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“HARVESTING AND BIODIVERSITY: A QUANTITATIVE APPROACH”

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Firma dello studente

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Introduction

This thesis deals with a problem of environmental economics, where a farmer wants to maximize the profit deriving from the cultivation of her field. The harvest, and therefore profit, depend on three factors: labor, capital, and the level of biodiversity. The greater the quantities of factors used, the higher the harvest. However, capital is polluting and damages biodiversity. The farmer must therefore find the right combination of production factors that maximize her profits, taking into account the trade-off between capital and biodiversity.

The problem translates into an optimal control problem with an infinite horizon. To address it, we analyze two models using as basis the Ramsey model (Novales, 2009). The two differ according to how biodiversity contributes to production: a *multiplicative model* where the biodiversity factor is multiplied to production function, and an *additive model* where the biodiversity is added.

The work is inspired by a research project by Professor Alessandra Buratto of the University of Padua, in collaboration with Professor Lucia Sbragia of the University of Durham.

The mathematical resolution of the optimal control problems has been performed by applying the *Pontryagin Maximum Principle* (Seierstad, 1987) by two mathematics graduands of the University of Padua: Matteo Pozzan and Valentina Viscovich. It has not been solved analytically since strongly non-linear equations are obtained from the necessary conditions. Consequently, the problem has been resolved by analyzing and classifying the stationary points of the system, to qualitatively characterize its solutions. With these, some economic explanations and interpretations have been given. In particular: what the differences of the controls are between the two models, which policy is better to implement to have higher profit, what the differences between the two production functions imply, if the logistic function is well suited to represent biodiversity and a possible interpretation of a particular case of the additive model, using the concept of externality.

This work is structured as follows: in the first chapter, we present the literature on which the models are based: the Ramsey model, the production function, and the theory of externalities. In the second chapter, the theoretical notions about optimal control problems and their resolution are discussed. The third chapter describes the farmer's problem and its formulation in terms of optimal control problem. Furthermore, its necessary and sufficient optimality conditions are analyzed. The fourth chapter reports some economic observations on the aforementioned models.

Chapter 1

Literature

1.1 Ramsey model

The Ramsey model was proposed by F. Ramsey in the paper *A Mathematical Theory of Savings* (Ramsey 1928). It is a model of economic growth that studies the optimal rate of savings in order to obtain the maximum utility through consumption

Using Ramsey's words, the question the model answers is: "how much of its income should a nation save?" The model assumes that the nation produces capital, that can be immediately consumed or invested. Only the consumption of capital generates some utility to the individuals of the nation. The capital saved for the future gives zero utility in the present, but is essential for future utilities: a share of the capital becomes constantly obsolete and must be substitute, otherwise total capital would go to zero over time, and no consumption would be possible (Ramsey, 1928).

The model makes some assumptions: individuals contribute to the same extent to the production and consumption of goods: they are equal in every aspect; there is no difference between goods, or, there is one single type of good; the economy is closed: no foreign trade, lending or borrowing allowed; the social planner (who rules the economy) of the nation is altruistic: they cares about the utility of future generations (but cares about them less than the present generation); there is full employment in the economy and all individuals work: the labor force coincides with the population of the nation; no technological improvements are introduced in the economy (Ramsey, 1928).

The Ramsey problem is formulated in a dynamic continuous time contest. The capital is the state variable, taken as given at the initial time. It cannot be directly modified by the social planner but it can only be affected through the control variable, that is the consumption. The capital increases or decreases following a path called "law of motion". The social planner has perfect foresight and their objective is to maximize the discounted utilities, optimizing constantly the consumption (Barro, 2004).

In per capita terms, the problem has the following form:

$$max \int_0^\infty e^{-\theta t} u(c_t)\ dt \quad t \in [0, \infty) \tag{1.1}$$

$$\text{s.t.} \quad \dot{k} = f(k_t) - c_t - (n + \delta)k_t \tag{1.2}$$

The function 1.1 is the objective function: the maximization of the present and future utilities, discounted by an impatience factor. The constant θ measures the impatience of the economy: the higher the θ, the less importance is given to the future generations. The constraint 1.2 is the *equation of motion* (state equation): it describes the variation of capital as the difference between the production of capital of today and the sum of the capital consumed today and the total depreciation (Novales, 2009).

The interval of time goes from zero to infinity. Considering an infinite horizon means that what happens in a distant future compared to the one we are interested in has almost no influence on the optimal solution.

In optimal control, the Pontryagin's Maximum Principle (Seierstad, 1987) identifies the necessary conditions for the optimality of a solution. This principle provides candidate solutions: there are no other possible solutions than those proposed by the theorem. However, the principle cannot by itself say whether a candidate optimal solution is optimal or not, or whether an optimal solution actually exists (Seierstad, 1987).

To use formulas, we introduce the *Hamiltonian function*, that is a linear combination of the integral we want to maximize and the function of the equation of motion. The Hamiltonian for this problem is:

$$H(c_t, k_t, \mu_t, t) = e^{-\theta t}u(c_t) + \mu_t \left[f(k_t) - c_t - (n+\delta)k_t\right]$$

where μ_t are called *co-state variables* (Seierstad, 1987).

Pontryagin's Maximum Principle formulates three necessary conditions for optimality: the first one is used to set consumption such that the planner is indifferent between consuming one more unit of capital today and investing that unit for future benefit. The second one is used to control the state variable. The third one looks at the final position of the capital, when the time ends: at time T (the final time), if the third condition is satisfied, it will not be possible to increase utility by consuming an extra unit of capital. These three conditions are set as follows (Novales, 2009):

1. $\frac{\partial H}{\partial c} = 0$;
2. $\frac{\partial H}{\partial k} = -\dot{\mu}_t$;
3. $\lim_{t\to\infty} k_t e^{-\theta t}u'(c_t) = 0$.

To have a maximum point in the first condition, we assume that C is a maximal argument of the Hamiltonian. This means that if we calculate the function at point C, we get a maximum.

The system can be restated as a two ODEs:

$$\begin{cases} \frac{\dot{c}_t}{c_t} = -\frac{u'(c_t)}{c_t u''(c_t)}(f'(k_t) - (n+\delta+\theta)) \\ \dot{k}_t = f(k_t) - c_t - (n+\delta)k_t \end{cases}$$

The first equation is called the Keynes-Ramsey condition (Novales, 2009). We can derive the following observations: per capita consumption increases, decreases or remains constant if the marginal product of capital is greater, less, or equal to the time preference rate, net of total depreciation (population growth and depreciation of capital). The growth rate of consumption depends on the elasticity of the intertemporal substitution, that is $\sigma(c_t) = \frac{u'(c_t)}{c_t u''(c_t)}$. The second equation is the state equation: the central planner can modify it only by changing consumption, that is the control variable (Barro, 2004).

Over time, the dynamic system moves and converges to some point, that depends on the initial state variable (different initial quantities of capital move the economy towards different converging points). This point is called steady state: the state variable does not move anymore, i.e. $\dot{k}$ goes to zero. Also the consumption does not change over time, being strictly related to capital variations.

To explain visually the steady state, we use the graphs taken from the book *Economic Growth* (Novales, 2009). The following graphs can be found in chapter 3, *Optimal Growth. Continuous Time Analysis*, at the section Existence, "Uniqueness and Stability of Long-Run Equilibrium – A Graphical Discussion".

We first analyze the stability of consumption c. We consider the Keynes-Ramsey condition and impose $\dot{c} = 0$. We derive the equation $f'(k_t) = n + \delta + \theta$. k_{ss} is the value of capital at $\dot{c} = 0$. Along the capital path, when the stock of capital is below k_{ss} the marginal product of capital is greater than the steady state marginal product of capital, and the consumption will increase. Conversely, when the stock of capital is above k_{ss}, the consumption will decrease (Novales, 2009). The graphical representation is at fig. 1.1.

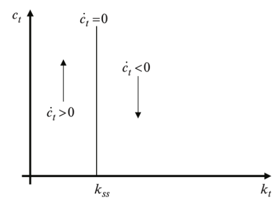

Figure 1.1: Consumption path

The farther away the stock of capital is from the optimal steady state, the faster the rate of consumption increases or decreases.

To study graphically the stability of capital, we impose $\dot{k} = 0$ in the equation of motion, and we obtain $c = f(k) - (n + \delta)k$: this function is everywhere concave, and goes through the origin. The maximum point of the function is at $f'(k) = (n + \delta)$. The graphical representation is at fig. 1.2.

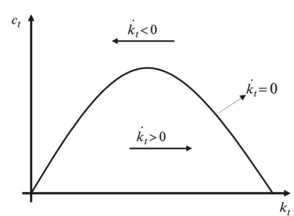

Figure 1.2: Capital path

At points below the line, consumption is such that $c < f(k) - (\delta + n)k$; from the Keynes-Ramsey condition we obtain $\dot{k} > 0$: the stock of capital increases. The opposite happens in any point above the curve: capital decreases, because the total depreciation of capital is greater that new invested capital.
Note that the farther away the capital is from the curve, the higher the accumulation rate of capital is.
We use the two precedent graphs to obtain the phase diagram at fig. 1.3.

We split the diagram into four regions and analyse how the system moves, depending on the position where we are.

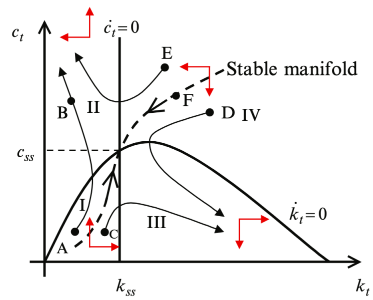

Figure 1.3: Model path

Starting from point A, we move from region I to region II. This path is not feasible: with high consumption, capital will decrease more and more, until the economy will have zero capital. Point B will move the economy to the same result.
Starting from point C, consumption and stock of capital will initially increase. However, once entered in region III, consumption will start to going down and the economy will move to a suboptimal steady state. From point B, the economy will reach the same result. From point D, the economy will reach the same result. From region IV, point E will bring the economy into region II, in the not feasible region.
Differently from the other cases, starting from point F, the economy will have those capital accumulation and consumption rates adequate to reach the optimal steady state. All those points as F that move the economy into the optimal steady state lie along a curve called saddle path, or stable manifold (Novales, 2009).

For any level of the stock of capital there is a unique level of consumption which will move the economy into the optimal steady state. That path is the solution of the Ramsey problem.

Since all individuals have equal preferences, the resource allocation is also Pareto-efficient: everybody within the same generation has equal quantities of resources, so it is impossible to increase the utility of an individual, without decreasing the utility of another one. Conversely, all the other trajectories are not Pareto-optimum (Novales, 2009).

Calculus of Variations

We now present the Ramsey model applying the *Calculus of Variations*, to determine and analyze its maximum and minimum points. We express the analysis referring to the book *Optimal control theory with economic applications* (Seierstad, 1987).

We assume a closed economy in the time interval $[0, T]$, where:

- $Y = Y(t)$ is the production at time t;
- $C = C(t)$ is the consumption at time t;
- $K = K(t)$ is the capital at time t.

Assuming a $\mathcal{C}^2$ function f, the production function is:

$$Y = f(K)$$

Production grows as capital increases, with decreasing returns to scale:

1. $f(K) \geq 0$
2. $f'(K) > 0.$
3. $f''(K) \leq 0$

The initial and final conditions on capital are:

$$K(0) = K^0 \quad \text{and} \quad K(T) \geq K^T$$

where K^T is the minimum quantity of capital required at the final time.
The output is divided by consumption and investment:

$$f(K(t)) = C(t) + \dot{K}(t) \quad t \in [0, T]$$

The flow of consumption is:

$$C(t) = f(K(t)) - \dot{K}(t), \quad t \in [0, T]$$

The utility function $U(C)$, a $\mathcal{C}^2$ function, values the consumption flow. The utility grows as consumption increases, with decreasing returns to scale:

$$U'(C) > 0 \qquad U''(C) < 0 \tag{1.3}$$

The utility function is discounted by a factor $e^{-\rho t}$ to weight differently the consumption over time. The aim of the Ramsey problem is to maximize the total discounted utility, having some constraints:

$$\begin{aligned} \text{maximize} \quad & J(K) = \int_0^T U(C(t))e^{-\rho t}\, dt \\ \text{subject to} \quad & \dot{K}(t) = f(K(t)) - C(t) \\ & K(0) = X^0 \\ & K(t) \geq K^T \\ & C(t) \geq 0 \end{aligned}$$

The state variable is $K(t)$. The necessary conditions for an optimal solution $K^*(t)$ are:

1. the Euler equation,
$$\frac{d}{dt}\frac{\partial F}{\partial \dot{x}} = \frac{\partial F}{\partial x}$$
2. the initial and final conditions
$$K(0) = K^0 \quad \text{and} \quad K(T) \geq K^T$$
3. the transversality conditions
$$-e^{-\rho T}U'(C(T)) \leq 0$$
$$e^{-\rho T}U'(C(T))\left(K(T) - K^T\right) = 0$$

We can rewrite the Euler equation as:

$$-\frac{U''(C)}{U'(C)}\dot{C} = f'(K) - \rho$$

Considering 1.3, we can derive the following observation:

$$\dot{C} \geq 0 \quad \Longleftrightarrow \quad f'(K) - \rho \geq 0$$

having an optimal consumption flow, the growth rate of consumption is positive if and only if the marginal productivity of capital is greater than the discount rate.

Moreover, knowing that $f''(K) \leq 0$, and defining K_ρ as

$$f'(K) = \rho$$

we can derive:

$$f'(K) \geq \rho \quad \Longleftrightarrow \quad K \leq K_\rho$$

Combining the two previous observations, we obtain:

$$\dot{C} \geq 0 \quad \Longleftrightarrow \quad K(t) \leq K_\rho$$

having an optimal consumption flow: when the capital is below K_ρ, the consumption grows; when the capital is above K_ρ, the consumption decreases.

The Euler equation can be written as

$$\frac{1}{U'(C)}\frac{d}{dt}U'(C(t)) = \rho - f'(K(t))$$

from which we can derive the following characteristics of the optimal solution:

the growth rate of the marginal utility is equal to the difference between the discount rate and the marginal productivity.

