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Hilbert schemes: construction and pathologies

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Introduction

The aim of this thesis is to present the construction of the Hilbert scheme and some of its pathologies. We will present its fundamental properties and some standard examples, concluding with a short discussion about unexpected behaviours and the Murphy's Law for Hilbert schemes.

The notion of Hilbert scheme was first introduced by Alexander Grothendieck (1928 - 2014) during a series of lectures given for the *Séminaire Bourbaki* between 1957 and 1962, which have been then collected under the famous name of *Fondements de la Géométrie Algébrique*, familiarly known as *FGA*. The intent of this collection was a "generalization" of algebraic geometry, which turned out into the introduction of several notions and techniques, such as *algebraic schemes*, *representable functors*, *descent theory* and *étale topology*, that are nowadays still central. Hilbert schemes were presented to the public during the Séminaire Bourbaki of 1960/61, in a talk called *"Technique de descente et théorèmes d'existence en géométrie algébrique IV. Les schémas de Hilbert"*, in continuity with the previous seminar *"Technique de descente et théorèmes d'existence en géométrie algébrique III. Préschémas quotients"*, which was held some months before. According to Grothendieck himself their intent was *"to replace the use of the Chow coordinates"*, a generalization of *Plücker coordinates*, that were formally introduced by Wei-Liang Chow (1911 - 1995) and Bertel Leendert van der Waerden (1903 - 1996) less than thirty years earlier, in [W.-L. Chow and B. L. van der Waerden, "Zur algebraischen Geometrie IX.", *Mathematische Annalen, vol. 113* (1937), pp. 692–704].

Hilbert schemes are named after David Hilbert (1862 - 1943), since one of the fundamental tools needed in order to define them, as we will see, is *Hilbert polynomial*, introduced by Hilbert himself using techniques coming both from commutative algebra (the study of modules) and complex analysis (the study of zeros and poles of a function). Such polynomials were widely studied first by Emmy Noether (1882 - 1935), Emanuel Lasker (1868 - 1941), B. L. van der Waerden and Pierre Samuel (1921 - 2009), and then by Jean-Pierre Serre (1926 -) and Alexander Grothendieck himself after the introduction of the *coherent sheaf cohomology* and *Čech cohomology*, to which the notion was adapted via the study of the *Euler characteristic*.

Together with Hilbert polynomials and a suitable cohomology theory, another fundamental ingredient for the definition of the Hilbert schemes, again coming from commutative algebra, is *flatness*. It was introduced by Serre for *modules over rings* in its groundwork [J.-P. Serre, "Géométrie algébrique et géométrie analytique", *Annales de l'Institut Fourier, tomes 6* (1956), pp. 1–42], widely known as *GAGA theory* and then generalized to *sheaves* with the use of the so called *Tor* functors, an homological tool that was developed to study abelian groups by Eduard Čech (1893 - 1960) and then adapted to the case of modules over rings by Henri Cartan (1904 - 2008) and Samuel Eilenberg (1913 - 1998).

The original construction of the Hilbert scheme made by Grothendieck was then slightly modified, thanks to the contribution of David Mumford (1937 -) and the introduction of the notion of *m-regularity*. However, Mumford acknowledged to Guido Castelnuovo (1865 - 1952) the first presentation of this concept, and so the notion is also known as *Castelnuovo-Mumford regularity*.

According to Grothendieck, the Hilbert scheme should become central in the further developement of algebraic geometry, but already few years after their introduction, Mumford partially took down this optimistic expectation. Indeed, in a series of articles during the 60's, he provided some examples of *"pathologies"* in algebraic geometry, including the first example of an Hilbert scheme that has a *bad behaviour* in an open dense subset, even though it parametrizes *well behaved* curves. This series was probably the starting point for a new research path involving *pathologies in algebraic geometry*, to which many famous modern mathematicians contributed, including Grothendieck's student Michel Raynaud (1938 - 2018), Heisuke Hironaka (1931 -), Philippe Ellia (1955 -), Fabrizio

Catanese (1950 -), Joe Harris (1951 -), Ian Morrison (1951 -) and, more recently, Ravi Vakil (1970 -).

Anyway, even if they turned out not to be always well behaved, Hilbert schemes are still considered very relevant, as Grothendieck claimed and desired, not only because of their nicer properties with respect to other parameter spaces. They are indeed often taken as a preliminary and first construction of a *Moduli space*, that are now largely studied both for their relevance in geometrical classification problems and, more recently, for their connection with theoretical physics, and also a standard example in the study of *deformation theory.* Moreover, they are a key example of the new *functorial approach* to algebraic geometry that became crucial after Grothendieck's work.

In the first chapter of this work we recall the definition and construction of the *Hilbert polynomial*, starting from the original approach dealing with modules over rings, and then moving to varieties and its interpretation using cohomology theory, with a large number of examples. We then introduce the base change operation and flatness and present the relations between these two notions and Hilbert polynomials.

The second chapter is devoted to the definition and costruction of *Hilbert schemes.* We start presenting briefly *Grassmannians*, since we will show the existence of the Hilbert scheme realizing it as a closed subscheme of a Grassmannian, and then go on with the discussion of *Castelnuovo-Mumford regularity* and *flattening stratifications*, that will directly lead us to the third section of the chapter, in which the construction of Hilbert schemes is completely provided. For the construction we will follow the approach by [Sernesi], who constructs directly the scheme, while Grothendieck, in [FGA], originally obtained it as a particular case of the construction of *Quot schemes.* At the end of the chapter we provide some first "easy" examples of Hilbert schemes, showing that the Grassmannians are a special case of this construction and considering the Hilbert scheme of hypersurfaces in the projective space. Some references to more recent generalizations of the notion are also given.

The third chapter discusses some useful properties of Hilbert schemes. In the first section we will follow the ideas presented by Robin Hartshorne (1938 -), who in his Ph.D. thesis ([H66]) proved that the Hilbert scheme of closed subschemes of projective space is connected. The second section ends with two results that characterize the *Zariski tangent space* to the Hilbert scheme and includes a short introduction to *deformation theory.* In particular, the last claim of the section will allow us to give a better comprehension of the *Hilbert scheme of n points* $X^{[n]}$, that is of the Hilbert scheme having Hilbert polynomial constantly equal to a positive integer n, which is the topic of the last section of the chapter. We will state that this scheme is always connected if X is connected and quasi-projective, generalizing Hartshorne's theorem, and that when X is an irreducible, nonsingular quasi-projective curve, or surface, also $X^{[n]}$ is nonsingular and irreducible.

Since the Hilbert scheme is a parameter space, our hope would be that it inherits at least some of the good properties of the objects it parametrizes, for example smoothness, irreducibility, connectedness or reducedness. We may also be guided to this idea by some results presented along the third chapter, but even in some of the simpler cases, such as the Hilbert scheme of points, we already find some first examples of *bad behaviours*, like singularity or a non reduced structure at its points. These brief considarations will naturally guide us to the fourth, and last, chapter of the thesis, in which *pathologies* take a central position. We will start the discussion, omitting direct proof, with the famous *Mumford's Example* of an Hilbert scheme parametrizing smooth space curves of a given degree and genus, that has more than one irreducible component of same dimension, and one of them is singular and nonreduced at its generic point. The second section shortly presents some further pathologies of Hilbert schemes of curves, following [HM]. The final section gives a small outline of the *Murphy's law for Hilbert schemes*, introduced informally in [HM] and then restated by Vakil in [Va2] into the *Murphy's law for Moduli spaces*, trying to point out the fundamental ideas of [Va2] regarding Hilbert schemes.

Terminology In this thesis, if it is not differently stated, we will use the following convention: rings will always be considered commutative with $1 \neq 0$; k will always denote a field, not necessarily algebrically closed, even though we will often have $\mathbb{C}$ in our minds; a variety will be an integral separated scheme of finite type over k, as in [H].
Moreover, just for clarity, we underline that in this work we call *scheme* what in some references, for example [H66] or versions of [FGA] published before 1971, used to be called *prescheme*, and we call *separated scheme* the object that used to be called *scheme.*

Chapter 1

Hilbert polynomial

1.1 Hilbert polynomial of Noetherian local rings

We start this chapter giving an outline of the first approach to Hilbert polynomials and the context in which they arose. The first results about such polynomials were provided by Hilbert at the end of the 19^{th} century, even though the name "Hilbert polynomial" was given by E. Lasker at the beginning of the following one, while summarazing the most important results produced by Hilbert in Invariant Theory (see [E. Lasker, "Zur Theorie der Moduln und Ideale", *Math. Ann., vol. 60* (1905), pp. 20–116]).
The original context was the study of dimension of Noetherian rings, as pointed out in [AM], and the tools developed in this first approach have been generalized and improved later on using both commutative algebra and category theory.
One of the basic notions behind dimension is the *additivity* of a function on short exact sequences of class of modules. Let $\mathcal{C}$ be a class of modules over a ring A. A function $\lambda : \mathcal{C} \to \mathbb{Z}$ is *additive on short exact sequences* if, for each short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

of A-modules in $\mathcal{C}$ it holds

$$\lambda(M') + \lambda(M'') = \lambda(M).$$

For example the map giving the dimension of finite-dimensional vector spaces over a field k is an additive function over the class of finite-dimensional vector spaces. This kind of functions is also as well behaved as possible on long exact sequences of modules. That is given a long exact sequence

$$0 \to M_0 \to M_1 \to \cdots \to M_n \to 0$$

of A-modules such that all the kernels of the morphisms and all the elements of the sequence belong to the same class $\mathcal{C}$, then

$$\sum_{i=0}^{n} (-1)^i \lambda(M_i) = 0.$$

After this general presentation without specifical assumptions on the rings and modules involved we move to the particular case of Noetherian rings and finitely generated modules. For the remainder of this section let A be a Noetherian ring and M a finitely generated A-module.
If A is also a graded ring, say $A = \bigoplus_{n=0}^{\infty} A_n$, it is a well-known fact that the subring of homogeneous elements is itself Noetherian and A can be seen as an A_0-algebra generated by homogeneous elements (see [AM 10.7]). Moreover, if M is a finitely generated graded A-module, then each homogeneous component M_n of degree n of M is finitely generated as an A_0-algebra. Thus, we can consider an additive function on the class of all the finitely generated A_0-modules.
We may then define the *Poincaré series of* M (w.r.t. a given additive function λ) as the generating function of $\lambda(M_n)$, i.e. the power series

$$P_\lambda(M, t) := \sum_{n=0}^{\infty} \lambda(M_n) t^n.$$

In the same setting given above, the Poincaré (or Hilbert-Poincaré) series turns out to be a rational function, as proved first by Hilbert in 1890 using its Syzygy Theorem, and then by Serre.

Theorem 1.1. *(Hilbert-Serre) Let A be generated, as an A_0-algebra, by homogeneous elements $x_1, \dots, x_s$ with degree $k_1, \dots, k_s$, let M be a finitely generated A-module and λ and additive function. Then $P_\lambda(M,t)$ is a rational function in t of the form*

$$P_\lambda(M,t) = \frac{f(t)}{\prod_{i=1}^{s}(1-t^{k_i})},$$

with $f(t) \in \mathbb{Z}[t]$.
Furthermore if we call $d(M)$ the order of the pole of $P_\lambda(M,t)$ at $t=1$ and we assume that $k_i = 1$ for all i then, for all sufficiently large n, $\lambda(M_n)$ is a polynomial in n of degree $d(M)-1$.

Proof. See [AM, 11.1]. □

Definition 1. The function (or polynomial) obtained in Theorem 1.1 is usually called *Hilbert-Samuel function (or polynomial) of M w.r.t. λ*. If we take as additive function the length, or dimension, of a module, we speak just of *Hilbert function (or polynomial) of M*.

In the environment of Noetherian local rings we may define the *characteristic polynomial* of an $\mathfrak{m}$-primary ideal $\mathfrak{q}$ and prove that it is a *numerical polynomial*, i.e. a polynomial $P(z) \in \mathbb{Q}[z]$ s.t. $P(n) \in \mathbb{Z}$ for all n sufficiently large.

Proposition 1.2. *Let $(A,\mathfrak{m})$ be a Noetherian local ring, $\mathfrak{q}$ an $\mathfrak{m}$-primary ideal, M a finitely generated A-module and $(M_n)_{n\in\mathbb{N}}$ a stable $\mathfrak{q}$-filtration of M. Call s the minimal number of generators of $\mathfrak{q}$. Then:*

i) M/M_n has finite length for all $n \geq 0$;

ii) for all sufficiently large n, the length of (M/M_n) is a polynomial $\chi_{\mathfrak{q},M}(n) \in \mathbb{Q}[n]$ of degree smaller than or equal to s;

iii) the degree and the leading monomial of $\chi_{\mathfrak{q},M}(n)$ depend only on M and $\mathfrak{q}$, not on the chosen filtration.

In particular, if we pick $M = A$, the polynomial correponding to the filtration $(\mathfrak{q}^n)_{n\in\mathbb{N}}$ is called the characteristic polynomial of the $\mathfrak{m}$-primary ideal $\mathfrak{q}$ and it is denoted by $\chi_{\mathfrak{q}}(n)$.

Proof. See [AM, 11.4] or [Eis, 12.2]. □

Once we have such results, with some additional work in commutative algebra, we may give a first characterisation of the dimension of a Noetherian local ring, and thus a local notion of dimension of a variety, linking three different objects:

1. the intuitive idea of dimension as "number of generators";
2. the topologial idea of dimension as length of chains of suitable subsets, which in this case are prime ideals;
3. the degree of the characteristic polynomial;

see e.g. [AM, §11] or [Bo, §2.4].

We proceed now considering Hilbert polynomials, with the goal of extending their defintion to algebraic subsets of projective spaces. In order to do that, we give an "algebraic version" of Hilbert-Serre's Theorem involving the annihilator of a module. We recall that, if M is a graded A-module, the *annihilator of M* is $\operatorname{Ann} M := \{s \in A \mid sM = 0\}$ and it is an homogeneous ideal in A. Furthermore, for a graded A-module M and for any $l \in \mathbb{Z}$ we recall that the *twisted module $M(l)$* is the A-module obtained by shifting "l places to the right each graded part", i.e. $M(l)_k = M_{k+l}$. Due to this "shift to the right", $M(l)$ will be isomorphic to M as an A-module, but will generally loose the grading isomorphism.

Proposition 1.3. *Let M be a finitely generated graded A-module. Then there exists a filtration by graded submodules*

$$0 = M^0 \subseteq M^1 \subseteq \ldots \subseteq M^r = M$$

such that for each i we have an isomorphism $M^i/M^{i-1} \simeq (A/\mathfrak{p}_i)(l_i)$, where $\mathfrak{p}_i$ is a homogeneous prime ideal of A and $l_i \in \mathbb{Z}$. Such a filtration is not unique, anyway for any such filtration the following statements hold:

i) if $\mathfrak{p}$ is a homogeneous prime ideal of A, then

$$\mathfrak{p} \supseteq \operatorname{Ann} M \iff \mathfrak{p} \supseteq \mathfrak{p}_i$$

for some i. In particular, the minimal elements of $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$ are exactly the minimal primes of $\operatorname{Ann} M$;

ii) for each minimal prime $\mathfrak{p}$ of M, the multiplicity of $\mathfrak{p}$ in the sequence $(\mathfrak{p}_1, \ldots, \mathfrak{p}_r)$ is equal to the length of $M_{\mathfrak{p}}$ over the local ring $A_{\mathfrak{p}}$, and thus it does not depend on the chosen filtration.

Proof. See [H I , 7.4]. □

Consider now $A = k[x_0, \ldots, x_n]$ and M a graded A-module. We denote the *Hilbert function of* M by

$$h_M(l) := \dim_k M_l$$

for each $l \in \mathbb{Z}$. All such dimensions are finite, as a finitely generated module over a Noetherian ring is itself Noetherian, and fields are trivially Noetherian, see e.g. [Eis, 1.4]. Furthermore, the Hilbert function is an additive function on the category of finitely generated modules.

Theorem 1.4. *(Hilbert-Serre algebraic version) Let $A = k[x_0, \ldots, x_n]$, and let M be a finitely generated graded A-algebra. Then there exists a unique polynomial $P_M(z) \in \mathbb{Q}[z]$ s.t. $h_M(l) = P_M(l)$ for all l sufficiently large. Moreover, it holds that $\deg P_M(z) = \dim Z(\operatorname{Ann} M)$, where $Z(\cdot)$ denotes the zero set of a homogeneous ideal in $\mathbb{P}^n$.*

Proof. See [H I, 7.5] or [Eis, 1.11]. □

Definition 2. The polynomial P_M given by Theorem 1.4 is called the *Hilbert polynomial of* M. If $Y \subseteq \mathbb{P}^n$ is an algebraic set of dimension r, we define the *Hilbert polynomial of* Y, denoted by P_Y, to be the Hilbert polynomial of its homogeneous coordinate ring $A(Y)$, which is indeed a polynomial of degree r again by Hilbert-Serre theorem.

We end this section introducing two numbers strictly related to P_M and that often appear while working with varieties.

Definition 3. Let Y be an algebraic set of dimension r. We set the *degree of* Y, denoted simply by $\deg Y$, to be $r!$ times the leading coefficient of P_Y. If $\mathfrak{p}$ is a minimal prime of a graded A-module M, we define the *multiplicity of M at* $\mathfrak{p}$, denoted by $\mu_{\mathfrak{p}}(M)$, as the length of $M_{\mathfrak{p}}$ over $A_{\mathfrak{p}}$.

In particular it can be proved that $\deg Y$ is a positive integer for any non empty algebraic subset of $\mathbb{P}^n$ and that the definition of degree given using the generating homogeneous ideal is consistent with the one given via Hilbert polynomials (see e.g. [H I, 7.6]).

1.2 Hilbert polynomial of Varieties

Once we have developed all such tools for modules and algebraic subsets of $\mathbb{P}^n$ we can leave behind the abstract setting and move to projective varieties.

Definition 4. We define the *Hilbert function* of any Zariski closed subset $X \subseteq \mathbb{P}^n$ to be

$$h_X(m) := \dim(A(X)_m)$$

where $A(X) = k[x_0, \ldots, x_n]/I(X)$ and the subscript m denotes the m-th graded piece.

Hence, we may see that it is the codimension, in the vector space of all homogeneous polynomials of degree m in $\mathbb{P}^n$, of the subspace of those polynomials vanishing on X.

Remark 1. If we consider a projective variety, i.e. an irreducible Zariski closed subset $X \subseteq \mathbb{P}^n$, we can give a "new" meaning to the *degree* of the variety X: since the leading term of the Hilbert polynomial is $(d/k!) \cdot m^k$, the degree d is exactly the number of points of intersection of X with a general $(n-k)$-plane (see [H I, 7.3 and 7.7], or [Sh1, §6.2]).

We show now some first easy examples of computation of Hilbert polynomials.

Example 1.1 Consider three points in $\mathbb{P}^2$. Then we have that

$$h_X(1) = \begin{cases} 2 & \text{the three points are collinear} \\ 3 & \text{if not} \end{cases}$$

as

$$h_X(1) = \dim_k(A(X)_1) = \dim_k(k[x_0, x_1, x_2]_1) - \dim_k(I(X)_1)$$

and $I(X)_1$ is the ideal consisting of all homogeneous linear polynomials vanishing at all three points and so it is 1-dimensional if they are collinear, and 0 else.
If we now want to evaluate $h_X(2)$, it turns out that it is equal to 3, as we can always find a homogeneous quadratic polynomial vanishing on two of the three points, but not on the third one. Indeed if we fix a representative for each point in $\mathbb{P}^2$ and define a map $\varphi : k[x_0, x_1, x_2] \to k^3$ given by the evaluation at those representatives, its kernel will be exactly $I(X)_2$ and the map is surjective for the above argument about homogeneous quadratic polynomials, hence

$$h_X(2) = \dim_k(A(X)_2) = \dim_k(k[x_0, x_1, x_2]) - \dim_k \ker \varphi = \dim_k \mathrm{im}\varphi = 3.$$

Similarly, for all $m \geq 3$ we conclude that $h_X(m) = 3$.

Example 1.2 We pick now $X \subseteq \mathbb{P}^2$ consisting of four points. If the four points are collinear, then

$$h_X(m) = \begin{cases} 2 & m = 1 \\ 3 & m = 2 \\ 4 & m \geq 3 \end{cases}$$

else

$$h_X(m) = \begin{cases} 3 & m = 1 \\ 4 & m \geq 2 \end{cases}$$

by the same argument as above.

Example 1.3 It is a general fact that given $X \subseteq \mathbb{P}^n$ consisting of d points, then for $m \geq d - 1$ we have $h_X(m) = d$, see [AG, Lemma 6.1.4].

The next example allows us to see that different closed subsets, such as two distinct points and a double-point, may have the same Hilbert function and Hilbert polynomial, which is not so good for our theory, as we would like to be able to distinguish the two different cases.

Example 1.4 Consider $X = V(x_0^2) \subset \mathbb{P}^1$. Then

$$S(X)_m = \begin{cases} \mathrm{span}_k\{1\} & m = 0 \\ \mathrm{span}_k\{x_0 x_1^{m-1}, x_1^m\} & m > 0 \end{cases}$$

so that

$$h_X(m) = \begin{cases} 1 & m = 0 \\ 2 & m > 0 \end{cases}$$

which is exactly the same Hilbert function as the one of two distinct points.

We move now from the almost elementary case of distinct points to the easiest varieties we are used to work with: curves in $\mathbb{P}^2$.

Example 1.5 Let $X \subseteq \mathbb{P}^2$ be a curve described as the zero locus of a suitable polynomial $f(x)$ of degree d. In this case $I(X)_m$ consists of all polynomials of degree m which are divisible by f, so that we may identify $I(X)_m$ with the space of polynomials of degree $m-d$, thus, it is a well known fact that

$$\dim(I(X)_m) = \binom{m-d+2}{2}.$$

Hence, for $m \geq d$,

$$h_X(m) = \binom{m+2}{2} - \binom{m-d+2}{2} = d \cdot m - \frac{d(d-3)}{2}.$$

Example 1.6 Let $X = \mathbb{P}^d_k$. In this case $A(X) = k[x_0, \ldots, x_d]$ and $A(X)_m$ is the vector space generated by all monomials of total degree m in d indeterminates, so that

$$h_{\mathbb{P}^d}(m) = \binom{m+d}{d}.$$

We produce now an example which shows that Hilbert functions and polynomials dramatically depend on the embedding into the given projective space.

Example 1.7 We compute the Hilbert polynomial of the Veronese embedding. Let $\nu_d : \mathbb{P}^1_k \hookrightarrow \mathbb{P}^d_k$ be the d-th Veronese embedding of $\mathbb{P}^1_k$ and call $X := \nu_d(\mathbb{P}^1_k)$.
The induced map ν_d^* on homogeneous coordinate rings identifies the graded piece $k[x,y]_m$ with $k[x,y]_{dm}$, thus

$$h_X(m) = \dim_k(A(X)_m) = \dim_k(k[x,y]_{dm})) = dm + 1$$

using previous examples.
We may generalize the previous account for the Veronese variety $X = \nu_d(\mathbb{P}^n) \subset \mathbb{P}^N$ by observing that polynomials of degree m on $\mathbb{P}^N$ pull back via ν_d to polynomials of degree dm on $\mathbb{P}^n$, exactly as they did in the $\mathbb{P}^1$ case. Thus the dimension of $A(X)_m$ is the one of the space of polynomials of degree dm on $\mathbb{P}^n$, hence by Example 1.6 we have that

$$h_X(m) = p_X(m) = \binom{md+n}{n}.$$

Example 1.8 Take now two disjoint projective varieties X and Y in $\mathbb{P}^n$. Recall that

$$I(X) \cap I(Y) = I(X \cup Y) \text{ and } V(I(X) + I(Y)) = V(I(X)) \cap V(I(Y)) = X \cap Y.$$

In particular, as we supposed X and Y to be disjoint, we have $V(I(X) + I(Y)) = \emptyset$. Therefore, if we set $A = k[x_0, \ldots, x_n]$, starting from a short exact sequence

$$0 \to A/(I(X) \cap I(Y)) \to A/I(X) \times A/I(Y) \to A/(I(X) + I(Y)) \to 0$$

where the first non trivial map is given by $f \mapsto (f, f)$ and the second one by $(f, g) \mapsto f - g$, we obtain that

$$h_{X \cap Y} + h_{X \cup Y} = h_X + h_Y.$$

Since X and Y were supposed to be disjoint, we conclude that

$$h_{X \cup Y} = h_X + h_Y.$$

So far we provided examples in which the smallest integer needed to obtain a polynomial in the Hilbert function was not so difficult to find, although in general such a bound isn't so simple to evaluate.

There's another way to compute Hilbert polynomials relying on twisted modules, following [Ha, §13], that could be useful in same cases.

Example 1.9 Consider $A := k[x_0, \ldots, x_n]$ and take $f \in A_d$, by which we define a degree d hypersurface $X := V(f)$. Moving to the exact sequence of A-modules

$$0 \to A(-d) \to A \to A(X) \to 0$$

and taking the m-th graded pieces and their dimensions, we find out that

$$h_X(m) = \dim_k(A_m) - \dim_k(A_{m-d}) = \binom{m+n}{n} - \binom{m+n-d}{n}$$

generalising what we saw in Example 1.5.

The idea used in the previous example is the standard process that can be extended to a wider collection of modules.

Definition 5. A map $\phi : M \to N$ of graded A-modules such that $\phi(M_k) \subset N_{k+d}$ is said to be *homogeneous of degree d*.

Consider now the ideal $I(X)$ of a variety $X \subseteq \mathbb{P}^n$. Such an ideal is generated by homogeneous polynomials f_α of degree d_α, i.e. there is a surjection

$$\bigoplus_\alpha A(-d_\alpha) \to I(X) \to 0$$

or, equivalently, the sequence

$$\bigoplus_{\alpha \leq r} A(-d_\alpha) \xrightarrow{\phi_1} A \to A(X) \to 0$$

is exact, where ϕ_1 is given by the vector $(\ldots, \cdot f_\alpha, \ldots)$. The kernel of ϕ_1 is the module M_1 of all r-tuples $(g_1, \ldots, g_r)$ such that $\Sigma g_\alpha \cdot f_\alpha = 0$ and it is called *module of relations*.

Remark 2. Such a module is a graded module and, avoiding the case in which X is a hypersurface, it is always non empty as it contains all the relations of the form $f_\alpha \cdot f_\beta - f_\beta \cdot f_\alpha$.

Anyway, the module of relations M_1 is also finitely generated, as it is a submodule of a finitely generated module, so that we may consider a set of generators $(f_{\beta,1} \ldots, f_{\beta,r})$ for $\beta \leq s$ and for each β and a suitable integer e_β we have $\deg(f_{\beta,1}) + d_1 = \cdots = \deg(f_{\beta,r}) + d_s = e_\beta$, so we can lengthen the previous exact sequence to

$$\bigoplus_{\beta \leq s} A(-e_\beta) \xrightarrow{\phi_2} \bigoplus_{\alpha \leq r} A(-d_\alpha) \xrightarrow{\phi_1} A \to A(X) \to 0$$

and so on.
This "weird procedure" comes to an end due to the *Hilbert syzygy theorem* (see [Eis, 1.13]), thus we find a free resolution

$$0 \to N_k \to N_{k-1} \to \cdots \to N_1 \to A \to A(X) \to 0 \tag{1.1}$$

of $A(X)$, with $N_i = \oplus A(-a_{i,j})$, and 1.1 can be refined to a minimal resolution. If we consider

$$\binom{c}{n}_0 := \begin{cases} \frac{c\cdot(c-1)\cdots(c-n+1)}{n!} & c \geq n \\ 0 & c < n \end{cases}$$

for all $n \in \mathbb{N}$ and $c \in \mathbb{Z}$, we can provide, using resolution 1.1, a new description of the Hilbert function of X as

$$\dim(A(-a)_m) = \dim(A_{m-a}) = \binom{m-a+n}{n}_0.$$

Via this new construction it follows that, if $N_i = \oplus A(-a_{i,j})$,

$$\dim(A(X)_m) = \binom{m+n}{n}_0 + \sum_{i,j} (-1)^i \binom{m-a_{i,j}+n}{n}_0$$

and this particular binomial coefficient is a polynomial in c, for $c \geq 0$ and $m \geq \max(a_{i,j}) - n$, providing that the Hilbert function is a polynomial in m as already proved.

1.3 Hilbert polynomials via Cohomology

The last part of Section 1.2 makes us think that, in order to deal with Hilbert polynomials, a good setting might be cohomology theory. Indeed this is the most recent developement of the subject and one of the most used nowadays. Using such approach, Hilbert functions (and polynomials) are examples of a different object called *Euler characteristic.* The natural objects of cohomology theory, in algebraic geometry, are schemes and coherent sheaves on them, and algebraic varieties are a particular kind of schemes.
Let k be a field, X be a projective scheme over k and $\mathscr{F}$ a coherent sheaf on X. We set

$$h^i(X, \mathscr{F}) := \dim_k H^i(X, \mathscr{F}).$$

Once we have fixed such a notation we define the *Euler characteristic* to be

$$\chi(X, \mathscr{F}) := \sum_{i=0}^{\infty}(-1)^i h^i(X, \mathscr{F}).$$

and we immediately see that Euler characteristic is an additive function on exact sequences of coherent sheaves by the definition.
Moreover we can notice that for a fixed integer n and $m \geq 0$ we find

$$h^0(\mathbb{P}^n_k, \mathcal{O}_X(m)) = \binom{n+m}{m}$$

which has leading coefficient $\frac{m^n}{n!}$, even though such an equality does not hold for every m; indeed it breaks down for $m < -n$. Neverthless we might check that

$$\chi(\mathbb{P}^n_k, \mathcal{O}_X(m)) = \binom{m+n}{n}.$$

Definition 6. Given a coherent sheaf $\mathscr{F}$ on a projective k-scheme X, we define the *Hilbert function of* $\mathscr{F}$ as

$$h_{\mathscr{F}}(n) := h^0(X, \mathscr{F}(n))$$

and by *Hilbert function of* X we will mean the Hilbert function of its structure sheaf $\mathcal{O}_X$.

