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Tropical Theta functions for tropical abelian varieties

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1 Introduction

In this master thesis we study the real tori $\mathbb{R}^n/\Lambda$ endowed with a tropical structure, namely with an atlas of $\mathbb{Z}$-affine linear functions. The aim will be to prove, in analogy with the theory of complex abelian varieties, a result that gives conditions for a tropical torus to be embedded in the tropical projective space. The main tool for this will be the study of tropical theta functions, namely the global sections of tropical line bundles.
In section 2 we outline the case of polarized abelian varieties in the complex setting, which will serve as a motivation for the further sections: we start considering a complex torus $X = V/\Lambda$, where V is a complex vector space of dimension n and Λ a full rank lattice in V, then we define its Picard group $Pic(X)$, which is the group of its holomorphic line bundles up to isomorphisms. To an element L in $Pic(X)$ we can link a hermitian form H on V which is called the first Chern class of L, and its imaginary part E is an alternating form on V. Then the following theorem holds:

Theorem 1.1 (Elementary divisor theorem)**.** *Let A be a commutative principal ring and F a finite free A-module of rank m, let E be an alternating bilinear form on F. Then there is a base $\{e_1, \ldots, e_m\}$ of F and an even integer $r \leq m$ such that:*

$$E(e_1, e_2) = d_1, \ldots, E(e_{r-1}, e_r) = d_{r/2},$$

where the d_i are unique non-zero elements of A such that $d_i | d_{i+1}$ for every $1 \leq i \leq r/2 - 1$. Moreover, all other elements $E(e_i, e_j)$ with $i \leq j$ are zero.

This implies that there is a basis of Λ for which E has the form:

$$\begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix},$$

where D is the diagonal matrix $\mathrm{diag}(d_1, \ldots, d_n)$ with integer eigenvalues and $d_i | d_{i+1}$ for every $i = 1, \ldots, n-1$. We will call the tuple $(d_1, \ldots, d_n)$ the type of the line bundle L.
Using the global sections of L we can also define a rational function:

$$\phi_L : X \dashrightarrow \mathbb{P}^N,$$

then we will see conditions on the integer d_1 defined above for this function to be defined on the whole X, and to be an embedding, this last result is called Lefschetz's theorem.
In section 3 we switch to the tropical case, and begin defining the tropical semifield:

Definition 1.2. The *tropical semifield* $\mathbb{T}$ is the set $\mathbb{R} \cup \{-\infty\}$ with the operations of sum: "$x + y$" $= \max\{x, y\}$, and product "$x \cdot y$" $= x + y$. We use the convention that $\max\{x, -\infty\} = x$ and $x + (-\infty) = -\infty$ for every x in $\mathbb{T}$.

We are interested in defining the tropical analog of the notions of the complex case, in particular we will consider tropical manifolds, defined in the following way:

Definition 1.3. We call *tropical manifold* a manifold M with an atlas $\{(U_i, \phi_i)\}_i$ such that $\phi_i \circ \phi_j^{-1}$ is $\mathbb{Z}$-affine linear on $\mathbb{R}^n$ for every $U_i \cap U_j$ where n is the local dimension of M in $U_i \cap U_j$. A map $f : M_1 \to M_2$ between two tropical manifolds is called tropical if locally it can be written as a $\mathbb{Z}$-affine linear map.

Examples of tropical manifolds are real tori, that will be the object of the study of the rest of the thesis.
In section 4 we study the space of tropical theta functions, i.e. the global sections of tropical line bundles. The main difference with the complex case is that the first Chern classes of tropical line bundles are identified with symmetric bilinear forms on $\mathbb{R}^n$, hence the Elementary Divisors Theorem doesn't hold. We will see that a symmetric bilinear form with at least one negative eigenvalue gives rise to a line bundle with trivial set of global sections, therefore the strategy will be to distinguish the cases in which these forms are positive definite or positive semidefinite. We will see that in the positive definite case the space of theta functions can be identified with a convex finite dimensional polyhedron in $\mathbb{T}^N$ for some positive integer N.
In section 5.1 we will fix a positive definite line bundle L on a tropical torus X, similarly to the complex case, we are able to define a function:

$$\varphi_L : X \longrightarrow \mathbb{PT}^N$$

where the tropical projective space $\mathbb{PT}^N$ plays the role of the complex projective space. The advantage in the tropical setting is that this function is always well-defined, but the fact that the Elementary Divisors Theorem doesn't hold implies that to obtain the tropical analog of Lefschetz's theorem we have to add the hypothesis that there is a basis for the lattice Λ and a basis of $\mathbb{Z}^n$ with respect to which the positive definite bilinear form linked to the line bundle L has the form $\mathrm{diag}(d_1, \ldots, d_n)$, where $d_i | d_{i+1}$ for every $i = 1, \ldots, n-1$. The result of this is the following theorem:

Theorem 1.4. *Let X be an n-dimensional tropical torus and $L = L(Q, \alpha)$ a positive definite line bundle on X, suppose that there is a basis of Λ and a* *basis of $\mathbb{Z}^n$ such that the bilinear form Q is represented by a diagonal matrix*

$$\begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{pmatrix},$$

such that $d_1 | d_i$ for every $i = 1, \dots, n$, and $d_1 \geq 3$. Then the map φ_L is continuous and injective.

2 The complex abelian varieties case

The results of this thesis are mainly motivated by the theory of complex abelian varieties, more specifically by the theory of projective embeddings of polarized abelian varieties, in this preliminary section we summarise some of its results, which can be found in [1]. In the classical setting, let V denote a complex vector space of dimension g and Λ a lattice in V, i.e. a discrete subgroup which is also a free $\mathbb{Z}$-module of rank $2g$. The quotient $X = V/\Lambda$ is a *complex torus*. Since Λ is a discrete subgroup of V of full rank, we have that X is the image via the projection of a compact subset of V, and hence is compact. Moreover, by [1, Corollary A.7] it is a connected complex manifold.

Definition 2.1. Let X be a complex torus, we define a *holomorphic line bundle* to be the datum of a triple $(L, \pi, \{(U_i, \phi_i)\}_i)$, where L is a complex manifold, $\pi : L \to X$ is a surjective continuous map, $\{U_i\}_i$ is an open covering of X and $\phi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}$ are holomorphic maps such that the following properties hold:

1. For every open set U_i, denoting with $p_i : U_i \times \mathbb{C} \to U_i$ the first projection, we have that the following diagram commutes:

$$\begin{array}{ccc} \pi^{-1}(U_i) & \xrightarrow{\phi_i} & U_i \times \mathbb{C} \\ & \searrow^{\pi} & \downarrow p_i \\ & & U_i \end{array} .$$

2. For every open sets U_i, U_j there exists a holomorphic function $g_{ij} : U_i \cap U_j \to \mathbb{C} \setminus \{0\}$ such that $(\phi_j \circ \phi_i^{-1})(x, z) = (x, z \cdot g_{ij}(x))$ for every x in $U_i \cap U_j$ and every complex number z.

We call the maps ϕ_i *trivializations* and the maps g_{ij} *transition functions*.

The transition functions are holomorphic, nonvanishing, and satisfy the relations:

$$\begin{cases} g_{ij}g_{ji} = 1 \\ g_{ij}g_{jw}g_{wi} = 1 \end{cases} .$$

It turns out that, given a complex torus X, the datum of a set of functions $\{g_{ij}\}_{ij}$ together with an open covering $\{U_i\}_i$ satisfying these properties is sufficient to define a line bundle on X. As a consequence, we define a product on the set of holomorphic line bundles on X in the following way: suppose that L, L' are holomorphic line bundles given respectively by the transition functions $\{g_{ij}\}_{ij}$ and $\{f_{ij}\}_{ij}$ with respect to the same open covering (this can always be done by taking a common refinement), then the product $L \otimes L'$ is given by $\{g_{ij} f_{ij}\}_{ij}$. This product gives the set of line bundles the structure of an abelian group.

Definition 2.2. Let X be a complex torus, and let $(L, \pi, \{(U_i, \phi_i)\}_i)$ and $(L', \pi', \{(U_i, \phi'_i)\}_i)$ be two holomorphic line bundles on X. We say that L and L' are isomorphic if there exists a continuous function $f : L \to L'$ such that the following diagram commutes:

$$\begin{array}{ccc} L & \xrightarrow{f} & L' \\ & \searrow^{\pi} & \downarrow{\scriptstyle \pi'} \\ & & X \end{array},$$

moreover we request that for every i, j there is a holomorphic function $h_{ij} : U_i \cap U_j \to \mathbb{C} \setminus \{0\}$ such that the composite:

$$\phi'_j \circ f \circ \phi_i^{-1} : U_i \cap U_j \times \mathbb{C} \longrightarrow U_i \cap U_j \times \mathbb{C}$$

is given, for every (x, z) in $U_i \cap U_j \times \mathbb{C}$, by

$$(\phi'_j \circ f \circ \phi_i^{-1})(x, z) = (x, z h_{ij}(x)).$$

We call *Picard group* the set of line bundles on X up to isomorphisms and denote it with: $Pic(X)$.

Definition 2.3. Let $(L, \pi, \{(U_i, \phi_i\}_i)$ be a holomorphic line bundle on a complex torus $X = V/\Lambda$, and let $U \subseteq X$ be an open set. A *section* of L on U is a function:

$$s : U \longrightarrow \pi^{-1}(U),$$

such that $\pi \circ s$ is the identity map on U and $p_i \circ \phi_i \circ s_{|U_i \cap U}$ is a holomorphic function for every i. If $U = X$ we call s *global section*, and we denote with $H^0(L)$ the set of global sections of L.

It turns out that the space $H^0(L)$ of global sections is a $\mathbb{C}$-vector space of finite dimension. One can show that there is a base of $H^0(L)$ given by *theta functions*, i.e. holomorphic functions on V that are invariant under the action of Λ up to a multiplication for a common factor.

Definition 2.4. Let X and Y be complex tori and f a holomorphic function

$$f : X \longrightarrow Y.$$

Let L be a line bundle in $Pic(Y)$ given by the set of transition functions $\{g_{ij}\}_{ij}$ and the open covering $\{U_i\}_i$, we define its pullback $f^*(L)$ to be the line bundle defined by the set $\{g_{ij} \circ f\}_{ij}$ and the covering $\{f^{-1}(U_i)\}_i$.

Denoting with $\mathcal{O}_X$ the sheaf of holomorphic functions on X, and with $\mathcal{O}_X^*$ its subsheaf of nowhere zero functions, we have that $Pic(X)$ is naturally identified with the first cohomology group $H^1(X, \mathcal{O}_X^*)$. For any open subset $U \subseteq X$ of a complex torus, denote with $\mathbb{Z}(U)$ the group of locally constant functions on U with integer values, then recall that there is a short exact sequence of sheaves:

$$0 \to \mathbb{Z} \xrightarrow{i} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 1, \tag{1}$$

where i is the inclusion and $\exp(f) = e^{2\pi i f}$. Consider the following part of its long cohomology sequence:

$$H^1(X, \mathbb{Z}) \to H^1(\mathcal{O}_X) \to H^1(\mathcal{O}_X^*) \xrightarrow{c_1} H^2(X, \mathbb{Z}),$$

we call the map c_1 *first Chern class map*. The following lemma yields a generalization of Künneth's formula to n factors, and it can be proven just by using it.

Lemma 2.5. *The canonical map from the n-th exterior algebra over $H^1(X, \mathbb{Z})$ to $H^n(X, \mathbb{Z})$ induced by the cup product:*

$$\bigwedge^n H^1(X, \mathbb{Z}) \longrightarrow H^n(X, \mathbb{Z})$$

is an isomorphism for every $n \geq 0$.

Since the group $H^1(X, \mathbb{Z})$ can be identified via universal coefficient theorem with the group of homomorphisms $\mathrm{Hom}(H_1(X, \mathbb{Z}), \mathbb{Z}) = \mathrm{Hom}(\Lambda, \mathbb{Z})$, it follows that there is a canonical isomorphism

$$H^n(X, \mathbb{Z}) \cong Alt_n(\Lambda, \mathbb{Z}),$$

where $Alt_n(\Lambda, \mathbb{Z})$ is the group of alternating n-forms on Λ. As a consequence, we can identify the first Chern class $c_1(L)$ of a line bundle on X with an alternating $\mathbb{Z}$-valued 2-form on Λ.
Via an $\mathbb{R}$-linear extension we can identify a class $c_1(L)$ with an alternating form $E : V \times V \to \mathbb{R}$. Conversely we can determine which alternating forms come from line bundles in the following way:

Lemma 2.6. *Let $X = V/\Lambda$ be a complex torus and E an alternating form $E : V \times V \to \mathbb{R}$, then the following are equivalent:*

i) There is a holomorphic line bundle L in $Pic(X)$ such that E represents $c_1(L)$.

ii) $E(\Lambda, \Lambda) \subseteq \mathbb{Z}$ and $E(iv, iw) = E(v, w)$ for all v, w in V.

Recall that a hermitian form is a map $H : V \times V \to \mathbb{C}$ that is $\mathbb{C}$-linear in the first argument and such that:

$$H(v, w) = \overline{H(w, v)},$$

for every v, w in V. We have that alternating forms satisfying the second part of the lemma above are actually the imaginary parts of hermitian forms, we summarise this in the following lemma:

Lemma 2.7. *There is a one-to-one correspondence between alternating forms E on V such that $E(iv, iw) = E(v, w)$ for every v, w in V and hermitian forms H on V such that $\Im(H(\Lambda, \Lambda)) \subseteq \mathbb{Z}$. This correspondence is given by:*

$$E(v, w) = \Im(H(v, w)) \quad H(v, w) = E(iv, w) + iE(v, w),$$

where $\Im$ denotes the imaginary part.

Definition 2.8. Given a complex torus $X = V/\Lambda$, we define the Néron-Severi group $NS(X)$ to be the group of hermitian forms H on V such that $\Im(H(\Lambda, \Lambda)) \subseteq \mathbb{Z}$.

Notice that by *Lemma 2.7* and *Lemma 2.6* there is a canonical isomorphism between $NS(X)$ and the image of c_1.

Definition 2.9. A *semicharacter* χ for an hermitian form H in $NS(X)$ is a map

$$\chi : \Lambda \longrightarrow \{z \in \mathbb{C} | z\overline{z} = 1\}$$

such that for every λ, μ in Λ:

$$\chi(\lambda + \mu) = \chi(\lambda)\chi(\mu)\exp(\pi i \Im(H(\lambda, \mu))).$$

We denote with $\mathcal{P}(\Lambda)$ the set of pairs (H, χ) where $H \in NS(X)$ and χ is a semicharacter for H. $\mathcal{P}(\Lambda)$ is a group with structure given by the product: $(H, \chi) \cdot (H', \chi') = (H + H', \chi\chi')$.

Now, given a complex torus $X = V/\Lambda$ we will define a function

$$L(\cdot,\cdot) : \mathcal{P}(\Lambda) \longrightarrow Pic(X)$$

in the following way: for a pair (H,χ) in $\mathcal{P}(\Lambda)$, define a function:

$$a := a_{(H,\chi)} : \Lambda \times V \longrightarrow \mathbb{C} \setminus \{0\}$$

by setting

$$a(\lambda, v) := \chi(\lambda) \exp\Big(\pi H(v,\lambda) + \frac{\pi}{2} H(\lambda,\lambda)\Big).$$

The function a is called *canonical factor* for $L(H,\chi)$, we have that theta functions θ for L are defined using the following quasi-periodicity property for every x in X and λ in Λ:

$$\theta(x+\lambda) = a(\lambda, z)\theta(x).$$

We define the holomorphic line bundle $L(H,\chi)$ to be given by the quotient:

$$L(H,\chi) = (V \times \mathbb{C})/\Lambda,$$

where Λ acts on $V \times \mathbb{C}$ by:

$$\lambda \cdot (v,z) := (v+\lambda, a_{(H,\chi)}(\lambda, v)z).$$

One can prove that the axioms for a group action are satisfied, moreover considering the natural projection

$$(V \times \mathbb{C})/\Lambda \to X$$

we have that its fibers are copies of $\mathbb{C}$. This map defines an isomorphism:

$$\mathcal{P}(\Lambda) \longrightarrow Pic(X),$$

for the details see: [1, Section 2.2]. One can also notice that $c_1(L(H,\chi)) = H$.
Now, let v be an element of X, we can define an isomorphism $t_v : X \to X$ of X by setting:

$$t_v(x) = x + v.$$

The following lemma gives a way to compute the pullback of a line bundle via a translation t_v in terms of the map $L(\cdot,\cdot)$:

Lemma 2.10. *Let X be a complex torus and (H,χ) an element of $\mathcal{P}(\Lambda)$, then for every v in X:*

$$t_v^*(L(H,\chi)) = L(H, \chi \exp(2\pi i \Im(H(v,\cdot)))).$$

As a consequence we find the following:

Theorem 2.11 (Theorem of the Square)**.** *Let X be a complex torus, for every v, w in X and for every L in $Pic(X)$ we have that:*

$$t_{v+w}^* L = t_v^* L \otimes t_w^* L \otimes L^{-1}.$$

Now, let X be a complex torus and L an element of $Pic(X)$, denote with H the first Chern class of L and with E its imaginary part. From [2, Section 5.1] we get the following theorem:

Theorem 2.12 (Elementary divisor theorem)**.** *Let A be a commutative principal ring and F a finite free A-module of rank m, let G be an alternating bilinear form on F. Then there is a base $\{e_1, \ldots, e_m\}$ of F and an even integer $r \leq m$ such that:*

$$G(e_1, e_2) = d_1, \ldots, G(e_{r-1}, e_r) = d_{r/2},$$

where the d_i are unique non-zero elements of A such that $d_i | d_{i+1}$ for every $1 \leq i \leq r/2 - 1$. Moreover, all other elements $G(e_i, e_j)$ with $i \leq j$ are zero.

From the theorem above it follows that there is a base of Λ with respect to which E is given by the matrix:

$$\begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix},$$

where D is the diagonal matrix $\mathrm{diag}(d_1, \ldots, d_n)$ where the d_i are integers such that $d_i | d_{i+1}$ for every $1 \leq i \leq n-1$. We call the tuple $(d_1, \ldots, d_n)$ the *type* of L.