1.2 Cobb-Douglas functions

We now present the Cobb-Douglas production function because it is present in all three formulations of the farmer's problem. We have decided to write the Cobb-Douglas in its general form, because in the three models it has been reported with different numbers of inputs: with one, two or three inputs.

In economics, the production function gives the relationship between the maximum output obtainable with a given set of inputs:

$$y = f(x_i)$$

y is the output and x_i is the quantity of factor inputs. The production function $f(\cdot)$ is the technology (Varian, 1992).

The basic production function is defined on the production factors labor and capital:

$$y = (L, K)$$

Important production functions in microeconomics having labor and capital as factor inputs are the Cobb-Douglas, which take their name from mathematician Charles Cobb and economist Paul Douglas (Cobb, Douglas, 1928). The form presented in their study *A theory of production*, was:

$$f(x) = bL^k K^{1-k}$$

where L and K is respectively the labor and capital.

The general form of Cobb-Douglas is:

$$f(x) = A \prod_{i=1}^{n} x_i^{\alpha_i} \quad x = (x_1, ..., x_n)$$

where

- x is the series of factor inputs;
- α_i is the elasticity of f(x) with respect to x_i;
- A is the total factor productivity. It reflects the technology progress as well as the education or ability of the workforce.

Cobb-Douglas are widely used because of their properties, that economists consider desirable (Heathfield, 1987). The marginal product of the factors is positive. Taking the marginal product of x_i, i.e. the change in output due to a variation of x_i, ceteris paribus:

$$\begin{aligned} \frac{df(x)}{dx_n} &= \alpha_n A \prod_{i=1}^{n-1} x_i^{\alpha_i} x_n^{\alpha_n - 1} \\ &= \alpha_n A \prod_{i=1}^{n} x_i^{\alpha_i} x_n^{-1} \\ &= \alpha_n \frac{f(x)}{x_n} > 0 \end{aligned}$$

the marginal product of factor i is positive because $f(x)$, x_n and α_n are positive.
The factors have diminishing returns: ceteris paribus, as x_i increases, $f(x)$ increases at a diminishing rate:

$$\begin{aligned} \frac{d^2 f(x)}{d{x_n}^2} &= (\alpha_n - 1)\alpha_n A \prod_{i=1}^{n-1} x_i^{\alpha_i} x_n^{\alpha_n - 2} \\ &= (\alpha_n - 1)\alpha_n A \prod_{i=1}^{n} x_i^{\alpha_i} x_n^{-2} \\ &= (\alpha_n - 1)\alpha_n \frac{f(x)}{x_n^2} < 0 \end{aligned}$$

all terms are positive except for $(\alpha_n - 1)$, so the second derivative is negative.
The marginal product of a factor input increases if another factor input is increased. Taking the cross partial derivatives:

$$\begin{aligned}\frac{d^2 f(x)}{dx_n dx_{n-1}} &= \alpha_n A \prod_{i=1}^{n-2} x_i^{\alpha_i} x_n^{\alpha_n - 1} x_{n-1}^{\alpha_{n-1}-1} \\ &= \alpha_n A \prod_{i=1}^{n} x_i^{\alpha_i} x_n^{-1} x_{n-1}^{-1} \\ &= \alpha_n \frac{f(x)}{x_n x_{n-1}} > 0\end{aligned}$$

Cobb-Douglas are homogeneous functions, with degree of homogeneity being $\sum_{i=1} \alpha_i$ and the returns to scale can be easily calculated looking at the exponents of the function. The returns to scale tell us what happens to the output when the factor inputs change in the same proportion. If $\sum_i \alpha_i \lesseqgtr 1$ for all i , Cobb-Douglas will have respectively decreasing, constant and increasing returns to scale. To prove it, we assume the case of constant returns to scale, i.e. all the α_i sum to 1:

$$\begin{aligned}f(x) &= A \prod_{i=1}^{n} (\lambda x_i)^{\alpha_i} \\ &= \prod_{i=1}^{n} \lambda^{\alpha_i} A \prod_{i=1}^{n} (x_i)^{\alpha_i} \\ &= \lambda A \prod_{i=1}^{n} (x_i)^{\alpha_i}\end{aligned}$$

In a perfect competitive market, the remuneration of the factor inputs is strongly related with their exponents. In perfect competition, the monetary return of the factor inputs is equal to their marginal product. In Cobb-Douglas, the marginal product of x_1 is:

$$mpc = \alpha_i \frac{f(x)}{x_i}$$

every unit of factor receives the remuneration written above. The total monetary pay of a factor input is:

$$x_i \ pay = x_i \times \alpha_i \frac{f(x)}{x_i} = \alpha_i f(x)$$

In a competitive market, the remuneration of the factor inputs is equal to the x_i's share in total output $f(x)$ (Heathfield 1987).

The production function can be transformed into a linear model by taking the log of both sides of the equation:

$$\log(Y) = \log(A) + \sum_{i=1}^{n} \alpha_i \log(x_i)$$

In a linear form, Cobb-Douglas functions permit OLS regression methods, which is used in Economics to understand the link between factor inputs on production output (Heathfield, 1987).
Cobb-Douglas have constant and unitary elasticity of substitution. To explain this, we have to introduce the marginal rate of technical substitution (MRTS): is the amount by which an input must be increased when

another input decreases, in order to keep the output constant (Simon, 1994). The formula is

$$\text{MRTS}(x_1, x_2) = -\frac{\Delta x_2}{\Delta x_1}$$

The elasticity of substitution is:

$$\sigma = \frac{\partial \ln(x_i/x_j)}{\partial \ln MRTS_{x_i x_j}}$$

To demonstrate the constant and unitary elasticity of substitution of the function, for simplicity we consider a Cobb-Douglas with two inputs.

$$\sigma = \frac{d\ln(\frac{x_j}{x_i})}{d\ln(\frac{\alpha_i}{\alpha_j}\frac{x_j}{x_i})} = \frac{d(\frac{x_j}{x_i})}{d(\frac{\alpha_i}{\alpha_j}\frac{x_j}{x_i})}\frac{\frac{\alpha_i}{\alpha_j}\frac{x_j}{x_i}}{\frac{x_j}{x_i}} = 1$$

The elasticity of substitution measures the degree of substitution between two factor inputs. The term σ goes between zero (perfect substitute) and infinity (perfect complementary). In case of L and K as factors and under perfect competition, a constant value of σ equal to 1 implies that the ratio between income from work and capital remains constant over time. (Jehle, Reny, 2011).
In Cobb-Douglas, each factor is essential for production and no factor can be completely substituted with another (Murray Brown).

Why we used it

Cobb-Douglas are widely used to treat problems with natural resources. In the literature we can find many models dealing with human capital and natural capital, using Cobb-Douglas as production functions.

Valente (2010): the paper analyzes an OLG model with endogenous growth, exhaustible resources, and human capital externalities. The production function is a Cobb-Douglas:

$$Y_t = X_t^{\alpha} H_t^{1-\alpha}$$

with X being the natural capital and H the human capital with $0 < \alpha < 1$

Agnani Gutierrez Iza (2005): the paper studies a two-period overlapping generation model, with physical capital and exhaustible resources. As production function, the model uses a Cobb-Douglas with three inputs: labor, tangible capital and exhaustible resource:

$$Y_t = BK_t^{\alpha} N_t^{\beta} X_t^{\upsilon}$$

with $\alpha + \beta + \upsilon = 1$. N_t is the number of agents in the economy (input labor) at time t. In this case, constant returns to scale are considered.

Shou (2000): the paper studies an endogenous growth model with non-renewable resources and human capital accumulation. The non-renewable resources cause pollution, damaging the production process through a decrease of output:

$$Y = AK^{\alpha}(L)^{\beta} R^{\gamma} P^{-\delta} h^{\theta}$$

the term h is the average human capital level, R is the non-renewable resources, P is the pollution. Actually, in the model we have $L = uhN$, where N is the labor force, u is the share of time worked by the labor force and h is the average human capital level.

Solow (2016): the model is about a closed economy with one output and three inputs: labor, capital and non-renewable resource. The production function is CES type:

$$F(K, L, R) = (aR^{(\alpha-1)/\alpha} + bC^{(\alpha-1)/\alpha})^{\alpha/(\alpha-1)} \tag{1.4}$$

CES function can be seen as a general form of the Cobb-Douglas: in fact, if $\alpha = 1$ the function in 1.4 becomes a Cobb-Douglas.

In this thesis we propose two possible formulations of the farmer's problem, differing on the production functions. Cobb-Douglas are considered well suited to take into consideration the natural resources in production functions, that's why they are used both in the multiplicative and additive models.

In the *multiplicative model*, the Cobb-Douglas has three inputs: labor, capital and level of biodiversity:

$$y_t = \mu L_t^\alpha K_t^\beta X_t^\gamma$$

with $0 < \alpha, \beta, \gamma \leq 1$ and $\alpha + \beta < 1$ and $\mu > 0$. It can be seen that each factor is essential for production and no factor can be completely substituted with another. The return to scale is not bounded: the exponents can be smaller, bigger or equal to 1.

In the *additive model*, the biodiversity level is summed up to a Cobb-Douglas with L and K as input:

$$y_t = \mu L_t^\alpha K_t^\beta + bX_t^\gamma$$

with $0 < \alpha, \beta, \gamma \leq 1$ and $\alpha + \beta < 1$ and $\mu, b > 0$. Also the second part of the equation can be seen as a Cobb-Douglas with one single factor input. Unlike the multiplicative model, in this function no factor is essential: with zero labor and capital, the output is still positive. Vice versa, with zero natural input, the function become a basic Cobb-Douglas with decreasing return to scale.

1.3 Bionomic growth ratio

We now anticipate some parts of the models that will be further discussed in next chapters, to introduce the literature about a constraint we have imposed.

The model aims to maximize the farmer's revenues derived from the sale of the cultivated product over time. These revenue flows are discounted at a constant discount rate δ in order to consider the time preference of the farmer.

One of the inputs for production is biodiversity, that grows over time following a logistic function in the ODE:

$$\dot{X}(t) = gX(t)\left(1 - \frac{X(t)}{K}\right) - A(t),$$

where g is the intrinsic growth ratio of biodiversity, X is the biodiversity level, K is the carrying capacity of the system, and A is the damage caused by the use of pesticides.

Along the analysis of the farmer's problem we take as a reference the fishery models (for an example of it, see Clark 1975). Following those models, we have imposed the constraint:

$$g > \delta,$$

assuming that the intrinsic growth ratio g is greater with respect to the discount rate δ.

We report below a simplified version of the analysis present in the book *Mathematical Models in Population Biology and Epidemiology* (Brauer, 2001), with the sole purpose of giving an explanation about the reason behind the imposition of the constraint $g > \delta$.

If we want to determine the optimal trade-off between current and future harvest, we have to value the future harvest rent in present term using the discount rate δ. The higher the δ, the more weight will be given to the current harvesting income.

Considering a logistic function, the optimal harvesting rate in absence of a discount factor is represented by the point A in the figure 1.4, where the growth rate of the fish population is at its maximum point. The size population is represented in the abscissa axis, the growth rate of the population in the ordinate axis. The carrying capacity of the system is fixed $K = 10$. We set the harvest equal to the growth rate of the population, in order to obtain a sustainable harvest and a steady state.

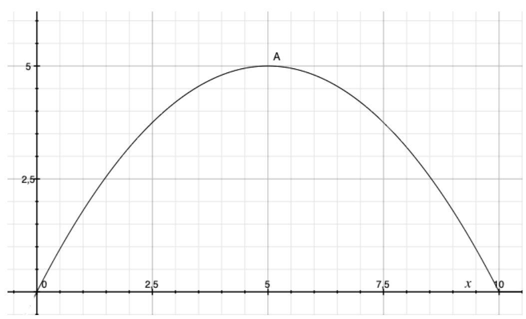

Figure 1.4: population growth

We now discuss the behavior of the optimal harvesting rate, depending on the value of the discount rate. In general, the higher the δ, the sooner the optimum harvesting rate will be reached. If we set $\delta = 0$, the optimum harvesting rate is at point A, as before. Conversely, if we set $\delta = \infty$, we are considering the future revenue without value and the optimum harvesting rate will be in $X = 0$.

For all the values $\delta \in (0, \infty)$, the optimal harvesting rate will be bewteen zero and point A. In this case, the optimal population level is:

$$X^* = \frac{K}{2}\left(1 - \frac{\delta}{g}\right) < X^A$$

where K is the carrying capacity of the population and X^A is the population level at point A. The term

$$\frac{\delta}{g}$$

is called *bionomic growth ratio*. If it exceed 1, then the optimal population level will go inevitably to $X^* = 0$. The optimal harvest policy to implement is to increase the harvest, leading to the fastest possible extinction of the population. This is the reason why we set the constraint $g > \delta$ in the farmer's problem, just to permit an equilibrium with a population level above $X^* = 0$.

1.4 Externalities

In economics, an externality is defined as a benefit or cost affecting a third party who did not choose to incur it (Lin, 1976).

With his behavior, an individual may affect others' welfare. This effect can be seen as a benefit if is welfare-increasing, or a cost if is welfare-decreasing.

Whenever the cost or benefit is fully captured by a change in prices, no market inefficiency is created: the economic agents modify their choices taking into account the change in prices. In this case, this cost or benefit is called a *pecuniary externality* (Balestrino, 2015).

Whenever the cost or benefit affecting the welfare of others does not cause a change in prices, we are in presence of a *technological externality.* This externality leads to an inefficient distribution of goods and services, and therefore to a market failure: a situation where each individual makes the optimal decision for themselves, but all these decisions are not the best ones for the group (Balestrino, 2015).

In this chapter, only technological externalities will be discussed. This concept will be useful for next chapters. For ease of writing, the term *externality* will be used instead of technological externality.

To better discuss externality and its economical implications we present the following analysis (Buchanan and Stubblebine, 1962).

We assume two agents, A and B, having the following utility functions:

$$U^A = U^A(X_1, X_2, \dots, X_m) \qquad U^B = U^B(Y_1, Y_2, \dots, Y_m)$$

The utility of the agent A depends on the series of activities X and the utility of the agent B depends on the series of activities Y. The activities X are under the control of the agent A, while the activities Y are under B's control. The agents act to maximize their utility.