As we already know in a special case, the Hilbert function agrees, for large enough n, with a polynomial, called *Hilbert polynomial.* After the introduction of Euler characteristic, we expect that this "eventual polynomiality" arises because the Euler characteristic actually is a polynomial and the higher cohomology vanishes for $n >> 0$.

Theorem 1.5. *Let $\mathscr{F}$ be a coherent sheaf on a projective k-scheme X embedded in the projective space $\mathbb{P}^n_k$. Then $\chi(X, \mathscr{F}(m))$ is a polynomial in m of degree equal to $\dim_k \operatorname{Supp} \mathscr{F}$. Thus for $m >> 0$ we have that $h^0(X, \mathscr{F}(m))$ is a polynomial of degree $\dim_k \operatorname{Supp} \mathscr{F}$. In particular $h^0(X, \mathcal{O}_X(m))$ is a polynomial of degree equal to $\dim X$.*

Proof. See [Va1, 12.1]. □

According to the notation we introduced before, if $\mathscr{F}$ is a coherent sheaf on a projective k-scheme X we define

$$p_{\mathscr{F}}(m) = \chi(X, \mathscr{F}(m))$$

and

$$p_X(m) = \chi(X, \mathcal{O}_X(m)) = p_{\mathcal{O}_X}(m).$$

By this definition we find out that

$$p_{\mathbb{P}^n}(m) = \binom{m+n}{n}.$$

If H is a degree d hypersurface in $\mathbb{P}^n$, using the additivity of the Euler characteristic on the exact sequence of closed subschemes

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-d) \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_H \to 0$$

we can calculate

$$p_H(m) = p_{\mathbb{P}^n}(m) - p_{\mathbb{P}^n}(m-d) = \binom{m+n}{n} - \binom{m+n-d}{n}$$

following almost the same procedure of Example 1.9.

A priori Euler characteristic needs not to be finite, but using the so called Vanishing Theorems (Serre, Grothendieck, Kodaira, see e.g. [H III]) we obtain some hypothesis under which higher cohomology groups are equal to 0 or, as we usually say, “higher cohomology groups vanish”.
Notice that we did not ask k to be algebraically closed. The assumption is actually almost irrelevant, as using base change theorems for affine morphism, or flat base change, we find that cohomology groups do not change under field extensions. This idea, which will be discussed in the next section, allows us to say that the Euler characteristic is invariant under base changes. Hence, we may compute Euler characteristic, and the Hilbert polynomials, either on k or on its algebraic closure, without the loss, or addition, of any information, which is very useful as for algebraically closed field (such as $\mathbb{C}$) we have more algebraic tools at our disposal.
Moreover, one can show that if $X \subset \mathbb{P}^n_k$ is a Zariski closed subset, then the function $d \mapsto \chi(\mathbb{P}^n_k, \mathcal{O}_X(d))$ coincides with the earlier definition of Hilbert polynomial. Indeed they are projective subschemes to which we may associate a coherent sheaf on the projective space $\mathbb{P}^n$ in a natural way using the structure sheaf. Hence, using Theorem 1.5, we may define the *Hilbert polynomial of a coherent sheaf on* $\mathbb{P}^n_k$ as the polynomial given by the map $d \mapsto \chi(\mathbb{P}^n_k, \mathscr{F}(d))$ for $d >> 0$.

Example 1.10 Let X be a degree-d hypersurface in $\mathbb{P}^n_k$ and consider its ideal sheaf exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-d) \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_X \to 0.$$

Twisting by $\mathcal{O}_{\mathbb{P}^n}(m)$, passing to the long exact sequence in cohomology, and then taking dimensions, we find out that

$$0 = h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m-d)) - h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) + h^0(\mathbb{P}^n, \mathcal{O}_X(m)) - h^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m-d)) + \cdots$$

which, by collecting terms, gives

$$\chi(\mathbb{P}^n, \mathcal{O}_X(m)) = h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) - h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m-d)) + (-1)^n h^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) - (-1)^n h^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m-d)).$$

We know the first two dimensions as they are given by the usual binomial coefficient, and for $m - d > -n - 1$ the terms h^n vanish, so we are left with the polynomial

$$\chi(\mathbb{P}^n, \mathcal{O}_X(m)) = \binom{m+n}{m} - \binom{m+n-d}{n}$$

which is exactly the Hilbert polynomial we found out in the previous computations.

1.4 Base Change and Flatness

In this section we introduce two important notions in algebraic geometry: base change, or extensions, and flatness. In order to do that we briefly recall some preliminary concepts in the category of schemes.

Let S be a scheme. We say that X is a *scheme over* S, shortly an S*-scheme*, if X is a scheme together with a morphism $X \to S$. If X and Y are two schemes over S, then a *morphism of* S*-schemes* is a morphism $X \to Y$ compatible with the given morphisms to S. The underlying scheme S is usually called *base scheme.* Once we are given two S-schemes X and Y we define the *fibered product of* X *and* Y *over* S, denoted by $X \times_S Y$, to be a scheme together with a pair of morphisms $p_1 : X \times_S Y \to X$ and $p_2 : X \times_S Y \to Y$ called *projections*, which make a commutative diagram with the given morphisms $X \to S$ and $Y \to S$, satisfying the following universal property: given any other S-scheme Z and morphisms $f : Z \to X$ and $g : Z \to Y$ that make a commutative diagram with the given morphisms $X \to S$ and $Y \to S$, there is a unique morphism $\theta : Z \to X \times_S Y$ such that $f = p_1 \circ \theta$ and $g = p_2 \circ \theta$.
As, a priori, not all categories admit fibered products, also known as pullbacks, or Cartesian squares, in category theory, we need a result ensuring us that, in our case, this a construction is possible.

Theorem 1.6. *For any two S-schemes X and Y the fibered product exists, and it is "essentially" unique.*

Proof. See [H II, 3.3]. □

The notion of fibered product is useful in order to translate the well-known analytical concept of *fibre of a function* in the scheme context.

Definition 7. Let $f : X \to Y$ be a morphism of schemes and fix a point $y \in Y$. Let $k(y) = \mathcal{O}_{y,Y}/\mathfrak{m}_y$ be te residue field of y on Y, where $\mathcal{O}_{y,Y}$ denotes the local ring at y and $\mathfrak{m}_y$ its maximal ideal, and consider the natural morphism $\operatorname{Spec} k(y) \to Y$. We define the *fibre* of the morphism f over y to be the scheme

$$X_y := X \otimes_Y \operatorname{Spec} k(y).$$

This definition allows us to regard a morphism as a family of schemes, its fibres, parametrized by the point of the image scheme. So we develop a useful tool to study such families of schemes, which Hartshorne calls *"a form of cohomology along the fibres"* in [H III, §9].

Definition 8. Let $f : X \to Y$ be a continuous map between two topological spaces. As the category of sheaves of abelian groups on a topological space X, denoted by $\mathfrak{Ab}(X)$, has enough injectives (see [H III, 2.3.]) and taking the direct image $f_* : \mathfrak{Ab}(X) \to \mathfrak{Ab}(Y)$ turns out to be a left exact functor, we define the *higher direct image functors*

$$R^i f_* : \mathfrak{Ab}(X) \to \mathfrak{Ab}(Y)$$

to be the right derived functor of the direct image functor.

In particular the following characterization holds.

Proposition 1.7. *For each $i \geq 0$ and each $\mathscr{F} \in \mathfrak{Ab}(X)$, the higher direct image sheaf $R^i f_*(\mathscr{F})$ is the sheaf associated to the presheaf*

$$V \mapsto H^i(f^{-1}(V), \mathscr{F}\mid_{f^{-1}(V)})$$

on Y. In particular, if $V \subseteq Y$ is any open susbset then

$$R^i f_*(\mathscr{F})\mid_V = R^i f'_*(\mathscr{F}\mid_{f^{-1}(V)}),$$

where $f' : f^{-1}(V) \to V$ is the restriction of f.

Proof. See [H III, 8.1 and 8.2]. □

Recalling that a sheaf $\mathscr{F}$ on a topological space X is said to be *flasque* if for every inclusion of open sets $V \subseteq U$, the restriction map $\mathscr{F}(U) \to \mathscr{F}(V)$ is surjective and relying on the fact that restrictions of flasque sheaves are flasque and they are acyclic for the globals section functor (see e.g. [H III, 2.4, 2.5, 6.1]), the characterization provided by Proposition 1.7 produces the following useful vanishing result.

Corollary 1.8. *If $\mathscr{F}$ is a flasque sheaf on X, then $R^i f_*(\mathscr{F}) = 0$ for all $i > 0$.*

Proof. See [H III, 8.3].

This last result makes us capable of computing the higher direct image functor of a morphism of ringed spaces not only on $\mathfrak{Ab}(X)$, but also on the category $\mathfrak{Mod}(X)$ of sheaves of $\mathcal{O}_X$-modules, since to calculate the right derived functors we need injective resolutions, and injective objects are flasque (see e.g. [H III, 2.4]), and thus acyclic for f_* by Corollary 1.8.
In the case of Noetherian schemes we may provide a further outlining of the higher direct image sheaf.

Proposition 1.9. *Let X be a Noetherian scheme, $f : X \to Y$ a morphism of schemes, with $Y = \operatorname{Spec} A$. Then, for any quasi-coherent sheaf $\mathscr{F}$ on X we have the following isomorphism*

$$R^i f_*(\mathscr{F}) \cong H^i(X, \mathscr{F})\tilde{},$$

where the notation $\tilde{}$ means taking the associated sheaf in $\mathfrak{Mod}(Y)$.
Moreover, in such a case the higher direct image sheaf is again quasi-coherent.

Proof. See [H III, 8.5 and 8.6]. □

After this brief discussion, we go back to fibered products and their connection with Hilbert polynomials.
From the concept of fibered product arises the notion of *base extension*, or *base change* ([H] uses the first name while many other authors such as [Bosch] or [GW] the second one). We have already used the idea of base when we said that for an S-scheme X, the scheme S is called "base scheme". The process of base extension will try to generalize the elementary concept of field extensions, which is a well-known useful tool in ring theory.

Definition 9. Let k be a field and consider the scheme $S = \operatorname{Spec} k$. If S' is another base scheme and if $S' \to S$ is a morphism fo schemes, then for any S-scheme X we let $X' = X \times_S S'$ be another scheme, defined using the fibered product, which is by construction an S'-scheme. We say that X' is obtained from X by a *base extension*, or a *base change*, $S' \to S$.

Taking base extension turns out to be transitive, as shown e.g. in [GW, Proposition 4.16].

Example 1.11 Linking this definition with the standard abstract algebra's idea of enlarging the field of the coefficients of polynomials introduced by van der Waerden, if $k \to k'$ is a field extension we may think to a base extension $S' \to S$, with $S' = \operatorname{Spec} k'$ and $S = \operatorname{Spec} k$, induced by the extension of fields.

Considering fibres and base extensions we shift the interest from studying one variety at a time to studying properties of the morphism that defines its family. In this new view it becomes important to study the behaviour of properties of f under base extensions and to relate properties of the morphism to properties of its fibres. Hence, our attention should be given to properties that are *invariant*, or *stable*, *under base extensions*, i.e. to properties P of a morphism of S-schemes f such that for a given a base extension $S' \to S$, also the induced map $f' : X' = X \times_S S' \to S'$ satisfies P.
Several well known properties turn out to be stable under base change, such as being "locally of finite type" or being "closed immersions". For a presentation of most of them see e.g. [H II, §4, §9 and §10], [GW, §9, §10, §12, §13 and §14] or [Bosch, §7, §8 and §9]), by the way we will focus on one particular property: being *flat*. The idea of flatness has been first introduced by Serre in an algebraic context in his fundamental article [GAGA] and has been then reinterpretated in the geometric context by the groundwork of Grothendieck [FGA].
We start defining flatness in the original case of modules and then we move on to schemes.

Definition 10. Let F be an A-module. If for every monomorphism $M' \to M$ of A-modules the induced map $M' \otimes_A F \to M \otimes_A F$ is again a monomorphism we say that F is *flat*. An equivalent condition is that tensoring by F on the right is exact.

By the equivalent definitions we gave, we get straightforwardly that flatness is a local property, i.e. F is a flat A-module if and only if $F_{\mathfrak{p}}$ is a flat $A_{\mathfrak{p}}$ module for each prime ideal $\mathfrak{p}$, if and only if $F_{\mathfrak{m}}$ is a flat $A_{\mathfrak{m}}$ module for each maximal ideal $\mathfrak{m}$.

To study flatness the most valuable context is category theory, since in that case flatness turns out to be equivalent to the vanishing of a specific functor, the so called *torsion functor*. Anyway we will try to avoid as much as possible an explicit use of this functor, which would require an important amount of space and time. Nevertheless, we will use some facts that can be proved using the category theory approach, referring for a further and more complete treatment of the subject to [Eis, §6], [Matsumura, §3.7 and §7], [Weibel, §3] or [Rotman, §7], even though some useful properties of flat modules may be proved avoiding the use of Tor functor (see e.g. [Rotman, §3] and [AM, §3]). Here we recall simply that if M is a projective A-module, and in particular a free one, as free implies projective, then it is flat (see [Rotman, 3.46]).
Even remaining in the context of modules, we may state a first result linking base change and flatness, claiming that flatness is preserved by change of rings. For the proof, which is an easy application of canonical morphisms between tensor products, see [Bosch 4.4, Proposition 1 *iii*)].

Proposition 1.10. *Let M be an R-module and consider $R \to R'$ a ring homomorphism. If M is flat as an R-module, then also $M \otimes_R R'$ is flat as an R'-module.*

We consider now as a special family of schemes one provided by a flat morphism, a so called *flat family*.

Definition 11. Let $f : X \to Y$ be a morphism of schemes, and let $\mathscr{F}$ be an $\mathcal{O}_X$-module. We say that the sheaf $\mathscr{F}$ is *flat at a point* $x \in X$ *over* Y if the stalk $\mathscr{F}_x$ is flat as an $\mathcal{O}_{y,Y}$-module, where $y = f(x)$ and the stalk has a module structure via the natural map $f^\sharp : \mathcal{O}_{y,Y} \to \mathcal{O}_{x,X}$. If $\mathscr{F}$ is flat at any point $x \in X$ we say briefly that $\mathscr{F}$ is *flat over* Y. The morphism f itself is said to be a *flat morphism* if the sheaf $\mathcal{O}_X$ is flat over Y.

Example 1.12 If Y is of the form $Y = \operatorname{Spec} k$, with k a field, then every morphism of schemes $X \to Y$ is flat.

Some properties of flat modules extend to properties of flat sheaves.

Proposition 1.11. *The following hold:*

i) An open immersion is a flat morphism;

ii) let $f : X \to Y$ be a morphism, take $\mathscr{F}$ a flat $\mathcal{O}_X$-module over Y and let $g : Y' \to Y$ be any morphism. Consider $X' = X \times_Y Y'$ and call $f' : X' \to Y'$ the second projection. Set $\mathscr{F}' = p_1^*(\mathscr{F})$. Then $\mathscr{F}'$ is flat over Y' (Base Change Property)*;*

iii) let $f : X \to Y$ and $g : Y \to Z$ be two morphisms. Suppose that $\mathscr{F}$ is a flat $\mathcal{O}_X$-module over Y and that Y is flat over Z. Then $\mathscr{F}$ is flat over Z (Transitivity)*;*

iv) let $A \to B$ be a ring homomorphism and M be a B-module. Consider

$$f : \operatorname{Spec} B = X \to \operatorname{Spec} A = Y$$

the corresponding morphism of affine schemes and set $\mathscr{F} = \tilde{M}$. Then $\mathscr{F}$ is flat over Y if and only if M is flat over A;

v) let X be a Noetherian scheme and take a coherent $\mathcal{O}_X$-module $\mathscr{F}$. Then $\mathscr{F}$ is flat if and only if it is locally free.

Proof. See [H III, 9.2]. □

The fundamental result we state below tells us that "cohomology commutes with flat base extensions", and that is why we may find more convenient to take flat morphism instead of general one.

Theorem 1.12. *Let $f : X \to Y$ be a separated morphism of finite type of Noetherian schemes and let $\mathscr{F}$ be a quasi-coherent sheaf on X. Let $u : Y' \to Y$ be a flat morphism of Noetherian schemes. Consider the following cartesian diagram:*

$$\begin{tikzcd} X' \arrow[r, "v"] \arrow[d, "g"] & X \arrow[d, "f"] \\ Y' \arrow[r, "u"] & Y \end{tikzcd}$$

Then for all $i \geq 0$ there are natural isomorphisms

$$u^* R^i f_*(\mathscr{F}) \cong R^i g_*(v^*\mathscr{F}).$$

Proof. See [H III, 9.3]. □

Remark 3. If we drop the flatness assumption on u we find anyway that there is a natural map

$$u^* R^i f_*(\mathscr{F}) \to R^i g_*(v^*\mathscr{F}),$$

but this map is not an isomorphism in general.

Corollary 1.13. *Let $f : X \to Y$ and $\mathscr{F}$ be as in Theorem 1.12 and assume that Y is affine. Consider for any point $y \in Y$ the fibre X_y and let $\mathscr{F}_y$ be the induced sheaf. Denote by $k(y)$ the constant sheaf $k(y)$ on the closed subset $\{y\}$ of Y. Then, for all $i \geq 0$ there are natural isomorphisms*

$$H^i(X_y, \mathscr{F}_y) \cong H^i(X, \mathscr{F}) \otimes k(y).$$

Proof. See [H III, 9.4]. □

We provide now a relation between flatness and dimension of fibres.

Theorem 1.14. *(Fibre dimension Theorem) Let $f : X \to Y$ be a flat morphism of schemes of finite type over a field k and for any point $x \in X$ set $y = f(x)$. Then*

$$\dim_x X_y = \dim_x X - \dim_y Y$$

where for any scheme X by $\dim_x X$ we mean the dimension of the local ring $\mathcal{O}_{x,X}$.
If moreover Y is also irreducible, then the following are equivalent:

i) every irreducible component of X has dimension $\dim Y + n$;

ii) for any point $y \in Y$, every irreducible component of the fibre X_y has dimension n.

Proof. See [H III, 9.5, 9.6]. □

We have now almost every tool needed to prove the link between flatness and Hilbert polynomials. We still need just two results about integral regular schemes of dimension 1.

Definition 12. Given a scheme X, a point $x \in X$ is said to be an *associated point of X* if the maximal ideal $\mathfrak{m}_x$ is an associated prime of 0 in $\mathcal{O}_{x.X}$.

Proposition 1.15. *Let $f : X \to Y$ be a morphism of schemes and Y be integral and regular of dimension 1. Then X is flat over Y if and only if every associated point of X maps to the generic point of Y.*

Proof. See [H III, 9.7]. □

Proposition 1.16. *Let Y be a regular integral scheme of dimension 1 and let $P \in Y$ be a closed point. Consider $X \subseteq \mathbb{P}^n_{Y-P}$ be a closed subscheme flat over $Y - P$. Then there exists a unique closed subscheme $\overline{X} \subseteq \mathbb{P}^n_Y$, flat over Y, such that its restriction to $\mathbb{P}^n_{Y-P}$ is X.*

Proof. See [H III, 9.8]. □

Now we are ready to state, and prove, the main result of the section.

Theorem 1.17. *Let T be an integral Noetherian scheme and consider a closed subscheme $X \subseteq \mathbb{P}^n_T$. For each point $t \in T$ consider $P_t \in \mathbb{Q}[z]$ the Hilbert polynomial of the fibre X_t taken as a closed subscheme of $\mathbb{P}^n_{k(t)}$. Then X is flat over T if and only if the Hilbert polynomial P_t is independent of t.*

Proof. For all $m >> 0$ let

$$P_t(m) := \dim_{k(t)} H^0(X_t, \mathcal{O}_{X_t}(m))$$

be the Hilbert polynomial of the fibre X_t. We now make two simplifications of the problem:

1. we replace $\mathcal{O}_X$ by any coherent sheaf $\mathscr{F}$ on $\mathbb{P}^n_T$ and use the Hilbert polynomial on $\mathscr{F}_t$, so that we may assume $X = \mathbb{P}^n_T$;
2. we point out that the question is local on T, as we may compare any point to the generic point, thus it is sufficient to consider $T = \operatorname{Spec} A$ for a local Noetherian ring $(A, \mathfrak{m})$.

Once we have done this switch we show that the following are equivalent:

i) $\mathscr{F}$ is flat over T;

ii) for all $m >> 0$, the cohomology group $H^0(X, \mathscr{F}(m))$ is a free A-module of finite rank;

iii) the Hilbert polynomial P_t of $\mathscr{F}_t$ on $X_t = \mathbb{P}^n_{k(t)}$ is independent on t for any $t \in T$.

"$i) \Rightarrow ii)$" Using the standard open affine cover $\mathfrak{U}$ of X we compute $H^i(X, \mathscr{F}(m))$ via Čech cohomology (see [H III, §4]), i.e. if $C^i(\mathfrak{U}, \mathscr{F}(m))$ is the Čech complex, we set

$$H^i(X, \mathscr{F}(m)) = h^i(C^i(\mathfrak{U}, \mathscr{F}(m))).$$

Since $\mathscr{F}$ is flat, also each term $C^i(\mathfrak{U}, \mathscr{F}(m))$ of the Čech complex is a flat A-module, while the left hand side is 0 for $m >> 0$ and $i > 0$ by Serre's Vanishing Theorem. Thus the complex $C^i(\mathfrak{U}, \mathscr{F}(m))$ is a resolution of the module $H^0(X, \mathscr{F}(m))$, indeed we have an exact sequence

$$0 \to H^0(X, \mathscr{F}(m)) \to C^0(\mathfrak{U}, \mathscr{F}(m)) \to C^1(\mathfrak{U}, \mathscr{F}(m)) \to \cdots \to C^n(\mathfrak{U}, \mathscr{F}(m)) \to 0.$$

If we split it into short exact sequences we may conclude that $H^0(X, \mathscr{F}(m))$ is a flat A-module as all the $C^i(\mathfrak{U}, \mathscr{F}(m))$ are. On top of that, Serre's Vanishing ensures us that it is also finitely generated, thus free of finite rank by properties of finitely generated flat modules.
"$ii) \Rightarrow i)$" Let $S = A[x_0, \ldots, x_n]$ and take the graded S-module

$$M := \bigoplus_{m \geq m_0} H^0(X, \mathscr{F}(m))$$

with m_0 chosen large enough to have $H^0(X, \mathscr{F}(m))$ free for all $m \geq m_0$. Then $\mathscr{F}$ coincides with the sheaf $\tilde{M}$ associated to M over $\operatorname{Proj} S$, which for $m \geq m_0$ is the same as the sheaf associated to $\Gamma_*(\mathscr{F})$, where $\Gamma_*(\mathscr{F})$ denotes the graded S-module associated to a sheaf $\mathscr{F}$ over $\operatorname{Proj} S$ (see [H II, §2 and §5]). Since M is a free, thus flat, A-module, $\mathscr{F}$ is also flat by point $iv)$ of Proposition 1.11.
"$ii) \iff iii)$" It will be enough to show that, for $m >> 0$,

$$P_t(m) = \mathrm{rk}_A H^0(X, \mathscr{F}(m)).$$

In order to get the equivalence, we will show that, for any $t \in T$ and for all $m >> 0$

$$H^0(X_t, \mathscr{F}_t(m)) \cong H^0(X, \mathscr{F}(m)) \otimes_A k(t).$$

Let us first denote by $\mathfrak{p}$ the prime ideal corresponding to $t \in T$ and consider $T' = \operatorname{Spec} A_{\mathfrak{p}}$. Now we consider the flat base extension given by $T' \to T$ so, as cohomology commutes with flat base changes by Theorem 1.12, we reduce to consider t to be the closed point of T.
Denote the closed fibre X_t by X_0, the fibre $\mathscr{F}_t$ by $\mathscr{F}_0$ and the field $k(t)$ by k. From a presentation of k over A

$$A^q \to A \to k \to 0 \tag{1.2}$$

we find an exact sequence of sheaves

$$\mathscr{F}^q \to \mathscr{F} \to \mathscr{F}_0 \to 0 \tag{1.3}$$

on X.
Now, for $m >> 0$, from (1.3) we may find an exact sequence

$$H^0(X, \mathscr{F}(m)^q) \to H^0(X, \mathscr{F}(m)) \to H^0(X_0, \mathscr{F}_0(m)) \to 0.$$

Tensoring (1.2) by $H^0(X, \mathscr{F}(m))$ and comparing the two sequences we deduce that for all $m >> 0$

$$H^0(X_0, \mathscr{F}_0(m)) \cong H^0(X, \mathscr{F}(m)) \otimes_A k.$$

For the converse it suffices to notice that the above argument is reversible as we can check the freeness of $H^0(X, \mathscr{F}(m))$ by comparing its rank at the generic point and at the closed point of T (see [H II, 8.9]). $\square$

From this result we have in particular that, for a connected Noetherian scheme T and a closed flat subscheme X over T, the dimension of the fibre and the degree of the scheme, as defined before, are independent of t. Notice moreover that we have proved something more general along the proof, i.e. that Euler characteristic itself is constant in flat families, which is *"a first sign that cohomology behaves well in flat families"*, as Vakil says in ([Va1, §24.7]).

The "second sign" will be the continuity of the function associating to a coherent sheaf its dimension at a point, but it turns out to be a too optimistic request. Anyhow it can be proved that such a function behaves not so badly in a neighbourhood of a point, under some not so strict assumptions. As such result requires some work with suitable functors on the category of A-modules we just provide the fundamental statement, omitting the proof which can be found in [H III, §12] (for some further aspects one might see [Va1, §28]).

Definition 13. Let Y be a topological space. A function $\varphi : Y \to Z$ is *upper semicontinuous* if for each $y \in Y$, there is an open neighbourhood U of y such that $\varphi(y') \leq \varphi(y)$ for all $y' \in U$ or, equivalently, if for any $n \in \mathbb{Z}$ the set $\{y \in Y \mid \varphi(y) \geq n\}$ is a closed subset of Y.

Theorem 1.18. *(Semicontinuity Theorem for flat schemes) Let $f : X \to Y$ be a projective morphism of Noetherian schemes and let $\mathscr{F}$ be a coherent sheaf on X, flat over Y. Then the function*

$$h^i(y, \mathscr{F}) = \dim_{k(y)} H^i(X_y, \mathscr{F}_y)$$

is an upper semicontinuous function on Y for all $i \geq 0$.
If moreover Y is integral and, for some i, the function $H^i(y, \mathscr{F})$ is constant on Y, then $R^i f_*(\mathscr{F})$ is locally free on Y and, for every y, the natural map $R^i f_*(\mathscr{F}) \otimes k(y) \to H^i(X_y, \mathscr{F}_y)$ is an isomorphism.

We end this first chapter stating one, last, theorem aboute the relations between cohomology and base change, as it will be needed in the next chapters.

Theorem 1.19. *(Cohomology and base change) Let $f : X \to Y$ be a projective morphism of Noetherian schemes and let $\mathscr{F}$ be a coherent sheaf on X, flat over Y and take $y \in Y$.*

a) If the natural map

$$\varphi^i(y) : R^i f_*(\mathscr{F} \otimes k(y)) \to H^i(X_y, \mathscr{F}_y)$$

is surjective, then it is an isomorphism in a suitable neighbourhood of y;

b) Assume that $\varphi^{i-1}(y)$ is surjective. Then, the following are equivalent:

- *i) $\varphi^{i-1}(y)$ is also surjective;*
- *ii) $R^i f_*(\mathscr{F})$ is locally free in a neighbourhood of y.*

Proof. See [H III, 12.11]. ☐

Chapter 2

Hilbert Schemes

The target of this second chapter is the construction of the Hilbert Schemes, that has been introduced by Grothendieck in [FGA], one of his fundamental work about scheme theory, as schemes representing a suitable functor. Since the contents of [FGA] were mostly schematic and sketched, many authors have rearranged his work, trying to give it more formal consistency. In particular, we will refer to [Sernesi] and to the collection of notes coming out from the "Advanced School in Algebraic Geometry", which was held at ICTP in Trieste (IT), July 7-18, 2003, collected by the speakers in [FGAE].
We start our discussion introducing an older, well-known object of algebraic geometry, which are Grassmannians; we go on discussiong a special regularity condition due to Mumford and a further analysis about flatness, ending into the the construction of Hilbert Schemes.

Starting from this chapter, if not otherwise stated by *a scheme* we will always mean a Noetherian separated scheme over a fixed field k, not necessarily algebraically closed.

2.1 Grassmannians

The ideas behind Grassmannians come from one of the first and most famous books written by Hermann Günther Grassmann in 1844: *Die Lineale Ausdehnungslehre ein neuer Zweig der Mathematik* and the construction of the projective space formalized by Julius Plücker.

While introducing an abstract notation for operations in a general set, including for the first time the exterior product, Grassmann starts also a first analysis of particular subspaces of what will be lately called a vector space over a field k: the linear subspaces of a given dimension d, which we call *d-planes*. His idea is partially related to the homogeneous coordinates introduced by Plücker for the space $\mathbb{P}^n$ of all lines through the origin in an $n+1$-dimensional affine space, reducing to the computation of some minors of a given matrix. The construction made by Plücker is still one of the techniques used today to introduce Grassmannians (or Grassmann varieties), as it provides directly some useful properties, for example their dimension, and has a direct application in Schubert calculus. However, this standard approach doesn't allow to get other important fact about Grassmannians, in particular that they are projective varieties, so that along the XIX century, other techniques were introduced in order to study them. One of the most recent is due to the groundwork by Grothendieck about scheme theory and the use of category theory. It turns out in fact that Grassmann varieties are objects that "represent" a special functor linking the categories of schemes to the one of sets.

