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Percolazione per insiemi di livello del Gaussian free field

Relatori:
Prof. Giambattista Giacomin
Université Paris Diderot

Prof. Paolo Dai Pra

Laureando:
1130557
Jacopo Borga
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Introduction

Existence of phase transition for the level-set percolation for the discrete Gaussian free field on $\mathbb{Z}^d$ (DGFF) is a problem that received much attention in the past year, in particular it was studied in the 80's by J. Bricmont, J.L. Lebowitz and C. Maes (see [3]). They showed that in three dimension the DGFF has a nontrivial percolation behavior: sites on which $\varphi_x \geq h$ percolate if and only if $h < h_*$ with $0 \leq h_* < \infty$. Moreover, they generalized the lower bound for h_* in any dimension $d \geq 3$, i.e. $h_*(d) \geq 0$, but they were not able to extend the proof of existence of a non trivial transition for any $d \geq 4$. Recently P.-F. Rodriguez and A.-S. Sznitman (see [11]) proved that $h_*(d)$ is finite for all $d \geq 3$ as a corollary of a more general result concerning the stretched exponential decay of the connectivity function when $h > h_{**}$, where h_{**} is a second critical parameter that satisfied $h_{**} \geq h_*$. In this thesis we tried to get acquainted with some of the techniques developed in the domain, notably to control the large excursions of these fields and to understand the entropic repulsion phenomena, and to comprehend the results on level set percolation in dimension three and larger. In particular, the main goal is to present the two works of Bricmont, Lebowitz and Maes and of Rodriguez and Sznitman. Finally, in the last two chapters we also present a new and original (but incomplete) generalization of the proof (due to J. Bricmont, J.L. Lebowitz and C. Maes) of the existence of a non trivial phase transition to any $d \geq 3$.

Introduzione

L'esistenza di una transione di fase per gli insiemi di livello del Gaussian Free Field discreto in $\mathbb{Z}^d$ (DGFF) è un problema che ha ricevuto molta attenzione negli anni passati, in particolar modo è stato studiato intorno agli anni '80 da J. Bricmont, J.L. Lebowitz and C. Maes (vedi [3]). In questo articolo dimostrarono che il DGFF in dimensione 3 presenta una transizione di fase non triviale: i siti nei quali $\varphi_x \geq h$ percolano se e solo se $h < h_*$ per $0 \leq h_* < \infty$. Inoltre generalizzarono il bound inferiore per h_* in ogni dimensione $d \geq 3$, cioè $h_*(d) \geq 0$, ma non furono in grado di estendere la dimostrazione per l'esistenza di una transizione di fase non triviale ad ogni $d \geq 4$. Recentemente P.-F. Rodriguez e A.-S. Sznitman (vedi [11]) hanno dimostrato che $h_*(d)$ è finito per ogni dimonsione $d \geq 3$ come corollario di un risultato più generale riguardante il semi-decadimento esponenziale della funzione di connettività quando $h \geq h_{**}$, dove h_{**} è un secondo parametro che soddisfa $h_{**} \geq h_*$. In questa tesi l'autore ha cercato di prendere familiarità con alcune tecniche sviluppate in questo dominio, in particolar modo a controllare le grandi escursioni di questi campi, capire il fenomeno di repulsione entropica e comprendere i risultati riguardanti la percolazione per insiemi di livello in dimensione tre o maggiore. In particolare l'obbiettivo principale è quello di presentare i due lavori di Bricmont, Lebowitz e Maes e di Rodriguez e Sznitman. Infine, negli ultimi due capitoli, presenteremo una nuova ed originale (ma incompleta) generalizzazione della dimostrazione (di Bricmont, Lebowitz e Maes) dell' esistenza di una transizione di fase ad ogni $d \geq 3$.

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Contents

Notation

We introduce some notation to be used in the sequel. First of all we explain our convection regarding constant: we denote by $c, c', \dots$ positive constants (different constants could have the same name). Numbered constants $c_0, c_1, \dots$ are defined at the place they first occur within the text and remain fixed from then on until the end of the section. In chapters 1, 3, 4, constants will implicitly depend on the dimension d. Throughout the entire paper, dependence of constants on additional parameters will appear in notation. On $\mathbb{Z}^d$, we respectively denote by $|\cdot|$ and $|\cdot|_\infty$ the Euclidean and ℓ^∞-norms. We denote by $i \sim j$ the couple of vertices i and j such that $|i-j| = 1$. Moreover, for any $x \in \mathbb{Z}^d$ and $r \geq 0$, we let $B(x,r) = \{y \in \mathbb{Z}^d; |y-x|_\infty \leq r\}$ and $S(x,r) = \{y \in \mathbb{Z}^d; |y-x|_\infty = r\}$ stand for the ℓ^∞-ball and the ℓ^∞-sphere of radius r centered at x. Given K and U subsets of $\mathbb{Z}^d$, $K^c = \mathbb{Z}^d \setminus K$ stands for the complement of K in $\mathbb{Z}^d$, $|K|$ for the cardinality of K, $K \subset\subset \mathbb{Z}^d$ means that $|K| < \infty$, and $d(K,U) = \inf\{|x-y|_\infty; x \in K, y \in U\}$ denotes the ℓ^∞-distance between K and U. If $K = \{x\}$, we simply write $d(x,U)$. Moreover, we define the inner boundary of K to be the set $\partial^i K = \{x \in K; \exists y \in K^c, |x-y| = 1\}$, and the outer boundary of K as $\partial K = \partial^i(K^c)$. We also introduce the diameter of any subset $K \subset \mathbb{Z}^d$, diam(K), as its $\ell^\infty-$diameter, i.e. diam(K)= $\sup\{|x-y|_\infty; x, y \in K\}$. Throughout the paper, vectors are taken to be row vectors, and a small t indicates transposition. The inner product between x and y in $\mathbb{R}^d$ is usually denoted by $x \cdot y$ and sometimes we will write for a vector $(t_i)_{i \in \Lambda}$, $\Lambda \subset \mathbb{Z}^d$, simply t_Λ.

For the symmetric simple random walk $X = (X_k)_{k \in \mathbb{N}}$ on $\mathbb{Z}^d$, which at each time step jumps to any one of its 2d nearest-neighbours with probability $\frac{1}{2d}$, we denote by P_i the distribution of the walk starting at $i \in \mathbb{Z}^d$, and with E_i the corresponding expectation. That is, we have $P_i(X_0 = i) = 1$, and $P_i(X_{n+1} = k | X_n = j) = 1/2d \cdot \mathbb{1}_{\{k \sim j\}} =: P(j,k)$, for all $k, j \in \mathbb{Z}^d$. Given $U \subset \mathbb{Z}^d$, we further denote the entrance time in U by $\tau_U = \inf\{n \geq 0 : X_n \in U\}$ and the hitting time in U by $\tilde{\tau}_U = \inf\{n \geq 1 : X_n \in U\}$.

Given two functions $f, g : \mathbb{Z}^d \to \mathbb{R}$, we write $f(x) \sim g(x)$, as $|x| \to \infty$, if they are asymptotic, i.e. if $\lim_{|x| \to \infty} f(x)/g(x) = 1$

Chapter 1

The DGFF

In this chapter we introduce the model studied in this paper, that is, the *Lattice Gaussian Free Field* or *Discrete Gaussian Free Field* (DGFF), also known as *Harmonic Crystal*.

1.1 The costruction of the model

We begin by defining the configuration space in finite and infinite volume as

$$\Omega_\Lambda := \mathbb{R}^\Lambda \qquad \text{and} \qquad \Omega := \mathbb{R}^{\mathbb{Z}^d},$$

where $\Lambda \subset\subset \mathbb{Z}^d$. The measurable structure on Ω_Λ (risp. Ω) is the σ-algebra $\mathcal{F}_\Lambda$ (risp. $\mathcal{F}$) generated by the cylinder sets, that is, the sets of the form $\{\omega \in \Omega_\Lambda : \omega_i \in A_i \text{ for every } i \in I\}$, with I a finite subset of Λ (risp. $\mathbb{Z}^d$) and A_i an open subset of $\mathbb{R}$.

Given a configuration $\omega \in \Omega$ we call the random variables $\varphi_i(\omega) = \omega_i$, $i \in \mathbb{Z}^d$, the *spin* or *height* at i. We consider the *Hamiltonian* (i.e. the energy associated to a given configuration $\omega \in \Omega_\Lambda$) defined by[1]

$$\mathcal{H}_{\Lambda,\beta,m}(\omega) := \frac{\beta}{4d} \sum_{\{i,j\} \in \mathcal{E}^b_\Lambda} (\varphi_i(\omega) - \varphi_j(\omega))^2 + \frac{m^2}{2} \sum_{i \in \Lambda} \varphi_i(\omega)^2, \qquad \omega \in \Omega_\Lambda, \tag{1.1}$$

where $\beta \geq 0$ is the *inverse temperature*, $m \geq 0$ is the *mass*[2] and $\mathcal{E}^b_\Lambda = \{\{i,j\} \cap \Lambda \neq \emptyset : i \sim j\}$. Once we have a Hamiltonian and a configuration $\eta \in \Omega$, we define the corresponding *Gibbs distribution* for the DGFF in Λ with boundary condition η, at inverse temperature $\beta \geq 0$ and mass $m \geq 0$,

[1]The constants $\beta/4d$ and $m^2/2$ will be very convinient later on.

[2]The terminology "mass" is inherited from quantum field theory, where the corresponding quadratic term in the Lagrangian indeed give rise to the mass of the associated particles.

as the probability measure $\mu^{\eta}_{\Lambda,\beta,m}$ on $(\Omega,\mathcal{F})$ defined by

$$\mu^{\eta}_{\Lambda,\beta,m}(d\omega) := \frac{\exp\left(-\mathcal{H}^{\eta}_{\Lambda,\beta,m}(\omega)\right)}{Z^{\eta}_{\Lambda,\beta,m}}\lambda^{\eta}_{\Lambda}(d\omega), \tag{1.2}$$

where[3]

$$\lambda^{\eta}_{\Lambda}(d\omega) := \prod_{i\in\Lambda} d\omega_i \prod_{i\in\Lambda^C} \delta_{\eta_i}(d\omega_i), \tag{1.3}$$

and $Z^{\eta}_{\Lambda,\beta,m}$ is a renormalization constant called *partition function*, that is of course (after some easy computation to show that is finite)

$$Z^{\eta}_{\Lambda,\beta,m} = \int \exp\left(-\beta\mathcal{H}^{\eta}_{\Lambda,\beta,m}(\omega)\right)\lambda^{\eta}_{\Lambda}(d\omega) < \infty. \tag{1.4}$$

Remark 1.1.1. We immediately observe that the scaling property of the Gibbs measure imply that one of the parameter, β or m, plays an irrelevant role when studying the DGFF. Indeed, the change of variables $\omega'_i = \beta^{1/2}\omega_i$, $i \in \Lambda$, leads to

$$Z^{\eta}_{\Lambda,\beta,m} = \beta^{-|\Lambda|/2} Z^{\eta'}_{\Lambda,1,m'}, \tag{1.5}$$

where $m' = \beta^{-1/2}m$ and $\eta' = \beta^{1/2}\eta$, and, similarly,

$$\mu^{\eta}_{\Lambda,\beta,m}(\cdot) = \mu^{\eta'}_{\Lambda,1,m'}(\cdot). \tag{1.6}$$

This shows that there is no loss of generality in assuming that $\beta = 1$. In particular our interest is on the *massless* model, that is when $m = 0$.

1.2 Some heuristic interpretations

We now give some heuristic interpretations of the DGFF.

First of all note that from the definition of the energy in (1.1) only spins located at nearest-neighbours vertices of $\mathbb{Z}^d$ interact. A second important remark is to note that from definition (1.2) we know that our measure gives higher weight to the configurations that have low energy. So to have a low energy we want both terms in (1.1) to be small, in particular

- for the first term to be small we need that all $(\varphi_i - \varphi_j)^2$ (that could be viewed as a sort of gradient) are small, and so that every vertex has a value similar to the neighbour vertices, namely the interaction favours agreement of neighbouring spins;

[3]Obviously $d\omega_i$ denotes the Lebesgue measure. Note that the term $\prod_{i\in\Lambda^C}\delta_{\eta_i}(d\omega_i)$ fixes the boundary condition equal to η.

- for the second term to be small we need that all $(\varphi_i)^2$ are small, and so that all vertex has a value close to zero, that is, the spins favour localization near zero.

One possible interpretation of this model is as follows. In $d = 1$, the spin at vertex $i \in \Lambda$, $\omega_i \in \mathbb{R}$, we can interpret as the height of a random line above the x-axis. The behaviour of the model in large volumes is therefore intimately related to the fluctuations of the line away from the $x-$axis. Similarly, in $d = 2$, ω_i can be interpreted as the height of a surface above the $(x, y)-$plane (see for an example the figure in the first page).

The model comes from the Quantum Field Theory. It is the basic model on top of which more interesting field theories are constructed. Indeed a lot of other model are constructed as pertubation of the DGFF, so it is a sort of building block.

Recently, the reason to study this model is that the continuum GFF is a sort of rescaling of the DGFF as the mesh of the lattice goes to zero. The GFF plays a very important role in relation of critical properties of critical systems, especially in dimension 2 (for example, we recall the remarkable works due to the two Fields medal Wendelin Werner and Stanislav Smirnov).

Finally, the DGFF could also be interpreted as a model which describes the small fluctuations of the positions of atoms of a crystal. That's why the DGFF is also called the *Harmonic Crystal.*

1.3 The random walk rappresentation for the massless model

When we look at the density distribution in (1.2) we immediately note an affinity with the Gaussian distribution. The goal of this section is to rewrite the measure $\mu^\eta_{\Lambda,1,0}(d\omega) =: \mu^\eta_\Lambda(d\omega)$ in the canonical form

$$\frac{1}{(2\pi)^{|\Lambda|/2}\sqrt{\det G_\Lambda}} \exp\big\{ - (x-u) \cdot G_\Lambda^{-1}(x-u)\big\}, \tag{1.7}$$

where $u = (u_i)_{i\in\Lambda}$, with $u_i = \mathbb{E}^\eta_\Lambda[\varphi_i]$, is the $|\Lambda|$-dimensional mean vector and $G_\Lambda(i,j) = \mathrm{Cov}^\eta_\Lambda(\varphi_i, \varphi_j)$ is the$|\Lambda| \times |\Lambda|$ covariance matrix.

Before locking at this representation we need some preliminary notions on harmonic functions.

1.3.1 Harmonic functions and the Discrete Green Identities

Given a collection $f = (f_i)_{i\in\mathbb{Z}^d}$ of real numbers, we define, for each pair $\{i, j\} \in \mathcal{E}_{\mathbb{Z}^d}$, the *discrete gradient*

$$(\nabla f)_{ij} := f_j - f_i, \tag{1.8}$$

and, for all $i \in \mathbb{Z}^d$, the *discrete Laplacian*

$$\frac{1}{2d}(\Delta f)_i \coloneqq \Big[\frac{1}{2d}\sum_{j\sim i} f_j\Big] - f_i = \frac{1}{2d}\sum_{j\sim i}(\nabla f)_{ij} = \frac{1}{2d}\sum_{j=1}^{d}(\nabla^2 f)_{i,i+e_j}, \tag{1.9}$$

where $(\nabla^2 f)_{i,i+e_j} = (f_{i+e_j} - f_i) + (f_{i-e_j} - f_i) = f_{i+e_j} - 2f_i + f_{i-e_j}$. The last term resembles the usual definition of the Laplacian of a function on $\mathbb{R}^d$, but the first expression is a more natural way to think of the Laplacian, the difference between the mean value of f over the neighbours of i and the value off at i.

We have the following discrete analogues of the classical Green identities.