Taking as example the agent B, necessary condition for utility maximization is:

$$\frac{u^B_{Y_i}}{u^B_{Yj}} = \frac{f^B_{Yi}}{f^B_{Y_j}}$$

where Y_j is a numeraire commodity which is available on equal terms to A.

The left-hand side is the marginal benefit of the activity Y_i to B. The right-hand side is the marginal rate of substitution in production, i.e. the marginal cost of the activity Y_1 to B.

The production function is,

$$f^B = f^B(Y_1, Y_2, \dots, Y_m)$$

An externality is present when,

$$U^A = U^A(X_1, X_2, X_m, \dots, Y_1) \qquad U^B = U^B(Y_1, Y_2, \dots, Y_m)$$

In this case, while all the activities affecting the utility of B are under B's control, the utility of A no longer depends exclusively on A's choices, but also on the activity Y_1 . Agent A will still act to maximize his utility, subject to the choice of B about Y_1 . As Y_1 moves, agent A will change the activities Xs to maintain the equilibrium.

The marginal externality faced by A and caused by Y_1 is,

$$u^A_{Y_1} = \frac{\partial U^A}{\partial Y_1}$$

This is the partial derivative of the utility function of agent A with respect to Y_1.

A marginal externality exists when,

$$u^A_{Y_1} \neq 0$$

A marginal externality can be positive, $u^A_{Y_1} > 0$, or negative, $u^A_{Y_1} < 0$.

Sometimes, the marginal externality is zero, $u^A_{Y_1} = 0$, but the externality continues to exist: $U^A = U^A(X_1, X_2, X_m, ..., Y_1)$. The utility of A has been modified by the total effect of Y_1, but any other incremental change of Y_1 does not affect A's utility. This is called an *infra-marginal* externality. An infra-marginal externality is positive when,

$$u^A_{Y_1} = 0 \quad \text{and} \quad \int_0^{Y_1} u^a_{Y_1} d_{Y_1} > 0$$

and negative when,

$$u^A_{Y_1} = 0 \quad \text{and} \quad \int_0^{Y_1} u^a_{Y_1} d_{Y_1} < 0$$

Externalities influence the total welfare of the affected individuals, but not all the affected agents have the desire to modify the behavior of the individual empowered to take action. Therefore, we introduce a distinction. *Potentially relevant* externalities are present if A has any desire to modify the behavior of B. Conversely, irrelevant externalities exist if A has no desire to influence the B's behavior. A potentially relevant externality exists when,

$$u^A_{Y_1}|_{Y_1=\bar{Y_1}} \neq 0$$

We consider $\bar{Y_1}$ as the equilibrium value for the Y_1.

In equilibrium, the marginal utility of A with respect to Y_1 is different from zero. Bigger than zero means positive externality, smaller than zero means negative externality. In any case, A wants to put some effort to change the behavior of B.

By definition, infra-marginal externalities are irrelevant for a small change in the scope of B's activity. A is motivated to change the behavior of B only when discrete changes are considered.
A is motivated in all cases except that for which,

$$u^A_{Y_1}|_{Y_1=\bar{Y_1}} = 0,$$

and

$$u^A(X_1, X_2, ..., X_m, \bar{Y_1}) \geq u^A(X_1, X_2, ..., X_m, Y_1) \qquad \text{for all} \;\; Y_1 \neq \bar{Y_1}.$$

If it holds, A has already achieved a maximum of utility with respect to changes over Y_1, i.e. A is satisfied with respect to Y_1. In all other cases, A wants to change the behavior of B.

In absence of a central authority, letting all the decisions be made by the economic agents, the inefficiency due to externality can be mitigated only through private solutions. The market can mitigate the inefficiency only if the externality is *Pareto-relevant*. An externality is defined to be Pareto-relevant if the activity may be modified in such a way that A can be made better off without B being made worse off. A and B can find a private solution to the problem of externality through the market, so “gains from trade” are possible.

A marginal externality is Pareto relevant when

$$(-)\frac{u^A_{\bar{Y}_1}}{u^A_{Xj}} > \left(\frac{u^B_{\bar{Y}_1}}{u^B_{Y_j}} - \frac{f^B_{\bar{Y}_1}}{f^B_{Y_j}}\right) \quad \text{and} \quad \frac{u^A_{\bar{Y}_1}}{u^A_{Xj}} < 0$$

$$\frac{u^A_{\bar{Y}_1}}{u^A_{Xj}} > (-)\left(\frac{u^B_{\bar{Y}_1}}{u^B_{Y_j}} - \frac{f^B_{\bar{Y}_1}}{f^B_{Y_j}}\right) \quad \text{and} \quad \frac{u^A_{\bar{Y}_1}}{u^A_{Xj}} > 0$$

X_i and Y_j are the activities in consuming some numeraire commodity, that is available on identical terms to A and B.

The equation means that the A’s marginal rate of substitution between Y_1 and X_j must exceed the B’s net marginal rate of substitution between $Y_1$and Y_j . If this equation is violated, “gains from trade” would be impossible. The mechanism of “gains from trade” is the following: A, in order to increase or reduce the externality, gives to B an amount of money equal to B’s loss utility. Having a marginal utility higher than B’s, A is better off while B has the same welfare as before the trade. Every time that B is in equilibrium, the term on the right-side goes to zero and some trade is always possible.

Pareto equilibrium, after the trade, is present when,

$$(-)\frac{u^A_{\bar{Y}_1}}{u^A_{Xj}} = \left(\frac{u^B_{\bar{Y}_1}}{u^B_{Y_j}} - \frac{f^B_{\bar{Y}_1}}{f^B_{Y_j}}\right) \quad \text{and} \quad \frac{u^A_{\bar{Y}_1}}{u^A_{Xj}} < 0$$

$$\frac{u^A_{\bar{Y}_1}}{u^A_{Xj}} = (-)\left(\frac{u^B_{\bar{Y}_1}}{u^B_{Y_j}} - \frac{f^B_{\bar{Y}_1}}{f^B_{Y_j}}\right) \quad \text{and} \quad \frac{u^A_{\bar{Y}_1}}{u^A_{Xj}} > 0$$

At the equilibrium, the Pareto relevant externality vanishes, but some externalities may still exist. The crucial thing is that the marginal rates of substitution in activities for the two persons are precisely offsetting.

Same reasoning for infra-marginal externality. Here, we no longer take into consideration marginal changes in behavior, but discrete changes: this discrete change is represented by ΔY_1.

$$(-)\frac{\Delta u^A_{\bar{Y}_1}}{\Delta u^A_{Xj}} > \left(\frac{\Delta u^B_{\bar{Y}_1}}{\Delta u^B_{Y_j}} - \frac{\Delta f^B_{\bar{Y}_1}}{\Delta f^B_{Y_j}}\right) \quad \text{and} \quad \frac{u^A_{\bar{Y}_1}}{u^A_{Xj}} < 0$$

$$\frac{\Delta u^A_{\bar{Y}_1}}{\Delta u^A_{Xj}} > (-)\left(\frac{\Delta u^B_{\bar{Y}_1}}{\Delta u^B_{Y_j}} - \frac{\Delta f^B_{\bar{Y}_1}}{\Delta f^B_{Y_j}}\right) \quad \text{and} \quad \frac{u^A_{\bar{Y}_1}}{u^A_{Xj}} > 0$$

We can extend the analysis to a vast number of affected people. In the case of various agents $(A_1, A_2, ..., A_n)$

affected by B's activity, the marginal externalities of the group are Pareto-relevant when,

$$(-)\sum_{i=1}^{n}\frac{u^A_{\bar{Y}_1}}{u^A_{Xj}} > \left(\frac{u^B_{\bar{Y}_1}}{u^B_{Y_j}} - \frac{f^B_{\bar{Y}_1}}{f^B_{Y_j}}\right) \quad \text{and} \quad \frac{u^A_{\bar{Y}_1}}{u^A_{Xj}} < 0$$

$$\sum_{i=1}^{n}\frac{u^A_{\bar{Y}_1}}{u^A_{Xj}} > (-)\left(\frac{u^B_{\bar{Y}_1}}{u^B_{Y_j}} - \frac{f^B_{\bar{Y}_1}}{f^B_{Y_j}}\right) \quad \text{and} \quad \frac{u^A_{\bar{Y}_1}}{u^A_{Xj}} > 0$$

We can reach the same conclusion as above. The only thing to add is that the group may incur some decisional or organizational cost that can influence the reachability of the equilibrium.

Chapter 2

Theoretical elements

2.1 Optimal control problems

In this chapter we took as reference the book *Optimal Control Theory with economic Applications* (Seierstad, Sydsæter, 1987).

An optimal control problem deals with dynamic systems. The aim of the problem is to find some optimal control variables that maximize or minimize an objective function over time.

The *vector of state variables* represents the state of the system at time $t \in [t_0, t_1]$, $x : [t_0, t_1] \longrightarrow \mathbb{R}^n$, that is a continuous function

$$x(t) = (x_1(t), \dots, x_n(t))' \in \mathbb{R}^n$$

The *vector of the control variables* represents the strategies put in place in the interval $t \in [t_0, t_1]$ in order to influence the evolution of the state function over time

$$u(t) = (u_1(t), \dots, u_n(t))' \in \mathbb{R}^r$$

with $u : [t_0, t_1] \longrightarrow \mathbb{R}^r$ being a continuous piecewise-defined function, subject to

$$u(t) \in \Omega$$

where $\emptyset \neq \Omega \subseteq \mathbb{R}^r$ is often convex and invariant over time. This function is called an *admissible control.*
The *equations of motion* define the path of the state function

$$\frac{dx(t)}{dt} = f(x(t), u(t), t)$$

where it is supposed that $f : \mathbb{R}^{n+r+1} \longrightarrow \mathbb{R}^n$ is continuous with continuous first partial derivatives in x.
For any admissible control $u(t)$, the initial conditions and the equations of motion form a Cauchy problem. A unique solution $x(t), t \in [t_0, t_1]$ is admitted. The Cauchy problem is

$$\begin{cases} \frac{dx(t)}{dt} = f(x(t), u(t), t) \\ x(t_0) = x_0 \end{cases} \tag{2.1}$$

A state function is admissible if is detected by the Cauchy problem for an admissible control $u(t)$ and the conditions at the final time are satisfied. So, an admissible solution is a couple of $u((t), x(t))$, both admissible. The *objective function* is $J : U \longrightarrow \mathbb{R}$ with admissible control set U

$$u \longrightarrow J(u) = \int_{t_0}^{t_1} f_0(x(t), u(t), t)dt$$

where $f_0(x, u, t)$ is a continuous function with continuous partial derivative in x.

2.2 Optimal control necessary conditions

The Pontryagin's Maximum Principle finds the necessary conditions for solving the optimal control problem. The control related to the optimal solution optimizes the Hamiltonian $H : \mathbb{R}^n \times \Omega \times \mathbb{R}^n \times [t_0, t_1] \longrightarrow \mathbb{R}$. To define the Hamiltonian we introduce the *adjoint variable* $\lambda(t) \in \mathbb{R}^n$ and the parameter λ_0.

$$H(x, u, p, t) = \lambda_0 f_0(x, u, t) + \sum_{i=1}^{n} \lambda_i f_i(x, u, t)$$

Theorem 1 (Pontryagin's Maximum Principle - Necessary conditions)**.** *Let $u^*(t)$ be a piecewise continuous optimal control, defined in the interval $[t_0, t_1]$, to which a state function $x^*(t)$ is associated. Then, there exists a constant $\lambda_0 \in \mathbb{R}$ and a piecewise continuous $\mathcal{C}^1$ function $\lambda(t)$ such that, for every $t \in [t_0, t_1]$, the following conditions hold:*

1. $(\lambda_0, \lambda(t)) \neq 0 \in \mathbb{R}^{n+1}$;
2. $u^*(t)$ *maximizes* $H(x^*, u, \lambda(t), t)$ *for any* $u \in \Omega$;
3. *except at the points where* $u^*(t)$ *is discontinuous,* $\lambda(t)$ *is differentiable and*

$$\dot{\lambda} = -\frac{\partial H(x^*(t), u^*(t), \lambda(t), t)}{\partial x}$$

4. $\lambda_0 \in \{0, 1\}$
5. $\lambda_i(t_1) \in \mathbb{R} \quad i = 1, \ldots, l$
   $\lambda_i(t_1) \geq 0 \;\; e \;\; \lambda_i(t_1)(x_i^*(t_1) - x_i^1) = 0, \quad i = l + 1, \ldots, m$
   $\lambda_i(t_1) = 0, \quad i = m + 1, \ldots, n$

2.3 Infinite horizon optimal control problems

A finite time horizon problem requires the determination of a final time T, and the final capital to be left after that period. However, the choice of the final time T is not always easy to determine, it is often arbitrary and there is not always a reason to choose a certain T, among the very possible ones. In growth theory, an infinite time horizon is therefore often used. The problem to face no longer requires the determination of a final time, nor the amount of capital to be left after this final time.

Dealing with an infinite time horizon makes crucial the introduction of a discount factor, that permits us to evaluate the utility according to its distribution over time. This discount factor is a decreasing continuous function $e^{-\delta t}$, where δ is the discount rate.

In infinite horizon, the generic form of an optimal control problem with a discount factor is

$$\begin{aligned}
\text{maximize} \quad & J(u) = \int_{t_0}^{+\infty} f^0(x(t), u(t), t) e^{-\delta t} dt \\
\text{subject to} \quad & \dot{x}(t) = f(x(t), u(t), t) \\
& x(t_0) = x^0 \\
& u(t) \in \Omega \\
& \lim_{t \to +\infty} x_i(t) = x_i^1 \quad i = 1, \dots, l \\
& \varliminf_{t \to +\infty} x_i(t) > x_i^1 \quad i = l+1, \dots, m \\
& \text{no condition on} \;\; x_i(t_1) \quad i = m+1, \dots, n.
\end{aligned}$$

Definition 1 (limit inferior)**.** Having a function $h\,[t_0, +\infty] \to \mathbb{R}$, the limit inferior of $h(t)$ is

$$\varliminf_{t \to +\infty} h(t) = \lim_{t \to +\infty} \inf\{h(s), s \in [t_0, +\infty]\}$$

The condition $\varliminf_{t \to +\infty} x(t_1) \geq x_i^1$ is equivalent to:

$$\text{for any } \varepsilon > 0, \text{ exists } t' \text{ such that } x_i(t) > x_i^1 - \varepsilon, \quad t' > t$$

In infinite horizon, the definition of optimality of a solution is different respect to the one in finite horizon, due to the fact that the objective function is an improper integral. We now discuss the catching-up optimality considering the infinite time horizon. Given an admissible solution $(u(t), x(t))$, and an optimal solution $(u^*(t), x^*(t))$, we define the function

$$D(t) = \int_{t_0}^{t} f_0(u^*(\tau), x^*(\tau), \tau) d\tau - \int_{t_0}^{t} f_0(u(\tau), x(\tau), \tau) d\tau$$

for any $t \geq t_0$, $D(t)$ is the difference between two terms that we can interpret as *goodness indices* until time t of the solutions $(u(t), x(t))$ and $(u^*(t), x^*(t))$, respectively.