As we will see along this chapter, Grassmann varieties are not only a particular case of Hilbert scheme, but they are actually used in order to construct it: we will indeed regard them as closed subschemes of a suitable Grassmannian. That's why we start defining Grassmannians. For this purpose we will follow [Ar], and just sketch other approaches, giving references along the text for a more specific discussion. The functorial definition will be anyway obtained in the following sections.

Let us consider a vector space V of dimension $n+1$ over a field, which for simplicity we may take to be $\mathbb{C}$, and consider the projective space $\mathbb{P}^n = \mathbb{P}(V)$ with homogeneous coordinates $x_0, \dots, x_n$. Usually this space is seen as the space of lines in V, but since Grothendieck's work, the viewpoint of $\mathbb{P}^n$ as the space of all hyperplanes of V has become the most common one. One can also see $\mathbb{P}(V)$ as the set of lines in the dual vector space V^*.

Definition 14. We set

$$G(k+1, n+1) := \{\, (k+1)\text{-dimensional linear subspaces of } V^*\},$$

or, equivalently,

$$G(k+1, n+1) := \{\, (n-k)\text{-dimensional quotients of } V\},$$

and we call it the *Grassmannian* of k-dimensional linear subspaces of $\mathbb{P}^n$. In [Ar] it is underlined that there is no standard notation for this object and that many authors denote it also by $G(k, n)$ or $G(n+1, n-k)$, so we need to pay attention to what we are refering to.

It is a well known fact that to a k-dimensional linear subspace Λ, we may associate a $(k+1)\times(n+1)$ matrix

$$\begin{pmatrix} a_{00} & \dots & a_{0n} \\ \vdots & & \vdots \\ a_{k0} & \dots & a_{kn} \end{pmatrix} \tag{2.1}$$

called *Plücker matrix*, where the rows are the coordinates of a basis of Λ and at least one of the $k+1$ minors is non-zero. Of course this representation for Λ changes if we change its basis, which implies that the Plücker matrix is multiplied on the left by a non-degenerate square matrix of order $k+1$ corresponding to such a change. This means that, once we provide that

$$\begin{vmatrix} a_{00} & \cdots & a_{0k} \\ \vdots & & \vdots \\ a_{k0} & \cdots & a_{kk} \end{vmatrix} \neq 0$$

Λ can be represented in a "unique way" by a matrix

$$\begin{pmatrix} 1 & \cdots & 0 & b_{0k+1} & \cdots & b_{0n} \\ & \ddots & & \vdots & & \vdots \\ 0 & \cdots & 1 & b_{kk+1} & \cdots & b_{kn} \end{pmatrix} \tag{2.2}$$

As at least one of the $k+1$ minors in (2.1) is non-zero, computing all maximal minors we find $N+1$ "coordinates", i.e. an $N+1$-tuple, with $N := \binom{n+1}{k+1}$. This is indeed a point in the projective space $\mathbb{P}^N = \mathbb{P}(\wedge^{k+1}V)$, and the coordinates of this point are called *Plücker coordinates*, usually denoted by $p_{i_0,\dots,i_k}$. The map

$$\varphi_{k,n} : G(k+1, n+1) \to \mathbb{P}^N$$

that associates to $\Lambda \in G(k+1, n+1)$ the point given by the Plücker coordinates is called *Plücker embedding*, this means that $\varphi_{k,n}$ associates to the space generated by the row vectors $v_0, \dots, v_k$ of (2.1) the points in $\mathbb{P}^N$ whose coordinates are the maximal minors of the matrix. It can be shown, see [KL] or[Sh1], that this map is indeed an embedding of $G(k+1, n+1)$ in $\mathbb{P}^N$ as an algebraic variety and that Plücker coordinates satisfy particular quadratic relations, called *Plücker relations* that provide homogeneous ideal of the projective variety (see again [KL] or [Ha, §6]).
Moreover, by (2.2), we see that $G(k+1, n+1)$ contains an open affine subset which is isomorphic to an affine space of dimension $(k+1)(n-k)$, of coordinates b_{0k+1}, ..., b_{kn} that can be described as the set of k-planes not meeting the $(n-k-1)$-plane of equations $x_0 = \ldots = x_k = 0$.
If we call $U_{i_0,\dots,i_k}$ those subsets and we set

$$V_{i_0,\dots,i_k} := \{p_{i_0,\dots,i_k} \neq 0\},$$

we have that

$$\varphi_{k,n}(G(k+1, n+1)) \cap V_{i_0,\dots,i_k} = \varphi_{k,n}(U_{i_0,\dots,i_k}).$$

By the same consideration about maximal minors, $G(k+1, n+1)$ can be covered by N affine pieces, which allows us to say that Grassmannians are actually manifolds of dimension $(k+1)(n-k)$.

Since the charts we obtained are affine spaces we find out also that Grassmannians are smooth, and moreover they are compact as they are projective.

These two properties might be obtained also using a different approach, as it is suggested in [Bar]. Smoothness is provided showing that for every two charts (U, φ) and (V, ψ) such that $U \cap V \neq \emptyset$ we have that

$$\varphi \circ \psi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$$

is a diffeomorphism, i.e. a bijection with smooth inverse. Such a result is obtained working with linear maps $A : P \to Q$, where P and Q are complementary subspaces of the vector space V such that $V = P \oplus Q$, considering maps of the form $\phi : L(P, Q) \to U_Q$ associating to each linear map A its graph $\Gamma(A)$, where $L(P, Q)$ is the vector space of linear maps from P to Q and U_Q denotes the subset of the Grassmannian $G(k+1, n+1)$ consisting of $(k+1)$-dimensional linear subspaces interacting trivially with Q, and taking (U_Q, ϕ^{-1}) as charts.

In order to achieve compactness it is used another approach involving orthogonal projections, leading to a map $\Phi : G(k+1, n+1) \to H(n+1)$, with $H(n+1)$ the space on symmetric $(n+1) \times (n+1)$-matrices, that turns out to be an homeomorphism onto its image.

We last remark that it can be shown that Grassmannians are proper over $\operatorname{Spec} \mathbb{Z}$ (see [FGA, §5]).

2.2 Castelnuovo-Mumford regularity and flattening stratifications

This second section will deal with a regularity condition on coherent sheaves over projective spaces introduced by Mumford in [D. Mumford, *Lectures on Curves on an Algebraic Surface*, Annals of Mathematics Studies vol. 59, Princeton University Press (1966)], recalling a result he attributes to Guido Castelnuovo, from which the name *Castelnuovo-Mumford regularity.* We then proceed with some further properties of flat families, keeping in mind what we have already seen in Section 1.4.

The following definition of Castelnuovo-Mumford regularity makes sense for a coherent sheaf $\mathscr{F}$ on any projective scheme X endowed with a very ample line bundle $\mathscr{O}(1)$, but for simplicity we will consider only the case $X = \mathbb{P}^r$, as done in [Sernesi], which will be the main reference for all this section.

Definition 15. Take $m \in \mathbb{Z}$. A coherent sheaf $\mathscr{F}$ on $\mathbb{P}^r$ is said to be *m-regular* if $H^i(\mathscr{F}(m-i)) = 0$ for all $i \geq 1$.

We certainly have sheaves that are m-regular, as by Serre's Vanishing Theorem, every coherent sheaf $\mathscr{F}$ on $\mathbb{P}^r$ is m-regular for some integer m.

Theorem 2.1. *(Mumford) Suppose that $\mathscr{F}$ is m-regular on $\mathbb{P}^r$ and set $\mathscr{O} = \mathcal{O}_{\mathbb{P}^r}$. Then:*

i) The natural map

$$H^0(\mathscr{F}(k)) \otimes H^0(\mathscr{O}) \to H^0(\mathscr{F}(k+1))$$

is surjective for all $k \geq m$;

ii) $H^i(\mathscr{F}(k)) = 0$ for all $i \geq 1$ and $k \geq m - i$; in particular, for all $n \geq m$ the sheaf $\mathscr{F}$ is n-regular;

iii) $\mathscr{F}(m)$, and thus $\mathscr{F}(k)$ for all $k \geq m$, is generated by its global sections.

Furthermore, if the sequence

$$0 \to \mathscr{F}(-1) \to \mathscr{F} \to \mathscr{G} \to 0$$

is an exact sequence, then also $\mathscr{G}$ is m-regular.

Proof. We show just the first two conditions. For the remaning part of the proof see [Sernesi, 4.1.1]. We proceed by induction on r.

We first recall that the *set of associated points to a quasi-coherent sheaf $\mathscr{F}$* on a scheme X, denoted by $\operatorname{Ass}(\mathscr{F})$, is the set of all points $x \in X$ which are *associated* to $\mathscr{F}$, i.e. whose maximal ideal $\mathfrak{m}_x$ is associated to the $\mathcal{O}_{x,X}$-module $\mathscr{F}_x$.

As in the case $r = 0$ we have nothing to prove, we assume $r \geq 1$.
Consider an hyperplane H not containing any point of $\mathrm{Ass}(\mathscr{F})$, which exists because $\mathrm{Ass}(\mathscr{F})$ is a finite set.
Tensoring now the exact sequence

$$0 \to \mathscr{O}(-H) \to \mathscr{O} \to \mathscr{O}_H \to 0$$

by $\mathscr{F}(k)$ and setting $\mathscr{F}_H := \mathscr{F} \otimes H$, we get another exact sequence

$$0 \to \mathscr{F}(k-1) \to \mathscr{F}(k) \to \mathscr{F}_H(k) \to 0.$$

Now, for each $i > 0$ we obtain an exact sequence

$$H^i(\mathscr{F}(m-i)) \to H^i(\mathscr{F}_H(m-i)) \to H^{i+1}(\mathscr{F}(m-i-1)),$$

which implies that $\mathscr{F}_H$ is m-regular on H, so that by induction both $i)$ and $ii)$ hold for $\mathscr{F}_H$.
Let us consider then the exact sequence

$$H^{i+1}(\mathscr{F}(m-i-1)) \to H^{i+1}(\mathscr{F}_H(m-i)) \to H^{i+1}(\mathscr{F}(m-i)).$$

The two extremes are zero, the left one by the previous step, the right one by m-regularity, for every $i \geq 0$. Therefore $\mathscr{F}$ is $(m+1)$-regular. By induction this proves condition $ii)$.
To prove the first condition consider the following commutative diagram

$$\begin{array}{ccccc}
 & & H^0(\mathscr{F}(k)) \otimes H^0(\mathscr{O}(1)) & \xrightarrow{u} & H^0(\mathscr{F}_H(k)) \otimes H^0(\mathscr{O}_H(1)) \\
 & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle t} \\
H^0(\mathscr{F}(k)) & \longrightarrow & H^0(\mathscr{F}(k+1)) & \xrightarrow{v} & H^0(\mathscr{F}_H(k+1))
\end{array}$$

The map u is surjective when $k \geq m$ as $H^1(\mathscr{F}(k-1)) = 0$; moreover, the map t is surjective for $k \geq m$ by condition $i)$ for $\mathscr{F}_H$. Hence $v \circ w$ is surjective. It follows that $H^0(\mathscr{F}(k+1))$ is generated by $\mathrm{Im}(w)$ and by $H^0(\mathscr{F}(k))$ again for $k \geq m$. But, $H^0(\mathscr{F}(k)) \subset \mathrm{Im}(w)$ because the inclusion $H^0(\mathscr{F}) \subset H^0(\mathscr{F}(k+1))$ is just the multiplication by H, hence w itself is surjective.

□

A first remarkable consideration coming out from the notion of m-regularity is the fact that if a sheaf $\mathscr{F}$ is m-regular, then the associated graded ring $\Gamma_*(\mathscr{F})$ can be generated by elements of degree smaller than or equal to m, as this condition is equivalent to the surjectivity of the map at point $i)$ of Proposition 2.1. In particular, if an ideal sheaf $\mathscr{I}$ in $\mathcal{O}_{\mathbb{P}^r}$ is m-regular, then the homogeneous ideal associated into the graded ring $k[x_0, \ldots, x_r]$ is again generated by elements of degree smaller than or equal to m.

It can be proved also a kind of converse result.

Proposition 2.2. *Let*

$$0 \to \mathscr{F}(-1) \to \mathscr{F} \to \mathscr{G} \to 0$$

be a short exact sequence of coherent sheaves on $\mathbb{P}^r$, and assume that $\mathscr{G}$ is m-regular. Then:

i) $H^i(\mathscr{F}(k)) = 0$ *for* $i \geq 2$ *and* $k \geq m - i$;

ii) $h^1(\mathscr{F}(k-1)) \geq h^i(\mathscr{F}(k))$ *for* $k \geq m - 1$;

iii) $H^1(\mathscr{F}(k)) = 0$ *for* $k \geq m - 1 + h^1(\mathscr{F}(m-1))$.

On top of that, $\mathscr{F}$ is $m + h^1(\mathscr{F}(m-1))$-regular.

Proof. See [Sernesi 4.1.3]. □

We now proceed providing a characterization of m-regularity, which will allow us to relate the definition to Hilbert polynomials via the introduction of a particular kind of resolution for coherent sheaves on $\mathbb{P}^r$.

Theorem 2.3. *A coherent sheaf $\mathscr{F}$ on $\mathbb{P}^r$ is m-regular if and only if it has a resolution of the form*

$$\cdots \to \mathcal{O}(-m-2)^{b_2} \to \mathcal{O}(-m-1)^{b_1} \to \mathcal{O}(-m)^{b_0} \to \mathscr{F} \to 0$$

for some nonnegative integers $(b_i)_{i\in\mathbb{Z}}$.

Proof. See [Sernesi, 4.1.4]. □

Definition 16. Consider a sequence $\sigma_1, \dots, \sigma_N$ of N sections of $\mathcal{O}_{\mathbb{P}^r}(1)$. This sequence will be called *$\mathscr{F}$-regular* if the sequences of sheaf homomorphisms induced by mutiplication by $\sigma_1, \dots, \sigma_N$ are exact.

As it can be shown that $\mathscr{F}$-regular sequences of any length exist by choosing a section σ_{i+1} not containing any poin of $\mathrm{Ass}(\mathscr{F}_i)$, we see that any general N-tuple $(\sigma_1, \dots, \sigma_N) \in H^0(\mathcal{O}_{\mathbb{P}^r}(1))^N$ is an $\mathscr{F}$-sequence.

Definition 17. Let $\mathscr{F}$ be a coherent sheaf on $\mathbb{P}^r$, and let $(\mathbf{b}) = (b_0, \dots, b_N)$ be a sequence of nonnegative integers such that $N > \dim[\mathrm{Supp}(\mathscr{F})]$. We say that $\mathscr{F}$ is a $(\mathbf{b})$-*sheaf* if there exists an $\mathscr{F}$-regular sequence $\sigma_1, \dots, \sigma_N$ of sections of the twisting sheaf of Serre of $\mathbb{P}^r$ such that $h^0(\mathscr{F}_i(-1)) \le b_i$, for $i = 0, \dots, N$, where $\mathscr{F}_0 = \mathscr{F}$ and $\mathscr{F}_i = \mathscr{F}/(\sigma_1, \dots, \sigma_i)\mathscr{F}(-1)$ for $i \ge 1$.

From the definition we directly find that if $\mathscr{F}$ is a $(\mathbf{b})$-sheaf, then $\mathscr{F}_1$ is a $(b_1, \dots, b_N)$-sheaf and it is almost clear that for every coherent sheaf $\mathscr{F}$ on $\mathbb{P}^r$ there is a sequence $(\mathbf{b})$ such that $\mathscr{F}$ is a $(\mathbf{b})$-sheaf. Moreover, a subsheaf of a $(\mathbf{b})$-sheaf is clearly a $(\mathbf{b})$-sheaf again, in particular, every ideal sheaf of $\mathscr{I} \subset \mathcal{O}_{\mathbb{P}^r}$ is a $(\mathbf{0})$-sheaf, as $\mathcal{O}_{\mathbb{P}^r}$ is itself a $(\mathbf{0})$-sheaf.
We link now $(\mathbf{b})$-sheaves to Hilbert polynomials.

Lemma 2.4. *Let*

$$0 \to \mathscr{F}(-1) \to \mathscr{F} \to \mathscr{G} \to 0$$

be a short exact sequence of coherent sheaves on $\mathbb{P}^r$. If the Hilbert polynomial of $\mathscr{F}$ is

$$p_{\mathscr{F}}(k) = \sum_{i=0}^{r} a_i \binom{k+i}{i}$$

then

$$p_{\mathscr{G}}(k) = \sum_{i=0}^{r-1} a_{i+1} \binom{k+i}{i}.$$

Proposition 2.5. *Let $\mathscr{F}$ be a $(\mathbf{b})$-sheaf, let $s = \dim[\mathrm{Supp}(\mathscr{F})]$ and consider the Hilbert polynomial of $\mathscr{F}$*

$$p_{\mathscr{F}}(k) = \sum_{i=0}^{r} a_i \binom{k+i}{i}.$$

Then:

- *i) for each $k \ge -1$ it holds $h^0(\mathscr{F}(k)) \le \sum_{i=0}^{s} b_i \binom{k+i}{i}$;*
- *ii) $a_s \le b_s$ and $\mathscr{F}$ is not only a $(\mathbf{b})$-sheaf, but also a $(b_0, \dots, b_{s-1}, a_s)$-sheaf.*

Proof. i) We proceed by induction on s.
If $s = 0$, then $a_0 = h^0(\mathscr{F}) = h^0(\mathscr{F}(-1)) \le b_0$.
Assume now $s \ge 1$. We have an exact sequence

$$0 \to \mathscr{F}(-1) \to \mathscr{F} \to \mathscr{F}_1 \to 0$$

with $\mathscr{F}_1$ being a $(b_1, \dots, b_N)$-sheaf, and $\dim[\mathrm{Supp}(\mathscr{F}_1)] = s-1$. Then

$$h^0(\mathscr{F}(k)) - h^0(\mathscr{F}(k-1)) \le h^0(\mathscr{F}_1(k))$$

and using the inductive hypothesis

$$h^0(\mathscr{F}_1(k)) \le \sum_{i=0}^{s-1} b_{i+1} \binom{k+1}{i}.$$

Since $h^0(\mathscr{F}(-1)) \leq b_0$, by induction on $k \geq -1$ we conclude.
ii) By Lemma 2.4 and proceeding by induction on s we conclude. □

Using this last result, we may produce a numerical criterion for m-regularity linked with the Hilbert polynomials involved, using the following notion.

Definition 18. The following polynomials, defined by induction for each integer $r \geq -1$ as

$$P_{-1} := 0$$

$$P_r(x_0, \ldots, x_r) := P_{r-1}(x_1, \ldots, x_r) + \sum_{i=0}^{r} x_i \binom{P_{r-1}(x_1, \ldots, x_r) - 1 + i}{i}$$

are called $(\mathbf{b})$*-polynomials*.

Theorem 2.6. *Let $\mathscr{F}$ be a $(\mathbf{b})$-sheaf on $\mathbb{P}^r$ and let*

$$p_{\mathscr{F}}(k) = \sum_{i=0}^{r} a_i \binom{k+i}{i}$$

be its Hilbert polynomial. Let $(c_0, \ldots, c_r)$ be a sequence of integers such that $c_i \geq b_i - a_i$, for $0 \leq i \leq r$ and $m = P_r(c_0, \ldots, c_r)$. Then $m \geq 0$ and $\mathscr{F}$ is m-regular. Furthermore, $\mathscr{F}$ is $P_{s-1}(c_0, \ldots, c_{s-1})$-regular if $s = \dim[\mathrm{Supp}(\mathscr{F})]$.

Proof. We prove the claim by induction on r.
If $r = 0$, then $m = 0$ and $\mathscr{F}$ is n-regular for every $n \in \mathbb{Z}$.
Assume now that $r \geq 1$. As in the proof of Proposition 2.5 take the exact sequence

$$0 \to \mathscr{F}(-1) \to \mathscr{F} \to \mathscr{F}_1 \to 0$$

with $\mathscr{F}_1$ a $(b_1, \ldots, b_N)$-sheaf supported on $\mathbb{P}^{r-1}$. From Lemma 2.4 and using inductive hypothesis we deduce that $n = P_{r-1}(c_1, \ldots c_r)$ is greater than or equal to 0 and that $\mathscr{F}_1$ is n-regular. Now, from Proposition 2.2 we obtain that $\mathscr{F}$ is $[n + h^1(\mathscr{F}(n-1))]$-regular, and $h^i(\mathscr{F}(n-1)) = 0$ for $i \geq 2$. Thus

$$h^1(\mathscr{F}(n-1)) = h^0(\mathscr{F}(n-1)) - \chi(\mathscr{F}(n-1)) \leq \sum_{i=0}^{r} (b_i - a_i) \binom{n-1+i}{i}$$

by point *i*) of Proposition 2.5. It follows that $\mathscr{F}$ is $[n + \sum_{i=0}^{r} c_i \binom{n-1+i}{i}]$-regular by Theorem 2.1, which proves the first assertion.
The second part is just a direct consequence of part *ii*) of Proposition 2.5, using the fact that

$$P_r(x_0, \ldots, x_t, 0, \ldots, 0) = P_t(x_0, \ldots, x_t).$$

□

The following corollary follows directly from the facts that a sheaf of ideals of $\mathcal{O}_{\mathbb{P}^r}$ is a $(\mathbf{0})$-sheaf and the use of $F_r(x_0, \ldots, x_r) = P_r(-x_0, \ldots, -x_r)$ as polynomials.

Corollary 2.7. *For each $r \geq 0$ there exists a polynomial $F_r(x_0, \ldots, x_r)$ such that every sheaf of ideals $\mathscr{I} \subset \mathcal{O}_{\mathbb{P}^r}$ having the Hilbert polynomial*

$$p_{\mathscr{I}}(k) = \sum_{i=0}^{r} a_i \binom{k+r}{i}$$

is m-regular, where $m = F_r(a_0, \ldots, a_r)$, and $m \geq 0$.

We will now add this new ingredient to go further in the study of flat families, as done in [Sernesi, §4.2].

We start fixing some notations. Fix a scheme S and a coherent sheaf $\mathscr{F}$ on $\mathbb{P}^r \times S$. Consider a morphism of schemes $g : T \to S$ and the diagram

$$\begin{array}{ccc} \mathbb{P}^r \times T & \xrightarrow{h} & \mathbb{P}^r \times S \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ T & \xrightarrow{g} & S \end{array}$$

where $h = id \times g$. For every open set $U \subset S$ we have homomorphisms

$$H^j(\mathbb{P}^r \times U, \mathscr{F}) \to H^j(\mathbb{P}^r \times g^{-1}(U), h^*\mathscr{F}) \to H^0(g^{-1}(U), R^j q_*(h^*\mathscr{F}))$$

and therefore a homomorphism

$$R^j p_* \mathscr{F} \to g_*[R^j q_*(h^*\mathscr{F})]$$

which corresponds to a homomorphism

$$g^*(R^j p * \mathscr{F}) \to R^j q_*(h^*\mathscr{F}).$$

The following asymptotic result holds for $j = 0$.

Proposition 2.8. *For $m >> 0$ the homomorphism $g^*(p_*\mathscr{F}(m)) \to q_*(h^*\mathscr{F}(m))$ is an isomorphism and, if T is Noetherian, we have that $R^j q_*(h^*\mathscr{F}(m)) = 0$ for all $j \geq 1$.*

Proof. See [Sernesi, 4.2.4]. □

In particular, in [FGAE, Lemma 5.1], Nitin Nirsure emphasizes that we do not need any flatness hypothesis in order to get this result, but we have to pay a little price, which is the fact that the condition $m >> 0$ may depend on the morphism $g : T \to S$, producing also a different proof from the one given in [Sernesi, 4.2.4], following [D. Mumford, *Lectures on Curves on an Algebraic Surface*, Annals of Mathematics Studies vol. 59, Princeton University Press (1966)].
If we consider the special case of $g : \operatorname{Spec} k(s) \to S$ being the inclusion in S of a point $s \in S$, setting $\mathscr{F}(s) := \mathscr{F} \mid_{X \times \{s\}}$ and given a scheme $\mathcal{X}$ denoting by $\mathcal{X}(s)$ its fibre at s, for the homomorphism of Proposition 2.8 we will use the notation

$$t^j(s) : R^j p_*(\mathscr{F})_s \otimes k(s) \to H^j(\mathbb{P}^r(s), \mathscr{F}(s)).$$

Adapting now Theorem 1.19 to our particular case we have the following results.

Theorem 2.9. *Let $\mathscr{F}$ be coherent on $\mathbb{P}^r \times S$ and flat over S, let $s \in S$ and $j \geq 0$. Then:*

i) if $t^j(s)$ is surjective, then it is an isomorphism;

ii) if $t^{j+1}(s)$ is an isomorphism then $R^{j+1}p_*(\mathscr{F})$ is free at s if and only if $t^j(s)$ is an isomorphism;

iii) if $R^j p_*(\mathscr{F})$ is free at s for all $j \geq j_0 + 1$, then $t^j(s)$ is an isomorphism for all $j \geq j_0$.

Proof. See [Sernesi, 4.2.5]. □

Corollary 2.10. *Let $\mathcal{X} \to S$ be a projective morphism and consider a coherent sheaf $\mathscr{F}$ on $\mathcal{X}$, flat over S. Then:*

i) if $H^{j+1}(\mathcal{X}(s), S(s)) = 0$ for some $s \in S$ and $j \geq 0$, then $R^{j+1}p_*(\mathscr{F})_s = 0$, and

$$t^j(s) : R^j p_*(\mathscr{F})_s \otimes k(s) \to H^j(\mathcal{S}(s), S(s))$$

is an isomorphism;

ii) let j_0 be an integer such that $H^j(\mathcal{X}(s), S(s)) = 0$ for all $j \geq j_0 + 1$ and $s \in S$, then $t^{j_0}(s)$ is an isomorphism for all $s \in S$;

iii) let $j_0 \geq 0$ be an integer. Then there is a nonempty open set $U \subset S$ such that $t^{j_0}(s)$ is an isomorphism for all $s \in U$.

Proof. See [Sernesi 4.2.6]. □

After this brief recall about the first chapter, we move on with the notion of *stratification* of a scheme, and in particular, *flattening* one.

Definition 19. Let S be a scheme. A *stratification* of S consists of a set of finitely many locally closed subschemes $\{S_1, \dots, S_n\}$, called *strata*, which are pairwise disjoint and satisfy the following condition: $S = \cup_{i=1}^n S_i$, that is we have a surjective morphism

$$\coprod_{i=1}^{n} S_i \to S.$$

Definition 20. Let $\mathscr{F}$ be a coherent sheaf on a scheme S. For each $s \in S$ set

$$e(s) := \dim_{k(s)}[\mathscr{F}_s \otimes k(s)].$$

If we fix a point $s \in S$ and call briefly $e = e(s)$, we may consider $a_1, \dots, a_e \in \mathscr{F}_s$ such that their images in $\mathscr{F}_s \otimes k(s)$ form a basis. As a consequence of Nakayama's lemma (see e.g. [Bo, §1.4] or [Eis, §4.1]), it follows that the morphism $f_s : \mathscr{O}_{S,s}^e \to \mathscr{F}_s$ defined by the elements a_i is surjective, so that we find an open neighbourhood U of s to which f extends, defining a surjective homomorphism $f : \mathscr{O}_U^e \to \mathscr{F}|_U$. Applying a similar argument to $\ker(f_s)$ we may find an affine open neighbourhood $U(s)$ of s contained in U and an exact sequence

$$\mathscr{O}_{U(s)}^d \xrightarrow{g} \mathscr{O}_{U(s)}^e \xrightarrow{f} \mathscr{F}|_{U(s)} \to 0.$$

Using this easy construction, with some more work (see [Sernesi, 4.2.7]), it might be proved the following result.

Theorem 2.11. *Let S be a scheme and $\mathscr{F}$ be a coherent sheaf on S. There is a unique stratification $\{Z_e\}_{e\geq 0}$ of S such that if $q : T \to S$ is a morphism, the sheaf $q^*(\mathscr{F})$ is locally free if and only if the morphism q factors through the disjoint union of the Z_e's, i.e. we have a sequence $T \to \coprod_e Z_e \to S$. Moreover, the strata are indexed so that for each $e = 0, 1, \dots$ the restriction of $\mathscr{F}$ to Z_e is locally free of rank e. Furthermore, for a given e, we have that*

$$\overline{Z}_e \subset \bigcup_{e' \geq e} Z_{e'}.$$

In particular, if E is the highest integer such that $Z_E \neq \emptyset$, then Z_E is closed.
On top of that, the stratification $\{Z_e\}_{e\geq 0}$ commutes with base change.

The above theorem describes a natural way to construct stratifications on schemes, and the family of strata $\{Z_e\}_{e\geq 0}$ is usually called *the stratification defined by the sheaf $\mathscr{F}$*. Furthermore, as we can construct a stratification, we find out immediately that we are dealing with an object that indeed exists.
Nevertheless, as we said before, we are interested in a particular type of stratifications, having some further properties linked to flatness.

