \cdots & B_1 \\ B_1 & B_0 & \cdots & \cdots & B_n^\top & 0 & \cdots & 0 & \ddots & \vdots \\ \vdots & & \ddots & & & \ddots & \ddots & & \ddots & B_n \\ B_n & & & \ddots & & & \ddots & 0 & & 0 \\ 0 & \ddots & & & B_0 & & & B_n^\top & \ddots & \vdots \\ \vdots & \ddots & \ddots & & & \ddots & & & \ddots & 0 \\ 0 & & 0 & B_n & & & \ddots & & & B_n^\top \\ B_n^\top & \ddots & & \ddots & \ddots & & & \ddots & & \vdots \\ \vdots & \ddots & \ddots & & \ddots & \ddots & & & \ddots & B_1^\top \\ B_1^\top & \cdots & B_n^\top & 0 & \cdots & 0 & B_n & \cdots & B_1 & B_0 \end{bmatrix}.$$

Using the cyclic-shift matrix (2.1), we can equivalently rewrite the subspace $\mathcal{B}$ as

$$\mathcal{B} := \left\{ \mathbf{B} \in \mathcal{C} : \mathbf{B} = \sum_{k=-n}^{n} U^{-k} \otimes B_k, \quad B_{-k} = B_k^\top \right\}.$$

With $\bar{\mathcal{B}}$ we denote the subset of $\mathcal{C}$ containing matrices with support that is complementary with respect to the support of the matrices in $\mathcal{B}$ namely, an arbitrary $\bar{\mathbf{B}} \in \bar{\mathcal{B}}$ has the following structure

$$\bar{\mathbf{B}} = \begin{bmatrix} 0 & \cdots & 0 & \bar{B}_{n+1}^\top & \cdots & \bar{B}_{n+1} & 0 & \cdots & 0 \\ \vdots & 0 & \cdots & \ddots & \bar{B}_{n+1}^\top & \cdots & \bar{B}_{n+1} & \ddots & \vdots \\ 0 & & \ddots & & 0 & \ddots & \vdots & \ddots & 0 \\ \bar{B}_{n+1} & \ddots & & \ddots & & \ddots & \bar{B}_{n+1}^\top & & \bar{B}_{n+1} \\ \vdots & \ddots & \ddots & & 0 & & 0 & \ddots & \vdots \\ \bar{B}_{n+1}^\top & & \ddots & 0 & & \ddots & & \ddots & \bar{B}_{n+1}^\top \\ 0 & \ddots & & \bar{B}_{n+1} & \ddots & & \ddots & & 0 \\ \vdots & \ddots & \ddots & \vdots & \ddots & 0 & \cdots & 0 & \vdots \\ 0 & \cdots & 0 & \bar{B}_{n+1}^\top & \cdots & \bar{B}_{n+1} & 0 & \cdots & 0 \end{bmatrix}.$$

Since block-circulant matrices are completely characterized by their first column-blocks, it is useful to introduce here the set $\mathcal{M}$, containing block-matrices of dimension $m \times mN$, defined in the following way: if N is even

$$\mathcal{M} := \left\{ M \in \mathbb{R}^{m \times mN} : M = [M_0 \quad \dots \quad M_{N-1}],\, M_i \in \mathbb{R}^{m \times m},\, M_0 = M_0^\top,\, M_i = M_{N-i}^\top \text{ for } i > \frac{N}{2} \right\},$$

otherwise, if N is odd

$$\mathcal{M} := \left\{ M \in \mathbb{R}^{m \times mN} : M = [M_0 \quad \dots \quad M_{N-1}],\, M_i \in \mathbb{R}^{m \times m},\, M_0 = M_0^\top,\, M_i = M_{N-i}^\top \text{ for } i > \frac{N-1}{2} \right\}.$$

The corresponding scalar product is $\langle M, N\rangle_{\mathcal{M}} = \mathrm{tr}(MN^\top)$, in particular

$$\langle M, N\rangle_{\mathcal{M}} = \mathrm{tr}(M_0 N_0) + \mathrm{tr}(M_{\frac{N}{2}} N_{\frac{N}{2}}^\top) + 2\sum_{j=1}^{\frac{N}{2}-1} \mathrm{tr}(M_j N_j^\top), \qquad \text{if } N \text{ even},$$

$$\langle M, N\rangle_{\mathcal{M}} = \mathrm{tr}(M_0 N_0) + 2\sum_{j=1}^{\frac{N-1}{2}} \mathrm{tr}(M_j N_j^\top), \qquad \text{if } N \text{ odd}.$$

Notice that, for any $M \in \mathcal{M}$ it corresponds a unique (symmetric) block-circulant matrix $\mathbf{M} = \mathrm{circ}\{[M_0 \ \dots \ M_{N-1}]\}$, and viceversa. Accordingly, $\mathcal{M}$ and $\mathcal{C}$ are isomorphic. Finally, we introduce the subspace $\mathcal{M}_{\mathcal{B}}$ of $\mathcal{M}$, which contains all the banded block-matrices that belongs to $\mathcal{M}$, that is

$$\mathcal{M}_{\mathcal{B}} := \left\{ M \in \mathcal{M} : \ M = [M_0 \ M_1 \ \dots \ M_n \ 0 \ \dots \ 0 \ M_n^\top \ \dots \ M_1^\top] \right\}.$$

The subspace $\mathcal{M}_{\mathcal{B}}$ inherits the inner product defined in space $\mathcal{M}$. In particular, for any $M, N \in \mathcal{M}_{\mathcal{B}}$ we have

$$\langle M, N\rangle_{\mathcal{M}} = \mathrm{tr}(M_0 N_0) + 2\sum_{j=1}^{n} \mathrm{tr}(M_j N_j^\top).$$

3 Mathematical preliminaries

In this Chapter we review the main mathematical tools used throughout this thesis. More precisely, we will recall some basic facts on Gaussian random vectors and some results concerning Hilbert spaces of random variables with finite second moments. Finally we will summarize the main notions and results of convex optimization which will be used in the remainder of the thesis.

3.1 Gaussian random vectors

In this Section we review some basic facts on Gaussian random vectors that will be used in the next chapters. We state just the result we will need, the proofs can be easily found in any book that includes this topic.

Definition 2. We say that the random vector $\mathbf{z} \in \mathbb{R}^n$ is a *standard Gaussian random vector* if its components $\{\boldsymbol{z}_i\}_{i=1}^n$ are independent, identically distributed (i.i.d.) random variables with zero mean and unit variance, shortly $\mathcal{N}(0,1)$. The random vector $\mathbf{x} \in \mathbb{R}^n$ is said to be a *Gaussian random vector* if it can be represented as an affine combination of a standard Gaussian random vector,

$$\mathbf{x} = A\,\mathbf{z} + \mu,$$

for some: $k \in \mathbb{N}$, standard Gaussian random vector $\mathbf{z} \in \mathbb{R}^k$, matrix $A \in \mathbb{R}^{n\times k}$ and vector $\mu \in \mathbb{R}^n$. In this case we will write $\mathbf{x} \sim \mathcal{N}(\mu, \Sigma)$ where μ is the mean vector and $\Sigma := AA^\top \geq 0$ is the covariance matrix of $\mathbf{x}$.

From now on we will assume without loss of generality that $\mu = 0$ and $\Sigma > 0$. Then, if $\mathbf{x} \sim \mathcal{N}(0, \Sigma)$ it admits probability density

$$p(x) = \frac{1}{\sqrt{(2\pi)^n \det(\Sigma)}} \exp\left\{-\frac{1}{2}\, x^\top \Sigma^{-1} x\right\}, \qquad x \in \mathbb{R}^n.$$

Suppose now to partition the Gaussian random vector $\mathbf{x} \in \mathbb{R}^n$ into two components

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}, \qquad \mathbf{x}_1 \in \mathbb{R}^p,\, \mathbf{x}_2 \in \mathbb{R}^q,$$

where $p + q = n$. Then, its covariance matrix Σ and its *concentration matrix* $K := \Sigma^{-1}$ can be partitioned accordingly

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}, \qquad K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}.$$

It is well known that $\mathbf{x}_1 \sim \mathcal{N}(0, \Sigma_{11})$ and $\mathbf{x}_2 \sim \mathcal{N}(0, \Sigma_{22})$ being both vectors linear combinations of $\mathbf{x}$. At this point, the following Lemma will be useful.

Lemma 2. Let $X \in \mathbb{R}^{n\times n}$ be a non-singular, block-matrix partitioned as follows

$$X = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where $A \in \mathbb{R}^{p\times p}$ and $D \in \mathbb{R}^{q\times q}$. Then, the determinant of X can be factorized as $\det X = (\det D)(\det S)$ where

$$S := A - BD^{-1}C$$

is the *Schur complement* of D in X. Moreover,

$$X^{-1} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} S^{-1} & -S^{-1}G \\ -FS^{-1} & D^{-1} + FS^{-1}G \end{bmatrix},$$

where $F = D^{-1}C$ and $G = BD^{-1}$.

See e.g. [5] for a proof of this fact. The following Proposition is a consequence of the previous Lemma and it will be used in the following chapters.

Proposition 1. Let $\mathbf{x} \sim \mathcal{N}(0, \Sigma)$ where $\mathbf{x}$ and Σ are partitioned as above. Then, the conditional distribution of $\mathbf{x}_1$ given $\mathbf{x}_2 = x_2$ is $\mathcal{N}(0, \Sigma_{1|2})$ where

$$\Sigma_{1|2} := \Sigma_{11} - \Sigma_{12}\,\Sigma_{22}^{-1}\,\Sigma_{21} \tag{3.1}$$

The proof exploits the formula for the conditional probability density between two random vectors and the previous Lemma applied to the matrix Σ so that we can write

$$K_{11}^{-1} = \Sigma_{11} - \Sigma_{12}\,\Sigma_{22}^{-1}\,\Sigma_{21}. \tag{3.2}$$

See e.g. [3], Appendix C, for the complete proof.

3.2 Hilbert space of second order random variables

This Section is dedicated to recall some facts about Hilbert spaces of zero-mean random variables with finite second-order moments.

Random variables which have finite second moments are commonly called second order random variables. Throughout this thesis we work in the wide-sense setting of second-order, zero-mean random variables. It is well known that the set of real or complex-valued second-order random variables defined on the same probability space is an Hilbert space $\mathcal{H}$ equipped with the inner product

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle_{\mathcal{H}} = \mathbb{E}[\boldsymbol{x}\,\boldsymbol{y}], \qquad \forall\, \boldsymbol{x}, \boldsymbol{y} \in \mathcal{H}.$$

Since we are considering an Hilbert space, the concept of orthogonality makes sense. We say that two random variables $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{H}$ are *orthogonal* if $\mathbb{E}[\boldsymbol{x}\,\boldsymbol{y}] = 0$ in which case we will write $\boldsymbol{x} \perp \boldsymbol{y}$. Orthogonality of two random vectors $\mathbf{x} = [\boldsymbol{x}_1, \dots \boldsymbol{x}_n]^\top$ and $\mathbf{y} = [\boldsymbol{y}_1, \dots \boldsymbol{y}_n]^\top$,

whose components belong to $\mathcal{H}$, will be understood as component-wise uncorrelation, namely

$$\mathbf{x} \perp \mathbf{y} \quad \iff \quad \mathbb{E}[\mathbf{x}\mathbf{y}^\top] = 0.$$

Let $\hat{\mathbb{E}}[\cdot|\cdot]$ be the symbol that denotes the orthogonal projection (conditional expectation in the Gaussian case) onto the subspaces spanned by a family of finite variance random variables listed in the second argument.

Definition 3. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be subspaces of zero-mean second-order random variables in a certain common ambient Hilbert space $\mathcal{H}$. Then $\mathcal{X}$ and $\mathcal{Y}$ are said to be *conditionally orthogonal, given* $\mathcal{Z}$, and it can be denoted by $\mathcal{X} \perp \mathcal{Y} \mid \mathcal{Z}$, if

$$\Big(\boldsymbol{x} - \hat{\mathbb{E}}[\boldsymbol{x} \mid \mathcal{Z}]\Big) \perp \Big(\boldsymbol{y} - \hat{\mathbb{E}}[\boldsymbol{y} \mid \mathcal{Z}]\Big) \qquad \forall\, \boldsymbol{x} \in \mathcal{X}, \forall\, \boldsymbol{y} \in \mathcal{Y}, \tag{3.3}$$

or equivalently if

$$\langle \boldsymbol{x} - \hat{\mathbb{E}}[\boldsymbol{x} \mid \mathcal{Z}],\, \boldsymbol{y} - \hat{\mathbb{E}}[\boldsymbol{y} \mid \mathcal{Z}] \rangle_{\mathcal{H}} = 0 \qquad \forall\, \boldsymbol{x} \in \mathcal{X}, \forall\, \boldsymbol{y} \in \mathcal{Y}.$$

It will be useful the following fact about orthogonal projections in the second-order stationary processes framework. The proof can be found in [2], see also [11].

Proposition 2. The following statements are equivalent.

i) $\mathcal{X} \perp \mathcal{Y} \mid \mathcal{Z}$

ii) $(\mathcal{X} \vee \mathcal{Z}) \perp \mathcal{Y} \mid \mathcal{Z}$

iii) $\hat{\mathbb{E}}[\boldsymbol{y} \mid \mathcal{X} \vee \mathcal{Z}] = \hat{\mathbb{E}}[\boldsymbol{y} \mid \mathcal{Z}]$, for all $\boldsymbol{y} \in \mathcal{Y}$

where $\mathcal{A} \vee \mathcal{B}$ denotes the closure of $\mathcal{A} + \mathcal{B} = \{\mathbf{a} + \mathbf{b} \mid \mathbf{a} \in \mathcal{A},\, \mathbf{b} \in \mathcal{B}\}$.

Proof. The equivalence between i) and ii) follows from the definition. Indeed, starting from ii)

$$\langle \boldsymbol{x}+\boldsymbol{z} - \hat{\mathbb{E}}[\boldsymbol{x}+\boldsymbol{z} \mid \mathcal{Z}],\, \boldsymbol{y} - \hat{\mathbb{E}}[\boldsymbol{y} \mid \mathcal{Z}] \rangle_{\mathcal{H}} = \langle \boldsymbol{x}+\boldsymbol{z},\, \boldsymbol{y} - \hat{\mathbb{E}}[\boldsymbol{y} \mid \mathcal{Z}] \rangle_{\mathcal{H}} = 0, \qquad \forall\, \boldsymbol{x} \in \mathcal{X},\, \boldsymbol{y} \in \mathcal{Y},\, \boldsymbol{z} \in \mathcal{Z},$$

since $\hat{\mathbb{E}}[\boldsymbol{x} + \boldsymbol{z} \mid \mathcal{Z}] \in \mathcal{Z}$ and $\boldsymbol{y} - \hat{\mathbb{E}}[\boldsymbol{y} \mid \mathcal{Z}] \perp \mathcal{Z}$, meaning that ii) is equivalent to $\boldsymbol{y} - \hat{\mathbb{E}}[\boldsymbol{y} \mid \mathcal{Z}] \perp \mathcal{X} \vee \mathcal{Z}$. Then,

$$\begin{aligned} \langle \boldsymbol{x} + \boldsymbol{z},\, \boldsymbol{y} - \hat{\mathbb{E}}[\boldsymbol{y} \mid \mathcal{Z}] \rangle_{\mathcal{H}} &= \langle \boldsymbol{x} - \hat{\mathbb{E}}[\boldsymbol{x} \mid \mathcal{Z}] + \hat{\mathbb{E}}[\boldsymbol{x} \mid \mathcal{Z}] + \boldsymbol{z},\, \boldsymbol{y} - \hat{\mathbb{E}}[\boldsymbol{y} \mid \mathcal{Z}] \rangle_{\mathcal{H}} \\ &= \langle \boldsymbol{x} - \hat{\mathbb{E}}[\boldsymbol{x} \mid \mathcal{Z}],\, \boldsymbol{y} - \hat{\mathbb{E}}[\boldsymbol{y} \mid \mathcal{Z}] \rangle_{\mathcal{H}} \\ &= 0, \end{aligned}$$

because $\boldsymbol{z} + \hat{\mathbb{E}}[\boldsymbol{x} \mid \mathcal{Z}] \in \mathcal{Z}$. We have just proved that ii) is equivalent to $\boldsymbol{y} - \hat{\mathbb{E}}[\boldsymbol{y} \mid \mathcal{Z}] \perp \mathcal{X} \vee \mathcal{Z}$, that is

$$\hat{\mathbb{E}}[\boldsymbol{y} - \hat{\mathbb{E}}[\boldsymbol{y} \mid \mathcal{Z}] \mid \mathcal{X} \vee \mathcal{Z}] = 0 \quad \iff \quad \hat{\mathbb{E}}[\boldsymbol{y} \mid \mathcal{X} \vee \mathcal{Z}] = \hat{\mathbb{E}}[\hat{\mathbb{E}}[\boldsymbol{y} \mid \mathcal{Z}] \mid \mathcal{X} \vee \mathcal{Z}] = \hat{\mathbb{E}}[\boldsymbol{y} \mid \mathcal{Z}], \qquad \forall\, \boldsymbol{y} \in \mathcal{Y},$$

which proves the equivalence between ii) and iii). □

Conditional orthogonality is the same as conditional uncorrelatedness (and, hence, conditional independence) in the Gaussian case. When $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ are generated by finite-dimensional random vectors, condition (5.4) can equivalently be rewritten in terms of generating vectors, which we shall normally do in the following.

3.3 Elements of convex analysis

In this Section we will introduce the basic notions used in convex optimization. Then, we will face the rank minimization problem and the problem of inducing sparsity in the solution of an optimization problem, in the simplest possible settings. Finally, we will give a general picture with the main facts on Lagrange duality theory in optimization. The material presented in this section is mainly taken from [4] and [18].

3.4 Optimization problems

Consider the problem of finding $x \in \mathbb{R}^n$ that minimizes the objective function $f_0 : \mathbb{R}^n \to \mathbb{R} : x \mapsto f_0(x)$, among all x satisfying some constraints. In other words, consider the following *optimization problem*

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & f_0(x) \\ \text{subject to} \quad & f_i(x) \leq 0,\ i = 1, \dots, m \qquad (3.4a) \\ & h_i(x) = 0,\ i = 1, \dots, p, \qquad (3.4b) \end{aligned}$$

where we have m equality constraints (3.4a) and p inequality constraints (3.4b). We consider by default minimization problems because each maximization problem can be view as the minimization of the opposite of its objective function subject to the same constraints. The set of $x \in \mathbb{R}^n$ on which all the functions are defined,

$$\mathcal{D} = \left(\bigcap_{i=0}^{m} \operatorname{dom} f_i \right) \cap \left(\bigcap_{i=0}^{p} \operatorname{dom} h_i \right),$$

is called *domain* of the optimization problem (3.4).

Definition 4. A point $x \in \mathcal{D}$ is *feasible* for the optimization problem (3.4) if it satisfies the constraints (3.4a) and (3.4b). Problem (3.4) is said to be *feasible* if there exists at lest one feasible point in $\mathcal{D}$, *infeasible* otherwise.

The optimal point of (3.4) is defined as

$$p_o := \inf \{ f_0(x) :\ f_i(x) \leq 0,\ i = 1, \dots, m,\ h_i(x) = 0,\ i = 1, \dots, p \}$$

Notice that p_o can be $\pm\infty$. In particular, if there are feasible points $\{x_k\}_k$ so that $f_0(x_k)$ tends to $-\infty$ as $x_k \to \infty$ then $p_o = -\infty$ and the problem (3.4) is said to be *unbounded below*. We say that x_o is an *optimal point* for Problem (3.4) if $x_o \in \mathcal{D}$ and $f_0(x_o) = p_o$.

3.4.1 Convex optimization problems

A convex optimization problem is an optimization problem of the form

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & f_0(x) \\ \text{subject to} \quad & f_i(x) \leq 0,\ i = 1, \dots, m \qquad (3.5a) \\ & a_i^\top x = b_i,\ i = 1, \dots, p. \qquad (3.5b) \end{aligned}$$

With respect to the general optimization problem (3.4), the convex optimization Problem (3.5) has three additional requirements:

- the objective function f_0 must be convex,
- the inequality constraints functions $f_1, \dots, f_m$ must be convex,
- the equality constraints functions $h_i(x) = a_i^\top x - b_i$ must be affine.

The feasible set of the problem is the intersection of the domain of the problem

$$\mathcal{D} = \bigcap_{i=0}^{m} \operatorname{dom} f_i,$$

which is a convex set (because f_i, $i = 1, \dots, m$ are convex), with m convex sets $\{x : f_i(x) \leq 0\}$ and p hyperplanes $\{x : a_i^\top x = b_i\}$, thus it is convex.

3.5 Rank minimization problem

In this Section we will try to give an intuitive explanation behind the choice of the regularizers we will make in Section 6.2.2. More precisely, we want to show in the simplest possible settings, why the minimization of the trace and the minimization of the ℓ^1-norm induce, respectively, low-rank and sparsity on the solution. What showed in this Section can be also viewed as an example of application of the concepts we have introduced in the previous Section. We refer the interested reader to [18], from which is taken the material presented here.

Definition 5. Let $\mathcal{K}$ be a convex set. The *convex envelope* of $f : \mathcal{K} \to \mathbb{R}$ is the largest convex function $g(\cdot)$ such that $g(x) \leq f(x)$ for all $x \in \mathcal{K}$.

The definition only says that, among all convex functions, the convex envelope g of a function f is the one that is the closest to f pointwise. A pictorial representation of this fact is given in Figure 3.1.

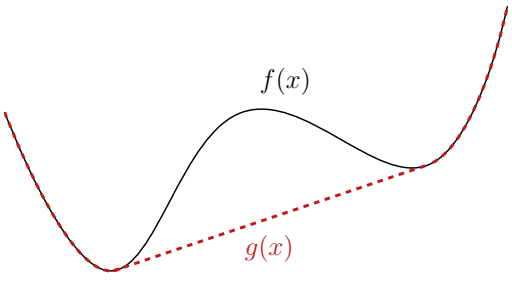

Figure 3.1: Explanation of the definition of convex envelope g of a function f.

Consider the following optimization problem

$$\begin{aligned} \underset{X \in \mathbb{R}^{m \times n}}{\text{argmin}} \quad & \operatorname{rank}(X) \\ \text{subject to} \quad & X \in \mathcal{K}, \end{aligned} \tag{3.6}$$

where $\mathcal{K}$ is the convex set representing the constraints. This is not a convex optimization problem because the objective function $\text{rank}(X)$ is not convex in X. In this situation, we can minimize the convex envelope of the $\text{rank}(\cdot)$ function which is actually a convex function and thus it may be minimized efficiently. The minimum of the convex envelope provides a lower-bound on the true minimum.