Definition 2 (Catching-up)**.** The solution $(u^*(t), x^*(t))$ is optimal if

$$\varliminf_{t \to +\infty} D(t) \geq 0 \quad \text{for any admissible solution } (u(t), x(t))$$

For the limit inferior, the definition of optimality in the sense of *catching-up* requires that for any admissible solution $(u(t), x(t))$ and for any $\varepsilon > 0$, a t' exists such that

$$D(t) > \varepsilon, \quad t > t',$$

that is

$$\int_{t_0}^{t} f_0(u^*(\tau), x^*(\tau), \tau)\, d\tau > \int_{t_0}^{t} f_0(u(\tau), x(\tau), \tau)\, d\tau - \varepsilon \quad t' > t$$

Considering the problem with an infinite horizon and a discount factor, we write the necessary conditions for optimality.

We define the Hamiltonian function at present value:

$$H^c(x, u, q, t) = p_0 f^0(x, u, t) + \sum_{i=1}^{n} q_i f_i(x, u, t)$$

Theorem 2 (Necessary condition)**.** *Let $u^*(t)$ be a piecewise continuous optimal control, defined on $[t_0, +\infty]$, to which a state function $x^*(t)$ is associated, and $(u^*(t), x^*(t))$ be an optimal solution according to the catching up criterion;*
then, there exist a constant $\lambda_0 \in \mathbb{R}$ and a piecewise continuous $\mathcal{C}^1$ function $q(t)$ ($q\ [t_0, +\infty] \longrightarrow \mathbb{R}^n$) such that, for any $t \in [t_0, +\infty]$, the following conditions hold:

1. $(\lambda_0, q(t)) \neq 0 \in \mathbb{R}^{n+1}$;

2. *$u^*(t)$ maximizes $H^c(x^*, u, q(t), t)$ for $u \in \Omega$;*

3. *except for the periods of time t where $u^*(t)$ is discontinuous, $q(t)$ is differentiable and*

$$\dot{q}_i(t) = -\frac{\partial H(x^*(t), u^*(t), q(t), t)}{\partial x_i} + \delta q_i(t), \ \ a.e. \quad i = 1, \ldots, n$$

4. $\lambda_0 \in \{0, 1\}$

Sufficient conditions

We now discuss the sufficient condition for an optimal solution. The Pontryagin Maximum Principle is only able to determine the candidate optimal solutions, but it cannot by itself tell us if those candidate solutions are actually optimal or not. So, we know that the optimal solutions, if they exist, will be among the candidate solutions. In order to determine if a candidate optimal solution is actually optimal, we study the sufficient condition for optimality.

Theorem 3 (Mangasarian theorem - Sufficient condition-infinite horizon). *Having the problem*

$$\begin{aligned}
\textit{maximize} \quad & J(u) = \int_{t_0}^{+\infty} f^0(x(t), u(t), t) e^{-\delta t} dt \\
\textit{subject to} \quad & \dot{x}(t) = f(x(t), u(t), t) \\
& x(t_0) = x^0 \\
& u(t) \in \Omega \in \mathbb{R}^r \\
& \lim_{t \to +\infty} x_i(t) = x_i^1 \quad i = 1, \dots, l. \\
& \underline{\lim}_{t \to +\infty} x_i(t) \geq x_i^1 \quad i = l+1, \dots, m \\
& \textit{no conditions on } x_i(t) \textit{ if } t \longrightarrow +\infty \quad i = m+1, \dots, n
\end{aligned}$$

Let the pair $(x^*(t), u^*(t))$ be admissible for the problem, Ω be a convex set and suppose that the catching up is used as the optimality criterion. We also suppose that the partial derivatives of $f_0, f_1, \dots, f_n$ respect to the control u exist and are continuous, and a piecewise continuous C^1 function $p(t)$ exists such that for any $\lambda_0 = 1$, and if for any $t \geq t_0$ the following conditions hold:

1. *u^* maximizes $H(x^*(t), u, p(t), t)$, $u \in \Omega$;*
2. *$\dot{p}(t) = -\frac{\partial H^*}{\partial x}$ a.e.;*
3. *$H(x, u, p(t), t)$ is concave in (x, u) $\forall t \geq t_0$;*
4. *$\underline{\lim}_{t \to +\infty} p(t)(x(t) - x^*(t)) \geq 0$ for any admissible $x(t)$*

Then, $(x^*(t), u^*(t))$ is the solution in the sense of catching-up. If in the condition 3 *the function is strictly concave, then $(x^*(t), u^*(t))$ is unique.*

Theorem 4 (Arrow theorem - Sufficient condition). *Having the problem*

$$\begin{aligned}
\textit{maximize} \quad & J(u) = \int_{t_0}^{+\infty} f^0(x(t), u(t), t) e^{-\delta t} dt \\
\textit{subject to} \quad & \dot{x}(t) = f(x(t), u(t), t) \\
& x(t_0) = x^0 \\
& u(t) \in \Omega \in \mathbb{R}^r \\
& \lim_{t \to +\infty} x_i(t) = x_i^1 \quad i = 1, \dots, l. \\
& \underline{\lim}_{t \to +\infty} x_i(t) \geq x_i^1 \quad i = l+1, \dots, m \\
& \textit{no conditions on } x_i(t) \textit{ if } t \longrightarrow +\infty \quad i = m+1, \dots, n
\end{aligned}$$

we suppose that $x(t)$ belongs to a given convex set $A(t)$ for any $t \geq t_0$ and that the catching up is used as the optimality criterion.
We suppose that the pair $(x^*(t), u^*(t))$ is admissible for the problem and $x^*(t)$ belongs within $A(t)$ for any $t \geq t_0$. If there exists a continuous function $p(t)$ such that for $\lambda_0 = 1$ and for all the periods $t \geq t_0$,

1. *u^* maximizes $H(x^*(t), u, p(t), t)$, $u \in \Omega$;*

2. *$\dot{p}(t) = -\frac{\partial H^*}{\partial x}$ a.e.;*

3. *$\hat{H}(x, p(t), t) = \max_{u\in\Omega} H(x, u, p(t), t)$ exists and is concave in x, for $x \in A(t)$ and $\forall t \geq t_0$;*

4. *$\underline{\lim}_{t\to+\infty} p(t)(x(t) - x^*(t)) \geq 0$*

Then, $(x^*(t), u^*(t))$ is optimal is the sense of catching up.

The Mangasarian (3) and Arrow theorems (4) hold even in an infinite horizon problem with a discount factor, with the substitution $p(t) = e^{-\delta t}q(t)$.

2.4 Phase-space analysis

The phase space is a description of a dynamic system in which we look at the state variables of the system, which form a vector space in which the system is represented. The phase space is useful when in optimal control problems, the differential equations obtained from the Maximum Principle are not analytically solvable. The phase space allows to analyze the equilibrium points of the system, helping us to qualitatively characterize the solutions.

Basis

Starting from a *vector field*, defined on an open set $U \subset \mathbb{R}^n$

$$X : U \longrightarrow \mathbb{R}^n$$

that associates a vector $X(z)$ of $\mathbb{R}^n$ at any point z in U. This equation is defined as a first order differential equation:

$$\dot{z} = X(z).$$

A solution of this equation is a curve

$$z : J \longrightarrow U, \quad J \subset \mathbb{R}$$

where its velocity vector is equal to the vector field at any time

$$\frac{d}{dt}z(t) = X(z(t)), \quad \forall t \in J.$$

If we fix a point z_0 for which the function passes at the initial time t_0, we can write the Cauchy Problem:

$$\begin{cases} \frac{d}{dt}z(t) = X(z(t)) \\ z(t_0) = z_0 \end{cases}$$

and the solution of this problem will be a function $z(t; z_0, t_0)$. As z_0 varies, the function $(t, z_0) \longrightarrow z(t; z_0, t_0)$ provides all the solutions of the system and is called "flow" of the differential equation.

Theorem 5 (existence and uniqueness theorem)**.** *We suppose that $X\colon U \longrightarrow \mathbb{R}^n$ is differentiable. Then:*

1. *for any $t_0 \in \mathbb{R}$ and for any $z_0 \in U$, there are an interval $J \subset \mathbb{R}$ containing t_0 and a solution $z\colon J \longrightarrow U$ of $\dot{z} = X(z)$, such that $z(t_0) = z_0$;*

2. *if $z\colon J \longrightarrow U$ and $z'\colon J' \longrightarrow U$ are two solutions with the same initial point, then $z'(t) = z(t)$ for $t \in J \cap J'$.*

Equilibrium points

In the theory of optimal control, sometimes it is not possible to solve analytically the differential equations involved. Consequently, it turns out to be complex or even impossible to obtain the flow $(t, z_0) \longrightarrow z(t; z_0, t_0)$ in an analytic form. One way to solve the problem is to find the *equilibrium points* of the system. A point $\bar{z}$ is an equilibrium point of $\dot{z}$, if $X(\bar{z}) = 0$. The equilibrium points represent the stationary states of the system.

According to Lyapunov, a definition of stability is:

Definition 3 (Lyapunov stability)**.** A point of equilibrium $\bar{z}$ of $\dot{z} = X(z)$ is called

1. stable if for every neighborhood U of $\bar{z}$ there exists a neighborhood U_0 of $\bar{z}$ such that

$$z_0 \in U_0 \quad \Longrightarrow \quad z(t; z_0) \in U \quad \forall t > 0;$$

2. unstable if is not stable;

3. asymptotically stable if is stable and if there is a neighborhood V di $\bar{z}$ such that

$$z_o \in V \quad \Longrightarrow \quad \lim_{t \to +\infty} z(t; z_0) = \bar{z}.$$

Autonomous systems of the plane

Given two C^1 functions, F and G, we consider the autonomous system of differential equations:

$$\begin{cases} \dot{X} = F(X, \lambda) \\ \dot{\lambda} = G(X, \lambda) \end{cases} \tag{2.2}$$

We define as phase space, the plane $X - \lambda$. A solution $(X(t), \lambda(t))$ will move forming a curve in the phase space. Its velocity $(\dot{X}(t), \dot{\lambda}(t))$ is tangent to the curve.
We want to analyze the phase space of the system (2.2) and how the solutions behave in the phase space.
Our aim is to find the equilibrium points

$$\dot{X} = F(X, \lambda) = 0, \quad \dot{\lambda} = G(X, \lambda) = 0$$

that are the points $E = (\bar{x}, \bar{\lambda})$ of intersection of the two curves $F = 0$ and $G = 0$, where $X(t) = \bar{x}$ and $\lambda(t) = \bar{\lambda}$ are the solutions of (2.2). Actually, the phase diagram cannot tell us accurately the stability of an

equilibrium, but it can suggest it. Using the linear approximation of the equation we can understand the stability of the equilibrium, looking at a neighborhood of the points of equilibrium.
The linear approximation of the system (2.2), close to a point of equilibrium $E = (\bar{x}, \bar{\lambda})$ is:

$$\begin{pmatrix} \dot{X} \\ \dot{\lambda} \end{pmatrix} = z = J(E)(z - E)$$

$$\text{with } J(E) = \begin{pmatrix} \frac{d}{dX}F(\bar{x}, \bar{\lambda}) & \frac{d}{d\lambda}F(\bar{x}, \bar{\lambda}) \\ \frac{d}{dX}G(\bar{x}, \bar{\lambda}) & \frac{d}{d\lambda}G(\bar{x}, \bar{\lambda}) \end{pmatrix} \tag{2.3}$$

The phase portrait of the non linear system (2.2) is similar to the phase portrait of the linearized system, if we consider a neighborhood close to the equilibrium point. So, the behaviour of the solutions close to the equilibrium point depends on the eigenvalues of the Jacobian matrix in the equilibrium that linearizes the system. A saddle point is a point which is a maximum considering one axial direction, and a minimum if we consider the crossing axis. It provides two solutions of the system, converging from opposite directions towards equilibrium.

Theorem 6 (Local saddle point theorem)**.** *Having the system*

$$\begin{cases} \dot{x} = F(x, \lambda) \\ \dot{\lambda} = G(x, \lambda) \end{cases}$$

we assume that the C^1 functions F and G are in a neighborhood of the equilibrium $E = (\bar{x}, \bar{\lambda})$, and that the two eigenvalues of the Jacobian matrix (2.3) *in the equilibrium are real and have opposite sign. Then, there exist two solutions $(x_1(t), \lambda_1(t))$ and $(x_2(t), \lambda_2(t))$ in an interval $[t_0, \infty]$, converging to $E = (\bar{x}, \bar{\lambda})$ from opposite directions. These solutions are tangent to a line passing through E, generated by the eigenvector relative to the negative eigenvalue. Such a point is called saddle point.*

2.5 Inada conditions

In Ramsey model, to ensure the production function is well defined, it has to satisfy the Inada conditions (Takayama, 1985).