Definition 2.13. We say that a holomorphic line bundle $L = L(H,\chi)$ is *positive definite* (respectively *positive semidefinite*) if H is positive definite (respectively positive semidefinite). If $L = L(H,\chi)$ is a positive definite line bundle, we call its first Chern class H *polarization*, and the pair (X, L) *polarized abelian variety*.

Suppose that $\{e_1, \ldots, e_{2n}\}$ is the base of Λ obtained with the Elementary Divisors Theorem, then if

$$ie_1 = e_2, \ldots, ie_{2n-1} = e_{2n},$$

from the equality:

$$H(e_i, e_i) = E(ie_i, e_i)$$

it follows that H is positive definite if and only if all the d_i are positive. However, this is not always true, and it will be a difference with the tropical setting.

Fixing a base $\sigma_0, \ldots, \sigma_N$ for $H^0(L)$, we obtain a rational map

$$\phi_L : X \dashrightarrow \mathbb{P}^N$$

defined by:

$$\phi_L(x) := (\sigma_0(x) : \cdots : \sigma_N(x)),$$

whenever there is an index i for which $\sigma_i(x) \neq 0$.

Definition 2.14. A line bundle L in $Pic(X)$ is said to be *very ample* if the map ϕ_L is an embedding, it is said *ample* if there is an $n \geq 1$ such that L^n is very ample.

The following lemma gives a condition for which ϕ_L is defined on the whole torus.

Lemma 2.15. *Let L be a positive definite line bundle on X of type $(d_1, \ldots, d_n)$, with $d_1 \geq 2$, then the map ϕ_L defined above is holomorphic.*

Now, recall that $|L|$ indicates the set of effective divisors on X that are linearly equivalent to L, using again *Lemma 2.10* we can prove the following:

Lemma 2.16. *Take n elements $v_1, \ldots, v_n$ in X such that $\sum_{i=1}^n v_i = 0$ and a holomorphic line bundle L on X, then:*

$$\bigotimes_{i=1}^{n} t_{v_i}^* L \cong L^n.$$

The question we ask at this point is when the function ϕ_L defines an embedding of the complex torus in the projective space, to answer we first need another technical lemma:

Lemma 2.17. *Let L be a positive definite line bundle in $Pic(X)$, then there is an open dense set U in $|L|$ such that for every divisor D in U:*

$$t_x^* D = D \iff x = 0.$$

Finally, we can state the following theorem:

Theorem 2.18 (Lefschetz's theorem)**.** *Let L be a positive definite line bundle in $Pic(X)$ of type $(d_1, \ldots, d_n)$ with $d_1 \geq 3$, then $\phi_L : X \to \mathbb{P}^N$ is an embedding.*

The proof of Lefschetz's theorem involves the results stated before, and as a direct consequence we get the following lemma:

Lemma 2.19. *For a line bundle L on X, the following statements are equivalent:*

1. *L is ample.*
2. *L is positive definite.*

3 The tropical semifield

3.1 Tropical manifolds

Definition 3.1. The *tropical semifield* $\mathbb{T}$ is the set $\mathbb{R} \cup \{-\infty\}$ with the operations of sum: "$x + y$" $= \max\{x, y\}$, and product "$x \cdot y$" $= x + y$. We use the convention that $\max\{x, -\infty\} = x$ and $x + (-\infty) = -\infty$ for every x in $\mathbb{T}$.

In the tropical semifield the element $-\infty$ is the additive unit, while 0 is the unit for the product, indeed we get a commutative monoid for addition. Notice that no element (except $-\infty$) has additive inverse, while every real number x has multiplicative inverse $-x$, hence we have that $\mathbb{T}^\times = \mathbb{R}$. We will consider on $\mathbb{T}$ the order topology, i.e. the topology generated by subsets of the form $\{x \in \mathbb{T} | x < a\}$ and $\{x \in \mathbb{T} | x > b\}$ for a, b in $\mathbb{R}$; here we use the convention that $-\infty < x$ for every x in $\mathbb{R}$. We give the following:

Definition 3.2. Let $(V, \oplus)$ be a commutative monoid with unit $-\infty$ and a map $\cdot : \mathbb{T} \times V \to V$ such that the following properties hold for every x, y in $\mathbb{T}$ and for every u, v in V:

1. ("$x + y$") $\cdot v = x \cdot v \oplus y \cdot v$.

2. $x \cdot (u \oplus v) = x \cdot u \oplus x \cdot v$.

3. $x \cdot (y \cdot v) = (\text{“}x \cdot y\text{”}) \cdot v$.

4. $0 \cdot v = v$.

5. If $x \cdot v = y \cdot v$ then either $x = y$ or $v = -\infty$.

We say that V is a *tropical module* and $\cdot$ is its *scalar product*.

Given a tropical module V and any element v in V, we have that:

$$1 \cdot (-\infty \cdot v) = (\text{“}1 \cdot -\infty\text{”}) \cdot v = -\infty \cdot v = (\text{“}2 \cdot -\infty\text{”}) \cdot v = 2 \cdot (-\infty \cdot v),$$

hence axiom 5 of *Definition 3.2* implies that $-\infty \cdot v = -\infty$ for every v in V. As an example, the set $\mathbb{T}^n$ with the componentwise scalar product is a tropical module.

Definition 3.3. Let V, W be tropical modules, a function $f : V \to W$ is a *tropical linear function* if $f(v_1 \oplus v_2) = f(v_1) \oplus f(v_2)$ and $f(t \cdot v_1) = t \cdot f(v_1)$ for every t in $\mathbb{T}$ and v_1, v_2 in V. If f has an inverse which is also a tropical linear function, we say that f is an *isomorphism*.

We call a function $f : \mathbb{R}^n \to \mathbb{T}$ a *tropical polynomial* if it is the constant map to $-\infty$ or it has the form $\max_{j \in S}\{a_j + j \cdot x\}$ for a finite subset S of the dual space of $\mathbb{Z}^n_{\geq 0}$, which we denote with $(\mathbb{Z}^n_{\geq 0})^*$. Notice that such function is piecewise-linear, but not all piecewise linear function arise in this way (take as example the real function $x \mapsto -|x|$).
We call f a *tropical Laurent polynomial* if S is a subset of $(\mathbb{Z}^n)^*$ instead. We have that the set of tropical polynomials (resp. tropical Laurent polynomials) on $\mathbb{R}^n$ is a tropical module: indeed for every t in $\mathbb{T}$ and every tropical polynomial $f = \max_{j \in S}\{a_j + j \cdot x\}$ we can define $t \cdot f = \max_{j \in S}\{a_j + t + j \cdot x\}$, and one can check that this defines a scalar product. We call $\mathbb{Z}$-*affine linear* a map between two open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ which is given by the restriction of an affine map whose linear part is represented by an integral matrix.

Definition 3.4. We call *tropical manifold* a manifold M with an atlas $\{(U_i, \phi_i)\}_i$ such that $\phi_i \circ \phi_j^{-1}$ is $\mathbb{Z}$-affine linear on $\mathbb{R}^n$ for every $U_i \cap U_j$ where n is the local dimension of M in $U_i \cap U_j$. A map $f : M_1 \to M_2$ between two tropical manifolds is called tropical if locally it can be written as a $\mathbb{Z}$-affine linear map.

Example 3.5. The set $\mathbb{R}^n$ with the trivial atlas $\{(\mathbb{R}^n, id_{\mathbb{R}^n})\}$ is a tropical manifold.

Example 3.6. Let $\Lambda \subseteq \mathbb{R}^n$ be a lattice, then consider an open covering $\{U_i\}_{i\in I}$ of $\mathbb{R}^n$ for some set I such that for every i, j there exists an element a_{ij} in $\mathbb{R}^n$ such that $U_i = U_j + a_{ij}$. This induces a covering $\{\tilde{U}_i\}_{i\in I}$ of the quotient space $\mathbb{R}^n/\Lambda$ and we can suppose that each $\tilde{U}_i$ is homeomorphic to U_i via the projection map. For every i, fix a map $s_i : \tilde{U}_i \to U_i$ such that s_i is a local section of the projection $\mathbb{R}^n \to \mathbb{R}^n/\Lambda$, then $\{(\tilde{U}_i, s_i)\}_{i\in I}$ is an atlas for $\mathbb{R}^n/\Lambda$ which endows it with a tropical structure.

Definition 3.7. Let M be a tropical manifold and U an open subset of M. A continuous function $f : U \to \mathbb{T}$ is called *regular* if it can be written locally as the restriction of a tropical Laurent polynomial.
A continuous function $h : U \to \mathbb{T}$ is called *rational* if it can be written locally as tropical quotient of two regular functions, i.e. for every x in U, there exists an open neighborhood V of x and two regular functions f, g such that the restriction of h to V is given by "f/g" $= f - g$.
We define the *structure sheaf* O_U of U to be the sheaf of regular functions on U. The sheaf O_U^* is its subsheaf of invertible regular functions.

Example 3.8. Consider the union X of the sides of the square with vertices the four points (0,0), (0,1), (1,1), (1,0) in $\mathbb{R}^2$. Then X is a tropical manifold, consider the function f on X that has value 0 on the top and right edge of the square, and is given by $-1 + x_1 + x_2$ on the bottom and left edge. We have that f is a continuous function, moreover f is locally the restriction of a tropical Laurent polynomial: near (0,1) $f(x, y)$ is given by $x_2 - 1$, while near $(1, 0)$ it's given by $x_1 - 1$. Notice that $-f$ is given by 0 on the top and right edge and by $1 - x_1 - x_2$ on the bottom and left edge, hence $-f$ is regular and it's the tropical inverse of f.

Notice that the sheaf O_U^* is equal to the sheaf of $\mathbb{Z}$-affine linear functions: indeed any $\mathbb{Z}$-affine linear function on $\mathbb{R}^n$ can be written as $x \mapsto x \cdot j + a_j$ for some j in $\mathbb{Z}^n$ and a_j in $\mathbb{R}^n$, therefore its inverse is given by $x \mapsto x \cdot (-j) - a_j$ which is still a $\mathbb{Z}$-affine linear function. Conversely, the inverse of a non-monomial function

$$x \mapsto \max_{j\in S}\{a_j + j \cdot x\},$$

where S is a subset of $(\mathbb{Z}^n)^*$, is given by

$$x \mapsto -\max_{j\in S}\{a_j + j \cdot x\} = \min_{j\in S}\{-a_j - j \cdot x\},$$

which is not a tropical Laurent polynomial.
In the following definition we see how the set $\mathbb{T}^\times = \mathbb{R}$ takes over the role of $\mathbb{C}^\times$ from the complex manifold setting.

3.2 Tropical line bundles

Definition 3.9. Let M be a tropical manifold. A *tropical line bundle* on M is a tuple $(L, \pi, \{(U_i, \Psi_i)\}_i)$, where L is a topological space, $\pi : L \to M$ a continuous surjection, $\{U_i\}_i$ is an open covering of M and $\Psi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{T}$ are homeomorphisms such that the following properties hold:

1. If we denote with p_i the first projection $U_i \times \mathbb{T} \to U_i$, then the following diagram commutes:

$$\begin{array}{ccc} \pi^{-1}(U_i) & \xrightarrow{\Psi_i} & U_i \times \mathbb{T} \\ & \searrow^{\pi} & \downarrow{\scriptstyle p_i} \\ & & U_i \end{array} .$$

2. For every i, j there exists a $\mathbb{Z}$-affine linear function

$$\phi_{ij} : U_i \cap U_j \to \mathbb{R} = \mathbb{T}^\times$$

such that $(\Psi_i \circ \Psi_j^{-1})(x, t) = (x, \text{“}t \cdot \phi_{ij}(x)\text{”})$ for every x in $U_i \cap U_j$.

We call $\{U_i\}_i$ *trivializing cover*, the maps Ψ_i *trivializations* and the maps ϕ_{ij} *transition functions*.

Notice that the transition functions satisfy the cocycle condition “$\phi_{ij} \cdot \phi_{jk}$” $= \phi_{ik}$. One can also check that $\phi_{ij} = -\phi_{ji}$.

Given a tropical manifold M and an open covering $\{U_i\}_i$ with a set of transition functions $\phi_{ij} : U_i \cap U_j \to \mathbb{R}$, we can build a tropical line bundle on M by taking the quotient $\coprod_i U_i \times \mathbb{T}/\sim$ where $(x_i, t_i) \sim (x_j, t_j)$ for x_i in U_i and x_j in U_j if $x_i = x_j$ and $t_j =$ “$t_i \cdot \phi_{ij}(x_i)$”. One can show that a tropical line bundle $(L, \pi, \{(U_i, \Psi_i)\})$ is isomorphic to the line bundle built as stated above starting from the trivializing cover and transition functions of L, so in general we can denote a tropical line bundle up to isomorphisms just by the datum of the trivializing cover and transition functions.
We identify two line bundles $(L, \pi, \{(U_i, \Psi_i)\}_{i \in I})$ and $(L, \pi, \{(U_j, \Psi_j)\}_{j \in J})$, if $(L, \pi, \{(U_i, \Psi_i)\}_{i \in I \cup J})$ is a line bundle.
In this way we can refine the trivializing covering of a line bundle L with trivializing cover $(\{U_i\}_i, \Psi_i)$ by taking a refinement $\{V_j\}_j$ of $\{U_i\}_i$, then the

trivializations Ψ_i induce trivializations Ψ_j on $\{V_j\}_j$ and the new line bundle is identified with the first.
Notice that, given two tropical line bundles on a manifold M, up to taking a refinement we can assume that they have the same trivializing covering, in particular the following definition makes sense for every couple of line bundles on a tropical manifold.

Definition 3.10. Let $(L, \pi, \{(U_i, \Psi_i)\}_i)$ and $(L', \pi', \{(U_i, \Phi_i)\}_i)$ be two tropical line bundles on the tropical manifold M. We say that L and L' are *isomorphic* if there exists a continuous map $f : L \to L'$ and $\mathbb{Z}$-affine linear maps $h_{ij} : U_i \cap U_j \to \mathbb{T}$ such that $\Phi_j \circ f \circ \Psi_i^{-1} : U_i \cap U_j \times \mathbb{T} \to U_i \cap U_j \times \mathbb{T}$ is given by $(x, t) \mapsto (x, \text{“}t \cdot h_{ij}(x)\text{”})$.

Definition 3.11. Let L and L' be two line bundles on a tropical manifold M with transition functions ϕ_{ij} and ψ_{ij}. We define their tensor product to be the line bundle with same trivializing covering and transition functions $\phi_{ij} + \psi_{ij}$, and we denote this by $L \otimes L'$.

Notice that the trivial line bundle (i.e. with transition functions equal to 0) is the identity for the tensor product, and that each tropical line bundle admits an inverse, i.e. the line bundle whose transition functions are the opposite.

Definition 3.12. Given a tropical manifold M, we define $Pic(M)$ to be the abelian group of the tropical line bundles on M up to isomorphisms, with product given by the tensor product.

Definition 3.13. Let $(L, \pi, \{U_i, \Psi_i\}_i)$ be a tropical line bundle on M, then for every open set U of M a function $s : U \to \pi^{-1}(U)$ is a *regular section* on U if $\pi \circ s$ is the identity on U, and $p_i \circ \Psi_i \circ s_{|U_i \cap U}$ is a regular function, where $p_i : U_i \times \mathbb{T} \to \mathbb{T}$ is the second projection. If $U = M$ we say that s is a *global section*. We denote with $\Gamma(L, U)$ the set of regular sections of L on U, and with $H^0(L)$ the set of global sections of L.

For a line bundle $(L, \pi, \{U_i, \Psi_i\}_i)$ on M and an open subset U of M, we have that $\Gamma(L, U)$ is naturally endowed with a tropical module structure: given two of its elements s, s', we first define their sum $s \oplus s'$ in the following way: for every i, the restriction of $s \oplus s'$ to $U \cap U_i$ is given by $\Psi_i^{-1}(u, \text{“}p_i(\Psi_i(s(u)) + p_i(\Psi_i(s'(u))))\text{”})$ for every u in $U \cap U_i$. Indeed, one sees that

$$p_i \circ \Psi_i \circ (s \oplus s')_{|U_i \cap U} = \text{“}p_i \circ \Psi_i \circ s_{|U_i \cap U} + p_i \circ \Psi_i \circ s'_{|U_i \cap U}\text{”}$$

which is regular, moreover $\pi \circ (s \oplus s') = \mathrm{id}_U$ holds by definition. One checks that this operation gives $\Gamma(L, U)$ the structure of a commutative semigroup, with identity given by the identically $-\infty$ section. Moreover, we define the product $\cdot : \mathbb{T} \times \Gamma(L, U)$ by setting for every i and for every x in $\mathbb{T}$, u in $U \cap U_i$ and s in $\Gamma(L, U)$:

$$x \cdot s(u) := \Psi_i^{-1}(u, \text{“}x \cdot p_i(\Psi_i(s(u)))\text{”}),$$

and one checks that it satisfies the axioms of the scalar product. If we denote with $\mathrm{Aff}_{\mathbb{Z}}$ the sheaf of $\mathbb{Z}$-affine linear functions on M, we identify $Pic(M)$ with $H^1(M, \mathrm{Aff}_{\mathbb{Z}}) = H^1(M, O_M^*)$.
Let $f : M \to M'$ be a tropical map and L a line bundle on M' with covering $\{U_i\}_i$ and transition functions $\{\phi_{ij}\}$, we define the pull-back of L to be the line bundle f^*L on M defined by the covering $\{f^{-1}(U_i)\}_i$ and transition functions $\{\phi_{ij} \circ f\}$.