Theorem 2. Let $\mathcal{X}_M := \{X \in \mathbb{R}^{m\times n} : \|X\| \leq M\}$ be the set of matrices with bounded norm. Then the convex envelope of the function $f(X) = \text{rank}(X)$ is

$$g(X) = \frac{1}{M}\|X\|_* = \frac{1}{M}\sum_{i=1}^{\min\{m,n\}} \sigma_i(X), \tag{3.7}$$

namely, the *nuclear norm* $\|X\|_*$ of X, up to a constant. In particular, if X is symmetric and positive semi-definite, the convex envelope of the rank function is $g(X) = \text{tr}(X)$, up to a constant.

Note that the nuclear norm is the tightest convex lower-bound approximation for the rank function over the set $\mathcal{X}_M$ thus, the problem

$$\begin{aligned} \underset{X\in\mathbb{R}^{m\times n}}{\text{argmin}} \quad & \|X\|_* \\ \text{subject to} \quad & X \in \mathcal{K}, \end{aligned} \tag{3.8}$$

yields the tightest global lower-bound on $\text{rank}(X)$ over $\mathcal{K}$. In the following we give an intuitive explanation of what we have said so far. Introducing the indicator function of the positive real line

$$I_+(x) := \begin{cases} 1 & \text{if} \quad x > 0, \\ 0 & \text{if} \quad x \leq 0, \end{cases}$$

we can rewrite the rank of a matrix $X \in \mathbb{R}^{m\times n}$ as the sum of its singular values greater than zero

$$\text{rank}(X) = \sum_{i=1}^{\min\{m,n\}} I_+(\sigma_i(X)) \tag{3.9}$$

The simplest way to understand why this approximation works is to consider the scalar case $n = m = 1$. In this case the only singular value of X is $\sigma(X) = |X|$ and $\text{rank}(X) = I_+(|X|)$, which are depicted in Figure 3.2.

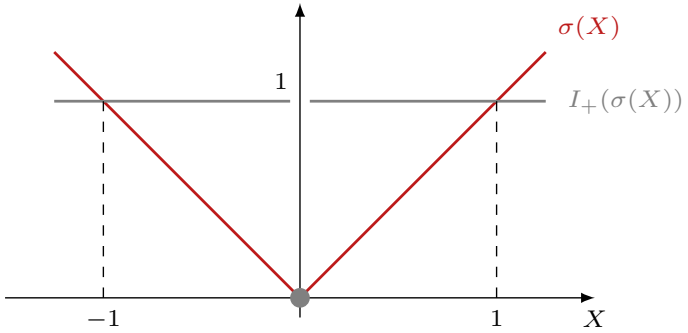

Figure 3.2: The functions $\sigma(X)$ and $\text{rank}(X)$ in the scalar example.

The Figure suggests that the convex envelope of the rank over the set of matrices with norm bounded by M may be obtained by replacing each $I_+(\sigma_i(X))$ term in (3.9) by $1/M\,\sigma_i(X)$. This procedure leads precisely to the general result (3.7).

3.5.1 Cardinality minimization problem

The cardinality minimization problem is the problem of minimizing the number of non-zero components of a vector $x \in \mathbb{R}^n$ and it can be seen as a special instance of rank minimization problem when the matrix X is diagonal $X = \text{diag}\{x_1, \dots, x_n\}$ where x_i, $i = 1, \dots, n$ are the components of the vector x. In other words we want to find the sparsest vector in a convex set $\mathcal{K}$:

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{argmin}} \quad & \text{card}(x) \\ \text{subject to} \quad & x \in \mathcal{K}. \end{aligned} \tag{3.10}$$

In this case, it can be proven that the ℓ^1-norm of x, is the convex envelope of $\text{card}(x)$ over the set of the vectors with bounded infinity norm $\{x \in \mathbb{R}^n : \|x\|_\infty \leq 1\}$. Exploiting the same reasoning used for the rank minimization problem in the previous Section, we can conclude that the ℓ^1-norm minimization encourages sparsity on the solution, in fact the ℓ^1-norm represents the best convex approximation of the cardinality of a vector over a suitable set. The intuitive idea behind this result is that the ℓ^1-norm is a sharp-cornered norm. Consider the following least-squares problem

$$\begin{aligned} \underset{\theta \in \Theta}{\text{argmin}} \quad & \|y - \Phi\,\theta\|^2 \\ \text{subject to} \quad & \sum_{i=1}^{p} |\theta_i| \leq t, \end{aligned} \tag{3.11}$$

and suppose that $p = 2$. With reference to Figure 3.3, the ℓ^1-norm constraint $|\theta_1| + |\theta_2| \leq t$ describes the blue diamond, the red ellipses are the level-curves of the objective function and $\hat{\theta}$ is the solution of the unconstrained least-squares problem.

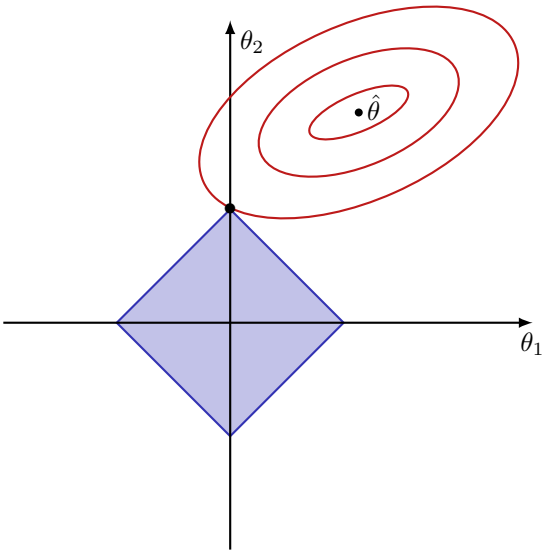

Figure 3.3: Objective function contours and constraints set for the least-squares problem considered above with $p = 2$.

The solution of problem (3.11) is the first point where the elliptical level-curves hit the constraints region. If the meeting point, i.e. the solution, occurs at a corner then it has one parameter $\theta_i = 0$. When $p > 2$ the diamond becomes a rhomboid and it has more corners, that is, more sparse the estimated parameter vector will be.

3.6 Lagrange duality

Consider the general optimization problem (3.4) with domain $\mathcal{D}$, we do not need to assume convexity. We define the *Lagrangian* $\mathcal{L} : \mathcal{D} \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$ associated with Problem (3.4) as

$$\mathcal{L}(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i \, f_i(x) + \sum_{i=1}^{p} \nu_i \, h_i(x)$$

where $\lambda_i \geq 0$ is the Lagrange multiplier associated to the i-th inequality constraint $f_i(x) \leq 0$, ν_i is the Lagrange multiplier associated to the i-th equality constraint $h_i(x) = 0$ and

$$\lambda = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{bmatrix} \in \mathbb{R}^m, \qquad \nu = \begin{bmatrix} \nu_1 \\ \vdots \\ \nu_p \end{bmatrix} \in \mathbb{R}^p,$$

are the *Lagrange multipliers* vectors. The *Lagrange dual function* $g : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$ associated to Problem (3.4) is defined as

$$g(\lambda, \nu) = \inf_{x \in \mathcal{D}} \mathcal{L}(x, \lambda, \nu).$$

Remark 1. Being a point-wise infimum of a family of affine functions of (λ, ν) *it is always concave* even when the Problem (3.4) is not convex.

Clearly, if the Lagrangian is unbounded below in x, $g(\lambda, \nu) = -\infty$. The following Proposition states a simple but fundamental result.

Proposition 3. The dual function g yields a lower bound on the optimal value p_o of Problem (3.4). In fact, for any $\lambda \in \mathbb{R}^m$, $\lambda \geq 0$, and any $\nu \in \mathbb{R}^p$ we have

$$g(\lambda, \nu) \leq p_o, \tag{3.12}$$

where the inequalities have to be understood component-wise.

The proof is very simple and can be found in [4], pp. 216. Of course the dual function gives a non-trivial lower-bound to p_o only when the pair (λ, ν) is *dual feasible*, i.e. $\lambda \geq 0$ and $(\lambda, \nu) \in \operatorname{dom} g$ so that $g(\lambda, \nu) > -\infty$.

3.6.1 The Lagrange dual problem

As we have just seen in (3.12), the Lagrange dual function gives a lower-bound on the optimal value p_o of the optimization problem (3.4). This lower-bound clearly depends on (λ, ν), $\lambda \geq 0$ so the natural question is: what is the best lower-bound that can be obtained from the Lagrange dual function varying $(\lambda, \nu) \in \mathbb{R}^m \times \mathbb{R}^p$, $\lambda \geq 0$? The question leads to the following optimization problem called *Lagrange dual problem* associated to Problem (3.4)

$$\begin{aligned} \max_{(\lambda,\nu) \in \mathbb{R}^m \times \mathbb{R}^p} \quad & g(\lambda, \nu) \\ \text{subject to} \quad & \lambda \geq 0 \end{aligned} \tag{3.13}$$

Accordingly, Problem (3.4) is often called *primal problem*. If λ_o, ν_o are optimal for the dual problem, they are often called *optimal Lagrange multipliers*.

Remark 2. The dual problem (3.13) is a convex optimization problem whether or not the primal problem (3.4) is convex. Indeed, (3.13) is a maximization of a concave function (see Remark 1) under a convex constraint.

Let $d_o := g(\lambda_o, \nu_o)$ be the optimal value of the Lagrange dual problem, i.e. the best lower-bound on p_o that can be obtained from the Lagrange dual function. Since (3.12) holds for any $\lambda \in \mathbb{R}^m$, $\lambda \geq 0$, and for any $\nu \in \mathbb{R}^p$, it must be true also for (λ_o, ν_o), namely

$$d_o \leq p_o, \tag{3.14}$$

and this property is called *weak-duality*. In particular, if $p_o = -\infty$ (primal unbounded below) it must be that $d_o = -\infty$ (dual infeasible) and conversely, if $d_o = +\infty$ (dual unbounded above) it must be $p_o = +\infty$ (primal infeasible). The difference $p_o - d_o \geq 0$ is called *duality gap* of the original problem.The inequality (3.14) can be used to find a lower bound for the optimal value of the primal problem when this is difficult to solve it. Indeed, the dual problem is always convex and in many cases it can be solved efficiently.

3.6.2 Strong duality

We say that there is *strong duality* between the primal problem (3.4) and its dual (3.13) when the duality gap is zero, i.e. $d_o = p_o$. Suppose that strong duality holds and let

$$p_o = f_0(x_o), \qquad d_o = g(\lambda_o, \nu_o).$$

Then,

$$\begin{aligned} f_0(x_o) = g(\lambda_o, \nu_o) &= \inf_{x \in \mathcal{D}} \left\{ f_0(x) + \sum_{i=1}^{m} \lambda_{o_i} f_i(x) + \sum_{i=1}^{p} \nu_{o_i} h_i(x) \right\} \\ &\leq f_0(x_o) + \sum_{i=1}^{m} \lambda_{o_i} f_i(x_o) + \sum_{i=1}^{p} \nu_{o_i} h_i(x_o) \\ &\leq f_0(x_o), \end{aligned}$$

where the last inequality follows from the fact that $\lambda_o \geq 0$ and $h_i(x_o) = 0$, $i = 1, \ldots, p$. Therefore, the inequalities are in fact equalities, and in particular we have that

$$\inf_{x \in \mathcal{D}} \left\{ f_0(x) + \sum_{i=1}^{m} \lambda_{o_i} f_i(x) + \sum_{i=1}^{p} \nu_{o_i} h_i(x) \right\} = f_0(x_o) + \sum_{i=1}^{m} \lambda_{o_i} f_i(x_o) + \sum_{i=1}^{p} \nu_{o_i} h_i(x_o).$$

By definition of the Lagrangian function this means that

$$\inf_{x \in \mathcal{D}} \mathcal{L}(x, \lambda_o, \nu_o) = \mathcal{L}(x_o, \lambda_o, \nu_o),$$

namely, under strong duality, if x_o is optimum for the primal problem (3.4), then x_o minimizes $\mathcal{L}(x, \lambda_o, \nu_o)$ over $x \in \mathcal{D}$. Of course, the Lagrangian can have other minimizers, in general: x_o is simply *a* minimizer.

Remark 3 (Solving the primal via the dual)**.** We have just seen that if strong duality holds and a dual solution (λ_o, ν_o) exists, then any optimal point x_o is also a minimizer of $\mathcal{L}(x, \lambda_o, \nu_o)$. This fact sometimes allows to compute a primal optimal solution from a dual optimal solution. More precisely, suppose that we have strong duality and a dual

optimal (λ_o, ν_o) is known. If the minimizer of $\mathcal{L}(x, \lambda_o, \nu_o)$ is *unique and primal feasible*, then it must be the primal optimal. If it is not primal feasible, then no primal optimal point exists, i.e. we can conclude that the primal optimum is not attained. Of course, this observation is interesting when the dual problem is easier to solve than the primal problem.

Slater's conditions. In the case of convex optimization problems, sufficient conditions for strong duality can be established. Consider the convex optimization problem

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & f_0(x) \\ \text{subject to} \quad & f_i(x) \leq 0,\ i = 1, \ldots, m \\ & Ax = b, \end{aligned}$$

where we recall that f_i, $i = 1, \ldots, m$, are convex functions and the domain $\mathcal{D}$ of the problem is convex. Assume that $\mathcal{D}$ has a non-empty interior, i.e. $\text{int}\mathcal{D} \neq \emptyset$. If there exists $x \in \text{int}\mathcal{D}$ such that

$$\begin{aligned} & f_i(x) < 0, \quad i = 1, \ldots, m \\ & Ax = b \end{aligned} \tag{3.15}$$

and the problem is convex, then strong duality holds between the problem and its dual. Conditions (3.15) are called *Slater's conditions*. If some of the inequality functions are affine, let's say the first k functions, $f_1, \ldots, f_k$, then strong duality holds if the following (weaker) *refined Slater's conditions* hold

$$\begin{aligned} & f_i(x) \leq 0, \quad i = 1, \ldots, k \\ & f_i(x) < 0, \quad i = k+1, \ldots, m \\ & Ax = b. \end{aligned} \tag{3.16}$$

4 Foundations

The purpose of this Chapter is to present the main ingredients behind the problem of the identification of a sparse plus low-rank graphical model for a discrete-time, periodic, Gaussian reciprocal process addressed in this thesis. First of all, we will briefly review the historical development of maximum entropy problems by emphasizing some of those that are particularly relevant. Given their importance in our problem, we will review in more detail *Dempster's covariance selection method* and the classical *covariance extension problem* widely studied in literature. We will see that these two problems are strictly connected each other. Next, we summarize the bases of the theory of graphical models for Gaussian stochastic processes which are of fundamental importance in the remainder of this thesis.

4.1 Maximum entropy problems

There is an endless literature concerning maximum entropy problems, see e.g. [14], [16], [15], [17] and references therein. In the following we recall three important classic maximum entropy problems by reviewing in more detail Dempster's covariance selection problem and the covariance extension problem. As it will be clear in a moment, all the maximum entropy problems consist in the maximization of an entropic functional (not necessarely the entropy) under linear constraints. The material presented in this Section is for the most taken from [17].

First of all we mention the *Boltzmann's loaded dice problem* (1877) because this is, for the best of our knowledge, the first maximum entropy problem in the history but still, it capture the essence of all the maximum entropy problems. In his paper Boltzmann refers to molecules, microstates, macrostates and so on but, roughly speaking, the essence of this problem is gathered in the following question: among a given set of probability distributions, which is the one that can be realized in more ways? Or, in other words, which is the most probable one? It turns out that solving this problem is equivalent (under mild assumptions) to find the probability distribution, called Boltzmann distribution, that maximizes the entropy. We refer the interested reader to [17] and its references for more details.
Other important maximum entropy problems have been considered in the history (e.g. Schrödinger's bridges). Among those problem it is particularly useful to review the Dempster's covariance selection method and the covariance extension problem because in the following (see Chapter 6) we will use a generalization of those problems for the case of block-circulant covariance matrices.

4.1.1 Dempster's covariance selection method

Back in 1972, in the seminal paper [16], A. P. Dempster introduced a general method for completing a partially specified covariance matrix. Formally, the problem can be formulated as follows. Consider a zero-mean, multivariate Gaussian distribution with density

$$p(x) = (2\pi)^{-\frac{n}{2}} \det(\Sigma)^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}\, x^\top \Sigma^{-1} x\right\}, \qquad x \in \mathbb{R}^n. \tag{4.1}$$

Suppose that the elements $\{\sigma_{ij} : 1 \leq i \leq j \leq n, (i,j) \in \bar{\mathcal{I}}\}$ have been specified, where $\sigma_{ij} := (\Sigma)_{ij}$ and $\bar{\mathcal{I}}$ is a set of pairs (i,j) that always contains the main diagonal, i.e. $\{(i,j) : i = j, i = 1, \ldots, n\} \subseteq \bar{\mathcal{I}}$. At this point, the covariance selection problem boils down in the following question: how should Σ be completed in a simple way? At first look, in order to have a model with low-complexity, one can think to set the unspecified elements of Σ equal to zero. Even if this choice would lead to a positive definite completion (which is not true in general), it is not the best choice. Indeed, it is apparent from $p(x)$ that the natural parameters of the model are not the entries of Σ but the elements σ^{ij} of the concentration matrix Σ^{-1}. For this reason, Dempster resorts to a form of the Principle of Parsimony in parametric model fitting: set to zero all the elements σ^{ij} of Σ^{-1} for $1 \leq i \leq j \leq n$, $(i,j) \notin \bar{\mathcal{I}}$.

Definition 6. A positive definite completion $\hat{\Sigma}$ of Σ is called *Dempster's Completion* if

$$\hat{\sigma}^{ij} = (\hat{\Sigma}^{-1})_{ij} = 0, \qquad \forall\, (i,j) \in \bar{\mathcal{I}}.$$

Of course, this choice seems less natural than setting all the unspecified elements of Σ equal to zero however, it has considerable advantages compared to the latter, see [16]. The existence of a symmetric, positive definite completion of Σ is not always guaranteed but, if it does exist, the following fundamental result can be established [16].

Proposition 4. Assume that a symmetric, positive definite completion of Σ exists. Then there exists a unique Dempster's completion Σ_o. This completion maximizes the (differential) entropy

$$H(p) = \int_{\mathbb{R}^n} p(x) \log p(x)\, dx = \frac{1}{2} \log(\det \Sigma) + \frac{1}{2} n(1 + \log(2\pi)) \tag{4.2}$$

among all zero-mean, Gaussian distributions whose covariances have the prescribed elements $\{\sigma_{ij} : 1 \leq i \leq j \leq n, (i,j) \in \bar{\mathcal{I}}\}$.

This Theorem basically says that the Dempster's completion solves a maximum entropy problem, namely, the maximization of an entropic functional (in this case precisely the entropy) under linear constraints. In order to see this, let e_k, $k = 1, \ldots, n$, be the k-th vector of the canonical basis of $\mathbb{R}^n$. Then, the covariance completion problem formalizes in the following optimization problem

$$\begin{aligned} &\underset{\Sigma \in \mathcal{S}_n^+}{\text{argmax}} && \log(\det \Sigma) \\ &\text{subject to} && e_i^\top \Sigma\, e_j = \sigma_{ij}, \qquad \forall\, (i,j) \in \bar{\mathcal{I}}, \end{aligned} \tag{4.3}$$

where $\mathcal{S}_n^+$ is the set of positive definite $n \times n$ matrices. We will show how to solve this problem by resorting to Lagrange duality theory. The Lagrangian function for this problem is

$$\mathcal{L}(\Sigma, \Lambda) = \mathcal{J}(\Sigma) - \sum_{(i,j)\in\bar{\mathcal{J}}} \lambda_{ij}\, \sigma_{ij},$$

where Λ is the vector containing the Lagrange multipliers λ_{ij} for all $(i,j) \in \bar{\mathcal{J}}$, and

$$\begin{aligned}\mathcal{J}(\Sigma) &:= \log(\det \Sigma) + \sum_{(i,j)\in\bar{\mathcal{J}}} \lambda_{ij}\, e_i^\top \Sigma\, e_j \\ &= \log(\det \Sigma) + \sum_{(i,j)\in\bar{\mathcal{J}}} \lambda_{ij}\ \mathrm{tr}(e_j e_i^\top \Sigma).\end{aligned}$$

Notice that, minimizing $\mathcal{L}$ over $\mathcal{S}_n^+$, is equivalent to minimize $\mathcal{J}(\Sigma)$ over $\mathcal{S}_n^+$ which is a strictly concave function of Σ. Let $\mathcal{J}'(\Sigma_o, \delta\Sigma)$ be the first variation of $\mathcal{J}$ in direction $\delta\Sigma \in \mathcal{S}_n$ computed in Σ_o. Accordingly, necessary and sufficient condition for Σ_o to be the minimum of $\mathcal{L}$ over $\mathcal{S}_n^+$ is that

$$\mathcal{J}'(\Sigma_o, \delta\Sigma) = \mathrm{tr}(\Sigma_o^{-1}\delta\Sigma) + \sum_{(i,j)\in\bar{\mathcal{J}}} \lambda_{ij}\ \mathrm{tr}(e_j e_i^\top \delta\Sigma) = 0, \qquad \forall\, \delta\Sigma \in \mathcal{S}_n.$$

This condition is satisfied if and only if

$$\Sigma_o^{-1} = \sum_{(i,j)\in\bar{\mathcal{J}}} \lambda_{ij}\, e_j e_i^\top,$$

namely, if and only if Σ_o is a Dempster's completion of Σ. Finally, it is worth noting that, besides symmetry and positive definiteness, we are not requiring further properties for the matrix completion we are looking for.

4.1.2 The covariance extension problem

Depending on the nature of the process we are studying, a number of interesting alternative formulations of the Dempster's covariance selection problem have been considered. In this class of problems we want to find a covariance matrix $\mathbf{\Sigma}$ that, besides being symmetric and positive definite, enjoys further properties such as having a (block) Toeplitz structure, if the covariance matrix originates from a stationary time series, or having a (block) Toeplitz and circulant structure if we are considering a stationary process (vector) on the discrete circle. Concerning the first case, here we consider the so-called *covariance extension problem*, or *Burg's maximum entropy covariance extension problem*, introduced by J. Burg in 1967 while it was working on spectral estimation from geophysical data [14]. More precisely, in the following we consider the *multivariate* covariance extension problem which readily follows from the scalar version introduced by Burg.