Definition 4. Given a continuously differentiable function $F \longrightarrow Y$, with $X = \{x \colon x \in \mathbb{R}^n_+\}$ and $Y = \{y \colon y \in \mathbb{R}_+\}$, the Inada conditions are:

1. $f(x)|_{x=0} = 0$
2. $\frac{\partial f(x)}{\partial x_i} > 0 \qquad \frac{\partial^2 f(x)}{\partial x_i^2} < 0$
3. $\lim_{x_i \to 0} \frac{\partial f(x)}{\partial x_i} = +\infty$
4. $\lim_{x_i \to +\infty} \frac{\partial f(x)}{\partial x_i} = 0$

Chapter 3

The Farmer's problem

Let us consider a farmer who wants to maximize the profit deriving from her arable land. The production, i.e. the harvest, depends on three factor inputs: labor, pesticides and biodiversity present in the soil. Labor and biodiversity have the only result to positively contribute to the harvest, while pesticides have two opposite effects: on one hand this input positively contributes to the harvest, but on the other it damages the biodiversity, causing a decrease in production. So, the farmer faces the problem to find the optimal allocation of inputs that maximizes her profit, considering the trade-off of using pesticides.
The production function in its functional form is:

$$y(t) = f(L(t), A(t), X(t))$$

where L is labor, A is pesticides and X is biodiversity. We assume an infinite demand for the final good, at a constant price p. In the cost function, only labor and pesticides enter, with constant prices. The unit costs are ω and c, respectively to L and A.
The biodiversity growth depends on the level of biodiversity itself and on the level of pesticides used for the production. The growth, or the reduction, of the biodiversity level is expressed by the following differential equation:

$$\dot{X} = g(X(t), y(t))$$

The farmer's problem can be formalized through an optimal control problem with an infinite horizon with a discount factor. The aim is to find L(t) and A(t) that maximize the discounted profit:

$$\begin{aligned} \text{maximize} \quad & \int_0^{+\infty} (py(t) - \omega L(t) - cA(t))e^{-\delta t} dt \\ \text{subject to} \quad & \dot{X}(t) = g(X(t), y(t)) \\ & X(t_0) = \bar{X}_0 \\ & L(t) \geq 0 \\ & A(t) \geq 0 \end{aligned}$$

3.1 Path of biodiversity: the logistic function

The models we are going to analyze use a logistic function to describe the dynamics of the biodiversity in absence of human intervention.

The logistic is a sigmoid function, i.e. a s-curve, that has found application in many fields. In economics for example, it is used to describe the diffusion of an innovation, a technological progress or a new market standard. It explains the growth of a population with resource constraints, that can be a spacial constraint or a constraint due to the lack of food. The logistic function derives from the need to impose an upper limit to the growth of an exponential function (Kingsland, 1982).

Dealing with the logistic function, we need to consider the following assumptions (Kingsland, 1982):

1. all individuals are equal: there is no difference in age, genotype, size etc.;

2. the density of the species is homogeneous throughout the field;

3. the rate of reproduction is both proportional to the existing population and to the amount of available resources.

4. the function has an upper limit called carrying capacity.

The logistic function is described by the following equation:

$$X(t) = \frac{K}{1 + \frac{K - X_0}{X_0} e^{-gt}}$$

where g is the growth rate, X_0 is the level of the biodiversity at time $t = 0$ and K is the carrying capacity. Both g and K are assumed to be constant.

Over time, the system tends to the carrying capacity:

$$\lim_{t \to +\infty} X(t) = K$$

Assuming X_0 close to zero, initially $\left(1 - \frac{X(t)}{K}\right)$ can be approximated to one and the biodiversity will grow exponentially. For values of X close to the carrying capacity, the fraction $\frac{X(t)}{K}$ will tend to one and $\left(1 - \frac{X(t)}{K}\right)$ to zero, causing a slower and slower growth.
The equation of motion used in the farmer's models is a logistic function in its differential form, where the biodiversity $X(t)$ get damaged by the human use of pesticides $A(t)$:

$$X'(t) = gX(t)\left(1 - \frac{X(t)}{K}\right) - A(t)$$

3.2 Production function

In what follows, we will consider two models: a multiplicative and an additive one. These models differ in their production function:

- in the *multiplicative* model, the production function is a Cobb-Douglas with three inputs: labor L, pesticides A and level of biodiversity X:

$$y(t) = \mu L(t)^{\alpha} A(t)^{\beta} X(t)^{\gamma}$$

- in the *additive* model, the more original respect to the literature, the biodiversity level is summed up to a Cobb-Douglas with L and A as inputs:

$$y(t) = \mu L(t)^{\alpha} A(t)^{\beta} + bX(t)^{\gamma}$$

The parameters and variables that characterize the production are:

- $y(t)$ is the production at time t
- $L(t)$ is labor
- $A(t)$ is the quantity of pesticides used at time t
- $X(t)$ is the level of biodiversity at time t
- μ and b are the total productivity factors, both positive and constant.
- α, β, γ are the output elasticity of labor, pesticides and biodiversity, respectively. Both models suppose: $0 < \alpha, \beta, \gamma \leq 1$ and $\alpha + \beta < 1$

To be sure that the production functions are well defined, we verify that the Inada conditions are satisfied for both functions.
We first analyze the multiplicative production function. The Inada conditions (for the factor L) are:

1. $y(t)|_{L=0} = 0$
2. $\frac{\partial y}{\partial L}(t) = \mu\alpha L_t^{\alpha-1} A_t^{\beta} X_t^{\gamma} > 0 \qquad \frac{\partial^2 y}{\partial L^2}(t) = \mu\alpha(\alpha-1) L_t^{\alpha-2} A_t^{\beta} X_t^{\gamma} < 0$
3. $\lim_{L\to 0} \frac{\partial y}{\partial L}(t) = \lim_{L\to 0} \mu\alpha L_t^{\alpha-1} A_t^{\beta} X_t^{\gamma} = +\infty$
4. $\lim_{L\to +\infty} \frac{\partial y}{\partial L}(t) = \lim_{L\to +\infty} \mu\alpha L_t^{\alpha-1} A_t^{\beta} X_t^{\gamma} = 0$

Analogous for A and X.
We now verify the Inada conditions for the additive production function (for the factor L):

1. $y(t)|_{L=0} = bX(t)^{\gamma} > 0$
2. $\frac{\partial y}{\partial L}(t) = \mu\alpha L_t^{\alpha-1} A_t^{\beta} > 0 \qquad \frac{\partial^2 y}{\partial L^2}(t) = \mu\alpha(\alpha-1) L_t^{\alpha-2} A_t^{\beta} < 0$
3. $\lim_{L\to 0} \frac{\partial y}{\partial L}(t) = \lim_{L\to 0} \mu\alpha L_t^{\alpha-1} A_t^{\beta} = +\infty$
4. $\lim_{L\to +\infty} \frac{\partial y}{\partial L}(t) = \lim_{L\to +\infty} \mu\alpha L_t^{\alpha-1} A_t^{\beta} = 0$

Analogous for A.
Regarding the factor X:

1. $y(t)|_{X=0} = \mu L(t)^\alpha A(t)^\beta > 0$

2. $\frac{\partial y}{\partial X}(t) = b\gamma X^{\gamma-1} > 0 \qquad \frac{\partial^2 y}{\partial X^2}(t) = b\gamma(\gamma-1)X^{\gamma-2} < 0$

3. $\lim_{X\to 0} \frac{\partial y}{\partial X}(t) = \lim_{X\to 0} b\gamma X^{\gamma-1} = +\infty$

4. $\lim_{X\to +\infty} \frac{\partial y}{\partial X}(t) = \lim_{X\to +\infty} b\gamma X^{\gamma-1} = 0$

The main difference between the two functions concern the case in which a factor goes to zero. In the multiplicative model, any factor input is essential for production:

$$L(t) = 0 \text{ or } A(t) = 0 \text{ or } X(t) = 0 \Longrightarrow y(t) = 0$$

Conversely, in the additive model neither the economic activity (labor and pesticides) nor biodiversity are essential for production:

$$X(t) = 0 \Longrightarrow y(t) = \mu L(t)^\alpha A(t)^\beta$$
$$L(t) = 0 \text{ or } A(t) = 0 \text{ or } X(t) = 0 \Longrightarrow y(t) = bX(t)^\gamma$$

Another difference between the two production functions regards the impact of the marginal product of pesticides when the biodiversity level increases. In the multiplicative production function, an increase in the level of biodiversity has a positive impact on the marginal production of pesticides:

$$\frac{\partial y}{\partial A} = \mu\beta L_t^\alpha A_t^{\beta-1} X^\gamma$$
$$\frac{\partial^2 y}{\partial A \partial X} = \gamma\mu\beta L_t^\alpha A_t^{\beta-1} X^{\gamma-1}$$

Conversely, in the additive production function, an increase in the level of biodiversity has no impact on the marginal production of pesticides:

$$\frac{\partial y}{\partial A} = \mu\beta L_t^\alpha A_t^{\beta-1}$$
$$\frac{\partial^2 y}{\partial A \partial X} = 0$$

3.3 Resolution of the model

To consider both models at the same time, we keep the generic production function as $f(L, A, X)$ where possible, instead of making it explicit as *additive* production function and *multiplicative* production function.

The problem we want to solve is an optimal control problem according to the catching-up criterion in the time interval $[0, +\infty]$, formulated as follows:

$$\begin{aligned}
\max_{L(t),A(t)} \quad & \int_0^{+\infty} (p(f(L,A,X)) - \omega L(t) - cA(t))e^{-\delta t} dt \\
\text{subject to} \quad & \dot{X} = gX(t)\left(1 - \frac{X(t)}{K}\right) - A(t) \\
& X_0 = X_0 \\
& L(t) \geq 0 \\
& A(t) \geq 0
\end{aligned}$$

Necessary conditions

First, we analyze the necessary conditions for the optimality given by Pontryagin Maximum Principle (Seierstad, 1987). The Hamiltonian function at current value is so defined:

$$H^c(X^*, L^*, A^*, \lambda(t), t) = \lambda_0(p(f(L,A,X)) - \omega L(t) - cA(t)) + \lambda(t)\left(gX(t)\left(1 - \frac{X(t)}{K}\right) - A(t)\right)$$

Maximum Principle in infinite horizon Considering an optimal solution according to catching-up criterion $(u^*(t), x^*(t))$, and a piecewise continuous optimal control, defined on $[0,+\infty) \times [0,+\infty)$, to which a state function $x^*(t)$ is associated. Then, there exists a constant $\lambda_0 \in \mathbb{R}$ and a piecewise continuous C^1 function $\lambda : [0,+\infty) \longrightarrow \mathbb{R}$. For any $t \in \mathbb{R}$, the following conditions hold:

1. $(\lambda_0, \lambda(t)) \neq 0$;
2. $u = (A^*, L^*)$ maximizes $H(X^*, L^*, A^*, \lambda(t), t)$ for any $u \in \Omega$;
3. except for times t where $u^*(t)$ is discontinuous, $\lambda(t)$ is differentiable and
$$\dot{\lambda} = -\frac{\partial H(X^*, L^*, A^*, \lambda(t), t)}{\partial X} + \delta\lambda(t)$$
4. $\lambda_0 \in \{0,1\}$

Case $\lambda_0 = 0$

For the 4th condition $\lambda_0 \in \{0,1\}$, we first consider $\lambda_0 = 0$. For the 1st condition $\lambda(t) \neq 0$ and the Hamiltonian becomes:

$$H^c = \lambda\left(gX(t)\left(1 - \frac{X(t)}{K}\right) - A(t)\right)$$

For the 2nd condition:

$$H^c(X, A^*(t), L, \lambda) \geq H^c(X, A, L, \lambda)$$

and then:

$$\lambda\left(gX(t)\left(1 - \frac{X(t)}{K}\right) - A^*(t)\right) \geq \lambda\left(gX(t)\left(1 - \frac{X(t)}{K}\right) - A\right)$$

Assuming $\lambda(t) \geq 0\ \forall t$, the optimal solution requires $A^*(t) = 0$, because of the constraint $A(t) \geq 0$. Consequently, $L^*(t) = 0$.

Here, the two models diverge. For the sake of clarity, we introduce a notation: the subscript a refers to the additive model, while the subscript m to the multiplicative model. The two subscripts have been put only where necessary, in order to keep the writing clear.

In the multiplicative model the production function, the cost function and the profit go to zero. This is a degenerate case, not interesting for the economy. So, we no longer consider it.
Conversely, in additive model the production still remains positive, despite the fact that the labor and pesticides are zero. The cost function goes to zero, and the farmer makes positive profit. So, the candidate optimal control of the problem is $u^* = (L^*, A^*) = (0, 0)$. The function we have to maximize becomes:

$$J_0 = \int_0^{+\infty} pbX(t)^\gamma e^{-\delta t}$$

With $A^*(t) = 0$, the equation of motion is:

$$\dot{X}_a(t) = gX(t)\left(1 - \frac{X(t)}{K}\right)$$

There are two equilibrium point:

- the first one is in $X_a = 0$ and is an unstable equilibrium (the derivative is positive);
- the second one is in $X_a = K$ and is a stable equilibrium (the derivative is negative).

The system now depends on the initial level of biodiversity: if $X_0 = 0$, the biodiversity cannot grow and $X(t) = 0$ is a stationary solution. Any input is null and consequently also the profit is null. If the initial level of biodiversity is positive, the system will tend to the carrying capacity K. The profit, in this case will be:

$$J_0 = \int_0^{+\infty} \left(pb\left(\frac{e^{gt}KX_0}{K - X_0 + X_0 egt}\right)\right) \tag{3.1}$$

Although this case is a limit case, it is interesting from an economic point of view. If the farmer has a technology that leads to a production function similar the one in this model, the profit will never be zero, as long as biodiversity is not null. Indeed, biodiversity will continue to generate income, without any cost, for ever. We know that from an economic point of view is a very strong assumption, but it remains interesting considering it.