4 Theta functions

In this section we will consider a fixed lattice Λ in $\mathbb{R}^n$ of dimension n, and the tropical torus $X = \mathbb{R}^n/\Lambda$. We denote with $\underline{\mathbb{R}}$ the constant sheaf on X with stalks $\mathbb{R}$. Then $\underline{\mathbb{R}}$ is a subsheaf of $\mathrm{Aff}_{\mathbb{Z}}$, and hence we can consider the quotient sheaf $\mathcal{T}_{\mathbb{Z}}^* \cong \underline{(\mathbb{Z}^n)}^*$. The natural map $\mathrm{Aff}_{\mathbb{Z}} \to \mathcal{T}_{\mathbb{Z}}^*$ induces a map in cohomology:

$$c_1 : H^1(X, \mathrm{Aff}_{\mathbb{Z}}) \to H^1(X, \mathcal{T}_{\mathbb{Z}}^*)$$

that we call *Chern class map.* We have that $H^1(X, \mathcal{T}_{\mathbb{Z}}^*)$ is isomorphic to $\Lambda^* \otimes (\mathbb{Z}^n)^*$, where with Λ^* we denote the group $\mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$. We get the following proposition:

Proposition 4.1. *With the above notations, the image of the map c_1 in $\Lambda^* \otimes (\mathbb{Z}^n)^*$ is the set of elements which extend to symmetric bilinear forms on $\mathbb{R}^n$.*

For the proof of this proposition, see [3], section 5.1.

It follows that an element of $c_1(H^1(X, \mathrm{Aff}_{\mathbb{Z}}))$ can be either seen as a map $\Lambda \to (\mathbb{Z}^n)^*$ or as a symmetric bilinear form on $\mathbb{R}^n$.

Definition 4.2. Let $[c]$ be an element in $\mathrm{im}(c_1)$, if the bilinear form on $\mathbb{R}^n$ induced by $[c]$ is positive definite, we say that $[c]$ is a *polarization.* We say that a tropical line bundle L is positive definite (resp. positive semidefinite) if $c_1(L)$ is positive definite (resp. positive semidefinite). Given a polarization $[c]$, since as a bilinear form it is symmetric and positive definite, we get that the image $[c](\Lambda)$ in $(\mathbb{Z}^n)^*$ is a lattice of rank n, hence the quotient $(\mathbb{Z}^n)^*/[c](\Lambda)$ has a finite number of elements. We call *degree* of $[c]$ its cardinality.

Definition 4.3. Let $X = \mathbb{R}^n/\Lambda$ be a tropical torus and L an element of $Pic(X)$ such that $c_1(L)$ is a polarization, then we call the pair $(X, c_1(L))$ *polarized tropical abelian variety* and the torus X *tropical abelian variety.*

Recall that in Chapter 1 we obtained a Λ-action on $V \times \mathbb{C}$ by defining a function $a_{(H,\chi)}$, the following definition is the tropical analogue of such notion, defining a Λ-action on $\mathbb{R}^n \times \mathbb{T}$.

Definition 4.4. Let $\alpha : \mathbb{R}^n \to \mathbb{R}$ be a linear map, and Q an element of $\Lambda^* \otimes (\mathbb{Z}^n)^*$ which extends to a symmetric bilinear form $Q_\mathbb{R}$ on $\mathbb{R}^n$. We first define a Λ-action on $\mathbb{R}^n \times \mathbb{T}$ by:

$$\lambda \cdot (x,t) = (x + \lambda, t + Q_\mathbb{R}(\lambda, x) + \frac{1}{2}Q_\mathbb{R}(\lambda,\lambda) + \alpha(\lambda)).$$

Indeed, we have that $0 \cdot (x,t) = (x,t)$ for every (x,t) in $\mathbb{R}^n \times \mathbb{T}$, moreover we have that for every λ, μ in Λ:

$$\begin{aligned}\mu \cdot (\lambda \cdot (x,t)) &= \mu \cdot (x + \lambda, t + Q_\mathbb{R}(\lambda, x) + \frac{1}{2}Q_\mathbb{R}(\lambda,\lambda) + \alpha(\lambda)) = \\ &= (x + \lambda + \mu, t + Q_\mathbb{R}(\lambda + \mu, x) + \frac{1}{2}Q_\mathbb{R}(\mu + \lambda, \mu + \lambda) + \alpha(\mu + \lambda) = \\ &= (\mu + \lambda) \cdot (x,t).\end{aligned}$$

We define the map $L(\cdot,\cdot) : \mathrm{Im}c_1 \times (\mathbb{R}^n)^* \to H^1(X, \mathrm{Aff}_\mathbb{Z})$ by defining the line bundle $L(Q,\alpha)$ on X to be the line bundle $(\mathbb{R}^n \times \mathbb{T})/\Lambda$. Consider an open covering of $\mathbb{R}^n$ as in *Example 3.6* such that for every open set U in the covering the projection $U \to U/\Lambda$ is a homeomorphism, then the trivialization of $L(Q,\alpha)$ corresponding to U is given by the commutative diagram:

$$\begin{array}{ccc} U \times \mathbb{T} & \longrightarrow & (U \times \mathbb{T})/\Lambda \\ \downarrow & & \downarrow \\ U/\Lambda \times \mathbb{T} & \longrightarrow & U/\Lambda \end{array}$$

where all maps are natural projections. Notice that the two maps $U \times \mathbb{T} \to U/\Lambda \times \mathbb{T}$ and $U \times \mathbb{T} \to (U \times \mathbb{T})/\Lambda$ are homeomorphisms by the choice of U.

It's interesting to notice that every line bundle on a tropical torus is of the form as in definition 4.4, we obtain this in the following proposition:

Proposition 4.5. *The map $L(\cdot,\cdot)$ defined above is a surjective group homomorphism. Furthermore, its kernel is $0 \times (\mathbb{Z}^n)^*$.*

For a proof see [4], *Proposition 28.*

Now, given a line bundle $L := L(Q, \alpha)$ on X, a global section s of L is a function $s : X \to L$ such that, for every set U/Λ of the covering, the composition with its restriction:

$$p_U \circ \Psi_U \circ s_U$$

is a regular function, let's denote it with h_U. By the homeomorphism $U \cong U/\Lambda$ we have that h_U descends to a regular function $f_U : U \to \mathbb{T}$. Patching together all these functions we obtain a regular function

$$\Theta : \mathbb{R}^n \longrightarrow \mathbb{T}.$$

Now take an element x in $\mathbb{R}^n$ and a λ in Λ, suppose that $x + \lambda$ belongs to the open set U of the covering. Since we have that

$$s([x+\lambda]) = [(x+\lambda, h_U([x+\lambda])] = [(x+\lambda, \Theta(x+\lambda)] = [(x, \Theta(x))],$$

it follows that Θ satisfies the following quasi-periodicity with respect to the lattice Λ:

$$\Theta(x+\lambda) = \Theta(x) + Q_{\mathbb{R}}(\lambda, x) + \frac{1}{2}Q_{\mathbb{R}}(\lambda, \lambda) + \alpha(\lambda) \tag{2}$$

for every x in $\mathbb{R}^n$ and λ in Λ. By regularity, we have that Θ is either the constant map with value $-\infty$ or has only real values, in the latter case we have that such function can be locally expressed as a tropical Laurent polynomial, and hence is "locally convex" in the sense that for every point x in $\mathbb{R}^n$ there is a convex open set containing x such that the restriction of Θ to that neighborhood is convex. The following proposition holds:

Proposition 4.6. *Every locally convex function on a convex subset of a normed vector space is convex.*

For a proof of the previous proposition see [5, Corollary 2]. As a consequence, we have the following Corollary:

Corollary 4.7. *Given a positive definite tropical line bundle $L = L(Q, \alpha)$ on the tropical torus $X = \mathbb{R}^n/\Lambda$, we have that every regular function Θ obtained with the above procedure is convex.*

It follows that we can then identify the tropical module $\Gamma(X, L(Q,\alpha))$ with the set of all regular functions on $\mathbb{R}^n$ satisfying the quasi-periodicity in 2, as well as the map which is constantly $-\infty$. We call every element of $\Gamma(X, L(Q,\alpha))$ different from $-\infty$ a *tropical Theta function.*

Observation 4.8. Suppose that Q has a negative eigenvalue with eigenvector λ, then the term

$$Q_{\mathbb{R}}(\lambda,\lambda) + \alpha(\lambda) + Q_{\mathbb{R}}(\lambda, x)$$

diverges quadratically to $-\infty$ for $|\lambda| \to +\infty$, because $\alpha(\lambda) + Q_{\mathbb{R}}(\lambda, x)$ is linear in λ. It follows that any function in $\Gamma(X, L(Q,\alpha))$ different from $-\infty$ diverges to $-\infty$ in the direction of λ, this contradicts the convexity of the function.

4.1 Positive definite case

In the following we fix an element Q of $\Lambda^* \otimes (\mathbb{Z}^n)^*$ and a linear function $\alpha : \mathbb{R}^n \to \mathbb{R}$. Suppose that Q extends to a positive definite and symmetric bilinear form on $\mathbb{R}^n$. Denote with q the map $\Lambda \to (\mathbb{Z}^n)^*$ given by $q(\lambda) = Q(\lambda, \cdot)$, and denote with $q_{\mathbb{R}}$ its extension to $\mathbb{R}^n$. Since $Q_{\mathbb{R}}$ is positive definite, we have that $q_{\mathbb{R}}$ is invertible, hence there is a unique r in $\mathbb{R}^n$ such that $q_{\mathbb{R}}(r) = \alpha$. From now on, we will assume to have extended Q to $Q_{\mathbb{R}}$ in the following way: consider the canonical basis $\{e_1^*, \dots, e_n^*\}$ of $(\mathbb{Z}^n)^*$, for every $i = 1, \dots, n$ define λ_i to be a primitive vector in Λ such that $q(\lambda_i) = a_i e_i^*$ for some positive integer a_i. This is possible because the group $(\mathbb{Z}^n)^*/q(\Lambda)$ is torsion. It follows that the set $\mathcal{B} = \{\lambda_1, \dots, \lambda_n\}$ is a basis for Λ, and let $\{e_1, \dots, e_n\} \subseteq \mathbb{Z}^n$ be the canonical basis. Then the bilinear form $Q_{\mathbb{R}}$ is defined by:

$$Q_{\mathbb{R}}(b_1\lambda_1 + \dots + b_n\lambda_n, c_1e_1 + \dots + c_ne_n) := \sum_{i,j} b_i c_j Q(\lambda_i, e_j).$$

In particular, the matrix of $Q_{\mathbb{R}}$ with respect to the basis $\{\lambda_1, \dots, \lambda_n\}$ and $\{e_1, \dots, e_n\}$ is diagonal with integer coefficients.

Definition 4.9. Let $B \subseteq (\mathbb{Z}^n)^*$ be a complete set of representatives of $(\mathbb{Z}^n)^*/q(\Lambda)$, for every b in B we define the function $\Theta_b : \mathbb{R}^n \to \mathbb{R}$ by:

$$\Theta_b(x) := \max_{\lambda\in\Lambda} \Big\{ (b + q(\lambda)) \cdot x - \frac{1}{2} Q_{\mathbb{R}}(\lambda + q_{\mathbb{R}}^{-1}(b) - r, \lambda + q_{\mathbb{R}}^{-1}(b) - r) \Big\}.$$

We also define $a(b,\lambda)$ to be the term $\frac{1}{2}Q_{\mathbb{R}}(\lambda + q_{\mathbb{R}}^{-1}(b) - r, \lambda + q_{\mathbb{R}}^{-1}(b) - r)$ in the maximum.

Notice that the cardinality of a set B as in definition 4.9 is by definition the same as the degree of the polarization Q.

Proposition 4.10. *The functions Θ_b of definition 4.9 are Theta functions.*

Proof. Let b be an element of B, then we first check that $\Theta_b(x)$ satisfies the quasi-periodicity with respect to the lattice Λ. Given any λ in Λ, one has that:

$$\Theta_b(x+\lambda) = \max_{\mu\in\Lambda}\left\{(b+q(\mu))\cdot(x+\lambda) - \frac{1}{2}Q_{\mathbb{R}}(\mu+q_{\mathbb{R}}^{-1}(b)-r, \mu+q_{\mathbb{R}}^{-1}(b)-r)\right\}$$

and this value remains the same when we make the replacement $\mu \mapsto \mu+\lambda$ in the maximum. This leads to:

$$\Theta_b(x+\lambda) = \max_{\mu\in\Lambda}\left\{(b+q(\lambda)+q(\mu))\cdot(x+\lambda) - \frac{1}{2}Q_{\mathbb{R}}(\mu+\lambda+q_{\mathbb{R}}^{-1}(b)-r, \mu+\lambda+q_{\mathbb{R}}^{-1}(b)-r)\right\}$$

which is equal to:

$$\max_{\mu\in\Lambda}\Big\{(b+q(\mu))\cdot x + b\cdot\lambda + Q_{\mathbb{R}}(\lambda,\lambda) + Q_{\mathbb{R}}(\mu,\lambda) + Q_{\mathbb{R}}(\lambda,x) - \frac{1}{2}Q_{\mathbb{R}}(\lambda,\lambda) -$$
$$-\frac{1}{2}Q_{\mathbb{R}}(\mu+q_{\mathbb{R}}^{-1}(b)-r, \mu+q_{\mathbb{R}}^{-1}(b)-r) - Q_{\mathbb{R}}(\lambda, \mu+q_{\mathbb{R}}^{-1}(b)-r)\Big\},$$

and this is equal to:

$$\Theta_b(x) + Q_{\mathbb{R}}(\lambda,x) + \frac{1}{2}Q_{\mathbb{R}}(\lambda,\lambda) + \alpha(\lambda).$$

This shows that if $b+q(\lambda)$ is a slope of Θ_b, then $b+q(\lambda)+q(\mu)$ is also a slope of Θ_b for every μ in Λ, it follows that the set of slopes of Θ_b is exactly $b+q(\Lambda)$. Now, let D_λ^b be the closure of the subset of $\mathbb{R}^n$ on which Θ_b is an affine-linear function with slope $b+q(\lambda)$, by the quasi-periodicity we have that

$$D_{\lambda+\lambda'}^b = \lambda' + D_\lambda^b \tag{3}$$

for every λ, λ' in Λ. It follows that D_λ^b is a fundamental domain for the quotient $\mathbb{R}^n/\Lambda$ for every λ, hence Θ_b is fully determined by its values on D_λ^b. It follows that Θ_b is regular because near every point of $\mathbb{R}^n$ it can be written as the maximum of a finite number of $\mathbb{Z}$-affine functions. □

As a consequence of proposition 4.10 we have that in the positive definite case there always exist theta functions for $L(Q,\alpha)$. Then take a theta function Θ and consider its Legendre transform:

$$\widehat{\Theta}(a) = \max_{x\in\mathbb{R}^n}\{a\cdot x - \Theta(x)\}$$

for every a in $(\mathbb{Z}^n)^*$. One can see that $\Theta(x)$ diverges to $+\infty$ quadratically as $|x|$ tends to infinity: this can be shown by taking $x = x' + \lambda$, where x' lives in a bounded set of representatives of $\mathbb{R}^n/\Lambda$ and $|\lambda|$ tends to infinity and then use the quasi-periodicity and the positive definiteness of Q. As a consequence, the maximum above exists. One can check directly that the Legendre transform is convex in the sense that $\widehat{\Theta}(a) \leq \sum_{i\in I} t_i \widehat{\Theta}(a_i)$ for a finite set I, some a, a_i in $(\mathbb{Z}^n)^*$, and real numbers $0 \leq t_i \leq 1$ such that $\sum_{i\in I} t_i a_i = a$ and $\sum_{i\in I} t_i = 1$. Moreover the Legendre transform satisfies the following quasi-periodicity:

$$\widehat{\Theta}(a + q(-\lambda)) = \widehat{\Theta}(a) - a \cdot \lambda + \frac{1}{2} Q_{\mathbb{R}}(\lambda, \lambda) + \alpha(\lambda).$$

Moreover, by *Corollary 4.7*, the Legendre transform of $\widehat{\Theta}$ is Θ, therefore:

$$\begin{aligned}
\Theta(x) &= \max_{a\in(\mathbb{Z}^n)^*} \{a \cdot x - \widehat{\Theta}(a)\} = \\
&= \max_{b\in B} \max_{\lambda\in\Lambda} \{(b + q(-\lambda)) \cdot x - \widehat{\Theta}(b + q(-\lambda))\} = \\
&= \max_{b\in B} \max_{\lambda\in\Lambda} \{(b + q(-\lambda)) \cdot x - \widehat{\Theta}(b) + b \cdot \lambda - \frac{1}{2} Q_{\mathbb{R}}(\lambda, \lambda) - Q_{\mathbb{R}}(\lambda, r)\} = \\
&= \max_{b\in B} \{\Theta_b(x) + \frac{1}{2} Q_{\mathbb{R}}(q_{\mathbb{R}}^{-1}(b) - r, q_{\mathbb{R}}^{-1}(b) - r) - \widehat{\Theta}(b)\}
\end{aligned}$$

where in the last equality we had to replace λ with $-\lambda$. Then we have that every Theta function Θ of $\Gamma(X, L(Q, q_{\mathbb{R}}(r)))$ can be written as:

$$\Theta(x) = \max_{b\in B} \{\Theta_b(x) + s_b\}$$

for some s_b in $\mathbb{T}$, therefore the functions $\Theta_b(x)$ generate $\Gamma(X, L(Q, q_{\mathbb{R}}(r)))$, i.e. every theta function can be written as a tropical linear combination of the Θ_b.

Proposition 4.11. *The Legendre transform induces a bijection between the set of convex functions*

$$\eta : (\mathbb{Z}^n)^* \to \mathbb{R}$$

that satisfy the following quasi-periodicity for every a in $(\mathbb{Z}^n)^*$ and λ in Λ:

$$\eta(a + q(\lambda)) = \eta(a) + a \cdot \lambda + \frac{1}{2} Q_{\mathbb{R}}(\lambda, \lambda) - Q_{\mathbb{R}}(r, \lambda),$$

and the set $\Gamma(X, L) \setminus \{-\infty\}$. Furthermore, the set above is convex.