Let $\mathbf{y} := \{\mathbf{y}(t),\, t \in \mathbb{Z}\}$ be a $\mathbb{R}^m$-valued, zero-mean, Gaussian, stationary purely non-deterministic process of full-rank. Under these assumption the process admits power spectral density

$$\mathbf{\Phi}_{\mathbf{y}} := \sum_{k=-\infty}^{+\infty} \Sigma_k\, e^{j\theta k},$$

where $\theta \in (0, 2\pi)$ and

$$\Sigma_k = \mathbb{E}[\mathbf{y}(t+k)\,\mathbf{y}(t)^\top], \qquad \Sigma_{-k} = \Sigma_k^\top,$$

are the covariance lags of the process $\mathbf{y}$, for any $k \in \mathbb{Z}$. Note that $\mathbf{\Phi_y}$ is a positive definite function, Fourier transform of the sequence $\{\Sigma_k\}$, $k \in \mathbb{Z}$. At this point we can formulate the following matrix completion problem.

Problem 1 (Covariance extension problem)**.** Given $n+1$ initial data matrices $C_0, C_1, \ldots, C_n$, complete it with a sequence $\Sigma_{n+1}, \Sigma_{n+2}, \ldots$ in such a way that the Fourier transform of the extended (infinite) sequence is a power spectral density.

Remark 4. The initial matrices $C_0, C_1, \ldots, C_n$ are not known in practice. On the other hand, given a finite-length realization $y(1), \ldots, y(N)$, $y(t) \in \mathbb{R}^m$, of the process $\mathbf{y}$, an estimate of the k-th lag C_k with $k = 0, \ldots, n$ is given by

$$C_k = \frac{1}{N} \sum_{t=k}^{N} y(t)\, y(t-k)^\top, \qquad k = 0,\, 1,\, \ldots,\, n. \tag{4.4}$$

It is well known that n should be such that $n \ll N$ in order to have sufficiently reliable estimates.

In other words we want to find the stationary process having spectral density that matches the first n covariance lags. Burg suggested the following approach based on the maximization of entropy: choose $\Sigma_{n+1}, \Sigma_{n+2}, \ldots$ maximizing the *entropy rate* of the process. The solution corresponds to an autoregressive (AR) process of the form

$$\hat{\mathbf{y}}(t) = \sum_{k=0}^{n-1} A_k^o\, \hat{\mathbf{y}}(t-k) + \mathbf{w}(t),$$

where $\mathbf{w}$ is a zero-mean, Gaussian noise with covariance matrix $\mathbf{\Sigma_w}$. The parameters $A_0^o, \ldots, A_{n-1}^o$ and $\mathbf{\Sigma_w}$ are such that the first n covariance lags are matched. Accordingly, the solution of the problem consists in a constrained maximization of an entropic functional, the entropy rate of the process, under linear constraints, i.e. the first n covariance lags have to match the estimated ones. We refer to [14] for the complete discussion of this problem. The solution proposed by Burg perfectly reconciles with Dempster's covariance selection method once observed that, searching the infinite sequence that gives a certain power spectrum is equivalent to search a completion that leads to an infinite block-Toeplitz covariance matrix whose first block-row is completely specified by this infinite sequence. In this setting, Proposition 4 reads as follows.

Proposition 5. Assume feasibility of the covariance extension problem. Among all covariance extensions of the data $C_0, \ldots, C_n$, there exists a unique extension whose inverse-matrix block-entries are zero in all the positions complementary to those where the elements of the covariance are assigned. This extension corresponds to the Gaussian distribution with maximum entropy.

In Chapter 6 we will consider in detail a re-parametrization of the same problem for the stationary reciprocal processes on the discrete circle, introduced in Chapter 5, in that case $\mathbf{\Sigma}$ has a block-circulant Toeplitz structure.

4.2 Elements of theory of graphical models

Graphical (interaction) models have their origins in several scientific areas such as statistical physics and genetics. Their popularity is increased a lot in the recent years because of their capability of giving a pictorial explanation of the underlying relations between the interesting quantities in a certain application. During the recent years customized graphical models have been developed for different applications, in particular, in this thesis we will deal with *latent-variable graphical models* for Gaussian stochastic processes which have been studied, for instance, in [7], [9] and in [10] for fMRI (functional Magnetic Resonance Imaging) applications. In the following we consider only random variables or random vectors that admit probability density. This Section basically follows the book [3] to which we refer the interested reader.

4.2.1 Basic notions of graph theory

Here we give some basic notions on graph theory in order to fix the notation and terminology which will be used in the next subsection.

Consider a graph $\mathcal{G} = (V, E)$ where V, with cardinality $|V| = n$, is the vertex set and $E \subseteq V \times V$ is the set of edges. First of all, given $A \subseteq V$, it is useful to introduce the following subsets of V

- the *boundary of* A, denoted as $\mathrm{bd}(A)$, is the set of vertices in $V \setminus A$ that are neighbours to vertices in A;

- the *closure of* A is defined as $\mathrm{cl}(A) := A \cup \mathrm{bd}(A)$.

In order to simplify the notation, if $A = \{\alpha\}$, we will write $\mathrm{cl}(\alpha)$ in place of $\mathrm{cl}(\{\alpha\})$ and $\mathrm{bd}(\alpha)$ in place of $\mathrm{bd}(\{\alpha\})$.

Definition 7. A *path* of length n from a vertex α to a vertex β is a sequence $\alpha = \alpha_0, \alpha_1, \ldots, \alpha_n = \beta$ of distinct vertices such that $(\alpha_{i-1}, \alpha_i) \in E$ for all $i = 1, \ldots, n$. If there is a path from α to β we say that α *leads* to β and we write $\alpha \mapsto \beta$.

If both $\alpha \mapsto \beta$ and $\beta \mapsto \alpha$ thus we say that α and β are *connected* and we write $\alpha \rightleftharpoons \beta$. Clearly $\rightleftharpoons$ is an equivalence relation and the corresponding equivalence classes $[\alpha]$, where

$$\beta \in [\alpha] \quad \Longleftrightarrow \quad \alpha \rightleftharpoons \beta,$$

are the *connected components* of $\mathcal{G}$. If $\alpha \in A \subseteq V$, the symbol $[\alpha]_A$ denotes the connected components of α in $\mathcal{G}_A := (A, E_A)$ where the edges set $E_A := E \cap (A \times A)$ is obtained from $\mathcal{G}$ by keeping only the edges with both endpoints in A.

Definition 8. A subset $C \subseteq V$ is said to be an (α, β)-*separator* if all paths from α to β intersect C. Hence, in an undirected graph, C is an (α, β)-separator if and only if

$$[\alpha]_{V \setminus C} \neq [\beta]_{V \setminus C}.$$

We say that the subset C *separates* A from B if it is an (α, β)-separator for every $\alpha \in A$ and every $\beta \in B$.

Roughly speaking, C separates A from B if the removal of C from the graph, separates A and B into two distinct connected components $\mathcal{G}_A$ and $\mathcal{G}_B$. The following example illustrates the concepts we have just introduced.

Example 1. Consider the graph $\mathcal{G} = (V, E)$ with $n = |V| = 7$ vertices, depicted in Figure 4.1.

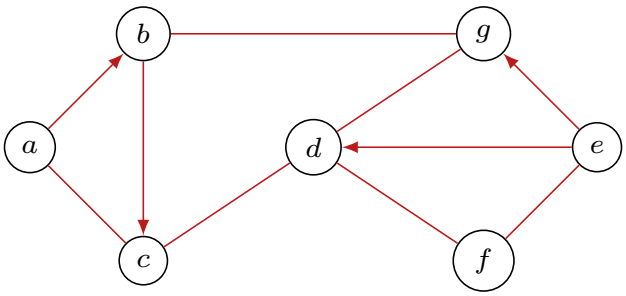

Figure 4.1: Example of a graph with 7 vertices.

If we pick for instance $D = \{d\}$ we have that $\mathrm{bd}(D) = \mathrm{bd}(d) = \{c, e, f, g\}$ and $\mathrm{cl}(D) = \mathrm{cl}(d) = \{d\} \cup \mathrm{bd}(d)$. Moreover, notice that $a \mapsto c$ because a, b, c is a path from a to c and also that $f \mapsto b$ through the path f, d, g, b. The same reasoning can be applied to see that also $c \mapsto a$ and $b \mapsto f$. Finally, we can say that $C = \{d, g\}$ separates $A = \{a, b, c\}$ and $B = \{f, e\}$ into the two distinct connected components $\mathcal{G}_A$ and $\mathcal{G}_B$ depicted in Figure 4.2.

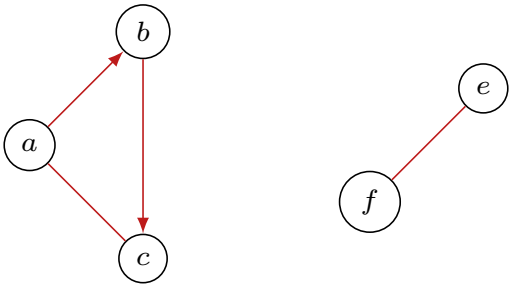

Figure 4.2: The two connected components of the graph depicted in Figure 4.1 for the above choice of A and B.

In this example we have considered some directed edges only for explaining all the definitions given above. It is worth to pointing out that in the remaining of this thesis only undirected graphs will be considered.

4.2.2 Conditional independence and Markov properties

In this subsection we introduce the crucial concept of conditional independence between two random variables and we state the Markov properties of a graphical model associated to a family of random variables. These concepts are of fundamental importance in this thesis because, roughly speaking, they are precisely the tools that tell us how to construct a graphical model for a certain random vector. We start with a definition.

Definition 9. Let $\mathbb{P}$ be a probability measure and let $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$ be three random variables

with joint probability density $p_{\boldsymbol{xyz}}(x, y, z)$. We say that $\boldsymbol{x}$ is *conditionally independent of* $\boldsymbol{y}$ *given* $\boldsymbol{z}$, and we write $\boldsymbol{x} \perp\!\!\!\perp \boldsymbol{y} \mid \boldsymbol{z}$, when

$$p_{\boldsymbol{xy}|\boldsymbol{z}}(x, y \mid z) = p_{\boldsymbol{x}|\boldsymbol{z}}(x \mid z)\, p_{\boldsymbol{y}|\boldsymbol{z}}(y \mid z),$$

almost surely with respect to the probability measure $\mathbb{P}$.

Markov properties. Let $\mathcal{G} = (V, E)$ be an undirected graph. Consider a collection of random variables $\{\boldsymbol{x}_\alpha\}_{\alpha \in V}$ taking values in the probability spaces $\{\mathcal{X}_\alpha\}_{\alpha \in V}$. If $A \subseteq V$ we let

$$\mathcal{X}_A := \times_{\alpha \in A} \mathcal{X}_\alpha$$

be the cartesian product of the probability spaces $\{\mathcal{X}_\alpha\}_{\alpha \in A}$ and further, we define $\mathcal{X} := \mathcal{X}_V$. Accordingly, the collection of random variables $\boldsymbol{x}_A := \{\boldsymbol{x}_\alpha\}_{\alpha \in A}$ is defined on the probability space $\mathcal{X}_A$. Adopting the short-hand notation

$$A \perp\!\!\!\perp B \mid C \qquad \text{for} \qquad \boldsymbol{x}_A \perp\!\!\!\perp \boldsymbol{x}_B \mid \boldsymbol{x}_C,$$

a probability measure $\mathbb{P}$ on $\mathcal{X}$ is said to obey

(P) the *pairwise Markov property*, relative to $\mathcal{G}$, if for any pair (α, β) of non-adjacent vertices

$$\alpha \perp\!\!\!\perp \beta \mid V \setminus \{\alpha, \beta\};$$

(L) the *local Markov property*, relative to $\mathcal{G}$, if for any vertex $\alpha \in V$

$$\alpha \perp\!\!\!\perp \{V \setminus \mathrm{cl}(\alpha)\} \mid \mathrm{bd}(\alpha);$$

namely, every random variable is conditional independent from the remaining, given its neighbors;

(G) the *global Markov property*, relative to $\mathcal{G}$, if for any triple (A, B, S) of disjoint subsets of V such that S separates A from B in $\mathcal{G}$,

$$A \perp\!\!\!\perp B \mid S.$$

Example 2. In order to illustrate the three properties justly stated, consider the undirected graph $\mathcal{G} = (V, E)$ depicted in Figure 4.3.

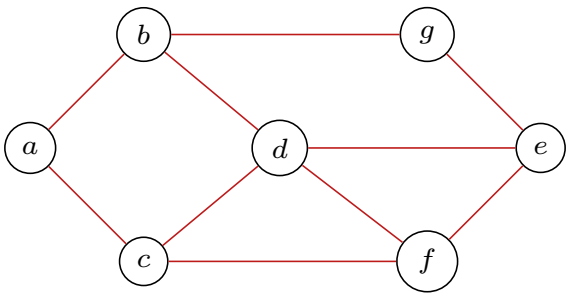

Figure 4.3

The *pairwise Markov property* simply says that any non-adjacent pair of random variables are conditionally independent given the remaining. For example, we have

$$a \perp\!\!\!\perp d \mid \{b, c, e, f, g\} \quad \text{or} \quad g \perp\!\!\!\perp f \mid \{a, b, c, d, e\}.$$

The *local Markov property* means instead that every random variable is conditional independent from the remaining, given its neighbors. For instance,

$$d \perp\!\!\!\perp \{a, g\} \mid \{b, c, e, f\} \quad \text{or} \quad e \perp\!\!\!\perp \{a, b, c\} \mid \{d, f, g\}.$$

The *global Markov property* requires to look to separating sets in the graph, such as $\{b, c\}$, $\{d, f, g\}$ or $\{b, d, f\}$. Examples can be

$$a \perp\!\!\!\perp e \mid \{b, d, f\} \quad \text{or} \quad b \perp\!\!\!\perp f \mid \{c, d, g\}.$$

Notice that the Global Markov property (G) is very important because it gives a general criterion for deciding when two groups of variables are A and B are conditional independent given a third group of variables S. It can be proven that, see [3] Proposition 3.4. on pp. 33, for any undirected graph $\mathcal{G}$ and any probability distribution on $\mathcal{X}$,

$$(\mathsf{G}) \implies (\mathsf{L}) \implies (\mathsf{P}),$$

but much more is true: it can be shown that if $\mathbb{P}$ admits a positive and continuous density with respect to a product measure μ, *the Markov properties are all equivalent* namely,

$$(\mathsf{G}) \iff (\mathsf{L}) \iff (\mathsf{P}),$$

see [3], on pp. 34.

4.2.3 Gaussian graphical models

The graphical interaction models for the multivariate Gaussian distribution are called *Gaussian graphical models* or *covariance selection models*. It will be clear very soon that this models are strongly connected with the Dempster's covariance selection method presented in the previous Section.

Let $\mathcal{G} = (V, E)$ be an undirected graph with, $|V| = n$, $E \subseteq V \times V$. Consider a zero-mean, Gaussian random vector $\mathbf{x} \sim \mathcal{N}(0, \Sigma)$ taking values in $\mathbb{R}^n$, where $\Sigma = \Sigma^\top > 0$ so that the concentration matrix $K := \Sigma^{-1}$ is well defined. Conditional independence for the multivariate Gaussian distribution is simply reflected in the concentration matrix of the distribution through zero entries. This fact is formalized in the following Proposition.

Proposition 6. Let $\mathbf{x} \sim \mathcal{N}(0, \Sigma)$ and $\mathcal{G} = (V, E)$ as above. Then, for any α, $\beta \in V$ with $\alpha \neq \beta$ it holds that

$$\boldsymbol{x}_\alpha \perp\!\!\!\perp \boldsymbol{x}_\beta \mid \{\boldsymbol{x}_\gamma\}_{\gamma \neq \alpha, \beta} \iff k_{\alpha\beta} = 0,$$

where $k_{ij} := (K)_{ij}$ is the element in position (i, j) in the concentration matrix of $\mathbf{x}$, for any i, $j \in V$.

Proof. Consider the 2×2 matrix

$$K_{\{\alpha,\beta\}} = \begin{bmatrix} k_{\alpha\alpha} & k_{\alpha\beta} \\ k_{\alpha\beta} & k_{\beta\beta} \end{bmatrix}$$

extracted from K. Comparing equations (3.1) and (3.2) it is apparent that $K_{\{\alpha,\beta\}}$ is the concentration matrix of the vector $\boldsymbol{x}_{\{\alpha,\beta\}}$ given the random vector $\boldsymbol{x}_{V\setminus\{\alpha,\beta\}}$ composed by the remaining random variables in $\mathbf{x}$. The covariance matrix of this conditional distribution is therefore equal to

$$\Sigma_{\alpha,\beta|V\setminus\{\alpha,\beta\}} = K^{-1}_{\{\alpha,\beta\}} = \frac{1}{\det K_{\{\alpha,\beta\}}} \begin{bmatrix} k_{\beta\beta} & -k_{\alpha\beta} \\ -k_{\alpha\beta} & k_{\alpha\alpha} \end{bmatrix}$$

from which follows that $\boldsymbol{x}_\alpha \perp\!\!\!\perp \boldsymbol{x}_\beta \mid \{\boldsymbol{x}_\gamma\}_{\gamma\neq\alpha,\beta}$ if and only if $k_{\alpha\beta} = 0$. □

Thus, the Gaussian graphical model for a random vector $\mathbf{x}$ with graph $\mathcal{G}$ is given by assuming that $\mathbf{x}$ follows a multivariate Gaussian distribution which obeys the undirected pairwise Markov property (P) with respect to $\mathcal{G}$. If we denote as $\mathcal{S}^+(\mathcal{G})$ the set of positive definite, symmetric matrices A satisfying

$$(i,j) \notin E \implies (A)_{ij} = 0,$$

then, in the light of Proposition 6, the Gaussian graphical model for $\mathbf{x}$ can be compactly described as

$$\mathbf{x} \sim \mathcal{N}(0,\Sigma), \qquad \Sigma^{-1} \in \mathcal{S}^+(\mathcal{G}).$$

In this case $\mathbf{x}$ is a collection of Gaussian random variables $\{\boldsymbol{x}_i,\, i = 1,\ldots,n\}$ defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$, that admits a positive and continuous joint probability density

$$p(x) = (2\pi)^{-\frac{n}{2}} \det(\Sigma)^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}\, x^\top \Sigma^{-1} x\right\}, \qquad x \in \mathbb{R}^n.$$

Accordingly, the Markov properties for the graph $\mathcal{G}$ associated to $\mathbf{x}$ are all equivalent and thus the pairwise Markov property coincides with the local and the global Markov property.

5 Periodic reciprocal processes

In the first Section of this Chapter we present conditions under which a wide-sense stationary process defined on a finite interval $[1,\, N] \subseteq \mathbb{Z}$ can be extended by periodicity over the whole integer line $\mathbb{Z}$. The second Section is devoted to the introduction of a class of random processes on a finite-interval which are a natural generalization of reciprocal processes: a characterization of such processes in terms of the sparsity pattern of their concentration matrix is also provided. Finally, we show how their dynamics can be described by a generalization of the classical autoregressive (AR) model. The material presented in this Chapter is mainly taken from [12] and [11]. In particular we refer to [12] for the omitted proofs of the results presented in this Chapter.

5.1 Stationary processes on a finite interval

A m-dimensional stochastic process on a finite interval $[1, N]$, is just an ordered collection of (zero-mean) m-dimensional random vectors $\mathbf{y} := \{\mathbf{y}(k),\, k = 1, 2, \dots, N\}$ which will be written as a column vector with N, m-dimensional components. We say that $\mathbf{y}$ is stationary if the covariances $\mathbb{E}[\mathbf{y}(k)\mathbf{y}(j)^\top]$ depend only on the difference of the arguments, namely

$$\mathbb{E}[\mathbf{y}(k)\mathbf{y}(j)^\top] = \Sigma_{k-j}, \qquad k, j = 1, \dots, N. \tag{5.1}$$

In that case the covariance matrix of $\mathbf{y}$ has a symmetric block-Toeplitz structure

$$\mathbf{\Sigma_y} := \mathbb{E}[\mathbf{y}\mathbf{y}^\top] = \begin{bmatrix} \Sigma_0 & \Sigma_1^\top & \dots & \Sigma_{N-1}^\top \\ \Sigma_1 & \Sigma_0 & \Sigma_1^\top & \dots \\ \vdots & \ddots & \ddots & \ddots \\ \Sigma_{N-1} & \dots & \Sigma_1 & \Sigma_0 \end{bmatrix}. \tag{5.2}$$

Processes $\mathbf{y}$ which have a positive definite covariance matrix $\mathbf{\Sigma_y}$ are called *full-rank processes*. In the following we assume to deal with full-rank processes. Consider now a stationary process $\tilde{\mathbf{y}}$ on the integer line $\mathbb{Z}$, which is *periodic of period* T, i.e.

$$\tilde{\mathbf{y}}(k + nT) = \tilde{\mathbf{y}}(k) \quad \text{almost surely}, \qquad \forall\, n \in \mathbb{Z}.$$

We can think of $\tilde{\mathbf{y}}$ as a process indexed on the discrete circle group $\mathbb{Z}_T := \{1, 2, \dots, T\}$ with arithmetic $\bmod T$. Clearly, its covariance function $\tilde{\mathbf{\Sigma}}_\mathbf{y}$ must also be periodic of period T, namely $\tilde{\Sigma}_{k+T} = \tilde{\Sigma}_k$ for all $k \in \mathbb{Z}$. Hence, we may also see the covariance sequence as a function on the discrete group $\tilde{\mathbb{Z}}_T := \{0, \dots, T-1\}$ with arithmetic $\bmod T$. The following Proposition states a more stronger result.

Proposition 7. A (second-order) stochastic process $\mathbf{y}$ on $[1, T]$ is the restriction to the interval $[1, T]$ of a wide-sense stationary periodic process $\tilde{\mathbf{y}}$ of period T defined on $\mathbb{Z}$, if and only if its covariance matrix $\mathbf{\Sigma_y}$ is symmetric block-circulant.

Remark 5. The periodic extension to the whole line $\mathbb{Z}$ of deterministic signals originally given on a finite interval $[1, T]$ is a common device in (deterministic) signal processing. This simple periodic extension does, however, not preserve the structure of a stationary random process since the covariance of a periodically extended process will not be, in general, symmetric block-circulant, as stated by the previous Proposition. It is easy to see that the fact that $\mathbf{\Sigma_y}$ is symmetric block circulant is equivalent to the fact that the sequence of covariance lags $\tilde{\Sigma}_k$, $k = 0, \dots, T-1$, of the extended process $\tilde{\mathbf{y}}$ is symmetric with respect to the midpoint of the interval $[1, T]$.