Lemma 1.3.1 (Discrete Green Identities)**.** *Let* $\Lambda \subset\subset \mathbb{Z}^d$*. Then, for all collections of real numbers* $f = (f_i)_{i\in\mathbb{Z}^d}$, $g = (g_i)_{i\in\mathbb{Z}^d}$,

$$\sum_{\{i,j\}\in\mathcal{E}^b_\Lambda} (\nabla f)_{ij}(\nabla g)_{ij} = -\sum_{i\in\Lambda} g_i(\Delta f)_i + \sum_{i\in\Lambda, j\notin\Lambda, i\sim j} g_j(\nabla f)_{ij}, \tag{1.10}$$

and

$$\sum_{i\in\Lambda}\Big\{f_i(\Delta g)_i - g_i(\Delta f)_i\Big\} = \sum_{i\in\Lambda, j\notin\Lambda, i\sim j}\Big\{f_i(\Delta g)_{ij} - g_j(\Delta f)_{ij}\Big\}. \tag{1.11}$$

Proof. See, for example, [7], Lemma 8.7. □

We can write the action of the Laplacian on $f = (f_i)_{i\in\mathbb{Z}^d}$, as:

$$(\Delta f)_i = \sum_{j\sim i}(\nabla f)_{ij} = \sum_{j\in\mathbb{Z}^d} \nabla_{ij} f_j, \tag{1.12}$$

where

$$\nabla_{ij} = \begin{cases} -2d & \text{if } i = j, \\ 1 & \text{if } i \sim j, \\ 0 & \text{otherwise.} \end{cases} \tag{1.13}$$

Moreover we introduce the restriction of Δ to Λ, defined by

$$\Delta_\Lambda = (\Delta_{i,j})_{i,j\in\Lambda}. \tag{1.14}$$

Note that $f\cdot\Delta_\Lambda g = \sum_{i\in\Lambda} f_i(\Delta_\Lambda g)_i = \sum_{i,j\in\Lambda} f_i\Delta_{ij}g_j = g\cdot\Delta_\Lambda f$ and $(\Delta_\Lambda f) = \sum_{j\in\Lambda}\Delta_{ij}f_j$.

Returning to the density of the DGFF and remembering that $\Lambda \subset\subset \mathbb{Z}^d$, $f = (f_i)_{i\in\mathbb{Z}^d}$, $f_i = \eta_i$ for all $i \notin \Lambda$, we have, applying (1.10) with $f = g$

$$\begin{aligned}\sum_{\{i,j\}\in\mathcal{E}^b_\Lambda} (f_j - f_i)^2 = \sum_{\{i,j\}\in\mathcal{E}^b_\Lambda} (\nabla f)^2_{ij} &= -\sum_{i\in\Lambda} f_i(\Delta f)_i + \sum_{i\in\Lambda, j\notin\Lambda, i\sim j} f_j(\nabla f)_{ij} \\ &= -\sum_{i\in\Lambda} f_i(\Delta_\Lambda f)_i - 2\sum_{i\in\Lambda, j\notin\Lambda, i\sim j} f_i f_j + B_\Lambda, \\ &= f\cdot\Delta_\Lambda f - 2\sum_{i\in\Lambda, j\notin\Lambda, i\sim j} f_i f_j + B_\Lambda,\end{aligned} \tag{1.15}$$

where in the last inequality we used that $(\Delta f)_i = (\Delta_\Lambda f)_i + \sum_{j\notin\Lambda} f_j$, for all $i \in \Lambda$ and $f_j(\nabla f)_{ij} = f_j^2 - f_i f_j = \eta_j^2 - f_i f_j$ for all $i \in \Lambda, j \notin \Lambda, i \sim j$, and B_Λ is a boundary term. One can then introduce $u = (u_i)_{i\in\mathbb{Z}^d}$, to be determined later, depending on η and Λ, and playing the role of the mean of f.

Our aim is to rewrite (1.15) in the form $-(f-u)\cdot\Delta_\Lambda(f-u)$, up to boundary terms. We can, in particular, include in B_Λ any expression that depends only on the values of u. We have

$$\begin{aligned}(f-u)\cdot\Delta_\Lambda(f-u) &= f\cdot\Delta_\Lambda f - 2f\cdot\Delta_\Lambda u + u\cdot\Delta_\Lambda u\\ &= f\cdot\Delta_\Lambda f - 2\sum_{i\in\Lambda} f_i(\Delta_\Lambda u)_i + B_\Lambda\\ &= f\cdot\Delta_\Lambda f - 2\sum_{i\in\Lambda} f_i(\Delta u)_i + 2\sum_{i\in\Lambda}\sum_{j\notin\Lambda, j\sim i} f_i u_j + B_\Lambda.\end{aligned} \tag{1.16}$$

Comparing the two expressions for $f\cdot\Delta_\Lambda f$ in (1.16) and (1.15), we deduce that

$$\sum_{\{i,j\}\in\mathcal{E}_\Lambda^b}(f_j-f_i)^2 = -(f-u)\cdot\Delta_\Lambda(f-u) - 2\sum_{i\in\Lambda} f_i(\Delta u)_i + 2\sum_{\substack{i\in\Lambda,\\ j\notin\Lambda\\ j\sim i}} f_i(u_j - f_j) + B_\Lambda. \tag{1.17}$$

A look at the second term in this last display indicates exactly the restrictions one should impose on u in order for $-(f-u)\cdot\Delta_\Lambda(f-u)$ to be the one and only contribution to the Hamiltonian (up to boundary terms). To cancel the non-trivial terms that depend on the values of f inside Λ, we need to ensure that:

- u is harmonic in Λ, that is
$$(\Delta u)_i = 0, \qquad \text{for all } i\in\Lambda; \tag{1.18}$$
- u coincides with f (hence with η) outside Λ, that is
$$u_i = \eta_i, \qquad \text{for all } i\notin\Lambda. \tag{1.19}$$

We have thus proved

Lemma 1.3.2. *Assume that $u = (u_i)_{i\in\mathbb{Z}^d}$ solves the Dirichlet problem in Λ with boundary condition η :*

$$\begin{cases}(\Delta u)_i = 0, & i\in\Lambda,\\ u_i = \eta_i, & i\notin\Lambda.\end{cases} \tag{1.20}$$

then

$$\sum_{\{i,j\}\in\mathcal{E}_\Lambda^b}(f_j-f_i)^2 = -(f-u)\cdot\Delta_\Lambda(f-u) + B_\Lambda. \tag{1.21}$$

Existence of a solution to the Dirichlet problem will be proved later. Uniqueness can be verified easily.

Let us consider the massless Hamiltonian $\mathcal{H}^{\eta}_{\Lambda,1,0} =: \mathcal{H}^{\eta}_{\Lambda}$, expressed in terms of the variables $\varphi = (\varphi_i)_{i\in\mathbb{Z}^d}$, which are assumed to satisfy $\varphi_i = \eta_i$ for all $i \notin \Lambda$. We apply Lemma 1.3.2 with $f = \varphi$, assuming for the moment that one can find a solution u to the Dirichlet problem (in Λ, with boundary condition η). Since it does not alter the Gibbs distribution, the constant B_Λ in (1.21) can always be subtracted from the Hamiltonian. We get

$$\mathcal{H}^{\eta}_{\Lambda} = \frac{1}{2}(\varphi - u)\cdot(-\frac{1}{2d}\Delta_\Lambda)(\varphi - u). \tag{1.22}$$

Our next tasks are, first, to invert the matrix $-\frac{1}{2d}\Delta_\Lambda$, in order to obtain an explicit expression for the covariance matrix, and, second, to find an explicit expression for the solution u to the Dirichlet problem.

1.3.2 The random walk representation

We begin by writing

$$-\frac{1}{2d}\Delta_\Lambda = \mathbb{I}_\Lambda - P_\Lambda, \tag{1.23}$$

where $\mathbb{I}_\Lambda = (\delta_{ij})_{i,j\in\Lambda}$ and $P_\Lambda = (P(i,j))_{i,j\in\Lambda}$ with elements

$$P(i,j) = \begin{cases} \frac{1}{2d} & \text{if } i \sim j, \\ 0 & \textit{otherwise}. \end{cases} \tag{1.24}$$

We immediately recognise that the numbers $(P(i,j))_{i,j\in\mathbb{Z}^d}$ are the transition probabilities of the symmetric simple random walk $X = (X_k)_{k\in\mathbb{N}}$ on $\mathbb{Z}^d$, which at each time step jumps to any one of its 2d nearest-neighbours with probability $\frac{1}{2d}$, as explained in the introduction. We denote by P_i the distribution of the walk starting at $i \in \mathbb{Z}^d$. That is, we have $P_i(X_0 = i) = 1$, and $P_i(X_{n+1} = k|X_n = j) = P(j,k)$ for all $k, j \in \mathbb{Z}^d$.

The next lemma shows that the matrix $\mathbb{I}_\Lambda - P_\Lambda$ is invertible, and provides a probabilistic interpretation for its inverse:

Lemma 1.3.3. *The matrix $\mathbb{I}_\Lambda - P_\Lambda$ is invertible. Moreover, its inverse $G_\Lambda = (\mathbb{I}_\Lambda - P_\Lambda)^{-1}$ is given by $G_\Lambda = (G_\Lambda(i,j))_{j\in\Lambda}$, the Green function in Λ of the simple random walk on $\mathbb{Z}^d$, defined by*

$$G_\Lambda(i,j) := E_i\Big[\sum_{n=0}^{\tau_{\Lambda^c}-1} \mathbb{1}_{\{X_n=j\}}\Big]. \tag{1.25}$$

The Green function $G_\Lambda(i,j)$ represents the average number of visits at j made by a walk started at i, before it leaves Λ.

Proof. First of all, observe that (below, P^n denotes the nth power of a matrix P)

$$(\mathbb{I}_\Lambda - P_\Lambda)(\mathbb{I}_\Lambda + P_\Lambda + P_\Lambda^2 + \cdots + P_\Lambda^n) = (\mathbb{I}_\Lambda - P_\Lambda^{n+1}). \tag{1.26}$$

Rewriting

$$\begin{aligned} P_\Lambda^k(i,j) = \sum_{i_1,\dots,i_{k-1}\in\Lambda} P_\Lambda(i,i_1)P_\Lambda(i_1,i_2)\cdot\dots\cdot P_\Lambda(i_{k-1},j) = \\ = P_i\big(X_k = j, \tau_{\Lambda^C} > k\big) \le P_i\big(\tau_{\Lambda^C} > k\big), \end{aligned} \tag{1.27}$$

and using the classical bound on the probability that the walk exits a finite region in a finite time $P_i\big(\tau_{\Lambda^C} > k\big) \le e^{-ck}$, we can take the limit $n \to \infty$ in (1.26) obtaining

$$(\mathbb{I}_\Lambda - P_\Lambda)\Big(\sum_{k\ge 0} P_\Lambda^k\Big) = \mathbb{I}_\Lambda, \tag{1.28}$$

that is,

$$G_\Lambda = (\mathbb{I}_\Lambda - P_\Lambda)^{-1} = \sum_{k\ge 0} P_\Lambda^k, \tag{1.29}$$

since by symmetry we have also that $(G_\Lambda)(\mathbb{I}_\Lambda - P_\Lambda) = \mathbb{I}_\Lambda$. Finally

$$\sum_{k\ge 0} P_\Lambda^k(i,j) = \sum_{k\ge 0} P_i\big(X_k = j, \tau_{\Lambda^C} > k\big) = E_i\Big[\sum_{n=0}^{\tau_{\Lambda^c}-1} \mathbb{1}_{\{X_n = j\}}\Big], \tag{1.30}$$

gives the desired expression for $G_\Lambda(i,j)$. □

Let us now prove the existence of a solution to the Dirichlet problem, also expressed in terms of the simple random walk. Let $X_{\tau_{\Lambda^c}}$ denote the position of the walk at the time of first exit from Λ.

Lemma 1.3.4. *The solution to the Dirichlet problem in (1.20) is given by the function $u = (u_i)_{i\in\mathbb{Z}^d}$ defined by*

$$u_i = E_i\Big[\eta_{X_{\tau_{\Lambda^c}}}\Big], \qquad \text{for all } i \in \mathbb{Z}^d. \tag{1.31}$$

Proof. When $j \notin \Lambda$, $P_j(\tau_{\Lambda^c} = 0) = 1$ and, thus, $u_j = E_j\Big[\eta_{X_{\tau_{\Lambda^c}}}\Big] = E_j\big[\eta_{X_0}\big] = \eta_j$. When $i \in \Lambda$, by conditioning on the first step of the walk and using the Markov's property,

$$u_i = E_i\Big[\eta_{X_{\tau_{\Lambda^c}}}\Big] = \sum_{j\sim i} E_i\Big[\eta_{X_{\tau_{\Lambda^c}}} | X_1 = j\Big] P_i\big(X_1 = j\big) = \frac{1}{2d}\sum_{j\sim i} u_j, \tag{1.32}$$

which implies $(\Delta u)_i = 0$. □

We finally have the desired representation.

Theorem 1.3.5. *Under μ_Λ^η, $\varphi = (\varphi_i)_{i\in\Lambda}$ is Gaussian, with mean $u = (u_i)_{i\in\Lambda}$ defined by*

$$u_i = E_i\Big[\eta_{X_{\tau_{\Lambda^c}}}\Big], \qquad \text{for all } i \in \Lambda, \tag{1.33}$$

and positive definite covariance matrix $G_\Lambda = (G_\Lambda(i,j))_{,j\in\Lambda}$, given by the Green function

$$G_\Lambda(i,j) := E_i\Big[\sum_{n=0}^{\tau_{\Lambda^c}-1} \mathbb{1}_{\{X_n=j\}}\Big]. \tag{1.34}$$

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The reader should note the remarkable fact that the distribution of $\varphi = (\varphi_i)_{i\in\Lambda}$ under μ_Λ^η depends on the boundary condition η only through its mean; the covariance matrix is only sensitive to the choice of Λ.

1.4 The infinite volume extension

In this section, we will present the problem of existence of infinite-volume Gibbs measures for the massless DGFF.

In order to do that we need to introduce the notion of *Gibbs state* and we will do that in the particular case of the DGFF measure[4].

Definition 1.4.1. Let $f : \Omega = \mathbb{R}^{\mathbb{Z}^d} \to \mathbb{R}$ be a function. We say that f is *local* if exists $A \subset\subset \mathbb{Z}^d$ such that $f(\omega) = f(\omega')$ as soon as $\omega_i = \omega'_i$ for all $i \in A$. The smallest such set A is called the *support* of f and it is denoted by *supp*(f).

Definition 1.4.2. A *state* is a map $f \to \langle f\rangle$ acting on a local function $f : \Omega \to \mathbb{R}$ satisfying the following three properties:

1. $\langle 1\rangle = 1$;

2. if $f \geq 0$ then $\langle f\rangle \geq 0$;

3. for all $\lambda \in \mathbb{R}$, $\langle f + \lambda g\rangle = \langle f\rangle + \lambda\langle g\rangle$.

Definition 1.4.3. Let $(\Lambda)_{n\geq1}$be a sequence of finite set such that increase to $\mathbb{Z}^d$, then the sequence $\mu_{\Lambda_n}^\eta$ is said to *converge to the state* $\langle\cdot\rangle$, if $\mathbb{E}_{\Lambda_n}^\eta[f] \to \langle f\rangle$ for all f local functions. Then the state $\langle\cdot\rangle$ is called a *Gibbs state.*

[4]The notion of Gibbs state could be developed in a very more general framework but we prefer to chose a simpler presentation of this notion since the dissertation of this argument is not the goal of this paper.

Remark 1.4.4. This notion is really natural from a mathematical point of view. The reason is that as soon as you have a functional with the properties of Definition 1.4.2 you can prove that there exists an underlying probability measure μ such that $\langle f \rangle = \int f d\mu$ and then the notion of convergence given in Definition 1.4.3 it is just the notion of *weak convergence* of the sequence $\mu^{\eta}_{\Lambda_n}$ to μ.

Definition 1.4.5. We characterize the space of *infinite-volume Gibbs measures* by

$$\mathcal{G} = \{\mu \in \mathcal{M}_1(\Omega) \big| \mu(A|\mathcal{F}_{\Lambda^c})(\omega) = \mu^{\omega}_{\Lambda}(A) \text{ for all } \Lambda \subset\subset \mathbb{Z}^d \text{ and all } A \in \mathcal{F}\},$$

where we denote by $M_1(\Omega)$ the set of probability measures on Ω.

Of course in the particular case of the DGFF, since we are dealing with sequences of Gaussian measures, any limit point of $\mu^{\eta}_{\Lambda_n}$ is in any case a Gaussian measure and this convergence takes place if and only if both covariance and mean converge (to finite limits). We note, by a standard monotone argument and remembering that the random walk on $\mathbb{Z}^d$ is transient if and only if $d \geq 3$, that

$$\lim_{n\to\infty} G_{\Lambda_n}(i,j) = E_i\Big[\sum_{n\geq 0} \mathbb{1}_{\{X_n = j\}}\Big] = \begin{cases} < +\infty & \text{if } d \geq 3, \\ = +\infty & \text{if } d = 1 \text{ or } d = 2. \end{cases} \tag{1.35}$$

This has the following consequence:

Theorem 1.4.6. *When $d = 1$ or $d = 2$, the massless Gaussian Free Field has no infinite-volume Gibbs measures.*

When $d \geq 3$, transience of the symmetric simple random walk implies that the limit in (1.35) is finite. This will allow us to construct infinite-volume Gibbs measures. We say that $\eta = (\eta_i)_{i\in\mathbb{Z}^d}$ is harmonic (in $\mathbb{Z}^d$) if $(\Delta\eta)_i = 0$ for all $i \in \mathbb{Z}^d$.