Proof. The first part of the statement follows from what was observed above. For convexity, let η_1, η_2 be two of its elements, then given two real numbers $0 \leq a, b \leq 1$ such that $a + b = 1$ it follows that the sum $a\eta_1 + b\eta_2$ defines a function from $(\mathbb{Z}^n)^*$ to $\mathbb{R}$, and it's easy to see that it satisfies the requested quasi-periodicity. $\square$

Proposition 4.12. *Let Q, α and B be as before. Then the set $\{\Theta_b\}_{b \in B}$ is a minimal set of generators of $\Gamma(X, L(Q, \alpha))$, and hence the minimal number of generators is $|B|$.*

Proof. In the proof of proposition 4.10 we showed that any function Θ_b is fully determined by its values on a fundamental domain D^b_λ, on which it is affine-linear with slope $b + q(\lambda)$. Hence Θ_b cannot be written as a tropical linear combination of two or more Theta functions, because if it was the case, then their maximum on D^b_λ would be equal to only one of them. It follows that the set $\{\Theta_b\}_{b \in B}$ is a minimal set of generators of $\Gamma(X, L(Q, q_{\mathbb{R}}))$. $\square$

Let's now see what happens when instead of B we choose a different subset B' of $(\mathbb{Z}^n)^*$ as set of representatives of $(\mathbb{Z}^n)^*/q(\Lambda)$. Clearly for every b' in B' there is an element $\lambda_{b'}$ in Λ and a unique b in B such that $b' = b + q(\lambda_{b'})$. We can make the following:

Observation 4.13. For every x in $\mathbb{R}^n$ and b, b' as above, we have that $\Theta_{b'}(x) = \Theta_b(x)$. Indeed, one just needs to consider the maximum defining $\Theta_b(x)$ with respect to an element μ running through the lattice Λ, and then make the substitution $\mu = \lambda + \lambda_{b'}$, where λ runs through the lattice Λ. As a consequence, we have that considering two different complete sets of representatives of $(\mathbb{Z}^n)^*/q(\Lambda)$, say B and B', the set $\{\Theta_{b'}\}_{b' \in B'}$ is equal to the set $\{\Theta_b\}_{b \in B}$.

We can say something more about the relation between a generator Θ_b and the translate $\Theta_{b+q(\tilde{\lambda})}$ for an element $\tilde{\lambda}$ in $\tilde{\Lambda} := q_{\mathbb{R}}^{-1}((\mathbb{Z}^n)^*)$: namely, it's straightforward to prove that:

$$\Theta_{b+q_{\mathbb{R}}(\tilde{\lambda})}(x) = \Theta_b(x - \tilde{\lambda}) + Q_{\mathbb{R}}(\tilde{\lambda}, x) - \frac{1}{2} Q_{\mathbb{R}}(\tilde{\lambda}, \tilde{\lambda}) + \alpha(\tilde{\lambda}), \tag{4}$$

and, as a consequence:

$$\Theta_{b+q_{\mathbb{R}}(\tilde{\lambda})}(x + \tilde{\lambda}) = \Theta_b(x) + Q_{\mathbb{R}}(\tilde{\lambda}, x) + \frac{1}{2} Q_{\mathbb{R}}(\tilde{\lambda}, \tilde{\lambda}) + \alpha(\tilde{\lambda}). \tag{5}$$

As a consequence of the above relations, we get the following Lemma:

Lemma 4.14. *Consider the line bundle $L(Q,\alpha)$ on the n-dimensional tropical torus $X = \mathbb{R}^n/\Lambda$, and let B be a complete set of representatives for $(\mathbb{Z}^n)^*/q(\Lambda)$. Fix an element $\tilde{\lambda}$ in $\tilde{\Lambda}$, then, for every λ in Λ and b, b' in B, one has that $D^b_\lambda + \tilde{\lambda} = D^{b+q_{\mathbb{R}}(\tilde{\lambda})}_\lambda$. Moreover, all the fundamental domains $D^{b'}_\lambda$ are translates of D^b_λ.*

Proof. We have that for every x in $D^b_\lambda + \tilde{\lambda}$, the element $x - \tilde{\lambda}$ belongs to D^b_λ, and therefore the slope of $\Theta_b(x - \tilde{\lambda})$ is $b + q(\lambda)$. Using formula 4 we obtain that:

$$\Theta_{b+q_{\mathbb{R}}(\tilde{\lambda})}(x) = (b + q(\lambda)) \cdot (x - \tilde{\lambda}) - a(b, \lambda) + Q_{\mathbb{R}}(\tilde{\lambda}, x) - \frac{1}{2}Q_{\mathbb{R}}(\tilde{\lambda}, \tilde{\lambda}) + \alpha(\tilde{\lambda}),$$

which shows that the slope of $\Theta_{b+q(\tilde{\lambda})}$ in x is $b+q_{\mathbb{R}}(\tilde{\lambda})+q(\lambda)$, hence x belongs to $D^{b+q_{\mathbb{R}}(\tilde{\lambda})}_\lambda$. The reverse inclusion is proven similarly using formula 5. Now, take an element b' of B, there is an element $\tilde{\lambda}$ in $\tilde{\Lambda}$ such that $q_{\mathbb{R}}(\tilde{\lambda}) + b = b'$. It follows that

$$D^{b'}_\lambda = D^{b+q_{\mathbb{R}}(\tilde{\lambda})}_\lambda = D^b_\lambda + \tilde{\lambda}.$$

□

Thanks to *Observation 4.13* we can suppose that the element 0 belongs to B. Now, following [6], we will give a result useful for the description of D^0_0.

Definition 4.15. Let G be a positive definite symmetric bilinear form on $\mathbb{R}^n$, we denote with N the norm on $\mathbb{R}^n$ defined in the following way:

$$N(x) := G(x, x) \quad \forall x \in \mathbb{R}^n.$$

Let $\Lambda \subseteq \mathbb{R}^n$ be a full rank lattice, then for every λ in Λ the *Voronoi cell* $V(\lambda)$ is defined to be the set of points:

$$V(\lambda) := \{x \in \mathbb{R}^n | N(x - \lambda) \leq N(x - \lambda') \quad \forall \lambda' \in \Lambda\}.$$

Definition 4.16. Let G be a positive definite symmetric bilinear form on $\mathbb{R}^n$ and $\Lambda \subseteq \mathbb{R}^n$ a full rank lattice. An element $\lambda \neq 0$ in Λ is called *Voronoi vector* if the hyperplane

$$\left\{x \in \mathbb{R}^n | G(x, \lambda) = \frac{1}{2}G(\lambda, \lambda)\right\}$$

has a non-empty intersection with the Voronoi cell $V(0)$. A Voronoi vector is *strict* if this intersection is a $(n-1)$-dimensional face of $V(0)$, otherwise it is *lax*.

The following theorem holds:

Theorem 4.17. *Let G be a positive definite symmetric bilinear form on $\mathbb{R}^n$ and $\Lambda \subseteq \mathbb{R}^n$ a full rank lattice. A non-zero vector λ in Λ is a Voronoi vector if and only if λ is a shortest vector with respect to the norm N in the class $\lambda + 2\Lambda$. It is a strict Voronoi vector if and only if λ and $-\lambda$ are the only shortest vectors in $\lambda + 2\Lambda$.*

Proof. Suppose that λ is a Voronoi vector and there is a λ' in Λ such that $\lambda - \lambda' \in 2\Lambda$ and $N(\lambda') < N(\lambda)$. Then the vectors:

$$t := \frac{1}{2}(\lambda + \lambda'),$$

$$u := \frac{1}{2}(\lambda - \lambda')$$

both belong to Λ. Let x be an element of $\mathbb{R}^n$ such that $G(x, \lambda) = \frac{1}{2}G(\lambda, \lambda)$ and the following inequalities hold:

$$G(x, t) \leq \frac{1}{2}N(t),$$

$$G(x, u) \leq \frac{1}{2}N(u).$$

From these inequalities we deduce the following

$$\begin{aligned} N(\lambda) &= G(\lambda, \lambda) = 2G(x, \lambda) = 2G(x, t + u) = \\ &= 2G(x, t) + 2G(x, u) \leq N(t) + N(u) = \\ &= \frac{1}{4}G(\lambda + \lambda', \lambda + \lambda') + \frac{1}{4}G(\lambda - \lambda', \lambda - \lambda') = \\ &= \frac{1}{2}N(\lambda) + \frac{1}{2}N(\lambda'), \end{aligned}$$

which implies that $N(\lambda) \leq N(\lambda')$, which is a contradiction. Conversely, suppose that λ is a shortest vector in $\lambda + 2\Lambda$ but not a Voronoi vector. Suppose that for every λ' in Λ we have that $G(\lambda, \lambda') \leq N(\lambda')$, then it follows that the point $x = \lambda/2$ belongs to the hyperplane

$$\left\{ x \in \mathbb{R}^n | G(x, \lambda) = \frac{1}{2}G(\lambda, \lambda) \right\}$$

and to $V(0)$, contradicting the fact that λ is not a Voronoi vector. Then there is a λ' in Λ such that $G(\lambda, \lambda') > N(\lambda')$, but then the following holds:

$$N(\lambda - 2\lambda') = N(\lambda) + 4N(\lambda') - 4G(\lambda, \lambda') < N(\lambda),$$

contradicting the fact that λ is a shortest vector in $\lambda + 2\Lambda$.
The last part of the theorem can be proven in a similar way. $\square$

Corollary 4.18. *Let $X = \mathbb{R}^n/\Lambda$ be a tropical torus with a positive definite line bundle $L = L(Q, \alpha)$, and define N to be the norm on $\mathbb{R}^n$ given by $N(x) := Q_{\mathbb{R}}(x, x)$ for every x in $\mathbb{R}^n$. Then we have that D_0^0 is equal to the translation by $-r$ of the Voronoi cell $V(0)$ with respect to the norm N. Moreover we have that D_0^0 can have at most $2(2^n - 1)$ facets.*

Proof. On D_0^0 the theta function Θ_0 is identically equal to $-\alpha(r)/2$, therefore we must have that for every λ in Λ and x in D_0^0:

$$-\alpha(r)/2 \geq Q_{\mathbb{R}}(\lambda, x) - \frac{1}{2} Q_{\mathbb{R}}(\lambda - r, \lambda - r),$$

which is true if and only if:

$$N(\lambda) \geq 2Q_{\mathbb{R}}(x + r, \lambda),$$

and this implies that $x + r$ belongs to $V(0)$. The second part follows from *Theorem 4.17* and the fact that there are $2^n - 1$ non-zero classes in $\Lambda/2\Lambda$. $\square$

Example 4.19. Let $L := L(Q, \alpha)$ be a positive definite tropical line bundle on the torus $X = \mathbb{R}^n/\mathbb{Z}^n$, suppose that Q with respect to the canonical basis $\{e_1, \dots, e_n\}$ of $\mathbb{Z}^n$ is diagonal, with eigenvalues $d_1, \dots, d_n$. Suppose that 0 belongs to the set B of representatives of $(\mathbb{Z}^n)^*/q(\mathbb{Z}^n)$, then we have that $D_0^0 = [-1/2, 1/2]^n$. To see this, first notice that

$$\Theta_0(0) = \max_{\lambda \in \Lambda} \Big\{ -\frac{1}{2} Q_{\mathbb{R}}(\lambda, \lambda) \Big\} = 0,$$

moreover

$$\Theta_0\Big(\frac{1}{2}, \dots, \frac{1}{2}\Big) = \max_{\substack{a_1, \dots, a_n \in \mathbb{Z} \\ \lambda = a_1 e_1 + \dots + a_n e_n}} \Big\{ q(\lambda) \cdot \Big(\frac{1}{2}, \dots, \frac{1}{2}\Big) - \frac{1}{2} Q_{\mathbb{R}}(\lambda, \lambda) \Big\},$$

which is equal to:

$$\frac{1}{2} \cdot \max_{\substack{a_1, \dots, a_n \in \mathbb{Z} \\ \lambda = a_1 e_1 + \dots + a_n e_n}} \Big\{ d_1(a_1 - a_1^2) + \dots + d_n(a_n - a_n^2) \Big\} = 0.$$

In a similar way one proves that $\Theta_0(a_1 e_1 + \dots a_n e_n) = 0$ for every $(a_1, \dots, a_n)$ in $\{-1/2, 1/2\}^n$, and then the convexity of Θ_0 implies the statement.

Now, let $\pi : \mathbb{T}^B \to \Gamma(X, L(Q, \alpha))$ be the map given by $\pi((s_b)_{b\in B})(x) = \max_{b\in B}\{\Theta_b(x) + s_b\}$, by *Observation 4.12*, π is a surjective tropical morphism.

Definition 4.20. We define the map $\phi : \Gamma(X, L(Q, \alpha)) \to \mathbb{T}^B$ by

$$\phi(\Theta) = (\phi^b(\Theta))_{b\in B}$$

where $\phi^b(\Theta) = \min_{x\in\mathbb{R}^n}\{\Theta(x) - \Theta_b(x)\}$.

Notice that the minimum in the definition of $\phi^b(\Theta)$ always exists because by quasi-periodicity the difference $\Theta(x) - \Theta_b(x)$ is periodic with respect to the lattice Λ, hence we can compute this minimum on a compact subset of $\mathbb{R}^n$.

Observation 4.21. Let $\Theta(x) := \max_{b\in B}\{\Theta_b(x) + s_b\}$ for an element $(s_b)_b$ in $\mathbb{T}^B$. We have that for every x in $\mathbb{R}^n$ and every b' in B:

$$\Theta(x) - \Theta_{b'}(x) \geq \Theta_{b'}(x) + s_{b'} - \Theta_{b'}(x) = s_{b'},$$

hence $\phi^{b'}(\Theta) \geq s_{b'}$.

We have the following:

Proposition 4.22. *Let $\Theta(x) := \max_{b\in B}\{\Theta_b(x) + s_b\}$ for an element $(s_b)_b$ in $\mathbb{T}^B$. Then for all b' in B we have that $\phi^{b'}(\Theta) = s_{b'}$ if and only if there exists a y in $\mathbb{R}^n$ such that $\Theta(y) = \Theta_{b'}(y) + s_{b'}$.*

Proof. Suppose that there exists an y in $\mathbb{R}^n$ such that $\Theta(y) = \Theta_{b'}(y) + s_{b'}$, then the set $A := \{z \in \mathbb{R}^n | \Theta(z) = \Theta_{b'}(z) + s_{b'}\}$ is non-empty. We have that:

$$\phi^{b'}(\Theta) = \min_{x\in\mathbb{R}^n}\{\Theta(x) - \Theta_{b'}(x)\} \leq \min_{x\in A}\{\Theta(x) - \Theta_{b'}(x)\} = s_{b'}.$$

The converse inequality is true by *Observation 4.21.* □

Notice that it's not always true that $\phi^b(\Theta) = s_b$, for example when $s_b = -\infty$.

Observation 4.23. The composition $\pi\circ\phi$ is the identity map on $\Gamma(X, L(Q, \alpha))$. Indeed, let $\Theta(x) = \max_{b\in B}\{\Theta_b(x) + s_b\}$ be a Theta function, then by *Observation 4.21* we have that for every b in B and x in $\mathbb{R}^n$:

$$\Theta_b(x) + \phi^b(\Theta) \geq \Theta_b(x) + s_b,$$

hence taking the maximum with respect to b we find that $\pi(\phi(\Theta))(x) \geq \Theta(x)$. Conversely, we have that

$$\Theta_b(x) + \phi^b(\Theta) = \Theta_b(x) + \min_{y \in \mathbb{R}^n} \{\Theta(y) - \Theta_b(y)\} \leq \Theta(x),$$

hence taking the maximum of the left side with respect to b we get the reverse inequality. An immediate consequence of this is that the map ϕ is injective.

Via the map ϕ we can define a topology on $\Gamma(X, L)$, more precisely: if the space $\mathbb{T}$ is equipped with the order topology, and $\mathbb{T}^B$ is equipped with the product topology, then the topology on $\Gamma(X, L)$ is generated by all preimages of open sets in $\mathbb{T}^B$. We will need the following proposition as a useful way of computing the map ϕ:

Proposition 4.24. *Let X be a tropical torus, $L = L(Q, \alpha)$ a tropical line bundle on X, and let Θ be a theta function in $\Gamma(X, L)$, then for all b in B we have:*

$$\phi^b(\Theta) = -\widehat{\Theta}(b) + a(b, 0).$$

Proof. Recall that in the proof of *Proposition 4.10* we have seen that the function Θ_b is fully determined by its values on D^b_λ for every b in B and λ in Λ. As a consequence we can rewrite the minimum defining ϕ^b in the following way

$$\begin{aligned}\phi^b(\Theta) &= \min_{x \in \mathbb{R}^n} \{\Theta(x) - \Theta_b(x)\} = \\ &= \min_{x \in D^b_0} \{\Theta(x) - \Theta_b(x)\} = \\ &= \min_{x \in D^b_0} \{\Theta(x) - b \cdot x + a(b, 0)\}\end{aligned}$$

by the quasi-periodicity. Now notice that from the inequality $\Theta_b(x) \geq b \cdot x - a(b, 0)$, which holds by definition, one deduces that

$$\Theta(x) - b \cdot x + a(b, 0) \geq \Theta(x) - \Theta_b(x),$$

and taking the minimum with respect to x in $\mathbb{R}^n$ one finds that

$$\min_{x \in \mathbb{R}^n} \{\Theta(x) - b \cdot x + a(b, 0)\} \geq \min_{x \in \mathbb{R}^n} \{\Theta(x) - \Theta_b(x)\} = \min_{x \in D^b_0} \{\Theta(x) - \Theta_b(x)\}.$$

The reverse inequality holds, therefore one can write:

$$\begin{aligned}\phi^b(\Theta) &= \min_{x \in \mathbb{R}^n} \{\Theta(x) - b \cdot x + a(b, 0)\} = \\ &= -\max_{x \in D^b_0} \{-\Theta(x) + b \cdot x\} + a(b, 0) = \\ &= -\widehat{\Theta}(b) + a(b, 0).\end{aligned}$$

□

Observation 4.25. A consequence of *Proposition 4.24* is that the image $\phi(\Gamma(X,L))$ is always a convex set.