In many applications to signal and image processing, the signals under study naturally live on a finite interval of time (or space) variable and modeling them as functions defined on the whole line appears just as an artifice introduced in order to use the standard tools of (causal) time-invariant systems and harmonic analysis on the line. It may indeed be more logical to describe these data as stationary processes $\mathbf{y}$ defined on a finite-interval $[1, T]$. The covariance function, say $\mathbf{\Sigma_y}$, of such a process will be a symmetric positive definite block-Toeplitz matrix which has in general no block-circulant structure. It is however always possible to extend the covariance function of $\mathbf{y}$ to a larger interval so as to make it center-symmetric. This can be achieved by simply letting $\Sigma_{T+\tau} = \Sigma_{T-1-\tau}^{\top}$ for $\tau = 0, \dots, T-1$. In this way $\mathbf{\Sigma_y}$ is extended to a symmetric block-circulant matrix $\tilde{\mathbf{\Sigma}}_{\mathbf{y}}$ of dimension $(2T-1) \times (2T-1)$. This operation does not necessarily preserve positivity. Positivity of a symmetric, block-circulant extension, however, can always be guaranteed provided the extension is done on a suitably large interval.
The original process $\mathbf{y}$ can then be seen as the restriction to the interval $[1, T]$ of an extended process, say $\tilde{\mathbf{y}}$, with symmetric block-circulant covariance matrix $\tilde{\mathbf{\Sigma}}_{\mathbf{y}}$, which lives on an interval $[1, N]$ of length $N \geq 2T-1$. Since the extended covariance is, in any case, completely determined by the entries of the original covariance matrix $\mathbf{\Sigma_y}$, any statistical estimate thereof can be computed from the variables of the original process $\mathbf{y}$ in the interval $[1, T]$ (or from their sample values). Hence, there is no need to know what the random vectors $\{\tilde{\mathbf{y}}(k);\, k = T+1, \dots, N\}$ look like. Indeed, as soon as we are given the covariance of the process $\mathbf{y}$ defined on $[1, T]$, even if we may not ever see (sample values of) the "external" random vectors $\{\tilde{\mathbf{y}}(k);\, k = T+1, \dots, N\}$, we would in any case have a completely determined second-order description (covariance function) of $\tilde{\mathbf{y}}$. In this sense, one can think of any stationary process $\mathbf{y}$ given on a finite interval $[1, T]$ as the restriction to $[1, T]$ of a wide-sense stationary *periodic* process, $\tilde{\mathbf{y}}$, of period $N \geq 2T-1$, defined on the whole integer line $\mathbb{Z}$. This process naturally live on the discrete circle $\mathbb{Z}_N$. Hence, dealing in our future study with the periodic extension $\tilde{\mathbf{y}}$, instead of the original process $\mathbf{y}$, will entail no loss of generality.

5.2 AR-type reciprocal processes

In this Section we describe a class of random processes on a finite-interval which are natural generalization of the reciprocal processes. In a sense, they are an acausal "symmetric" generalization of autoregressive (AR) processes on the integer line.

Let $\mathbf{y}$ be a zero-mean m-dimensional stationary process on $[1, N]$ and let $\mathbf{\Sigma_y}$ denotes its $mN \times mN$ symmetric, block-circulant covariance matrix, so that $\mathbf{y}$ may be seen as a process on the discrete circle $\mathbb{Z}_N$. As we said above, we may imagine that the matrix $\mathbf{\Sigma_y}$

was obtained by extending a positive block-Toeplitz matrix as (5.2) to make it symmetric block-circulant. Then $[1, N]$ will have to be identified with an enlarged interval on which $\mathbf{y}$ is the periodic extension of some underlying stationary process defined in a smaller interval.
Let n be a natural number such that $N > 2n \geq 0$ (it will always be assumed in the following). We introduce the notation

$$\mathbf{y}_{[t-n,t)} := \begin{bmatrix} \mathbf{y}(t-n) \\ \vdots \\ \mathbf{y}(t-1) \end{bmatrix}, \qquad \mathbf{y}_{(t,t+n]} := \begin{bmatrix} \mathbf{y}(t+1) \\ \vdots \\ \mathbf{y}(y+n) \end{bmatrix},$$

where sums $t-k$ and $t+k$ are to be understood modulo N. Consider a subinterval $(t_1, t_2) \subset [1, N]$ where $(t_1, t_2) := \{t : t_1 < t < t_2\}$ and $(t_1, t_2)^c$ denotes the complementary set in $[1, N]$.

Definition 10. A reciprocal process of order n on $[1, N]$ is characterized by the property that the random variables of the process in the interval (t_1, t_2) are conditionally orthogonal to the random variables in the exterior, $(t_1, t_2)^c$, given the $2n$ boundary values $\mathbf{y}_{(t_1-n,t_1]}$ and $\mathbf{y}_{[t_2,t_2+n)}$.

Equivalently, it must hold that

$$\hat{\mathbb{E}}\left[\mathbf{y}_{(t_1,t_2)} \mid \mathbf{y}(s),\, s \in (t_1,t_2)^c\right] = \hat{\mathbb{E}}\left[\mathbf{y}_{(t_1,t_2)} \mid \mathbf{y}_{(t_1-n,t_1]} \vee \mathbf{y}_{[t_2,t_2+n)}\right]$$

for $t_1,\, t_2 \in [1, N]$. To see this, simply apply Proposition 2, with $\mathcal{X} = [1, t_1-n] \cup [t_2+n, N]$, $\mathcal{Y} = (t_1, t_2)$, $\mathcal{Z} = (t_1-n, t_1] \vee [t_2, t_2+n)$ so that $\mathcal{X} \vee \mathcal{Z} = (t_1, t_2)^c$. In particular, we should have

$$\hat{\mathbb{E}}\left[\mathbf{y}(t) \mid \mathbf{y}(s),\, s \neq t\right] = \hat{\mathbb{E}}\left[\mathbf{y}(t) \mid \mathbf{y}_{[t-n,t)} \vee \mathbf{y}_{(t,t+n]}\right]$$

for $t \in [1, N]$, where the estimation error

$$\mathbf{d}(t) := \mathbf{y}(t) - \hat{\mathbb{E}}\left[\mathbf{y}(t) \mid \mathbf{y}_{[t-n,t)} \vee \mathbf{y}_{(t,t+n]}\right], \qquad t \in [1, N], \tag{5.3}$$

must clearly be orthogonal to all random variables $\{\mathbf{y}(s),\, s \neq t\}$, that is

$$\mathbb{E}[\mathbf{y}(t)\mathbf{d}(s)^\top] = \Delta\, \delta_{ts}, \qquad t, s \in [1, N], \tag{5.4}$$

where δ is the Kronecker delta and Δ is a square matrix (its meaning will be clear in a moment). Since $\mathbf{d}(t+k)$ is a linear combination of the components of the random vector $\mathbf{y}_{[t+k-n,t+k+n]}$, it follows from (5.4) that both $\mathbf{d}(t+k)$ and $\mathbf{d}(t-k)$ are orthogonal to $\mathbf{d}(t)$ as soon as $k > n$. Hence, the process $\{\mathbf{d}(t)\}$ has *correlation bandwidth* n, i.e.

$$\mathbb{E}[\mathbf{d}(t+k)\mathbf{d}(t)^\top] = 0 \quad \text{for} \quad n < |k| < N-n,\, k \in [0, N-1]. \tag{5.5}$$

It follows from (5.3) that a reciprocal process of order n on $[1, N]$, can always be described by a linear double-sided recursion of the form

$$\sum_{k=-n}^{n} A_k\, \mathbf{y}(t-k) = \mathbf{d}(t), \qquad t \in [1, N], \tag{5.6}$$

where A_k are $m \times m$ matrices, in general dependent on t, with $A_0 = I_m$, and $\mathbf{d}$ is a process of correlation bandwidth n called *conjugate process* of $\mathbf{y}$, orthogonal to $\mathbf{y}$ in the sense of

(5.4). From (5.4) it follows that $\mathbb{E}[\mathbf{d}(t)\mathbf{d}(t)^\top] = \Delta$, and hence, Δ is the covariance matrix of $\mathbf{d}(t)$, symmetric and positive semi-definite. It can be proven that if $\mathbf{y}$ is stationary, the matrices $\{A_k\}$ in the representation (5.6) do not depend on t, moreover, if $\mathbf{y}$ is full-rank, they are uniquely determined by the covariance lags of the process up to order $2n$. Equation (5.6) requires the specification of the boundary values, specified by the following Theorem.

Theorem 3. A stationary reciprocal process $\mathbf{y}$ of order n on $\mathbb{Z}_N$ satisfies a linear, constant-coefficients difference equation of the type (5.6), associated to the $2n$ cyclic boundary conditions

$$\mathbf{y}(k) = \mathbf{y}(N+k), \qquad k = -n+1, \dots, n. \tag{5.7}$$

The model can be rewritten in matrix form as

$$\mathbf{A}\,\mathbf{y} = \mathbf{d} \tag{5.8}$$

where $\mathbf{A}$ is the N-block banded circulant matrix of bandwidth n

$$\mathbf{A} := \operatorname{circ}\{I, A_1, \dots, A_n, 0, \dots, 0, A_{-n}, \dots, A_{-1}\}$$

and $\mathbf{y}$ and $\mathbf{d}$ are understood as

$$\mathbf{y} := \begin{bmatrix} \mathbf{y}(1) \\ \vdots \\ \mathbf{y}(N) \end{bmatrix}, \qquad \mathbf{d} := \begin{bmatrix} \mathbf{d}(1) \\ \vdots \\ \mathbf{d}(N) \end{bmatrix}.$$

If the process is full rank this description is unique.

Proposition 8. A stationary reciprocal process $\mathbf{y}$ is full rank if and only if the covariance matrix Δ of the conjugate process $\mathbf{d}$ is positive definite.

Proof. Suppose $\Delta > 0$. Multiplying both members of (5.8) from the right by $\mathbf{y}^\top$ and taking the expectations, in virtue of the orthogonality relation (5.4), we get

$$\mathbf{A}\,\mathbf{\Sigma_y} = \mathbf{A}\,\mathbb{E}[\mathbf{y}\mathbf{y}^\top] = \mathbb{E}[\mathbf{d}\mathbf{y}^\top] = \operatorname{diag}\{\Delta, \dots, \Delta\}. \tag{5.9}$$

Thus, $\Delta > 0$ implies that the square matrices $\mathbf{A}$ and $\mathbf{\Sigma_y}$ are invertible which, combined with the positive semidefinitness of $\mathbf{\Sigma_y}$, implies $\mathbf{\Sigma_y} > 0$. Conversely, suppose now that Δ in only positive semi-definite. This implies that there exists $a \in \mathbb{R}^m$, $a \neq 0$, such that

$$\mathbb{E}[a^\top \mathbf{d}(t)\mathbf{d}(t)^\top a] = 0$$

almost surely, that is such that $a^\top \mathbf{d}(t) = 0$ almost surely. This means that the scalar components of $\mathbf{d}(t)$ are linearly dependent, which, by (5.6), implies that $\mathbf{y}(t-n)$, …, $\mathbf{y}(t)$, …, $\mathbf{y}(t+n)$ are linearly dependent. Thus, $\mathbf{\Sigma_y}$ must be singular, which contradicts the assumption $\mathbf{\Sigma_y} > 0$. □

By (5.9) we can express the inverse of $\mathbf{\Sigma_y}$ as

$$\mathbf{M_y} := \mathbf{\Sigma_y}^{-1} = \operatorname{diag}\{\Delta^{-1}, \dots, \Delta^{-1}\}\,\mathbf{A}$$

so that $\mathbf{M_y}$ is symmetric block-circulant and positive definite, being the inverse of a matrix with the same properties. Furthermore, $M_k := \Delta^{-1}A_k$, $k = -n, \dots, n$ and $M_0 = \Delta^{-1}$, must form a *center-symmetric sequence* of bandwidth n, i.e.

$$M_{-k} = M_k^\top, \qquad k = 1, \dots, n.$$

If we normalize the conjugate process by setting

$$\mathbf{e}(t) := \Delta^{-1}\mathbf{d}(t)$$

so that $\mathrm{Var}\{\mathbf{e}(t)\} = \Delta^{-1}$, model (5.6) can be rewritten as

$$\sum_{k=-n}^{n} M_k\, \mathbf{y}(t-k) = \mathbf{e}(t), \qquad t \in \mathbb{Z}_N,$$

for which the orthogonality relation (5.4) is replaced by

$$\mathbb{E}[\mathbf{y}\mathbf{e}^\top] = I_{mN}. \tag{5.10}$$

By the normalization of (5.8), our reciprocal process $\mathbf{y}$ satisfies the linear equation

$$\mathbf{M_y}\,\mathbf{y} = \mathbf{e} \tag{5.11}$$

which implicitly includes the cyclic boundary conditions (5.7). Multiplying this equality from the right by $\mathbf{e}^\top$ and taking expectations on both members we get $\mathbf{M_y}\,\mathbb{E}[\mathbf{y}\mathbf{e}^\top] = \mathbb{E}[\mathbf{e}\mathbf{e}^\top]$ which, in force of (5.10), yields

$$\mathrm{Var}\{\mathbf{e}\} = \mathbf{M_y},$$

thus $\mathbf{M_y}$ is actually the covariance matrix of the normalized conjugate process $\mathbf{e}$. So we see that the inverse of the covariance matrix of a full-rank stationary reciprocal process of order n, must be a banded block-circulant matrix of bandwidth n. The following result states that this is in fact a fundamental characterization of a stationary reciprocal process of order n namely, also the inverse statement holds: to an autoregressive model of the form (5.6) associated to the proper cyclic boundary conditions, determines uniquely a process $\mathbf{y}$ which is stationary and reciprocal of order n.

Theorem 4. A non-singular $mN \times mN$-dimensional matrix $\mathbf{\Sigma_y}$ is the covariance matrix of a reciprocal process of order n on the discrete group $\mathbb{Z}_N$ if and only if its inverse is a positive definite symmetric block-circulant matrix which is banded of bandwidth n.

Note that the second-order statistics of both $\mathbf{y}$ and $\mathbf{e}$ are encapsulated in the covariance matrix $\mathbf{M_y}$. In other words, the whole autoregressive model of $\mathbf{y}$ is defined in terms of the matrix $\mathbf{M_y}$. Accordingly, the problem of identify a model for the periodic reciprocal process $\mathbf{y}$ can be formulate as follows.

Problem 2. Given the observations of the reciprocal process $\mathbf{y}$ of (known) order n, estimate the parameters (M_0, M_1, ..., M_n) of the banded inverse of the covariance matrix $\mathbf{\Sigma_y}$.

This Problem will be address in Section 6.2 in order to introduce the classical block-circulant band extension problem.

6 Sparse plus low-rank identification

In this Chapter we introduce latent-variable graphical models for zero-mean, Gaussian, periodic reciprocal processes of order n, and we derive a procedure for the identification of such models. This procedure involves an optimization problem which is a regularized version of the maximum entropy dual problem that solves the block-circulant band extension problem and it is used to estimate a sparse plus low-rank decomposition of the concentration matrix of the process. The latter is basically a generalization of the Dempster's covariance extension problem presented in Section 4.1.1 and exploits the particular structure of the problem: according to Theorem 4 in Chapter 5, the concentration matrix of the process is a symmetric, banded block-circulant matrix of bandwidth n. It is worth noting that the derivation of the model as well as the identification paradigm follow the one presented in [7].

6.1 Problem set-up

In this Section we lay the foundations for the identification procedure that will be developed in the sequel. In particular, we will introduce the assumptions needed in order to develop a sparse plus low-rank graphical model for the process we are considering and we will give some preliminary results that we will use in the solution of the optimization problem.

Let $\mathbf{y} := \{\mathbf{y}(k),\, k = 1, \dots, N\}$ be a real-valued, Gaussian, zero-mean, m-dimensional stationary reciprocal process of order n, defined on a finite interval $[1, N]$, with $N > 2n$. It is worth noting that $\mathbf{y}$ is extended to the whole integer line $\mathbb{Z}$ as a periodic stationary process with period N, i.e. such that $\mathbf{y}(t + kN) = \mathbf{y}(t)$ almost surely, where $k \in \mathbb{N}$. Moreover, $\mathbf{y}$ may be seen as a process defined on the discrete circle group $\mathbb{Z}_N := \{1, 2, \dots, N\}$ with arithmetic mod N. In what follows, for simplicity we assume that N is an even number.

Remark 6. The corresponding results for N odd are very similar and their derivations follow the same line of what we will do for N even, having in mind the small differences in the notation highlighted in Chapter 2.

If we consider only the interval $[1, N]$ we can write the process as column vector with N, m-dimensional components:

$$\mathbf{y} := \begin{bmatrix} \mathbf{y}(1) \\ \vdots \\ \mathbf{y}(N) \end{bmatrix} \in \mathbb{R}^{mN}, \qquad \text{where} \qquad \mathbf{y}(k) = \begin{bmatrix} \boldsymbol{y}_1(k) \\ \vdots \\ \boldsymbol{y}_m(k) \end{bmatrix} \in \mathbb{R}^m.$$

It is also useful to define the j-th component of the process $\mathbf{y}$ as the $\mathbb{R}^N$-valued vector

$\mathbf{y}_j := [\boldsymbol{y}_j(1) \dots \boldsymbol{y}_j(N)]^\top$, obtained by stacking all the j-th components of the process for each time $k = 1, \dots, N$. Let $\mathbf{\Sigma_y} > 0$ denotes the $mN \times mN$ covariance matrix of $\mathbf{y}$. By assumption, the process $\mathbf{y}$ is a reciprocal process of order n on the discrete group $\mathbb{Z}_N$ hence, by Theorem 4, $\mathbf{M_y} = \mathbf{\Sigma_y}^{-1}$ must belong to $\mathcal{B}$ namely,

$$\mathbf{M_y} = \begin{bmatrix} M_0 & M_1^\top & \cdots & M_n^\top & 0 & \cdots & 0 & M_n & \cdots & M_1 \\ M_1 & M_0 & \cdots & \cdots & M_n^\top & 0 & \cdots & 0 & \ddots & \vdots \\ \vdots & & \ddots & & & \ddots & \ddots & & \ddots & M_n \\ M_n & & & \ddots & & & \ddots & 0 & & 0 \\ 0 & \ddots & & & M_0 & & & M_n^\top & \ddots & \vdots \\ \vdots & \ddots & \ddots & & & \ddots & & & \ddots & 0 \\ 0 & & 0 & M_n & & & \ddots & & & M_n^\top \\ M_n^\top & \ddots & & \ddots & \ddots & & & \ddots & & \vdots \\ \vdots & \ddots & \ddots & & \ddots & \ddots & & & \ddots & M_1^\top \\ M_1^\top & \cdots & M_n^\top & 0 & \cdots & 0 & M_n & \cdots & M_1 & M_0 \end{bmatrix}. \tag{6.1}$$

Now we start with the construction of the latent-variable graphical model. Suppose that we can only observe the manifest (i.e. accessible to observations) process $\mathbf{y}$ and consider a relatively small number l of latent (i.e. hidden, not accessible to observations) variables $\mathbf{x} := [\boldsymbol{x}_1, \dots, \boldsymbol{x}_l]^\top$, which cannot be observed, such that $\mathbf{z} := [\mathbf{y}^\top \mathbf{x}^\top]^\top$ is a Gaussian random vector.

Remark 7. These latent variables can be seen either as variables that are actually present in the system we are observing but we cannot measure them or as fictitious variables introduced in order to explain as best as possible the interactions between the components of $\mathbf{y}$ using the smallest possible number of variables. Both point of views make sense. In any case, notice that the latent variables $\boldsymbol{x}_i$, $i = 1, \dots, l$, are just random variables, therefore they do not introduce any dynamics.

According with the previous assumptions, the covariance matrix of vector $\mathbf{z}$ can be partitioned as follows

$$\mathbf{\Sigma_z} := \mathbb{E}[\mathbf{z}\mathbf{z}^\top] = \begin{bmatrix} \mathbf{\Sigma_y} & \mathbf{\Sigma_{yx}} \\ \mathbf{\Sigma_{yx}}^\top & \mathbf{\Sigma_x} \end{bmatrix} \in \mathbb{R}^{(mN+l)\times(mN+l)}.$$

By partitioning the corresponding inverse as

$$\mathbf{\Sigma_z}^{-1} = \begin{bmatrix} \mathbf{S} & \mathbf{A} \\ \mathbf{A}^\top & \mathbf{R} \end{bmatrix}$$

and applying the Schur complement (see Section 3.1), we obtain the relation

$$\mathbf{\Sigma_y} = (\mathbf{S} - \mathbf{L})^{-1}, \tag{6.2}$$

where $\mathbf{S}$ is the concentration matrix of the process $\mathbf{y}$ conditioned on $\mathbf{x}$, and $\mathbf{L}$ is defined as $\mathbf{L} := \mathbf{A}\,\mathbf{R}^{-1}\,\mathbf{A}^\top$. We assume that $\mathbf{S} \in \mathcal{C}$ is a positive definite matrix while $\mathbf{L} \in \mathcal{C}$ is a

positive semi-definite matrix, that is

$$\mathbf{S} = \begin{bmatrix} S_0 & S_1^\top & \cdots & S_{\frac{N}{2}}^\top & \cdots & S_1 \\ S_1 & S_0 & \cdots & \cdots & \ddots & \vdots \\ \vdots & S_1 & \cdots & \cdots & \cdots & S_{\frac{N}{2}}^\top \\ S_{\frac{N}{2}} & \cdots & \ddots & S_0 & \cdots & \vdots \\ \vdots & \ddots & & & \ddots & S_1^\top \\ S_1^\top & \cdots & S_{\frac{N}{2}} & \cdots & S_1 & S_0 \end{bmatrix}, \qquad \mathbf{L} = \begin{bmatrix} L_0 & L_1^\top & \cdots & L_{\frac{N}{2}}^\top & \cdots & L_1 \\ L_1 & L_0 & \cdots & \cdots & \ddots & \vdots \\ \vdots & L_1 & \cdots & \cdots & \cdots & L_{\frac{N}{2}}^\top \\ L_{\frac{N}{2}} & \cdots & \ddots & L_0 & \cdots & \vdots \\ \vdots & \ddots & & & \ddots & L_1^\top \\ L_1^\top & \cdots & L_{\frac{N}{2}} & \cdots & L_1 & L_0 \end{bmatrix},$$

so that $\mathbf{M_y} \in \mathcal{B}$. Notice that we did not assume any banded structure for $\mathbf{S}$ and $\mathbf{L}$, we will see that we will get $\mathbf{\Sigma_y}$ banded of bandwidth n anyway. In Chapter 7 we consider a particular instance of this setting, i.e. we choose both $\mathbf{S}$ and $\mathbf{L}$ to be banded matrices of bandwidth n in order to ensure that $\mathbf{M_y} \in \mathcal{B}$. Of course, this is only *one choice*. In fact, it is an admissible choice because there are examples in which the optimization problem we are going to set-up for the estimation of $\mathbf{S}$ and $\mathbf{L}$ admits a solution under this setting.