Theorem 1.4.7. *In dimensions $d \geq 3$, the massless Gaussian Free Field possesses infinitely many infinite-volume Gibbs measures. More precisely, given any harmonic function η on $\mathbb{Z}^d$, there exists a Gaussian Gibbs measure μ^{η} with mean η and covariance matrix given by the Green function*

$$G(i,j) = E_i\Big[\sum_{n\geq 0} \mathbb{1}_{\{X_n = j\}}\Big]. \tag{1.36}$$

The proof of the last theorem is the topic of the next section.

1.5 The Gibbs-Markov property.

From now till the end we tacitly suppose that $d \geq 3$. In this section we want to show that every Gaussian Gibbs measure μ^η, given by Theorem 1.4.7, satisfies the *Gibbs-Markov property*, that is, for all $\Lambda \subset\subset \mathbb{Z}^d$, and for all $A \in \mathcal{F}$,

$$\mu^\eta\big(A|\mathcal{F}_{\Lambda^c}\big)(\omega) = \mu^\omega_\Lambda(A), \quad \text{for } \mu^\eta\text{-almost all } \omega. \tag{1.37}$$

For that, we will verify that the field $\varphi = (\varphi_i)_{i\in\mathbb{Z}^d}$ with mean $\mathbb{E}^\eta[\varphi_i] = \eta_i$ and covariance

$$\mathrm{Cov}^\eta(\varphi_i\varphi_j) = G(i,j), \tag{1.38}$$

when conditioned on $\mathcal{F}_{\Lambda^c}$, remains Gaussian (Lemma 1.5.2 below) and that, for all t_Λ,

$$\mathbb{E}^\eta[e^{it_\Lambda\cdot\varphi_\Lambda}|\mathcal{F}_{\Lambda^c}](\omega) = e^{it_\Lambda\cdot a_\Lambda(\omega)-\frac{1}{2}t_\Lambda\cdot G_\Lambda t_\Lambda}, \tag{1.39}$$

where $a_i(\omega) = E_i[\omega_{X_{\tau_{\Lambda^c}}}]$ is the solution of the Dirichlet problem in Λ with boundary condition ω.

Remark 1.5.1. We want to remark the difference between the probability measure $\mathbb{P}$ (with the associated expectation $\mathbb{E}$) and P (with the associated expectation E): the first acts on the random field φ as just defined, the second acts on the simple random walk on $\mathbb{Z}^d$ as defined in the initial section about notation.

Lemma 1.5.2. *Let φ be the Gaussian field construct below. Let, for all $i \in \Lambda$,*

$$a_i(\omega) := \mathbb{E}^\eta[\varphi_i|\mathcal{F}_{\Lambda^c}](\omega). \tag{1.40}$$

Then, μ^η-a.s., $a_i(\omega) = E_i[\omega_{X_{\tau_{\Lambda^c}}}]$. In particular, each $a_i(\omega)$ is a finite linear combination of the variables φ_j and $(a_i)_{i\in\mathbb{Z}^d}$ is a Gaussian field.

Proof. When $i \in \Lambda$, we use the following characterization of the conditional expectation: up to equivalence, $\mathbb{E}^\eta[\varphi_i|\mathcal{F}_{\Lambda^c}]$ is the unique $\mathcal{F}_{\Lambda^c}-$measurable random variable ψ for which

$$\mathbb{E}^\eta[(\varphi_i - \psi)\varphi_j] = 0, \quad \text{for all } j \in \Lambda^c. \tag{1.41}$$

We verify that this condition is indeed satisfied when $\psi = E_i[\omega_{X_{\tau_{\Lambda^c}}}]$. By (1.38),

$$\begin{aligned}\mathbb{E}^\eta\Big[\big(\varphi_i - E_i[\varphi_{X_{\tau_{\Lambda^c}}}]\big)\varphi_j\Big] &= \mathbb{E}^\eta[\varphi_i\varphi_j] - \mathbb{E}^\eta\Big[E_i[\varphi_{X_{\tau_{\Lambda^c}}}]\varphi_j\Big] = \\ &= G(i,j) + \eta_i\eta_j - \mathbb{E}^\eta\Big[E_i[\varphi_{X_{\tau_{\Lambda^c}}}]\varphi_j\Big].\end{aligned}$$

Using again (1.38),

$$\begin{aligned}\mathbb{E}^\eta\Big[E_i[\varphi_{X_{\tau_{\Lambda^c}}}]\varphi_j\Big] &= \sum_{k\in\partial\Lambda}\mathbb{E}^\eta[\varphi_k\varphi_j]P_i(X_{\tau_{\Lambda^c}} = k) = E_i\Big[\mathbb{E}^\eta[\varphi_{X_{\tau_{\Lambda^c}}}\varphi_j]\Big] \\ &= E_i\big[G(X_{\tau_{\Lambda^c}}, j)\big] + E_i\Big[\mathbb{E}^\eta[\varphi_{X_{\tau_{\Lambda^c}}}]\mathbb{E}^\eta[\varphi_j]\Big].\end{aligned} \tag{1.42}$$

On the one hand, since $i \in \Lambda$ and $j \in \Lambda^c$, any trajectory of the random walk that contributes to $G(i,j)$ must intersect $\partial\Lambda$ at least once, so the Markov property gives

$$G(i,j) = E_i\Big[\sum_{h\geq 0} \mathbb{1}_{\{X_h=j\}}\Big] = \sum_{k\in\partial\Lambda} P_i(X_{\tau_{\Lambda^c}} = k)G(k,j) = E_i\big[G(X_{\tau_{\Lambda^c}}, j)\big]. \tag{1.43}$$

On the other hand, since φ has mean η and η is solution of the Dirichlet problem in Λ with boundary condition η, we have

$$E_i\Big[\mathbb{E}^\eta[\varphi_{X_{\tau_{\Lambda^c}}}]\mathbb{E}^\eta[\varphi_j]\Big] = E_i\big[\eta_{X_{\tau_{\Lambda^c}}}\eta_j\big] = E_i[\eta_{X_{\tau_{\Lambda^c}}}]\eta_j = \eta_i\eta_j. \tag{1.44}$$

This shows that $a_i(\omega) = E_i[\omega_{X_{\tau_{\Lambda^c}}}]$. In particular, the latter is a linear combination of the $\omega_j s$:

$$a_i(\omega) = \sum_{k\in\partial\Lambda} \omega_k P(X_{\tau_{\Lambda^c}} = k), \tag{1.45}$$

which implies that also $(a_i)_{i\in\mathbb{Z}^d}$ is a Gaussian field. $\square$

Corollary 1.5.3. *Under μ^η, the random vector $(\varphi_i - a_i)_{i\in\Lambda}$ is independent of $\mathcal{F}_{\Lambda^c}$.*

Proof. We know that the variables $\varphi_i - a_i$, $i \in \Lambda$, and φ_j, $j \in \Lambda^c$ form a Gaussian field. Therefore, a classical result implies that $(\varphi_i - a_i)_{i\in\Lambda}$, which is centered, is independent of $\mathcal{F}_{\Lambda^c}$ if and only if each pair $\varphi_i - a_i$ $(i \in \Lambda)$ and φ_j $(j \in \Lambda^c)$ is uncorrelated. But this follows from (1.41). $\square$

Let $a_\Lambda = (a_i)_{i\in\Lambda}$. By Corollary 1.5.3 and since a_Λ is $\mathcal{F}_{\Lambda^c}$-measurable,

$$\mathbb{E}^\eta[e^{it_\Lambda\cdot\varphi_\Lambda}|\mathcal{F}_{\Lambda^c}] = e^{it_\Lambda\cdot a_\Lambda}\mathbb{E}^\eta[e^{it_\Lambda\cdot(\varphi_\Lambda - a_\Lambda)}|\mathcal{F}_{\Lambda^c}] = e^{it_\Lambda\cdot a_\Lambda}\mathbb{E}^\eta[e^{it_\Lambda\cdot(\varphi_\Lambda - a_\Lambda)}]. \tag{1.46}$$

We know that the variables $\varphi_i - a_i$, $i \in \Lambda$, form a Gaussian vector under μ^η. Since it is centered, we need only to compute its covariance. For $i, j \in \Lambda$, write

$$(\varphi_i - a_i)(\varphi_j - a_j) = \varphi_i\varphi_j - (\varphi_i - a_i)a_j - (\varphi_j - a_j)a_i - a_ia_j. \tag{1.47}$$

Using Corollary 1.5.3 again, we see that $\mathbb{E}^\eta[(\varphi_i - a_i)a_j] = 0$ and $\mathbb{E}^\eta[(\varphi_j - a_j)a_i] = 0$ (since a_i and a_j are $\mathcal{F}_{\Lambda^c}$-measurable). Therefore

$$\mathrm{Cov}^\eta\big((\varphi_i - a_i), (\varphi_j - a_j)\big) = \mathbb{E}^\eta[\varphi_i\varphi_j] - \mathbb{E}^\eta[a_ia_j] = G(i,j) + \eta_i\eta_j - \mathbb{E}^\eta[a_ia_j]. \tag{1.48}$$

Proceeding as in (1.42)

$$\mathbb{E}^\eta[a_ia_j] = E_{i,j}\Big[G\big(X_{\tau_{\Lambda^c}}, X'_{\tau'_{\Lambda^c}}\big)\Big] + E_{i,j}\Big[\mathbb{E}^\eta\big[\varphi_{X_{\tau_{\Lambda^c}}}\big]\mathbb{E}^\eta\big[\varphi_{X'_{\tau'_{\Lambda^c}}}\big]\Big], \tag{1.49}$$

where X and X' are two independent symmetric random walks, starting respectively at i and j, $P_{i,j}$ denotes their joint distribution, and τ'_{Λ^c} is the first exit time of X' from Λ. As was done earlier,

$$E_{i,j}\Big[\mathbb{E}^\eta\big[\varphi_{X_{\tau_{\Lambda^c}}}\big]\mathbb{E}^\eta\big[\varphi_{X'_{\tau'_{\Lambda^c}}}\big]\Big] = E_{i,j}\big[\eta_{X_{\tau_{\Lambda^c}}}\eta_{X'_{\tau'_{\Lambda^c}}}\big] = E_i\big[\eta_{X_{\tau_{\Lambda^c}}}\big]E_j\big[\eta_{X_{\tau_{\Lambda^c}}}\big] = \eta_i\eta_j. \tag{1.50}$$

Let us then define the modified Green function

$$K_\Lambda(i,j) := E_i\Big[\sum_{n\geq\tau_{\Lambda^c}} \mathbb{1}_{\{X_n=j\}}\Big] = G(i,j) - G_\Lambda(i,j). \tag{1.51}$$

Observe that $K_\Lambda(i,j) = K_\Lambda(j,i)$ since G and G_Λ are both symmetric; moreover, $K_\Lambda(i,j) = G(i,j)$ if $i \in \Lambda^c$. We can thus write

$$\begin{aligned}
E_{i,j}\Big[G\big(X_{\tau_{\Lambda^c}}, X'_{\tau'_{\Lambda^c}}\big)\Big] &= \sum_{k,l\in\partial\Lambda} P_i\big(X_{\tau_{\Lambda^c}} = k\big)P_j\big(X_{\tau_{\Lambda^c}} = l\big)G(k,l) \\
&= \sum_{l\in\partial\Lambda} P_j\big(X_{\tau_{\Lambda^c}} = l\big)K_\Lambda(i,l) \\
&= \sum_{l\in\partial\Lambda} P_j\big(X_{\tau_{\Lambda^c}} = l\big)K_\Lambda(l,i) \\
&= \sum_{l\in\partial\Lambda} P_j\big(X_{\tau_{\Lambda^c}} = l\big)G(l,i) \\
&= K_\Lambda(j,i) = G(i,j) - G_\Lambda(i,j).
\end{aligned} \tag{1.52}$$

We have thus shown $\mathrm{Cov}^\eta\big((\varphi_i - a_i), (\varphi_j - a_j)\big) = G_\Lambda(i,j)$, which implies that

$$\mathbb{E}^\eta\big[e^{it_\Lambda\cdot\varphi_\Lambda}|\mathcal{F}_{\Lambda^c}\big] = e^{it_\Lambda\cdot a_\Lambda}e^{-\frac{1}{2}t_\Lambda\cdot G_\Lambda t_\Lambda}. \tag{1.53}$$

This shows that, under $\mu^\eta(\cdot|\mathcal{F}_{\Lambda^c})$, φ_Λ is Gaussian with distribution given by $\mu^\eta_\Lambda(\cdot)$. We have therefore (1.39).

Remark 1.5.4. All the computations done in this section can be generalized conditioning on the σ-algebra $\mathcal{F}_\Lambda$, for $\emptyset \neq \Lambda \subset\subset \mathbb{Z}^d$. In this case we obtain the following version of Lemma 1.5.2

Lemma 1.5.5. *Let φ be the Gaussian field construct at the beginning of this section. Let, for all $i \in \Lambda^c$,*

$$u_i(\omega) := \mathbb{E}^\eta[\varphi_i|\mathcal{F}_\Lambda](\omega). \tag{1.54}$$

Then, μ^η-a.s., $u_i(\omega) = E_i[\omega_{X_{\tau_\Lambda}}, \tau_\Lambda < \infty]$. In particular, each $u_i(\omega)$ is a finite linear combination of the variables φ_j, $(u_i)_{i\in\mathbb{Z}^d}$ is a Gaussian field, and $(\varphi_i - u_i)_{i\in\Lambda^c}$ is independent from $\mathcal{F}_\Lambda$.

We now define, for $U \subset \mathbb{Z}^d$, the probability measure $\mu_U^\eta(\cdot)$ on $\mathbb{R}^{\mathbb{Z}^d}$ of the Gaussian field with mean $\mathbb{E}_U^\eta[\varphi_i] = \eta_i$ and covariance equal to the Green function $G_U(\cdot,\cdot)$ killed outside U, that is

$$\mathrm{Cov}_U^\eta[\varphi_i, \varphi_j] = G_U(i,j) \coloneqq \sum_{n \geq 0} P_x\big(X_n = y, n < T_U\big) \tag{1.55}$$

where $T_U \coloneqq \inf\{n \geq 0, X_n \notin U\}$ is the exit time from U. We have the following

Lemma 1.5.6. *Let $\emptyset \neq K \subset\subset \mathbb{Z}^d$, $U = \Lambda^c$. Every Gaussian Gibbs measure μ^η, given by Theorem 1.4.7, satisfies the "exterior Gibbs-Markov property", that is, for all $\Lambda \subset\subset \mathbb{Z}^d$, and for all $A \in \mathcal{F}$,*

$$\mu^\eta\big(A|\mathcal{F}_\Lambda\big)(\omega) = \mu_U^\omega(A), \quad \textit{for } \mu^\eta\textit{-almost all } \omega. \tag{1.56}$$

Proof. See [11], Lemma 1.2 and Remark 1.3 $\square$

Chapter 2

Some useful general tools

In this and in the following chapter we present some important general tools that we are going to use in the sequel. In this first chapter we present some useful general probability tools.

2.1 Extended version for non-negative increasing functions of the Markov's inequality

We state an easy but important generalization of the classical *Markov's inequality.*

Theorem 2.1.1. *Let X be any random variable, and f a non-negative increasing function. Then, supposing that $\mathbb{E}[f(X)] < \infty$,*

$$\mathbb{P}(X \geq \epsilon) \leq \mathbb{E}[f(X)]f(\epsilon). \tag{2.1}$$

Proof. Since $X \geq \varepsilon$ if and only if $f(X) \geq f(\varepsilon)$ then the basic Markov inequality gives the result. □

2.2 BTIS-inequality

The BTIS-inequality, is a result bounding the probability of a deviation of the uniform norm of a centred Gaussian stochastic process above its expected value. The inequality has been described (in [1]) as "the single most important tool in the study of Gaussian processes." We now present this important tool.