Example 4.26. Let $X = \mathbb{R}/\mathbb{Z}$ be a 1-dimensional tropical torus, let L be the line bundle $L(2,0)$ and define B to be the set $B := \{0,1\}$. We easily see that in this case $r = 0$ and

$$\Theta_0(x) = \max_{n\in\mathbb{Z}}\{2nx - n^2\},$$

$$\Theta_1(x) = \max_{n\in\mathbb{Z}}\{(2n+1)x - (n+1/2)^2\}.$$

One can compute the maximums and find that:

$$\Theta_0(x) = 2x[x] - [x]^2 + (2(x-[x]) - 1)^+,$$

$$\Theta_1(x) = 2x[x-1/2]+x-[x-1/2]^2-1/4-[x-1/2]+(2(x-1/2-[x-1/2])-1)^+,$$

where $[x]$ denotes the integer part of x and $x^+ = \max\{0,x\}$. Now consider the following cases:

- If $x-[x] < 1/2$ then $[x-1/2] = [x]-1$, and hence: $x-1/2-[x-1/2] = x-[x]+1/2 \geq 1/2$. It follows that:

  $$\Theta_0(x) = 2x[x] - [x]^2,$$

  $$\Theta_1(x) = 2x[x] - [x]^2 + x - [x] - 1/4,$$

  and therefore $\Theta_0(x) - \Theta_1(x) = x - [x] - 1/4$.

- If $x - [x] \geq 1/2$ then $[x] = [x-1/2]$, and therefore one finds that:

  $$\Theta_0(x) = 2x[x] - [x]^2 + 2(x-[x]) - 1,$$

  $$\Theta_1(x) = 2x[x] + x - [x]^2 - 1/4 - [x],$$

  and hence $\Theta_0(x) - \Theta_1(x) = x - [x] - 3/4$.

As a consequence, by *Proposition 4.22* we have that $\phi^0(\Theta_0) = 0$, while

$$\phi^1(\Theta_0) = \min_{x\in\mathbb{R}^n}\{\Theta_0(x) - \Theta_1(x)\} = -1/4.$$

This proves that $\phi(\Theta_0) = (0,-1/4)$, and this implies that $\Theta_0(x) = \pi(0,-1/4) = \max\{\Theta_0(x), \Theta_1(x) - 1/4\}$. Moreover $\phi^1(\Theta_0) = -1/4 > -\infty$.

In this way we can also construct theta functions with unexpected images through ϕ: for example define

$$\Theta(x) := \max\{\Theta_0(x) + 3/2, \Theta_1(x) + 1\},$$

then since we have that $\Theta_0(x) - \Theta_1(x) \geq -1/4 > -1/2 = 1 - 3/2$ it follows that $\Theta_0(x) + 3/2 > \Theta_1(x) + 1$ for every x in $\mathbb{R}^n$. As a consequence we have that $\phi^1(\Theta) = \min_{x\in\mathbb{R}^n}\{\Theta_0(x) + 3/2 - \Theta_1(x)\} = 3/2 - 1/4 = 5/4 > 1$. This shows that in general it's possible that $\phi^b(\Theta) > s_b$ if $\Theta(x) = \max_{b'\in B}\{\Theta_{b'}(x) + s_{b'}\}$.

We state the following lemma, which will be useful later, for the proof see [4, Lemma 35]:

Lemma 4.27. *Let D be a compact convex polyhedron in $\mathbb{R}^n$ and*

$$f : D \times \mathbb{R}^m \to \mathbb{R}$$

a convex piecewise-linear function with finitely many slopes, i.e. it can be written as the maximum of finitely many affine-linear functions. Define $g : \mathbb{R}^m \to \mathbb{R}$ by: $g(y) := \min_{x\in D}\{f(x, y)\}$, then g is a convex piecewise-linear function with finitely many slopes.

Now, a series of observations will lead to the proof of the main theorem of this section. First of all, fix a tropical line bundle L on a tropical abelian variety $X = \mathbb{R}^n/\Lambda$ such that $Q := c_1(L)$ is positive definite, furthermore fix a complete set B of representatives for $(\mathbb{Z}^n)^*/q(\Lambda)$ as done before.

Notice that an element $(s_b)_{b\in B}$ of $\mathbb{T}^B$ belongs to the image of ϕ only if there exists a Theta function $\Theta(x) = \max_{b\in B}\{\Theta_b(x) + t_b\}$ in $\Gamma(X, L)$ such that $\phi(\Theta) = (s_b)_{b\in B}$, and therefore $\pi((s_b)_{b\in B}) = \max_{b\in B}\{\Theta_b + s_b\} = \Theta$, so one only needs to check that $\phi(\max\{\Theta_b + s_b\}) = (s_b)_{b\in B}$.
Moreover, take a Theta function $\Theta(x) := \max_{b'\in B}\{\Theta_{b'}(x) + t_{b'}\}$, we have that for every b in B, asking that $\phi^b(\Theta) = t_b$ is equivalent to ask that:

$$t_b \geq \min_{x\in\mathbb{R}^n}\{\max_{b'\in B\setminus\{b\}}(\Theta_{b'}(x) + t_{b'}) - \Theta_b(x)\}. \tag{6}$$

Indeed, if $\phi^b(\Theta) = t_b$ then we have that

$$t_b = \min_{x\in\mathbb{R}^n}\{\Theta(x) - \Theta_b(x)\} \geq \min_{x\in\mathbb{R}^n}\{\max_{b'\in B\setminus\{b\}}(\Theta_{b'}(x) + t_{b'}) - \Theta_b(x)\}.$$

For the converse, first notice that for every x in $\mathbb{R}^n$ one has that $\Theta(x) - \Theta_b(x) \geq t_b$, therefore taking the minimum one deduces that $\phi^b(\Theta) \geq t_b$. Moreover, from the inequality

$$t_b \geq \min_{x \in \mathbb{R}^n} \{ \max_{b' \in B \setminus \{b\}} (\Theta_{b'}(x) + t_{b'}) - \Theta_b(x) \}$$

one deduces that

$$t_b \geq \min\{ \min_{x \in \mathbb{R}^n} \{ \max_{b' \in B \setminus \{b\}} (\Theta_{b'}(x) + t_{b'}) - \Theta_b(x) \}, \Theta_b(x) + t_b - \Theta_b(x) \},$$

and since the right hand side of the inequality is equal to $\min_{x \in \mathbb{R}^n} \{\Theta(x) - \Theta_b(x)\}$ one can conclude that $\phi^b(\Theta) \leq t_b$ and hence $\phi^b(\Theta) = t_b$.

Now, using the notation of *Proposition 4.10*, and using the periodicity of the difference between two theta functions, we can rewrite the minimum in 6 in the following way:

$$t_b \geq \min_{x \in D_0^b} \{ \max_{b' \in B \setminus \{b\}} (\Theta_{b'}(x) + t_{b'}) - \Theta_b(x) \},$$

which is then equivalent to:

$$t_b \geq \min_{x \in D_0^b} \{ \max_{b' \in B \setminus \{b\}} (\Theta_{b'}(x) + t_{b'}) - (b \cdot x - a(b, 0)) \}.$$

Consider the right-hand side of the above inequality, and define a function $f : D_0^b \times \mathbb{R}^B \to \mathbb{R}$ by setting

$$f(x, (t_{b'})_{b' \in B}) = \max_{b' \in B \setminus \{b\}} (\Theta_{b'}(x) + t_{b'}) - b \cdot x + a(b, 0).$$

Notice that some of the $t_{b'}$ might be equal to $-\infty$, in that case the corresponding sum $\Theta_{b'}(x) + t_{b'}$ is trivial and doesn't give a contribution to the maximum above. Thus, one has to ignore this term and consider a space of smaller dimension than $\mathbb{R}^B$. If all the $t_{b'}$ are equal to $-\infty$ except t_b, then the condition $(t_{b'})_{b' \in B} \in \mathrm{im}\phi$ is automatically fulfilled. The function f is the maximum of a finite number of affine linear functions, we can apply *Lemma 4.27* and deduce that the right-hand side of the above inequality is a convex function, say g_b, that can be written as the maximum of a finite number of affine linear functions. Now, we give the following definition:

Definition 4.28. A *rational polyhedron* in $\mathbb{R}^n$ is a subset of the form

$$\bigcap_{i \in I} \{x \in \mathbb{R}^n | x \cdot a_i \geq b_i\}$$

where I is a finite set, the a_i belong to $\mathbb{Z}^n$ and b_i are real numbers.

We can summarise what we found so far in the following way: let $\{e_b\}_{b\in B}$ be the canonical basis for $\mathbb{R}^B$, then

$$\mathrm{im}\phi \cap \mathbb{R}^B = \bigcap_{b\in B} \{x \in \mathbb{R}^B | x \cdot e_b \geq g_b(x)\},$$

and by what we have noticed above, there is a finite set J_b and some a_j in $\mathbb{Z}^B$ with real parameters b_j for every j in J_b, such that $g_b(x) = \max_{j\in J}\{a_j \cdot x + b_j\}$. Then the set $\mathrm{im}\phi \cap \mathbb{R}^B$ can be written as a finite intersection of closed half spaces with rational slopes as in *Definition 4.28*, therefore it is a rational polyhedron. Now recall that in *Observation 4.25* we have seen that the image of ϕ is convex, we can deduce that its restriction to $\mathbb{R}^B$ is a convex polyhedron. To compute its dimension, let's look at what points are certainly contained in this image. First of all, consider the theta function $\Xi(x) := \max_{b\in B}\{\Theta_b(x)\}$ and define the lattice $\tilde{\Lambda} := q_{\mathbb{R}}^{-1}(\mathbb{Z}^n)^*$. For every η in $\tilde{\Lambda}$, b in B and x in $\mathbb{R}^n$:

$$\Xi(x + \eta) = \max_{b\in B}\{\Theta_b(x + \eta)\},$$

and remember that by formula 4 this is equal to:

$$\max_{b\in B}\{\Theta_{b+q_{\mathbb{R}}(\eta)}(x)\} + Q_{\mathbb{R}}(\eta, x) + \frac{1}{2}Q_{\mathbb{R}}(\eta, \eta) + \alpha(\eta).$$

Recall that by *Observation 4.13* the sets $\{\Theta_b\}_{b\in B}$ and $\{\Theta_{b+q_{\mathbb{R}}(\eta)}\}_{b\in B}$ are equal, hence:

$$\Xi(x + \eta) = \Xi(x) + Q_{\mathbb{R}}(\eta, x) + \frac{1}{2}Q_{\mathbb{R}}(\eta, \eta) + \alpha(\eta).$$

As a consequence, one sees that the set of slopes of Ξ is exactly $(\mathbb{Z}^n)^*$. Now, since the set of slopes is $(\mathbb{Z}^n)^*$ and the sets $b + q(\Lambda)$ are pairwise disjoint, then every Θ_b is relevant in the tropical sum, this implies that $\phi^b(\Xi) = 0$ for every b in B, and hence the origin $(0, \dots, 0)$ is an element of the image of ϕ. Now, fix an element b in B, by what we just noticed it's clear that for every b' in B different from b there is an $y_{b'}$ in $\mathbb{R}^n$ such that

$$\Xi(y_{b'}) = \Theta_{b'}(y_{b'}) > \Theta_b(y_{b'}),$$

because if this wasn't true it wouldn't be possible for $\Xi(x)$ to have $b' + q(\Lambda)$ as a subset of the set of slopes. Therefore, one can find a positive real number $\varepsilon_{b'}$ such that

$$\Theta_{b'}(y_{b'}) > \Theta_b(y_{b'}) + \varepsilon_{b'}.$$

Let $\tilde{\varepsilon}_b$ be the minimum of all the $\varepsilon_{b'}$, now exchanging b with another element b'' of B we find a number $\tilde{\varepsilon}_{b''}$ defined with a similar procedure as before. Finally, we define:

$$\varepsilon := \min_{\overline{b} \in B} \tilde{\varepsilon}_{\overline{b}},$$

which by construction is positive and it has the following property: for every b, b' in B there exists an element y in $\mathbb{R}^n$ such that

$$\Xi(y) = \Theta_{b'}(y) > \Theta_b(y) + \varepsilon.$$

As a consequence, for every b in B consider the theta function

$$\Theta^b(x) := \max\{\Xi(x), \Theta_b(x) + \varepsilon\},$$

by *Proposition 4.22* we have that $\phi^b(\Theta^b) = \varepsilon$, because of the positivity of ε and the fact that $\phi^b(\Xi) = 0$ trivially imply that the maximum that defines $\Theta^b(x)$ is realised by $\Theta_b(x)+\varepsilon$ for some x in $\mathbb{R}^n$. Moreover, the construction of ε implies in the same way that $\phi^{b'}(\Theta^b) = 0$ for every b' different from b. This means that all the points $(\varepsilon, 0, \ldots, 0), (0, \varepsilon, 0, \ldots, 0), \ldots, (0, \ldots, 0, \varepsilon)$ belong to the image of ϕ, and together with $\phi(\Xi)$ we have a total of $|B| + 1$ points in general position. By the convexity of the image of ϕ, it follows that their convex hull is contained in the image, which is then a convex polyhedron of dimension $|B|$.
We summarise the conclusions of the previous argument in the following theorem, due to Sumi, one can find a similar proof in [4]:

Theorem 4.29. *Let $L = L(Q, \alpha)$ be a tropical line bundle on a tropical abelian variety $X = \mathbb{R}^n/\Lambda$ such that Q extends to a positive definite bilinear form on $\mathbb{R}^n$. Then $\Gamma(X, L)$ is generated by $|B| = |Cok(q)|$ elements as a $\mathbb{T}$-module. Moreover, the map ϕ identifies $\Gamma(X, L)$ with a convex $|B|$-dimensional polyhedron in $\mathbb{T}^B$.*

4.2 Positive semidefinite case.

In the following section we will consider a line bundle $L := L(Q, \alpha)$ on $X = \mathbb{R}^n/\Lambda$ such that Q extends to a positive and symmetric semidefinite bilinear form on $\mathbb{R}^n$. The first observation is that in this case the kernel of $q_{\mathbb{R}}$ is not necessarily 0, and there is a set of generators which are all in Λ or in $\mathbb{Z}^n$ because of the symmetry (we call this properties respectively Λ-rationality and $\mathbb{Z}$-rationality). Remember that, by *Proposition 4.5*, for every r in $\mathbb{R}^n$ and γ in $(\mathbb{Z}^n)^*$ the line bundles $L(Q, q_{\mathbb{R}}(r))$ and $L(Q, q_{\mathbb{R}}(r) + \gamma)$ are isomorphic. Therefore, the study of every element belonging to $\mathrm{im} q_{\mathbb{R}} + (\mathbb{Z}^n)^*$

can be reduced to that of an element in im$q_{\mathbb{R}}$. So let's first study the case $\alpha = q_{\mathbb{R}}(r)$ for r in $\mathbb{R}^n$, and let λ be any element of $\ker(q) \subseteq \Lambda$, then for every theta function Θ in $\Gamma(X, L)$ we have that:

$$\Theta(x+\lambda) = \Theta(x) + Q_{\mathbb{R}}(\lambda, x) + \frac{1}{2}Q_{\mathbb{R}}(\lambda,\lambda) + \alpha(\lambda) = \Theta(x),$$

by the definition of λ. The formula above together with the convexity of Θ, imply that $\Theta(x)$ is constant along the directions of $\ker(q)$, hence for every u in $\ker(q_{\mathbb{R}})$:

$$\Theta(x+u) = \Theta(x).$$

As a consequence, we have that theta functions in $\Gamma(X, L)$ descend to functions on $\mathbb{R}^n/\ker(q_{\mathbb{R}})$. Define then the lattice $\overline{\Lambda}$ to be the lattice $\Lambda/\ker(q)$ in $\mathbb{R}^n/\ker(q_{\mathbb{R}})$, as well as $\overline{Q}_{\mathbb{R}}$ to be the element of $(\mathbb{R}^n/\ker(q_{\mathbb{R}}))^* \otimes (\mathbb{R}^n/\ker(q_{\mathbb{R}}))^*$ induced by $Q_{\mathbb{R}}$ and $\overline{\alpha}$ the linear map on $\mathbb{R}^n/\ker(q_{\mathbb{R}})$ induced by α. Similarly, call $\overline{\Theta}$ a function descending from a theta function in $\Gamma(X, L)$, it satisfies the following quasi-periodicity for every λ in $\overline{\Lambda}$ and x in $\mathbb{R}^n/\ker(q_{\mathbb{R}})$:

$$\overline{\Theta}(x+\lambda) = \overline{\Theta}(x) + \overline{Q}_{\mathbb{R}}(x,\lambda) + \frac{1}{2}\overline{Q}_{\mathbb{R}}(\lambda,\lambda) + \overline{\alpha}(\lambda).$$

One can check that the slopes of $\overline{\Theta}$ lie in $(\mathbb{Z}^n)^*/(\ker(q_{\mathbb{R}}))$. Now, we can identify $(\mathbb{R}^n/\ker(q_{\mathbb{R}}))^*$ with im$q_{\mathbb{R}}$, and similarly we identify $(\mathbb{Z}^n/(\ker q_{\mathbb{R}} \cap \mathbb{Z}^n))^*$ with im$q_{\mathbb{R}} \cap \mathbb{Z}^n$.
Define $\overline{q} : \overline{\Lambda} \to$im$q_{\mathbb{R}} \cap (\mathbb{Z}^n)^*$ to be the linear map induced by q, since now $\overline{q}$ is positive definite we can apply the results of the previous section and obtain that $\Gamma(X, L)$ is generated by $\ell := |\mathrm{Cok}(\overline{q})|$ elements and is identified with an ℓ-dimensional convex polyhedron in $\mathbb{T}^{\ell}$.
Now let's analyze the case in which α doesn't belong to im$q_{\mathbb{R}} + (\mathbb{Z}^n)^*$: first suppose that $\alpha(\ker(q_{\mathbb{R}}) \cap \mathbb{Z}^n) \subseteq \mathbb{Z}$, i.e. α is integral on $\ker(q_{\mathbb{R}}) \cap \mathbb{Z}^n$. Then translating α by some element in $(\mathbb{Z}^n)^*$ we can suppose that $\alpha(\ker(q_{\mathbb{R}}) \cap \mathbb{Z}^n) = 0$, but then the $\mathbb{Z}$-rationality of $\ker(q_{\mathbb{R}})$ implies that α is a linear map on $\mathbb{R}^n/\ker(q_{\mathbb{R}})$, hence it belongs to im$q_{\mathbb{R}}$, which is a contradiction. It follows that α isn't integral on $\ker(q_{\mathbb{R}}) \cap \mathbb{Z}^n$. Now, take any theta function Θ and notice that given an element λ' in $\Lambda \cap \ker(q_{\mathbb{R}})$ the quasi-periodicity implies that:

$$\Theta(x+\lambda') = \Theta(x) + \alpha(\lambda').$$

Then the Λ-rationality of $\ker(q_{\mathbb{R}})$ implies that Θ is affine linear on $\ker(q_{\mathbb{R}})$ with slope α, but since α isn't integral on $\ker(q_{\mathbb{R}}) \cap \mathbb{Z}^n$, it would follow that the function isn't regular, which is absurd. Hence for such α the set $\Gamma(X, L)$ is just $\{-\infty\}$. We can then summarise the previous argument in the following:

Theorem 4.30. *Let X be a tropical torus of dimension n and $L = L(Q, \alpha)$ be a tropical line bundle on X such that Q extends to a positive semidefinite bilinear form on $\mathbb{R}^n$. Then:*

1. *If α belongs to $imq_{\mathbb{R}} + (\mathbb{Z}^n)^*$, then $\Gamma(X, L)$ is generated by ℓ elements, where ℓ is the cardinality of the torsion part of Cok(q), as a $\mathbb{T}$-module and $\Gamma(X, L)$ embeds into $\mathbb{T}^\ell$ via the map ϕ. Moreover, $\Gamma(X, L)$ is identified with an ℓ-dimensional convex polyhedron in $\mathbb{T}^\ell$.*

2. *If α is not in $imq_{\mathbb{R}} + (\mathbb{Z}^n)^*$, then $\Gamma(X, L) = \{-\infty\}$.*

4.3 Divisors on tropical tori.

In this section we give some notions about divisors on tropical tori: remember that we have defined a rational polyhedron in $\mathbb{R}^n$ to be a set of the form

$$\cap_{i \in I} \{x \in \mathbb{R}^n | a_i \cdot x \geq b_i\}$$

for a finite set I, some a_i belonging to $(\mathbb{Z}^n)^*$ and some real numbers b_i. Now let σ be a rational polyhedron, we define a *face of* σ to be a set of the form

$$\sigma \cap \{x \in \mathbb{R}^n | a \cdot x = b\}$$

where a belongs to $(\mathbb{Z}^n)^*$, b is a real number, and the following condition holds:

$$\sigma \subseteq \{x \in \mathbb{R}^n | a \cdot x \geq b\}.$$

We can then give the following definitions:

Definition 4.31. Let M be a tropical manifold of dimension n with atlas $\{(U_i, \phi_i)\}$.

- A *rational polyhedron of* M is a subset $\sigma \subseteq M$ such that for every chart: $\phi_i(\sigma \cap U_i)$ is the intersection between imϕ_i and a rational polyhedron ρ_i in $\mathbb{R}^n$ for every i. We define a *face of* σ to be a subset $\tau \subseteq \sigma$ such that im$\phi_i(\tau \cap U_i)$ is the intersection of imϕ with a face of ρ_i for every i.

- A *rational polyhedral complex on* M is a set Σ of rational polyhedra on M that satisfies the following:

  1. Σ is locally finite, i.e. each point of its support is contained in a finite number of polyhedra in Σ.

2. Any face of a rational polyhedron in σ belongs to Σ.
3. For every σ, τ in Σ such that $\sigma \cap \tau \neq \emptyset$, their intersection is a face of both σ and τ.

A rational polyhedral complex is said to be *pure k dimensional* if its maximal cells have dimension k.

- Let Σ be a pure $n-1$ dimensional polyhedral complex in M, a *weight* of σ is a function
$$f : \Sigma_{n-1} \to \mathbb{Z},$$
where Σ_{n-1} is the set of $n-1$-dimensional cells in Σ. We call the pair (Σ, f) a *weighted rational polyhedral complex.*

The k-dimensional polyhedral complexes that we want to study must satisfy another condition, namely to be balanced at every $k-1$-dimensional cell in the following sense:

Definition 4.32. Let (Σ, f) be a weighted rational polyhedral complex in $\mathbb{R}^n$, and let P be a $n-2$-dimensional cell of Σ. We denote with $L(P)$ the tangent space of P, we also denote with $S_1, \ldots, S_k$ the $n-1$-dimensional cells adiacent to P. Consider the primitive outgoing vectors $v_1, \ldots, v_k$ that are respectively parallel to $S_1/L(P), \ldots, S_k/L(P)$, notice that for dimensional reasons the choice of these vectors is unique. We define (Σ, f) to be *balanced at P* if $\sum_{i=1}^{k} f(S_i)v_i = 0$. We say that (Σ, f) is *balanced* if it's balanced at every $n-2$-cell.
A weighted rational polyhedral complex on a tropical manifold is balanced if its restriction to each chart is a balanced polyhedral complex.

A way to construct balanced polyhedral complexes is the following: let
$$f : \mathbb{R}^n \longrightarrow \mathbb{R}$$
be a piecewise-linear function with integer slopes. We define $D(f)$ to be the subset of $\mathbb{R}^n$ given by the points of non-differentiability of f. If we consider the set K defined by:
$$K := \{(x, y) \in \mathbb{R}^{n+1} | y = f(x), f \text{ non differentiable in } x\},$$
we have that, by the definition of f, each point of K has a neighborhood in which K is the intersection of a finite number of hyperplanes, therefore it's homeomorphic to the restriction of a rational polyhedron in $\mathbb{R}^{n+1}$. Since the projection $K \to \mathbb{R}^n$ given by $(x, y) \mapsto x$ is a homeomorphism onto its image

$D(f)$, we have that $D(f)$ is a rational polyhedral complex. We can also define a weight on such complex in the following way: let S be a maximal cell of $D(f)$, for dimensional reasons, we can find two open sets U_1, U_2 such that the intersection of their closures is S, and the restriction of f to these subsets is linear with integer slope. We define the *lattice length* ℓ_S of the difference between the slopes to be the largest positive integer that divides such difference. Then the weight of S is defined to be ℓ_S if f is convex near S, and $-\ell_S$ otherwise. One can prove that this gives $D(f)$ the structure of a balanced polyhedral complex.
We can finally state the following:

Definition 4.33. Let M be a tropical manifold, a *divisor* D on M is a balanced polyhedral complex on M. Notice that one can always write D locally as $D(f)$ for a rational function f on an open subset of X. This means that a divisor is given by pairs $\{(U_i, f_i)\}$ where $\{U_i\}$ is an open covering of M and the $f_i : U_i \to \mathbb{R}$ are rational functions.

One can check changing charts that for a divisor $\{(U_i, f_i)\}$ the differences $f_i - f_j$ are $\mathbb{Z}$-affine linear on $U_i \cap U_j$, therefore we can always associate to a divisor D a line bundle $O(D)$ on X defining the transition function ϕ_{ij} to be the difference $f_i - f_j$. We denote with $\mathrm{Div}(X)$ the set of all divisors on X, we can define a group structure on $\mathrm{Div}(X)$ by setting:

$$\{(U_i, f_i)\} + \{(V_j, g_j)\} = \{(U_i \cap V_j, f_i + g_j)\}.$$

Definition 4.34. We say that a divisor D is *effective* if it's given by a global regular section of $O(D)$, we say that two divisors D_1, D_2 are *linearly equivalent* if $O(D_1)$ and $O(D_2)$ are isomorphic.

Notice that the definition of effective divisor is different from the classical one, found for example in [6, page 37], where we define an effective divisor to be a divisor given by pairs $\{(U_i, f_i)\}$ where all f_i are regular. This because it's not always true that the set of non-differentiable points of the f_i is equal to the set of non-differentiable points of some theta function. Indeed, recall *Example 4.26*, in that situation we have that a divisor can have as support any finite subset of $\mathbb{R}/\mathbb{Z}$, but a theta function $\Theta = \max\{\Theta_0 + s_0, \Theta_1 + s_1\}$ can have at most four points of non-differentiability, since the generators have one point of non differentiability each. Despite this we have that the converse holds, namely every theta function gives rise to a divisor:

Lemma 4.35. *Let X be a tropical torus with a positive semidefinite line bundle L, and let Θ be a theta function in $\Gamma(X, L)$. Then there is an effective* *divisor D in $|L|$ whose support is equal to the set of non-differentiable points of Θ.*

Proof. We will construct an effective divisor D given by Θ in such way that $O(D) = L$ and the thesis holds. Suppose that the line bundle L is given by the covering $\{U_i\}$ and the set of transition functions $\{\phi_{ij}\}$, in the following argument we will assume that each transition function ϕ_{ij}, which is $\mathbb{Z}$-affine linear and defined only on $U_i \cap U_j$, can be extended to a $\mathbb{Z}$-affine linear function on U_i, and hence on any set U_k via the homeomorphisms $U_k \cong \mathbb{R}^n \cong U_i$. Let's start fixing two open sets U_i, U_j with non-empty intersection, and define the function $f_i : U_i \to \mathbb{R}$ by $f_i(p) = \phi_{ij}(p) + \Theta(p)$, also define $f_j : U_j \to \mathbb{R}$ by $f_j(p) = \Theta(p)$. As a consequence we have that their difference $f_i - f_j$ defined on U_{ij} is equal to ϕ_{ij}. As second step we consider every open set U_k of the covering such that $U_k \cap U_i \neq \emptyset$, and we define a function $f_k : U_k \to \mathbb{R}$ by $f_k(p) = \phi_{ki}(p) + \phi_{ij}(p) + \Theta(p)$, in this way we have that $f_k - f_i = \phi_{ki}$, and moreover, given two such subsets U_k, U_w:

$$f_k - f_w = \phi_{ki} + \phi_{ij} + \Theta - \phi_{wi} - \phi_{ij} - \Theta = \phi_{ki} - \phi_{wi},$$

and remember that $\phi_{iw} = -\phi_{wi}$ and that transition functions satisfy the cocycle condition, hence this is equal to

$$\phi_{ki} + \phi_{iw} = \phi_{kw}.$$

Moreover, suppose that $U_k \cap U_j \neq \emptyset$, then one computes the difference

$$f_k - f_j = \phi_{ki} + \phi_{ij} + \Theta - \Theta = \phi_{ki} + \phi_{ij} = \phi_{kj}.$$

The further steps consist in considering sets U_r for which f_r is still not defined and that have non-empty intersection with an open set U_ℓ for which f_ℓ is defined, then we put: $f_r := \phi_{r\ell} + f_\ell$. In such way we have that the difference $f_r - f_\ell = \phi_{r\ell}$, moreover for each $f_r, f_{r'}$ defined in such way we have similarly as before that:

$$f_r - f_{r'} = \phi_{r\ell} + f_\ell - \phi_{r'\ell} - f_\ell = \phi_{r\ell} + \phi_{\ell r'} = \phi_{rr'}.$$

Furthermore, for every open set $U_{\ell'}$ for which $f_{\ell'}$ was already defined and such that $U_{\ell'} \cap U_r \neq \emptyset$ we have that:

$$f_r - f_{\ell'} = \phi_{r\ell} + f_\ell - f_{\ell'} = \phi_{r\ell} + \phi_{\ell\ell'} = \phi_{r\ell'}$$

because by construction we have that $f_\ell - f_{\ell'} = \phi_{\ell\ell'}$. Compactness of X implies that this procedure ends in a finite number of steps, then we can define a divisor D given by the covering $\{U_t\}$ and the set of regular functions $\{f_t\}$, by construction we have that the line bundle $O(D)$ is equal to L, moreover we have by construction that every regular function f_t is the sum of Θ and a $\mathbb{Z}$-affine linear function, therefore the set of points of non-differentiability of f_t is just the set of points of non-differentiability of Θ in U_t, hence D is effective. □

Given a divisor D on X, we denote with $|D|$ the set of all effective divisors linearly equivalent to D.

5 Embedding in the projective space.

In this section we give the conditions for a tropical torus to be embedded in a tropical projective space, most of the results included are original, although strongly inspired by the theory of complex abelian varieties that one can study from [1]. In section 2 of this thesis one can find a complete set of the results of which we find a tropical analogue in this section.

5.1 The function φ_L

The notion of projective space extends to a tropical analogue in a way that we give in the following definition:

Definition 5.1. Let $\sim$ be the equivalence relation on $\mathbb{T}^{n+1}$ given by:

$$x \sim y \iff \exists t \in \mathbb{R} \text{ such that } x = \text{“}t \cdot y\text{”},$$

where “$t \cdot y$” stands for the scalar product giving $\mathbb{T}^{n+1}$ the structure of a tropical module. Let's still denote with $-\infty$ the $n+1$-tuple of $\mathbb{T}^{n+1}$: $(-\infty, \dots, -\infty)$, then we define *n-dimensional tropical projective space* the quotient:

$$\mathbb{TP}^n := (\mathbb{T}^{n+1} \setminus \{-\infty\}) / \sim .$$

We consider this space to be equipped with the quotient topology.

We will now study the way and the conditions to give an embedding of a tropical abelian variety in the tropical projective space, namely we consider a tropical torus $X = \mathbb{R}^n/\Lambda$ with a positive semi-definite line bundle $L = L(Q, \alpha)$. As done before we consider a set B of representatives of $(\mathbb{Z}^n)^*/q(\Lambda)$ and a basis $\{\Theta_b\}_{b \in B}$ of $\Gamma(X, L)$, then we define a function:

$$\varphi_L : X \longrightarrow \mathbb{TP}^{|B|-1}$$

by:

$$\varphi_L(x) = (\Theta_b(x))_{b\in B}.$$

We first notice that if $y = x + \lambda$ for an element λ in Λ, the quasi-periodicity of theta functions implies that $\varphi_L(x) = \varphi_L(y)$, and hence φ_L is well-defined. This is a difference with the complex analog of the function φ_L, indeed according to *Lemma 2.15* we have that the function ϕ_L defined in section 2 is well defined on a complex torus if every eigenvalue is bigger or equal to 2.

Denote respectively with p and p' the projections $\mathbb{R}^n \to X$ and $\mathbb{T}^{|B|} \to \mathbb{TP}^{|B|-1}$, moreover denote with f the function $\mathbb{R}^n \to \mathbb{T}^{|B|} \setminus \{-\infty\}$ defined by $f(x) := (\Theta_b(x))_{b\in B}$. By definition the following diagram commutes:

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{\;f\;} & \mathbb{T}^{|B|} \setminus \{-\infty\} \\ \Big\downarrow{\scriptstyle p} & & \Big\downarrow{\scriptstyle p'} \\ X & \xrightarrow{\;\varphi_L\;} & \mathbb{PT}^{|B|-1} \end{array}.$$

By the universal property of the projection this implies that φ_L is a continuous function, we would like to study when this it is injective, the following example tells us that in general this is not true:

Example 5.2. Let's recall *Example 4.26*, in which we had found that for the 1-dimensional torus X the space $\Gamma(X, L(2,0))$ has generators:

$$\Theta_0(x) = 2x[x] - [x]^2 + (2(x - [x]) - 1)^+,$$

$$\Theta_1(x) = 2x[x-1/2]+x-[x-1/2]^2-1/4-[x-1/2]+(2(x-1/2-[x-1/2])-1)^+.$$

We want to find to points x, y and a real number t such that $\varphi_L(x) =$ "$t \cdot \varphi_L(y)$", by periodicity of the difference between theta functions we just have to look at the case in which x, y belong to [0,1). There are two different cases:

- If $x, y < 1/2$ then we have that $\Theta_0(x) = 0 = \Theta_0(y)$, so we must have $t = 0$, but then the condition $\varphi_L(x) =$ "$t \cdot \varphi_L(y)$" becomes:

  $$x - 1/4 = \Theta_1(x) = \Theta_1(y) = y - 1/4,$$

  which implies $x = y$.

- If $x < 1/2$ and $y \geq 1/2$ we have that $\Theta_0(x) = 0$ and $\Theta_0(y) = 2y - 1$ so we put $t := 1 - 2y$. Then the condition $\varphi_L(x) =$ "$t \cdot \varphi_L(y)$" becomes:

$$x - 1/4 = \Theta_1(x) = \Theta_1(y) + 1 - 2y = y - 1/4 + 1 - 2y,$$

  namely:

$$y = 1 - x.$$

  It follows that all points x, y in [0,1) satisfying this condition have the same image via φ_L, which therefore is not injective.