Case $\lambda_0 = 1$
We now cosider the case where $\lambda_0 = 1$. The Hamiltonian function becomes:

$$H^c = \lambda_0(p(f(L, A, X)) - \omega L(t) - cA(t)) + \lambda(t)\left(gX(t)\left(1 - \frac{X(t)}{K}\right) - A(t)\right)$$

From the 2nd condition, we obtain the stationary points:

$$\begin{cases} \frac{\partial H^c}{\partial L} = 0 \\ \frac{\partial H^c}{\partial A} = 0 \end{cases} \tag{3.2}$$

The two models now diverge another time. In the multiplicative model, we obtain the following equations:

$$\begin{cases} \alpha p\mu L^{\alpha-1}A^{\beta}X_t^{\gamma} = \omega \\ \beta p\mu L^{\alpha}A^{\beta-1}X_t^{\gamma} = c+\lambda_t \end{cases} \tag{3.3}$$

In order to have positive values for L and A, we have to assume $\omega > 0$, that is positive by definition, and $c + \lambda(t) > 0$. If we combine the two equations, we can isolate A and L:

$$A_m = \frac{\beta}{\alpha}\frac{\omega}{c+\lambda_t}L \qquad L_m = \frac{\alpha}{\beta}\frac{c+\lambda_t}{\omega}A \tag{3.4}$$

Inserting 3.4 into 3.3, we can isolate L as a function of X_t and λ_t:

$$L_m = \max\left\{0\ ,\ \left((\beta p\mu)^{\frac{1}{1-\alpha-\beta}}\left(\frac{\alpha}{\beta\omega}\right)^{\frac{1-\beta}{1-\alpha-\beta}}\right)\left(\frac{1}{c+\lambda_t}\right)^{\frac{\beta}{1-\alpha-\beta}}X_t^{\frac{\gamma}{1-\alpha-\beta}}\right\} \tag{3.5}$$

Then, we can isolate A as a function of X_t and λ_t, from 3.3 and 3.4:

$$A_m = \max\left\{0\ ,\ \left((\beta p\mu)^{\frac{1}{1-\alpha-\beta}}\left(\frac{\alpha}{\beta\omega}\right)^{\frac{\alpha}{1-\alpha-\beta}}\right)\left(\frac{1}{c+\lambda_t}\right)^{\frac{1-\alpha}{1-\alpha-\beta}}X_t^{\frac{\gamma}{1-\alpha-\beta}}\right\} \tag{3.6}$$

In the additive model, we obtain the following equations:

$$\begin{cases} \alpha p\mu L^{\alpha-1}A^{\beta} = \omega \\ \beta p\mu L^{\alpha}A^{\beta-1} = c+\lambda_t \end{cases} \tag{3.7}$$

As in the multiplicative, to have positive L and A, we assume $\omega > 0$, that is positive by definition, and $c + \lambda(t) > 0$. If we combine the two equations, we can isolate A and L:

$$A_a = \left(\frac{\omega L^{1-\alpha}}{p\mu\alpha}\right)^{\frac{1}{\beta}} \qquad L_a = \left(\frac{p\mu\alpha A^{\beta}}{\omega}\right)^{\frac{1}{1-\alpha}} \tag{3.8}$$

We can then isolate A as a function of λ_t:

$$A_a = \max\left\{0\ ,\ \left(\beta(p\mu)^{\frac{1}{1-\alpha}}\left(\frac{\alpha}{\omega}\right)^{\frac{\alpha}{1-\alpha}}\right)^{\frac{1-\alpha}{1-\alpha-\beta}}\left(\frac{1}{c+\lambda_t}\right)^{\frac{1-\alpha}{1-\alpha-\beta}}\right\} \tag{3.9}$$

and L as a function of λ_t:

$$L_a = \max\left\{0\ ,\ \left(\frac{\alpha(c+\lambda_t)}{\beta\omega}\right)\left(\beta(p\mu)^{\frac{1}{1-\alpha}}\left(\frac{\alpha}{\omega}\right)^{\frac{\alpha}{1-\alpha}}\right)^{\frac{1-\alpha}{1-\alpha-\beta}}\left(\frac{1}{c+\lambda_t}\right)^{\frac{1-\alpha}{1-\alpha-\beta}}\right\} \tag{3.10}$$

We thus found for both models, L and A as a function of X_t and λ_t. In particular, in the additive model, L and A are only in function of λ_t. We can now write down the system of differential equations describing the model:

$$\begin{cases} \dot{X}(t) = f(X(t), \lambda(t), L, A) \\ \dot{\lambda}(t) = g(X(t), \lambda(t), L, A) \end{cases}$$

The equation of motion is

$$\dot{X} = gX(t)\left(1 - \frac{X(t)}{K}\right) - A(t)$$

where $A(t)$ in the multiplicative is:

$$A_m(t) = \left((\beta p\mu)^{\frac{1}{1-\alpha-\beta}} \left(\frac{\alpha}{\beta\omega}\right)^{\frac{\alpha}{1-\alpha-\beta}}\right) \left(\frac{1}{c+\lambda_t}\right)^{\frac{1-\alpha}{1-\alpha-\beta}} X_t^{\frac{\gamma}{1-\alpha-\beta}}$$

and in the additive is:

$$A_a(t) = \left(\beta(p\mu)^{\frac{1}{1-\alpha}} \left(\frac{\alpha}{\omega}\right)^{\frac{\alpha}{1-\alpha}}\right)^{\frac{1-\alpha}{1-\alpha-\beta}} \left(\frac{1}{c+\lambda_t}\right)^{\frac{1-\alpha}{1-\alpha-\beta}}$$

For the 3rd condition of the Pontryagin Maximum Principle, the adjoint equation in multiplicative is:

$$\dot{\lambda_m}(t) = 2\frac{g}{K}\lambda_t X_t + (\delta - g)\lambda_t - \left(\gamma(p\mu)^{\frac{1}{1-\alpha-\beta}} \beta^{\frac{\alpha+\beta}{1-\alpha-\beta}} \left(\frac{\alpha}{\beta\omega}\right)^{\frac{\alpha}{1-\alpha-\beta}}\right) \left(\frac{1}{c+\lambda_t}\right)^{\frac{1-\alpha}{1-\alpha-\beta}} X_t^{\frac{\gamma}{1-\alpha-\beta}-1}$$

and in the additive is:

$$\dot{\lambda_a}(t) = -\frac{pb\gamma}{X_t^{1-\gamma}} - g\lambda_t\left(1 - \frac{2X_t}{K}\right) + \delta\lambda_t$$

We can write down the system of differential equations in X_t and λ_t:

$$\begin{cases} \dot{X}(t) = f(X(t), \lambda(t)) \\ \dot{\lambda}(t) = g(X(t), \lambda(t)) \end{cases} \tag{3.11}$$

Phase space analysis

The solutions of the system of differential equations 3.11 cannot be found analytically. We therefore proceed with an analysis of the phase space: the aim is to find the equilibria of the system 3.11, i.e. the stationary points (X_{ss}, λ_{ss}) such that:

$$\begin{cases} f(X_{ss}, \lambda_{ss}) = 0 \\ g(X_{ss}, \lambda_{ss}) = 0 \end{cases} \tag{3.12}$$

We now start analyzing $\dot{X}(t) = 0$ and $\dot{\lambda}(t) = 0$.

- for $\dot{X}(t) = 0$, the two models differ because of their different $A(t)$.

$$gX(t)\left(1 - \frac{X(t)}{K}\right) = A(t)$$

For the multiplicative model:

$$A_m(t) = \left((\beta p\mu)^{\frac{1}{1-\alpha-\beta}} \left(\frac{\alpha}{\beta\omega} \right)^{\frac{\alpha}{1-\alpha-\beta}} \right) \left(\frac{1}{c+\lambda_t} \right)^{\frac{1-\alpha}{1-\alpha-\beta}} X_t^{\frac{\gamma}{1-\alpha-\beta}}$$

this term is always positive: on one hand because the farmer's problem says that $A(t)$ cannot be negative by definition, on the other hand because all terms to the right of the equal sign are positive: all the exponents are positive, the biodiversity level $X(t)$ cannot be negative, and the term $c + \lambda(t)$ must be positive for 3.3 in order to have positive control.
For the additive model:

$$A_a(t) = \left(\beta(p\mu)^{\frac{1}{1-\alpha}} \left(\frac{\alpha}{\omega} \right)^{\frac{\alpha}{1-\alpha}} \right)^{\frac{1-\alpha}{1-\alpha-\beta}} \left(\frac{1}{c+\lambda_t} \right)^{\frac{1-\alpha}{1-\alpha-\beta}}$$

as in the multiplicative, all the terms to the right of the equal sign are positive: we have assumed $c + \lambda(t) > 0$, and the exponent of the first factor

$$\frac{1-\alpha}{1-\alpha-\beta}$$

is bigger than 1 because $1 - \alpha > 1 - \alpha - \beta$.

- for $\dot{\lambda}(t) = 0$, we have to consider the two model separately.
  In the multiplicative model, the analysis is easy:

$$\dot{\lambda}_m(t) = 2\frac{g}{K}\lambda_t X_t + (\delta - g)\lambda_t - \left(\gamma(p\mu)^{\frac{1}{1-\alpha-\beta}} \beta^{\frac{\alpha+\beta}{1-\alpha-\beta}} \left(\frac{\alpha}{\beta\omega} \right)^{\frac{\alpha}{1-\alpha-\beta}} \right) \left(\frac{1}{c+\lambda_t} \right)^{\frac{1-\alpha}{1-\alpha-\beta}} X_t^{\frac{\gamma}{1-\alpha-\beta}} = 0 \quad (3.13)$$

  What makes the shape of the equation change are:

  1. the exponent of $X(t)$

$$\exp = \frac{\gamma}{1-\alpha-\beta} - 1 \qquad \begin{cases} \exp > 0, & \text{if } \gamma > 1 - \alpha - \beta \\ \exp < 0, & \text{if } \gamma < 1 - \alpha - \beta \end{cases}$$

  2. $(\delta - g) < 0$ or $(\delta - g) > 0$
     Here, according to the literature about the bionomic growth ratio, we consider economically relevant only the case $(g - \delta) > 0$.

  In the additive model, for $\dot{\lambda}_a(t) = 0$, we have:

$$\frac{pb\gamma}{X(t)^{1-\gamma}} = \left(-(g - \delta) - \frac{2X(t)}{K} \right) \lambda_a(t) \quad (3.14)$$

  As before, we exclude from the analysis the case in which $(g - \delta) < 0$. Biodiversity must be positive, and $\lambda(t) > 0$ because $c + \lambda > 0$ with $c > 0$. The coefficient of $\lambda(t)$ is positive when:

$$X > \frac{K(g-\delta)}{2g} =: x_2$$

For both models, we associate the values L_{ss}, A_{ss} and profit_{ss} to an equilibrium (X_{ss}, λ_{ss}):

- for the multiplicative:

$$A_{ss} = \max\left\{0\ ,\ \left(\frac{1}{c+\lambda ss}\right)^{\frac{1-\alpha}{1-\alpha-\beta}} (\beta\mu p)^{\frac{1}{1-\alpha-\beta}} \left(\frac{\alpha}{\beta\omega}\right)^{\frac{\alpha}{1-\alpha-\beta}} X_{ss}^{\frac{\gamma}{1-\alpha-\beta}}\right\}$$
$$L_{ss} = \max\left\{0\ ,\ \left(\frac{1}{c+\lambda ss}\right)^{\frac{\beta}{1-\alpha-\beta}} (\beta\mu p)^{\frac{1}{1-\alpha-\beta}} \left(\frac{\alpha}{\beta\omega}\right)^{\frac{1-\beta}{1-\alpha-\beta}} X_{ss}^{\frac{\gamma}{1-\alpha-\beta}}\right\}$$
$$\text{profit}_{ss} = \delta\left(p(f(L_{ss}, A_{ss}, X_{ss}) - \omega L_{ss} - cA_{ss}\right)$$

- for the additive:

$$A_{ss} = \max\left\{0\ ,\ \left(\frac{\beta(p\mu)^{\frac{1}{1-\alpha}}\left(\frac{\alpha}{\omega}\right)^{\frac{\alpha}{1-\alpha}}}{c+\frac{bpK_\gamma}{X_{ss}^{1-\gamma}(K(\delta-g)+2gX_{ss})}}\right\}\right)^{\frac{1-\alpha}{1-\alpha-\beta}}$$
$$L_{ss} = \max\left\{0\ ,\ \frac{p\mu\alpha}{\omega}\left(\left(\frac{\beta(p\mu)^{\frac{1}{1-\alpha}}\left(\frac{\alpha}{\omega}\right)^{\frac{\alpha}{1-\alpha}}}{c+\frac{bpK_\gamma}{X_{ss}^{1-\gamma}(K(\delta-g)+2gX_{ss})}}\right)^{\frac{1-\alpha}{1-\alpha-\beta}}\right)^{\frac{1}{1-\alpha}}\right\}$$
$$\text{profit}_{ss} = \delta^{-1}\left(p(f(L_{ss}, A_{ss}, X_{ss}) - \omega L_{ss} - cA_{ss}\right)$$

We can note that while in the multiplicative it has not been possible to make both λ_{ss} and X_{ss} explicit, in the additive model, it has been possible to make λ_{ss} explicit:

$$\lambda_{ss_a} = \frac{bpK_\gamma}{X_{ss}^{1-\gamma}(K(\delta-g)+2gX_{ss})}$$

Equilibrium points

We now study the equilibrium points of the system, i.e. the points (λ_{ss}, X_{ss}) such that:

$$\begin{cases} f(X_{ss}, \lambda_{ss}) = 0 \\ g(X_{ss}, \lambda_{ss}) = 0 \end{cases}$$

We will consider only the points economically relevant, i.e. those points where the profit is not null and the parameters respect the constraints present in the literature.

In the multiplicative model, there are two equilibrium points. The first one is in $(0,0)$: considering $(g-\delta) > 0$ from literature, it is a saddle point, but is not economically relevant. The second one has both controls positive and was found by Pozzan through various simulations using *Wolfram Mathematica*. The nature of the equilibrium varies according to the cases 1. and 2. explained at page 39, and can be an unstable equilibrium or a saddle point (sometimes two equilibrium points are found, besides the point $(0,0)$). Four cases are then found, depending on the positivity or negativity of $(g-\delta)$ and if $\alpha+\beta+\gamma$ is bigger or smaller than 1. We have chosen the case where $(g-\delta)>0$, following the literature, and where $\alpha+\beta+\gamma<1$: these three are the exponents of the inputs of the production function. Being the function a Cobb-Douglas, imposing them to be smaller than one, it means considering decreasing returns to scale. Otherwise, we would be in presence of increasing returns to scale and this is quite a strong assumption from an economic point of view. In conclusion, the equilibrium found according to the selected parameters is a saddle point.