Consider a Gaussian random variable $X \sim N(0, \sigma^2)$. The following two important bounds hold for every $u > 0$ and become sharp very quickly as x *grows*:

$$\left(\frac{\sigma}{\sqrt{2\pi}u} - \frac{\sigma^3}{\sqrt{2\pi}u^3}\right)e^{-\frac{1}{2}u^2/\sigma^2} \leq \mathbb{P}(X > u) \leq (\frac{\sigma}{\sqrt{2\pi}u})e^{-\frac{1}{2}u^2/\sigma^2}. \tag{2.2}$$

In particular the upper bound follows from the observation that

$$\begin{aligned}\mathbb{P}(X > u) &= \int_u^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}\, dx \leq \\ &\leq \int_u^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \frac{x}{u} e^{-\frac{x^2}{2\sigma^2}}\, dx = \Big(\frac{\sigma}{\sqrt{2\pi} u}\Big) e^{-\frac{1}{2}u^2/\sigma^2}.\end{aligned} \tag{2.3}$$

For the lower bound, make the substitution $x \mapsto u + y/u$ to note that

$$\begin{aligned}\mathbb{P}(X > u) &= \int_u^{-\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}\, dx = \frac{1}{\sqrt{2\pi\sigma^2}} \int_0^{+\infty} \frac{e^{-\frac{u+y/u}{2\sigma^2}}}{u} dy \\ &= \frac{e^{-\frac{u^2}{2\sigma^2}}}{u\sqrt{2\pi\sigma^2}} \int_0^{+\infty} e^{-(y^2/u^2+2y)/2\sigma^2} dy \\ &\geq \frac{e^{-\frac{u^2}{2\sigma^2}}}{u\sqrt{2\pi\sigma^2}} \int_0^{+\infty} e^{-(y/\sigma^2)} \Big(1 - \frac{y^2}{2u^2\sigma^2}\Big) dy = \frac{e^{-\frac{u^2}{2\sigma^2}}}{u\sqrt{2\pi\sigma^2}} \Big(\sigma^2 - \frac{\sigma^4}{u^2}\Big),\end{aligned} \tag{2.4}$$

where the inequality is given by the fact that $e^{-z} \geq 1 - z$ for all $z \geq 0$. One immediate consequence of (2.2) is that

$$\lim_{u\to\infty} u^{-2} \ln \mathbb{P}(X > u) = -\frac{1}{2\sigma^2}. \tag{2.5}$$

Now we state a classical result related to (2.5), but for the supremum of a general centered Gaussian process $(X_t)_{t\in T}$. Assume that $(X_t)_{t\in T}$ is a.s. bounded, then

$$\lim_{u\to\infty} u^{-2} \ln \mathbb{P}\Big(\sup_{t\in T} X_t > u\Big) = -\frac{1}{2\sigma_T^2}, \tag{2.6}$$

where

$$\sigma_T^2 \coloneqq \sup_{t\in T} \mathbb{E}[X_t^2]. \tag{2.7}$$

An immediate consequence of (2.6) is that for all $\varepsilon > 0$ and u large enough,

$$\mathbb{P}\Big(\sup_{t\in T} X_t > u\Big) \leq e^{\varepsilon u^2 - u^2/2\sigma_T^2}. \tag{2.8}$$

Since $\varepsilon > 0$ is arbitrary, comparing (2.8) with (2.2), we reach the rather surprising conclusion that the supremum of a centered, bounded Gaussian process behaves much like a single Gaussian variable with a suitable chosen variance.

Now we want to see from where (2.8) comes. In fact, (2.8) and its consequences are all special cases of a ”nonasymptotic” result due independently, and with very different proofs, to Borell (B) and Tsirelson, Ibraginov and Sudakov (TIS).

Theorem 2.2.1 (BTIS-inequality). *Let $(X_t)_{t\in T}$ be a centered Gaussian process, a.s. bounded on T. Write $\|X\| = \|X\|_T = \sup_{t\in T} X_t$. Then*

$$\mathbb{E}[\|X\|] < \infty,$$

and for all $u > 0$,

$$\mathbb{P}\big(\|X\| - \mathbb{E}[\|X\|] > u\big) \leq e^{-u^2/2\sigma_T^2}. \tag{2.9}$$

Before looking at the proof, we take a moment to look at an immediate and trivial consequence of (2.9), that is, for all $u > \mathbb{E}(\|X\|)$,

$$\mathbb{P}(\|X\| > u) \leq e^{-(u-\mathbb{E}[\|X\|])^2/2\sigma_T^2}, \tag{2.10}$$

so that (2.6) and (2.8) follows from the BTIS-inequality.

We now turn to the proof of the BTIS-inequality. There are essentially three quite different ways to tackle this proof:

- The Borell's original proof [2] relied on isoperimetric inequalities;
- The proof of Tsirelson, Ibragimov, Sudakov [15] relied on Ito's formula;
- The proof reported in the collection of exercises [5] (although it root is much older).

We choose, as in [1], the third and more direct route. The first step in this route involves the following two lemmas.

Lemma 2.2.2. *Let X and Y be independent k-dimesional vectors of centered, unit-variance, independent, Gaussian variables. If $f, g : \mathbb{R}^k \to \mathbb{R}$ are bounded C^2 function then*

$$Cov(f(X), g(X)) = \int_0^1 \mathbb{E}\big[\nabla f(X) \cdot \nabla g(\alpha X + \sqrt{1-\alpha^2}Y)\big] d\alpha, \tag{2.11}$$

where $\nabla f(X) \coloneqq \big(\frac{\partial}{\partial x_i} f(x)\big)_{i=1,\dots,k}$.

Proof. It suffices to prove the lemma with $f(x) = e^{i(t\cdot x)}$ and $g(x) = e^{i(s\cdot x)}$ with $s, t, x \in \mathbb{R}^k$. Standard approximation arguments (which is where the requirement that f is C^2 appears) will do the rest. Write

$$\varphi(t) \coloneqq \mathbb{E}[e^{i(t\cdot x)}] = \exp\{it \cdot 0 - \frac{1}{2} t\mathbb{I}t^t\} = e^{|t|^2/2}, \tag{2.12}$$

since X is a k-dimesional vectors of centered, unit-variance, independent, Gaussian variables. It is then trivial that

$$\begin{aligned} \mathrm{Cov}(f(X), g(X)) &= \mathbb{E}[e^{i(t\cdot X)}e^{i(s\cdot Y)}] - \mathbb{E}[e^{i(t\cdot X)}]\mathbb{E}[e^{i(s\cdot Y)}] \\ &= \mathbb{E}[e^{i((t+s)\cdot X)}] - \mathbb{E}[e^{i(t\cdot X)}]\mathbb{E}[e^{i(s\cdot X)}] = \varphi(t+s) - \varphi(t)\varphi(s), \end{aligned} \tag{2.13}$$

where the second line follows from the fact that X and Y are independent with the same distribution. On the other hand, computing the integral in (2.11), using $\frac{\partial}{\partial x_i} f(x) = it_i e^{i(t\cdot x)}$, gives

$$\begin{aligned}
&\int_0^1 \mathbb{E}\Big[\nabla f(X)\cdot \nabla g(\alpha X+\sqrt{1-\alpha^2}Y)\Big]d\alpha\\
&\qquad = \int_0^1 \mathbb{E}\Big[\sum_{j=1}^d it_j e^{i(t\cdot x)} is_j e^{i(s\cdot(\alpha X+\sqrt{1-\alpha^2}Y))}\Big]d\alpha\\
&\qquad = -\int_0^1 \sum_{j=1}^d s_j t_j \mathbb{E}\big[e^{i((t+\alpha s)\cdot x)}\big]\mathbb{E}\big[e^{i(s\cdot(\sqrt{1-\alpha^2}Y))}\big]d\alpha\\
&\qquad = -\int_0^1 (s\cdot t)e^{(|t|^2+2\alpha|t||s|+|s|^2)/2}d\alpha\\
&\qquad = -\varphi(s)\varphi(t)(1-e^{s\cdot t}) = \varphi(s+t)-\varphi(s)\varphi(t),
\end{aligned} \tag{2.14}$$

which is all that we need. □

Lemma 2.2.3. *Let X be a k-dimensional vector of centered, unit-variance, independent, Gaussian variables. If $h:\mathbb{R}^k\to\mathbb{R}$ is C^2 with Lipschitz constant 1 and if $\mathbb{E}[h(X)]=0$, then for all $t>0$,*

$$\mathbb{E}\big[e^{th(X)}\big] \le e^{t^2/2}. \tag{2.15}$$

Proof. Let Y be an independent copy of X and α a uniform random variable on $[0,1]$. Define the pair (X,Z_α) via

$$(X,Z_\alpha) \coloneqq (X,\alpha X+\sqrt{1-\alpha^2}Y) \tag{2.16}$$

Take h as in the statement of the lemma, $t\ge 0$ fixed and define $g=e^{th}$. Applying (2.11) and using $\nabla e^{th} = te^{th}\nabla h$, gives

$$\begin{aligned}
\mathbb{E}\big[h(X)e^{th(X)}\big] = \mathbb{E}\big[h(X)g(X)\big] &= \int_0^1 \mathbb{E}\big[\nabla g(X)\cdot\nabla h(Z_\alpha)\big]d\alpha\\
&= t\int_0^1 \mathbb{E}\big[\nabla h(X)\cdot\nabla h(Z_\alpha)e^{th(X)}\big]d\alpha\\
&\le t\mathbb{E}\big[e^{th(X)}\big],
\end{aligned} \tag{2.17}$$

where we used the Cauchy-Schwarz inequality and the Lipschitz property of h. Let u be the function defined by

$$e^{u(t)} = \mathbb{E}\big[e^{th(X)}\big], \tag{2.18}$$

then derivating in the both sides

$$\mathbb{E}\big[h(X)e^{th(X)}\big] = u'(t)e^{u(t)}, \tag{2.19}$$

so that from the preceding inequality, $u'(t) \le t$. Since $u(0) = 0$ it follows that $u(t) \le t^2/2$ and we are done. $\square$

The following lemma gives the crucial step toward proving the BTIS inequality.

Lemma 2.2.4. *Let X be a k-dimensional vector of centered, unit-variance, independent, Gaussian variables. If $h : \mathbb{R}^k \to \mathbb{R}$ has Lipschitz constant σ, then for all $u > 0$,*

$$\mathbb{P}\Big(h(X) - \mathbb{E}[h(X)] > u\Big) \le e^{-\frac{1}{2}u^2/\sigma^2}. \tag{2.20}$$

Proof. By considering $\widetilde{h}(x) = h(x)/\sigma$ it suffices to prove the result for $\sigma = 1$. Assume for the moment that $h \in C^2$. Then, for every $t, u > 0$,

$$\begin{aligned}\mathbb{P}\big(h(X) - \mathbb{E}[h(X)] > u\big) &\le \int_{h(X)-\mathbb{E}[h(X)]>u} e^{t(h(X)-\mathbb{E}[h(X)]-u)} dP(x)\\ &\le e^{-tu}\mathbb{E}\big[e^{t(h(x)-\mathbb{E}[h(X)])}\big]\\ &\le e^{\frac{1}{2}t^2-tu},\end{aligned} \tag{2.21}$$

the last inequality following from (2.15). Taking the optimal choice of $t = u$ gives (2.20) for $h \in C^2$.

To remove the C^2 assumption, take a sequence of C^2 approximations to f each one of which has Lipschitz coefficient no grater than σ (we recall that if a function f has Lipschitz constant σ the regularized function $\Phi_\epsilon(f)(x)$ has Lipschitz constant smaller than σ) and apply Fatou's inequality. This complete the proof. $\square$

We now have all we need to prove Theorem 2.2.1.

Proof of Theorem 2.2.1. There will be two stages to the proof. Firstly, we shall establish Theorem 2.2.1 for finite T. We than lift the result from finite to general T.

Thus, let T be finite, so that we can write it as $\{1, 2, \ldots, k\}$. In this case we can replace sup by max, which has Lipshitz constant 1. Let C the $k \times k$ covariance matrix of X on the finite set T, with components $c_{i,j} = \mathbb{E}[X_i X_j]$ so that

$$\sigma_T^2 = \max_{1\le i\le k} c_{ii} = \max_{1\le i\le k} \mathbb{E}[X_i^2]. \tag{2.22}$$

Let W a vector of independent, standard Gaussian variables, and A such that $A^t A = C$. Thus $X \overset{d}{=} AW$ and $\max_i X_i \overset{d}{=} \max_i(AW)_i$, where $\overset{d}{=}$ indicates equivalence in distribution. Consider the function $h(x) = \max_i(Ax)_i$, which is trivially C^2. Then

$$\begin{aligned} |\max_i (Ax)_i - \max_i (Ay)_i| &= |\max_i (e_i Ax) - \max_i (e_i Ay)| \\ &\le \max_i |e_i A(x-y)| \\ &\le \max_i |e_i A| \cdot |x-y|, \end{aligned} \tag{2.23}$$

where, as usual, e_i is the vector with 1 in position i and zeros elsewhere. The first inequality above is elementary, and the second is Cauchy–Schwarz. But

$$|e_i A|^2 = e_i^t A^t A e_i = e_i^t C e_i = c_{ii}, \tag{2.24}$$

so that

$$|\max_i (Ax)_i - \max_i (Ay)_i| \le \sigma_T |x-y|. \tag{2.25}$$

In view of the equivalence in law of $\max_i X_i$ and $\max_i (AW)_i$ and Lemma 2.2.4, this establishes the theorem for finite T.

We now turn to lifting the result from finite to general T. For each $n > 0$, let T_n be a finite subset of T such that $T_n \subset T_{n+1}$ and T_n increases to a dense subset of T. By separability,

$$\sup_{t \in T_n} X_t \xrightarrow{a.s.} \sup_{t \in T} X_t, \tag{2.26}$$

and since the convergence is monotone, we also have that

$$\mathbb{P}\Big(\sup_{t \in T_n} X_t > u\Big) \to \mathbb{P}\Big(\sup_{t \in T} X_t > u\Big) \quad \text{and} \quad \mathbb{E}\Big[\sup_{t \in T_n} X_t\Big] \to \mathbb{E}\Big[\sup_{t \in T} X_t\Big]. \tag{2.27}$$

Since $\sigma_{T_n}^2 \to \sigma_T^2 < \infty$ (again monotonically), we would be enough to prove general version of the BTIS-inequality from the finite-T version if only we knew that the term, $\mathbb{E}[\sup_{t \in T} X_t]$, were definitely finite, as claimed in the statement of the theorem. Thus if we show that the assumed a.s. finiteness of $\|X\|$ implies also the finiteness of its mean, we shall have a complete proof to both parts of the theorem.

We proceed by contradiction. Thus, assume $\mathbb{E}[\|X\|] = \infty$, and chose $u_0 > 0$ such that

$$e^{-u_0^2/\sigma_T^2} \le \frac{1}{4} \quad \text{and} \quad \mathbb{P}\Big[\sup_{t \in T} X_t < u_0\Big] \ge \frac{3}{4}. \tag{2.28}$$

Now chose $n \ge 1$ such that $\mathbb{E}[\|X\|_{T_n}] > 2u_0$, which is possible since $\mathbb{E}[\|X\|_{T_n}] \to \mathbb{E}[\|X\|_T] = \infty$. The BTIS-inequality on the finite space T_n then gives

$$\begin{aligned} \frac{1}{2} \ge 2e^{-u_0^2/\sigma_T^2} \ge 2e^{-u_0^2/\sigma_{T_n}^2} &\ge \mathbb{P}\Big(\big|\, \|X\|_{T_n} - \mathbb{E}[\, \|X\|_{T_n}\,]\big| > u_0\Big) \\ &\ge \mathbb{P}\Big(\mathbb{E}[\, \|X\|_{T_n}\,] - \|X\|_T > u_0\Big) \ge \mathbb{P}\Big(\, \|X\|_T < u_0\Big) \ge \frac{3}{4}. \end{aligned} \tag{2.29}$$

This provides the required contradiction, and so we are done. □

Chapter 3

Some useful tools for the DGFF

From now till the end of this paper our object of study is the Discrete Gaussian Free Field (DGFF) on $\mathbb{Z}^d$, with the canonical law $\mathbb{P}$ on $\Omega = \mathbb{R}^{\mathbb{Z}^d}$ such that under $\mathbb{P}$, the canonical field $\varphi = (\varphi_x)_{x\in\mathbb{Z}^d}$ is a centered Gaussian field with covariance $\mathbb{E}[\varphi_x\varphi_y] = G(x,y)$, for all $x, y \in \mathbb{Z}^d$, where $G(\cdot,\cdot)$ denotes the Green function of simple random walk on $\mathbb{Z}^d$ as defined in (1.36). Again, we will use the same notation for the probabilities $\mathbb{P}$ and P as explained in Remark 1.5.1.