6.1.1 Latent-variable graphical models

As explained in Chapter 4 (see Section 4.2.3) it is possible to associate a graph to the Gaussian random vector $\mathbf{z}$ whose edges are determined by the conditional dependence relations among the components of $\mathbf{z}$. We assume that l is sufficiently small (in particular $l \ll mN$) and that conditional dependence relations among the observed variables $\boldsymbol{y}_i$, $i = 1, \ldots, m$, are mostly through this limited number of latent variables, for *any* time. Since $\mathbf{S}$ contains the conditional dependence relations among the observed variables given the latent variables (see Section 4.2.3), the latter assumption implies that $\mathbf{S}$ will be a sparse matrix. The fact that we assume that l is small implies instead that $\mathbf{L}$ will be a low-rank matrix whose rank coincides with the number of latent variables l. It is now clear why, under the present assumptions, we will refer to equation (6.2) as *sparse plus low-rank decomposition* of the inverse covariance matrix (concentration matrix) of the process $\mathbf{y}$. Moreover, we assume that the blocks S_0, S_1, …, $S_1^\top$ of matrix $\mathbf{S}$ have common support $\Omega \subseteq \{(i,j) : i,j = 1, \ldots, m\}$ namely,

$$(S_k)_{ij} = (S_k)_{ji} = 0, \qquad k = 0, \ldots, N/2, \quad \forall\, (i,j) \in \Omega^c. \tag{6.3}$$

It is worth noting that *the common support* Ω *always contains the pairs* (i, i), $i = 1, \ldots, m$, as in the Dempster's problem. This of course make sense because conditional independence is not defined between one variable and itself and thus Ω^c in (6.3) cannot contain the pairs (i, j) with $i = j$.

Remark 8. In order to clarify the whole picture, it may be useful for the reader to have a rough but clear graphical interpretation of the assumptions we have stated so far. In the light of what we have seen in Section 4.2, our assumptions already hint how the graph associated to the graphical model of the vector $\mathbf{z}$ will look like. First of all, since $\mathbf{S}$ is sparse, we will have very few edges between the nodes representing the components of $\mathbf{y}$, i.e. $\mathbf{y}_1, \ldots, \mathbf{y}_m$. Moreover, the fact that we have assumed that the S_ks have common support means that we do not have an edge between the nodes $\mathbf{y}_i$ and $\mathbf{y}_j$, $i \neq j$, if and

only if the components $\boldsymbol{y}_i(k)$ and $\boldsymbol{y}_j(k)$ are conditional independent, given the remaining components of $\mathbf{z}$, at *any time* k. To be very clear, since the process is periodic of period N, it suffices to restrict our attention for $k = 0, 1, \ldots, N/2$ and if for some k we have that $\boldsymbol{y}_i(k)$ and $\boldsymbol{y}_j(k)$ become conditional dependent, then we will have an edge between $\mathbf{y}_i$ and $\mathbf{y}_j$. Finally, the nodes representing the latent components in the graph will be very few with respect to the nodes concerning the observed variables because we have assumed that the number of latent variables l must be small with respect to the number of the observed variables.

According to the characterization of the conditional independence of the components of $\mathbf{z}$ in terms of concentration matrix of the conditioned process $\mathbf{y}|\mathbf{x}$ given by Proposition 6, (6.3) is equivalent to

$$\mathbb{E}\left[\boldsymbol{y}_i(t_1)\,\boldsymbol{y}_j(t_2) \mid \boldsymbol{y}_h(s),\, h \neq i, j,\, s = 1, \ldots, N,\, \boldsymbol{y}_i(s_1),\, s_1 \neq t_1,\, \boldsymbol{y}_j(s_2),\, s_2 \neq t_2,\, \mathbf{x}\right] = 0, \tag{6.4}$$

for any $t_1,\, t_2 \in [1, N]$ and for any pair $(i, j) \in \Omega^c$. This condition is equivalent to

$$\mathbb{E}\left[\boldsymbol{y}_i(t_1)\,\boldsymbol{y}_j(t_2) \mid \boldsymbol{y}_h(s),\, h \neq i, j,\, s = 1, \ldots, N,\, \mathbf{x}\right] = 0, \qquad \forall\, t_1,\, t_2 \in [1, N],\, \forall\, (i, j) \in \Omega^c, \tag{6.5}$$

which reflects the fact that the random variables $\{\boldsymbol{y}_i(s_1), \boldsymbol{y}_j(s_2),\, s_1 \neq t_1,\, s_2 \neq t_2\}$ do not play any role in conditioning $\boldsymbol{y}_i(t_1) \cdot \boldsymbol{y}_j(t_2)$ for any $t_1,\, t_2 \in [1, N]$ and for any pair $(i, j) \in \Omega^c$. To see this, define the set

$$\mathcal{A}(i, j, t_1, t_2) := \left\{\boldsymbol{y}_h(s),\, h \neq i, j,\, s = 1, \ldots, N,\, \boldsymbol{y}_i(s_1),\, s_1 \neq t_1,\, \boldsymbol{y}_j(s_2),\, s_2 \neq t_2,\, \mathbf{x}\right\}$$

and notice that

$$\mathcal{A}(i, j, t_1, t_2) = \tilde{\mathcal{A}}(i, j, t_1, t_2) \cup \bar{\mathcal{A}}(i, j)$$

where

$$\tilde{\mathcal{A}}(i, j, t_1, t_2) := \left\{\boldsymbol{y}_i(s_1),\, s_1 \neq t_1,\, \boldsymbol{y}_j(s_2),\, s_2 \neq t_2,\, \mathbf{x}\right\},$$
$$\bar{\mathcal{A}}(i, j) := \left\{\boldsymbol{y}_h(s),\, h \neq i, j,\, s = 1, \ldots, N,\, \mathbf{x}\right\}.$$

According with these definitions equation (6.4) can be rewritten as

$$\mathbb{E}\left[\boldsymbol{y}_i(t_1)\,\boldsymbol{y}_j(t_2) \mid \mathcal{A}(i, j, t_1, t_2)\right] = \mathbb{E}\left[\boldsymbol{y}_i(t_1)\,\boldsymbol{y}_j(t_2) \mid \tilde{\mathcal{A}}(i, j, t_1, t_2),\, \bar{\mathcal{A}}(i, j)\right] = 0, \tag{6.6}$$

for any $t_1,\, t_2 \in [1, N]$ and for all $(i, j) \in \Omega^c$. Observe now that the quantities at the left-hand side of (6.5) are contained in the matrix

$$\mathbf{V} := \mathbb{E}\left[\begin{bmatrix}\mathbf{y}_i \\ \mathbf{y}_j\end{bmatrix}[\mathbf{y}_i^\top\ \mathbf{y}_j^\top]^\top \,\middle|\, \mathbf{y}_h,\, h \neq i, j,\, \mathbf{x}\right] = \mathbb{E}\left[\begin{bmatrix}\mathbf{y}_i \\ \mathbf{y}_j\end{bmatrix}[\mathbf{y}_i^\top\ \mathbf{y}_j^\top]^\top \,\middle|\, \bar{\mathcal{A}}(i, j)\right]$$

which is the 2×2 block matrix

$$\mathbf{V} = \left[\begin{array}{ccc|ccc} * & \cdots & * & \mathbb{E}[\boldsymbol{y}_i(1)\boldsymbol{y}_j(1) \mid \bar{\mathcal{A}}(i,j)] & \cdots & \mathbb{E}[\boldsymbol{y}_i(1)\boldsymbol{y}_j(N) \mid \bar{\mathcal{A}}(i,j)] \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ * & \cdots & * & \mathbb{E}[\boldsymbol{y}_i(N)\boldsymbol{y}_j(1) \mid \bar{\mathcal{A}}(i,j)] & \cdots & \mathbb{E}[\boldsymbol{y}_i(N)\boldsymbol{y}_j(N) \mid \bar{\mathcal{A}}(i,j)] \\ \hline \mathbb{E}[\boldsymbol{y}_j(1)\boldsymbol{y}_i(1) \mid \bar{\mathcal{A}}(i,j)] & \cdots & \mathbb{E}[\boldsymbol{y}_j(1)\boldsymbol{y}_i(N) \mid \bar{\mathcal{A}}(i,j)] & * & \cdots & * \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \mathbb{E}[\boldsymbol{y}_j(N)\boldsymbol{y}_i(1) \mid \bar{\mathcal{A}}(i,j)] & \cdots & \mathbb{E}[\boldsymbol{y}_j(N)\boldsymbol{y}_i(N) \mid \bar{\mathcal{A}}(i,j)] & * & \cdots & * \end{array}\right].$$

It is clear by its definition that $\mathbf{V}$ is the covariance matrix of the random vector $[\mathbf{y}_i^\top \ \mathbf{y}_j^\top]^\top$ conditioned on $\{\mathbf{y}_h,\, h \neq i, j,\, \mathbf{x}\}$, which is still a Gaussian random vector. As explained in Chapter 4, the entry in position (h, k) in the concentration matrix of a Gaussian random vector describes the dependence relation between the components h an k of the random vector, conditioned on the other components, hence

$$\mathbf{K} := \mathbf{V}^{-1} = \left[\begin{array}{c|c} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \hline \mathbf{K}_{12}^\top & \mathbf{K}_{22} \end{array}\right]$$

where

$$\mathbf{K}_{12} = \begin{bmatrix} \mathbb{E}[\boldsymbol{y}_i(1)\boldsymbol{y}_j(1) \mid \bar{\mathcal{A}}(i,j), \tilde{\mathcal{A}}(i,j,1,1)] & \cdots & \mathbb{E}[\boldsymbol{y}_i(1)\boldsymbol{y}_j(N) \mid \bar{\mathcal{A}}(i,j), \tilde{\mathcal{A}}(i,j,1,N)] \\ \vdots & \ddots & \vdots \\ \mathbb{E}[\boldsymbol{y}_i(N)\boldsymbol{y}_j(1) \mid \bar{\mathcal{A}}(i,j), \tilde{\mathcal{A}}(i,j,N,1)] & \cdots & \mathbb{E}[\boldsymbol{y}_i(N)\boldsymbol{y}_j(N) \mid \bar{\mathcal{A}}(i,j), \tilde{\mathcal{A}}(i,j,N,N)] \end{bmatrix}$$

$$= \begin{bmatrix} \mathbb{E}[\boldsymbol{y}_i(1)\boldsymbol{y}_j(1) \mid \mathcal{A}(i,j,1,1)] & \cdots & \mathbb{E}[\boldsymbol{y}_i(1)\boldsymbol{y}_j(N) \mid \mathcal{A}(i,j,1,N)] \\ \vdots & \ddots & \vdots \\ \mathbb{E}[\boldsymbol{y}_i(N)\boldsymbol{y}_j(1) \mid \mathcal{A}(i,j,N,1)] & \cdots & \mathbb{E}[\boldsymbol{y}_i(N)\boldsymbol{y}_j(N) \mid \mathcal{A}(i,j,N,N)] \end{bmatrix}$$

$$= \begin{bmatrix} (S_0)_{ij} & \cdots & (S_{N-1})_{ij} \\ \vdots & \ddots & \vdots \\ (S_{N-1})_{ji} & \cdots & (S_0)_{ij} \end{bmatrix}.$$

By (6.3) we have clearly that $\mathbf{K}_{12} = 0$ and thus $\mathbf{V}^{-1}$ is a block diagonal matrix. Accordingly, also $\mathbf{V}$ must be a block-diagonal matrix which means that

$$\mathbb{E}\left[\boldsymbol{y}_i(t_1)\, \boldsymbol{y}_j(t_2) \mid \boldsymbol{y}_h(s),\, h \neq i, j,\, s = 1, \ldots, N,\, \mathbf{x}\right] = 0, \qquad \forall\, t_1,\, t_2 \in [1, N],\, \forall\, (i,j) \in \Omega^c,$$

as we wanted to show. Such conditional independence relations define a graphical model for the Gaussian random vector $\mathbf{z}$ which is referred to as *latent-variable graphical model*. The latter admits a two-layers structure where

Nodes. The nodes in the upper layer represent the latent random variables $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_l$, while in the bottom layer we have the nodes corresponding to the observed variables. Differently from the classical case, each node in the bottom layer does not represent a random variable $\boldsymbol{y}_i$ but represents a vector $\mathbf{y}_i$, including all the i-th components for each time.

Edges. The edges are defined by the conditional dependence relationships between the components of the vector $\mathbf{z}$. If we look at the edges in terms of the information they are carrying we have the following distinction

- an edge between two latent random variables, let's say $\boldsymbol{x}_h$ and $\boldsymbol{x}_k$, is a scalar quantity described by the element in position (h, k) of the sub-matrix $\mathbf{R}$ of $\boldsymbol{\Sigma}_{\mathbf{z}}^{-1}$.

- an edge between a latent variable $\boldsymbol{x}_k$ and an observed vector $\mathbf{y}_j$ carries a vectorial information and it is described by the N elements of $\boldsymbol{\Sigma}_{\mathbf{z}}^{-1}$ contained in column $mN+k$

in rows j, $j+(N-1)$, $j+2(N-1)$, $\dots$, $j+m(N-1)$, as can be easily seen by writing explicitly the vector $\mathbf{z}$. Clearly, if all of this numbers are zero, we do not have such an edge.

- an edge between two observed vectors $\mathbf{y}_i$ and $\mathbf{y}_j$, $i \neq j$, contains a matrix information related to the conditional dependence relations between the components $\{\boldsymbol{y}_i(k)\}_k$ and $\{\boldsymbol{y}_j(h)\}_h$. According to the characterization of conditional independence, such an edge is described by the quantities

$$\mathbb{E}\left[\boldsymbol{y}_i(t_1)\,\boldsymbol{y}_j(t_2) \mid \boldsymbol{y}_h(s),\, h \neq i,j,\, s = 1,\dots,N,\, \mathbf{x}\right]$$

for any t_1, $t_2 \in [1, N]$ and for $i, j = 1, \dots, m$, all contained in the sub-matrix $\mathbf{S}$ of $\boldsymbol{\Sigma}_{\mathbf{z}}^{-1}$. However, given the circulant structure of $\mathbf{S}$, without loss of generality, this matrix information can be represented by the vector

$$\left[(S_0)_{ij} \quad (S_1)_{ij} \quad \dots \quad (S_{\frac{N}{2}-1})_{ij} \quad (S_{\frac{N}{2}})_{ij} \quad (S_{\frac{N}{2}-1})_{ji} \quad \dots \quad (S_1)_{ji}\right].$$

Notice that, the graphical model of the process $\mathbf{z}$ is completely determined by the concentration matrix $\boldsymbol{\Sigma}_{\mathbf{z}}^{-1}$, as we have seen.

Example 3. Consider the case in which $N = 2$, $m = 7$, $l = 2$ latent variable, and suppose that the graphical model associated to the vector $\mathbf{z}$ is the one depicted in Figure 6.1.

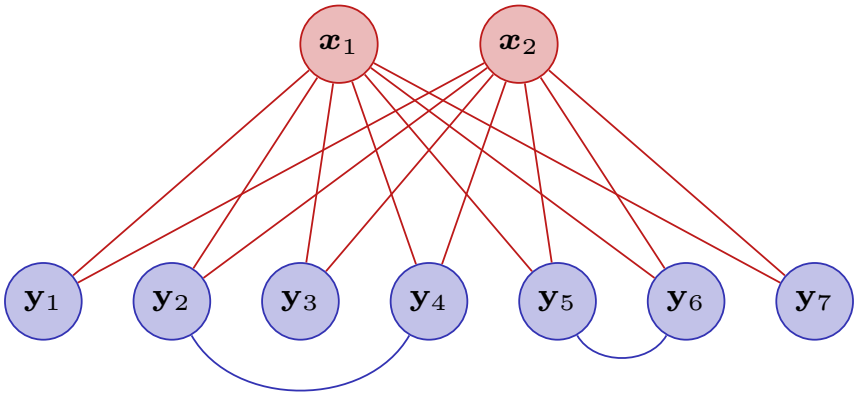

Figure 6.1: Example of a latent-variable graphical model: $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ are the latent-variables and $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_7$ are the manifest variables.

In this case, the concentration matrix of vector $\mathbf{z}$ will have the following structure

$$\boldsymbol{\Sigma}_{\mathbf{z}}^{-1} = \left[\begin{array}{c|c} \mathbf{S} & \mathbf{A} \\ \hline \mathbf{A}^\top & \mathbf{R} \end{array}\right] = \left[\begin{array}{cc|cc} S_0 & S_1^\top & & \\ S_1 & S_0 & \multicolumn{2}{c}{*} \\ \hline & & r_{11} & 0 \\ \multicolumn{2}{c|}{*} & 0 & r_{22} \end{array}\right]$$

where S_0 and S_1 are 7×7 matrices. In particular, $r_{12} = 0$ because we have no edge between $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. Moreover, the presence of the edge between $\mathbf{y}_2$ and $\mathbf{y}_4$ implies that at least one between $(S_0)_{24}$ and $(S_1)_{24}$ is different from zero, while the presence of an edge between $\mathbf{y}_5$ and $\mathbf{y}_6$ means that at least one between $(S_0)_{56}$ and $(S_1)_{56}$ is different from zero. In other words, $\Omega = \{(i,i) : i = 1,\dots,7\} \cup \{(2,4), (5,6)\}$ because the diagonal elements of S_0 and S_1 have to be specified.

Some useful operators

We introduce now the linear operator $\mathsf{C} : \mathcal{M} \to \mathcal{C}$ defined as follows

$$\mathsf{C}(A) = \mathrm{circ}\{[A_0 \quad \dots \quad A_{N-1}]\} = \begin{bmatrix} A_0 & A_1^\top & \cdots & A_{\frac{N}{2}}^\top & \cdots & A_1 \\ A_1 & A_0 & \cdots & \cdots & \ddots & \vdots \\ \vdots & A_1 & \cdots & \cdots & \cdots & A_{\frac{N}{2}}^\top \\ A_{\frac{N}{2}} & \cdots & \cdots & A_0 & \cdots & \vdots \\ \vdots & \ddots & & & \ddots & A_1^\top \\ A_1^\top & \cdots & A_{\frac{N}{2}} & \cdots & A_1 & A_0 \end{bmatrix} = \sum_{k=0}^{N-1} U^{-k} \otimes A_k,$$

where we recall that $A_0 = A_0^\top$ and $A_k = A_{N-k}^\top$ for $k > N/2$. Notice that, by defining the matrix $I_{\mathcal{M}} \in \mathcal{M}$ as

$$I_{\mathcal{M}} := [I_m \quad 0 \quad \dots \quad 0],$$

we have that $\mathsf{C}(I_{\mathcal{M}}) = I_{mN}$ where we recall that

$$I_{mN} = \begin{bmatrix} I_m & 0 & \cdots & 0 & 0 \\ 0 & I_m & 0 & \cdots & 0 \\ \vdots & \cdots & I_m & \cdots & \vdots \\ \vdots & \ddots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & I_m \end{bmatrix}$$

is the $mN \times mN$ block identity matrix. Moreover, using Theorem 1 and Lemma 1 we show that the inner product in $\mathcal{C}$ is equivalent (up to a constant) to the inner product in $\mathcal{M}$ as stated by the following Proposition.

Proposition 9. Consider two symmetric sequences $A, B \in \mathcal{M}$ and let

$$A(\zeta) = \sum_{k=0}^{N-1} A_k \, \zeta^{-k}, \qquad A_k = A_{N-k}^\top \text{ for } k > \frac{N}{2},$$
$$B(\zeta) = \sum_{k=0}^{N-1} B_k \, \zeta^{-k}, \qquad B_k = B_{N-k}^\top \text{ for } k > \frac{N}{2},$$

the corresponding discrete-time Fourier transforms, i.e. their symbols evaluated at $\zeta \in \mathbb{T}_N$. Let $\mathbf{A} := \mathsf{C}(A)$ and $\mathbf{B} := \mathsf{C}(B)$ be the corresponding symmetric, block-circulant matrices. Then,

$$\langle \mathbf{A}, \mathbf{B} \rangle_{\mathcal{C}} = N \, \langle A, B \rangle_{\mathcal{M}} .$$

Proof. By Theorem 1, we have

$$\langle A, \, B \rangle_{\mathcal{M}} = \mathrm{tr} \sum_{k=0}^{N-1} A_k B_k^\top = \frac{1}{N} \, \mathrm{tr} \sum_{j=0}^{N-1} A(\zeta^j) B(\zeta^j),$$

and therefore

$$\langle A,\, B\rangle_{\mathcal{M}} = \frac{1}{N}\ \mathrm{tr}\left(\begin{bmatrix} A(\zeta^0) & & & \\ & A(\zeta^1) & & \\ & & \ddots & \\ & & & A(\zeta^{N-1}) \end{bmatrix} \begin{bmatrix} B(\zeta^0) & & & \\ & B(\zeta^1) & & \\ & & \ddots & \\ & & & B(\zeta^{N-1}) \end{bmatrix}\right)$$

$$= \frac{1}{N}\ \mathrm{tr}\left(\mathbf{F}^* \begin{bmatrix} A(\zeta^0) & & & \\ & A(\zeta^1) & & \\ & & \ddots & \\ & & & A(\zeta^{N-1}) \end{bmatrix} \mathbf{F}\mathbf{F}^* \begin{bmatrix} B(\zeta^0) & & & \\ & B(\zeta^1) & & \\ & & \ddots & \\ & & & B(\zeta^{N-1}) \end{bmatrix} \mathbf{F}\right).$$

Finally, by Lemma 1

$$\langle A,\, B\rangle_{\mathcal{M}} = \frac{1}{N}\ \mathrm{tr}(\mathbf{A}\,\mathbf{B}) = \frac{1}{N}\ \langle \mathbf{A}, \mathbf{B}\rangle_{\mathcal{C}}\,.$$

□

The previous Proposition allows the straightforward computation of the adjoint operator of C. We know that

$$\langle \mathsf{C}(A), \mathbf{B}\rangle_{\mathcal{C}} = N\,\langle A, B\rangle_{\mathcal{M}}\,.$$

On the other hand, let $\mathsf{C}^* : \mathcal{C} \to \mathcal{M}$ be the adjoint operator of C so that

$$\langle \mathsf{C}(A), \mathbf{B}\rangle_{\mathcal{C}} = \langle A, \mathsf{C}^*(\mathbf{B})\rangle_{\mathcal{M}}\,, \qquad \forall\, A \in \mathcal{M},\, \mathbf{B} \in \mathcal{C}.$$

By simply comparing the previous two expressions, we conclude that

$$\mathsf{C}^*(\mathbf{B}) = N\,B, \qquad \forall\, B \in \mathcal{M}. \tag{6.7}$$

It is useful to introduce also the projection operator $\mathsf{P}_{\mathscr{B}} : \mathcal{M} \to \mathcal{M}_{\mathscr{B}}$, defined for a generic symmetric sequence $A \in \mathcal{M}$ as

$$\mathsf{P}_{\mathscr{B}}(A) := [\mathsf{P}_{\mathscr{B}}(A_0)\ \mathsf{P}_{\mathscr{B}}(A_1)\ \dots\ \mathsf{P}_{\mathscr{B}}(A_{N-1})]$$

where for the generic matrix $A_k \in \mathbb{R}^{m\times m}$, $k = 0, \dots, N-1$,

$$\mathsf{P}_{\mathscr{B}}(A_k) := \begin{cases} A_k, & \text{for } k \in \mathscr{B} \\ 0, & \text{otherwise} \end{cases}$$

and $\mathscr{B} := [0,\, n] \cup [N-n,\, N-1]$ is the subset of $[0,\, N-1]$ containing the indexes of the non-zero blocks in the banded structure. In particular, we have

$$\begin{aligned} \mathsf{P}_{\mathscr{B}}(A) &= \mathsf{P}_{\mathscr{B}}([A_0\ \dots\ A_n\ A_{n+1}\ \dots\ A_{\frac{N}{2}}\ \dots\ A_{n+1}^\top\ A_n^\top\ \dots\ A_1^\top]) \\ &= [A_0\ \dots\ A_n\ 0\ \dots\ 0\ \dots\ 0\ A_n^\top\ \dots\ A_1^\top], \end{aligned}$$

that is, $\mathsf{P}_{\mathscr{B}}(A)$ maps $A \in \mathcal{M}$ onto its corresponding banded sequence in $\mathcal{M}_{\mathscr{B}}$. Accordingly, we denote with $\mathsf{P}_{\mathscr{B}}(\mathbf{A})$ the symmetric, block-circulant matrix associated to the projection $\mathsf{P}_{\mathscr{B}}(A)$, that is $\mathsf{P}_{\mathscr{B}}(\mathbf{A}) := \mathsf{C}(\mathsf{P}_{\mathscr{B}}(A))$. Finally, we observe that $\mathsf{P}_{\mathscr{B}}$ is a self-adjoint operator, namely

$$\langle \mathsf{P}_{\mathscr{B}}(A), B\rangle_{\mathcal{M}} = \langle A, \mathsf{P}_{\mathscr{B}}(B)\rangle_{\mathcal{M}}$$

for any $A, B \in \mathcal{M}$.