Definition 21. Let S be a scheme and $\mathscr{F}$ be a coherent sheaf on $\mathbb{P}^r \times S$. A stratification $\{S_1, \dots, S_n\}$ of S for $\mathscr{F}$ such that for every morphism $g : T \to S$ the sheaf

$$\mathscr{F}_g := (1 \times g)^*(\mathscr{F})$$

on $\mathbb{P}^r \times T$ is flat if and only if g factors through $\coprod S_i$, is called *a flattening stratification for $\mathscr{F}$.*

A priori, this kind of stratifications needs not to exist. We surely gain the existence of a flattening stratification if $r = 0$, as in this case the notion of flattening stratification and the stratification defined in Theorem 2.11 coincide. We can also notice that, if such a stratification exists, then it is unique. So, the tough part of the problem is proving that a flattening stratification exists for all $r \geq 1$.

Theorem 2.12. *For every coherent sheaf $\mathscr{F}$ on $\mathbb{P}^r \times S$, a flattening stratification exists.*

Since the proof of this result is rather technical, involving also a result on generic flatness (see [FGAE, Lemma 5.11]), we would like to avoid it, refering to [Sernesi, 4.2.11] or [FGAE, §5.4] for the complete argument.
What is anyway remarkable is that, along the explicit construction of the flattening stratification that is provided in the proof of Theorem 2.12, we find out that there are finitely many locally closed subsets $Y^1, \dots, Y^k$ of S such that, for each i, if we consider on Y^i the reduced scheme structure, then $\mathscr{F} \otimes \mathcal{O}_{Y^i \times \mathbb{P}^r}$ is flat over Y^i. This fact, joint with Theorem 1.17, ensures us that only finitely many polynomials $P^1, \dots, P^h$ occur as Hilbert polynomials of the sheaves $\mathscr{F}(s)$, for $s \in S$, and the strata $\{Z^1, \dots, Z^h\}$ obtained at the end of the proof are indexed by the Hilbert polynomials of the sheaves $\mathscr{F}(s)$. This indexing on the Hilbert polynomials is the motivation for the name "Hilbert schemes" of the structure we are going to build in Section 2.3.

2.3 Construction of Hilbert Schemes

Now we have developed all the tools we need in order to construct the so called *Hilbert schemes*. Complete self-contained references on the construction of Hilbert schemes are rather rare, and always refer to [FGA] and to [A. Grothendieck, "Techniques de construction et thèorémes d'existence en géométrie algébriques IV: les schémas de Hilbert", *Séminaire Bourbaki, vol. 221* (1860/61)].

In [FGAE], Nitin Nisture underlines that the original result by Grothendieck relied on Chow coordinates, and that the introduction of the notion of Castelnuovo - Mumford regularity led to a simplification in the construction of Hilbert schemes. The underlying idea is a generalization of the construction of Grassmannians (see Section 2.1) to a wider case of families of subschemes, so we start defining what Hilbert schemes are, and then we will show that they actually exist.

Consider a projective scheme Y and a closed embedding of Y into $\mathbb{P}^r$. Let us fix a numerical polynomial (see p. 2) of degree smaller than or equal to r, say

$$P(t) = \sum_{i=0}^{r} a_i \binom{t+r}{i}$$

where $P(t) \in \mathbb{Q}[t]$ and $a_i \in \mathbb{Z}$ for all i.

Definition 22. For every scheme S we define

$$Hilb^Y_{P(t)}(S)$$

to be the set of all flat families $\mathcal{X} \subset Y \times S$ of closed subschemes of Y, parametrized by S, with fibres having Hilbert polynomial $P(t)$.

As in the first chapter we saw that flatness is preserved by base change, this association defines a contravariant functor

$$Hilb^Y_{P(t)} : \text{Schemes}^{op} \to \text{Sets}$$

called the *Hilbert functor of* Y *relative to* $P(t)$, where "Schemes" denotes the category of locally Noetherian separated k-schemes.
If such a functor is representable by a scheme X, i.e. there is a scheme X and an isomorphism

$$\xi : \text{Hom}(X, \cdot) = h_X \to Hilb^Y_{P(t)}(\cdot),$$

then the scheme X representing it will be called the *Hilbert scheme of* Y *relative to* $P(t)$, will be denoted by $\text{Hilb}^Y_{P(t)}$ and thus it exists. In the case $Y = \mathbb{P}^r$ we may use the notation $Hilb^r_{P(t)}$ and $\text{Hilb}^r_{P(t)}$. Hence, we have now, theoretically, defined the object we are interested in.
By this way of presenting the notion, it is clear that the hard part is not the idea behind the structure, but the proof that such an idea is consistent. Before approaching this tough part, we introduce some related concepts, assuming that Hilbert schemes truly exist.

Definition 23. There is a flat family of closed subschemes of Y having Hilbert polynomial equal to $P(t)$, say $\mathcal{W} \subset Y \times \text{Hilb}^Y_{P(t)}$, parametrized by $\text{Hilb}^Y_{P(t)}$ and possessing the following universal property:

for each scheme S and each flat family $\mathcal{X} \subset Y \times S$ of closed subschemes of Y having Hilbert polynomial $P(t)$, there is a unique morphism $S \to \mathrm{Hilb}^Y_{P(t)}$, called the *classifying morphism*, such that

$$\mathcal{X} = S \times_{\mathrm{Hilb}^Y_{P(t)}} \mathcal{W} \subset Y \times S.$$

The family $\mathcal{W}$ is called the *universal family*, and the pair $(\mathrm{Hilb}^Y_{P(t)}, \mathcal{W})$ represents the functor $Hilb^Y_{P(t)}$.

This family is the universal element of the Hilbert functor, namely the element corresponding to the identity under the identification

$$\mathrm{Hom}(\mathrm{Hilb}^Y_{P(t)}, \mathrm{Hilb}^Y_{P(t)}) = Hilb^Y_{P(t)}(\mathrm{Hilb}^Y_{P(t)}).$$

If we recall that, given two covariant (contravariant) functors F and G, we may consider

$$\mathrm{Nat}(F, G) := \{\text{ natural transformations } F \to G\}$$

we may also provide a further, well-known fact: given the existence, we immediately gain the uniqueness of this object, by applying the following famous result (see e.g. [Rotman, 1.17])

Theorem 2.13. *(Yoneda Lemma) Let $\mathcal{C}$ be a category, let $A \in Obj(\mathcal{C})$, and let $G : \mathcal{C} \to Sets$ be a covariant functor. Then there is a bijection*

$$y : \mathrm{Nat}(\mathrm{Hom}_{\mathcal{C}}(A, \cdot), G) \to G(A)$$

between the natural trasformations $\mathrm{Hom}_{\mathcal{C}}(A, \cdot) \to G$ and the set $G(A)$, that associates to a functor τ the object $\tau_A(1_A)$, with $\tau_A : \mathrm{Hom}_{\mathcal{C}}(A, A) \to G(A)$.

Indeed, the Yoneda lemma ensures us that given an object $X \in \mathcal{C}$ and given the functor $h_X : \mathcal{C}^{op} \to Sets$ defined as $h_X(S) = \mathrm{Hom}_{\mathcal{C}}(S, X)$ the functor $X \mapsto h_X$ is fully faithful. Thus, from now on we may identify an object X with the functor h_X.

So, we need a way to prove that the Hilbert functor is representable, which is not so immediate. In order to reach this goal, we have to briefly introduce some notions and results, mostly due to Grothendieck, concerning *Zariski sheaves* and *covering by open functors*. Our guiding text along this part will be [GW, §8], as it focuses on our same purpose, but for a deepest insight in the theory behind the results and the complete proof of many of them, we will refer to [FGAE, §1], in which descent theory and Grothendieck topologies are treated.

Definition 24. A morphism $f : F \to G$ of contravariant functors from S-Schemes$^{op} \to Sets$ is called *representable* if for all schemes X and all morphisms $g : X \to G$ in S-Schemes$^{op} \to Sets$, the functor $F \times_G X$ is representable.

Let $F : S$-Schemes$^{op} \to Sets$ be a functor. If $i : U \to X$ is an open immersion of S-schemes and $\xi \in F(X)$ we will write, following what is usually done for sections of sheaves, simply $\xi|_U$ instead of $F(i)(\xi)$.

Definition 25. A functor $F : S$-Schemes$^{op} \to Sets$ is a *sheaf for the Zariski topology*, or simply a *Zariski sheaf* (on S-Schemes) if for every S-scheme X and for every open covering $X = \cup_{i \in I} U_i$ we have the following condition:
given $\xi_i \in F(U_i)$ for all $i \in I$ such that $\xi_i|_{(U_i \cap U_j)} = \xi_j|_{(U_i \cap U_j)}$ for all $i, j \in I$, there exists a unique element $\xi \in F(X)$ such that $\xi|_{U_i} = \xi_i$ for all $i \in I$.

The usual technique of gluing together morphisms allows us to state the following result.

Lemma 2.14. *Every representable functor $F : S$-Schemes$^{op} \to Sets$ is a sheaf for the Zariski topology.*

Proof. See [GW, Proposition 8.8]. □

We will now claim that every Zariski sheaf having a suitable Zariski covering by representable functors is itself representable, providing a representability criterion for S-Schemes, which will allow us to say that Hilbert schemes do exist.

Definition 26. Suppose that the functor $F : S\text{-Schemes}^{op} \to Sets$ is contravariant. An *open subfunctor* F' of F is a representable morphism $f : F' \to F$ that is an open immersion, i.e. the second projection $F' \times_F X \to X$ is an open immersion of schemes for every morphism $g : X \to F$ and every S-scheme X. If we have a family $(f_i : F_i \to F)_{i \in I}$ of open subfunctors such that for every S-scheme X and every morphism $g : X \to F$ the images of the second projections $F_i \times_F X \to X$ form a covering of X, then the family is said to be a *Zariski open covering of* F.

Theorem 2.15. *(Grothendieck) Let $F : S\text{-Schemes}^{op} \to Sets$ be a functor such that*

i) F is a sheaf for the Zariski topology;

ii) F has a Zariski open covering $(f_i : F_i \to F)_{i \in I}$ consisting of representable functors;

then, F is itself representable.

Proof. See [GW, Theorem 8.9]. □

So, we now state the main result of this chapter, again due to Grothendieck.

Theorem 2.16. *(Grothendieck) For every projective scheme $Y \subset \mathbb{P}^r$ and every numerical polynomial $P(t)$, the Hilbert scheme $\mathrm{Hilb}^Y_{P(t)}$ exists and is a projective scheme.*

The proof of Theorem 2.16 is provided via the use of Grassmannians, that's why we outlined their construction in Section 2.1 and we also stop now to prove that the so called *Grassmann functor* is representable. Once we have this result, we will use it to gain the representability of the Hilbert functor by realizing it as a subscheme of a suitable Grassmannian.
Moreover, as Grassmann varieties parametrize linear spaces of a fixed dimension n in k^N, which are the closed subschemes with Hilbert polynomials of the form $\binom{t+n-1}{n-1}$, they are a particular case of Hilbert schemes too, as already hinted in section 2.1. For a more complete study of Grassmannians of schemes see [Kleiman].

Definition 27. Fix a k-vector space V of dimension N and let $1 \leq n \leq N$. Let

$$\mathbf{G}_{V,n}(S) = \{\text{locally free quotients of rank } n \text{ of the free sheaf } V^\vee \otimes_k \mathcal{O}_S \text{ on } S\}.$$

We define a contravariant functor

$$\mathbf{G}_{V,n} : \mathrm{Schemes}^{op} \to Sets$$

and call it the *Grassmann functor*. If no confusion arise, we will denote it simply by $\mathbf{G}$.

Theorem 2.17. *The Grassmann functor $\mathbf{G}$ is represented by a scheme $G_n(V)$, together with a locally free quotient of rank n*

$$V^\vee \otimes_k \mathcal{O}_{G_n(V)} \to \mathcal{Q}.$$

The locally free quotient of rank n

$$V^\vee \otimes_k \mathcal{O}_{G_n(V)} \to \mathcal{Q}$$

is called the *universal quotient bundle* of the Grassmann functor and the object $G_n(V)$ representing it is called the *Grassmannian of n-dimensional subspaces of V*, or also the *Grassmannian of $(n-1)$-dimensional projective subspaces of $\mathbb{P}(V)$*.

Proof. Take a scheme S and an open cover $\{U_i\}$ of S. To give a locally free quotient of rank n of $V^\vee \otimes_k \mathcal{O}_S$ is equivalent to give one such a quotient over each open $\{U_i\}$, so that they patch together on the intersection $U_i \cap U_j$. Therefore $\mathbf{G}$ is a sheaf, satisfying this way the first assumption we need to apply Theorem 2.15.
Let us fix now a basis $\{e_k\}$ of the dual vector space $V^\vee$ and choose a set J of n distinct indices in $\{1, \ldots, N\}$. This way we have an induced decomposition $V^\vee = E' \oplus E''$, with E' a vector subspace of rank n and E'' a vector subspace of rank $N - n$. Using this set of indices we define a subfunctor $\mathbf{G}_J(S)$ of $\mathbf{G}(S)$ as the collection of locally free rank n quotients of $V^\vee \otimes_k \mathcal{O}_S \to \mathscr{F}$ such that the induced map $E' \otimes_k \mathcal{O}_S \to \mathscr{F}$ is surjective.
Let S be any scheme and $f : \mathrm{Hom}(\cdot, S) \to \mathbf{G}$ be a morphism of functors corresponding to a locally free rank n quotient $V^\vee \otimes_k \mathcal{O}_S \to \mathscr{F}$. The fibered product $S_J := \mathrm{Hom}(\cdot, S) \times_{\mathbf{G}} \mathbf{G}_J$ is represented

by the open subscheme of S supported on the points where $E' \otimes_k \mathcal{O}_S \to \mathscr{F}$ is surjective, as it can be indeed identified with the set of such points, which is an open set in S (see [GW, Proposition 8.4] for the proof of this fact). But now, as the S_J's cover S, we also have that the family of subfunctors provided by $\mathbf{G}_J$ is an open covering of the functor $\mathbf{G}$. So, it lasts to prove the representabilty of each $\mathbf{G}_J$.
If

$$q : V^\vee \otimes_k \mathcal{O}_S \to \mathscr{F}$$

is an element of $\mathbf{G}(S)$, then the induced map

$$\eta : E' \otimes_k \mathcal{O}_S \to \mathscr{F}$$

is surjective if and only if it is an isomorphism (see [GW, Corollary 8.12]). In this case the composition

$$\eta^{-1} \circ q : V^\vee \otimes_k \mathcal{O}_S \to E' \otimes_k \mathcal{O}_S$$

restricts to the identity on $E' \otimes_k \mathcal{O}_S$, hence it is determined by the composition

$$E'' \otimes_k \mathcal{O}_S \to V^\vee \otimes_k \mathcal{O}_S \to E' \otimes_k \mathcal{O}_S.$$

Thus we can identify the following objects:

$$\mathbf{G}_J(S) = \mathrm{Hom}(E'' \otimes_k \mathcal{O}_S, E' \otimes_k \mathcal{O}_S) = \mathrm{Hom}(E'', E') \otimes_k \mathcal{O}_S.$$

This proves that the funtor $\mathbf{G}_J$ is isomorphic to the functor $\mathrm{Hom}(\cdot, \mathbb{A}^{n(N-n)})$, hence it is representable by Theorem 2.15. $\square$

By the construction above it is also clear that $G_n(V)$ is smooth over $\mathrm{Spec}\,\mathbb{Z}$ and has relative dimension $n(N-n)$.
When $V = k^N$, the Grassmannian $G_n(k^N)$ is denoted by $G(n, N)$, recalling the notation we introduced before, furthermore, if $n = 1$, the functor $\mathbf{G}_{V,1}$ is represented by

$$G_1(V) = \mathrm{Proj}(\mathrm{Sym}(V^\vee)) = \mathbb{P}(V),$$

the $(N-1)$-dimensional projective space associated to V and in this case $\mathcal{Q} = \mathcal{O}_{\mathbb{P}(V)}(1)$.
From the theorem it follows that for all schemes S, the morphisms $f : S \to G_n(V)$ are in one-to-one correspondence via $f \leftrightarrow f^*\mathcal{Q}$ with the locally free quotients $V^\vee \otimes_k \mathcal{O}_S \to \mathscr{F}$. This property is called the *universal property of* $G_n(V)$.
The universal quotient bundle defines also an exact sequence of locally free sheaves on $G_n(V)$

$$0 \to \mathcal{K} \to V^\vee \otimes_k \mathcal{O}_{G_n(V)} \to \mathcal{Q} \to 0$$

called the *tautological exact sequence* and $\mathcal{K}$ is called the *universal subbundle*.
Also in this environment one can introduce the *Plücker morphism* and use it in order to show several properties, see again [Kleiman].

We are now ready to prove Theorem 2.16, following [Sernesi, 4.3.4]. The idea of the proof is the following:

1. prove the claim assuming that $Y = \mathbb{P}^r$ realizing Hilb^r_P as a closed subscheme of a Grassmannian, which we know to be representable from Theorem 2.17, using the flattening stratification given by Hilbert polynomials obtained from Theorem 2.12. In this first step we also get that the Hilbert scheme is quasi-projective;

2. prove that the Hilbert scheme Hilb^r_P is projective by proving that it is proper using the valuative criterion for properness, see e.g. [H II, 4.7];

3. move to a general closed subscheme Y of $\mathbb{P}^r$ and show that the functor $Hilb^Y_P$ is represented by a closed subscheme of Hilb^r_P, that is representable and projective by the previous steps;

Proof. (of Theorem 2.16)
We first prove the theorem for $Y = \mathbb{P}^r$.
By Corollary 2.7 it follows that there is an $m_0 \in \mathbb{Z}$ such that for every closed subscheme $X \subset \mathbb{P}^r$ having Hilbert polynomial $P(t)$, the sheaf of ideals $\mathscr{I}_X$ is m_0-regular, indeed it suffices to take $m_0 := F_r(-a_0, \ldots, -a_{r-1}, 1 - a_r)$. Hence for every $k \geq m_0$

$$h^i(\mathbb{P}^r, \mathscr{I}_X(k)) = 0 \tag{2.3}$$

for $i \geq 1$ and

$$h^0(\mathbb{P}^r, \mathscr{I}_X(k)) = \binom{k+r}{r} - P(k). \tag{2.4}$$

We may notice now that if $\mathscr{I}$ is the sheaf of ideals of the closed subscheme $X \subset \mathbb{P}^r$ and it is m-regular for $m \geq 0$, then the sheaf $\mathcal{O}_X$ is $(m-1)$-regular. Conversely, if $\mathcal{O}_X$ is $(m-1)$-regular and the restriction map

$$H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(m-1)) \to H^0(X, \mathcal{O}_X(m-1))$$

is surjective, then $\mathscr{I}$ is m-regular, as the sequence

$$0 \to \mathscr{I}(k) \to \mathcal{O}_{\mathbb{P}^r}(k) \to \mathcal{O}_X(k) \to 0$$

is exact for $k \geq m-1$. Using this remark we have that for all $k \geq m_0$ and all $i \geq 1$

$$h^i(X, \mathcal{O}_X(k)) = 0. \tag{2.5}$$

Set now $N := \binom{m_0+r}{r} - P(m_0)$, $V := H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(m_0))$ and consider $G = G_N(V)$ the Grassmann variety of N-dimensional vector subspaces of V, with $V^\vee \otimes_k \mathcal{O}_G \to \mathcal{Q}$ its universal quotient bundle, which is locally free of rank N on G. Call $p : \mathbb{P}^r \times G \to G$ the projection on the second component, so that we may identify

$$V \otimes_k \mathcal{O}_G = p_*[\mathcal{O}_{\mathbb{P}^r \times G}(m_0)].$$

Consider the composition

$$\begin{tikzcd}
p^*\mathcal{Q}^\vee(-m_0) \arrow[r] & V \otimes_k \mathcal{O}_{\mathbb{P}^r \times G}(-m_0) \arrow[r] \arrow[d, equal] & \mathcal{O}_{\mathbb{P}^r \times G} \\
& p^*p_*[\mathcal{O}_{\mathbb{P}^r \times G}(m_0)] \otimes \mathcal{O}_{\mathbb{P}^r \times G}(-m_0) &
\end{tikzcd}$$

The image of this composition is a sheaf of ideals, say $\mathbf{J}$.
We will see that $\mathrm{Hilb}^r_{P(t)}$ is a subscheme of the Grassmannian G, and we will do that using the stratification defined by the Hilbert polynomials in Theorem 2.12. Let $\mathcal{Z} \subset \mathbb{P}^r \times G$ be the closed subscheme defined by $\mathbf{J}$ and denote by $q : \mathcal{Z} \to G$ the restriction of the projection p to such a scheme.
Consider a flattening stratification

$$\coprod_i G^i \subset G$$

for $\mathcal{O}_{\mathcal{Z}}$ and let H be the stratum relative to the given polynomial $P(t)$. Our purpose is now to show that $H = \mathrm{Hilb}^r_{P(t)}$ with universal family $\mathcal{W} := H \times_G \mathcal{Z}$ given by the pullback of q to H

$$\begin{tikzcd}
H \times_G \mathcal{Z} \arrow[r] \arrow[d, "\pi"] & \mathcal{Z} \arrow[d, "q"] \\
H \arrow[r] & G
\end{tikzcd}$$

From the choice of H we have that $\mathcal{W}$ defines a flat family of closed subschemes of $\mathbb{P}^r$ with Hilbert polynomial equal to $P(t)$, thus we need to prove that it satisfies the universal property. Consider a flat family $\mathcal{X} \subset \mathbb{P}^r \times S$ of closed subschemes of $\mathbb{P}^r$ with Hilbert polynomial $P(t)$, with $f : \mathbb{P}^r \times S \to S$. From (2.3) and (2.5), using Theorem 2.9 and Corollary 2.10, it follows that

$$R^1 f_* \mathscr{I}_{\mathcal{X}}(m_0) = 0 = R^1 f_* \mathcal{O}_{\mathcal{X}}(m_0).$$

In particular, we find the following exact sequence on S:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & f_*\mathscr{I}_{\mathcal{X}}(m_0) & \longrightarrow & f_*\mathcal{O}_{\mathbb{P}^r\times S}(m_0) & \longrightarrow & f_*\mathcal{O}_{\mathbb{P}^r\times G} & \longrightarrow & 0 \\ & & & & \| & & & & \\ & & & & V\otimes_k \mathcal{O}_S & & & & \end{array}$$

If we apply again Theorem 2.9 and Corollary 2.10 taking $j=-1$ we find out that $f_*\mathscr{I}_{\mathcal{X}}(m_0)$ and $f_*\mathcal{O}_{\mathcal{X}}(m_0)$ are locally free and moreover the first one has rank N.
From the universal property of G now we get a unique morphism $g: S\to G$ such that

$$f_*\mathscr{I}_{\mathcal{X}}(m) = g^*\mathcal{Q}^\vee. \tag{2.6}$$

We then claim that for all $m >> m_0$

$$f_*\mathcal{O}_{\mathcal{X}}(m) = g^* p_*\mathcal{O}_{\mathcal{Z}}(m). \tag{2.7}$$

Indeed, for all $m >> m_0$ the sequence

$$0\to p_*\mathbf{J}(m)\to p_*\mathcal{O}_{\mathbb{P}^r\times G}(m)\to q_*\mathcal{O}_{\mathcal{Z}}\to 0 \tag{2.8}$$

is exact on G, while the sequence

$$0\to f_*\mathscr{I}_{\mathcal{X}}(m)\to f_*\mathcal{O}_{\mathbb{P}^r\times S}(m)\to f_*\mathcal{O}_{\mathcal{Z}}\to 0 \tag{2.9}$$

is exact on S. Since by definition of the morphisms g, p and f it follows that

$$g^*p_*\mathcal{O}_{\mathbb{P}^r\times G}(m) = f_*\mathcal{O}_{\mathbb{P}^r\times S}(m), \tag{2.10}$$

by (2.8) and (2.9) we only need to show that

$$f_*\mathscr{I}_{\mathcal{X}}(m) \cong g^*p_*\mathbf{J}(m) \tag{2.11}$$

for all $m >> m_0$.
From the surjection of sheaves $p^*\mathcal{Q}^\vee(m-m_0)\to\mathbf{J}(m)$ on $\mathbb{P}^r\times G$ we may obtain the following equality on G

$$p_*\mathbf{J}(m) = \mathrm{Im}[\mathcal{Q}^\vee\otimes p_*\mathcal{O}(m-m_0)\to p_*\mathcal{O}_{\mathbb{P}^r\times G}(m)],$$

hence, for all $m\geq m_0$, by applying g^* we have

$$\begin{aligned} g^*p_*\mathbf{J}(m) &= g^*\mathrm{Im}[\mathcal{Q}^\vee\otimes p_*\mathcal{O}_{\mathbb{P}^r\times G}(m-m_0)\to p_*\mathcal{O}_{\mathbb{P}^r\times G}(m)] \\ &\overset{(2.10)}{=} \mathrm{Im}[g^*\mathcal{Q}^\vee\otimes p_*\mathcal{O}_{\mathbb{P}^r\times S}(m-m_0)\to f_*\mathcal{O}_{\mathbb{P}^r\times S}(m)] \\ &\overset{(2.6)}{=} \mathrm{Im}[f_*\mathscr{I}_{\mathcal{X}}(m_0)\otimes f_*\mathcal{O}_{\mathbb{P}^r\times S}(m-m_0)\to f_*\mathcal{O}_{\mathbb{P}^r\times S}(m)]) \\ &= f_*\mathscr{I}_{\mathcal{X}}(m) \end{aligned}$$

from which (2.11), and thus (2.7) holds.
Relation (2.7) implies the following two facts:

i) g factors through H.

Indeed, from Proposition 2.8 it follows that for all $m >> m_0$

$$g^*q_*\mathcal{O}_{\mathcal{Z}}(m) = f_*(1\times g)^*\mathcal{O}_{\mathcal{Z}}(m)$$

and since the first member of (2.7) is a locally free sheaf of rank $P(m)$ for all such m, using Theorem 1.17 we deduce that $(1\times g)^*\mathcal{O}_{\mathcal{Z}}$ is flat over S and has Hilbert polynomial $P(t)$, so that g factors by the definition of H itself.

ii) $\mathcal{X} = S \times_H \mathcal{W}$.

Indeed we have that

$$\begin{aligned}
\mathcal{X} &= \mathrm{Proj}[\bigoplus_{m>>0} f_*\mathcal{O}_{\mathcal{X}}(m)] \\
&= \mathrm{Proj}[\bigoplus_{m>>0} g^* q_* \mathcal{O}_{\mathcal{Z}}(m)] \\
&= \mathrm{Proj}[\bigoplus_{m>>0} g^* \pi_* \mathcal{O}_{\mathcal{W}}(m)] \\
&= S \times_H \mathrm{Proj}[\bigoplus_{m>>0} \pi_* \mathcal{O}_{\mathcal{W}}(m)] \\
&= S \times_H \mathcal{W}
\end{aligned}$$

So, these two properties verify that $H = \mathrm{Hilb}^r_{P(t)}$ and π is the universal family we were looking for.
Up to now, we have provided the existence of a scheme $\mathrm{Hilb}^r_{P(t)}$ being quasi-projective. In order to prove that it is projective, it suffices to show that it is proper over k and to check properness we will use the valuative criterion for properness.
Let A be a discrete valuation k-algebra, with quotient field Q and residue field L, and let

$$\varphi : \mathrm{Spec}\, Q \to \mathrm{Hilb}^r_{P(t)}$$

be any morphism. The condition to apply the criterion is that φ extends to a morphism

$$\tilde{\varphi} : \mathrm{Spec}\, A \to \mathrm{Hilb}^r_{P(t)}.$$

Pulling back the universal family by φ we obtain a flat family

$$\mathcal{X} \subset \mathbb{P}^r \times \mathrm{Spec}\, Q$$

made by closed subschemes of the r-projective space with Hilbert polynomial $P(t)$.
Since $\mathrm{Spec}\, A$ is nonsingular of dimension 1 and $\mathrm{Spec}\, L = \mathrm{Spec}\, A \setminus \{\text{closed point}\}$, by [H III, 9.8] we get the existence of a flat family

$$\mathcal{X}' \subset \mathbb{P}^r \times \mathrm{Spec}\, A$$

extending $\mathcal{X}$. But now we may use the universal property of $\mathrm{Hilb}^r_{P(t)}$, which tells us that the family $\mathcal{X}'$ corresponds to a morphism $\tilde{\varphi} : \mathrm{Spec}\, A \to \mathrm{Hilb}^r_{P(t)}$ that extends φ. Thus, $\mathrm{Hilb}^r_{P(t)}$ is projective and this concludes the claim for $Y = \mathbb{P}^r$.