3.1 Maximum for the Lattice Gaussian Free Field

We now state a very useful bound for the expectation of the maximum of the DGFF in a fixed bounded subset of $\mathbb{Z}^d$.

Proposition 3.1.1. *Let $\emptyset \neq K \subset\subset \mathbb{Z}^d$ then there exists a constant $c > 0$ such that*

$$\mathbb{E}\Big[\max_K \varphi\Big] \leq c\sqrt{\log|K|}. \tag{3.1}$$

Proof. In order to bound $\mathbb{E}[\max_K \varphi]$, we write, using Fubini's theorem in the third relation,

$$\begin{aligned}
\mathbb{E}\big[\max_K \varphi\big] \leq \mathbb{E}\big[\max_K \varphi^+\big] &= \mathbb{E}\Big[\int_0^{+\infty} \mathbb{1}_{\{y\leq \max_K \varphi^+\}} dy\Big] \\
&= \int_0^{+\infty} \mathbb{E}\Big[\mathbb{1}_{\{y\leq \max_K \varphi^+\}}\Big] dy \\
&= \int_0^{+\infty} \mathbb{P}\Big(y \leq \max_K \varphi^+\Big) dy \\
&\leq A + \int_A^{+\infty} \mathbb{P}\Big(\max_K \varphi^+ > y\Big) dy,
\end{aligned} \tag{3.2}$$

for arbitrary $A \geq 0$. Now using the following claim (that we will prove at the end)

$$\mathbb{P}\Big(\max_K \varphi^+ > y\Big) \leq |K| e^{-u^2/2G(0)} \tag{3.3}$$

and inserying it into (3.2) yields, for arbitrary $A > 0$,

$$\mathbb{E}\big[\max_K \varphi\big] \leq A + \int_A^{+\infty} |K| e^{-u^2/2G(0)} dy \leq A + c|K| \cdot e^{-A^2/2G(0)}. \tag{3.4}$$

We select A such that $e^{-A^2/2G(0)} = |K|^{-1}$ (i.e. $A = (2G(0)\log|K|)^{1/2}$), by which means (3.4) readily implies that

$$\mathbb{E}\big[\max_K \varphi\big] \leq c\sqrt{\log|K|}, \qquad \text{for all } \emptyset \neq K \subset\subset \mathbb{Z}^d. \tag{3.5}$$

We now prove the claim (3.3). Recalling that $\mathbb{E}[\varphi_x^2] = G(x,x) = G(0)$ for all $x \in \mathbb{Z}^d$, using (in the third relation) the translation invariance of the probability $\mathbb{P}$, we can easily obtain the following bound

$$\mathbb{P}\Big(\max_K \varphi^+ > y\Big) = \mathbb{P}\Big[\bigcup_{x\in K} \{\varphi_x^+ > y\}\Big] \leq \sum_{x\in K} \mathbb{P}[\varphi_x^+ > y] = |K|\mathbb{P}[\varphi_0 > y]. \tag{3.6}$$

Introducing an auxiliary variable $\psi \sim N(0,1)$ and recalling that the partition function $F_{N(\mu,\sigma^2)}(x) = F_{N(0,1)}(\frac{x-\mu}{\sigma})$, we have

$$|K|\mathbb{P}(\varphi_0 > y) = |K|\ \mathbb{P}\big(\psi > G(0)^{1/2} y\big). \tag{3.7}$$

Using the extended version for non-negative increasing functions of the Markov's inequality (see section 2.1) with $f(a) = e^{\lambda a}$, we have

$$\mathbb{P}(\psi > a) \leq \mathbb{P}(|\psi| > a) \leq \min_{\lambda>0} e^{-\lambda a}\mathbb{E}[e^{\lambda\psi}] = \min_{\lambda>0} e^{-\lambda a + \lambda^2/2} = e^{-a^2/2} \tag{3.8}$$

since the minimum is attained at $\lambda = a$. Applying (3.8) to the last term of (3.7) we obtain

$$|K|\mathbb{P}(\psi > G(0)^{1/2} y) \leq |K|\ e^{-u^2/2G(0)}. \tag{3.9}$$

Summarizing (3.6),(3.7) and (3.9) we finally obtain

$$\mathbb{P}\Big(\max_K \varphi^+ > y\Big) \leq |K|\ e^{-u^2/2G(0)}. \tag{3.10}$$

□

3.2 Asymptotics for the Green function

We state here one important and very well known result regarding the behaviour of the Green function defined in (1.36).

First of all we recall that, due to translation invariance, $G(x, y) = G(x - y, 0) =: G(x - y)$.

Lemma 3.2.1. *If $d \geq 3$, as $|x| \to \infty$, there exists a constant $c(d) > 0$ such that*

$$G(x) \sim c(d)|x|^{2-d}. \tag{3.11}$$

Proof. See [9], Theorem 1.5.4. □

3.3 Potential theory

In this section we will introduce some very useful aspects of potential theory. The main reference for this topic is Chapter 2 §2 of [9].

Given $K \subset\subset \mathbb{Z}^d$, we define the escape probability $e_K : K \to [0, 1]$ by

$$e_K(x) = P_x[\tilde{\tau}_K = \infty], \quad x \in K. \tag{3.12}$$

We also define the capacity of K as

$$\text{cap}(K) = \sum_{x \in K} e_K(x). \tag{3.13}$$

It immediately follows from the two definitions above that the capacity is a monotone and subadditive set function (see [9], Prop 2.2.1 for a proof), in particular

$$\text{cap}(K) \leq \text{cap}(K'), \qquad \text{for all } K \subset K' \subset\subset \mathbb{Z}^d; \tag{3.14}$$

$$\text{cap}(K) + \text{cap}(K') \geq \text{cap}(K \cup K') + \text{cap}(K \cap K'), \qquad \text{for all } K, K' \subset\subset \mathbb{Z}^d. \tag{3.15}$$

Further the probability to enter in K may be expressed in terms of $e_K(x)$ (see [13], Theorem 25.1, for a derivation) as

$$P_x(\tau_K < \infty) = \sum_{y \in K} G(x, y) \cdot e_K(y), \tag{3.16}$$

where $G(\cdot, \cdot)$ denotes the Green function defined in (1.36). Moreover, the following bounds on $P_x(\tau_K < \infty)$, $x \in \mathbb{Z}^d$ holds (c.f (1.9) of [14])

$$\frac{\sum_{y \in K} G(x, y)}{\sup_{z \in K} \left(\sum_{y \in K} G(z, y) \right)} \leq P_x(\tau_K < \infty) \leq \frac{\sum_{y \in K} G(x, y)}{\inf_{z \in K} \left(\sum_{y \in K} G(z, y) \right)}. \tag{3.17}$$

Finally from (3.16) and (3.17), togheter with classical bounds on the Green function (see Lemma (3.2.1)) for $|x| \to \infty$, we obtain

$$\sup_{z\in K}\Big(\sum_{y\in K} G(z,y)\Big) \geq \frac{|K|}{\mathrm{cap}(K)} \geq \inf_{z\in K}\Big(\sum_{y\in K} G(z,y)\Big). \tag{3.18}$$

A trivial consequence of the last inequalities, together with classical bounds on the Green function (c.f. Lemma 3.2.1), is the following useful bound for the capacity of a box:

$$\mathrm{cap}\big(B(0,L)\big) \leq cL^{d-2}, \qquad \text{for all } L \geq 1. \tag{3.19}$$

3.4 Recurrent and transient sets: The Wiener's Test

We say that a set $A \subset \mathbb{Z}^d$ is recurrent if $P_0(X_k \in A$ for infinitely many $k) = 1$, transient otherwise. We now state a very important criterion to determine whether a subset $A \subset \mathbb{Z}^d$ is either recurrent or transient for a symmetric simple random walk on $\mathbb{Z}^d$.

Theorem 3.4.1 (Wiener's Test)**.** *Suppose $A \subset \mathbb{Z}^d$, $(d \geq 3)$ and let*

$$A_n = \{z \in A;\ 2^n \leq |z|_\infty < 2^{n+1}\}. \tag{3.20}$$

Then A is a recurrent set if and only if

$$T[A] = \sum_{n=0}^{\infty} \frac{cap(A_n)}{2^{n(d-2)}} = \infty. \tag{3.21}$$

Proof. See [9], Theorem 2.2.5. □

We finally remark (See [9], p.57) that the function

$$f_A(x) := P_x(\tau_A < \infty), \tag{3.22}$$

is harmonic for $x \in A^c$ and $f_A(x) = 1$, for all $x \in A$. Moreover, if A is finite then

$$\lim_{|x|\to\infty} f_A(x) = 0. \tag{3.23}$$

3.5 Notation for the DGFF

We now turn to the Gaussian free field on $\mathbb{Z}^d$ using the notation defined at the beginning of this section.

Given any subset $K \subset \mathbb{Z}^d$, we frequently write φ_K to denote the family $(\varphi_x)_{x\in K}$. For arbitrary $a \in \mathbb{R}$ and $K \subset\subset \mathbb{Z}^d$, we also use the shorthand

$\{\varphi_{|K} > a\}$ for the event $\{\min\{\varphi_x; x \in K\} > a\}$ and similarly $\{\varphi_{|K} < a\}$ instead of $\{\max\{\varphi_x; x \in K\} < a\}$. Next, we introduce certain crossing events for the Gaussian free field. To this end, we first consider the space $\tilde{\Omega} = \{0,1\}^{\mathbb{Z}^d}$ endowed with its canonical σ-algebra and define, for arbitrary disjoint subsets $K, K' \subset \mathbb{Z}^d$, the event (subset of $\tilde{\Omega}$)

$$\{K \longleftrightarrow K'\} = \{\text{there exists an open path (i.e. along which the configuration has value 1) conneting } K \text{ and } K'\}. \tag{3.24}$$

For any level $h \in \mathbb{R}$, we introduced the (random) subset of $\mathbb{Z}^d$

$$E_\varphi^{\geq h} = \{x \in \mathbb{Z}^d; \varphi_x \geq h\}, \tag{3.25}$$

and we write ϕ^h for the measurable map from $\Omega = \mathbb{R}^{\mathbb{Z}^d}$ into $\tilde{\Omega} = \{0,1\}^{\mathbb{Z}^d}$ which send

$$\omega \in \Omega \longmapsto (\mathbb{1}_{\{\varphi_x(\omega) \geq h\}})_{x \in \mathbb{Z}^d} \in \tilde{\Omega}, \tag{3.26}$$

and define

$$\{K \overset{\geq h}{\longleftrightarrow} K'\} = (\phi^h)^{-1}(\{K \longleftrightarrow K'\}) \tag{3.27}$$

(a measurable subset of $\mathbb{R}^{\mathbb{Z}^d}$ endowed with its canonical σ-algebra $\mathcal{F}$), which is the event that K and K' are connected by a (nearest-neighbour) path in $E_\varphi^{\geq h}$. Note that $\{K \overset{\geq h}{\longleftrightarrow} K'\}$ is an increasing event upon introducing on $\mathbb{R}^{\mathbb{Z}^d}$ the natural partial order (i.e. $f \leq f'$ when $f_x \leq {f'}_x$ for all $x \in \mathbb{Z}^d$).

3.6 Density and uniqueness for the infinite cluster

In this section we want to investigate the properties of the infinite cluster. In particular we will state two very important results due to C.M. Newman and L.S. Schulman (see [10]) and R.M. Burton and M. Keane (see [4]), that can be summarized in the folliwing statement:

"If an infinite cluster exists, then it is unique and have positive density with probability one".

All the results stated in this section hold for site percolation models in the d-dimensional cubic lattice with nearest-neighbor bonds and so, for this section, our notation is not refered to the notation for the DGFF. The models we consider may be defined by a lattice of (site percolation) random variables, $\{X_k;\ k \in \mathbb{Z}^d\}$, where each X_k either takes the value 1 (corresponding to site k occupied) or the value 0 (corresponding to site k not occupied). Such a model may equivalent be defined by the joint distribution $\mathbb{P}$ of $\{X_k\}$ which is a probability measure on the configuration space,

$$\Omega = \{0,1\}^{\mathbb{Z}^d} = \{\omega = (\omega_k : k \in \mathbb{Z}^d) : \text{each } \omega_k = 0 \text{ or } 1\}$$

(with the standard definition of measurable sets); we assume (without loss of generality) that $(\Omega,\ \mathbb{P})$ is the underlying space with $X_k(\omega) = \omega_k$.

For any $j \in \mathbb{Z}^d$ we consider the shift operator T_j which acts either on configuration $\omega \in \Omega$, or on events (i.e., measurable sets) $W \subset \Omega$, or on measures $\mathbb{P}$ on Ω, or else on random variables X on Ω according to the following rules:

$$\begin{aligned} &(T_j\omega)_k = \omega_{k-j}, && T_jW = \{T_j\omega : \omega \in W\}, \\ &(T_j\mathbb{P})(W) = \mathbb{P}(T_{-j}W), && (T_jX)(\omega) = X(T_{-j}\omega). \end{aligned} \tag{3.28}$$

For each $k \in \mathbb{Z}^d$ and $\eta = 0$ or 1 we consider the measure $\mathbb{P}^\eta_k$ on $\Omega_k = \{0,1\}^{\mathbb{Z}^d \setminus \{k\}}$, defined so that for $U \subset \Omega_k$,

$$\mathbb{P}^\eta_k(U) = \frac{\mathbb{P}(U \times \{\omega_k = \eta\})}{\mathbb{P}(\{\omega_k = \eta\})}, \tag{3.29}$$

$\mathbb{P}^\eta_k$ is the conditional distribution of $\{X_j;\ j \neq k\}$ conditioned on $X_k = \eta$. Finally, throughout this section, we assume the following two hypotheses on $\mathbb{P}$, or equivalently on $\{X_k\}$:

1. $\mathbb{P}$ is translation invariant; i.e., for any $j \in \mathbb{Z}^d$, $T_j\mathbb{P} = \mathbb{P}$.
2. $\mathbb{P}$ has the finite energy property; i.e., for any $k \in \mathbb{Z}^d$, $\mathbb{P}^0_k$ and $\mathbb{P}^1_k$ are equivalent measures; i.e., if $U \subset \Omega_k$, $\eta = 0$ or 1, and $\mathbb{P}^\eta_k(U) \neq 0$ then $\mathbb{P}^{1-\eta}_k(U) \neq 0$.

Remark 3.6.1. In the original article of C.M. Newman and L.S. Schulman, it is required a third hypothesis, that is, $\mathbb{P}$ is translation ergodic; i.e., if $j \neq 0$ and W is an event such that $T_jW = W$, then $\mathbb{P}(W) = 0$ or 1. Thanks to the ergodic decomposition theorem we can omit this third hypothesis, as done in the article of R.M. Burton and M. Keane.

Similarly to the definition given in (3.24) for the DGFF, we say that i is connected to j if $\{i \longleftrightarrow j\}$. Moreover, we define $C(j)$, the cluster belonging to j, as $C(j) = \{i : i \text{ is connected to } j\}$; note that $C(j) \neq \emptyset$ if j is not occupied. A set $C \subset \mathbb{Z}^d$ is called cluster if $C = C(j)$ for some j and is called infinite cluster if in addition $|C| = \infty$. The percolation probability is $\rho = \mathbb{P}(|C(K)| = \infty)$ (which is independent of j). We define for any $F \subset \mathbb{Z}^d$ its lower density, $\underline{D}(F)$ and upper density $\overline{D}(F)$ as

$$\underline{D}(F) = \liminf_{n\to\infty} \frac{|F \cap V_n|}{n^d}, \qquad \overline{D}(F) = \limsup_{n\to\infty} \frac{|F \cap V_n|}{n^d} \tag{3.30}$$

where

$$V_n = \left\{x \in \mathbb{Z}^d;\ -n/2 \leq x_i < n/2, \quad i \in \{1,\dots,d\}\right\} \tag{3.31}$$

is the cubic box of size lenght n in $\mathbb{Z}^d$. F is said to have a density $D(F)$ if $\underline{D}(F) = \overline{D}(F)$. We denote by H the set of all cluster C and define

$$H_0 = \{C \in H;\ |C| = \infty\}, \quad N_0 = |H_0|, \tag{3.32}$$

$$F_0 = \{j \in \mathbb{Z}^d; j \in C \text{ for some } C \in H_0\}, \tag{3.33}$$

H_0 is the set of infinite clusters, N_0 the number of infinite clusters, and F_0 the union of infinite clusters.

We are know ready to state the two main result of this section.