We now state the following lemma:

Lemma 5.3. *Let $X = \mathbb{R}^n/\Lambda$ be a tropical torus and $L = L(Q, \alpha)$ a positive definite line bundle such that there is an eigenvalue of Q which is at least 2. Let y_1, y_2 be two points in X such that $\varphi_L(y_1) = \varphi_L(y_2)$, then for every divisor D in $|L|$, y_1 belongs to D if and only if y_2 does.*

Proof. In this proof we will consider the basis $\mathcal{B}$ on Λ for which the matrix of $Q_{\mathbb{R}}$ is in diagonal form. Take any divisor D in $|L|$, we have that it is given by a global section of $O(D) \cong L$, therefore by a theta function of $\Gamma(X, L)$, let it be:

$$\Theta = \max_{b \in B}\{\Theta_b + s_b\},$$

for some s_b in $\mathbb{T}$. The hypothesis on the eigenvalues tells us that there are at least two Theta functions. Assume that y_1 belongs to D, let's first suppose that there are two different elements b, b' in B such that:

$$\Theta(y_1) = \Theta_b(y_1) + s_b = \Theta_{b'}(y_1) + s_{b'},$$

and since $\Theta(y_1)$ cannot be $-\infty$ it follows that $s_b \neq -\infty$, hence:

$$\Theta_b(y_1) - \Theta_{b'}(y_1) = s_{b'} - s_b.$$

Now, the fact that $\varphi_L(y_1) = \varphi_L(y_2)$ tells us that:

$$\Theta_b(y_2) - \Theta_{b'}(y_2) = \Theta_b(y_1) - \Theta_{b'}(y_1) = s_{b'} - s_b,$$

so $\Theta_b(y_2) + s_b = \Theta_{b'}(y_2) + s_{b'}$. It's not obvious that this number realises the maximum in $\Theta(y_2)$, but we can notice that for every $\overline{b}$ in B: $\Theta_b(y_1) - \Theta_{\overline{b}}(t_1) \geq s_{\overline{b}} - s_b$, so using once again the fact that $\varphi_L(y_1) = \varphi_L(y_2)$ we get:

$$\Theta_b(y_2) - \Theta_{\overline{b}}(y_2) = \Theta_b(y_1) - \Theta_{\overline{b}}(y_1) \geq s_{\overline{b}} - s_b,$$

which implies the thesis. Now let's suppose that there is a b^* in B such that $\Theta(y_1) = \Theta_{b^*}(y_1) + s_{b^*}$, and it's the only function realising the maximum. Since y_1 belongs to D it follows that Θ_{b^*} is non-differentiable in y_1. By what was noticed before we also have that $\Theta(y_2) = \Theta_{b^*}(y_2) + s_{b^*}$, and this is the only function realising the maximum. To study better this case, suppose that 0 is an element of B, and consider two representatives p, q of y_1, y_2 in D_0^0. Suppose that in a neighborhood U of p the divisor D is given by the function $\Theta_{b^*} + f$, with f $\mathbb{Z}$-affine on U, and suppose that p belongs to the facet of the boundary of $D_\lambda^{b^*}$ with slope $q(\lambda')$ for a λ in Λ. Consider another theta function of the form Θ_b, and a point x on the facet of the boundary of D_0^b with slope $q(\lambda')$. Then the function $F(y) := \Theta_b(y+p-x)+\Theta_{b^*}(p)-\Theta_b(2p-x)$ is regular and satisfies the quasi-periodicity that characterizes theta functions, hence it's a theta function, and it's different from Θ_{b^*} because it has different slopes. As a consequence we can redefine D by taking on U the function $F + f$, because it has the same non-differentiable points as before, then we conclude using a similar argument as before. □

A consequence of the proof of this lemma is that if two elements x, y have the same image via φ_L, then for every $\Theta = \max_{b\in B}\{\Theta_b + s_b\}$ in $\Gamma(X, L)$ the numbers $\Theta(x)$ and $\Theta(y)$ are written in the same way, i.e. there is one b' in B such that $\Theta_{b'}(x)+s_{b'}$ and $\Theta_{b'}(y)+s_{b'}$ realise the maximums in the definition of $\Theta(x)$ and $\Theta(y)$.

We have that the converse of the previous lemma is also true:

Lemma 5.4. *With the conditions of the previous lemma, take two points y_1, y_2 in X such that for every divisor D in $|L|$:*

$$y_1 \in D \iff y_2 \in D,$$

then $\phi_L(y_1) = \phi_L(y_2)$.

Proof. Let's first suppose that in y_1, y_2 every Θ_b is differentiable. Fix two distinct elements b, b' of B and define D to be the divisor given by $\max\{\Theta_b + s_b, \Theta_{b'} + s_{b'}\}$, where $s_b, s_{b'}$ are chosen in such way that

$$\Theta_b(y_1) - \Theta_{b'}(y_1) = s_{b'} - s_b.$$

Clearly y_1 belongs to D and hence y_2 does too, therefore

$$\Theta_b(y_2) - \Theta_{b'}(y_2) = s_{b'} - s_b = \Theta_b(y_1) - \Theta_{b'}(y_1),$$

and repeating the same procedure for every b, b' gives the thesis. Now suppose that a generator Θ_b is non-differentiable in y_1, y_2, then consider two representatives p, q for y_1, y_2 in D_0^0 and suppose that $\Theta_b(q) - \Theta_b(p)$ is positive. Define the theta function Θ by:

$$\Theta(x) := \max\{\Theta_b, \Theta_0 + \Theta_b(q) + \alpha(r)/2\},$$

then Θ is non-differentiable in q, and hence p belongs to the divisor defined by Θ. Since up to replacing $\Theta_0 + \Theta_b(q) + \alpha(r)/2$ with $\Theta_0 + \Theta_b(q) + \alpha(r)/2 + \epsilon$, for an ϵ small enough, in the definition of Θ, we find a contradiction, because $\Theta_0 + \Theta_b(q) + \alpha(r)/2$ is differentiable in p. □

Lemma 5.3 and *Lemma 5.4* imply the following corollary:

Corollary 5.5. *Let $X = \mathbb{R}^n/\Lambda$ be a tropical torus and $L = L(Q, \alpha)$ a positive definite line bundle such that there is an eigenvalue of Q which is at least 2. Then for every two points y_1, y_2 in X, we have that $\phi_L(y_1) = \phi_L(y_2)$ if and only if for every divisor D in $|L|$:*

$$y_1 \in D \iff y_2 \in D.$$

Let's now look at an example in which things go differently than in *Example 5.2*:

Example 5.6. Let X be the 1-dimensional torus $\mathbb{R}/\mathbb{Z}$, and consider the line bundle $L := L(3, 0)$ and the set $B := \{0, 1, 2\}$. For every b in B we have that the generators of $\Gamma(X, L)$ have the form:

$$\Theta_b(x) = \max_{n \in \mathbb{Z}} \Big\{ (3n + b)x - \frac{3}{2}\Big(\frac{3n + b}{3}\Big)^2 \Big\},$$

and we can compute this maximum, obtaining that it is equal to:

$$\Theta_b(x) = \max\Big\{ (3[x - b/3] + b)x - \frac{3}{2}\Big(\frac{3[x - b/3] + b}{3}\Big)^2, \\ (3[x - b/3] + 3 + b)x - \frac{3}{2}\Big(\frac{3[x - b/3] + 3 + b}{3}\Big)^2 \Big\}.$$

To understand this better let's compute the difference between the two elements I am considering, it is equal to:

$$3x - 3[x - b/3] - 3/2 - b,$$

therefore the maximum above is equal to the second element of the set if and only if:

$$3(x - b/3 - [x - b/3]) - 3/2 \geq 0,$$

which is true if and only if:

$$x - b/3 - [x - b/3] \geq 1/2.$$

Suppose that x belongs to [0,1), this implies the following facts:

1. If $x < 1/2$ then $\Theta_0(x) = 0$, otherwise $\Theta_0(x) = 3x - 3/2$.
2. If $1/3 \leq x < 5/6$ then $[x - 1/3] = 0$ and $x - 1/3 < 1/2$, hence $\Theta_1(x) = x - 1/6$. Otherwise, if $x \geq 5/6$ we have that $\Theta_1(x) = 4x - 8/3$, while if $0 \leq x < 1/3$ we have that $\Theta_1(x) = x - 1/6$.
3. If $1/6 \leq x < 2/3$ we have that $[x - 2/3] = -1$, hence $x - 2/3 - [x - 2/3] = x + 1/3 \leq 1/2$, and therefore $\Theta_2(x) = 2x - 2/3$. Otherwise, if $2/3 \leq x$ we have that $x - 2/3$ cannot be bigger than $1/2$, and hence we still have that $\Theta_2(x) = 2x - 2/3$, while if $0 \leq x < 1/6$ it follows that $\Theta_2(x) = -x - 1/6$.

Now, let's suppose to have two points x, y in [0,1) such that $\varphi_L(x) = \varphi_L(y)$, there are different possibilities:

- If they are both smaller than $1/2$, then $\Theta_0(x) = 0 = \Theta_0(y)$, and hence we must have $x - 1/6 = \Theta_1(x) = \Theta_1(y) = y - 1/6$, which implies that $x = y$.
- If $x, y \geq 1/2$, then $\Theta_0(x) = 3x - 3/2$ and $\Theta_0(y) = 3y - 3/2$, while $\Theta_2(x) = 2x - 2/3$ and $\Theta_2(y) = 2y - 2/3$, which implies that

  $$3(x - y) = 2(x - y),$$

  i.e. $x = y$.
- If $x \geq 1/2$ and $y < 1/2$, then $\Theta_0(y) = 0$ and $\Theta_0(x) = 3x - 3/2$, the following system of equalities must hold:

  $$\begin{cases} \Theta_1(x) - \Theta_1(y) = \Theta_0(x) \\ \Theta_2(x) - \Theta_2(y) = \Theta_0(x) \end{cases}.$$

  Let's first suppose that $x < 5/6$, the system becomes:

  $$\begin{cases} x - y = 3x - 3/2 \\ 2x - 2/3 - \Theta_2(y) = 3x - 3/2 \end{cases},$$

and one sees that for all values that $\Theta_2(y)$ can assume one either gets that $x = y$ or a contradiction.
Suppose that $x \geq 5/6$, then the equation

$$\Theta_1(x) - \Theta_1(y) = \Theta_0(x)$$

becomes

$$4x - 8/3 - y + 1/6 = 3x - 3/2,$$

which yields:

$$x = y + 1,$$

which is absurd. It follows that in this case the map φ_L is injective.

5.2 The tropical version of Lefschetz's theorem.

Now, consider a tropical torus $X = \mathbb{R}^n/\Lambda$ of dimension n, and a tropical line bundle $L = L(Q, \alpha)$ on X. Recall that by definition L is given by the quotient $\mathbb{R}^n \times \mathbb{T}/\Lambda$ where Λ acts on $\mathbb{R}^n \times \mathbb{T}$ by: $\lambda \cdot (x, t) = (x + \lambda, t + Q_{\mathbb{R}}(x, \lambda) + \frac{1}{2}Q_{\mathbb{R}}(\lambda, \lambda) + \alpha(\lambda))$. It follows that we can compute directly the transition functions: take two open subsets U, V of $\mathbb{R}^n$ such that they are respectively homeomorphic to $U/\Lambda, V/\Lambda$ and suppose that $U/\Lambda \cap V/\Lambda \neq \emptyset$. It follows that there exists (and it's unique) a λ in Λ such that $x + \lambda = y$ for every x in U and y in V such that $[x] = [y]$ in $U/\Lambda \cap V/\Lambda$. Then we have that the transition function linked to $U/\Lambda, V/\Lambda$ is equal to:

$$\phi_{UV}([x]) = Q_{\mathbb{R}}(x, \lambda) + \frac{1}{2}Q_{\mathbb{R}}(\lambda, \lambda) + \alpha(\lambda)$$

for every x in U such that $[x]$ belongs to $U/\Lambda \cap V/\Lambda$.
Then consider an element $[v]$ in X, it defines a translation:

$$t_v : X \longrightarrow X$$

by $t_v([x]) = [x] + [v]$. As a consequence of what we just noticed we have that the transition functions of $t_v^* L$ all look like:

$$\phi([x]) = Q_{\mathbb{R}}(x + v, \lambda) + \frac{1}{2}Q_{\mathbb{R}}(\lambda, \lambda) + \alpha(\lambda)$$

for some λ in Λ. Since $Q_{\mathbb{R}}(v, \lambda)$ is linear in λ, we deduce the following formula:

$$t_v^*(L(Q, \alpha)) = L(Q, \alpha + Q_{\mathbb{R}}(-, v)). \tag{7}$$

This formula allows us to prove the tropical version of the Theorem of the Square:

Theorem 5.7. *Let v, w be two elements of $\mathbb{R}^n$, X a tropical torus of dimension n, and $L = L(Q, \alpha)$ a line bundle on X. Then the following formula holds:*

$$t_{v+w}^* L \cong t_v^* L \otimes t_w^* L \otimes L^{-1}.$$

Proof. The formula is consequence of the following chain of equalities, in which we just use formula 7 and the fact that $L(-,-)$ is a homomorphism:

$$\begin{aligned} t_{v+w}^* L &= L(Q, \alpha + Q_{\mathbb{R}}(-, v+w)) = \\ &= L(Q, \alpha + Q_{\mathbb{R}}(-, v) + Q_{\mathbb{R}}(-, w))) = \\ &= L(Q + Q - Q, \alpha + Q_{\mathbb{R}}(-, v) + \alpha + Q_{\mathbb{R}}(-, w) - \alpha) = \\ &= L(Q, \alpha + Q_{\mathbb{R}}(-, v)) \otimes L(Q, \alpha + Q_{\mathbb{R}}(-, w)) \otimes L(-Q, -\alpha) = \\ &= t_v^* L \otimes t_w^* L \otimes L^{-1}. \end{aligned}$$

□

Observation 5.8. Notice that for every divisor D on X and for every x in X, we have by definition that $t_x^* D = D - x$, and therefore:

$$y \in t_x^* D \iff x \in t_y^* D. \tag{8}$$

We now need to give some definitions:

Definition 5.9. Let X be an n-dimensional tropical torus and $L = L(Q, \alpha)$ a line bundle on X, we define the group $\Lambda(L)$ to be the set:

$$\Lambda(L) = \{x \in \mathbb{R}^n | Q_{\mathbb{R}}(x, \mathbb{Z}^n) \subseteq \mathbb{Z}\},$$

then we define the subgroup of X, $K(L) := \Lambda(L)/\Lambda$.

Lemma 5.10. *Let X be an n-dimensional tropical torus and $L = L(Q, \alpha)$ a positive definite line bundle on X, then $K(L)$ is a finite group isomorphic to the cokernel of Q.*

Proof. Let $\{e_1, \ldots, e_n\}$ be the canonical base of $\mathbb{Z}^n$ and $\{\lambda_1, \ldots, \lambda_n\}$ a base of Λ such that the matrix of Q with respect to this bases has the form $\mathrm{diag}(d_1, \ldots, d_n)$. Take an element x of $\Lambda(L)$, we can write

$$x = a_1\lambda_1 + \cdots + a_n\lambda_n$$

for real numbers $a_1, \ldots, a_n$. It follows that for every $i = 1, \ldots, n$

$$Q_{\mathbb{R}}(x, e_i) = a_i d_i,$$

which is an integer if and only if a_i belongs to $1/d_i\mathbb{Z}$. As a consequence we have that x belongs to the group

$$\mathbb{Z}\{\lambda_1/d_1\} \oplus \cdots \oplus \mathbb{Z}\{\lambda_n/d_n\},$$

which modulo Λ is finite and isomorphic to $Coker(Q)$. □

From now on we will denote with $Pic^0(X)$ the subgroup of $Pic(X)$ given by the line bundles whose first Chern class is 0, in the following lemma we state some basic facts on the group $K(L)$.

Lemma 5.11. *Let X be an n-dimensional tropical torus and $L = L(Q, \alpha)$ a line bundle on X, denote with m_X the endomorphism of X induced by the multiplication by m, where m is an integer. The following properties hold:*

1. *For every P in $Pic^0(X)$, $K(L \otimes P) = K(L)$.*
2. *$K(L) = X$ if and only if L belongs to $Pic^0(X)$.*
3. *For every m in $\mathbb{Z}$ we have that $K(L^m) = m_X^{-1}(K(L))$.*
4. *For every m in $\mathbb{Z} \setminus \{0\}$, we have that $K(L) = m_X(K(L^m))$.*

Proof. The first two points follow directly from the definitions. For the third point, observe that $L^m = L(mQ, m\alpha)$, hence:

$$\Lambda(L^m) = \{x \in \mathbb{R}^n | mQ_{\mathbb{R}}(x, \mathbb{Z}^n) \subseteq \mathbb{Z}\} = \{x \in \mathbb{R}^n | Q_{\mathbb{R}}(mx, \mathbb{Z}^n) \subseteq \mathbb{Z}\},$$

so, making the substitution $y = mx$ we get:

$$\Lambda(L^m) = \{y/m \in \mathbb{R}^n | Q_{\mathbb{R}}(y, \mathbb{Z}^n) \subseteq \mathbb{Z}\} = m_X^{-1}(K(L)).$$

The last point follows from the third. □

The following theorem will be a useful tool for the study of the function φ_L:

Theorem 5.12. *Let X be an n-dimensional tropical torus and $M = L(Q, \alpha)$ a line bundle on X, then there exists a line bundle L on X such that $M = L^m$ if and only if the kernel of m_X is contained in $K(M)$.*

Proof. First suppose that $M = L^m$, by the third point of *Lemma 5.11* we have that

$$K(M) = K(L^m) = m_X^{-1}(K(L)),$$

and since 0 belongs to $K(L)$, we have that $m_X^{-1}(0)$ is contained in $K(M)$. For the converse, suppose that $\ker(m_X) \subseteq K(M)$, this implies that $1/m\Lambda \subseteq \Lambda(M)$. As a consequence we can define: $\alpha' : \mathbb{R}^n \to \mathbb{R}$ by $\alpha'(x) = \alpha(x)/m$ for every x in $\mathbb{R}^n$, and $Q' : \Lambda \to (\mathbb{Z}^n)^*$ by $Q'(\lambda, z) := Q_{\mathbb{R}}(\lambda/m, z)$ for every λ in Λ and z in $\mathbb{Z}^n$. We have that α' is a linear map on $\mathbb{R}^n$, while by hypothesis Q' is well-defined, it follows that taking $L := L(Q', \alpha')$ we have that:

$$L^m = L(mQ', m\alpha') = L(Q, \alpha) = M.$$

□

We now state two technical lemmas which we will need later:

Lemma 5.13. *Let X be an n-dimensional tropical torus and $L = L(Q, \alpha)$ a positive definite line bundle on X, suppose we have t points of X: $p_1, \dots, p_t$ such that $\sum_{i=1}^t p_i = 0$, then for every divisor D in $|L|$ we have that:*

$$\sum_{i=1}^{t} t_{p_i}^* D \sim t \cdot D,$$

or, equivalently:

$$\bigotimes_{i=1}^{t} t_{p_i}^* L = L^t.$$

Proof. The thesis is a consequence of the following computation, in which we only use formula 7:

$$\begin{aligned}\bigotimes_{i=1}^{t} t_{p_i}^* L &= \bigotimes_{i=1}^{t} L(Q, \alpha + Q_{\mathbb{R}}(-, p_i)) = \\ &= L\Big(t \cdot Q, t \cdot \alpha + \sum_{i=1}^{t} Q_{\mathbb{R}}(-, p_i)\Big) = \\ &= L(t \cdot Q, t \cdot \alpha)) = \\ &= L^t.\end{aligned}$$

□

Lemma 5.14. *Let X be an n-dimensional tropical torus and $L = L(Q, \alpha)$ a positive definite line bundle on X, if $[x]$ belongs to X, then there exists a divisor D in $|L|$ such that $t_x^* D = D$ if and only if $[x] = 0$.*

Proof. By *Lemma 4.35* we can consider the divisor D that rises from a theta function of the form Θ_b. Now, consider an element x in $\mathbb{R}^n \setminus \Lambda$, and recall that by *Proposition 4.12* the function Θ_b is fully determined by its values on a fundamental domain D^b_Λ, for a λ in Λ. The points of $\mathbb{R}^n$ on which Θ_b isn't linear are all the Λ-translates of the boundary of D^b_λ, and since x doesn't belong to Λ it follows that the translation by x cannot send the boundary of D^b_λ to the boundary of a $D^b_{\lambda'}$ for a λ' in Λ. □

Finally, we are able to prove the following theorem, which is an original result of this master thesis:

Theorem 5.15. *Let X be an n-dimensional tropical torus and $L = L(Q, \alpha)$ a positive definite line bundle on X, suppose that there is a basis of Λ and a basis of $\mathbb{Z}^n$ such that the bilinear form Q is represented by a diagonal matrix*

$$\begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{pmatrix},$$

such that $d_1 | d_i$ for every $i = 1, \dots, n$, and $d_1 \geq 3$. Then the map φ_L is continuous and injective.