We move now to the general case: assume that Y is an arbitrary closed subscheme of $\mathbb{P}^r$; it will suffice to show that the functor $Hilb^Y_{P(t)}$ is represented by a closed subscheme of $\mathrm{Hilb}^r_{P(t)}$, that we proved to be projective.
If we apply twice Corollary 2.7 we can find an integer μ such that $\mathscr{I}_Y \subset \mathcal{O}_{\mathbb{P}^r}$ is μ-regular and such that for every closed subscheme X of $\mathbb{P}^r$ having Hilbert polynomial $P(t)$ the ideal sheaf $\mathscr{I}_X \subset \mathcal{O}_{\mathbb{P}^r}$ is μ-regular. Consider $V := H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(\mu))$ and $U := H^0(\mathbb{P}^r, \mathscr{I}_Y(\mu))$. By Theorem 2.9 and its Corollary 2.10 it follows that $\pi_*\mathscr{I}_{\mathcal{W}}$ is a locally free subsheaf of $V \otimes_k \mathcal{O}_{\mathrm{Hilb}^{\mathrm{r}}_{\mathrm{P(t)}}}$, with locally free cokernel.
Now, on $\mathrm{Hilb}^{\mathrm{r}}_{\mathrm{P(t)}}$ consider the composition

$$\psi : U \otimes_k \mathcal{O}_{\mathrm{Hilb}^{\mathrm{r}}_{\mathrm{P(t)}}} \to V \otimes_k \mathcal{O}_{\mathrm{Hilb}^{\mathrm{r}}_{\mathrm{P(t)}}} \to V \otimes_k \mathcal{O}_{\mathrm{Hilb}^{\mathrm{r}}_{\mathrm{P(t)}}} / \pi_*\mathscr{I}_{\mathcal{W}}(\mu).$$

Let $Z \subset \mathrm{Hilb}^{\mathrm{r}}_{\mathrm{P(t)}}$ be the closed subscheme defined by the condition

$$U \otimes_k \mathcal{O}_Z \subset \pi_*\mathscr{I}_{\mathcal{W}}(\mu) \otimes \mathcal{O}_Z \tag{2.12}$$

and let $j : Z \to \mathrm{Hilb}^r_{P(t)}$ be the inclusion. By condition (2.12) we have that

$$\mathscr{I}_{Y \times Z} \subset (1 \times j)^*\mathscr{I}_{\mathcal{W}} \subset \mathcal{O}_{\mathbb{P}^r \times Z},$$

hence

$$Z \times_{\mathrm{Hilb}^{\mathrm{r}}_{\mathrm{P(t)}}} \mathcal{W} \subset Y \times Z \subset \mathbb{P}^r \times Z.$$

The inclusions above provide the universal family and $Z = \mathrm{Hilb}^Y_{P(t)}$. □

For any projective scheme Y in the r-projective space it is often convenient to consider the functor

$$Hilb^Y : Schemes \to Sets$$

defined as

$$Hilb^Y(S) := \coprod_{P(t)} Hilb^Y_{P(t)}(S).$$

This functor is naturally represented, using the construction above, by the disjoint union

$$\mathrm{Hilb}^Y := \coprod_{P(t)} \mathrm{Hilb}^Y_{P(t)}$$

which is called *the Hilbert scheme of* Y. It is a scheme locally of finite type, but not of finite type, since it has (possibly) infinitely many connected components. This viewpoint is sometimes convenient as this scheme doesn't depend on the projective embedding of Y in $\mathbb{P}^r$, even though its indexing by $P(t)$ does.

Remark 4. If our aim is to deal just with varieties, i.e. integral subschemes of a projective space, there is a simpler way to parametrize them, taking as starting points their subvarieties.
The key ingredient to obtain this different structure is the notion of *cycle*, that is a formal linear combination $\sum_\alpha n_\alpha [V_\alpha]$ of subvarieties V_α of a given variety V, with just a finite number of the coefficients n_α being non-zero. If we have a cycle $\sum_\alpha n_\alpha [V_\alpha]$, we can associate to it a number, called *degree of the cycle*, given by $\sum_\alpha n_\alpha d_\alpha$, where d_α is the degree of the subvariety V_α. If all the varieties of the cycle have the same dimension r, then we usually speak of *r-cycles*, or *cycles of dimension r*. Cycles of dimension 0 are just linear combinations of distinct points on the variety V. The *divisors* of a variety V are the cycles of dimension $\dim V - 1$.
Recall from Section 2.1 that, given a d-dimensional linear subspace L of $\mathbb{P}^n$, we can write it as the intersection of $n-d$ hyperplanes and the maximal minors of the associated $(n-d)\times(n+1)$-matrix determine L uniquely and return the so called Plücker coordinates, and that the set of all these d-planes thus coordinatized is the Grassmannian $G(d+1, n+1)$. The same d-plane L can also be written as the span of $d+1$ points, obtaining this time an associated $(d+1)\times(n+1)$-matrix whose maximal minors are called *dual Plücker coordinate*, to distinguish them from the above one. Standard Plücker coordinates and dual one with complementary indexing coincide up to a sign change.
Then, the r-cycles of degree d of a k-variety $V \subseteq \mathbb{P}^n_k$ can be parametrized by a projective algebraic variety over k, called *Chow variety*, see [Rydh1, 8.27]. This variety can be regarded as the set of all $(n-d-1)$-planes L of $\mathbb{P}^n$ such that $V \cap L$ is nonempty and it can be proved to be an hypersurface in the Grassmannian $G(n-d, n+1)$ whose coordinates are called *Chow coordinates*. Since the Chow variety is an hypersurface in a Grassmannian, it can be expressed as the zero of a unique polynomial in those coordinates, which is called the *(Cayley-)Chow form for the variety* V. So, each Chow coordinate identify uniquely an r-cycle of V, and those coordinates are nothing else than the coefficients of the Chow form for the variety. In particular, if V is itself a linear subspace of $\mathbb{P}^n$, then the Chow coordinates coincides with the dual Plücker coordinates of V.
This kind of construction is somehow easier then the one of Hilbert schemes, and often even more handy, at least for calculations. So, why Hilbert schemes are generally prefered to Chow varieties, even though they are much more complicated? The answer is rather simple, according to [HM, p. 10]: *"the most important difference is that the Hilbert scheme has a natural scheme structure whereas the Chow variety does not"*. Many authors tried to attach to the Chow variety a scheme structure, even getting different characterizations of the obtained scheme. As pointed out in [Rydh2, p. 1], "families" of cycles parametrized by a variety may have several "problems". The main issues are given by the following facts:

- the obtained family is not flat, so if ν is the cycle on $V \times S$ representing a family $\{\nu_s\}$, then ν_s is not simply the fibre of ν over s;
- even though it can be proved that the Chow variety is independent from the chosen projective embedding in $\mathbb{P}^n$ in characteristic 0, this independence fails in positive characteristic;

- we will see in Section 3.2 that the Hilbert scheme carries a suitable deformation theory, while the Chow variety does not, preventing the study of its infinitesimal structure.

Anyway, they are largely studied and investigated, and a suitable notion of *Chow schemes* has been recently produced. For a detailed and complete construction of the Chow variety and some of its first properties, including a study of 0-cycles which we will meet in Section 3.3 (we will provide further references in it) see [Rydh1], [DS] and [Ha, §21]. For the various approaches and the formal notion of Chow scheme see [Rydh2]. For a more complete study on cycles and their operation we refer to [Weil].

2.4 First examples of Hilbert schemes

We see now two easy examples of Hilbert schemes that are important in the general geometric theory. We start discussing the case of linear systems, passing then to Grassmann varieties in order to formalize what we often said about their relation with Hilbert schemes.

If $X \subset \mathbb{P}^r$ is a hypersurface of degree d then its Hilbert polynomial has the form

$$h(t) = \binom{t+r}{r} - \binom{t+r-d}{r} = \frac{d}{(r-1)!}t^{r-1} + \cdots . \tag{2.13}$$

Conversely, suppose to have a projective scheme Y in $\mathbb{P}^r$ with Hilbert polynomial given by (2.13). Then Y has dimension $r-1$, so $Y = Y_1 \cup Z$, where Y_1 is a hypersurface and Z has dimension strictly smaller than the dimension of Y. Consider now the ideal sheaves $\mathscr{I}_Y$ of Y and $\mathscr{I}_{Y_1}$ of Y_1 in $\mathcal{O}_{\mathbb{P}^r}$. The sequence

$$0 \to \mathscr{I}_{Y_1}/\mathscr{I}_Y \to \mathcal{O}_Y \to \mathcal{O}_{Y_1} \to 0$$

is short exact, hence we deduce that

$$h(t) = h_1(t) + k(t),$$

where $h_1(t)$ is the Hilbert polynomial of Y_1 while $k(t)$ is the Hilbert polynomial of $\mathscr{I}_{Y_1}/\mathscr{I}_Y$. But now, the latter sheaf is supported on Z, so the degree of the polynomial $k(t)$ has to be strictly smaller than $r-1$, thus the claim as Y_1 turns out to be a hypersurface of degree d and $k(t) = 0$. Therefore $\mathrm{Hilb}^r_{h(t)}$ parametrizes a universal family of hypersurfaces of degree d in $\mathbb{P}^r$. Let's describe it.

Let $V := H^0(\mathbb{P}^r, \mathcal{O}(d))$ and in $\mathbb{P}(V)$ take homogeneous coordinates

$$(\ldots, c_{i(0),\ldots,i(r)}, \ldots)_{i(0)+\ldots+i(r)=d}.$$

Remark 5. Notice that if we write $\mathbb{P}^r$ as $\mathbb{P}^r = \mathrm{Proj}(S)$ for a suitable graded algebra S, then $V = S_d = \mathrm{Sym}^d(S_1)$. If else $\mathbb{P}^r = \mathbb{P}(W)$, then $V \cong \mathrm{Sym}^d(W^\vee)$. These statements follow quite easily by the definition of $H^0(\mathbb{P}^r, \mathcal{O}(d))$ and [H III, 5.1.].

The hypersurface $\mathcal{H} \subset \mathbb{P}^r \times \mathbb{P}(V)$ defined by the equation

$$\sum c_{i(0),\ldots,i(r)} x_0^{i(0)} \cdots x_r^{i(r)} = 0$$

projects onto $\mathbb{P}(V)$ with degree d fibres. Denote by p the projection $\mathcal{H} \to \mathbb{P}(V)$ and consider $\mathscr{I}_{\mathcal{H}} \subset \mathcal{O}_{\mathbb{P}^r \times \mathbb{P}(V)}$ the ideal sheaf of $\mathcal{H}$.
If we take $t \in \mathbb{P}(V)$ we have that

$$1 = h^0(\mathbb{P}^r(t), \mathscr{I}_{\mathcal{H}(t)}(d)) = h^0(\mathbb{P}^r(t), \mathscr{I}_{\mathcal{H}}(d)(t))$$

and

$$0 = h^i(\mathbb{P}^r(t), \mathscr{I}_{\mathcal{H}(t)}(d)) = h^i(\mathbb{P}^r(t), \mathscr{I}_{\mathcal{H}}(d)(t))$$
$$0 = h^i(\mathcal{H}(t), \mathcal{O}_{\mathcal{H}(t)}(d))$$

where the last equations hold for all $i \geq 1$.

Now, using as usual Theorem 2.9 and Theorem 1.17, we obtain the following three facts:

i) $R^1 p_* \mathscr{I}_{\mathcal{H}}(d) = 0$;

ii) $p_* \mathscr{I}_{\mathcal{H}}(d)$ is an invertible subsheaf of $p_* \mathcal{O}_{\mathbb{P}^r \times \mathbb{P}(V)}(d) = V \otimes_k \mathcal{O}_{\mathbb{P}(V)}$;

iii) $p_* \mathcal{O}_{\mathbb{P}^r \times \mathbb{P}(V)}(d) / p_* \mathscr{I}_{\mathcal{H}}(d) = p_* \mathcal{O}_{\mathcal{H}}(d)$ is locally free.

Hence we deduce that $p_* \mathscr{I}_{\mathcal{H}}(d)$ is the tautological bundle of $\mathbb{P}(V)$, and that the natural map

$$p^* p_* \mathscr{I}_{\mathcal{H}}(d) \to \mathscr{I}_{\mathcal{H}}(d)$$

is an isomorphism. Therefore

$$\mathscr{I}_{\mathcal{H}} = [p^* \mathcal{O}_{\mathbb{P}(V)}(-1)](d). \tag{2.14}$$

Let us prove that the family $\mathcal{H}$ is a universal family.
Suppose that

$$\begin{array}{lcl} \mathcal{X} & \subset & \mathbb{P}^r \times S \\ \downarrow f & & \\ S & & \end{array}$$

is a flat family of closed subscheme of $\mathbb{P}^r$ having Hilbert polynomial given by (2.13) and let $\mathscr{I}_{\mathcal{X}}$ be its ideal sheaf. Proceeding as above we find that $f_* \mathscr{I}_{\mathcal{X}}(d)$ is an invertible subsheaf of $V \otimes_k \mathcal{O}_S$ with locally free cokernel $f_* \mathcal{O}_{\mathcal{X}}(d)$ and that

$$\mathscr{I}_{\mathcal{X}} = [f^* f_* \mathscr{I}_{\mathcal{X}}(d)](-d). \tag{2.15}$$

We have an induced morphism $g : S \to \mathbb{P}(V)$ such that

$$f_* \mathscr{I}_{\mathcal{X}}(d) = g^*[\mathcal{O}_{\mathbb{P}(V)}(-1)]. \tag{2.16}$$

The subscheme $S \times_{\mathbb{P}(V)} \mathcal{H} \subset \mathbb{P}^r \times S$ is defined by the ideal sheaf

$$(1 \times g)^* \mathscr{I}_{\mathcal{H}} = (1 \times g)^* [\mathcal{O}_{\mathbb{P}(V)}(-1)(-d)]. \tag{2.17}$$

where the equality follows by (2.14). Since

$$(1 \times g)^* [\mathcal{O}_{\mathbb{P}(V)}(-1)(-d)] = f^*[g^* \mathcal{O}_{\mathbb{P}(V)}(-1)](-d)$$

by relation (2.16) and (2.15) we obtain that

$$f^*[g^* \mathcal{O}_{\mathbb{P}(V)}(-1)](-d) = [f^* f_* \mathscr{I}_{\mathcal{X}}(d)](-d) = \mathscr{I}_{\mathcal{X}}, \tag{2.18}$$

hence joining (2.15) to (2.18) we proved that

$$(1 \times g)^* \mathscr{I}_{\mathcal{H}} = \mathscr{I}_{\mathcal{X}},$$

thus $S \times_{\mathbb{P}(V)} \mathcal{H} = \mathcal{X}$.
To conclude we should verify that the function g acting as desired is unique, which is a consequence of pullback's properties.
In the end $\mathcal{H} \subset \mathbb{P}^r \times \mathbb{P}(V)$ is a universal family, and finally

$$\mathrm{Hilb}^r_{h(t)} = \mathbb{P}(V).$$

As we said in Section 2.1, Grassmannians are a generalization of projective spaces and linear systems. We pointed out along the construction of Hilbert schemes, that Grassmannians are a special case of Hilbert scheme, in which the involved polynomials are of the form $\binom{r+n-1}{n-1}$, with $n-1$ being the dimension of the closed subschemes we are considering in $\mathbb{P}^r$. Let us describe their universal family. For some $1 \leq n \leq r$ let $G = G(n+1, r+1)$ be the Grassmannian of n-dimensional projective subspaces of $\mathbb{P}^r$ and call $\mathcal{Q}$ the universal quotient bundle on G. Define a projective bundle on G as

$$\mathbf{I} := \mathbb{P}(\mathcal{Q}^\vee) = \mathrm{Proj}(\mathrm{Sym}(\mathcal{Q})).$$

Using the surjection ${\mathcal{O}_G}^{r+1} \to \mathcal{Q}$ we find a closed embedding

$$\begin{array}{lcl} \mathbf{I} & \subset & \mathbb{P}^r \times G \\ \downarrow p & & \\ G & & \end{array}$$

and remark that

$$\mathcal{Q}^\vee = p_* \mathscr{I}_{\mathbf{I}}(1) \subset p_* \mathcal{O}_{\mathbb{P}^r \times G}(1) = {\mathcal{O}_G}^{r+1}$$

For every closed point $v \in G$ the fibre $\mathbf{I}(v)$ results to be the projective space $\mathbb{P}(v) \subset \mathbb{P}^r$, that's why $\mathbf{I}$ is usually called the *incidence relation.* Since all fibres of the morphism p have the same Hilbert polynomial given by $\binom{t+n}{n}$, using Theorem 1.17 we deduce that p is a flat family.
Suppose now to have another flat family

$$\begin{array}{lcl} \Lambda & \subset & \mathbb{P}^r \times S \\ \downarrow q & & \\ S & & \end{array}$$

having Hilbert polynomial $\binom{t+n}{n}$ at its fibres. We thus have an inclusion of sheaves on S

$$q_* \mathscr{I}_\Lambda(1) \subset q_* \mathcal{O}_{\mathbb{P}^r \times S}(1) = {\mathcal{O}_S}^{r+1}$$

having locally free cokernel $q_* \mathcal{O}_\Lambda(1)$. As we are dealing with the Grassmannian G, using its universal property from the above inclusion we find a unique induced morphism $g : S \to G$ such that

$$g^*(\mathcal{Q}^\vee) = q_* \mathscr{I}_\Lambda(1).$$

But now, since $\Lambda = \mathbb{P}(q_* \mathscr{I}_\Lambda(1))$ it follows that

$$\Lambda = S \times_G \mathbf{I}$$

which means that the family q is obtained by base change via the morphism g starting from the incidence relation $\mathbf{I}$, making us conclude that

$$G(n+1, r+1) = \mathrm{Hilb}^r_{\binom{t+n}{n}}.$$

2.5 Generalization of Hilbert Schemes

We conclude the chapter refering to some generalizations of Hilbert schemes that appeared in the last fifty years.

A first object, of which Hilbert schemes are a particular case, was already introduced by Grothendieck himself in [FGA] and relies on a change of viewpoint in the construction. If we look at a family Y of subschemes of $\mathbb{P}^n$ parametrised by a locally Noetherian scheme S as a coherent quotient sheaf $q : \mathcal{O}_{\mathbb{P}^n_S} \to \mathcal{O}_Y$ on $\mathbb{P}^n_S$, with $\mathcal{O}_Y$ being flat over S, we may enlarge the idea of Hilbert schemes to *families of quotients* having some properties, obtaining the *Quot Schemes.* In the second chapter of [FGAE] Nitin Nitsure provides the construction of Quot schemes, the view of Grassmannians and Hilbert schemes as Quot schemes and some variants, while in [Sernesi, §4.4] their presentation is related to the construction of *relative Hilbert schemes.* In the same work, Grothendieck uses quotients of open subschemes of Hilbert schemes in order to construct the *Picard scheme*, whose history and relevance is explained in details in [FGAE, §9].

A further generalization are *Hilbert-Flag schemes*, introduced by Jan O. Kleppe in [J. O. Kleppe, "The Hilbert-Flag scheme, its properties and its connection with the Hilbert scheme. Applications to curves in 3-space", Preprint, Inst. of Math. Univ. Oslo (1981)]. They are defined using length m chains of S-flat closed subschemes of $\mathbb{P}^r \times S$ and such a definition clearly coincides with the classical one for $m = 1$. See [Sernesi, §4.5] for a first approach to these objects and their first properties.

More recently, Mark Haiman and Bernd Sturmfels developed the concept of *Multigraded Hilbert scheme* in [M. Haiman, B. Sturmfels, "Multigraded Hilbert schemes", *J. Algebraic Geom., vol. 13* (2004), no. 4, pp. 725–769], that again has the Hilbert scheme by Grothendieck as a special case.

Furthermore, we should notice that a non-commutative version of Hilbert schemes has been introduced in the 70's in [M.V. Nori, *Appendix to an article "Desingularization of the moduli varieties of vector bundles over curve" by C.S. Seshadri*, Proceedings of the International Symposium on Algebraic Geometry, Kyoto, 1977, pp. 155–184].

Chapter 3

Properties of Hilbert Schemes

Truly little is still known about the properties that are satisfied by a general Hilbert scheme. If we focus on the case of Hilbert schemes on projective spaces, the only one we know that holds always is connectedness, as proved by Robin Hartshorne in his Ph.D. thesis in 1963 ([H66]), under the supervision of John Coleman Moore and Oscar Ascher Zariski. On top of that, Hartshorne himself notices in the introduction to [H66], that *"It also appears that the Hilbert scheme is never actually needed in the proof"*, introducing thus the notion of *connected functor* and proving that the Hilbert functor is not only connected, but *linearly connected*.

If we move to Hilbert schemes on a general scheme Y even this small sparkle of good property goes lost; if we look for a simple example we may consider Y to be two distinct points and $p = 1$ be the constant polynomial, then $\mathrm{Hilb}_p^Y = Y$, which is clearly disconnected. Recently there have been a lot of examples of connected, or irreducible, projective schemes whose Hilbert scheme is disconnected, see e.g. [See-Hak Seong, "The Hilbert scheme of the Grassmannian is not connected", *Comm. in Algebra, vol. 48* (2020), Issue 8, pp. 3439–3446].

Starting from the description by Grothendieck and the theorem by Hartshorne, many other mathematicians tried to formulate and prove results about this kind of schemes, achieving just results in special cases. One of the most studied, due to its use in several theories, is the *Hilbert scheme of points*; for this particular Hilbert scheme we have some more informations.

In the first section we will briefly introduce the connectedness theorem by Hartshorne, without a complete proof; in the second one we are going to introduce a little of deformation theory in order to characterize the tangent space to the Hilbert scheme, while in the final one we will outline the construction of the Hilbert scheme of points and some of its known properties.

3.1 Hartshorne's Connectedness Theorem

The Ph.D. thesis by Robin Hartshorne is expanded and presented in [H66], and has the following structure. He first defines *connected* and *linearly connected functors* and then recalls the definition of Hilbert scheme and the notion of *specialization* of a subscheme. The central part of the work deals with a specific kind of schemes called *fans*, the use of a special technique from commutative algebra called *distractions* and some invariants linked to the support of a scheme. The last chapter is devoted to the proof of the connectedness theorem as a corollary of a theorem about specializations of a closed subscheme to a special type of fan called *tight fan*.

Recall that, given an arbitrary topological space X, a point $x \in X$ is *closed* if the set $\{x\}$ is closed, a point $\eta \in X$ is a *generic point* of the space if $\overline{\{\eta\}} = X$. In particular, a point is called a *maximal point* if its closure is an irreducible component of the space. If we take instead two points, say x and x', we say that x' is a *specialization* of x, or that x is a *generalization* of x', if $x' \in \overline{\{x\}}$.

All such definitions clearly apply to the case of schemes and have their immediate correspondence with relations between ideals associated to points, saying that x' is a specialization of x if $\mathfrak{p}_{x'} \subseteq \mathfrak{p}_x$. In his discussion Hartshorne uses the following, more general, notion of specialization adapted to the scheme case.

Definition 28. Let x and x' be two points of a scheme X. We say that *x specializes linearly to x'* if there exists an extension field k_1 of k, and a morphism $f : \mathrm{Spec}\,(k_1[t]_{(t)}) \to X$, which sends the generic point to x and the special point to x'. We denote this operation by $x' \rightsquigarrow x$. Moreover, we say that the x and x' can be *connected by a sequence of linear specializations* if there is a sequence of points $x_1 = x\,,\, x_2\,, \dots\, x_n = x'$, with $x_i \in X$ for all i, such that, for each i, either x_i is a linear specialization of x_{i+1} or x_{i+1} is a linear specialization of x_i. We say that a scheme X is *linearly connected* if any two points on X can be connected by a sequence of linear specializations.

Hartshorne goes on analysing the notions of connectedness by working with rational curves over a fixed field k, which are one-dimensional integral schemes of finite type over k, whose function field is a pure transcendental extension of k

Definition 29. Let X be a scheme over k. Two points x_1 and x_2 of X are said to be *connected by a rational curve* if there exists an extension field k_1 of k, a rational curve Y over k_1, a morphism $f : X \to Y$ and points y_1 and y_2 in Y, rational over k_1, such that $f(y_1) = x_1$ and $f(y_2) = x_2$. As for the linear specialization we may enlarge the notion to a sequence of rational curves; we say that x_1 and x_2 are *connected by a sequence of rational curves* if there is a sequence of points of X starting at x_1 and ending at x_2 that can be connected orderly by rationals curves.

It can be proved that rational curves over a field are linearly connected, see [H66, Lemma 1.5], and that given a scheme X over k, if any two of its points can be connected by a sequence of rational curves, then X is linearly connected. Furthermore the converse holds for X of finite type over k, see [H66, Proposition 1.6.]. On top of that, in [H66, Proposition 1.7] it is proved that any open subset of $\mathbb{P}^r_k$ is linearly connected.
We need the following definition to move our attention from schemes to morphisms of schemes.

Definition 30. A scheme X over k is *geometrically connected* if for every extension k' of k, the scheme $X_{k'} = X \otimes_k k'$ is connected.

Definition 31. A morphism of k-schemes $f : X \to Y$ is *universally submersive* if it is surjective, the image space has the quotient topology and these two facts are stable under base extensions. A morphism of k-schemes that is universally submersive and such that its fibres are geometrically connected is said to be *connected*.

Also for geometrical connectedness Hartshorne introduces the generalization of *geometrically linearly connected scheme* and of *linearly connected morphism* following the same ideas.
What is fundamental in Hartshorne's work is showing that the Hilbert scheme represents a *connected functor* in the category of locally Noetherian schemes. This notion is not so simple to introduce and requires some preliminar work involving *relatively representable functors* and *disjoint sums of functors*. We avoid the introduction of all such definitions by using one of the three equivalent characterization of connected functors provided in [H66], to which we anyway refer for a complete analysis of the notion. To introduce it we shall explain what it is meant by a *scheme over a functor*.

Definition 32. If F is a functor, then a *scheme over* F is a pair (X, ξ) where X is a scheme and $\xi \in F(X)$; a *morphism* $(X, \xi) \to (Y, \eta)$ *of schemes over* F is a morphism $X \to Y$ for which the map $F(Y) \to F(X)$ sends η to ξ.

Remark 6. If F is a functor and X is a scheme, then the sets $F(X)$ and $\mathrm{Hom}(h_X, F)$ are canonically identified. Hence, to give a scheme X over a functor F is the same as to give a morphism of functors $h_X \to F$, so if X is a scheme over a functor G, by $X \times_G F$ we will mean the product functor $h_X \times_G F$, always using the identification given by the Yoneda Lemma (Theorem 2.13). Following this way, if F is, for example, a functor of points of a scheme Y, a scheme X over h_Y turns out to be a morphism of functors of points $h_X \to h_Y$, thus a morphism between the two schemes involved.

Definition 33. Let F be a functor of locally Noetherian schemes, then F is said to be a *connected functor* if, whenever X and X' are two non-empty connected schemes over F, there exists a sequence $X_1, \dots, X_n$ of non-empty connected schemes over F, with $X_1 = X$ and $X_n = X'$, such that for each i there is a morphism either $X_i \to X_{i+1}$ or $X_{i+1} \to X_i$ of schemes over F.

If F is representable, then F is connected if and only if the scheme representing it is connected. As above, it is possible also to introduce the notion of *linearly connected functor* just replacing "connected" with "linearly connected" in Definition 33, deducing also that a linearly connected functor is always connected and that, again, if F is a representable functor, then it is linearly connected if and only if the scheme representing it is linearly connected.

Now, we need to fit all this discussion into the Hilbert scheme, and we will do it introducing specializations. First of all, if X is a scheme over S, and $\operatorname{Spec} k \to S$ is a morphism, we call a *generalized fibre of X over S* the product $X_k := X \otimes_S \operatorname{Spec} k$.

Remark 7. As a morphism $\operatorname{Spec} k \to S$ is the same as a k-valued point, we may give a geometrical meaning to this construction. If we fix a point $P \in S$ and k is the residue field $k(P)$ of the point P, then we have a "natural" morphism $\operatorname{Spec} k \to S$ that maps the unique point of $\operatorname{Spec} k$ to the point P, while it pulls back a given section $\varphi \in \mathcal{O}_S(U)$, with $P \in U$, to the element of the residue field determined by the map $\mathcal{O}_S \to \mathcal{O}_{P,S} \to k(P)$. So, with the described morphism, if we have a morphism $X \to S$, X_k turns out to be the inverse image, or scheme-theoretic fibre, of $X \to S$ over the point $P \in S$.

Definition 34. Let X be a scheme over S, and let $Z_1 \subseteq X_{k_1}$ and $Z_2 \subseteq X_{k_2}$ be closed subscheme of some generalized fibres of X over S. We say that Z_1 *specializes to* Z_2, and we denote it by $Z_1 \rightsquigarrow Z_2$ as in the case of points, if either

a) Z_1 is obtained from Z_2 by a field extension $k_2 \subseteq k_1$, or

b) there exist a local domain A, with quotient field k_1 and residue field k_2, a morphism $\operatorname{Spec} A \to S$ and a closed subscheme Z of $X_A = X \otimes_S \operatorname{Spec} A$, flat over A, whose fibre over the generic point of $\operatorname{Spec} A$ is Z_1 and whose fibre over the closed point of $\operatorname{Spec} A$ is Z_2.

If moreover S is a scheme over k, we say that Z_1 *specializes linearly to* Z_2 if either

a) Z_1 is obtained from Z_2 by a field extension $k_2 \subseteq k_1$, or

b) there exist a local domain A, with quotient field k_1 and residue field k_2, a morphism $\operatorname{Spec} A \to S$ and a closed subscheme Z of $X_A = X \otimes_S \operatorname{Spec} A$, flat over A, whose fibre over the generic point of $\operatorname{Spec} A$ is Z_1 and whose fibre over the closed point of $\operatorname{Spec} A$ is Z_2 and $\operatorname{Spec} A$ is linearly connected.

Definition 35. Let X be a scheme over S. A *connected sequence of specializations in* X is a sequence of closed subschemes $Z_1, Z_2, \dots, Z_n$ of generalized fibres X_{k_i} of X over S, where for each i either Z_i specializes to Z_{i+1}, or vice versa. Similarly, if S is a scheme over a field k, one may define a *connected sequence of linear specializations in* X following the same procedure as above.