Theorem 3.6.2 (Newman and Schulman)**.** *Exactly one of the following three event has probability* 1*:*

1. $N_0 = 0$*;*
2. $N_0 = 1$*;*
3. $N_0 = \infty$*.*

If $N_0 \neq 0$ *(*$\mathbb{P}$*-a.s.), then* $\rho > 0$ *and* $D(F_0) = \rho$ *(*$\mathbb{P}$*-a.s.).*

Theorem 3.6.3 (Burton and Keane)**.** $\mathbb{P}(N_0 = \infty) = 0$.

Chapter 4

The two main results

4.1 Purpose of the thesis

We are interested in the event that the origin lies in an infinite cluster of $E_{\varphi}^{\geq h}$, which we denote by $\{0 \overset{\geq h}{\longleftrightarrow} \infty\}$ (we also denote by $C^h(0) = \{i : i \overset{\geq h}{\longleftrightarrow} 0\}$, the cluster containing the origin), and ask for which values of h this event occurs with positive probability. Since

$$\eta(h) := \mathbb{P}\big(0 \overset{\geq h}{\longleftrightarrow} \infty\big) \tag{4.1}$$

is decreasing in h, it is sensible to define the critical point for level-set percolation as

$$h_*(d) = \inf\{h \in \mathbb{R};\ \eta(h) = 0\} \in [-\infty, \infty] \tag{4.2}$$

(with the convection $\inf \emptyset = \infty$). A non-trivial phase transition is then said to occur if h_* is finite. In the next two sections we will present the proofs of the following two results:

- $h_*(d) \geq 0$ for all $d \geq 3$ and $h_*(3) < \infty$, proved by J. Bricmont, J.L. Lebowitz and C. Maes (BLM), see [3];
- $h_*(d) < \infty$ for all $d \geq 3$ proved by P.-F. Rodriguez and A.-S. Sznitman (RS), see [11].

Finally in chapter 5 we present a new and original (but incomplete) generalization of the BLM proof of the existence of a phase transition for $d = 3$ to any $d \geq 3$, that gives the same result due to Sznitman-Rodriguez with completely different arguments.

4.2 The proof of J. Bricmont, J.L. Lebowitz and C. Maes

In all this section we fix $d = 3$. See remark 4.2.4 to understand where the proof fails for $d \geq 4$.

4.2.1 Definitions and notation

We start this section by introducing some definition and notation. Let V and Λ be cubes centered around the origin, that is sets of the type $[-\alpha, \alpha]^d$, for some $\alpha \in \mathbb{N}$. In particular we suppose that $|V| \gg |\Lambda|$. Remember that $C^h(0)$ denote the (random) cluster containing the origin and define the event

$$C_V^h = \Big\{\varphi \in \Omega \colon C^h(0) \cap \partial^i V \neq \emptyset\Big\}. \tag{4.3}$$

Let $\mathcal{S}_V$ be the following collection of subsets of V

$$\mathcal{S}_V = \Big\{K \subseteq V : 0 \in K,\ K \text{ is connected},\ K \cap \partial^i V \neq \emptyset\Big\}. \tag{4.4}$$

Define for a paricular $K \in \mathcal{S}_V$ the event

$$E_K^h = \Big\{\varphi \in \Omega : \varphi_x \geq h, \forall x \in K \text{ and } \varphi_x < h, \forall x \in \partial_V K\Big\}, \tag{4.5}$$

where $\partial_V K = \partial K \cap V$.

Note that the family $\mathcal{S}_V$ is a collection of the possible shapes for the infinite cluster $C^h(0)$ inside V (see Figure 4.1).

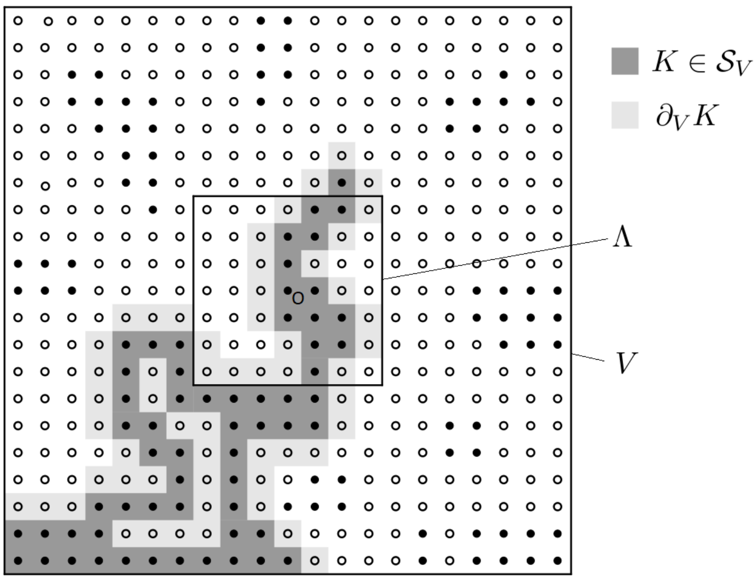

Figure 4.1: In this example we fix a level $h > 0$ and we paint in black the sites over the level h. Moreover we highlight the sets V, Λ, $K \in \mathcal{S}_V$ and $\partial_V K$. Note that $C^h(0) \cap V \supseteq K$.

4.2.2 The technical lemmas

We are now ready to state the four technical lemmas that we need to prove our statement.

Lemma 4.2.1. *Let C_V^h be the event defined in (4.3). Then*

1. *C_V^h is the disjoint union of the events E_K^h, i.e.,*

$$C_V^h = \bigsqcup_{K\in\mathcal{S}_V} E_K^h. \tag{4.6}$$

and if $K, K' \in \mathcal{S}_V$ and $K \neq K'$, then $E_K^h \cap E_{K'}^h = \emptyset$.

2. *$\mathbb{P}(C_V^h) \geq \eta(h)$ for all V and $\mathbb{P}(E_K^h) > 0$ for all $K \in \mathcal{S}_V$, all finite V.*

Proof. We start proving that $C_V^h = \bigsqcup_{K\in\mathcal{S}_V} E_K^h$. If $C^h(0)$ intersects the inner boundary of V, that is $C^h(0)\cap\partial^i V \neq \emptyset$, then the intersection of $C^h(0)$ with V contains a set $K \in \mathcal{S}_V$ and E_K^h occours (see Figure 4.1). Note that it is not always true that there exist a set $K \in \mathcal{S}_V$ such that $C^h(0) \cap V = K$ since $C^h(0) \cap V$ could not be connected. On the other side, if E_K^h occours for some $K \in \mathcal{S}_V$ then K is a subset of $C^h(0)$ and C_V^h occours. The events E_K^h are disjoint by definition. Part (2) is obvious. □

Lemma 4.2.2. *The function $x \mapsto \mathbb{E}[\varphi_x|E_K^h]$ is a harmonic function outside $\overline{K} = K \cup \partial K$.*

Proof. For all $K \subset\subset \mathbb{Z}^d$, applying Lemma 1.5.5 we have, for all $x \in \mathbb{Z}^d \setminus \overline{K}$,

$$\mathbb{E}[\varphi_x|\mathcal{F}_{\overline{K}}](w) = E_x[\omega_{X_{\tau_{\overline{K}}}}, \tau_{\overline{K}} < \infty], \quad \mathbb{P}\text{-a.s.}\ , \tag{4.7}$$

where $\mathcal{F}_{\overline{K}} = \sigma(\varphi_x;\ x \in \overline{K})$. Obviously $E_K^h \in \mathcal{F}_{\overline{K}}$ and in particular we have

$$\mathbb{E}[\varphi_x|E_K^h] = \mathbb{E}[E_x[\varphi_{X_{\tau_K}}]|E_K^h]. \tag{4.8}$$

Using the Markov property and the fact that, if $x \in \overline{K}^c$ and $y \sim x$ then $y \notin K$, we have

$$\begin{aligned}\frac{1}{2d}\sum_{y\sim x} E_y[\varphi_{X_{\tau_K}}] &= \sum_{y\sim x}\sum_{k\in\partial K} \frac{1}{2d} P_y(X_{\tau_K} = k)\varphi_k \\ &= \sum_{k\in\partial K} \varphi_k \sum_{y\sim x} P_x(X_1 = y) P_y(X_{\tau_K} = k) \\ &= \sum_{k\in\partial K} \varphi_k P_x(X_{\tau_K} = k) = E_x[\varphi_{X_{\tau_K}}].\end{aligned} \tag{4.9}$$

This implies that the function $x \mapsto \mathbb{E}[\varphi_x|E_K^h]$ is a harmonic function outside $\overline{K} = K \cup \partial K$. □

Lemma 4.2.3. *For some $0 < u \leq 1$ and for all Λ, we can take $V = V(\Lambda)$ large enough so that*

$$\frac{1}{|\Lambda|}\sum_{x\in\Lambda} f_{\overline{K}}(x) \geq u, \qquad \text{for all } K \in \mathcal{S}_V \text{ and for all } x \in \Lambda, \tag{4.10}$$

where $f_{\overline{K}}(x)$ is defined in (3.22).

Proof. In this proof we use the same notation used for the Winer's test (see 3.4.1). The proof of this test implies that, as a set A grows (in the sense of inclusion) such that $T[A] \to \infty$, then the function $f_A(x) \to 1$. Therefore, since the set Λ is a fixed and bounded region in $\mathbb{Z}^3$, it is sufficient to show that $T[K]$ can be made arbitrary large for all $K \in \mathcal{S}_V$ and V large enough. One has thus to verify condition (3.21) for an arbitrary infinite connected set A in $d = 3$. We will do this in two steps: first we reduce the volume of A and then we show that no set A is worse than a straight line. By the monotonicity property (3.14), $T[A] \geq T[a]$ where $a \subset A$ is obtained by keeping (in a nonunique but arbitrary fashion) for each $i = 0, 1, 2\ldots$ only one point in the intersection of A with the i-th shell$= \{y : y \text{ is on the boundary of the cube of size } 2i\}$. The volume $|a_n| = 2^n$, and by (3.18)

$$\mathrm{Cap}(a_n) \geq 2^n/M_n, \tag{4.11}$$

where

$$M_n = \max_{x\in a_n}\Big(\sum_{y\in a_n} G(x,y)\Big). \tag{4.12}$$

Fix $x \in a_n$. Then order $x \in a_n$, $y \neq x$, according to their distance from x. The $k-$th point in that order is at a distance at least k from x. Thus, using Lemma (3.2.1), we get

$$M_n \leq c\cdot\sum_{k=1}^{2^n}\frac{1}{k} \leq c'\cdot n. \tag{4.13}$$

Combining the inequalities (4.11)-(4.13) we thus get that $T[A] \geq c\cdot\sum 1/n$, hence the desired divergence. □

Remark 4.2.4. We underline that in the last proof there is the key point where the reasoning fails for $d \geq 4$. The main reason the result is restricted to $d = 3$ is that they use, in the previous lemma, the fact that $T[A] = \infty$ for any infinite, connected set A. This fails in $d = 4$, as can be seen explicitly by considering the set A equal to a lattice axis.

Lemma 4.2.5. *For $h < \infty$ large enough there is a constant $c > 0$ such that for all V_n large enough*

$$\mathbb{E}[\varphi_x|E_K] \geq c, \qquad \text{for all } x \in \partial K, \text{ all } K \in \mathcal{S}_{V_n}. \tag{4.14}$$

Proof. This lemma is proved in [3], Lemma 3, p. 1264. We are not able to follow the last part of the proof where the Ruelle's superstability estimate is applied. □

4.2.3 Conclusion of the proof of the Theorem

Lemma (4.2.2) says that $\mathbb{E}\big[\varphi_x|E_K^h\big]$ is a harmonic function in $\mathbb{Z}^d\backslash\overline{K}$. Lemma (4.2.5) says that this function is larger than a strictly positive constant c for all $x\in\overline{K}$, for h large enough, and zero at infinity. Hence, by the principle of domination for harmonic functions

$$\mathbb{E}\big[\varphi_x|E_K^h\big]\geq c\cdot f_{\overline{K}}(x),\quad \text{for all}\quad x\in\mathbb{Z}^d. \tag{4.15}$$

For $d=3$ we can apply Lemma 4.2.3 and combine it with (4.15): there is a constant $\tilde{\mu}>0$ such that for all Λ, we can choose $V=V(\Lambda)$ large enough such that

$$\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\mathbb{E}[\varphi_x|E_K^h]\geq\tilde{u},\qquad \text{for all } K\in\mathcal{S}_V. \tag{4.16}$$

By Lemma 4.2.1, denoting $S_\Lambda:=\sum_{x\in\Lambda}\varphi_x$,

$$\mathbb{E}\big[(S_\Lambda)^2\big]\geq\mathbb{E}\big[(S_\Lambda)^2\mathbb{1}_{C_V^h}\big]=\sum_{K\in\mathcal{S}_V}\mathbb{E}\big[(S_\Lambda)^2\mathbb{1}_{E_K}\big]=\sum_{K\in\mathcal{S}_V}\mathbb{E}\big[(S_\Lambda)^2|E_K\big]\mathbb{P}[E_K] \tag{4.17}$$

and by the Schwartz inequality,

$$\geq\sum_{K\in\mathcal{S}_V}\mathbb{E}\big[S_\Lambda|E_K\big]^2\mathbb{P}[E_K]\geq\sum_{K\in\mathcal{S}_V}\tilde{u}^2|\Lambda|^2\mathbb{P}[E_K], \tag{4.18}$$

where we used (4.16) for the last inequality. Now, by Lemma 4.2.1 again,

$$=\tilde{u}^2|\Lambda|^2\mathbb{P}[C_V^h]\geq\tilde{u}^2|\Lambda|^2\eta(h). \tag{4.19}$$

Since this chain of inequalities holds for all Λ and $\mathbb{E}\Big[\big((1/|\Lambda|)S_\Lambda\big)^2\Big]\to 0,$, for $\Lambda\to\mathbb{Z}^3$, we obtain that $\eta(h)=0$. This completes the proof.

4.3 The proof of P.-F. Rodriguez and A.-S. Sznitman

The RS proof is based on two key ingredients:

- The "Renormalization scheme" (see section 4.3.1);
- A "recursive bounds" for the probabilities of some specifics events (see Proposition 4.3.7).

Before analysing the two key ingredients we state two technical results for the conditional distribution of the DGFF.

Lemma 1.5.6 yelds a choice of regular conditional distributions for $(\varphi_x)_{x\in\mathbb{Z}^d}$ conditioned on the variables $(\varphi_x)_{x\in\Lambda}$. Namely, $\mathbb{P}$-almost surely,

$$\mathbb{P}((\varphi_x)_{x\in\mathbb{Z}^d} \in \cdot\,|\mathcal{F}_\Lambda) = \tilde{\mathbb{P}}((\tilde{\varphi}_x + u_x)_{x\in\mathbb{Z}^d} \in \cdot\,), \tag{4.20}$$

where $u_x = E_x[\omega_{X_{\tau_\Lambda}}, \tau_\Lambda < \infty]$, $x \in \mathbb{Z}^d$, $\tilde{\mathbb{P}}$ does not act on $(u_x)_{x\in\mathbb{Z}^d}$, and $(\tilde{\varphi}_x)_{x\in\mathbb{Z}^d}$ is a centered Gaussian field under $\tilde{\mathbb{P}}$, with $\tilde{\varphi}_x = 0$, $\tilde{\mu}-$almost surely for $x \in \Lambda$.

The explicit form of the conditional distribution in (4.20) readily yields the following result, which can be viewed as a consequence of the FKG-inequality for the free field (see for example [8], Appendix B.1.).

Lemma 4.3.1. *Let $\alpha \in \mathbb{R}$, $\emptyset \neq K \subset\subset \mathbb{Z}^d$, and assume $A \in \mathcal{F}$ (the canonical σ-algebra on $\mathbb{R}^{\mathbb{Z}^d}$) is an increasing event. Then*

$$\mathbb{P}[A|\varphi_K = \alpha] \leq \mathbb{P}[A|\varphi_{|_K} \geq \alpha], \tag{4.21}$$

where the left-hand side is defined by the version of the conditional expectation in (4.20).

Intuitively, augmenting the field can only favour the occurence of A, an increasing event.