6.2 The sparse plus low-rank block-circulant band extension problem

In this Section, Problem 2 is rephrased in terms of a matrix extension problem for block-circulant covariance matrices, i.e. the classical *block-circulant band extension problem*. The latter is a generalization of the covariance extension Problem 1 that takes into account the symmetric, block-circulant structure of the covariance matrix of the process $\mathbf{y}$ and the fact that its inverse must be symmetric, block-circulant, banded of bandwidth n, being $\mathbf{y}$ a periodic, stationary reciprocal process of order n. Once we have done that, we will see that the optimization problem for the estimation of the sparse and the low-rank component of the inverse covariance is a straightforward re-parametrization of the block-circulant band extension dual problem plus some regularization terms introduced in order to enforce sparsity and low-rank in the respective matrices.

6.2.1 The block-circulant band extension problem

The derivation of the problem basically follows the one in [12], Section IV. Consider the process $\mathbf{y}$ as before and suppose that a finite length realization of the process in one period is available $y(1), \dots, y(N)$ so that we estimate the covariance lags $C_0, C_1, \dots, C_n$ as

$$C_k = \frac{1}{N} \sum_{t=k}^{N} y(t)\, y(t-k)^\top, \qquad k = 0, 1, \dots, n,$$

where now n is the order of the reciprocal process $\mathbf{y}$. By exploiting maximum likelihood arguments one can prove that the identification Problem 2 is equivalent to the following matrix completion problem, see [12].

Problem 3 (Block-circulant band extension problem)**.** Given $n+1$ initial data $m \times m$ matrices $C_0, C_1, \dots, C_n$, complete them with a sequence $\Sigma_{n+1}, \Sigma_{n+2}, \dots, \Sigma_{N-1}$, in such a way to form a symmetric, positive definite block-circulant matrix $\mathbf{\Sigma_y}$ with a block-circulant banded inverse of bandwidth n (6.1).

According to Proposition 5, the solution of block-circulant band extension problem boils down in the maximization of the differential entropy (4.2), i.e. in a constrained optimization problem, very similar to the one we have consider in Section 4.1.1. Formally, let $C \in \mathcal{M}_{\mathcal{B}}$ defined as

$$C := [C_0\ C_1\ \dots\ C_n\ 0\ \dots\ 0\ C_n^\top\ \dots\ C_1^\top],$$

and let $\Sigma \in \mathcal{M}$,

$$\Sigma := [\Sigma_0\ \Sigma_1\ \dots\ \Sigma_n\ \Sigma_{n+1}\ \dots\ \Sigma_{\frac{N}{2}}\ \dots\ \Sigma_{n+1}^\top\ \Sigma_n^\top\ \dots\ \Sigma_1^\top],$$

such that $\mathsf{C}(\Sigma) = \mathbf{\Sigma_y}$. Consider now the following Gaussian maximum entropy problem on the discrete circle $\mathbb{Z}_N$ which is equivalent to the one considered in [12] (see also [13] where this problem is retrieved using Burg's technique for maximum entropy spectral

estimation):

$$\begin{aligned} \underset{\Sigma \in \mathcal{M}}{\text{argmax}} \quad & \log\det \mathsf{C}(\Sigma) \\ \text{subject to} \quad & \mathsf{C}(\Sigma) > 0 \\ & \mathsf{P}_{\mathscr{B}^c}(\Sigma - C) = 0. \end{aligned} \tag{6.8}$$

Recalling the definition of the operator C and the definition of the space $\mathcal{M}$, saying that $\Sigma \in \mathcal{M}$ and $\mathsf{C}(\Sigma) > 0$ is equivalent to restrict our attention to the symmetric, block-circulant matrices $\mathsf{C}(\Sigma)$. Moreover, by linearity of $\mathsf{P}_{\mathscr{B}}$,

$$\mathsf{P}_{\mathscr{B}}(\Sigma - C) = [\Sigma_0 - C_0 \ \Sigma_1 - C_1 \ \dots \ \Sigma_n - C_n \ 0 \ \dots \ 0 \ \dots \ 0 \ \Sigma_n^\top - C_n^\top \ \dots \ \Sigma_1^\top - C_1^\top],$$

thus the constraint $\mathsf{P}_{\mathscr{B}}(\Sigma - C) = 0$ means only that the covariance lags $\Sigma_0, \Sigma_1, \dots, \Sigma_n$ have to match the estimated ones $C_0, C_1, \dots, C_n$. Notice that we are not imposing that the inverse of the solution of (6.8) should have a banded structure: whenever the solution exists, this property will be automatically guaranteed [12]. Concerning the feasibility of Problem (6.8), here we say only that having the matrix

$$\mathbf{T}_n := \begin{bmatrix} C_0 & C_1^\top & \cdots & C_n^\top \\ C_1 & C_0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & C_1^\top \\ C_n & \cdots & \cdots & C_0 \end{bmatrix}$$

positive definite is a *necessary condition* for the existence of Σ solving problem (6.8). In general it turns out that, under such necessary condition, feasibility holds for N large enough, see Theorem 5.1. in [12]. In order to solve problem (6.8) we follow the same line we have followed in Section 4.1.1 for Dempster's problem, i.e. we exploit Lagrange duality theory. The Lagrangian function for the problem is

$$\mathcal{L}(\Sigma, \bar{X}) = \log\det \mathsf{C}(\Sigma) - \left\langle \mathsf{P}_{\mathscr{B}}(\bar{X}), \Sigma \right\rangle_{\mathcal{M}} + \left\langle \mathsf{P}_{\mathscr{B}}(\bar{X}), C \right\rangle_{\mathcal{M}}$$

where $\bar{X} \in \mathcal{M}$ is the Lagrange multiplier. Defining $X \in \mathcal{M}$ as

$$X := \frac{1}{N} \cdot \mathsf{P}_{\mathscr{B}}(\bar{X}) = \frac{1}{N} \cdot [\bar{X}_0 \ \bar{X}_1 \ \dots \ \bar{X}_n \ 0 \ \dots \ 0 \ \dots \ 0 \ \bar{X}_n^\top \ \dots \ \bar{X}_1^\top],$$

such that $\mathbf{X} = \mathsf{C}(X) > 0$, we can re-parametrize the Lagrangian as

$$\mathcal{L}(\Sigma, X) = \log\det \mathsf{C}(\Sigma) - N \langle X, \Sigma \rangle_{\mathcal{M}} + N \langle X, C \rangle_{\mathcal{M}}.$$

By Proposition 9 we can now rewrite the Lagrangian with respect to the corresponding circulant matrices $\mathbf{X} = \mathsf{C}(X)$, $\mathbf{\Sigma} = \mathsf{C}(\Sigma)$ and $\mathbf{C} = \mathsf{C}(C)$

$$\mathcal{L}(\mathbf{\Sigma}, \mathbf{X}) = \log\det \mathbf{\Sigma} - \langle \mathbf{X}, \mathbf{\Sigma} \rangle_{\mathcal{C}} + \langle \mathbf{X}, \mathbf{C} \rangle_{\mathcal{C}}$$

where $\mathbf{C} \in \mathcal{B}$ is the symmetric, banded block-circulant matrix of bandwidth n

$$\mathbf{C} = \begin{bmatrix} C_0 & C_1^\top & \cdots & C_n^\top & 0 & \cdots & 0 & C_n & \cdots & C_1 \\ C_1 & C_0 & \cdots & \cdots & C_n^\top & 0 & \cdots & 0 & \ddots & \vdots \\ \vdots & & \ddots & & & \ddots & \ddots & & \ddots & C_n \\ C_n & & & \ddots & & & \ddots & 0 & & 0 \\ 0 & \ddots & & & C_0 & & & C_n^\top & \ddots & \vdots \\ \vdots & \ddots & \ddots & & & \ddots & & & \ddots & 0 \\ 0 & & 0 & C_n & & & \ddots & & & C_n^\top \\ C_n^\top & \ddots & & \ddots & \ddots & & & \ddots & & \vdots \\ \vdots & \ddots & \ddots & & \ddots & \ddots & & & \ddots & C_1^\top \\ C_1^\top & \cdots & C_n^\top & 0 & \cdots & 0 & C_n & \cdots & C_1 & C_0 \end{bmatrix},$$

containing the covariance lags estimated from the data, and $\mathbf{X} \in \mathcal{B}$ has the same structure.

Remark 9. Notice that $\mathbf{T}_n$ is a *principal sub-matrix* of matrix $\mathbf{C}$, thus a necessary condition for $\mathbf{T}_n$ to be positive definite, i.e. for the feasibility of problem (6.8), is that $\mathbf{C}$ is positive definite.

The minimum of the Lagrangian over the cone $\mathcal{C}^+$ is given by setting its directional derivative $\mathcal{L}'$ in direction $\boldsymbol{\delta\Sigma}$ equal to zero, for any direction $\boldsymbol{\delta\Sigma} \in \mathcal{C}$ namely,

$$\mathcal{L}'(\boldsymbol{\Sigma}, \mathbf{X}; \boldsymbol{\delta\Sigma}) = \operatorname{tr}(\boldsymbol{\Sigma}^{-1}\boldsymbol{\delta\Sigma}) - \operatorname{tr}(\mathbf{X}\,\boldsymbol{\delta\Sigma}) = 0, \qquad \forall\, \boldsymbol{\delta\Sigma} \in \mathcal{C}. \tag{6.9}$$

Condition (6.9) must be satisfied in particular when

$$\boldsymbol{\delta\Sigma} = \boldsymbol{\Sigma}^{-1} - \mathbf{X} \in \mathcal{C},$$

hence it holds if and only if $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_o := \mathbf{X}^{-1}$. Substituting this value in the Lagrangian we get

$$\mathcal{L}(\boldsymbol{\Sigma}_o, \mathbf{X}, \bar{X}) = \log\det \mathbf{X}^{-1} + \langle \mathbf{X},\, \mathbf{C}\rangle_{\mathcal{C}} + \left\langle \mathbf{X},\, \mathbf{X}^{-1} \right\rangle_{\mathcal{C}}.$$

Since the last term is constant with respect to $\mathbf{X}$, the dual problem of Problem (6.8) can be formulated as follows

$$\begin{aligned} \underset{\mathbf{X}\in\mathcal{C}}{\operatorname{argmin}} \quad & -\log\det \mathbf{X} + \langle \mathbf{X},\, \mathbf{C}\rangle_{\mathcal{C}} \\ \text{subject to} \quad & \mathbf{X} > 0 \\ & \mathbf{X} \in \mathcal{B}. \end{aligned} \tag{6.10}$$

In the next Section we will solve a regularized version of this dual problem in which $\mathbf{X}$ is understood as the sparse plus low-rank decomposition (6.2) of the concentration matrix of the process $\mathbf{y}$.

6.2.2 Sparse plus low-rank optimization problem

In this Section we set-up and solve an optimization problem for the estimation of the sparse plus low-rank decomposition of the concentration matrix $\boldsymbol{\Sigma}_\mathbf{y}$ starting from Problem (6.10)

retrieved in the previous Section. Concerning the non-reciprocal and non-periodic case, a similar identification procedure has been developed, see [7] and [9].

Regularizers. In order to induce sparsity on $S \in \mathcal{M}$ and $\mathbf{S} \in \mathcal{C}$ respectively, inspired by [6] (see also [7]), we propose the following regularizer

$$h_\infty(S) = \sum_{k>h} \max\left\{ |(S_0)_{hk}|,\, 2 \max_{j=1,\dots,\frac{N}{2}} |(S_j)_{hk}|,\, 2 \max_{j=1,\dots,\frac{N}{2}} |(S_j)_{kh}| \right\},$$
$$h_\infty(\mathbf{S}) = \sum_{k>h} \max\left\{ |(S_0)_{hk}|,\, 2 \max_{j=1,\dots,\frac{N}{2}} |(S_j)_{hk}|,\, 2 \max_{j=1,\dots,\frac{N}{2}} |(S_j)_{kh}| \right\}.$$

Basically, this is a generalization of the ℓ^∞-norm used to induce sparsity on vectors. Concerning the low-rank regolarization it is well known that for a symmetric, positive semi-definite matrix $\mathbf{L}$, the trace is the convex envelope of the rank function. For this reason we will use the trace function as low-rank regularizer. See Section 3.5 for a motivation for the choice of these regularizers.

The following optimization problem for the estimation of the sparse plus low-rank components of the concentration matrix $\mathbf{\Sigma}_\mathbf{y}^{-1}$ now directly follows from (6.10) by setting $\mathbf{\Sigma}_\mathbf{y}^{-1} = \mathbf{S} - \mathbf{L}$, with $\mathbf{L} \geq 0$, and by adding the regularizers just introduced:

$$\begin{aligned} \underset{\mathbf{S},\mathbf{L}\in\mathcal{C}}{\operatorname{argmin}} \quad & -\log\det(\mathbf{S}-\mathbf{L}) + \operatorname{tr}(\mathbf{C}(\mathbf{S}-\mathbf{L})) + \lambda_S\, h_\infty(\mathbf{S}) + \lambda_L \operatorname{tr}(\mathbf{L}) \\ \text{subject to} \quad & \mathbf{S}-\mathbf{L} > 0, \quad \mathbf{L} \geq 0 \\ & \mathbf{S}-\mathbf{L} \in \mathcal{B} \end{aligned} \tag{6.11}$$

where λ_L, $\lambda_S > 0$ are two regularization parameters. Since $\mathbf{S}, \mathbf{L} \in \mathcal{C}$, we can rewrite them as

$$\mathbf{S} = \sum_{k=0}^{N-1} U^{-k} \otimes S_k, \qquad S_k = S_{N-k}^\top \quad \text{for} \quad k > \frac{N}{2},$$
$$\mathbf{L} = \sum_{k=0}^{N-1} U^{-k} \otimes L_k, \qquad L_k = L_{N-k}^\top \quad \text{for} \quad k > \frac{N}{2},$$

therefore we can re-parametrize the problem in terms of the symmetric sequences $S, L \in \mathcal{M}$,

$$S = [S_0 \quad \dots \quad S_{N-1}], \qquad L = [L_0 \quad \dots \quad L_{N-1}],$$

in order to have a smaller number of parameters involved in the optimization problem. Indeed, in this way we are considering only the first row of the symmetric, block-circulant matrices involved in the problem instead of the complete matrices and this means that we have less parameters to be optimized. Introducing the new variable $\mathbf{X} := \mathbf{S} - \mathbf{L}$, and the associated sequence $X = S - L \in \mathcal{M}$, problem (6.11) becomes

$$\begin{aligned} \underset{X,L\in\mathcal{M}}{\operatorname{argmin}} \quad & -\log\det(\mathsf{C}(X)) + \langle \mathbf{C}, \mathsf{C}(X)\rangle_\mathcal{C} + \lambda_S\, h_\infty(X+L) + \lambda_L \operatorname{tr}(\mathsf{C}(L)) \\ \text{subject to} \quad & \mathsf{C}(X) > 0, \quad \mathsf{C}(L) \geq 0 \\ & \mathsf{P}_{\mathscr{B}^c}(X) = 0 \end{aligned} \tag{6.12}$$

which is equivalent to the problem

$$\underset{X,L,Y\in\mathcal{M}}{\operatorname{argmin}} \quad -\log\det(\mathsf{C}(X)) + \langle \mathbf{C},\, \mathsf{C}(X)\rangle_{\mathcal{C}} + \lambda_S\, h_\infty(Y) + \lambda_L \operatorname{tr}(\mathsf{C}(L)) \tag{6.13}$$

$$\text{subject to} \quad \mathsf{C}(X) > 0, \quad \mathsf{C}(L) \geq 0 \tag{6.14}$$

$$\mathsf{P}_{\mathscr{B}^c}(X) = 0, \quad Y = X + L. \tag{6.15}$$

Now we show that problem (6.12) admits a solution by exploiting duality theory.

Remark 10. Notice that, since the objective function (6.13) is convex and the operators C and $\mathsf{P}_{\mathscr{B}}$ are linear, Problem (6.12) is a convex optimization problem. Moreover, Problem (6.12) is strictly feasible, for instance, pick $X = I_{\mathcal{M}}$ and $L = I_{\mathcal{M}}$. Hence, Slater's condition holds and the duality gap between (6.12) and its dual is equal to zero (see Chapter 3) namely, we have strong duality for (6.12) and its dual. As a consequence, we can try to solve the primal problem (6.12) by solving its dual.

Dual problem solution

We address the previous constrained optimization problem using the Lagrange multipliers theory.

1. The Lagrangian function for this problem is

$$\begin{aligned}\mathcal{L}(X, L, Y, \mathbf{V}, \bar{Z}, \bar{W}) = &-\log\det(\mathsf{C}(X)) + \langle \mathbf{C},\, \mathsf{C}(X)\rangle_{\mathcal{C}} + \lambda_S\, h_\infty(Y) + \lambda_L \operatorname{tr}(\mathsf{C}(L)) \\ &- \langle \mathbf{V},\, \mathsf{C}(L)\rangle_{\mathcal{C}} + \left\langle \bar{Z}, X + L - Y \right\rangle_{\mathcal{M}} + \left\langle \bar{W},\, \mathsf{P}_{\mathscr{B}^c}(X) \right\rangle_{\mathcal{M}}\end{aligned}$$

where $\mathbf{V} \in \mathcal{C}$, $\mathbf{V} \geq 0$, while $\bar{Z}, \bar{W} \in \mathcal{M}$. Exploiting the fact that $\mathsf{P}_{\mathscr{B}}$ is a self-adjoint operator we can equivalently parametrize the Lagrangian function in terms of the multipliers $W := 1/N \cdot \mathsf{P}_{\mathscr{B}^c}(\bar{W})$ and $Z = 1/N \cdot \bar{Z}$. The resulting Lagrangian is

$$\begin{aligned}\mathcal{L}(X, L, Y, \mathbf{V}, Z, W) = &-\log\det(\mathsf{C}(X)) + \langle \mathbf{C},\, \mathsf{C}(X)\rangle_{\mathcal{C}} + \lambda_S\, h_\infty(Y) + \lambda_L \operatorname{tr}(\mathsf{C}(L)) \\ &- \langle \mathbf{V},\, \mathsf{C}(L)\rangle_{\mathcal{C}} + N\,\langle Z, X + L - Y\rangle_{\mathcal{M}} + N\,\langle W,\, X\rangle_{\mathcal{M}}\,.\end{aligned}$$

Remark 11. Obviously $W \in \mathcal{M}$ but we highlight the fact that it is a *sparse* matrix, indeed, explicitly we have

$$W = \frac{1}{N} \cdot \mathsf{P}_{\mathscr{B}^c}(\bar{W}) = \frac{1}{N} \cdot [0 \ \ldots \ 0 \ \bar{W}_{n+1} \ \ldots \ \bar{W}_{\frac{N}{2}} \ \ldots \ \bar{W}_{n+1}^{\top} \ 0 \ \ldots \ 0].$$

Thus, by definition of C, the corresponding symmetric, block-circulant matrix $\mathbf{W} := \mathsf{C}(W)$ has complementary support with respect to the one of a banded matrix with bandwidth n, i.e. $\mathbf{W} \in \bar{\mathcal{B}}$.