So, we finally reached the first fundamental result of [H66]

Theorem 3.1. *Let X be a projective scheme over a locally Noetherian scheme S and let $p := p(z) \in \mathbb{Q}[z]$ be a polynomial. Then the functor $Hilb_p^{X/S}$ is connected, respectively linearly connected, if and only if whenever $Z' \subseteq X_{k'}$ and $Z'' \subseteq X_{k''}$ are closed subschemes of generalized fibres of X over S, having Hilbert polynomials equal to p, then there exists a connected sequence of specializations, respectively connected sequence of linear specializations, $Z' = Z_1, Z_2, \dots, Z_n = Z''$ in X*

Proof. See [H66, Proposition 1.12]. □

After this introduction to various connectedness properties for schemes and functor and their relations, Hartshorne presents a collection of integers $n_i(\mathscr{F})$ for i smaller than the dimension of the ambient space, attached to a coherent sheaf $\mathscr{F}$, whose intent is to measure the sections of $\mathscr{F}$ whose support is of dimension i. In particular for an integral subscheme $Z \subseteq \mathbb{P}^r_k$ of dimension q it is proved that, taking $\mathscr{F} = \mathcal{O}_Z$, it holds that $n_i(\mathscr{F}) = 0$ for $i \neq q$, and $n_q(\mathscr{F})$ is the degree of the subscheme Z. If we work, as in the last example, in a projective space $\mathbb{P}^r_k$, the collection we obtain is an $(r+1)$-tuple $n_r(\mathscr{F}), \dots, n_0(\mathscr{F})$ and it is denoted by $n_*(\mathscr{F})$. For the complete definition, further details and properties, including the stability under base extensions, we refer to [H66, §2].

The third fundamental object Hartshorne presents are a special type of subschemes of the projective space $\mathbb{P}^r_k$, called *fans*.

Definition 36. Fix a set of homogeneous coordinates $x_1, \dots, x_r$ of $\mathbb{P}^r_k$. A *fan* $X \subset \mathbb{P}^r_k$ is a subscheme of $\mathbb{P}^r_k$ whose ideal $\mathfrak{a}$ can be written as an intersection of prime ideals $\mathfrak{p}$ having the form

$$\mathfrak{p} = (x_1 - a_1x_0, x_2 - a_2x_0, \dots, x_q - a_qx_0) \tag{3.1}$$

for various $q \in \mathbb{N}$ and $a_i \in k$. If moreover the prime ideals involved have the form

$$\mathfrak{p} = (x_1, x_2, \dots, x_q - a_q x_0) \tag{3.2}$$

for various q and $a_q \in k$, then X is said to be a *tight fan*.

This algebraic definition has the following geometrical interpetation.

Remark 8. If X is a fan, then its ideal can be written as intersection of prime ideals $\mathfrak{p}_i$ of the form (3.1), so X is a reduced subscheme of the projective space, having irreducible components that are all linear subspaces and, for each $q \in \mathbb{N}$ that appears in the decomposition of the ideal $\mathfrak{a}$, all of the q-dimensional components of the subscheme contain a common $(q-1)$-dimensional linear subspace. That is the reason for the name "fan". By (3.2), a tight fan is a fan having an additional geometrical property: for each $q \in \mathbb{N}$, all of its q-dimensional components are also contained in a common $(q+1)$-dimensional linear subspace.

What is relevant about tight fans is their behaviour with respect to their Hilbert polynomial and linear specializations, as stated in the following result.

Theorem 3.2. *Let X_1 and X_2 be two tight fans in $\mathbb{P}^r_k$. Then, the following conditions are equivalent:*

i) $n_*(X_1) = n_*(X_2)$*;*

ii) X_1 *and* X_2 *have the same Hilbert polynomial;*

iii) there exists a subscheme X_3 of $\mathbb{P}^r_K$, for a suitable field K and linear specializations $X_3 \rightsquigarrow X_1$ and $X_3 \rightsquigarrow X_2$.

On top of that, the Hilbert polynomial of a tight fan $X \subset \mathbb{P}^r_k$ with $n_*(X) = (n_{r-1}, \dots, n_0)$ is

$$f(z) = \sum_{t=0}^{r-1} g(n_t + \cdots + n_{r-1}, t), \tag{3.3}$$

where, for any $n \in \mathbb{Z}$ and $t \in \mathbb{N}$, the polynomial $g(n,t)$ is defined by

$$g(n,t) := \binom{z+t}{i+1} - \binom{z+t-n}{i+1}. \tag{3.4}$$

Proof. See [H66, 3.3]. □

The proof of Theorem 3.2 is constructive, and actually proves something stronger in the implication *i)* $\Rightarrow$ *ii)*. Infact, it shows that the tight fans X_1 and X_2 have the same Hilbert function (see [H66, 3.2]). Relation (3.3) is obtained using the same strategy from the proof of the implication *i)* $\Rightarrow$ *ii)* using the Hilbert polynomial of hypersurfaces of degree n in $\mathbb{P}^r_k$ (see Example 1.9) and some suitable functions and computations with binomial coefficients.

We stopped to hint some details about this result as it actually provides a necessary and sufficient criterion for a numerical polynomial to be the Hilbert polynomial of a tight fan, since in [M. Nagata, *Local Rings*, Interscience tracts in pure and applied math, 13, Wiley, New York (1962), p. 69] is proved that any numerical polynomial $f(z)$ of degree s can be written in the form

$$f(z) = \sum_{k=0}^{s} g(m_k, k)$$

with $m_k \in \mathbb{Z}$ and $g(m_k, k)$ given by (3.4). On top of that such a form is unique.
So, we may rephrase the last part of Theorem 3.2 as follows.

Corollary 3.3. *A necessary and sufficient condition for a numerical polynomial to be the Hilbert polynomial of a tight fan is that when expressed in the form*

$$f(z) = \sum_{k=0}^{s} g(m_k, k)$$

with $g(m_k, k)$ given by (3.4), we have that $m_0 \geq m_1 \geq \ldots \geq m_s \geq 0$.

We should keep in mind this result as we will state later that this condition will allow us to obtain "something more" then a tight fan.

Chapter 4 of [H66] is then devoted to the discussion of *grading* and properties of *monomial ideals* and to the developement of a particular commutative algebra's tool dealing with them, called *distractions*, which result to link monomial ideals to fans via [H66, 4.9 and 4.10].

The last Chapter, after some brief remainders and results about group actions on schemes and the triangular group schemes of matrices, presents the following, fundamental, results that leads to Hartshorne's Connectedness Theorem.

Proposition 3.4. *Let $p \in \mathbb{Q}[z]$ be a numerical polynomial of degree at most r such that when we write*

$$p(z) = \sum_{t=0}^{\infty} g(m_t, t),$$

with $g(m_t, t)$ given by (3.4), we have $m_1 \geq m_2 \geq \ldots \geq m_{r-1} \geq 0$. Then there exists a proper separated subscheme X of $\mathbb{P}^r_{\mathbb{Z}}$, flat over $\mathbb{Z}$, whose fibre at every point of $\mathbb{Z}$ has Hilbert polynomial equal to p.

Proof. Let k be an infinite field. Then there is a tight fan $X'' \subseteq \mathbb{P}^r_k$ with

$$n_*(X'') = (m_{r-1}, m_{r-2} - m_{r-1}, \ldots, m_0 - m_1)$$

by [H66, 3.9 and 3.10]. Now, by Corollary 3.3 X'' has Hilbert polynomial equal to p.
Applying [H66, 5.3] we can find a second subscheme $X' \subseteq \mathbb{P}^r_k$ having the same Hilbert polynomial p, as specializations preserves Hilbert polynomials, and whose ideal in the polyomial ring $k[x_0, \ldots, x_r]$ is generated by monic monomials in the coordinates by [H66, 5.4].
Let now $\mathfrak{a} \subseteq \mathbb{Z}[x_0, \ldots x_r]$ be the ideal generated by the same monomials as the one of X'. So, $\mathfrak{a}$ defines a closed subscheme of $\mathbb{P}^r_{\mathbb{Z}}$, flat over $\mathbb{Z}$, whose Hilbert polynomial at every point is p, because the Hilbert polynomials of the quotient of a polynomial ring by an ideal generated by monomials is independent of the base field. □

Theorem 3.5. *Let S be a scheme, $r > 0$ an integer and $p \in \mathbb{Q}[z]$ a numerical polynomial satisfying the following property: whenever p is written in the form*

$$p(z) = \sum_{t=0}^{\infty} g(m_t, t) = \sum_{t=0}^{\infty} \binom{z+r}{r+1} - \binom{z+r-m_t}{r+1}$$

it satisfies

$$m_0 \geq m_1 \geq \ldots \geq m_{r-1} \geq 0$$

and

$$m_j = 0 \, for\ all\, j \geq r.$$

Then the morphism

$$f : Hilb_p^{\mathbb{P}^r_S/S}(S) \to S$$

is a linearly connected morphism of functors.

Proof. We give the idea of the proof, that is not so long but really deep.
Let X be a closed subscheme of $\mathbb{P}^r$. It is proved that there is a connected sequence of linear specializations from X to a subscheme X' that is stable under a particular action of the triangular groups scheme of matrices $T(r+1)$. The ideal this scheme is generated by monomials and balanced, so we find a fan X_2, and a linear specialization $X_2 \rightsquigarrow X_1$, sucht that $n_*(X_1) = n_*(X_2)$. Now, using Theorem 3.2, we may find another connected sequence of linear specializations joining X_2 to a third subscheme $X_3 \subseteq \mathbb{P}^r_k$. At this point there are two options: if X_3 is a tight fan we are done, else we can repeat the process, that will necessarely terminates in a finite number of steps by [H66, 3.10]. So, given any scheme X with Hilbert polynomial p we can joint it to a tight fan having the same Hilbert polynomial by a connected sequence of linear specializations. For a detailed proof of this fact see [H66, 5.6].

This discussion, together with Theorem 3.1, proves that for any field k, the functor $Hilb_p^{\mathbb{P}^r_k/k}$ is linearly connected over k, thus the fibres of the morphism f are geometrically linearly connected functors by definition. Using now Proposition 3.4 and considering the base extension $S \to \operatorname{Spec} \mathbb{Z}$, we obtain that the morphism f has a section, hence it is linearly connected. □

Corollary 3.6. *(Connectedness Theorem for Hilbert schemes) Let S be a connected Noetherian scheme, $r > 0$ and $p \in \mathbb{Q}[z]$ a numerical polynomial satisfying the following property: whenever p is written in the form*

$$p(z) = \sum_{t=0}^{\infty} g(m_t, t) = \sum_{t=0}^{\infty} \binom{z+r}{r+1} - \binom{z+r-m_t}{r+1}$$

it satisfies

$$m_0 \geq m_1 \geq \ldots \geq m_{r-1} \geq 0$$

and

$$m_j = 0 \, for \; all \, j \geq r.$$

Then $\mathrm{Hilb}_p^{\mathbb{P}^r_S}(S)$ is a connected non-empty scheme if and only if S in non empty.

Proof. It is a direct consequence of Theorem 3.5 and the properties of connected morphisms of schemes under base extension, see [H66, 5.9]. □

Remark 9. In the previous result the word "connected" can always be replaced by one of the following: "geometrically connected", "linearly connected" and "geometrically linearly connected".

Remark 10. The condition on the numerical polynomial given in Theorem 3.5 and 3.6 is a necessary and sufficient condition for the polynomial p to be the Hilbert polynomial of a proper closed subscheme of $\mathbb{P}^r_k$, as proved in [H66, Corollary 5.7], generalizing what we stated in Corollary 3.3.

This result has been generalised in a paper by Alyson Reeves [A. A. Reeves, "The radius of the Hilbert scheme", *J. Algebraic Geom., vol. 4* (1995), no. 4, pp. 639–657] using the notion of *radius* of the *component-graph* of a scheme and *lexicographic ideals*.

3.2 Tangent Space to Hilbert Schemes

The main references for this section will be [Sernesi, §1] and [FGAE, §6]. Within this section we will denote by $k-$Art the category of local Artinian k-algebras with residue field k, by $k-$Noeth the category of local Noetherian k-algebras having residue field k and by $k-$Loc the category of local k-algebras having residue field k. Moreover, notice that, geometrically, a k-algebra A lies in $k-$Art if $S := \operatorname{Spec} A$ is a k-scheme of finite type such that $S_{red} = \operatorname{Spec} k$. Finally, recall that if $R \in k-$Loc, then the tangent space $T_R := (\mathfrak{m}_R/\mathfrak{m}_R^2)^\vee$ is a finite dimensional k-vector space.

We need to recall shortly some algebraic tools in order to define deformation functors and spaces. We start by defining extensions of algebras and than we move to extensions of schemes.

Definition 37. Let $A \to R$ be a ring homomorphism. An *A-extension of R (by I)* is a short exact sequence

$$0 \to I \to R' \xrightarrow{\varphi} R \to 0$$

denoted shortly by (R', φ) such that R' is an A-algebra and φ is a homomorphism of A-algebras having kernel I satisfying $I^2 = (0)$. The A-extension (R', φ) is called *trivial* (or it is said that *it splits*) if there exists an A-algebras homomorphism $\sigma : R \to R'$, called *section* (or *splitting*) of φ, such that $\varphi\sigma = 1_R$.

Example 3.1 Every A-extension of A itself is trivial. In particular if we take an indeterminate t, the A-extension $A[t]/(t^2)$ is trivial, it is denoted by $A[\epsilon]$, with $\epsilon \equiv t \pmod{t^2}$ satisfying $\epsilon^2 = 0$, and it is called the *algebra of dual numbers over A*. The corresponding short exact sequence is

$$0 \to (\epsilon) \to A[\epsilon] \to A \to 0.$$

Example 3.2 Take $R \in k-$Loc and denote by $\mathfrak{m}_R$ its maximal ideal. A k-extension R' of R by k is called a *small extension* of R, and its corresponding short exact sequence is

$$0 \to \ker(f) \to R' \xrightarrow{f} R \to 0,$$

i.e. the map $R' \to R$ is a surjection whose kernel satisfies the following equality: $\ker(f) \cdot \mathfrak{m}_{R'} = (0)$. If (R', f) is a small k-extension, then every $t \in \mathfrak{m}_{R'}$ is annihilated by $\mathfrak{m}_{R'}$, so that the ideal (t) is a k-vector space of dimension one.

Definition 38. For every A-algebra R and for every R-module I we define $\mathrm{Ex}_A(R, I)$ to be the set of isomorphism classes of A-extensions of R by I, and we denote the class of an extension (R, φ) by $[R, \varphi]$. Using the operations of pullback and pushout it is possible to define an R-module structure on $\mathrm{Ex}_A(R, I)$, see e.g. [Sernesi, p. 13].
If we take the particular case $I = R$, then the R-module $T^1_{R/A} := \mathrm{Ex}_A(R, R)$ is called *first cotangent module of R over A*. If $A = k$ it is often abbreviated T^1_R.

We have the following result.

Lemma 3.7. *Let A be a ring, $f : S \to R$ a homomorphism of A-algebras and I an R-module. If we denote by $\mathrm{Der}_A(R, I)$ the module of A-derivations from R to I, then there is an exact sequence of R-modules*

$$0 \to \mathrm{Der}_S(R, I) \to \mathrm{Der}_A(R, I) \to \mathrm{Der}_A(S, I) \otimes_S R \xrightarrow{\rho}$$

$$\to \mathrm{Ex}_S(R, I) \xrightarrow{\nu} \mathrm{Ex}_A(R, I) \xrightarrow{f^*} \mathrm{Ex}_A(S, I) \otimes_S R.$$

Proof. Remark first that an A-extension

$$0 \to I \to R' \xrightarrow{\varphi} R \to 0$$

is also an S-extension if and only if it exists a morphism $f' : S \to R$ making the triangle

$$\begin{tikzcd} R' \arrow[r] & R \\ & S \arrow[u] \arrow[ul, "f'"] \end{tikzcd}$$

commute, which is equivalent to ask that $f^*(R', \varphi)$ is trivial.
So, ν is the application sending an S-extension to itself, regarded as an A-extension. By the last observation we have the exactness at $\mathrm{Ex}_A(R, I)$.
We have now to define ρ. We start considering the A-module $R \oplus I$, with multiplication given by $(r, i)(s, j) := (rs, rj + si)$, and turning it into an A-algebra $R\tilde{\oplus}I$. This A-algebra may be regarded also as an S-algebra via the homomorphism $s \mapsto (f(s), d(s))$, where $d : R \to I$ is an A-derivation. Sothat, the homomorphism ρ is defined by letting $\rho(d) = (R\tilde{\oplus}I, p)$, where $p : R\tilde{\oplus}I \to R$ is the first projection whose sections are nothing else then the A-derivations $d : R \to I$. By construnction we have that $\nu\rho = 0$.
For the proof of the exactness at $\mathrm{Ex}_S(R, I)$ and at $\mathrm{Der}_A(S, I)$ see [Sernesi 1.1.5]. □

Definition 39. Let $X \to S$ be a morphism of schemes. An *extension of X/S* is a closed immersion $X \subset X'$, where X' is an S-scheme defined by a sheaf of ideals $\mathcal{I} \subset \mathcal{O}_{X'}$ such that $\mathcal{I}^2 = 0$. It turns out that $\mathcal{I}$ has naturally a structure of sheaf of $\mathcal{O}_X$-modules coinciding with the conormal sheaf of $X \subset X'$. So, to give an extension $X \subset X'$ of X/S is the same as giving an exact sequence

$$\mathscr{E} : 0 \to \mathcal{I} \to \mathcal{O}_{X'} \xrightarrow{\varphi} \mathcal{O}_X \to 0$$

on X, where $\mathcal{I}$ is an $\mathcal{O}_X$-module such that $\mathcal{I}^2 = 0$ in $\mathcal{O}_{X'}$ and φ is a homomorphism of $\mathcal{O}_S$-algebras. Such a sequence $\mathscr{E}$ is called *an extension of X/S by $\mathcal{I}$*. In an analogous way to what we did in the case of A-algebras, we denote by $\mathrm{Ex}(X/S, \mathcal{I})$ the set of isomorphism classes of extensions of X/S having kernel $\mathcal{I}$ and it can be proved that $\mathrm{Ex}(X/S, \mathcal{I})$ has a structure of $\Gamma(X, \mathcal{O}_X)$-module with identity element the class of the *trivial extension*

$$0 \to \mathcal{I} \to \mathcal{O}_X\tilde{\oplus}\mathcal{I} \to \mathcal{O}_X \to 0.$$

The correspondence

$$\mathcal{I} \to \mathrm{Ex}(X/S, \mathcal{I})$$

defines a covariant functor from the category of $\mathcal{O}_X$-modules to the one of $\Gamma(X, \mathcal{O}_X)$-modules.

Now, by localizing as in [Sernesi, 1.1.8.], we find out that given a morphism of finite type of schemes $f : X \to S$ we may define a quasi-coherent sheaf $T^1_{X/S}$ on X with the following properties. If $U = \operatorname{Spec} A$ is an affine open subset of S and $V = \operatorname{Spec} B$ is an affine open subset of $f^{-1}(U)$, then

$$\Gamma(V, T^1_{X/S}) = T^1_{B/A}.$$

By the properties of the cotangent module the sheaf $T^1_{X/S}$ is indeed coherent and it is called the *first cotangent sheaf of* X/S, and similarly to the previous case, when $S = \operatorname{Spec} k$ we will write shortly T^1_X.

Remark 11. There are several properties that might be proved for the first cotangent sheaf of X/S, making clearer also some assumption we will do in the next results. We list here two of them, refering for more details to [Sernesi, 1.1.9].

i) If X is an algebraic scheme, then the first cotangent sheaf T^1_X is supported on the singular locus of X. If $X \to S$ is a morphism of finite type of algebraic schemes, then the first cotangent sheaf $T^1_{X/S}$ is supported on the locus where X is not smooth over S;

ii) If $f : X \to Y$ is a closed embedding of algebraic S-schemes, with Y nonsingular and $S = \operatorname{Spec} k$, then $T_{X/Y} = 0$, $N_{X/Y} = T^1_{X/Y}$ and we have an exact sequence of coherent sheaves on X

$$0 \to T_X \to T_{Y/X} \to N_{X/Y} \to T^1_X \to 0.$$

The following characterization of $\operatorname{Ex}(X/S, \mathcal{I})$ holds.

Theorem 3.8. *Let $X \to S$ be a morphism of finite type of algebraic schemes and let $\mathcal{I}$ be a coherent locally free sheaf on X. Suppose that X is reduced and has a dense open subset that is smooth over S. There exists then a canonical identification*

$$\operatorname{Ex}(X/S, \mathcal{I}) = \operatorname{Ext}^1_{\mathcal{O}_X}(\Omega^1_{X/S}, \mathcal{I})$$

associating to the isomorphism class of an extension

$$\mathscr{E} : 0 \to \mathcal{I} \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$$

the isomorphism class of the relative conormal sequence of $X \subset X'$ given by

$$c_{\mathscr{E}} : 0 \to \mathcal{I} \xrightarrow{\delta} (\Omega^1_{X'/S})|_X \to \Omega^1_{X/S} \to 0.$$

In particular, there is a canonical isomorphism

$$T^1_{X/S} \cong \operatorname{Ext}^1_{\mathcal{O}_X}(\Omega^1_{X/S}, \mathcal{O}_X).$$

Proof. See [Sernesi, 1.1.10.] and [Sernesi, 1.1.11.]. □

At this point we introduce the notion of *(local) deformation.*

Definition 40. Let X be an algebraic scheme. A cartesian diagram of morphism of schemes

$$\eta : \begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ \operatorname{Spec} k & \xrightarrow{s} & S \end{array}$$

with π flat, surjective and S connected, is called a *family of deformations*, or simply *a deformation* of X over S. The scheme S is called the *parameter scheme*, while the scheme $\mathcal{X}$ is called the *total scheme* of the deformation. The deformation η will also be denoted by (S, η) to emphasize the role of the parameter scheme, or by (A, η) in the case in which $S = \operatorname{Spec} A$.
For each k-rational point $t \in S$, the scheme-theoretic fibre $\mathcal{X}(t)$ is a *deformation of* X. If $S = \operatorname{Spec} A$ for $A \in k-\text{Art}$, and $s \in S$ is the closed point, we say to have a *local family of deformations*, or *a local deformation*, of X over A.

Definition 41. A local deformation (A, η) is said to be *infinitesimal* if $A \in k-\mathrm{Art}$, and it is said to be a *first order deformation* if $A = k[\epsilon]$.

Observe that, for every algebraic scheme X, given a k-pointed scheme (S, s), i.e. a pointed scheme (S, s) such that $k \cong k(s)$, there exists at least one family of deformations of X, over S, called the *product family*, given by

$$\begin{array}{ccc} X & \longrightarrow & X \times S \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \xrightarrow{s} & S \end{array}$$

Definition 42. A deformation (S, η) is *trivial* if it is isomorphic to the product family, and X is said to be *rigid* if every infinitesimal deformation of X over A is trivial for every $A \in k-\mathrm{Art}$.
An infinitesimal deformation of X is *locally trivial* if for every $x \in X$ there exists a neighbourhood $U_x \subset X$ such that

$$\begin{array}{ccc} U_x & \longrightarrow & \mathcal{X}|_{U_x} \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \xrightarrow{s} & S \end{array}$$

is a trivial deformation of U_x.

Example 3.3 It can be proved that a nonsingular variety X is rigid if and only if $H^1(X, T_X) = 0$, see e.g. [Sernesi 1.2.15]. Consider $\mathbb{P}^n$ for $n > 0$ and its Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}{}^{n+1} \to T_{\mathbb{P}^n} \to 0.$$

About this exact sequence there are several well known facts, one of which is that $H^1(\mathbb{P}^n, T_{\mathbb{P}^n}) = 0$, so that $\mathbb{P}^n$ is rigid for all $n > 0$.

Using these constructions we may now define a covariant functor $D : k-\mathrm{Art} \to \mathit{Sets}$ with the following idea: $D(k)$ is the object we want to deform and $D(A)$ is the set of isomorphism classes of deformations over $S = \operatorname{Spec} A$.

Definition 43. A *deformation functor*

$$\mathit{Def} : k-\mathrm{Art} \to \mathit{Sets}$$

is a covariant functor such that $\mathit{Def}(k)$ is a single point. If $\varphi : B \to A$ is a homomorphism in $k-\mathrm{Art}$ and $\alpha \in \mathit{Def}(A)$, we say that $\beta \in \mathit{Def}(B)$ is a *lifting* of α if $\varphi\beta = \alpha$.

Example 3.4 In [FGAE, §6], Barbara Fantechi and Lothar Göttsche remark that *"many interesting deformation functors arise as infinitesimal local versions of moduli functors"*, including Hilbert schemes. Indeed, if X is a scheme and Z is a closed subscheme we may define $H_{Z,X}(A)$ to be the set of deformations of Z in X over $S = \operatorname{Spec} A$, that is the set of S-flat closed subschemes Z_S of the product $X \times S$ having fibre over S_{red} in Z. Now, $H_{Z,X}(k) = Z$ and functoriality follows by flat base-change properties.
There is also an immediate generalization of this construction for a general covariant functor $F : \mathit{Schemes} \to \mathit{Sets}$ defined as follows: let $p \in F(\operatorname{Spec} k)$ be a point in F, then we can associate to the pair (F, p) a deformation functor $\mathit{Def}_{F,p}$ by letting

$$\mathit{Def}_{F,p}(A) := \{\alpha \in F(\operatorname{Spec} A) \mid \alpha|_{\operatorname{Spec} A/\mathfrak{m}_A} = p\}.$$

Indeed, the functor $H_{Z,X}$ we defined previously is associated to the point $[Z]$ of the Hilbert functor Hilb^X by letting $\mathrm{Hilb}^X(S)$ be the set of S-flat closed subschemes of $X \times S$.

In particular, if X is a scheme and we fix a point $p \in X$, we can also view p as a point in $h_X(\operatorname{Spec} k)$, and write $D_{X,p}$ instead of $\mathit{Def}_{h_X,p}$.

Remark 12. To every object $R \in k-\mathrm{Loc}$ we can associate a deformation functor in an almost natural way, taking $h_R(A) = Hom_k(R, A)$. We have a useful particular case if we consider a scheme X and a point $p \in X$ by letting $R = \mathcal{O}_{p,X}$. If A is an Artinian local ring, then $h_R(A)$ coincides with the set of morphisms $\operatorname{Spec} A \to X$ mapping the closed point to p, i.e. $h_R = Def_{X,p}$. Moreover, in this case $\dim T_R$ is the minimal dimension of a smooth scheme containing, as closed subscheme, an open neighbourhood of p in X. Such an integer is called *embedding dimension of R.*

We introduce a correspondence, named after two of the fathers of deformation theory, which provides the relation between certain first-order deformations and the first-cohomology of the tangent space to algebraic varieties.

Definition 44. Let X be an algebraic variety and recall that $T_X = Hom(\Omega^1_X, \mathcal{O}_X) = \mathrm{Der}_k(\mathcal{O}_X, \mathcal{O}_X)$. Then there is a 1-1 correspondence

$$\kappa : \{\text{isomorphism classes of first-order loc. trivial def. of } X\} \to H^1(X, T_X)$$

called *Kodaira-Spencer correspondence*, such that $\kappa(\xi) = 0$ if and only if ξ is the trivial deformation class. On top of that, if X is nonsingular, then κ is a correspondence

$$\kappa : \{\text{isomorphism classes of first-order deformations of } X\} \to H^1(X, T_X).$$

For the proof that such a 1-1 correspondence exists and is well defined we refer to [Sernesi, 1.2.6 and 1.2.9].
The next step is the introduction of a second particular space, called *obstruction space*, that is strictly related to the tangent one; the two together form what is called a *tangent-obstruction theory.*

Definition 45. A deformation functor Def is said to have a *(generalized) tangent-obstruction theory* if there exist (finite dimensional) k-vector spaces T_1, called *tangent space* and T_2, called *obstruction space*, such that the following holds:

1) For all small extensions $0 \to M \to B \to A \to 0$ there exists an exact sequence of sets

$$T_1 \otimes_k M \to Def(B) \to Def(A) \overset{ob}{\to} T_2 \otimes_k M; \tag{3.5}$$

2) If $A = k$, the sequence (3.5) becomes

$$0 \to T_1 \otimes_k M \to Def(B) \to Def(A) \overset{ob}{\to} T_2 \otimes_k M; \tag{3.6}$$

3) The exact sequences (3.5) and (3.6) are *functorial in small extensions* in the sense of [FGAE, 6.1.19 and 6.1.20].

If we now set

$$Def_X(A) := \{\text{isomorphism classes of first-order deformations of } X \text{ over } A\}$$

and

$$Def'_X(A) := \{\text{isomorphism classes of first-order loc. trivial def. of } X \text{ over } A\}$$

we can introduce two new functors related to the Kodaira-Spencer correspondence.

Definition 46. Let X be an algebraic scheme. Then, for every $A \in k-\mathrm{Art}$ the functor

$$Def_X(\cdot) : k-\mathrm{Art} \to Sets$$

is called the *local moduli functor of* X; if $X = \operatorname{Spec} B$, it is usually denoted by Def_B. The subfunctor

$$Def'_X(\cdot) : k-\mathrm{Art} \to Sets$$

is called the *locally trivial moduli functor of* X.

Both functors are *functors of Artinian rings* in the sense of [Sernesi, §2.2].

Now we are able to conclude this section stating some results about the tangent space to Hilbert schemes. Just for a matter of coherency, we claim here also one result about the tangent space to the particular case of the Hilbert scheme of n points, refering to Section 3.3 for its definition.

Theorem 3.9. *Let X be a scheme and Y a closed subscheme of X. The deformation functor $H_{Y,X}$ defined in Example 3.4 has a generalized tangent-obstruction theory, given by*

$$T_1 = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}_Y, \mathcal{O}_Y) \text{ and } T_2 = \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{I}_Y, \mathcal{O}_Y).$$

Proof. See [FGAE, 6.4.10]. □

Theorem 3.10. *Let Y be a projective scheme and consider a k-rational point $[X]$ in Hilb^Y parametrizing a closed subscheme $X \subset Y$ and call $\mathscr{I} \subset \mathcal{O}_Y$ the ideal sheaf of X in Y. Then, there is a canonical k-vector spaces isomorphism*

$$T_{[X]}\mathrm{Hilb}^Y \cong H^0(X, N_{X/Y})$$

where $N_{X/Y} = \mathcal{H}om_{\mathcal{O}_X}(\mathscr{I}/\mathscr{I}^2, \mathcal{O}_X)$ is the normal sheaf of X in Y.