Proof. See [11], Lemma 1.4. □

4.3.1 Renormalization scheme

The main goal of this section is to present the renormalization scheme. This technique is one of the main ingredients of the proof of theorem 4.3.4 and it is similar to the one developed by Sznitman and Sidoravicius in the context of random interlacements, see [14] and [12]. This scheme will be used to derive recursive estimates for the probability of certain crossing events and the resulting bounds constitute the main tool for the proof. We begin by defining on the lattice $\mathbb{Z}^d$ a sequence of length scales

$$L_n = l_0^n L_0, \qquad \text{for } n \geq 0, \tag{4.22}$$

where $L_0 \geq 1$ and $l_0 \geq 100$ are both assumed to be integers and will be specified below. Hence, L_0 represents the finest scale and $L_1 < L_2 < \dots$ correspond to increasingly coarse scales. We further introduce renormalized lattices

$$\mathbb{L}_n = L_n\mathbb{Z}^d \subset \mathbb{Z}^d, \qquad n \geq 0, \tag{4.23}$$

and note that $\mathbb{L}_k \supseteq \mathbb{L}_n$ for all $0 \leq k \leq n$. To each $x \in \mathbb{L}_n$, we attach the boxes

$$B_{n,x} := B_x(L_n), \qquad \text{for } n \geq 0,\ x \in \mathbb{L}_n, \tag{4.24}$$

where we define $B_x(L) = x + ([0, L) \cap \mathbb{Z})^d$, the box of side length L attached to x, for any $x \in \mathbb{Z}^d$ and $L \geq 1$ (not to be confused with $B(x, L)$), c.f. Figure 4.2 below (note that $B_{n,x}$ is closed just on the left-hand side in every direction). Moreover, we let

$$\widetilde{B}_{n,x} = \bigcup_{y \in \mathbb{L}_n : d(B_{n,y}, B_{n,x}) \leq 1} B_{n,y}, \qquad n \geq 0,\ x \in \mathbb{L}_n, \tag{4.25}$$

so that $\{B_{n,x}; x \in \mathbb{L}_n\}$ defines a partition of $\mathbb{Z}^d$ into boxes of side length L_n for all $n \geq 0$, and $\widetilde{B}_{n,x}; x \in \mathbb{L}_n$, is simply the union of $B_{n,x}$ and its $*$-neighbouring boxes at level n. Furthermore, for $n \geq 1$ and $x \in \mathbb{L}_n$, $B_{n,x}$ is the disjoint union of the l_0^d boxes $\{B_{n-1,y}; y \in B_{n,x} \cap \mathbb{L}_{n-1}\}$ at level $n-1$ it contains. We also introduce the indexing sets

$$\mathcal{I}_n = \{n\} \times \mathbb{L}_n, \qquad n \geq 0, \tag{4.26}$$

and given $(n, x) \in \mathcal{I}_n$, $n \geq 1$, we consider the sets of labels

$$\begin{aligned}
&\mathcal{H}_1(n,x) = \{(n-1, y) \in \mathcal{I}_{n-1}; B_{n-1,y} \subset B_{n,x} \text{ and } B_{n-1,y} \cap \partial^i B_{n,x} \neq \emptyset\}, \\
&\mathcal{H}_2(n,x) = \{(n-1, y) \in \mathcal{I}_{n-1}; B_{n-1,y} \cap \{z \in \mathbb{Z}^d;\ d(z, B_{n,x}) = \lfloor L_n/2 \rfloor\} \neq \emptyset\}.
\end{aligned} \tag{4.27}$$

Note that for any two indices $(n-1, y_i) \in \mathcal{H}_i(n,x)$, $i = 1, 2$, we have $\widetilde{B}_{n-1,y_1} \cap \widetilde{B}_{n-1,y_2} = \emptyset$ and $\widetilde{B}_{n-1,y_1} \cup \widetilde{B}_{n-1,y_2} \subset \widetilde{B}_{n,x}$. Finally, given $x \in \mathbb{L}_n, n \geq 0$, we introduce $\Lambda_{n,x}$, a family of subsets $\mathcal{T}$ of $\bigcup_{0 \leq k \leq n} \mathcal{I}_k$ (soon to be thought as a binary trees) defined as

$$\begin{aligned}
\Lambda_{n,x} = \Bigg\{ \mathcal{T} \subset \bigcup_{k=0}^{n} \mathcal{I}_k;\ & \mathcal{T} \cap \mathcal{I}_n = (n,x) \text{ and every } (k,y) \in \mathcal{T} \cap \mathcal{I}_k,\ 0 < k \leq n, \\
& \text{has two “descendants” } (k-1, y_i(k,y)) \in \mathcal{H}_i(k,y),\ i = 1, 2, \\
& \text{such that} \\
& \mathcal{T} \cap \mathcal{I}_{k-1} = \bigcup_{(k,y) \in \mathcal{T} \cap \mathcal{I}_k} \{(k-1, y_1(k,y)),\ (k-1, y_2(k,y))\} \Bigg\}.
\end{aligned} \tag{4.28}$$

Hence, any $\mathcal{T} \in \Lambda_{n,x}$ can naturally be identified as a binary tree having root $(n,x) \in \mathcal{I}_n$ and depth n. Since $|\mathcal{H}_1(n,x)| = c_1 {l_0}^{d-1}$ and $|\mathcal{H}_2(n,x)| = c_2 {l_0}^{d-1}$, the following bound on the cardinality of $\Lambda_{n,x}$ is easily obtained,

$$|\Lambda_{n,x}| \leq (c{l_0}^{d-1})^2 (c{l_0}^{d-1})^{2^2} \cdots (c{l_0}^{d-1})^{2^n} = (c{l_0}^{2(d-1)})^{2(2^n-1)} \leq (c_0 {l_0}^{2(d-1)})^{2^n}, \tag{4.29}$$

where $c_0 \geq 1$ is a suitable constant.

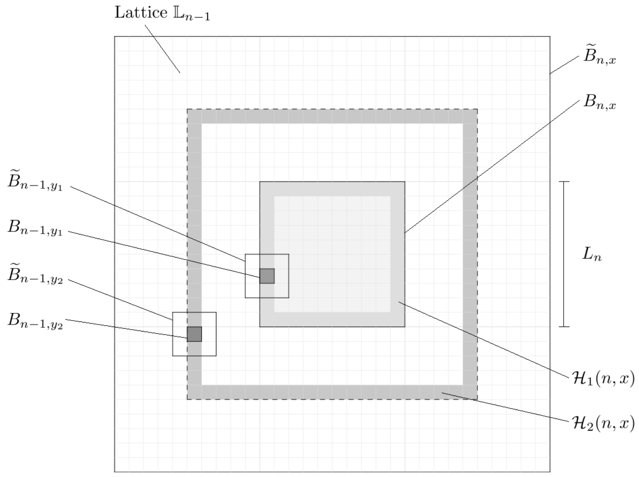

Figure 4.2: Renormalization scheme with $n = 2$, $L_0 = 1$, $l_0 = 10$.

4.3.2 Crossing events

We now consider the lattice Gaussian free field $\varphi = (\varphi_x)_{x\in\mathbb{Z}^d}$ defined in chapter 1 and introduce the crossing events

$$A^h_{n,x} = \{B_{n,x} \overset{\geq h}{\longleftrightarrow} \partial^i \widetilde{B}_{n,x}\}, \qquad \text{for } n \geq 0,\ x \in \mathbb{L}_n. \tag{4.30}$$

Three properties of the events $A^h_{n,x}$ will play a crucial role in what follows. Denoting by $\sigma(\varphi_y;\ y \in \widetilde{B}_{n,x})$ the σ-algebra on $\mathbb{R}^{\mathbb{Z}^d}$ generated by the random variables $\varphi_y, y \in \widetilde{B}_{n,x}$, we have

$$A^h_{n,x} \in \sigma(\varphi_y;\ y \in \widetilde{B}_{n,x}), \tag{4.31a}$$
$$A^h_{n,x} \text{ is increasing (in } \varphi) \qquad \text{(see the discussion below (3.27))}, \tag{4.31b}$$
$$A^h_{n,x} \supseteq A^{h'}_{n,x}, \quad \text{for all } h, h' \in \mathbb{R} \text{ with } h \leq h'. \tag{4.31c}$$

Indeed, the property (4.31c) that $A^h_{n,x}$ decreases with h follows since $E^{\geq h} \subseteq E^{\geq h'}$ for all $h \leq h'$ by definition, c.f. (3.25).

4.3.3 The structure of the proof

In this section we give all the P.-F. Rodriguez and A.-S. Sznitman's ideas to show that

$$h_*(d) < \infty, \qquad \text{for all } d \geq 3. \tag{4.32}$$

To prove (4.32) it enough to construct an explicit level $\overline{h}$ with $0 < \overline{h} < \infty$ such that

$$\mathbb{P}\big(B(0,L) \stackrel{\geq \overline{h}}{\longleftrightarrow} S(0,L)\big) \text{ decays in } L, \text{ as } L \to \infty. \tag{4.33}$$

Actually, the proof of P.-F. Rodriguez and A.-S. Sznitman will even show that $\mathbb{P}[B(0,L) \stackrel{\geq \overline{h}}{\longleftrightarrow} S(0,L)]$ has stretched exponential decay which implies a (seemingly) stronger result. A second critical parameter is defined

$$h_{**}(d) = \inf\{h \in \mathbb{R};\ \text{for some } \alpha > 0, \lim_{L\to\infty} L^{\alpha}\mathbb{P}[B(0,L) \stackrel{\geq h}{\longleftrightarrow} S(0,L)] = 0\}, \tag{4.34}$$

and the following stronger statement is proved:

$$h_{**}(d) < \infty, \qquad \text{for all } d \geq 3. \tag{4.35}$$

For the sake of clarity we investigate later the relevance of this second critical parameter (see Remark 4.3.3) and we now directly prove that $h_*(d) < \infty$, for all $d \geq 3$.

As stated before, it is enough to prove (4.33) and to understand why this implies that $h_*(d) < \infty$, for all $d \geq 3$. To this second end we note that for every $h \in \mathbb{R}$, given $L \geq 2L_0$, there exists $n \geq 0$ such that $2L_n \leq L \leq 2L_{n+1}$ and

$$\begin{aligned}\eta(h) = \mathbb{P}\big(0 \stackrel{\geq h}{\longleftrightarrow} \infty\big) &\stackrel{(1)}{\leq} \mathbb{P}\big(B(0,L) \stackrel{\geq h}{\longleftrightarrow} S(0,2L)\big) \stackrel{(2)}{\leq} \\ &\stackrel{(2)}{\leq} \mathbb{P}\Bigg(\bigcup_{x\in\mathbb{L}_n : B_{n,x}\cap S(0,L)\neq\emptyset} \{B_{n,x} \stackrel{\geq h}{\longleftrightarrow} \partial^i \widetilde{B}_{n,x}\}\Bigg).\end{aligned} \tag{4.36}$$

We now comment on the two inequality, helping out with some picture extracted from a simulation of the renormalization scheme that we realized during the master thesis.

1. Consider a realization of the event $\{0 \stackrel{\geq h}{\longleftrightarrow} \infty\}$, i.e. a path in $E_{\varphi}^{\geq h}$ connecting 0 to ∞, then this path must also connect the box $B(0,L)$ to the sphere $S(0,2L)$, c.f. Figure 4.3 (Step 1) below;

2. consider a realization of the event $\{B(0,L) \stackrel{\geq h}{\longleftrightarrow} S(0,2L)\}$, i.e. a path in $E_{\varphi}^{\geq h}$ connecting $B(0,L)$ to $S(0,2L)$, then this path must also cross the box $B_{n,x}$ for some $x \in \mathbb{L}_n : B_{n,x}\cap S(0,L) \neq \emptyset$ and so must connect the box $B_{n,x}$ to the boundary $\partial^i \widetilde{B}_{n,x}$ of its $\mathbb{L}_n-$neighbourhoods, c.f. Figure 4.3 (Step 2) below.

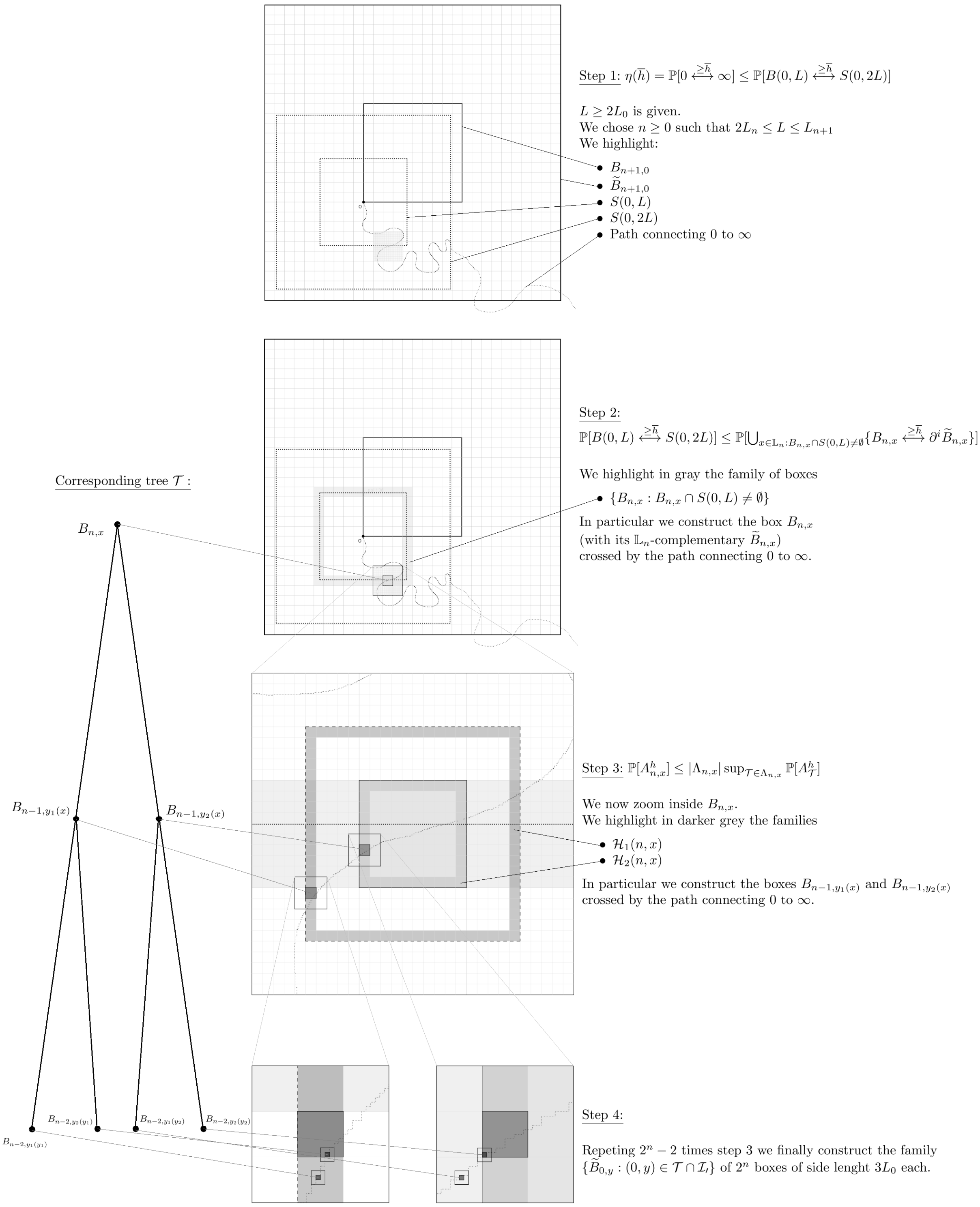

Figure 4.3: In this simulation we fix $L = 882$, $n = 2$, $L_0 = 1$, $l_0 = 10$.

Since the number of sets contributing to the union on the right-hand side of (4.36) is bounded by cl_0^{d-1} we obtain

$$\eta(h) \le cl_0^{d-1}\mathbb{P}\Big(B_{n,x} \overset{\ge h}{\longleftrightarrow} \partial^i \widetilde{B}_{n,x}\Big) = cl_0^{d-1}\mathbb{P}\Big(A^h_{n,x}\Big), \tag{4.37}$$

where the second term is well-defined (i.e independent of $x \in \mathbb{L}_n$) by translation invariance. Now we provide a lemma which separates the combinatorial complexity of the number of crossings in $A^h_{n,x}$ from probabilistic estimates, using $\Lambda_{n,x}$ as introduced in (4.28).