We can rearrange the Lagrangian function as follows in order to highlight that it depends separately from X, Y and L

$$\begin{aligned}\mathcal{L}(X, L, Y, \mathbf{V}, Z, W) &= -\log\det(\mathsf{C}(X)) + \langle \mathbf{C},\, \mathsf{C}(X)\rangle_{\mathcal{C}} + \lambda_S\, h_\infty(Y) + \langle \lambda_L I_{mN}, \mathsf{C}(L)\rangle_{\mathcal{C}} \\ &\quad - \langle \mathbf{V},\, \mathsf{C}(L)\rangle_{\mathcal{C}} + N\,\langle Z, X + L - Y\rangle_{\mathcal{M}} + N\,\langle W,\, X\rangle_{\mathcal{M}} \\ &= -\log\det(\mathsf{C}(X)) + \langle \mathbf{C},\, \mathsf{C}(X)\rangle_{\mathcal{C}} + N\,\langle Z,\, X\rangle_{\mathcal{M}} + N\,\langle W,\, X\rangle_{\mathcal{M}} \\ &\quad + \langle \lambda_L I_{mN} - \mathbf{V},\, \mathsf{C}(L)\rangle_{\mathcal{C}} + N\,\langle Z,\, L\rangle_{\mathcal{M}} + \lambda_S\, h_\infty(Y) - N\,\langle Z,\, Y\rangle_{\mathcal{M}} \\ &= -\log\det(\mathsf{C}(X)) + \langle \mathbf{C},\, \mathsf{C}(X)\rangle_{\mathcal{C}} + N\,\langle Z + W,\, X\rangle_{\mathcal{M}} \\ &\quad + \langle \mathsf{C}^*(\lambda_L I_{mN} - \mathbf{V}) + NZ,\, L\rangle_{\mathcal{M}} + \lambda_S\, h_\infty(Y) - N\,\langle Z,\, Y\rangle_{\mathcal{M}}\,.\end{aligned}$$

2. The second step is the unconstrained minimization of $\mathcal{L}$. The only term that depends on Y is $\lambda_S\, h_\infty(Y) - N\,\langle Z,\, Y\rangle_{\mathcal{M}}$ which is bounded below if and only if

$$\operatorname{diag}(Z_j) = 0, \quad j = 0,\, \dots,\, N/2 \tag{6.16}$$

$$2\,|(Z_0)_{kh}| + \frac{1}{2}|(Z_{\frac{N}{2}})_{kh}| + \frac{1}{2}|(Z_{\frac{N}{2}})_{hk}| + \sum_{j=1}^{N/2-1} |(Z_j)_{kh}| + |(Z_j)_{hk}| \leq \frac{\lambda_S}{N}, \quad k > h \tag{6.17}$$

in which case the infimum is zero. To see this, recall that

$$\langle Z, Y\rangle_{\mathcal{M}} = \operatorname{tr}(Z_0 Y_0) + \operatorname{tr}(Z_{\frac{N}{2}} Y_{\frac{N}{2}}^\top) + 2\sum_{j=1}^{N/2-1} \operatorname{tr}(Z_j Y_j^\top) \tag{6.18}$$

where each term is given by the usual explicit expression of the trace of the product of two matrices

$$\operatorname{tr}(Z_j Y_j^\top) = \sum_{h=1}^{m} (Z_j)_{hh}(Y_j)_{hh} + \sum_{k>h} (Z_j)_{kh}(Y_j)_{kh} + (Z_j)_{hk}(Y_j)_{hk}. \tag{6.19}$$

By substituting (6.19) in (6.18), under condition (6.16), we get

$$\begin{aligned}\langle Z,\, Y\rangle_{\mathcal{M}} = \sum_{k>h} \Big[& 2\,(Z_0)_{kh}(Y_0)_{kh} + (Z_{\frac{N}{2}})_{kh}(Y_{\frac{N}{2}})_{kh} + (Z_{\frac{N}{2}})_{hk}(Y_{\frac{N}{2}})_{hk} \\ & + 2\sum_{j=1}^{N/2-1} (Z_j)_{kh}(Y_j)_{kh} + (Z_j)_{hk}(Y_j)_{hk} \Big]\end{aligned} \tag{6.20}$$

where we have exploited the fact that h_∞ does not depend by the diagonal entries of Y and that $\langle Z,\, Y\rangle_{\mathcal{M}}$ is bounded below with respect to the diagonal entries of Y if and only if (6.16) holds true, in which case the minimum with respect to such entries is precisely (6.20). Now it remains minimize over the off-diagonal entries of Y. By substituting to each element in the summation its absolute value we obtain

$$\begin{aligned}\langle Z,\, Y\rangle_{\mathcal{M}} \leq \sum_{k>h} \Big[& \Big(2\,|(Z_0)_{kh}| + \frac{1}{2}|(Z_{\frac{N}{2}})_{kh}| + \frac{1}{2}|(Z_{\frac{N}{2}})_{hk}|\Big) \max\Big\{|(Y_0)_{hk},\, 2|(Y_{\frac{N}{2}})_{kh}|,\, 2|(Y_{\frac{N}{2}})_{hk}|\Big\} \\ & + \max_{j=1,\dots,\frac{N}{2}-1} \{\, 2|(Y_j)_{kh}|,\, 2|(Y_j)_{hk}|\} \sum_{j=1}^{N/2-1} |(Z_j)_{kh}| + |(Z_j)_{hk}|\Big] \\ & \leq \sum_{k>h} \max\left\{|(Y_0)_{hk}|,\, 2\max_{j=1,\dots,\frac{N}{2}} |(Y_j)_{kh}|,\, 2\max_{j=1,\dots,\frac{N}{2}} |(Y_j)_{hk}|\right\} \cdot \\ & \left[2\,|(Z_0)_{kh}| + \frac{1}{2}|(Z_{\frac{N}{2}})_{kh}| + \frac{1}{2}|(Z_{\frac{N}{2}})_{hk}| + \sum_{j=1}^{N/2-1} |(Z_j)_{kh}| + |(Z_j)_{hk}|\right].\end{aligned}$$

Finally, we have that

$$\begin{aligned}\lambda_S\, h_\infty(Y) - N\,\langle Z,\, Y\rangle_{\mathcal{M}} \geq \sum_{k>h} & \max\left\{|(Y_0)_{kh}|,\, 2\max_{j=1,\dots,\frac{N}{2}} |(Y_j)_{kh}|,\, 2\max_{j=1,\dots,\frac{N}{2}} |(Y_j)_{hk}|\right\} \cdot \\ & \left[\lambda_S - N\Big(2|(Z_0)_{kh}| + \frac{1}{2}|(Z_{\frac{N}{2}})_{kh}| + \frac{1}{2}|(Z_{\frac{N}{2}})_{hk}| + \sum_{j=1}^{N/2-1} |(Z_j)_{kh}| + |(Z_j)_{hk}|\Big)\right].\end{aligned}$$

It is immediate to see that the minimization over the off-diagonal entries of Y is bounded below only if (6.17) holds, in that case the minimum is zero. Hence

$$\inf_{Y} \mathcal{L} = \begin{cases} -\log\det(\mathsf{C}(X)) + \langle \mathbf{C},\, \mathsf{C}(X)\rangle_{\mathcal{C}} + N\,\langle Z+W,\, X\rangle_{\mathcal{M}} & \text{if} \quad (6.16),\ (6.17) \text{ hold,} \\ + \langle \mathsf{C}^*(\lambda_L I_{mN} - \mathbf{V}) + NZ,\, L\rangle_{\mathcal{M}} & \\ -\infty \quad \text{otherwise.} & \end{cases}$$

The only term that depends on L is $\langle \mathsf{C}^*(\lambda_L I_{mN} - \mathbf{V}) + NZ,\, L\rangle_{\mathcal{M}}$, which is linear in L, and therefore it is bounded below only if it is identically zero, i.e.

$$\mathsf{C}^*(\lambda_L I_{mN} - \mathbf{V}) + NZ = 0. \tag{6.21}$$

In this case, clearly, the minimum is zero. Accordingly,

$$\inf_{Y,L} \mathcal{L} = \begin{cases} -\log\det(\mathsf{C}(X)) + \langle \mathbf{C},\, \mathsf{C}(X)\rangle_{\mathcal{C}} + N\,\langle Z+W, X\rangle_{\mathcal{M}} & \text{if} \quad (6.16),\ (6.17),\ (6.21) \text{ hold,} \\ -\infty \quad \text{otherwise} & \end{cases}$$

The remaining terms only depend on X. According to Proposition 9, if (6.16), (6.17), (6.21) hold we can rewrite the infimum of the Lagrangian function as

$$\bar{\mathcal{L}}(\mathbf{X}) := \inf_{Y,L} \mathcal{L} = -\log\det(\mathbf{X}) + \langle \mathbf{C},\, \mathbf{X}\rangle_{\mathcal{C}} + \langle \mathbf{Z}+\mathbf{W}, \mathbf{X}\rangle_{\mathcal{C}}\,,$$

where $\mathbf{X} = \mathsf{C}(X)$, $\mathbf{Z} = \mathsf{C}(Z)$ and $\mathbf{W} = \mathsf{C}(W)$, according to the definition of the operator C. The function $\bar{\mathcal{L}}$ is strictly convex over the cone of the symmetric, positive definite, block-circulant matrices $\mathcal{C}^+$, in fact it is the sum of a strictly convex function and a linear function of $\mathbf{X}$. Assuming a priori that the solution lies in the interior of the cone (we have to check that a posteriori, i.e. once we have found the solution), a necessary and sufficient condition for $\mathbf{X}_o$ to be a minimum point for $\bar{\mathcal{L}}(\mathbf{X})$ is that its first Gateaux derivative $\bar{\mathcal{L}}'$ computed at $\mathbf{X} = \mathbf{X}_o$ is equal to zero in every direction $\boldsymbol{\delta}\mathbf{X}$ namely,

$$\bar{\mathcal{L}}'(\mathbf{X}_o;\, \boldsymbol{\delta}\mathbf{X}) = 0, \qquad \forall\, \boldsymbol{\delta}\mathbf{X} \in \mathcal{C}.$$

Remark 12. Notice that we do not require that $\boldsymbol{\delta}\mathbf{X}$ is positive definite, it is sufficient that it is symmetric, block-circulant. In fact, any point $\mathbf{X}_o$ in the interior of the cone has a whole neighborhood in the interior of the cone and thus $\mathbf{X}_o + \varepsilon\boldsymbol{\delta}\mathbf{X}$ belongs to the interior of the cone, for ε sufficiently small.

The Gateaux derivative of $\bar{\mathcal{L}}$ at $\mathbf{X}_o$ in direction $\boldsymbol{\delta}\mathbf{X}$ is

$$\begin{aligned} \bar{\mathcal{L}}'(\mathbf{X}_o;\, \boldsymbol{\delta}\mathbf{X}) &= -\operatorname{tr}(\mathbf{X}_o^{-1}\,\boldsymbol{\delta}\mathbf{X}) + \langle \mathbf{C},\, \boldsymbol{\delta}\mathbf{X}\rangle_{\mathcal{C}} + \langle \mathbf{Z}+\mathbf{W}, \boldsymbol{\delta}\mathbf{X}\rangle_{\mathcal{C}} \\ &= \operatorname{tr}\left[\left(-\mathbf{X}_o^{-1} + \mathbf{C} + \mathbf{Z} + \mathbf{W}\right)\boldsymbol{\delta}\mathbf{X}\right]. \end{aligned}$$

Assuming that $\mathbf{Z} \in \mathcal{C}$ and $\mathbf{W} \in \bar{\mathcal{B}}$, are s.t.

$$\mathbf{C} + \mathbf{Z} + \mathbf{W} > 0, \tag{6.22}$$

the directional derivative of $\bar{\mathcal{L}}$ at $\mathbf{X}_o$ is equal to zero in any direction $\boldsymbol{\delta}\mathbf{X} \in \mathcal{C}$ if and only if

$$\mathbf{X}_o = \mathsf{C}(X_o) = (\mathbf{C} + \mathbf{Z} + \mathbf{W})^{-1}\,. \tag{6.23}$$

Finally, we have that

$$\inf_{Y,L,X} \mathcal{L} = \begin{cases} \log\det\left(\mathbf{C}+\mathbf{Z}+\mathbf{W}\right)+mN & \text{if} \quad (6.16),\ (6.17),\ (6.21),\ (6.22) \text{ hold,} \\ \\ -\infty & \text{otherwise,} \end{cases}$$

3. The last step is to choose the multipliers so that the constraints (6.14) and (6.15) are satisfied. In order to do that, we solve the dual problem:

$$\begin{aligned} \underset{\mathbf{W}\in\bar{\mathcal{B}},\,\mathbf{Z}\in\mathcal{C}}{\operatorname{argmax}} \quad & \mathcal{V}(\mathbf{W},\mathbf{Z})+mN \\ \text{subject to} \quad & (6.16),\ (6.17),\ (6.21),\ (6.22) \end{aligned} \tag{6.24}$$

where we have defined

$$\mathcal{V}(\mathbf{W},\mathbf{Z}) := \log\det\left(\mathbf{C}+\mathbf{Z}+\mathbf{W}\right).$$

Observe now that using the explicit form (6.7) of the adjoint operator C, constraint (6.21) can be written as $V = \lambda_L I_{\mathcal{M}} + Z$. Thus, recalling that $\mathbf{V} \geq 0$ and exploiting the linearity of C, the constraint become

$$\lambda_L I_{mN} + \mathbf{Z} = \mathbf{V} \geq 0.$$

Accordingly, the dual problem takes the form

$$\begin{aligned} \underset{\mathbf{W},\,\mathbf{Z}\in\mathcal{C}}{\operatorname{argmax}} \quad & \mathcal{V}(\mathbf{W},\mathbf{Z})+mN \\ \text{subject to} \quad & \mathbf{W}\in\bar{\mathcal{B}} \\ & (6.16),\ (6.17) \\ & \lambda_L I_{mN}+\mathbf{Z}\geq 0 \\ & \mathbf{C}+\mathbf{Z}+\mathbf{W}>0. \end{aligned} \tag{6.25}$$

Proposition 10. Under the above assumptions, Problem (6.25) admits a unique solution.

Proof. The optimization problem (6.25) is feasible (it is sufficient to pick $\mathbf{W}=0$ and $\mathbf{Z}=0$ for instance)[1] namely, there exist $\bar{\mathbf{W}}$, $\bar{\mathbf{Z}} \in \mathcal{C}$ satisfying the constraints and such that

$$\left|\log\det\left(\mathbf{C}+\bar{\mathbf{Z}}+\bar{\mathbf{W}}\right)+mN\right| < \infty.$$

Let's introduce the set

$$\bar{\mathcal{K}} := \left\{ \mathbf{Z}\in\mathcal{C} \,\middle|\, (6.16),\ (6.17),\ \lambda_L I_{mN}+\mathbf{Z}\geq 0 \right\}$$

which is a closed and convex subset of $\mathcal{C}$. Accordingly, the above problem is equivalent to maximize $\mathcal{V}(\mathbf{W},\mathbf{Z})$ over the set

$$\mathcal{K} := \left\{ (\mathbf{W},\mathbf{Z})\in\mathcal{C}\times\bar{\mathcal{K}} \,\middle|\, \mathbf{C}+\mathbf{Z}+\mathbf{W}>0,\ \mathbf{W}\in\bar{\mathcal{B}},\ \mathcal{V}(\mathbf{W},\mathbf{Z})\geq\mathcal{V}(\bar{\mathbf{W}},\bar{\mathbf{Z}}) \right\}$$

indeed, we can get rid of the $(\mathbf{W},\mathbf{Z})$ for which $\mathcal{V}(\bar{\mathbf{W}},\bar{\mathbf{Z}}) \geq \mathcal{V}(\mathbf{W},\mathbf{Z})$ because obviously they cannot be solutions. We want to show that $\mathcal{K}$ is compact, i.e. closed and bounded

[1] Recall that $\mathbf{C}$ is assumed to be a positive definite matrix, see Remark 9.

(since we are working in finite dimension). In order to prove that $\mathcal{K}$ is bounded, let $\left\{(\mathbf{W}^{(k)}, \mathbf{Z}^{(k)})\right\}_{k\in\mathbb{Z}}$ be a sequence of elements in $\mathcal{K}$. Suppose that $\|\mathbf{Z}^{(k)}\|_{\mathcal{C}} \to \infty$ as $k \to \infty$. This implies that at least one eigenvalue of $\mathbf{Z}^{(k)}$ goes to infinity in absolute value. Indeed, since $\mathbf{Z}^{(k)}$ is symmetric,

$$\|\mathbf{Z}^{(k)}\|_{\mathcal{C}}^2 = \left\langle \mathbf{Z}^{(k)}, \mathbf{Z}^{(k)} \right\rangle_{\mathcal{C}} = \sum_{i=1}^{mN} \lambda_i^2$$

where λ_i, $i = 1, \ldots, mN$, are the eigenvalues of the matrix $\mathbf{Z}^{(k)}$ counted with their multiplicity. Moreover, since $\mathbf{Z}^{(k)} \in \bar{\mathcal{K}}$ we have that $\operatorname{diag}(Z_0^{(k)}) = 0$, thus

$$\operatorname{tr}(\mathbf{Z}^{(k)}) = \sum_{i=1}^{mN} \lambda_i = N \operatorname{tr}(Z_0^{(k)}) = 0,$$

hence, $\mathbf{Z}^{(k)}$ must have at least one eigenvalue that goes to $-\infty$. However, this is not possible because the fact that $\mathbf{Z}^{(k)} \in \bar{\mathcal{K}}$ implies also that $\mathbf{Z}^{(k)}$ must satisfy the inequality

$$\lambda_L I_{mN} + \mathbf{Z}^{(k)} \geq 0$$

where $\lambda_L I_{mN}$ is a constant matrix. We conclude that $\|\mathbf{Z}^{(k)}\|_{\mathcal{C}} < \infty$. In order to show that also $\|\mathbf{W}^{(k)}\|_{\mathcal{C}}$ is bounded, we recall that if a matrix $\mathbf{A}$ is positive definite, then all its principal sub-matrices must be positive definite. Suppose that $\|\mathbf{W}^{(k)}\|_{\mathcal{C}} \to \infty$ as $k \to \infty$, and recall that $\mathbf{W}^{(k)} \in \bar{\mathcal{B}}$ thus, in particular, $w_{ii}^{(k)} := (\mathbf{W}^{(k)})_{ii} = 0$. Then there exists an element $w_{ij}^{(k)}$ of $\mathbf{W}^{(k)}$ with $i \neq j$, so that $|w_{ij}^{(k)}| \to \infty$ as $k \to \infty$. In order to simplify the notation in the remaining of the proof define

$$\mathbf{A} := \mathbf{C} + \mathbf{Z} + \mathbf{W}$$

and consider the 2×2 sub-matrix

$$\mathbf{A}^{(k)}[i,j] := \begin{bmatrix} c_{ii} + z_{ii}^{(k)} + w_{ii}^{(k)} & c_{ij} + z_{ij}^{(k)} + w_{ij}^{(k)} \\ c_{ij} + z_{ij}^{(k)} + w_{ij}^{(k)} & c_{jj} + z_{jj}^{(k)} + w_{jj}^{(k)} \end{bmatrix} = \begin{bmatrix} c_{ii} + z_{ii}^{(k)} & c_{ij} + z_{ij}^{(k)} + w_{ij}^{(k)} \\ c_{ij} + z_{ij}^{(k)} + w_{ij}^{(k)} & c_{jj} + z_{jj}^{(k)} \end{bmatrix}$$

extracted from

$$\mathbf{A}^{(k)} = \mathbf{C} + \mathbf{Z}^{(k)} + \mathbf{W}^{(k)}.$$

We get that

$$\det(\mathbf{A}^{(k)}[i,j]) = \left(c_{ii} + z_{ii}^{(k)}\right)\left(c_{jj} + z_{jj}^{(k)}\right) - \left(c_{ij} + z_{ij}^{(k)} + w_{ij}^{(k)}\right)^2$$

thus, since $\mathbf{C}$ is fixed and $\|\mathbf{Z}^{(k)}\|_{\mathcal{C}} < \infty$ as $k \to \infty$, we have that $\det(\mathbf{A}^{(k)}[i,j]) \to -\infty$ as $k \to \infty$. Accordingly, $\mathbf{A}^{(k)}[i,j]$ and thus $\mathbf{A}^{(k)}$ cannot be positive definite as $k \to \infty$ which leads to a contradiction because $(\mathbf{W}^{(k)}, \mathbf{Z}^{(k)}) \in \mathcal{K}$ for any $k \in \mathbb{Z}$. We conclude that also $\|\mathbf{W}^{(k)}\|_{\mathcal{C}} < \infty$.

Now we show that $\mathcal{K}$ is closed. In doing this, denote with $\partial\mathcal{K}$ the boundary of $\mathcal{K}$. Since $\bar{\mathcal{K}}$ is a closed subset of $\mathcal{C}$, $\partial\mathcal{K}$ is at most the set of elements $(\mathbf{W}, \mathbf{Z})$ such that $\mathbf{A}$ is positive semidefinite and singular. Since

$$\lim_{(\mathbf{W},\mathbf{Z})\to\partial\mathcal{K}} \mathcal{V}(\mathbf{W}, \mathbf{Z}) = \lim_{(\mathbf{W},\mathbf{Z})\to\partial\mathcal{K}} \log\det \mathbf{A} = -\infty,$$

and the inequality $\mathcal{V}(\mathbf{W}, \mathbf{Z}) \geq \mathcal{V}(\bar{\mathbf{W}}, \bar{\mathbf{Z}})$ must hold, we conclude that $\partial\mathcal{K}$ is an empty set, i.e. $\mathcal{K}$ is closed. By what we have just proven, $\mathcal{K}$ is compact. Since $\mathcal{V}(\mathbf{W}, \mathbf{Z})$ is continuous over $\mathcal{K}$, it follows from Weierstrass' Theorem that $\mathcal{V}(\mathbf{W}, \mathbf{Z})$ admits maximum point in $\mathcal{K}$, namely, problem (6.25) admits a solution. Finally, since $\mathcal{V}(\mathbf{W}, \mathbf{Z})$ is strictly concave on $\mathcal{K}$ we conclude that the optimal solution $(\mathbf{W}_o, \mathbf{Z}_o)$ is unique. □

Proposition 11. Under the above assumptions, Problem (6.12) admits a solution $(X_o,\, L_o)$.

Proof. The strong duality between problems (6.12) and (6.25) (see Remark 10) and the existence of a unique optimum $(\mathbf{W}_o, \mathbf{Z}_o)$ for the dual problem (6.25), imply that there exists a unique $X_o \in \mathcal{M}$ so that

$$\mathbf{X}_o = \mathsf{C}(X_o) = (\mathbf{C} + \mathbf{Z}_o + \mathbf{W}_o)^{-1}$$

which solves the primal problem (6.12). It remains to show that there exists an $L_o \in \mathcal{M}$ solution of problem (6.12) with respect to L. More precisely, in view of (6.12) we have to show that there exists L_o that solves the optimization problem

$$\begin{aligned} \underset{L \in \mathcal{M}}{\text{argmin}} \quad & \lambda_S\, h_\infty(X_o + L) + \lambda_L \operatorname{tr}(\mathsf{C}(L)) \\ \text{subject to} \quad & \mathsf{C}(L) \geq 0. \end{aligned} \tag{6.26}$$

Notice that the objective function in (6.26) is continuous. Since $L = 0$ is a feasible point, the problem is equivalent to find $L \in \mathcal{M}$ that minimizes $\lambda_S\, h_\infty(X_o + L) + \lambda_L \operatorname{tr}(\mathsf{C}(L))$ over

$$\mathcal{Q} := \left\{ L \in \mathcal{M} \,\middle|\, \mathsf{C}(L) \geq 0,\ \lambda_S\, h_\infty(X_o + L) + \lambda_L \operatorname{tr}(\mathsf{C}(L)) \leq \lambda_S\, h_\infty(X_o) \right\}.$$

It is easy to see that $\mathcal{Q}$ is a closed and bounded (therefore compact) subset of $\mathcal{M}$, thus by Weierstrass' Theorem, problem (6.26) admits a solution L_o. At this point we can conclude that the primal problem (6.12) admits a solution $(X_o,\, L_o)$. □

7 A particular decomposition choice

In this Chapter we will basically retrace what we have done in the previous Chapter in the particular case in which both $\mathbf{S}$ and $\mathbf{L}$ are assumed symmetric, block-circulant, banded of bandwidth n. This is a special case of the scenario considered in the previous Chapter but it is worth studying it mainly for two reasons. First of all, with respect to the general choice $\mathbf{S}, \mathbf{L} \in \mathcal{C}$, with this choice we will have less parameters to be optimized in the identification procedure, which improves the efficiency of the identification algorithm. Moreover, it may be a consequence of a particular dynamics assumed for the latent component. Finally, it is worth noting that this is the equivalent choice done in [7] for the identification of sparse plus low-rank AR processes. These motivations will be better explained throughout the Chapter.