Proof. See [Sernesi, 4.3.5]. □

Theorem 3.11. *Denote by $X^{[n]}$ the Hilbert scheme of n points and take $Z \in X^{[n]}$. Then*

$$T_Z X^{[n]} = \mathrm{Hom}_{\mathcal{O}_X}(\mathscr{I}_Z, \mathcal{O}_Z).$$

Proof. For a short proof, which uses the local criterion for flatness, see [Bertin, Proposition 2.11 and 2.12]. For a "larger" idea of the proof, involving deformation theory see [Sernesi, §2.4], [HM, §1C] or [Lehn, §3.4]. □

3.3 Hilbert Scheme of Points

The "easiest" case of a Hilbert scheme might be the one in which the Hilbert polynomial is constant equal to $n \in \mathbb{Z}_{>0}$. This is the so called *Hilbert scheme of points* and is one of the few cases in which the Hilbert scheme is still almost well-behaved. The main references for this section will be [Nak] and [FGAE].

We start defining the Hilbert scheme of points and we see some first relevant properties.

Definition 47. Let n be a positive integer and consider the constant polynomial $P(m) = n$. We define

$$X^{[n]} := \mathrm{Hilb}^X_P$$

the Hilbert scheme corresponding to the polynomial P, and we call it the *Hilbert scheme of n points on X*.

If $x_1, \ldots, x_n$ are n distinct points in X, we may consider the closed subscheme $Z := \{x_1, \ldots, x_n\}$ in X. As its structure sheaf $\mathcal{O}_Z$ is the direct sum of the skyscraper sheaves of the points we find out that $\mathcal{O}_Z \otimes \mathcal{O}_X(m) = \mathcal{O}_Z$ for all $m \in \mathbb{Z}$, and thus $Z \in X^{[n]}$, from which the name "Hilbert scheme of points". From a more general point of view, the Hilbert scheme of n points in X parametrizes all 0-dimensional subschemes of X having length n, where by *length* we mean the length of a module over itself, i.e.

$$\mathrm{length}(Z) := \dim H^0(Z, \mathcal{O}_Z) = \sum_{p \in \mathrm{Supp}(Z)} dim_k(\mathcal{O}_{p,Z})$$

for a scheme Z.

In this special case an "elementary" proof of the existence of the Hilbert scheme has been produced in the paper, [T. Gustavsen, D. Laksov, R. Skjelnes, "An elementary, explicit, proof of the existence of Hilbert schemes of points", *Journal of Pure and Appplied Algebra, vol. 210* (2007), no. 3, pp. 705–720], for X projective over an arbitrary base scheme S, avoiding the notion of Castelnuovo-Mumford's regularity. Moreover, the following result states that the Hilbert scheme of point $X^{[n]}$ on any quasi-projective scheme X is always connected, generalizing Hartshorne's Connectedness theorem.

Lemma 3.12. *Let X be a a connected quasi-projective scheme. Then $X^{[n]}$ is connected for all $n \geq 0$.*

Proof. See [Lehn, Lemma 3.7]. The proof in the case of projective varieties over a field k is discussed in [FGAE, Lemma 7.2.1]. $\square$

Example 3.5 We show two easy examples.

1. Let X be a (quasi)projective scheme over k, call Z its universal family and suppose that $n = 0$. If $f : T \to \operatorname{Spec} k$ is a locally Noetherian k-scheme and $Y \to X \times_k T$ is a closed proper subscheme, flat over T having Hilbert polynomial 0, then Y is the empty subscheme. Thus f pullbacks Z to Y. On top of that, f is the only possible morphism over k, thus $X^{[0]} = \operatorname{Spec} k$ and its universal family Z is the empty subscheme of $X \times_k k = X$.

2. Let X be a projective scheme over k and suppose that $n = 1$ is the Hilbert polynomial. Take $Y \in \operatorname{Hilb}_1^{X/\operatorname{Spec} k}(T)$, with T a locally Nooetherian k-scheme. All the fibres of the map $f : Y \to T$ have Hilbert polynomial equal to 1, and so they consist of single points, so that f is a bijection. If f were an isomorphism, then composing f^{-1} with the projection of Y to X through its embedding in $X \times_k T$ gives a morphism from T to X. So we may construct the following diagram
$$\begin{tikzcd} Y \arrow[r] \arrow[d] & X \times_k T \arrow[r] \arrow[d] & T \arrow[d] \\ X \arrow[r, "\Delta"] & X \times_k X \arrow[r] \arrow[d] & X \arrow[d] \\ & X \arrow[r] & \operatorname{Spec} k \end{tikzcd}$$
It is then immediate that the pullback of the diagonal through this map is Y and the morphism $X \to \operatorname{Spec} k$ is projective, hence proper. This implies that f itself is proper, as this property is stable under base change, closed immersions are proper and composition of proper maps is again proper. Thus f is finite since T is locally Noetherian and f has finite fibres (see e.g. [Stacks, Lemma 30.21.1 (tag 02OG)]).

   Now, fix a point t of the scheme T, let $V \cong \operatorname{Spec} B$ be an affine open neighbourhood of t, and call $U := f^{-1}(V) \cong \operatorname{Spec} A$, for A a finite, flat B-module, and say that t corresponds to a point P of $\operatorname{Spec} B$. Take elements $a_1, \dots a_d \in A$ that map to a basis of $A \otimes_B B_P/P_P$, which is a finite B_P/P_P-vector space and call r its dimension. Notice that the images of these elements generate the module $A/PA \cong A \otimes_B B_P/P_P$. Let now $\varphi : B^d \to A$ be the module homomorphism sending the i-th base vector of B^d to a_i, and let N be the image of φ in A. By Nakayama's Lemma, as $P(A/N) = A/N$, there exists some $f \in 1 + P$ such that $N_f = A_f$, that is $\varphi_f : B_f^d \to A_f$ is surjective. By flatness we find an exact sequence
$$0 \to \ker(\varphi_f) \otimes_{B_f} B_P/P_P \to B_f^d \otimes_{B_f} B_P/P_P \to A_f \otimes_{B_f} B_P/P_P \to 0$$
   At this point, by the choice of a_i it turns out that $\ker(\varphi_f) \otimes_{B_f} B_P/P_P = 0$ and so, again using Nakayama's Lemma, we get an element $f' \in B_f$ such that $\ker(\varphi_f)_{f'} = 0$, and so $A_{ff'}$ is a free $B_{ff'}$-module of rank r. In other words we have an affine neighbourhood $V' \cong \operatorname{Spec} B'$ of t such that $U' := f^{-1}(V') \cong \operatorname{Spec} A'$ and A' is a free B'-module of rank r, which by definition was the dimension, as a k-vector space, of the global sections of the fibre over t, finding that $r = 1$, and so f is truly an isomorphism.

   This allows us to show that $X^{[1]} = X$ having the diagonal as universal family.

There exists also another space that parametrizes sets of n points in X: the *symmetric power of* X, which is defined as the quotient of the n-th power of X by the action of the symmetric group S_n, and is usually denoted by $X^{(n)}$, $S^n X$ or $\operatorname{Sym}^n X$, if it might be confused with $X^{[n]}$.
So, $X^{(n)}$ parametrizes effective 0-cycles of degree n on X. Recall from Remark 4 that 0-cycles are formal sums $\sum_i n_i[p_i]$, with $p_i \in X$, $n_i \in \mathbb{Z}_{>0}$ and $\sum_i n_i = n$. It can be proved that $S^n X$ is an algebraic variety and it is (quasi)projective if X does (see [Ha, Lecture 10]). In particular, it turns out that $S^n X$ may be regarded as the Chow variety of 0-cycles (see [Ha, Lecture 21]).

In some easier cases the relation between these the Hilbert scheme of points and the symmetric power of a scheme is nice, as we can see with the following example.

Example 3.6 Consider the Hilbert scheme of n points on the affine line $\mathbb{A}$. Then

$$\begin{aligned}\mathbb{A}^{[n]} &= \{\mathfrak{a} \subseteq k[z] \mid \mathfrak{a} \in \operatorname{Spec} k[z]\,,\, k[z]/\mathfrak{a} = n\} \\ &= \{f(z) \in k[z] \mid f(z) = z^n + a_1 z^{n-1} + \ldots + a_n\,,\, a_i \in k\} \\ &= S^n\mathbb{A}.\end{aligned}$$

Even if this example has a good behaviour, and it can be generalised to every nonsingular curve, as we will state below, the identification between the Hilbert scheme of points and the symmetric product of the scheme fails already in dimension two, as the following example shows.

Example 3.7 Let X be a nonsingular projective variety of dimension d and consider $X^{[2]}$. If $\{x_1, x_2\}$ are two distinct points, then $\{x_1, x_2\}$ is a point in $X^{[2]}$. What if the two points collide? As a nontrivial vector $v \in T_xX$ defines an ideal $\mathscr{I} = \{f \in \mathcal{O}_X \mid f(x) = 0\,,\, df_x(v) = 0\}$ of $\mathcal{O}_X$ for all $x \in X$, that has codimension 2, the quotient $\mathcal{O}_X/\mathscr{I}$ is a 0-dimensional subscheme Z in $X^{[2]}$. The geometric interpetation of this situation is that Z is a set of two infinitely near points in X along the direction of v. So the two cases have a different behaviour as soon as $\dim T_xX > 1$. In particular, if the two points coincide we find a non reduced structure. For a further, but rather sketchy, analysis of this case, including a global description of $X^{[2]}$ see [FGAE, Example 7.3.1].

The true relation between the symmetric product and the Hilbert scheme of points is provided by the so called *Hilbert-Chow morphism.*

Theorem 3.13. *There exists a (surjective) morphism*

$$\pi : X^{[n]}_{red} \to S^nX$$

defined by

$$\pi(Z) = \sum_{x\in X} \mathrm{length}(Z_x)[x]$$

which is called Hilbert-Chow morphism.

Proof. See e.g. [D. Mumford, J. Fogarty and F. Kirwan, *Geometric invariant theory, Third Enlarged Edition*, Springer-Verlag Berlin Heidelberg(1994), Theorem 5.4] as suggested in [Nak] or [FGAE, Theorem 7.1.14]. □

Moreover, the Hilbert-Chow morphism turns out to be projective, see Remark 2.19. of [Bertin]. We stated above that for curves we have an identification between $X^{[n]}$ and S^nX. Let us prove this claim.

Theorem 3.14. *Let C be a nonsingular quasiprojective curve. Then, the Hilbert-Chow morphism is an isomorphism.*

Proof. The local ring of the curve C at a point p is a discrete valuation ring, thus all ideals in $\mathcal{O}_{p,C}$ are powers of the maximal ideal $\mathfrak{m}_p$. Therefore, for all $[Z] \in C^{[n]}$ we have

$$\mathcal{O}_Z = \bigoplus_i \mathcal{O}_{p_i,C}/\mathfrak{m}_{p_i}^{n_i}$$

for $\sum_i n_i = n$. Then, $\pi(Z) = \sum_i n_i[p_i]$, and so it is bijective. As π is also birational, it turns out to be an isomorphism by Zariski's Main Theorem. □

Remark 13. As there are several versions of Zariski's Main Theorem we point out that we refer to [Oldfield, Theorem 2.7] for the version we need in the proof of Theorem 3.14, even if it is claimed without giving the proof. For a deeper insight on the argument, including the various formulation of Zariski's Main Theorem one may look at [Mumford III, §9].

Even though we lose the identification we have in dimension 1, also for 2-dimensional varieties we can prove that $X^{[n]}$ maintains some good properties of the underlying scheme.

Theorem 3.15. *Let X be an irreducible nonsingular quasiprojective variety of dimension $d \leq 2$ and take $n \geq 0$. Then $X^{[n]}$ is nonsingular, irreducible and has dimension dn.*

Proof. See [FGAE, Theorem 7.2.3] □

A crucial point of the proof of the previous theorem is the following remark.

Remark 14. Let X be a nonsingular quasiprojective variety of dimension d, and call $X_0^n \subset X^n$ the open dense set of $(p_1, \ldots, p_n)$, with $p_i \neq p_j$ for all $i \neq j$. Let $X_0^{(n)}$ denote its image in $X^{(n)}$, which parametrizes effective 0-cycles of the form $\sum_i [p_i]$, with the p_i distinct. This set is also dense and open. As S_n acts freely on X_0^n, we have that $X_0^{(n)}$ is nonsingular of dimension dn. Let now $X_0^{[n]}$ be the preimage of X_0^n in $X^{[n]}$. It can be proved that at any point of $X_0^{[n]}$ the dimension of the tangent space is dn and that the Hilbert-Chow morphism, restricted to $X_0^{[n]}$ is an isomorphism, thus the Hilbert scheme of n points in X contains a nonsingular open subset isomorphic to an open subset of $X^{(n)}$ (see [Oldfield, §5.4] for a general argument, or [Nak, Theorem 1.8] for the case of surfaces).

The following result, which was proved by John Fogarty in [J. Fogarty, "Algebraic families on an algebraic surface", *Amer. J. Math* (1968), pp. 511–521], restates Theorem 3.15 for $d = 2$, claiming something more about the Hilbert-Chow morphism.

Theorem 3.16. *Suppose that X is nonsingular and of dimension 2. Then:*

i) $X^{[n]}$ *is nonsingular of dimension $2n$;*

ii) *the Hilbert-Chow morphism $\pi : X^{[n]} \to S^n X$ is a resolution of singularities.*

Proof. i) Take $Z \in X^{[n]}$, consider the corresponding ideal $\mathscr{I}_Z$ and take the Zariski tangent space of $X^{[n]}$ at Z, which is given by

$$T_Z X^{[n]} = \mathrm{Hom}_{\mathcal{O}_X}(\mathscr{I}_Z, \mathcal{O}_X/\mathscr{I}_Z) = \mathrm{Hom}_{\mathcal{O}_X}(\mathscr{I}_Z, \mathcal{O}_Z)$$

by Theorem 3.11.

In order to prove the smoothness we need to show that the dimension of the Zariski tangent space doesn't depend on the point Z. By definition we have an exact sequence

$$0 \to \mathscr{I}_Z \to \mathcal{O}_X \to \mathcal{O}_Z \to 0.$$

Passing to the associated long exact sequence in cohomology and recalling that $\dim X = 2$ we find a sequence

$$\begin{aligned} 0 &\to \mathrm{Hom}(\mathcal{O}_Z, \mathcal{O}_Z) \to \mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_Z) \to \mathrm{Hom}(\mathscr{I}_Z, \mathcal{O}_Z) \\ &\to \mathrm{Ext}^1(\mathcal{O}_Z, \mathcal{O}_Z) \to \mathrm{Ext}^1(\mathcal{O}_X, \mathcal{O}_Z) \to \mathrm{Ext}^1(\mathscr{I}_Z, \mathcal{O}_Z) \\ &\to \mathrm{Ext}^2(\mathcal{O}_Z, \mathcal{O}_Z) \to \mathrm{Ext}^2(\mathcal{O}_X, \mathcal{O}_Z) \to \mathrm{Ext}^2(\mathscr{I}_Z, \mathcal{O}_Z) \to 0 \end{aligned}$$

where all the $\mathrm{Hom}(\cdot,\cdot)$ and $\mathrm{Ext}^i(\cdot,\cdot)$ are taken over $\mathcal{O}_X$.

Since $\displaystyle\sum_{i=0}^{n} (-1)^i \dim \mathrm{Ext}^i(\mathscr{I}_Z, \mathcal{O}_Z)$ in independent of Z, it is enough to prove that $\dim \mathrm{Ext}^i(\mathscr{I}_Z, \mathcal{O}_Z)$ in independent of Z for $i = 1, 2$.

Notice first that $\dim \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_Z, \mathcal{O}_Z) = n$ and also $\dim \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_Z) = n$, so that the first arrow is an isomorphism $k^n \to k^n$, and thus $\mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_Z) \cong \mathrm{Hom}(\mathcal{O}_Z, \mathcal{O}_Z)$. This fact also implies that $\mathrm{Hom}(\mathscr{I}_Z, \mathcal{O}_Z)$ is a natural subspace of $\mathrm{Ext}^1(\mathcal{O}_Z, \mathcal{O}_Z)$.

Now, by definition of Ext^i and by Serre's Vanishing theorem we have that

$$\mathrm{Ext}^i(\mathcal{O}_X, \mathcal{O}_Z) \cong H^i(X, \mathcal{O}_Z) \cong H^i(X, \mathcal{O}_Z(n)) = 0$$

for $i \geq 1$ and n sufficiently large. Hence

$$\mathrm{Ext}^1(\mathscr{I}_Z, \mathcal{O}_Z) \cong \mathrm{Ext}^2(\mathcal{O}_Z, \mathcal{O}_Z)$$

and

$$\mathrm{Ext}^2(\mathscr{I}_Z, \mathcal{O}_Z) = 0.$$

Using now Serre's Duality Theorem (see e.g. [H III, §7]) we find out that

$$\mathrm{Ext}^2(\mathcal{O}_Z, \mathcal{O}_Z) \cong (\mathrm{Hom}(\mathcal{O}_Z, \mathcal{O}_Z \otimes K_X))^\vee = (\mathrm{Hom}(\mathcal{O}_Z, \mathcal{O}_Z))^\vee$$

where K_X is the canonical sheaf of X.
Using the fact that $\mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_Z) \cong \mathrm{Hom}(\mathcal{O}_Z, \mathcal{O}_Z)$ we conclude that $\mathrm{Ext}^2(\mathcal{O}_Z, \mathcal{O}_Z) \cong k^n$, thus $\dim \mathrm{Ext}^2(\mathcal{O}_Z, \mathcal{O}_Z) = n$.
All these facts together show that $\dim \mathrm{Hom}(\mathscr{I}_Z, \mathcal{O}_Z)$ does not depend on Z, and thus the claim of point i). Furthermore, notice that we also obtained that $\mathrm{Hom}(\mathscr{I}_Z, \mathcal{O}_Z) \cong \mathrm{Ext}^1(\mathcal{O}_Z, \mathcal{O}_Z)$.

ii) By Remark 14 $X_0^{(n)}$ is nonsingular. We now show something more. $X_0^{(n)}$ is exactly the nonsingular locus of $X^{(n)}$ if X is a surface. Let $\Delta := \cup_{1 \leq i < j \leq n} \Delta_{i,j}$ be the *big diagonal* in X^n, where $\Delta_{i,j}$ denotes the set $\{(x_1, \ldots, x_n) \in X^n \mid x_i = x_j\}$, and let Δ_0 be the open subset where precisely two of the x_i coincide.
Take now $p \in \Delta_0$ and notice that we may suppose $p \in \Delta_{1,2}$ wlog. Then its stabilizer is $\{1, \tau\} \in S_n$, where τ denotes the trasposition of the first two entries. Hence, a formal neighbourhood of p in $X^{(n)}$, i.e. the completed local ring of $X^{(n)}$ at p, is isomorphic to the quotient $k[[u, v, w, x', y', x_3, y_3, \ldots, x_n, y_n]]/(uw - v^2)$ (see Example 7.1.3 of [FGAE]).
Now, the closure of the image of Δ_0 is $X^{(n)} \setminus X_0^{(n)}$ and finally, as the singular locus of $X^{(n)}$ is closed, we conclude that $X_0^{(n)}$ is the nonsingular locus of $X^{(n)}$.
Therefore π is a resolution of singularities, as $X^{[n]}$ is nonsingular and irreducible by Theorem 3.13 and the Hilbert-Chow morphism is an isomorphism over the open subset $X_0^{(n)}$ that is nonsingular in $X^{(n)}$ by Remark 14. □

We conclude the chapter providing an example that, together with Example 3.7, makes us aware of *"how things go easily wrong"*, as it will be the fundamental philosophy of the next chapter.

Example 3.8 Let X be a nonsingular variety of dimension 3 and let $[Z] \in X^{[4]}$ be the point $\mathcal{O}_Z = \mathcal{O}_P/\mathfrak{m}^2$, where $\mathfrak{m}$ is the maximal ideal at the point $P = Supp(Z)$ as in [FGAE, 7.2.5]. Then

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathscr{I}_Z, \mathcal{O}_Z) = \mathrm{Hom}_k(\mathfrak{m}^2/\mathfrak{m}^3, \mathfrak{m}/\mathfrak{m}^2) = k^{18}$$

and thus has dimension 18, that is clearly bigger than $dn = 12$, thus $X^{[4]}$ is singular. This example has the following "generalization": let $X = \mathrm{Spec}\, k[x_1, \ldots, x_d]$, for $d \geq 3$, and consider $\mathfrak{m} = (x_1, \ldots, x_d)$ and Z the closed subscheme determined by $\mathfrak{m}^2$. Then $X^{[d+1]}$ is singular at $[Z]$ as its tangent space, arguing similarly to the above equivalence, turns out to be isomorphic to $k^{d^2(d-1)}$, see e.g. Example 5.1. of [Oldfield].

Chapter 4

Pathologies and Murphy's Law for Hilbert Schemes

In the third chapter we encountered some first examples of Hilbert schemes which did not have a good behaviour, even if they belonged to one of the easiest cases of Hilbert schemes, the Hilbert schemes of points. The reason of this bad behaviour lies in the fact that the dimension of the deformation spaces related to those Hilbert schemes was not the expected one. So we may ask ourselves *"How bad can the deformation space of an object be?"*. This is the starting question of Ravi Vakil's paper *"Murphy's law in algebraic geometry: badly-behaved deformation spaces". Invent. Math., vol. 164 (2006), no.3, pp. 569–590* (refered to as [Va2]), and it turns out that *"unless there is some a priori reason, the deformation space may be as bad as possible"*.
The answer Vakil himself provides follows, and justifies, the philosophy David Mumford introduced in a series of papers

i) *D. Mumford, "Pathologies of Modular Algebraic Surfaces.", American Journal of Mathematics 83 (1961), no. 2, pp. 339–342* (refered to as [MumP1]),

ii) *D. Mumford, "Further Pathologies in Algebraic Geometry.", American Journal of Mathematics 84 (1962), no. 4, pp. 642–648* (refered to as [MumP2]);

iii) *D. Mumford, "Pathologies III", American Journal of Mathematics 89 (1967), no. 1, pp. 94–104.* (refered to as [MumP3]);

in which he pointed out *pathologies* that appear already when considering moduli spaces of well-behaved objects. The ideas presented in these papers are so relevant that we will devote a first section just to present the gist of Mumford's example of a pathological Hilbert scheme which is singular, nonreduced and has multiple components of the same dimension. The second section will shortly present some further pathologies of Hilbert schemes of curves, while the third section will briefly present Vakil's ideas.

4.1 Mumford's Example

"There is no geometric possibility so horrible that it cannot be found generically on some component of some Hilbert scheme."

This so called *"Murphy's law for Hilbert schemes"* was formulated by Joe Harris and Ian Morrison in [HM, §1D], introducing the discussion about *extrinsic pathologies of moduli of curves.* The first and main example they use to justify such a statement comes from [MumP2, Section II]. In this section of the article the author provides the first example of a Hilbert scheme being nonreduced, by analyzing the family A of curves $\gamma \subset \mathbb{P}^3$ having degree 14 and arithmetic genus 24.

The problem studied by Mumford relies deeply on the study developed by Kunihiko Kodaira in [K. Kodaira, "A Theorem of Completeness of Characteristic Systems for Analytic Families of Compact Submanifolds of Complex Manifolds", *Annals of Mathematics Second Series, vol. 75* (1962), no. 1, pp. 146–162].

As a consequence of his construction, Mumford claims that the family A of curves γ consists of an example of Hilbert scheme which *"has a multiple component, i.e. is not reduced at one of its generic points"* and is also singular. Moreover, he remarks that the blow-up of γ to a surface E turns out to be a new three-dimensional variety V_3, which is nonsingular, projective and whose local moduli scheme is nowhere reduced, see [MumP2, pp. 643–644].
To present Mumford's Example we will try to follow the original step by step construction of [MumP2, Section 2], giving just some hints of the various proof, regarding also [HM, §1D] as a fundamental reference, as the original article is often too concise. By the way, the approaches used by the two reference are rather different, since in [MumP2] we start taking curves with the given degree and arithmetic genus at the highest level of generality, ending with the particular case we will see in Proposition 4.4, while in [HM, §1D] the authors procede from the particular case to the most general one.

Let γ be a nonsingular curve having degree 14 and arithmetic genus 24 in $\mathbb{P}^3$. Call h the divisor class on γ induced by plane sections, H the (Cartier) divisor class of a plane section of a cubic, or quartic, surface in $S \subset \mathbb{P}^3$. Denote by K_S and K_γ the canonical divisor on S and γ respectively.

The first step consists in showing that any nonsingular curve $\gamma \subset \mathbb{P}^3$ of degree 14 and arithmetic genus 24 is contained in a suitable family of quartic surfaces, confirming the previous classification obtained by Max Noether in 1882 ([M. Noether, "Zur Grundlegung der Theorie der algebraischen Raumcurven", *Journal für die reine un angewandte Mathematik, n.93* (1882), Heidelberg, pp. 271–318]).

Lemma 4.1. *Any nonsingular space curve γ of degree 14 and arithemtic genus 24 is contained in a pencil P of quartic surfaces.*

Proof. This follows easily by a degree computation, see [MumP2, (A)]. □

Then, the study splits up into two cases, distinguishing whether the obtained pencil P has a fixed component or not:

- Curves of type (a): the pencil P obtained in Lemma 4.1 has no fixed component;
- Curves of type (b): the pencil P obtained in Lemma 4.1 has a fixed component.

The second step of the construction establishes an upper bound on the dimension of an algebraic family of space curves of type (a).

Lemma 4.2. *Every algebraic family of space curves of degree 14 and arithmetic genus 24 of type (a) has dimension less than or equal to 56.*

Proof. Show first that if we are working with curves of type (a) and we call F and F' the two quartics that span the pencil P, then almost every quartic $S \in P$ is nonsingular everywhere along γ, see [MumP2, p. 644]. The result is now proved by showing that every family of pairs (γ, S) consisting of curves γ of type (a) and quartics $S \supset \gamma$ being non singular along γ has dimension at most 57, see [MumP2, (B)]. □

These two lemmas complete the study of curves of type (a).

Suppose now to work with curves of type (b). Since a space curve γ of this type should be contained in a reducible quartic, and it cannot be contained in a plane or in a quadric surface, it has to be contained in a cubic surface. Moreover, by a matter of degree, it turns out that such a cubic is unique and we call it, again, S. We have now two possibilities:

- Curves of type (b_0): the cubic surface S containing γ is smooth;
- Curves of type (b_1): the cubic surface S containing γ is singular.

The third step of the construction proves an estimate on the dimension of a *maximal algebraic family* of curves of type (b_0), i.e. of a family such that given any other family of curves B containing a γ of the given type, this family B can be obtained from A taking pull-backs.
As it used along the proof of the next result and it will be needed also in the future, we provide here the following definitions.

Definition 48. Let $\mathcal{D}$ be a linear system of (Cartier) divisors on a variety X. A *general member* of $\mathcal{D}$ is said to satisfy a property P if there is a Zariski dense open subset U of the projective space parametrizing the system such that all divisors corresponding to points of U satisfy P. The *generic element* of a linear system is the generic point of the projective space parametrizing it, and a given property P is called *generic* if it is a property of the generic point.

For example, if we talk about a *general curve* we will mean an element in fixed dense open subset of the component of the Hilbert scheme of curves we are dealing with.

Proposition 4.3. *Every maximal algebraic family of curves γ of type (b_0) has dimension 56.*

Proof. Noticing that $K_S \equiv -H$ and that $K_\gamma \equiv \gamma \cdot (\gamma + K_S)$, using Riemann-Roch's and Serre's Duality Theorems (see e.g. [H IV, §1] for the version of the Riemann-Roch's Theorem involving curves, which is the one we need here) the claim follows by explicit computations on the family of cubic and on the cohomology of $\mathcal{O}_S(\gamma)$, proving that a generic curve γ of the given type is contained in a generic cubic surface. See [MumP2, (C)]. □

Now, denote by C the Chow variety (see Remark 4) of nonsingular curves of degree 14 and arithmetic genus 24. Call $C_{\mathrm{b}} \subset C$ the locus of curves of type (b) and $C_{\mathrm{b}_1} \subset C$ the locus of curves of type (b_1). By definition it is clear that $C_{\mathrm{b}_1} \subset C_{\mathrm{b}} \subset C$ and that both are closed subvarieties of C. By Lemma 4.2 and Lemma 4.3 every component of $C \setminus C_{\mathrm{b}}$ has dimension less than or equal to 56, while every component of $C_{\mathrm{b}} \setminus C_{\mathrm{b}_1}$ has dimension exactly 56. So, if we call $C_0 := C \setminus (C_{\mathrm{b}_1} \cup \overline{C \setminus C_{\mathrm{b}}})$, it turns out that C_0 is open in C, of dimension 56, and parametrizes almost all curves of type (b_0).
In order to state the fourth step of the construction, we single out a set of components of C_0. Define now a curve γ to be of type (b_0') if it is of type (b_0) and there exists a line E on S such that $\gamma \equiv 4H + 2E$ on the nonsingular cubic S. The locus C_0' of curves of type (b_0') will be both open and closed in C_0, see [MumP2, p. 646].
Hence, Mumford claims the following property to hold for curves in C_0'.

Proposition 4.4. *If N is the normal sheaf to a curve $\gamma \subset S$ of type (b_0'), then* $\dim H^0(N) = 57$.