We write again the equation of motion for the additive model:

$$\dot{X}_a = gX(t)\left(1-\frac{X(t)}{K}\right) - \left(\frac{\beta(p\mu)^{\frac{1}{1-\alpha}}\left(\frac{\alpha}{\omega}\right)^{\frac{\alpha}{1-\alpha}}}{c+\frac{bpK_\gamma}{X_{ss}^{1-\gamma}(K(\delta-g)+2gX_{ss})}}\right)^{\frac{1-\alpha}{1-\alpha-\beta}}$$

The equation of motion depends on the positivity of the basis of $A(t)$.
For simplicity, we call

$$\bar{A} = \left(\frac{\beta(p\mu)^{\frac{1}{1-\alpha}}\left(\frac{\alpha}{\omega}\right)^{\frac{\alpha}{1-\alpha}}}{c+\frac{bpK_\gamma}{X_{ss}^{1-\gamma}(K(\delta-g)+2gX_{ss})}}\right)$$

We can see that:

$$\bar{A}>0 \Longleftrightarrow \frac{X^{1-\gamma}(K(\delta-g)+2gX)}{cX^{1-\gamma}(\delta-g)K+2cgX^{2-\gamma}+bp\gamma K}>0$$

The numerator is positive for

$$X > x_2 =: \frac{(g-\delta)K}{2g} < \frac{K}{2}$$

It is not possible to identify analytically the positivity of the denominator of $\bar{A}$. Therefore, some numerical simulation was made by Viscovich to study its positivity. Two cases emerged: in the first one, the denominator is always positive, while in the second one, the denominator becomes null in two points, forming two asymptotes. Consequently, we do not take into consideration the second case because does not fit the reality.

We are in the first case when one of these conditions (or both) are met:

- γ is sufficiently high
- $g-\delta$ is sufficiently small, so the bionomic growth ratio is close to 1.

in the other case, we say that the model with conditions different from the ones above does not represent well the reality.

So, the condition for the positivity of the control is:

$$\bar{A}>0 \iff X>x_2$$

The candidate optimal controls are then piecewise defined:

The control A^* is:

$$A_a^* = \begin{cases} 0 & \text{if } X \leq x_2 \\ \left(\frac{\beta(p\mu)^{\frac{1}{1-\alpha}} \left(\frac{\alpha}{\omega}\right)^{\frac{\alpha}{1-\alpha}}}{c + \frac{bpK_\gamma}{X_{ss}^{1-\gamma}(K(\delta-g)+2gX_{ss})}} \right)^{\frac{1-\alpha}{1-\alpha-\beta}} & \text{if } X > x_2 \end{cases} \tag{3.15}$$

and control L^* is:

$$L_a^* = \begin{cases} 0 & \text{if } X \leq x_2 \\ \left(\frac{p\mu\alpha}{\omega} \left(\frac{\beta(p\mu)^{\frac{1}{1-\alpha}} \left(\frac{\alpha}{\omega}\right)^{\frac{\alpha}{1-\alpha}}}{c + \frac{bpK_\gamma}{X_{ss}^{1-\gamma}(K(\delta-g)+2gX_{ss})}} \right)^{\frac{1-\alpha}{1-\alpha-\beta}} \right)^{\frac{1}{1-\alpha-\beta}} & \text{if } X > x_2 \end{cases} \tag{3.16}$$

The equation of motion becomes:

$$\dot{X}_a = \begin{cases} gX(1-\frac{X}{K}) & \text{if } X \leq x_2 \\ gX(1-\frac{X}{K}) - \left(\frac{\beta(p\mu)^{\frac{1}{1-\alpha}} \left(\frac{\alpha}{\omega}\right)^{\frac{\alpha}{1-\alpha}}}{c + \frac{bpK_\gamma}{X_{ss}^{1-\gamma}(K(\delta-g)+2gX_{ss})}} \right)^{\frac{1-\alpha}{1-\alpha-\beta}} & \text{if } X > x_2 \end{cases} \tag{3.17}$$

In this case, the equilibrium points are two:

- an unstable equilibrium in 0;
- a stable equilibrium close to K. This is due to the fact that in K we obtain $\dot{X}(K) < 0$, so there must be another equilibrium besides the one in 0.

To summarize

We briefly recap the equilibrium points found in the analysis, the path of the systems, the controls and the relative profits in steady state.

Equilibrium points

In the multiplicative model, two stationary points are found:

- a saddle point in 0, not relevant;
- a saddle point with $X > 0$.

In the additive model, two stationary points are found:

- an unstable equilibrium in 0 (relevant because the profit is positive);
- a stable equilibrium close to K.

Note that other points where found by the analysis, but they have not been considered because not economically relevant.

Path of the systems

In the multiplicative model:

- if the initial level of biodiversity is $X_0 = 0$, the system does not move and stays in $X = 0$;

- if the initial level of biodiversity is $X_0 > 0$, the system will converge to the second equilibrium.

In the additive model:

- as in the multiplicative model, if the initial level of biodiversity is $X_0 = 0$, the system does not move and stays in $X = 0$;
- if the initial level of biodiversity is $X_0 > 0$, the system will converge to the second equilibrium, close to the carrying capacity K.

Controls

For $(L_{ss}, A_{ss}) = (0, 0)$

	multiplicative model	additive model
profit	0	$\int_0^{+\infty}\left(pb\left(\frac{e^{gt}KX_0}{K-X_0+X_0egt}\right)\right)$

For $(L_{ss}, A_{ss}) > (0, 0)$

	multiplicative model	additive model
L_{ss}	$(\beta\mu p)^{\frac{1}{1-\alpha-\beta}}\left(\frac{\alpha}{\beta\omega}\right)^{\frac{1-\beta}{1-\alpha-\beta}} X_{ss}^{\frac{\gamma}{1-\alpha-\beta}}\left(\frac{1}{c+\lambda ss_m}\right)^{\frac{\beta}{1-\alpha-\beta}}$	$\frac{p\mu\alpha}{\omega}\left(\beta(p\mu)^{\frac{1}{1-\alpha}}\left(\frac{\alpha}{\omega}\right)^{\frac{\alpha}{1-\alpha}}\right)^{\frac{1}{1-\alpha-\beta}}\left(\frac{1}{c+\lambda_{ss_a}}\right)^{\frac{1}{1-\alpha-\beta}}$
A_{ss}	$(\beta\mu p)^{\frac{1}{1-\alpha-\beta}}\left(\frac{\alpha}{\beta\omega}\right)^{\frac{\alpha}{1-\alpha-\beta}} X_{ss}^{\frac{\gamma}{1-\alpha-\beta}}\left(\frac{1}{c+\lambda ss_m}\right)^{\frac{1-\alpha}{1-\alpha-\beta}}$	$\beta(p\mu)^{\frac{1}{1-\alpha-\beta}}\left(\frac{\alpha}{\omega}\right)^{\frac{\alpha}{1-\alpha-\beta}}\left(\frac{1}{c+\lambda_{ss_a}}\right)^{\frac{1-\alpha}{1-\alpha-\beta}}$
profit	$\delta\left(p(f(L_{ss}, A_{ss}, X_{ss}) - \omega L_{ss} - cA_{ss}\right)$	$\delta^{-1}\left(p(f(L_{ss}, A_{ss}, X_{ss}) - \omega L_{ss} - cA_{ss}\right)$

Chapter 4

Economic observations

In this chapter, we report some observations and discussions about the multiplicative and the additive model from an economic point of view. In particular, we comment the choices made on the production functions and the logistic function, the difference between the controls in the two models, and the best policy to implement in order to have the highest profit. We conclude this chapter discussing about the interpretation of a particular case of the additive model, where it is assumed that a subsidy can be given to the farmer in order to avoid a high damage of biodiversity.

4.1 Production function

The difference in the two models is on the production function. In the additive model, the biodiversity factor input is added to a Cobb-Douglas: the natural input and the non-natural inputs are linked together through an additive relation. In the multiplicative model, the biodiversity forms with labor and pesticides a Cobb-Douglas with three inputs. This leads to a crucial difference. In the multiplicative model, labor and pesticides must always be positive in order to make profit; if one of them goes to zero, the production goes to zero. Differently, in the additive model the profit will always be positive, as long as biodiversity is not null. Indeed, if the farmer stops using labor and pesticides, biodiversity will continue to generate income, thank to the additive link between the natural input and the two non-natural inputs.

As in Ramsey model (Novales, 2009), where the consumption is set to be always positive, here it also turns out important to impose labor and pesticides greater than zero in the multiplicative model, in order to keep production positive. Being a problem with infinite horizon of time, the additive model is even more interesting: the biodiversity gives to the farmer a perpetual income with zero costs. If either labor or capital became overly expensive, a solution could be to let the biodiversity grow without human intervention: the cost function would be null, as null would be the pollution.

The production function formulated in the additive model deviates from the literature related to biodiversity, where a pure Cobb-Douglas is generally considered as production function, as in the multiplicative model. The natural capital is always multiplied to the human capital, that is composed of labor and physical capital in the majority of cases. In the additive model, the production function can be split into two subfunctions. The first one is composed by labor and pesticides, is linked to a positive cost function and is probable the greater source of income. The second one is made up of the natural resource. It will never have zero output, unless the biodiversity goes to zero but is a rare and strong assumption; it continues to produce output even if the farmer makes no effort. There is no cost function related to this part of the production function. The only cost in letting the biodiversity grow is the cost-opportunity of having a constraint to the quantity of A: the more pesticides are used, the less biodiversity is present in the land.

In the multiplicative model, two cases emerge, making the difference: when the production function has decreasing returns to scale and when it has increasing returns to scale. In the former case, two equilibrium points of the system are found: a saddle point in $X = 0$ and a saddle point with positive controls. In the latter case, two equilibria of the system are found. The first one is at $X = 0$, an unstable equilibrium. The second one is a saddle point with positive controls.

It is easy to see that in the multiplicative model the production function with increasing returns to scale gives the farmer a higher profit. In order to identify which parameters of the problem and which exponents of the Cobb-Douglas generate the highest profit, some simulations were made by Pozzan (Pozzan, 2020). The following conditions on the parameters have been found: $g > \delta$, $\alpha > \beta$.

It is interesting to compare the exponents of the production function that give the highest profit between the two models. Looking at the simulations made by Viscovich for the additive model (Viscovich, 2020), the conditions on parameters that lead to the highest profit: $g > \delta$, $\alpha > \beta$, $\alpha + \beta + \gamma > 1$. We can see that the constraints on the parameters in order to have highest profits are the same for the two models. This finding complies with our forecasts. We expected to see higher profit with increasing returns to scale and in $\alpha > \beta$, meaning that the non-polluting factor L contributes more than the polluting one. In this case, the quantity of L can be increased without causing damage to biodiversity.

In closing, we can note that the simulations with the multiplicative production functions had higher profit than the additive one. This is another finding we already expected.

4.2 Labor and pesticides

As we expected, the condition $\alpha > \beta$ leads to higher profit in both models. The assumption is that the labor does not damage the biodiversity, while only the capital (pesticides in the models) is responsible for pollution. Generalizing, the solutions of the models tell us that it is better to invest in those lands where the polluting human action is less productive than the non-polluting human action, and let the biodiversity grow to high levels. In the multiplicative model with the production function having decreasing returns to scale, and in the additive model with $\gamma < 1$, in order to have the profit maximized, the biodiversity factor has to strongly contribute to the production, and the pollution human factor must be the least productive. In the multiplicative model with the production function having increasing returns to scale, the biodiversity, although contributing less, remains the most productive factor. The pollution human factor still has to be the least productive.

An interesting economic analysis about labor and pesticides is to verify, if possible, which of the two controls in steady state is the bigger. In the multiplicative model L_{ss} and A_{ss} are:

$$L_{ss_m} = (\beta\mu p)^{\frac{1}{1-\alpha-\beta}} \left(\frac{\alpha}{\beta\omega}\right)^{\frac{1-\beta}{1-\alpha-\beta}} X_{ss}^{\frac{\gamma}{1-\alpha-\beta}} \left(\frac{1}{c+\lambda ss_m}\right)^{\frac{\beta}{1-\alpha-\beta}} \tag{4.1}$$

$$A_{ss_m} = (\beta\mu p)^{\frac{1}{1-\alpha-\beta}} \left(\frac{\alpha}{\beta\omega}\right)^{\frac{\alpha}{1-\alpha-\beta}} X_{ss}^{\frac{\gamma}{1-\alpha-\beta}} \left(\frac{1}{c+\lambda ss_m}\right)^{\frac{1-\alpha}{1-\alpha-\beta}} \tag{4.2}$$

we can simplify and remove the equal terms from the equations, ending up to:

$$\tilde{L}_{ss} = \left(\frac{\alpha}{\beta\omega}\right)^{\frac{1-\beta}{1-\alpha-\beta}} \left(\frac{1}{c+\lambda ss_m}\right)^{\frac{\beta}{1-\alpha-\beta}} \tag{4.3}$$

$$\tilde{A}_{ss} = \left(\frac{\alpha}{\beta\omega}\right)^{\frac{\alpha}{1-\alpha-\beta}} \left(\frac{1}{c+\lambda ss_m}\right)^{\frac{1-\alpha}{1-\alpha-\beta}} \tag{4.4}$$

Equations 4.3 and 4.4 have same bases but different exponents. Consequently, we focus our analysis to the latter. If $\alpha + \beta < 1$, the first term of the right side of the equations will be bigger in L_{ss}, while the second term will be bigger in A_{ss}. Conversely, if $\alpha + \beta > 1$, the first term will be bigger in A_{ss}, while the second term will be bigger in L_{ss}. So, we can say that there is not a control that turns out to be greater than the other in any case.

In the additive model, the two controls are:

$$L_{ss_a} = \frac{p\mu\alpha}{\omega} \left(\beta(p\mu)^{\frac{1}{1-\alpha}} \left(\frac{\alpha}{\omega}\right)^{\frac{\alpha}{1-\alpha}}\right)^{\frac{1}{1-\alpha-\beta}} \left(\frac{1}{c+\lambda_{ss_a}}\right)^{\frac{1}{1-\alpha-\beta}} \tag{4.5}$$

$$A_{ss_a} = \beta(p\mu)^{\frac{1}{1-\alpha-\beta}} \left(\frac{\alpha}{\omega}\right)^{\frac{\alpha}{1-\alpha-\beta}} \left(\frac{1}{c+\lambda_{ss_a}}\right)^{\frac{1-\alpha}{1-\alpha-\beta}} \tag{4.6}$$

The analysis leads to the same result of the one in the multiplicative model, but it has not been reported in this thesis because is longer and does not add anything to our observations.