Let us consider the same set-up of the previous Chapter: the observed process $\mathbf{y}$ is a real-valued, Gaussian, zero-mean, m-dimensional, stationary reciprocal process of order n, defined on a finite interval $[1, N]$ with $N > 2n$, and extended by periodicity over the whole integer line $\mathbb{Z}$. The latent component $\mathbf{x} = [\boldsymbol{x}_1 \ \dots \ \boldsymbol{x}_l]^\top$ is a zero-mean Gaussian random vector with values in $\mathbb{R}^l$ with $l \ll mN$. As explained in the previous Chapter, the concentration matrix of the observed process can be expressed by its sparse plus low-rank decomposition

$$\boldsymbol{\Sigma}_{\mathbf{y}} = (\mathbf{S} - \mathbf{L})^{-1}, \tag{7.1}$$

where $\mathbf{S} \in \mathcal{C}$ is a positive definite matrix while $\mathbf{L} \in \mathcal{C}$ is a positive semi-definite matrix. In particular, to ensure that $\boldsymbol{\Sigma}_{\mathbf{y}}^{-1}$ will be symmetric, block-circulant, banded of bandwidth n, here we choose both $\mathbf{S}$ and $\mathbf{L}$ to be symmetric, block-circulant, banded of bandwidth n, i.e. $\mathbf{S}, \mathbf{L} \in \mathcal{B}$.

Remark 13. Of course, also in this case, all the assumptions made in the previous Chapter make $\mathbf{S}$ to be a sparse matrix and $\mathbf{L}$ to be a low-rank matrix. Consequently, the graphical model described by (7.1) will be a latent-variable graphical model, as in Chapter 6. Recall that $\mathbf{S}$ is the concentration matrix of the conditioned process $\mathbf{y}|\mathbf{x}$. Accordingly, combining the characterization of conditional independence we have given in Section 4.2 (Proposition 6) and Theorem 4, the assumption $\mathbf{S} \in \mathcal{B}$ is equivalent to assume that the conditioned process $\mathbf{y}|\mathbf{x}$ is a stationary, Gaussian reciprocal process of order n, periodic of period N.

7.1 Sparse plus low-rank optimization problem

In this Section we will rewrite and solve, in the particular case of $\mathbf{S}, \mathbf{L} \in \mathcal{B}$, the optimization problem we have derived in the general setting of $\mathbf{S}, \mathbf{L} \in \mathcal{C}$ in the previous Chapter.

The optimization problem for the estimation of the sparse plus low-rank components of the concentration matrix $\mathbf{\Sigma}_{\mathbf{y}}^{-1}$ directly follows from (6.10) by adding the regularizers:

$$\begin{aligned} \underset{\mathbf{S},\mathbf{L}\in\mathcal{B}}{\operatorname{argmin}} \quad & -\log\det(\mathbf{S}-\mathbf{L}) + \operatorname{tr}(\mathbf{C}(\mathbf{S}-\mathbf{L})) + \lambda_S\,\bar{h}_\infty(\mathbf{S}) + \lambda_L\operatorname{tr}(\mathbf{L}) \\ \text{subject to} \quad & \mathbf{S}-\mathbf{L}>0, \quad \mathbf{L}\geq 0. \end{aligned} \tag{7.2}$$

where λ_L, $\lambda_S > 0$ are two regularization parameters. As expected, we have obtained just a simplified version of Problem (6.11), without the constraint $\mathbf{S}-\mathbf{L}\in\mathcal{B}$. Since $\mathbf{S},\mathbf{L}\in\mathcal{B}$, we can rewrite them as

$$\begin{aligned} \mathbf{S} &= \sum_{k=-n}^{n} U^{-k}\otimes S_k, \qquad S_{-k}=S_k^\top, \\ \mathbf{L} &= \sum_{k=-n}^{n} U^{-k}\otimes L_k, \qquad L_{-k}=L_k^\top, \end{aligned}$$

thus, the problem can be rewritten in terms of the symmetric sequences $S, L\in\mathcal{M}_{\mathcal{B}}$,

$$S=[S_0\ S_1\ \dots\ S_n\ 0\ \dots\ 0\ S_n^\top\ \dots\ S_1^\top], \qquad L=[L_0\ L_1\ \dots\ L_n\ 0\ \dots\ 0\ L_n^\top\ \dots\ L_1^\top],$$

in order to have a smaller number of parameters involved in the optimization problem.

Remark 14. Notice that here we have even less parameters than in the previous case, indeed here we have also $S_k = L_k = 0$ for $k = n+1, \dots, N/2$ and the same for the corresponding transposes.

Introducing the new variable $\mathbf{X} := \mathbf{S}-\mathbf{L}$, and the associated sequence $X = S - L \in \mathcal{M}_{\mathcal{B}}$, problem (7.2) becomes

$$\begin{aligned} \underset{X,L\in\mathcal{M}_{\mathcal{B}}}{\operatorname{argmin}} \quad & -\log\det(\mathsf{C}(X)) + \langle \mathbf{C},\, \mathsf{C}(X)\rangle_{\mathcal{C}} + \lambda_S\, h_\infty(X+L) + \lambda_L\operatorname{tr}(\mathsf{C}(L)) \\ \text{subject to} \quad & \mathsf{C}(X)>0, \quad \mathsf{C}(L)\geq 0, \end{aligned} \tag{7.3}$$

which is equivalent to the problem

$$\begin{aligned} \underset{\substack{X\in\mathcal{M}\\ L,Y\in\mathcal{M}_{\mathcal{B}}}}{\operatorname{argmin}} \quad & -\log\det(\mathsf{C}(X)) + \langle \mathbf{C},\, \mathsf{C}(X)\rangle_{\mathcal{C}} + \lambda_S\, h_\infty(Y) + \lambda_L\operatorname{tr}(\mathsf{C}(L)) \\ \text{subject to} \quad & \mathsf{C}(X)>0, \quad \mathsf{C}(L)\geq 0 \\ & Y = X+L. \end{aligned} \tag{7.4}$$

By following a similar reasoning carried out in Remark 10 we conclude that Problem (7.3) is strictly feasible and Slater's condition holds for it and its dual. As a consequence, we can try to solve the primal problem (7.3) by solving its dual.

Dual problem solution

We address the previous constrained optimization problem using the Lagrange multipliers theory.

1. The Lagrangian function for this problem is

$$\begin{aligned}\mathcal{L}(X, L, Y, \mathbf{V}, \bar{Z}) = &- \log\det(\mathsf{C}(X)) + \langle \mathbf{C}, \mathsf{C}(X)\rangle_{\mathcal{C}} + \lambda_S\, h_\infty(Y) + \lambda_L \operatorname{tr}(\mathsf{C}(L)) \\ &- \langle \mathbf{V}, \mathsf{C}(L)\rangle_{\mathcal{C}} + \left\langle \bar{Z}, X + L - Y \right\rangle_{\mathcal{M}}\end{aligned}$$

where $\mathbf{V} \in \mathcal{B}$, $\mathbf{V} \geq 0$, while $\bar{Z} \in \mathcal{M}$. The same computations done in the previous Chapter allow to reparametrize the Lagrangian in terms of the multiplier $Z := 1/N \cdot \bar{Z}$ and to separate the terms depending on X, Y and L

$$\begin{aligned}\mathcal{L}(X, L, Y, \mathbf{V}, Z) = = &- \log\det(\mathsf{C}(X)) + \langle \mathbf{C}, \mathsf{C}(X)\rangle_{\mathcal{C}} + N\,\langle Z, X\rangle_{\mathcal{M}} \\ &+ \langle \mathsf{C}^*(\lambda_L I_{mN} - \mathbf{V}) + NZ, L\rangle_{\mathcal{M}} + \lambda_S\, h_\infty(Y) - N\,\langle Z, Y\rangle_{\mathcal{M}}\,.\end{aligned}$$

2. The second step is the unconstrained minimization of $\mathcal{L}$. The only term that depends on Y is $\lambda_S\, h_\infty(Y) - N\,\langle Z, Y\rangle_{\mathcal{M}}$ which is bounded below if and only if

$$\operatorname{diag}(Z_j) = 0, \quad j = 0, \ldots, n \tag{7.5}$$

$$2|(Z_0)_{kh}| + \sum_{j=1}^{n} |(Z_j)_{kh}| + |(Z_j)_{hk}| \leq \frac{\lambda_S}{N}, \quad k > h \tag{7.6}$$

in which case the infimum is zero. It is worth noting, since $Y \in \mathcal{M}_{\mathcal{B}}$, even if Z is not banded, the constraints (7.5) and (7.6), only involves the first $n+1$ blocks of Z. Hence

$$\inf_Y \mathcal{L} = \begin{cases} -\log\det(\mathsf{C}(X)) + \langle \mathbf{C}, \mathsf{C}(X)\rangle_{\mathcal{C}} + N\,\langle Z, X\rangle_{\mathcal{M}} \quad \text{if} \quad (7.5),\ (7.6) \text{ hold,} \\ + \langle \mathsf{C}^*(\lambda_L I_{mN} - \mathbf{V}) + NZ, L\rangle_{\mathcal{M}} \\ \\ -\infty \quad \text{otherwise.} \end{cases}$$

The only term that depends on L is $\langle \mathsf{C}^*(\lambda_L I_{mN} - \mathbf{V}) + NZ, L\rangle_{\mathcal{M}}$. Recalling that $L, V \in \mathcal{M}_{\mathcal{B}}$, by exploiting the linearity of the projection operator $\mathsf{P}_{\mathscr{B}}$, we have that

$$\langle \mathsf{C}^*(\lambda_L I_{mN} - \mathbf{V}) + NZ, L\rangle_{\mathcal{M}} = N\,\langle \lambda_L I_{\mathcal{M}} - V + \mathsf{P}_{\mathscr{B}}(Z), L\rangle_{\mathcal{M}}$$

which is linear in L, and therefore it is bounded below if and only if

$$\mathsf{C}^*(\lambda_L I_{mN} - \mathbf{V}) + N\,\mathsf{P}_{\mathscr{B}}(Z) = 0. \tag{7.7}$$

In this case, clearly, the minimum is zero. Accordingly,

$$\inf_{Y,L} \mathcal{L} = \begin{cases} -\log\det(\mathsf{C}(X)) + \langle \mathbf{C}, \mathsf{C}(X)\rangle_{\mathcal{C}} + N\,\langle Z, X\rangle_{\mathcal{M}} \quad \text{if} \quad (7.5),\ (7.6),\ (7.7) \text{ hold,} \\ \\ -\infty \quad \text{otherwise} \end{cases}$$

The remaining terms only depend on X. According to Proposition 9, if (7.5), (7.6), (7.7) hold we can rewrite the infimum of the Lagrangian function as

$$\bar{\mathcal{L}}(\mathbf{X}) := \inf_{Y,L} \mathcal{L} = -\log\det(\mathbf{X}) + \langle \mathbf{C}, \mathbf{X}\rangle_{\mathcal{C}} + \langle \mathbf{Z}, \mathbf{X}\rangle_{\mathcal{C}}\,,$$

where $\mathbf{X} = \mathsf{C}(X)$ and $\mathbf{Z} = \mathsf{C}(Z)$, according to the definition of the operator C. For the minimization with respect to $\mathbf{X}$, we proceed as in Chapter 6. The function $\bar{\mathcal{L}}$ is a strictly

convex function of $\mathbf{X}$ over the cone of the symmetric, positive definite, banded block-circulant matrices $\mathcal{C}^+$. Assuming a priori that the solution lies in the interior of the cone (we have to check that a posteriori, i.e. once we have found the solution), a necessary and sufficient condition for $\mathbf{X}_o$ to be a minimum point for $\bar{\mathcal{L}}(\mathbf{X})$ is that its first Gateaux derivative computed at $\mathbf{X} = \mathbf{X}_o$ is equal to zero in every direction $\boldsymbol{\delta}\mathbf{X}$ namely,

$$\bar{\mathcal{L}}'(\mathbf{X}_o;\, \boldsymbol{\delta}\mathbf{X}) = 0, \qquad \forall\, \boldsymbol{\delta}\mathbf{X} \in \mathcal{C}.$$

As explained before, there is no need for $\boldsymbol{\delta}\mathbf{X} \in \mathcal{C}$ to be positive definite. The Gateaux derivative of $\bar{\mathcal{L}}$ at $\mathbf{X}_o$ in direction $\boldsymbol{\delta}\mathbf{X}$ is in this case

$$\bar{\mathcal{L}}'(\mathbf{X}_o;\, \boldsymbol{\delta}\mathbf{X}) = \operatorname{tr}\left[\left(-\mathbf{X}_o^{-1} + \mathbf{C} + \mathbf{Z}\right)\boldsymbol{\delta}\mathbf{X}\right].$$

Assuming that $\mathbf{Z} \in \mathcal{C}$ is s.t.

$$\mathbf{C} + \mathbf{Z} > 0, \tag{7.8}$$

the directional derivative of $\bar{\mathcal{L}}$ at $\mathbf{X}_o$ is equal to zero in any direction $\boldsymbol{\delta}\mathbf{X} \in \mathcal{C}$ if and only if

$$\mathbf{X}_o = \mathsf{C}(X_o) = (\mathbf{C} + \mathbf{Z})^{-1}. \tag{7.9}$$

Finally, we have that

$$\inf_{Y,L,X} \mathcal{L} = \begin{cases} \log\det(\mathbf{C} + \mathbf{Z}) + mN & \text{if} \quad (7.5),\ (7.6),\ (7.7),\ (7.8) \text{ hold}, \\ \\ -\infty & \text{otherwise}, \end{cases}$$

3. The last step is to choose the multipliers so that the constraints (7.4) are satisfied. In order to do that, we solve the dual problem:

$$\begin{aligned} \underset{\mathbf{Z}\in\mathcal{C}}{\operatorname{argmax}} \quad & \mathcal{V}(\mathbf{Z}) + mN \\ \text{subject to} \quad & (7.5),\ (7.6),\ (7.7),\ (7.8) \end{aligned} \tag{7.10}$$

where we have defined

$$\mathcal{V}(\mathbf{Z}) := \log\det(\mathbf{C} + \mathbf{Z}).$$

Observe now that using the explicit form (6.7) of the adjoint operator C, constraint (7.7) can be written as $V = \lambda_L I_{\mathcal{M}} + \mathsf{P}_{\mathscr{B}}(Z)$. Thus, recalling that $\mathbf{V} \geq 0$ and exploiting the linearity of C, the constraint become

$$\lambda_L I_{mN} + \mathsf{P}_{\mathscr{B}}(\mathbf{Z}) = \mathbf{V} \geq 0.$$

Accordingly, the dual problem takes the form

$$\begin{aligned} \underset{\mathbf{Z}\in\mathcal{C}}{\operatorname{argmax}} \quad & \mathcal{V}(\mathbf{Z}) + mN \\ \text{subject to} \quad & (7.5),\ (7.6) \\ & \lambda_L I_{mN} + \mathsf{P}_{\mathscr{B}}(\mathbf{Z}) \geq 0 \\ & \mathbf{C} + \mathbf{Z} > 0. \end{aligned} \tag{7.11}$$

Since we are considering a particular case of the one studied in the previous Chapter, according to Proposition 10 we can say that problem (7.11) admits a solution. For the same reason, Proposition 11 ensures instead that Problem (7.3) admits a solution. Indeed, the proofs of these two results in this settings can be obtained by slightly adjusting the proofs of Proposition 10 and Proposition 11 presented in the previous Chapter.

Remark 15. At first sight, the choice $\mathbf{S}, \mathbf{L} \in \mathcal{B}$ may appear too ambitious, i.e. it may seems that we are requiring to much to hope that the optimization problem admits a solution. However, examples in which the problem admits a solution $(\mathbf{S}, \mathbf{L})$ can be built thus also in this sense this seems a reasonable choice.

8 Conclusions

In this thesis, an identification procedure for a latent-variable graphical model associated to a stationary, Gaussian, zero-mean, periodic reciprocal process has been proposed.

Such a procedure basically concerns an optimization problem that follows from the maximum entropy principle. In order to explain more clearly why we have chosen such approach, in Chapter 4, Section 4.1, we have presented some classical maximum entropy problems, by given particular emphasis to the Dempster's covariance selection problem and the covariance extension problem: these are the two starting points of our reasoning. It was also pointed out that the two problems are slightly different versions of the same problem. In Section 4.2 we have explained how to associate a graphical model to a random vector and, in particular, to a Gaussian random vector. More precisely, with Proposition 6, we have given the fundamental characterization of conditional independence in terms of concentration matrix of the random vector, which was crucial throughout the thesis. We have preferred to explain these fundamental ingredients in the simplest possible settings in order to ensure a better understanding of these tools.

In Chapter 5, we have introduced stationary processes defined on a finite interval (of time or space) and then Gaussian, periodic reciprocal processes of order n have been presented. Moreover, we have reported the derivation of a model for such processes showing that their dynamics can be modeled by a generalized autoregressive model. Finally, a fundamental characterization of these processes in terms of their concentration matrix was given by Theorem 4.

Chapter 6 contains the original contributions of this work. First of all, the problem was setted-up by defining the observed process and the latent-component modeling our data and consequently, the sparse plus-low rank decomposition of the covariance matrix of the observed process was introduced. In complete generality, both the sparse and the low-rank component were assumed to be only symmetric, block-circulant: no banded structure was assumed. Then, we have provided a detailed explanation of the construction of a latent-variable graphical model for the process. In order to introduce the optimization problem of our identification procedure, in Section 6.2.1, the block-circulant band extension problem recently proposed in the literature was introduced, as a sequel of the basic maximum entropy problems presented in the previous Chapter and was re-written according to our notation. Together with the sparse plus low-rank identification procedure available in the literature, this allowed to easily introduce our starting optimization problem for the estimation of the sparse and the low-rank components of the concentration matrix of the reciprocal process: it became apparent that it was only a re-parametrization, according to the sparse plus low-rank decomposition of the concentration matrix of the process, of the classical block-circulant band extension problem, plus some regularization terms. At this point, exploiting the Lagrange duality theory we have showed that the problem admits a solution.

Finally, in Chapter 7, we have retraced what we have done in Chapter 6 for a particular choice of the sparse plus low-rank components of the concentration matrix of the process: both the matrices have been chosen to have a banded, block-circulant structure. The main motivations behind this choice were explained and all the steps required by the solution developed in Chapter 6 have been particularized according with this choice.

8.1 Future work

There are a number of different points that may deserve to be investigated. Firstly, one can approximate the continuous-time, non-reciprocal case results with these discrete-time, reciprocal settings. This will imply to study the goodness of the approximation and, if the approximation is sufficiently good, we will gain scalability of the identification algorithm because of all the peculiar properties (circularity, banded structure) induced by our settings that are not present in the classic case. Secondly, the intuition tells us that the banded, block-circulant structure of the sparse and the low-rank components must be a consequence of the additional assumption that the latent-component is a periodic, reciprocal process, just like the observed process. It would be interesting to prove this fact and to understand if the choice of working with banded sparse and low-rank matrices implies the preclusion of some solutions with respect to the general case addressed in Chapter 6.

Bibliography

[1] E. Fornasini, *Appunti di teoria dei sistemi*. Libreria progetto, 2013.

[2] A. G. Lindquist and G. Picci, *Linear stochastic systems: a geometric approach to modeling, estimation and identification*. Ser. contemporary mathematics, Berlin, Germany: Springer, 2015.

[3] S. Lauritzen, *Graphical models*. Oxford, U.K.: Oxford university press, 1996.

[4] S. Boyd and L. Vandenberghe, *Convex optimization*. Cambridge, U.K.: Cambridge university press, 2004.

[5] R. Horn and C. Johnson, *Matrix analysis*. Cambridge, U.K.: Cambridge university press, 1990.

[6] J. Songsiri and L. Vandenberghe, "Topology selection in graphical models of autoregressive processes". *J. Mach. Lear. Res.*, Vol. 11, pp. 2671-2705, 2010.

[7] M. Zorzi and R. Sepulchre, "AR-identification of latent-variable graphical models". *IEEE Trans. Aut. Control*, Vol. 61, No. 9, pp. 2327-2340, September 2016.

[8] M. Zorzi and A. Chiuso, "Sparse plus Low-rank network identification: A nonparametric approach". *Automatica*, Vol. 53, No. 2, pp. 355-366, 2017.

[9] V. Chandrasekaran, P. Parillo and A. Willsky, "Latent variable graphical model selection via convex optimization". *Annal. Statist. (with discussion)*, Vol. 40, no. 4, pp. 1935-2013, April 2010.

[10] R. Liegeois, B. Mishra, M. Zorzi and R. Sepulchre, "Sparse plus low-rank autoregressive identification in neuroimaging time series". *IEEE CDC*, 2015.

[11] J. A. Sand, "Four papers in stochastic realization theory". PhD thesis, Dept. of Math., Royal Institute of Technology (KTH), Stockholm, Sweden, 1994.

[12] F. P. Carli, A. Ferrante, M. Pavon and G. Picci, "A maximum entropy solution of the covariance extension problem for reciprocal processes". *IEEE Trans. Aut. Control*, Vol. 56, No. 9, pp. 1999-2012, September 2011.

[13] A. G. Lindquist and G. Picci, "The circulant rational covariance extension problem: a complete solution". *IEEE Trans. Aut. Control*, Vol. 58, No. 11, pp. 2848-2861, November 2013.

[14] J. Burg, “Maximum entropy spectral analysis”. Ph.D. dissertation, Stanford Univ., Dept. of Geophysics, Stanford, CA, 1975.

[15] J. Burg, D. Luenberger, and D. Wenger, “Estimation of structured covariance matrices,” *Proc. IEEE*, vol. 70, pp. 963–974, 1982.

[16] A. P. Dempster, “Covariance selection,” *Biometrics*, vol. 28, pp. 157–175, 1972.

[17] M. Pavon and A. Ferrante, "On the geometry of maximum entropy problems", *SIAM Review*, Vol. 55-3: 415-439, Sept. 2013.

[18] M.Fazel, "Matrix rank minimization with applications". Ph.D. dissertation, Stanford Univ., Dept. of electrical eng., Stanford, CA, 2002.