Proof. It is again a matter of computation of dimensions using Riemann-Roch's Theorem and some suitable intersection number. See [MumP2, (D)]. □

Proposition 4.4 implies that the Hilbert scheme of space curves of degree 14 and arithmetic genus 24 is singular at γ and nonreduced, as γ is a generic element of the family of curves of type (b_0'). This means that, at least at first order, we may find deformations of the curve γ not lying on cubics.
The last step of the construction proposed by Mumford shows that, if S is a smooth cubic, E one of the 27 lines on it and H is the (Cartier) divisor class of a plane on S, then a curve which is linearly equivalent to $4H + 2E$ exists and is a nonsingular curve of degree 14 and arithmetic genus 24.

Proposition 4.5. *Let S be any nonsingular cubic surface and $E \subset S$ any line. Then, there exist nonsingular curves $\gamma \in \mid 4H + 2E \mid$ and they have degree 14 and arithmetic genus 24.*

Proof. For the explicit computation of degree and genus see [HM, §1D], for the proof of the existence see [MumP2, (E)]. □

Moreover one may also ask oneself if the curves of type (b_0') are the only one lying on a smooth cubic surface S. The answer to this question is negative.

Proposition 4.6. *There exists exactly one other component of the Hilbert scheme of space curves of degree 14 and genus 24 whose general member lies on a smooth cubic surface.*

Proof. We should prove existence and uniqueness of the component of the Hilbert scheme.
The key point to get the existence is that, again by a matter of dimensional computation, a curve γ with the given degree and arithmetic genus that lies on a smooth cubic has to lie on a sextic surface T not containing the given cubic S and is residual to a second curve γ', of degree 4 in the intersection of S with a sextic, and there is no chance that a generic γ lies on a surface of higher degree than 6. Moreover, it can be proved that such a curve γ' has arithmetic genus -1 and self-intersection 0 on the cubic S, hence it is reducible. If now γ' contains two disjoint conics, then we obtain Mumford's component. Otherwise γ' has to contain a, possibly multiple, line, from which we see that the component given by this type of curves is different from Mumford's one.

For a sketch of the proof of uniqueness and for further details on the proof of existence we refer to [HM, pp. 21–23]. □

In particular one can show that also this second component has dimension 56, so the Hilbert scheme of curves of degree 14 and arithmetic genus 24 has at least two components of dimension 56.

4.2 Further pathologies on Hilbert schemes of curves

The example we cited in Section 4.1 is, probably, the most famous and well-known one in which "things go bad", but one may ask oneself if these unpleasant situations revealed by Mumford are the exception, rather then the norm. Murphy's law tells us that not only it is not an exception, but it is indeed the general rule. On top of that Joe Harris and Ian Morrison teach us that we need not to look for awful objects. In [HM, §1D], [HM, §1E] and in [HM, §2D], apart from the previous example, they provide other examples of badly-behaved Hilbert schemes of curves, often focusing to the study of the *restricted Hilbert scheme*.

Definition 49. Consider the Hilbert scheme $\mathcal{H} := \mathrm{Hilb}^r_P$ for a given Hilbert polynomial P. The *restricted Hilbert scheme of* $\mathcal{H}$ is the open subscheme $\mathcal{R} \subset \mathcal{H}$ consisting of those points $[X]$ such that every component $\mathcal{D}$ of $\mathcal{H}$ on which the point $[X]$ lies has smooth, irreducible and nondegenerate general element.

In all the following discussion denote by $\mathcal{H}_{d,g}$ the Hilbert scheme of curves of degree d and arithmetic genus g.
The following problems are proposed:

- The locus of smooth curves in a Hilbert scheme can form a disconnected subvariety, see [HM, Exercise (1.41)];
- Consider the Hilbert scheme $\mathcal{H}_{d,g}$ and its restricted Hilbert scheme $\mathcal{R}_{d,g}$. Is it true that if every curve γ on a component of $\mathcal{R}_{d,g}$ lies on a hypersurface S of degree d, then, for general curve, we may choose S to be smooth? The answer, following Murphy's law, should be clearly "no". A counterexample may be produced considering simply the scheme X of a double line in $\mathbb{P}^3$. By direct a direct computation, in [HM, pp. 24–25] it is proved that that a general curve in the component of $\mathcal{R}_{d,g}$ contaning a curve that is residual to X in a suitable complete intersection of a quartic S and a surface T of degree n has to lie on a quartic and that this quartic is always nonsingular for $n \geq 7$;
- It is possible to construct a smooth, reduced and irreducible curve γ lying in the intersection of two components of a Hilbert scheme $\mathcal{H}_{d,g}$, so that in this case the deformation space is reducible as a subscheme of $\mathbb{P}^r$. An explicit example, for $r \geq 4$, may be performed by considering a cone S over a rational normal curve in $\mathbb{P}^{r-1}$, a collection $L_1, \dots L_{r-2}$ of lines on S and $T \subset \mathbb{P}^r$ a general hypersurface of degree m containing $L_1, \dots L_{r-2}$. The curves that will satisfy the desired properties are the curves γ that are the residual intersection of T with S. For the complete argument that involves a further deeper insight on the study of curves using *Castelnuovo theory* see [HM, p. 25];
- We may provide a lower bound on the dimension of the Hilbert scheme $\mathcal{H}_{d,g}$, given by

  $$h_{d,g,r} := (r+1)d - (r-3)(g-1)$$

  which is called *Hilbert number*, at least at those points of $\mathcal{H}_{d,g}$ parametrizing curves that are locally complete intersections, in particular for smooth one. It is an immediate fact that the Hilbert number $h_{d,g,r}$ is independent on g if $r = 3$, and that for $r \geq 4$ it clearly decreases with g. There are, by the way, examples in which we find only components having dimension exactly the Hilbert number, while in some other all components have larger dimension. Hence, about the dimension of $\mathcal{H}_{d,g}$, or of its components, there are still open questions. For example, does the lower bound given by the Hilbert number $h_{d,g}$ hold for any component of $\mathcal{H}_{d,g}$? Fixing the dimension of the projective space r, can we find an upper bound on the dimension of the restricted Hilbert scheme $\mathcal{R}_{d,g}$? For further remarks on the Hilbert number and some other open problems about the dimension of the components of $\mathcal{H}_{d,g}$ see [HM, §1E];

- There are some conjectures about lower cohomology of the Hilbert scheme $\mathcal{H}_{d,g}$ that *"do seem to hold"* for $\mathbb{P}^1$ and $\mathbb{P}^2$, while they are surely false in general, see [HM, §2D].

4.3 Murphy's Law for Hilbert schemes

"The moral of Murphy's Law is as follows. We know that some moduli spaces of interest are well behaved, often because they are constructed as Geometric Invariant Theory quotients of smooth spaces, [...], (such as) the Hilbert scheme of divisors on projective space [...] In other cases, there has been some effort to try to bound how bad the singuralities can get. Murphy's Law in essence states that these spaces can be arbitrarly singular, and gives a means of constructing an example where any given behaviour happens". This is the philosophy behind [Va2], supported by several examples provided by researchers at the end of the nineteenth-century, not only the one by Mumford.
As mentioned in the introduction, singularity types are one of the two fundamental ingredients to formulate Vakil's results.

Definition 50. Consider the equivalence relation on pointed schemes generated by the following condition: $(X, p) \sim (Y, q)$ if $(X, p) \to (Y, q)$ is a smooth morphism. Under this viewpoint, pointed schemes will be called *singularities*, even if the point itself is regular, and the equivalence classes under the above relation will be called *singularity types.*

The second fundamental ingredient is a particular "scheme-theoretic version" of Mnëv's Universality Theorem, which requires the introduction of a further kind of schemes: the *incidence schemes of points and lines in* $\mathbb{P}^2$.
The original technique behind this definition and result is due to Mnëv himself and can be found in his Ph.D. thesis, as he states on his academic web page (http://www.pdmi.ras.ru/∼mnev/bhu.html), in which he summarizes the history of this problem. The study made bu Mnëv relies on the theory of *matroids*, a fundamental concept in the modern approach to combinatorics. The results formulated in the case of matroids have been then restated and applied to some particular cases of varieties, first by Mnëv himself in [N. E. Mnëv, "The universality theorems on the classification problem of configuration varieties and convex polytopes varieties", *Topology and geometry, Rohlin Semin. 1984-1986, Lect. Notes Math. 1346*, pp. 527–543, (1988)] and then by other authors, providing a combinatorial interpretation of the given geometrical data in $\mathbb{P}^2$. For a first easily available discussion, including pictures of some simple incidence schemes of points and lines, see [J. Richter-Gebert "Mnëv's Universality Theorem Revisited", *Séminaire Lotharingien de Combinatoire, vol. B34h* (1995)] and [J. Richter-Gebert, "The universality theorems for oriented matroids and polytopes", *Contemporary Mathematics, vol. 223* (1999), pp. 269–292].

Definition 51. Define an *incidence scheme of points and lines in* $\mathbb{P}^2$ to be a locally closed subscheme of $(\mathbb{P}^2)^m \times (\mathbb{P}^{2*})^n = \{p_1, \dots, p_m, l_1, \dots, l_n\}$ parameterizing $m \geq 4$ marked points and n marked lines, satisfying the following conditions:

- Let $p_1 = [1; 0; 0]$, $p_2 = [0; 1; 0]$, $p_3 = [0; 0; 1]$ and $p_4 = [1; 1; 1]$;
- For each pair (p_i, l_j) either p_i is required to lie on l_j or p_i is required not to lie on l_j;
- The marked points and the marked lines are required to be distinct;
- Given any two marked lines, there is a marked point required to be on both of them;
- Each marked line contains, at least, three marked points.

Note that, as we have no lower bound to the value of n, we could also have configurations made just by marked points for $m \geq 4$ and $n = 0$. Moreover, since the last condition of the definition requires that each marked line contains (at least) three marked points, for $n = 1, 2, 3$ we need to consider $m \geq 5, 6, 7$ respectively. On top of that, the case $m = 7$ marked points is the first one that allows the number n of marked lines to increase without adding extra marked points, as shown by the following examples.

Example 4.1 An easy example of an incidence scheme of points and lines in $\mathbb{P}^2$ can be elementarly constructed as follows.

- Take $p_1 = [1;0;0]$, $p_2 = [0;1;0]$, $p_3 = [0;0;1]$ and $p_4 = [1;1;1]$;
- Take $l_1 := \overline{p_2p_3}$, $l_2 := \overline{p_1p_3}$ and $l_3 := \overline{p_1p_2}$, where by $\overline{p_ip_j}$ we denote the line passing through the points p_i and p_j;
- Mark three other points taking $p_5 \in l_3$, $p_6 \in l_2$ and $p_7 \in l_1$, for example $p_5 = [1;1;0]$, $p_6 = [1;0;1]$ and $p_7 = [0;1;1]$;
- For each pair (p_i, l_j) it is obvious by construction that either p_i lies on l_j or p_i doesn't lie on l_j. We may notice that p_4 doesn't lie on any line;
- By construction, for each pair of lines there is at least one marked point on both of them;
- Each marked line contains, in this case, exactly three points.

Example 4.2 Let $p_1, \dots, p_7$ be the same points and l_1, l_2, l_3 be the same lines of Example 4.1.

- To the lines l_1, l_2 and l_3 add the following: $l_4 := \overline{p_4p_1}$, $l_5 := \overline{p_4p_2}$ and $l_6 := \overline{p_4p_3}$;
- Again, for each pair (p_i, l_j) it is immediate that either p_i lies on l_j or p_i doesn't lie on l_j;
- Again, for each pair of lines there is at least one marked point on both of them;
- Again, each marked line contains exactly three marked points;
- This time, each marked point lies on at least one of the marked lines.

Now we are able to state the version of Mnëv's theorem needed for the purpose.

Theorem 4.7. *(Mnëv-Sturmfels Theorem) Every singularity type of finite type over $\mathbb{Z}$ appears on some incidence scheme of points and lines in $\mathbb{P}^2$.*

A first schematic proof of this fact was given in [Va2], but the author complained that several readers of [Va2] couldn't obtain this result so easily as he claims. Thus he provided a complete proof of the result and the explicit construction of the desired incidence scheme, in a separated paper written together with the South Korean mathematician Seok Hyeong Lee in 2012: [Seok-Hyeong Lee, R. Vakil, "Mnëv-Sturmfels universality for schemes." (English summary) A celebration of algebraic geometry, *Clay Math. Proc., vol 18, Am.er. Math. Soc.*, Providence, RI, 2013, pp. 457–468].
With Theorem 4.7 Vakil has a fundamental tool that allows him to give a new, formal and rigorous formulation of *"Murphy's Law"*, providing a statements that adapts for a larger collection of objects, of which Hilbert schemes are a special case: moduli spaces.

A moduli space satisfies "Murphy's Law" if every singularity type of finite type over $\mathbb{Z}$ appears on that moduli space.

In [Va2, 1.1] it is claimed that about 15 "well-known" objects of algebraic geometry satisfy such a formulation of Murphy's Law, including some particular Hilbert schemes, and the various proof are achieved *"by drawing connections among various moduli spaces"*, taking as starting point the result by Mnëv.
This new viewpoint showed that Mumford's philosophy is truly consistent, furthermore, even though

"our experience and intuition tells us that pathologies of moduli spaces occur on the boundary, and that moduli spaces of good objects are also good, Murphy's Law shows that this intuition is incorrect; we should expect pathologies even where objects being parametrized seem harmless".
Starting from the fact that surfaces in $\mathbb{P}^3$ have a well-behaved Hilbert scheme (see Section 2.4), one may hope that also Hilbert schemes of surfaces, i.e. varieties of dimension 2, in $\mathbb{P}^4$, or in higher-dimensional projective spaces, are well-behaved. Unfortunately, this is not the case.

Theorem 4.8. *The following Hilbert schemes satisfy Murphy's Law for moduli spaces:*

i) the Hilbert scheme of nonsingular surfaces in $\mathbb{P}^5$*;*

ii) the Hilbert scheme of surfaces in $\mathbb{P}^4$*.*

The idea of the proof is the following, see [Va2, **M2**]:

1) Fix a singularity type. By Theorem 4.7, there exists an incidence scheme exhibiting such a singularity type at some configuration of m points and n lines. One can then show that there is a suitable morphism from that incidence scheme to the *moduli of surfaces with marked smooth divisors*, that is the moduli space whose points are given by pairs (S, C), where S is a smooth surface and C is a family of smooth curves contained in S, both being flat on the base or, from a functorial point of view, the moduli functor that associates to a base scheme the flat family of smooth curves, with the embedding into a flat family of smooth surfaces. This object can be proved to be well defined. The obtained morphism is in particular étale, so it doesn't change the type of the singularity;

2) Using *Abelian covers* and some intermediate deformation spaces (see the proof of **M2a-c**) obtain a regular, nonsingular *surface of general type* $\tilde{\mathcal{S}}$ presenting the same singularity type of (S, C) inside the moduli space of surfaces of general type;

3) Show that the Hilbert scheme of nonsingular surfaces in $\mathbb{P}^5$ satisfies Murphy's law by taking six general sections of a sufficiently positive multiple of the canonical bundle, which is very ample and with vanishing higher cohomology, and using this to embed the nonsingular surface $\tilde{\mathcal{S}}$ obtained at the previous point in $\mathbb{P}^5$ ([Va2, 4.6]);

4) Show that the Hilbert scheme of surfaces in $\mathbb{P}^4$ satisfies Murphy's law by taking five general sections of the bundle to map the nonsingular surface $\tilde{\mathcal{S}}$ to $\mathbb{P}^4$ and by reducing to the previous case if there are nonregular points ([Va2, 4.6]).

We will prove only that the morphism obtained at point 1) is étale, up to some results about *étale cohomology*, and we will give a sketch of the proof of point 2), as a complete one would require further investigation on the various moduli and deformation spaces involved in [Va2, 1.1].

Proof. (of point 1)
First of all we need to recall the definition of *étale morphism.* A morphism $f : X \to Y$ of schemes of finite type over a field k is *smooth of relative dimension* n if:

- f is a flat morphism (see Definition 11);
- if $X' \subseteq X$ and $Y' \subseteq Y$ are two irreducible components such that $f(X') \subseteq Y'$, then we have that $\dim(X') = \dim(Y') + n$;
- for each point x of X we have $\dim_{k(x)}(\Omega_{X/Y} \otimes k(x)) = n$, where we recall that $\Omega_{X/Y}$ is the sheaf of relative differentials of X over Y (see [H II, 8]). If X is also integral, this condition is equivalent to $\Omega_{X/Y}$ being locally free on X of rank n.

In particular, if $f : X \to Y$ is smooth of dimension 0, then is said to be an *étale morphism.* For some first properties see [H III, 10] and for a comparison and motivations from differential geometry see [Va1, §12.6]. For a more detailed reference that includes further different characterizations of the notion see [Milne1, §1] or the open source notes of Milne's course about étale cohomology [Milne2].
Fix now a singularity type. By Theorem 4.7 there is an incidence scheme exhibiting this singularity type at a certain configuration of points and lines, say $\{p_1, \ldots, p_m, l_1, \ldots, l_n\}$. Consider the surface S given by the blow-up of $\mathbb{P}^2$ at the points $\{p_i\}$ and let C be the proper transform of the union of the lines $\{l_j\}$, which is a smooth curve as it is a union of $\mathbb{P}^1$'s. This construction induces a morphism from the incidence scheme $(\mathbb{P}^2, \{p_i\}, \{l_j\})$ to the moduli space of surfaces with marked smooth divisors.

What we claim now is that this morphism is étale at $(\mathbb{P}^2, \{p_i\}, \{l_j\}) \mapsto (S, C)$. What we will produce is an étale local inverse near (S, C). Once we have proved this we obtain that, locally, the two have the same behaviour. Consider a deformation

$$\begin{tikzcd} (S,C) \arrow[r, hook] \arrow[d] & (\mathcal{S},\mathcal{C}) \arrow[d] \\ \mathrm{pt} \arrow[r, hook] & B \end{tikzcd} \tag{4.1}$$

of (S, C).
Next, pull back to an étale neighbourhood of pt (see [Milne1, p. 12]), so that the components of the curve C are labeled. Then, Vakil claims that the Hilbert scheme of (-1)-curves is étale over the base (for the proof of this fact Vakil referes to [Z. Ran, "Deformations of maps in Algebraic Curves and Projective Geometry", *Lect. Notes in Math. 1389* (1989), Springer-Verlag, pp. 246–253], in particular to [loc. cit, Theorem 3.2], even though he admits to be aware of the fact that it is not the original reference, and its generalization to the holomorphic category in [K. Kodaira, "On stability of compact submanifolds of complex manifolds", *Amer. J. Math., vol. 85* (1963), pp. 79–94]).
Call E_i the (-1)-curve corresponding to the point p_i. Pull back to an étale neighbourhood so that the points of the Hilbert scheme corresponding to the (-1)-curve E_i extend to sections, that is there are divisors $\mathcal{E}_i$ on the total space of the family that are (-1)-curves on the fibres, and by abuse of notation denote the resulting family again as in (4.1).
Now, Vakil claims that the surface $\mathcal{S}$ can be blown down along the divisors $\mathcal{E}_i$ obtaining a smooth surface, with marked sections extending the points $\{p_i\}$, using Castelnuovo's criterion over Artin local schemes (again he says that he is unaware of an explicit reference for the fact, and refers to [H V, 5.7] for Castelnuovo's criterion over closed points, claiming that it can be extended either directly, or using [Va2, 5.1]).
It turns out that the special fibre of this last family is $\mathbb{P}^2$, which is rigid (see Example 3.3), thus the family is locally trivial. The marked points $p_1, \ldots, p_4$ give a canonical isomorphism with $\mathbb{P}^2$, up to restriction to smaller neighbourhood in order to get that those points are in general position. Hence we are allowed to conclude since the lines $\{l_j\}$ pass through the necessary p_i as their preimages $\{C_j\}$ in C necessarily meet several (-1)-curves by construction. □

Once that we have moved from the incidence scheme to the moduli space of surfaces with marked divisors, we take another step connecting such marked surfaces to abelian covers, that form the gist of point 2) of the proof.

Definition 52. Let G be a finite abelian group and Y an n-dimensional smooth variety. An *abelian cover of Y with the group G*, or shortly a *G-cover*, is a finite map $\pi : X \to Y$, together with a faithful action of G on X, such that π exhibits Y as the quotient of X via G.

We focus on the case in which $G = (\mathbb{Z}/p)^3$, with $p = 2$ or $p = 3$ being prime to the characteristic of the residue field of the fixed singularity type. Denote by $G^\vee$ the group of characters, or dual group of G, and let $\langle \cdot, \cdot \rangle : G \times G^\vee \to \mathbb{Z}/p$ be the pairing defined in [Pa, Proposition 2.1], after choice of a root of unity. This pairing may also be extended to $\langle \cdot, \cdot \rangle : G \times G^\vee \to \mathbb{Z}$ by requiring $\langle \sigma, \chi \rangle \in \{0, \ldots, p-1\}$. For a smooth variety S, denote by $\mathrm{Div}(S)$ the free abelian group generated by the *prime divisors*, i.e. by irreducible subvarieties of S of codimension 1, and by $\mathrm{Pic}(S)$ the *Picard group of S*, which is given by $\mathrm{Div}(S)$ modulo the *principal divisors*.

Definition 53. (See [Pa, Proposition 2.1]) Suppose to have two maps $\mathrm{D} : G \to \mathrm{Div}(S)$ and $\mathrm{L} : G^\vee \to \mathrm{Pic}(S)$. We say that the pair (D, L) satisfies the *cover condition* if

1. $\mathrm{D}_0 = 0$;
2. $p\mathrm{L}_\chi = \sum_\sigma \langle \sigma, \chi \rangle \mathrm{D}_\sigma$ for all σ and χ, and the equality should be considered in $\mathrm{Pic}(X)$.

Using the notation introduced above, we claim the following result.

Proposition 4.9. *Suppose that the pair* (D, L) *satisfies the cover condition, and that the D_σ are nonsingular curves, no three meeting in a point, such that if* D_σ *and* $\mathrm{D}_{\sigma'}$ *meet, then they are transverse and the elements σ and σ' are linearly independent in G. Then:*

(i) There is a corresponding G-cover $\pi : \tilde{S} \to S$ *having branch divisor* $\mathrm{D} = \cup \mathrm{D}_\sigma$*;*

(ii) The surface $\tilde{S}$ *is nonsingular;*

(iii) We have that $\pi_* \mathcal{O}_{\tilde{S}} = \oplus_\chi \mathcal{O}_S(-\mathrm{L}_\chi)$*;*

(iv) Denote by $K_{\tilde{S}}$ *and* K_S *the canonical sheaf of* $\tilde{S}$ *and* S *respectively. Then* $\pi_* K_{\tilde{S}} \cong \oplus_\chi K_S(\mathrm{L}_\chi)$.

Proof. See [Pa, Proposition 2.1], [Pa, Proposition 3.1] and [Pa, p. 193]. □

We provide now two short key examples, which apply to the marked surface (S, C) produced at point 1).

Example 4.3 Consider a pair (S, C) as in point 1) of the proof of Theorem 4.8, take $p = 2$ and fix an element $\sigma_0 \in G$ different from 0. Let A be a sufficiently ample line bundle such that $A \equiv C \pmod 2$. We procede now defining the two maps D and L. Set $\mathrm{D}_{\sigma_0} = C$, $\mathrm{D}_0 = 0$ and let D_σ be a general section of A otherwise, satisfying the following condition: if $\sigma \neq \sigma'$, then D_σ and $\mathrm{D}_{\sigma'}$ meet transversely. Let $\mathrm{L}_0 = 0$, $\mathrm{L}_\chi = 2A$ if $\langle \sigma_0, \chi \rangle = 0$ and $\chi \neq 0$, and $\mathrm{L}_\chi = (3A + C)/2$ else. The pair (D, L) provided by the given data satisfies the hypothesis of Proposition 4.9 by construction.

Example 4.4 Consider a pair (S, C) as in point 1) of the proof of Theorem 4.8, take $p = 3$, fix an element $\sigma_0 \neq 0$ in G and a character $\chi_0 \in G^\vee$ such that $\langle \sigma_0, \chi_0 \rangle = 1$. Let again A be a sufficiently ample line bundle such that $A \equiv C \pmod 3$. We procede now defining the two maps D and L. Set again $\mathrm{D}_{\sigma_0} = C$ and let D_σ be a general section of A when $\langle \sigma_0, \chi_0 \rangle = 1$ and $\sigma \neq \sigma_0$, and $\mathrm{D}_\sigma = 0$ otherwise. Let now $\mathrm{L}_0 = 0$, $\mathrm{L}_{-\chi_0} = (16A + 2C)/3$, and for L_χ consider the following definition

$$\mathrm{L}_\chi = \begin{cases} (8A + C)/3 & \text{if } \langle \sigma_0, \chi_\rangle = 1 \\ 3A & \text{if } \langle \sigma_0, \chi_\rangle = 0 \text{ and } \chi \neq 0 \\ (7A + 2C)/3 & \text{if } \langle \sigma_0, \chi_\rangle = 2 \text{ and } \chi \neq \chi_0 \end{cases}$$

As in the case of Example 4.3, again the pair (D, L) provided by the given data satisfies the hypothesis of Proposition 4.9 by construction, as we may observe that if $\sigma \neq 0$, then at most one between D_σ and $D_{-\sigma}$ is nonzero.

Moreover, Vakil remarks that if the character of the residue field is 2, then only Example 4.4 applies, while if the character of the residue field is 3 only Example 4.3 does.

Now, some particular results apply to both the examples above.

Proposition 4.10. *Consider Example 4.3 and 4.4. If A is sufficiently ample, then:*

1. *the canonical sheaf* $K_{\tilde{S}}$ *is very ample;*
2. *the G-cover* $\tilde{S}$ *given by Proposition 4.9 is a surface of general type, i.e. an algebraic surface having Kodaira dimension 2;*
3. *the G-cover* $\tilde{S}$ *given by Proposition 4.9 is a regular surface, that is* $h^1(\tilde{S}, \mathcal{O}_{\tilde{S}}) = 0$*;*
4. *the deformations of the surface* $\tilde{S}$ *are the same as the deformations of the pairs* $(S, \{D_\sigma\})$*, and in particular deformations of G-covers are again G-covers;*
5. *the deformation space of* $\tilde{S}$ *has the same type as the deformation space of* (S, C).

Proof. See [Va2, 4.4 and 4.5]. □

For a more detailed study of these consequences in the case of surfaces and the definition of Kodaira dimension see [Perego, §3.2 and §4], and for the case of a variety of general type see the preliminar draft of [Kol].

For the remaining two points of the proof of Theorem 4.8 see [Va2, 4.6].

Vakil then goes on proving Murphy's law for some other deformation spaces, see [Va2, **M3**, **M5**]. Taking advantage of these facts and using the vanishing of some higher cohomology of the surface $\tilde{S}$ defined in Proposition 4.9, Vakil than obtains Murphy's Law for a second class of Hilbert schemes ([Va2, **M1**]), regarding as a fundamental tool in its proof a result presented by Barbara Fantechi and Rita Pardini in [B. Fantechi, R. Pardini, *"On the Hilbert scheme of curves in higher-dimensional projective space"*, Manuscripta Mathematica **90** (1996), no. 1, pp. 1–15], in which the authors proved that for $n \geq 3$ there exist infinitely many integers r and, for each one of them, a curve C_r lying exactly on n components of the Hilbert scheme of $\mathbb{P}^r$.

Theorem 4.11. *The Hilbert scheme of nonsingular curves in projective spaces satisfies Murphy's Law for moduli spaces. In particular the space of curves with the data of a linear system of degree d and projective dimension r does.*

After this fact, other moduli spaces are proved to be badly behaved, but again they are far from what we were interested to, see [Va2, §7].

As Vakil himself underlines in [Va2, 2.4], the results provided in [Va2], the philosophy behind it and the history of such an algebraic problem beg some further questions. In particular he raised the issue of whether the Hilbert scheme of curves in $\mathbb{P}^3$ and the Hilbert scheme of points on a smooth 3-fold do satisfy Murphy's Law. Other even relevant cases of moduli spaces were left without any answer. In the last twenty years many researchers focused their attention on Hilbert schemes, or some suitable variants and generalization of them like the one indicated in Section 2.5, constructing a huge literature on the subject.

Few years before the publication of [Va2], Robin Hartshorne provided a quick resume of what was known about connectedness of the Hilbert scheme of curves in $\mathbb{P}^3$ in [R. Hartshorne, "Questions of connectedness of the Hilbert scheme of curves in $\mathbb{P}^3$", *Algebra, arithmetic and geometry with applications (West Lafayette, IN, 2000)*, pp. 487–495, Springer, Berlin, 2004] and some further considerations about this problem, restricted to the case of *locally Cohen-Maculay curves* as proposed by Hartshorne, were reached by Paolo Lella and Enrico Schlesinger in [P. Lella, E. Schlesinger, "The Hilbert schemes of locally Cohen-Macaulay curves in $\mathbb{P}^3$ may after all be connected", *Collect. Math., vol. 64* (2013), no.3, pp. 363–372], while about its reducibility several authors have proved irreducibility in a wide range of cases for smooth and locally Cohen-Maculay curves in $\mathbb{P}^3$. Anyway, as far as we are aware at the moment, nobody has either established or disproved the validity of Murphy's Law for those Hilbert schemes, lefting the problem about the Hilbert scheme of curves in $\mathbb{P}^3$ posed by Vakil unsolved.

A different history involves Murphy's law for the Hilbert scheme of points on a smooth 3-fold, since recently Joachim Jelisiejew, in [J. Jelisiejew, "Pathologies on the Hilbert Scheme of Points", *Inventiones mathematicae, vol. 220* (2020), pp. 581–610] has proved that *"the Hilbert scheme of points on a higher dimensional affine space is non-reduced and has components lying entirely in characteristic p for all primes p. In fact, we show that Vakil's Murphy's Law holds up to retraction for this scheme."*

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