It would be interesting to compare the controls between the two models, trying to identify which one has the controls with the highest value. Unfortunately, we cannot make this analysis because it has not been possible in the multiplicative model to make explicit λ_{ss_m} as a function of X_{ss_m}.

4.3 Policy

According to the solutions, the optimal policy to reach the highest profit are the following.

In the additive model, when the biodiversity level is $X < x_2$, it is not convenient to invest in the land, letting the natural resource grow (according to 3.15). When $X > x_2$, the system tends over time towards the stable equilibrium point, that is the optimal solution of the problem, and the farmer will have the maximum profit.

In the multiplicative model, any quantity of biodiversity above $X = 0$, will converge to the optimal equilibrium. The optimal policy in this case is not whether and when starting making effort, but which land has the parameters that respect the following constraints: production function with increasing returns to scale and $\alpha > \beta$.

4.4 Logistic function

The question we want to try to answer is: does the logistic function represent the evolution of biodiversity well?

The logistic function is widely used in the field of resource economics. Resource economists study the natural environment through the population ecology: the focus is on the population and its size. A peculiar example is the models of fishery management, an harvesting problem where the fish population is set to follow the trend of a logistic function. Here, the objective is to find the optimal rate at which the fish population is harvested. The profit from harvest depends on both the harvesting rate and the population size. The higher the harvesting rate, i.e. the greater the quantity of fish offered, the lower the price per unit of fish. The population size affects the profit because the harvesting costs per unit are usually low when stocks are large and vice versa. Therefore, in resource economics, it is essential having a function as the logistic depending on the population size, that can be easily modified to take into consideration the rate at which the population is harvested (Perrings, 1995).

The biodiversity however is not the number of individuals within a species, but the variety of life in one or more regions or in a ecosystem (Gaston, 2004). Biodiversity plays an important role in environmental economics. Environmental economists base their studies on ecosystem ecology. Here, the objective is to value services that the ecosystems can provide, to the economy and to society. A good example to mention is the utilitarian value of biodiversity to pharmaceutical purposes. The genetic differentiation of the ecosystem is seen as a diverse portfolio of assets useful for making pharmaceuticals (assets with uncertain payoffs, depending on the technology that we currently own) (Perrings, 1995).

Population and ecosystem ecology differ by the state variables taken to characterize the system. In the first one, the typical state variable is the population size. In the second one, the state variable are usually indices regarding the "quality" of the ecosystem or the service provided (Perrings, 1995).

An index to measure the biodiversity is the *Simpson index*. It measures which species is prevalent in a ecosystem, measuring the probability that two individuals randomly chosen belong to the same species.

$$D = \sum \left(\frac{n_i}{N} \right)^2$$

where n is the number of individuals of the i-th species; N is the number of the individuals of the all species. The index goes from zero to one. $D = 0$ means that the biodiversity is at its maximum level, $D = 1$ means no diversity in the species (Simpson, 1949).

Regarding the models we analyzed, the intent was to build a model following the path of the Ramsey problem. The logistic function fits very well as the equation of motion. Moreover, population and ecosystem ecology are strongly interconnected. Certain ecological processes are dependent on the number of individuals of some specific species, so ecosystem ecology can take into account the population structures and their interaction among other species. At the same time, the environmental pollution or degradation can be quantified by the damage suffered by a species, so the dynamics of a population can be representative of the quality of an environment.

In conclusion, a logistic function that represents the biodiversity level can be consider a good approximation of the complexity of the reality. As mention above, a possible interpretation of the equation of motion used in the multiplicative and additive models is to consider the logistic function as the quantity index of a representative population that, with its path and population, describes and approximates well the quality of the biodiversity.

4.5 Externalities

A particular case proposed in the additive model (Viscovich, 2020) considers the exponent $\gamma = 1$. The constant γ represents the elasticity of output with respect to biodiversity. From an economic point of view, it is a strong assumption. The constant $\gamma = 1$ leads every additional quantity of biodiversity to increase the output in the same amount. However, we can try to give an interpretation of the model.

To do so, we briefly present the resolution of the model considering $\gamma = 1$. The aim is to locate the equilibrium points and the related profit. With $\gamma = 1$, the solutions can be collected into three scenario according to three different assumptions on parameters:

1. case with $g < \delta$;
2. case with $\delta < g < \frac{bp}{c} + \delta$;
3. case with $g > \delta + \frac{bp}{c}$.

We do not consider the first and the third cases as economically relevant, because neither of them fit the reality. In the first one, if we have a initial level of biodiversity below a certain threshold (i.e. an unstable equilibrium, close to zero), the system moves towards $-\infty$. In the third case, if the initial condition is

$$X_0 < \frac{(g-\delta)K}{2g},$$

the system moves to $-\infty$ because of the presence of an asymptote. We consider economically interesting only the second case, because is the only one that respects the constraint $g > \delta$ and the path of the system is well defined, for any X_0. In this case, two equilibrium points are found:

- the first one is an unstable equilibrium in $X = 0$;
- the second one is a stable equilibrium close to the carrying capacity K.

The system evolves as follows: for any level of biodiversity greater than zero, the system moves to the second equilibrium, otherwise it remains in zero.

We now present the application of the concept of externality on these results.

An important property of the model where the biodiversity is added to the production function is that we can separate the contributions

$$p\mu L(t)^{\alpha} A(t)^{\beta} \quad \text{and} \quad pbX(t).$$

We can consider these two parts of the production function as two separate sources of profit. The first one is the production function of the farmer. The second one is a source of profit due to biodiversity: the biodiversity level is multiplied by the price and the total productivity factor, b.

Taking into account only $pbX(t)$, we can end up to the following observations: the higher the level of biodiversity the higher the profit; the biodiversity input has constant marginal returns: any marginal increase of biodiversity increases the profit by the same amount; the function reaches its maximum contribution to the profit when no pesticides are used and the level of biodiversity is K (carrying capacity of the environment): the less the farmer pollutes the higher the output of the function (depending on X).

Considering the observations above, a possible interpretation of pbX is to consider it as a subsidy (Balestrino, 2015). It would be given in order to preserve biodiversity. A subsidy, in this case, would have the aim to add an external cost to the farmer's profit function, i.e. the reduction of the biodiversity level due to the use of pesticides.

This external cost is a negative externality, affecting the whole society. The society faces a further cost that has not chosen to have.

The profit function of the farmer can be rewritten as,

$$p(\mu L(t)^{\alpha} A(t)^{\beta}) - \omega L(t) - cA(t) + pbX(t).$$

The first part is the production function, and the second is the cost function. The third part can be seen as a subsidy, given to the farmer.

We take into consideration a land with no previous human intervention, so the level of biodiversity is K. Initially, the farmer's production function is

$$p\mu L(t)^{\alpha} A(t)^{\beta}$$

subject to the budget constraint. The motion equation, influenced by the pesticides, still exists,

$$\dot{X}(t) = gX(t)\left(1 - \frac{X(t)}{K}\right) - A(t),$$

but the farmer's profit does not change. The farmer is free to damage the biodiversity without repercussion. The use of A and L depends only on the respective costs and elasticity of output with respect to inputs.

We assume that the following contract is proposed: a subsidy will be given to the farmer if she reduces the use of A. The subsidy depends on the level of biodiversity, weighted by b and p.

The subsidy depends on the price of the final product. This condition avoids that the choice between the profit from subsidy and the production depends on the fluctuation of the market. Without being weighted by p, the subsidy would become less convenient as the output price increase.

After the contract, it is in the farmer's interests to keep the level of biodiversity high, in order to receive the subsidy. The farmer faces a trade-off between using pesticides and preserving biodiversity.

We assume that the externality is Pareto-relevant, i.e. the production may be modified in such a way that who faces the externality is better off, and the farmer is not made worse off. Due to the fact that the farmer can refuse the contract, we are sure that the farmer cannot be worse off. For the affected party, that could be an organization who cares about the environment, we assume that it has the willingness to pay till the maximum amount of subsidy, that is to pay pbK. Considering so, we are assuming that all the externality caused by the farmer turns out to be Pareto-relevant.

According to the literature (Buchanan and Stubblebine, 1962), the idea is that the choice of reducing the use of pesticides should be voluntary. So, the externality would be mitigated by "gains from trade": the affected party, i.e. an organization who cares about the environment, in order to reduce the externality, gives to the active party, that is the farmer, an amount of money that should be equal to farmer's utility lost for having reduced the production. In this analysis, we do not consider the intervention of coercive power. Being free to choose to accept the contract or not, the farmer will accept it only if she will have at least the same welfare as in the case of no contract. The cost of reducing the externality is all paid by the player that gives the subsidy, and consequently, by those who finance the organization. Actually, a contract would not even be necessary. The farmer has just to know that some organization has the intention to give her a subsidy. The farmer chooses then to care about biodiversity or not, freely. The organization sees the amount of biodiversity present in the land and gives the subsidy to the farmer. Such a subsidy, related to the current biodiversity, can also be given in the following period. In that case the profit function would be:

$$p(\mu L(t)^{\alpha} A(t)^{\beta}) - \omega L(t) - cA(t) + pbX(t-1)$$

and the problem would turn out to be an age dependent optimal control problem.

The ones who finance the organization should be all the economic agents that have the interest to protect the biodiversity. So, potentially, every citizen.

Referring to the analysis of the equilibrium points written above, if we assume a model respecting the constraints $\delta < g < \frac{bp}{c} + \delta$, the "*subsidy policy*" obtains the result wanted by the organization: for any initial level of biodiversity different from zero, the system will move to an equilibrium close to the carrying capacity of the population, with a low environmental damage. So, the farmer's optimal strategy is to preserve the biodiversity and get the subsidy. Note that in this case, the damage to the biodiversity is not zero. The farmer will maximize her utility freely and, probably will use some level of A.

The initial level of biodiversity can be at any point between zero and K, zero excluded. In fact, at that point biodiversity cannot grow anymore. A problem arises: the idea of offering the contract to farmers could probably work only with not previously cultivated lands or cultivated for a short period of time. Without a subsidy and with the production function equal to $p\mu L(t)^{\alpha} A(t)^{\beta}$, the farmer has no reason to care about biodiversity. If the budget constraint of the farmer is sufficiently high, there would be the risk that the level of biodiversity goes to zero, where all the biodiversity is extinct and consequently the policy of the subsidy would be useless.

Note that the rigorous analysis explained in chapter 1 cannot be applied in this context, because the analyzed models do not contain the utility function of the people who benefit from biodiversity. However, knowing the importance to preserve the biodiversity, I think that this *subsidy policy* depending on X is a good trade-off between rigorousness and feasibility.

Conclusions

This thesis deals with a problem of environmental economics. The aim is to give some economic interpretations to an optimal control problem, called *the farmer's problem*: a farmer needs to find the optimal allocation of input factors in order to maximize her profit, in an infinite horizon of time. The problem is formulated in two models. The two differ according to how biodiversity contributes to production: it is added to the production function in the *additive model*, and it is multiplied in the *multiplicative model*.

The mathematical resolution of the optimal control problems has been performed by two mathematics graduands of the University of Padua: Matteo Pozzan and Valentina Viscovich. The *Pontryagin Maximum Principle* (Seierstad, 1987) has been applied to find the candidate solutions of the problem. Since strongly non-linear equations are obtained from the principle, the problem has been resolved by finding the equilibria of the motion equation and the equilibrium points of the system of differential equations that characterize the model optimal solutions.

From all the equilibrium points and the related system paths found, we excluded the ones that we considered not economically relevant, i.e. the ones with zero profit or with a biodiversity path inconsistent with reality (such as an asymptote in the function).

The economically relevant equilibrium points found in the analysis are:

- In the multiplicative model:
  - an equilibrium point with $X > 0$;
- In the additive model:
  - an unstable equilibrium in 0 (relevant because of the positive profit);
  - a stable equilibrium close to K.

The related optimal paths of the systems are similar in both models:

- if the initial level of biodiversity is $X_0 = 0$, the system does not move and remains in X = 0;
- if the initial level of biodiversity is $X_0 > 0$, the system will converge to the positive equilibrium (that in the additive model we know it is close to the carrying capacity K).

According to the solutions, the optimal policy to reach the highest profit are the following. In the additive model, when the biodiversity level is sufficiently low ($X < x_2$), it is not convenient to invest in the land, letting the natural resource grow (see 3.15). On the other hand, when the biodiversity level is sufficiently high ($X > x_2$), the system tends over time towards the stable equilibrium point and the farmer will have maximum profit. In the multiplicative model, any positive quantity of biodiversity will converge to the optimal equilibrium. The optimal policy in this case is not whether and when starting making effort, but which land has the parameters that respect the following constraints: the output elasticity with respect to labor bigger than the output elasticity with respect to pesticides ($\alpha > \beta$) and production function with increasing returns to scale.

So, both models tell us that the optimal policy for higher profit is not using a large quantity of agro-chemicals, but rather respecting the environment caring about the health of biodiversity. In particular, in the additive model the non degenerate equilibrium point is close to the carrying capacity, so the damage to biodiversity is low.

The analyzed models consider the biodiversity strictly as an input. They do not take into consideration other economic relevance of the biodiversity. But the farmer, with her production, can for example affect the production function of another farmer close to her, or just pollute excessively for some economic agent having biodiversity in their utility function. To overcome this problem, and integrate the analysis we introduced the concept of externality and we tried to insert it in a case of the model. The additive model with $\gamma = 1$ lent itself well for that because its production function could be split in two parts, and doing so, it was possible to consider one of the parts as a subsidy given by a third party. This third party who cares about biodiversity could be an organization that has recognized the biodiversity as a public good, i.e. a good that is non-rivalrous and non-excludable, and that generates some benefit to someone that does not have to pay for enjoying it (Balestrino, 2015). The following scenario was considered: the farmer produces using labor and pesticides, but the pesticides damages the biodiversity. The farmer is free to damage the biodiversity without any income repercussion. This condition would probably lead to a high environmental damage and a low level of biodiversity in the field. To avoid this, a subsidy can be given the farmer, depending on the level of biodiversity present in the soil. According to the resolution of the additive model with $\gamma = 1$, the subsidy policy works: the farmer will decide to keep the biodiversity level high in order to receive a high subsidy, and the damage to the society would be low.

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