Proof. We have already noticed continuity, suppose we have two points $[y_1], [y_2]$ in X such that $\varphi_L([y_1]) = \varphi_L([y_2])$. Consider an element $[\lambda/d_1]$ belonging to $\ker d_{1X}$, the hypothesis on the diagonal form of Q implies that $[\lambda/d_1]$ belongs to $K(L)$, and hence by *Theorem 5.12* it follows that there is a positive definite line bundle M in $Pic(X)$ such that $M^{d_1} = L$. By *Lemma 5.14* we can now find a divisor D in $|M|$ such that $t^*_x D = D$ only if $[x] = 0$. Let's fix an element $[x_1]$ in $t^*_{y_1} D$, then take $d_1 - 1$ points $[x_2], \dots, [x_{d_1}]$ such that their sum is equal to $-[x_1]$. Now suppose that $[x_2]$ belongs to $t^*_{y_2} D$, since the topological dimension of this divisor is $n-1$, the open ball in X of centre $[x_2]$ and ray ϵ is not contained in $t^*_{y_2} D$ for some $\epsilon > 0$, it follows that there is a v in $\mathbb{R}^n$ such that $[x_2 + v]$ doesn't belong to $t^*_{y_2} D$. Since $d_1 \geq 3$, the points $[x_2], \dots, [x_{d_1}]$ are at least 2, so we can replace $[x_2]$ with $[x_2 + v]$ and $[x_3]$ with $[x_3 - v]$ obtaining $d_1 - 1$ points whose sum is still $-[x_1]$. We can repeat this procedure until we get to the point $[x_{d_1}]$: if it belongs to $t^*_{y_2} D$, analogously as before we can find a w in $\mathbb{R}^n$ such that $[x_{d_1} + w]$ doesn't belong to $t^*_{y_2} D$, and since now $[x_1]$ doesn't belong to $t^*_{y_2} D$, we can choose w small enough such that neither $[x_1 - w]$ does. Making these replacements we end up with $d_1 - 1$ points, whose sum is $-[x_1]$ and neither of which is in

$t^*_{y_2}D$. After a similar procedure and appropriate replacements, we can also suppose that these points don't belong to $t^*_{y_1}D$ either. Let's still denote these points with $[x_2], \dots, [x_{d_1}]$. Consider the divisor $D' := \sum_{i=1}^{d_1} t^*_{x_i}D$, by *Lemma 5.13* it is linearly equivalent to $d_1 \cdot D$, hence $O(D') \cong M^{d_1} \cong L$, and it's also effective because it's sum of effective divisors. Therefore D' belongs to $|L|$. Moreover, we have that $[y_1]$ belongs to $t^*_{x_1}D$, but by construction it doesn't belong to $t^*_{x_i}D$ for any $i = 2, \dots, d_1$, hence we have that $[y_1]$ belongs to D'. Now, *Lemma 5.3* implies that also $[y_2]$ belongs to D', so by construction we must have that it belongs to $t^*_{x_1}D$, i.e. $[x_1]$ belongs to $t^*_{y_2}D$. Since this holds for every point of $t^*_{y_1}D$, it follows that $t^*_{y_1}D \subseteq t^*_{y_2}D$, and by symmetry we get:

$$t^*_{y_1}D = t^*_{y_2}D.$$

Finally, this equality implies that $t^*_{y_2-y_1}D = D$, so we can conclude invoking *Lemma 5.14*. □

Definition 5.16. Let X be an n-dimensional tropical torus and $L = L(Q, \alpha)$ a positive semi-definite line bundle on X, we say that L is *very ample* if φ_L is injective, we say that L is *ample* if there is a positive integer m such that L^m is very ample.

The theorem we just proved implies the following Corollary:

Corollary 5.17. *Let X be an n-dimensional tropical torus and $L = L(Q, \alpha)$ a positive definite line bundle on X satisfying the same hypothesis as in* Theorem 5.15 *except that d_1 can assume every positive value, then L^m is very ample for every $m \geq 3$.*

5.3 Examples

In this subsection we collect a couple of examples that illustrate how *Theorem 5.15* works in the case of 2-dimensional tropical tori.

Example 5.18. Let $X = \mathbb{R}^2/\mathbb{Z}^2$ be a tropical torus and let $Q : \mathbb{Z}^2 \to (\mathbb{Z}^2)^*$ be the module homomorphism given by $Q(e_i) = 2e_i^*$, where $\{e_1, e_2\}$ is a canonical basis for $\mathbb{Z}^2$. Then the extension $Q_\mathbb{R}$ is represented by a scalar matrix with eigenvalue 2. Consider the line bundle $L := L(Q, 0)$, and define B to be the set: $\{(0,0), (0,1), (1,0), (1,1)\}$, from now on we will refer to the element $(0,0)$ just with 0. By *Example 4.19* we have that $D_0^0 = [-1/2, 1/2]^2$, so using *Lemma 4.14* we can reconstruct the behaviour of each generator on this fundamental domain for X. In particular, we obtain that $\Theta_{(0,1)}$ has slope $(0,1)$ on $D_0^0 + (0, 1/2)$, while it has slope $(0,-1)$ on $D_0^0 + (0, 1/2) - (0,1) =$

$D_0^0-(0,1/2)$. Similarly we have that $\Theta_{(1,0)}$ has slope $(1,0)$ on $D_0^0+(1/2,0)$, while it has slope $(-1,0)$ on $D_0^0-(1/2,0)$. As for $\Theta_{(1,1)}$, it has slope $(1,1)$ on $D_0^0+(1/2,1/2)$, while it has slopes $(-1,1),(1,-1),(-1,-1)$ respectively on $D_0^0+(-1/2,1/2), D_0^0+(1/2,-1/2), D_0^0+(-1/2,-1/2)$. We sum up these observations with the following images:

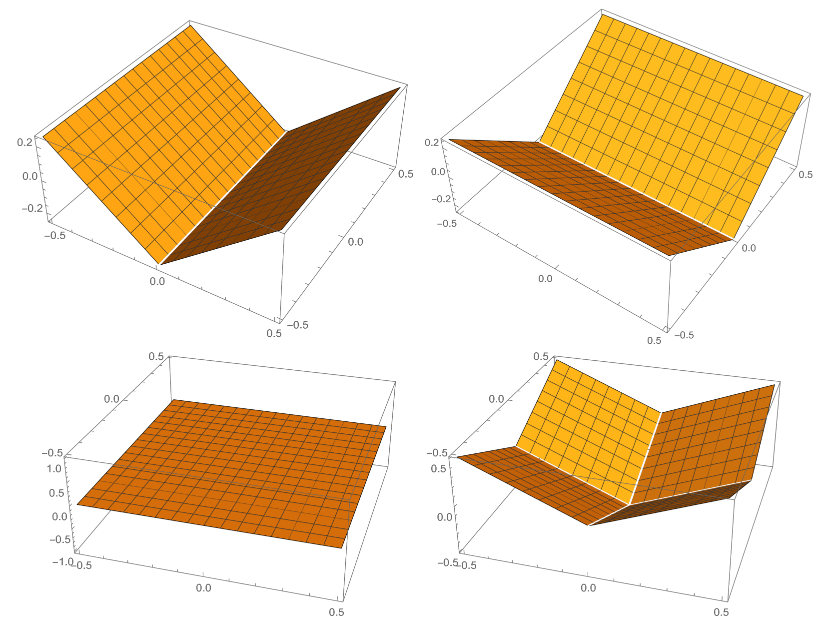
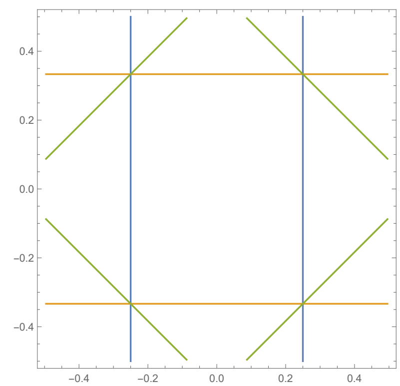


Figure 1: Generators of $\Gamma(X,L)$ on D_0^0, from the top left going clockwise we have: $\Theta_{(1,0)},\Theta_{(0,1)},\Theta_{(1,1)}$ and $\Theta_{(0,0)}$.

To see that ϕ_L in this case is not injective we need to find two points p_1,p_2 in X such that given two representatives x,y in D_0^0 we have that $\Theta_b(x)=\Theta_b(y)$ for every b in B. Let's fix a point p in D_0^0, say (1/4,1/3), then the conditions $\Theta_b(x)=\Theta_b(p)$,where we may suppose that $b\neq 0$, define a union of lines in D_0^0, which look like :



where lines of the same colour are the union of points satisfying the same equality. Since their intersection contains four points, this shows that ϕ_L is not injective.

Example 5.19. Let $X = \mathbb{R}^2/\mathbb{Z}^2$ be a tropical torus and let $Q : \mathbb{Z}^2 \to (\mathbb{Z}^2)^*$ be the module homomorphism given by $Q(e_i) = 3e_i^*$. Then the extension $Q_{\mathbb{R}}$ is represented by a scalar matrix with eigenvalue 3. Consider the line bundle $L := L(Q, 0)$, and define B to be the set:

$$\{(0,0),(0,1),(1,0),(1,1),(0,2),(2,0),(2,1),(1,2),(2,2)\}.$$

Once again, using *Example 4.19* we can compute the values of all generators on $D_0^0 = [-1/2, 1/2]^2$, in this case the hypotheses of *Theorem 5.15* are satisfied, hence we must have that ϕ_L is injective. Now the eight generators different from Θ_0 are represented by the following graphs:

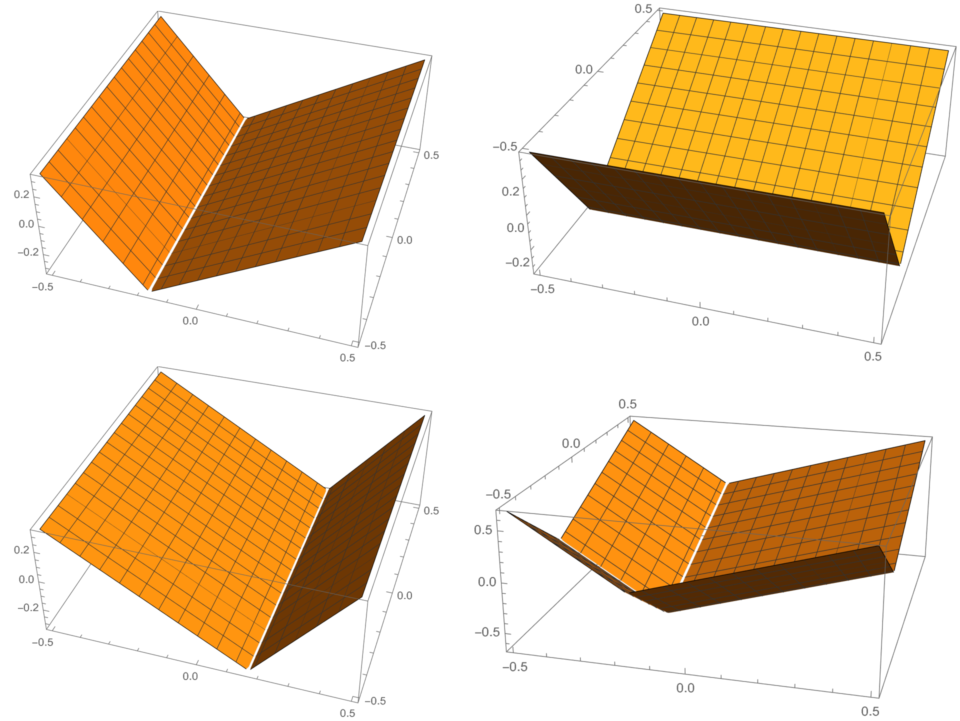


Figure 2: From the top left going clockwise we have: $\Theta_{(1,0)}, \Theta_{(0,1)}, \Theta_{(1,1)}$ and $\Theta_{(2,0)}$.

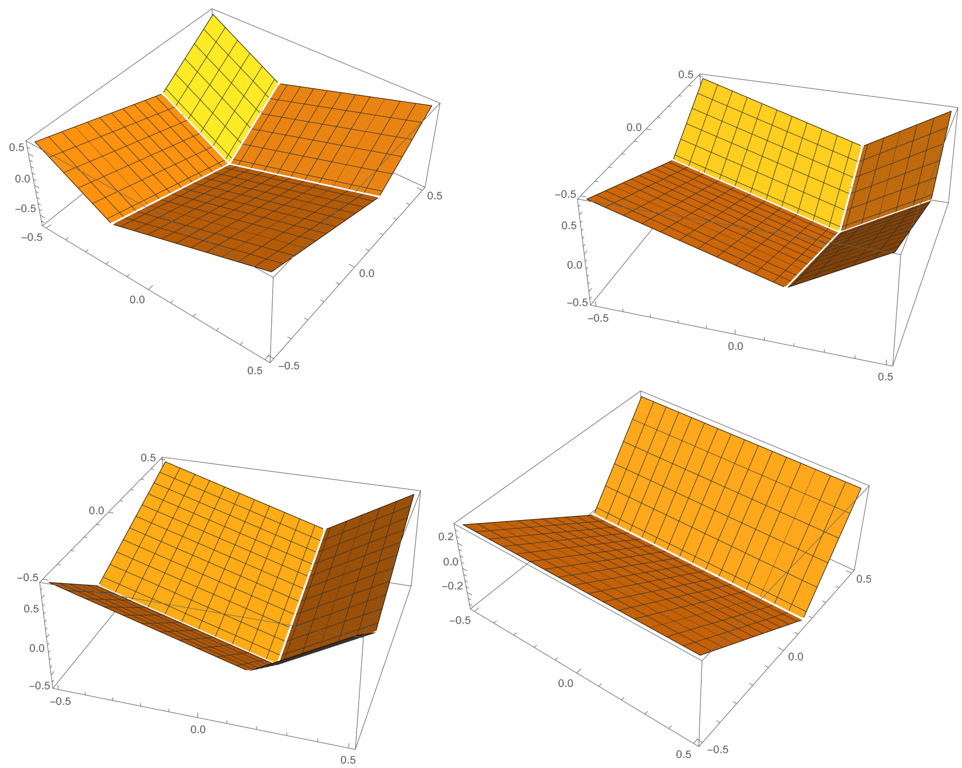
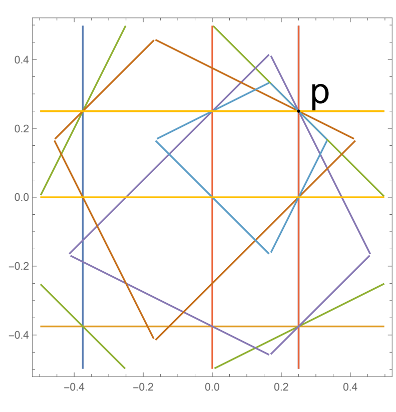


Figure 3: From the top left going clockwise we have: $\Theta_{(1,2)}, \Theta_{(2,2)}, \Theta_{(0,2)}$ and $\Theta_{(2,1)}$.

Clearly Θ_0 is identically zero on D_0^0. Now let's fix a point p in D_0^0, the sets of points in D_0^0 satisfying $\Theta_b(x) = \Theta_b(p)$ will look like:



and we notice that the intersection of all these sets is only $\{p\}$, verifying the injectivity of ϕ_L.

References

[1] Christina Birkenhake and Herbert Lange. *Complex abelian varieties*, volume 302. Springer, 2004.

[2] N. Bourbaki. *Algebra*, livre ii, chapitre ix, 1959.

[3] Grigory Mikhalkin and Ilia Zharkov. Tropical curves, their jacobians and theta functions. *Curves and abelian varieties*, 465:203–230, 2008.

[4] Ken Sumi. Tropical theta functions and Riemann–Roch inequality for tropical abelian surfaces. *Mathematische Zeitschrift*, 297(3):1329–1351, 2021.

[5] Yuan-Chuan Li and Cheh-Chih Yeh. Some characterizations of convex functions. *Computers & mathematics with applications*, 59(1):327–337, 2010.

[6] John Horton Conway and Neil J.A. Sloane. Low-dimensional lattices. vi. Voronoi reduction of three-dimensional lattices. *Proceedings of the Royal Society of London. Series A: Mathematical and Physical Sciences*, 436(1896):55–68, 1992.

[7] Phillip Griffiths and Joseph Harris. *Principles of algebraic geometry*, volume 19. Wiley Online Library, 1978.