Lemma 4.3.2. $(n \ge 0, (n,x) \in \mathcal{I}_n, h \in \mathbb{R})$

$$\mathbb{P}\Big(A^h_{n,x}\Big) \le |\Lambda_{n,x}| \sup_{\mathcal{T}\in\Lambda_{n,x}} \mathbb{P}\Big(A^h_{\mathcal{T}}\Big), \qquad \textit{where } A^h_{\mathcal{T}} = \bigcap_{(0,y)\in\mathcal{T}\cap\mathcal{I}_0} A^h_{0,y}. \tag{4.38}$$

Proof. We use induction on n to show that

$$A^h_{n,x} \subseteq \bigcup_{\mathcal{T}\in\Lambda_{n,x}} A^h_{\mathcal{T}}, \tag{4.39}$$

for all $(n,x) \in \mathcal{I}_n$, from which (4.38) immediately follows. When $n = 0$, (4.39) is trivial. Assume it holds for all $(n-1,y) \in \mathcal{I}_{n-1}$. For any $(n,x) \in \mathcal{I}_n$, a path in $E^{\ge h}_{\varphi}$ starting in $B_{n,x}$ and ending in $\partial^i \widetilde{B}_{n,x}$ must first cross the box B_{n-1,y_1} for some $(n-1,y_1) \in \mathcal{H}_1(n,x)$, and subsequently $B_{(n-1,y_2)}$ for some $(n-1,y_2) \in \mathcal{H}_2(n,x)$ before reaching $\partial^i \widetilde{B}_{n,x}$, c.f. Figure 4.3 (Step 3) below. Thus,

$$A^h_{n,x} \subseteq \bigcup_{\substack{(n-1, y_i) \in \mathcal{H}_i(n,x) \\ i=1,2}} A^h_{n-1,y_1} \cap A^h_{n-1,y_2}.$$

Upon applying the induction hypothesis to A^h_{n-1,y_1} and A^h_{n-1,y_2} separately, the claim (4.39) follows. $\square$

Before proceeding, we remark that the event $A^h_{\mathcal{T}}$, with $h \in \mathbb{R}$ and $\mathcal{T} \in \Lambda_{n,x}$ for some $(n,x) \in \mathcal{I}_n, n \ge 0$, defined in (4.38) depends on 2^n boxes of side $3L_0$ each, c.f. Figure 4.3 (Step 4) below, the first 2^{n-1} contained in $\mathcal{H}_1(n,x)$ and the remaining 2^{n-1} contained in $\mathcal{H}_2(n,x)$. Moreover, it follows from (4.31c) that for any two levels $h, h' \in \mathbb{R}$,

$$A^h_{\mathcal{T}} \supseteq A^{h'}_{\mathcal{T}}, \qquad \text{whenever } h \le h', \tag{4.40}$$

thus, upon introducing

$$p_n(h) = \sup_{\mathcal{T}\in\Lambda_{n,x}} \mathbb{P}\big(A^h_{\mathcal{T}}\big), \qquad \text{for } (n,x) \in \mathcal{I}_n, n \ge 0, \tag{4.41}$$

which is well-defined (i.e independent of $x \in \mathbb{L}_n$) by translation invariance, we obtain

$$p_n(h) \geq p_n(h'), \qquad \text{whenever } h \leq h'. \tag{4.42}$$

Returning to (4.37), using (4.38) and (4.41), we obtain

$$\eta(h) \overset{(4.38)}{\leq} cl_0^{d-1} |\Lambda_{n,x}| \sup_{\mathcal{T} \in \Lambda_{n,x}} \mathbb{P}[A_{\mathcal{T}}^h] \overset{(4.41)}{=} cl_0^{d-1} |\Lambda_{n,x}| p_n(h). \tag{4.43}$$

It remains to explicitly construct an increasing but bounded sequence $(h_n)_{n \geq 0}$, with finite limit h_∞, such that $p_n(h_n)$ decreases faster than $(c_0 {l_0}^{2(d-1)})^{-2^n}$, since $|\Lambda_{n,x}| \overset{(4.29)}{\leq} (c_0 {l_0}^{2(d-1)})^{2^n}$. This result appears in Theorem 4.3.4 where we show that such a sequence $(h_n)_{n \geq 0}$ exists and $p_n(h_n) \leq (2c_0 {l_0}^{2(d-1)})^{-2^n}$. With this result at hand we can conclude the proof. We set $\overline{h} = h_\infty$ and using (4.42), we obtain

$$\eta(\overline{h}) \overset{(4.42)}{\leq} cl_0^{d-1} |\Lambda_{n,x}| p_n(h_n) \leq c_0 l_0^{d-1} 2^{-2^n}. \tag{4.44}$$

We finally set $\rho = \log 2 / \log l_0$, whence $2^n = {l_0}^{n\rho} = (L_n/L_0)^\rho$. Then, by adjusting c, c', (4.44) readily implies

$$\eta(\overline{h}) \leq c \cdot e^{-c' L^\rho}, \qquad \text{for all } L \geq 1, \tag{4.45}$$

for suitable $c, c' > 0$ and $0 < \rho < 1$. It follows that $\overline{h} \geq h_*$ which completes the proof.

Remark 4.3.3. Note that (4.45) also implies that $\overline{h} \geq h_{**}$ and so $h_{**}(d) < \infty$, for all $d \geq 3$. An important open question is whether h_* equals h_{**} or not. In case the two differ, the decay of $\mathbb{P}\big(0 \overset{\geq h}{\longleftrightarrow} S(0,L)\big)$ as $L \to \infty$, for $h > h_*$, exhibits a sharp transition. Indeed, first note that by definition of h_*, for all $h > h_*$, $\mathbb{P}\big(0 \overset{\geq h}{\longleftrightarrow} S(0,L)\big) \to 0,$ as $L \to \infty$. If $h_{**} > h_*$, then by definition of h_{**},

$$\text{for } h \in (h_*, h_{**}) \text{ and any } \alpha > 0, \qquad \limsup_{n \to \infty} L^{d-1+\alpha} \mathbb{P}\big(0 \overset{\geq h}{\longleftrightarrow} S(0,L)\big) = \infty. \tag{4.46}$$

Hence $\mathbb{P}\big(0 \overset{\geq h}{\longleftrightarrow} S(0,L)\big)$ decays to zero with L, but with an at most polynomial decay for $h \in (h_*, h_{**})$. However, for $h > h_{**}$, $\mathbb{P}\big(0 \overset{\geq h}{\longleftrightarrow} S(0,L)\big)$ has a stretched exponential decay in L, since $\mathbb{P}\big(0 \overset{\geq h}{\longleftrightarrow} x\big) \leq \mathbb{P}\big(B(0,L) \overset{\geq h}{\longleftrightarrow} S(0,2L)\big) \overset{(4.45)}{\leq} c(h) e^{-c''(h)|x|^p}$ whenever $2L \leq |x|_\infty < 2(L+1)$. Recently A. Drewitz and P.-F. Rodriguez (see [6]) show that

$$h_*(d) \sim h_{**}(d), \qquad \text{as } d \to \infty \tag{4.47}$$

(we write $f(x) \sim g(x)$ as $x \to a$ if $\lim_{x \to a} f(x)/g(x) = 1$). It is at present an unresolved question whether both critical parameters are actually equal (in any dimension).

4.3.4 The main Theorem

We now state the aforementioned key result to complete the proof. Most of the smarter ideas are presented in Proposition 4.3.5 where Rodriguez and Sznitman are able to control the interactions between some crossing events.

Theorem 4.3.4. *There exist an increasing but bounded sequence $(h_n)_{n\geq 0}$, with finite limit h_∞, such that*

$$p_n(h_n) \leq (2c_0 {l_0}^{2(d-1)})^{-2^n}, \qquad \textit{for all } n \geq 0. \tag{4.48}$$

First of all we derive a "recursive bounds" for the probabilities $p_n(h_n)$, c.f. (4.41) below, along a suitable increasing sequence $(h_n)_{n\geq 0}$.

Proposition 4.3.5 ($L_0 \geq 1, l_0 \geq 100$)**.** *There exist positive constants c_1 and c_2 such that, defining*

$$M(n, L_0) = c_2\Big(\log(2^n(3L_0)^d)\Big)^{1/2}, \tag{4.49}$$

then,given any positive sequence $(\beta_n)_{n\geq 0}$ satisfying

$$\beta_n \geq (\log 2)^{1/2} + M(n, L_0), \qquad \textit{for all } n \geq 0, \tag{4.50}$$

and any increasing, real-valued sequence $(h_n)_{n\geq 0}$ satisfying

$$h_{n+1} \geq h_n + c_1\beta_n(2l_0^{-(d-2)})^{n+1}, \qquad \textit{for all } n \geq 0, \tag{4.51}$$

one has

$$p_{n+1}(h_{n+1}) \leq p_n(h_n)^2 + 3e^{-(\beta_n - M(n,L_0))^2}. \tag{4.52}$$

Remark 4.3.6. Note that the key-parameter β_n controls the size of the interval $h_{n+1} - h_n$ in a suitable way that the factor $e^{-(\beta_n - M(n,L_0))^2}$ in (4.52) can be small enough.

Before proceeding with the proof of the proposition, we recall again that the event $A^h_{\mathcal{T}}$, with $h \in \mathbb{R}$ and $\mathcal{T} \in \Lambda_{n,x}$ for some $(n, x) \in \mathcal{I}_n, n \geq 0$, defined in (4.38) depends on 2^n boxes of side $3L_0$ each, c.f. Figure 4.3 (Step 4), the first 2^{n-1} contained in $\mathcal{H}_1(n, x)$ and the remaining 2^{n-1} contained in $\mathcal{H}_2(n, x)$. In particular if we define the union of these boxes as

$$K_{\mathcal{T}} = \bigcup_{(0,y)\in\mathcal{T}\cap\mathcal{I}_0} \widetilde{B}_{0,y}, \tag{4.53}$$

immediately follows from the definition of $A^h_{\mathcal{T}}$ that

$$A^h_{\mathcal{T}} \in \sigma(\varphi_y\,;\, y \in K_{\mathcal{T}}). \tag{4.54}$$

We are now ready to prove the proposition.

Proof. We let $n \geq 0$, consider some $m = (n+1, x) \in \mathcal{I}_{n+1}$ and some tree $\mathcal{T} \in \Lambda_m$. We decompose

$$\mathcal{T} = \{m\} \cap \mathcal{T}_{n,y_1(m)} \cap \mathcal{T}_{n,y_2(m)}, \tag{4.55}$$

where $(n, y_i(m))$, $i = 1, 2$ are the two descendants of m in $\mathcal{T}$ and

$$\mathcal{T}_{n,y_i(m)} = \{(k,z) \in \mathcal{T} : \widetilde{B}_{k,z} \subseteq \widetilde{B}_{n,y_i(m)}\}, \quad \text{for } i = 1, 2, \tag{4.56}$$

that is $\mathcal{T}_{(n,y_i(m))}$ is the (sub-)tree consisting of all descendants of $(n, y_i(m))$ in $\mathcal{T}$ (in particular it is the left (i=1) or the right (i=2) (sub-)tree in Figure 4.3). Thus the union in (4.55) is over disjoint sets. Note in particular that $\mathcal{T}_{n,y_i(m)} \in \Lambda_{n,y_i(m)}$. By construction (see Figure 4.3), the subsets $K_{\mathcal{T}_{n,y_i(m)}}$ $\big(\subset \widetilde{B}_{n,y_i(m)}\big)$, for $i = 1, 2$, satisfy $K_{\mathcal{T}_{n,y_1(m)}} \cup K_{\mathcal{T}_{n,y_2(m)}} = \emptyset$. For sake of clarity, and since m and $\mathcal{T}$ will be fixed throughout the proof, we abbreviate

$$\mathcal{T}_{n,y_i(m)} = \mathcal{T}_i \quad \text{and} \quad K_{\mathcal{T}_{n,y_i(m)}} = K_i, \quad \text{for } i = 1, 2. \tag{4.57}$$

In order to estimate the probability of the event $A^h_{\mathcal{T}} = A^h_{\mathcal{T}_1} \cap A^h_{\mathcal{T}_2}$, $h \in \mathbb{R}$, we introduce a parameter $\alpha > 0$ (that will control the size of the interval $h_{n+1} - h_n$ to dominate the interactions) and write

$$\begin{aligned}
\mathbb{P}\big(A^h_{\mathcal{T}}\big) &= \mathbb{P}\Big(A^h_{\mathcal{T}_1} \cap A^h_{\mathcal{T}_2} \cap \big\{\max_{K_1} \varphi \leq \alpha\big\}\Big) + \mathbb{P}\Big(A^h_{\mathcal{T}_1} \cap A^h_{\mathcal{T}_2} \cap \big\{\max_{K_1} \varphi > \alpha\big\}\Big) \\
&= \mathbb{E}\Big[\mathbb{1}_{A^h_{\mathcal{T}_1}} \cdot \mathbb{1}_{\{\max_{K_1} \varphi \leq \alpha\}} \cdot \mathbb{E}\big[\mathbb{1}_{A^h_{\mathcal{T}_2}} | \varphi_{K_1}\big]\Big] + \mathbb{P}\Big(A^h_{\mathcal{T}_1} \cap A^h_{\mathcal{T}_2} \cap \big\{\max_{K_1} \varphi > \alpha\big\}\Big) \\
&\leq \mathbb{E}\Big[\mathbb{1}_{A^h_{\mathcal{T}_1}} \cdot \mathbb{1}_{\{\max_{K_1} \varphi \leq \alpha\}} \cdot \mathbb{P}\big[A^h_{\mathcal{T}_2} | \varphi_{K_1}\big]\Big] + \mathbb{P}\Big(\max_{K_1} \varphi > \alpha\Big),
\end{aligned} \tag{4.58}$$

where $\max_{K_1} \varphi = \max\{\varphi_x;\, x \in K_1\}$ and the second line follows because $A^h_{\mathcal{T}_1} \cap \big\{\max_{K_1} \varphi \leq \alpha\big\}$ is measurable with respect to $\sigma(\varphi_{K_1})$, c.f. (4.54).

We now split the proof in two steps to provide the following two bounds: <u>Bound 1</u>: On the event $\big\{\max_{K_1} \varphi \leq \alpha\big\}$, there exist a parameter $\gamma(\alpha)$ such that

$$\mathbb{P}\big[A^h_{\mathcal{T}_2} | \varphi_{|K_1}\big] \leq \mathbb{P}\big(A^{h-\gamma}_{\mathcal{T}_2}\big) \big(\mathbb{P}(\varphi_{|K_1} \geq -\alpha)\big)^{-1}. \tag{4.59}$$

Proof of Bound 1. Using (4.20) and (4.54) applied to $A^h_{\mathcal{T}_2}$, and with a slight abuse of notation, we find

$$\mathbb{P}[A^h_{\mathcal{T}_2} | \varphi_{K_1}] = \tilde{\mathbb{P}}[A^h_{\mathcal{T}_2}\big((\tilde{\varphi}_x + u_x)_{x \in K_2}\big)], \qquad \mathbb{P}\text{-almost surely}, \tag{4.60}$$

where $u_x = E_x[\varphi_{X_{\tau_{K_1}}}, \tau_{K_1} < \infty]$. On the event $\big\{\max_{K_1} \varphi \leq \alpha\big\}$, we have, for all $x \in K_2$,

$$u_x = \sum_{y \in K_1} \varphi_y P_x\big(X_{\tau_{K_1}} = y, \tau_{K_1} < \infty\big) \leq \alpha \cdot P_x(\tau_{K_1} < \infty) =: m_x(\alpha) \tag{4.61}$$

which is deterministic and linear in α. Moreover, we can bound $m_x(\alpha)$ as follows. By virtue of (3.16), $P_x(\tau_{K_1} < \infty) \leq \mathrm{cap}(K_1) \cdot \sup_{y \in K_1} g(x,y)$ for all $x \in K_2$. Since K_1 consists of 2^n disjoint boxes of side length $3L_0$, c.f. (4.57) and (4.53), its capacity can be bounded, using (3.15) and (3.19), as $\mathrm{cap}(K_1) \leq c 2^n L_0^{d-2}$. By Lemma 3.2.1 and (4.22) and the observation that $|x-y| \geq c' L_{n+1}$ whenever $x \in K_1$ and $y \in K_2$, it follows that

$$m_x(\alpha) \leq c_1 (2G(0))^{-1/2} \cdot \alpha \cdot 2^n l_0^{-(n+1)(d-2)} =: \frac{\gamma}{2}, \qquad \text{for } x \in K_2, \tag{4.62}$$

which defines the constant c_1 from (4.51), and the factor $(2G(0))^{-1/2}$ is kept for later convenience.

Returning to the conditional probability $\mathbb{P}\big[A^h_{\mathcal{T}_2} | \varphi_{K_1}\big]$, we first observe that, on the event $\big\{\max_{K_1} \varphi \leq \alpha\big\}$, and for any $x \in K_2$, the inequality $\tilde{\varphi}_x + u_x \geq h$ implies

$$\tilde{\varphi}_x - m_x(\alpha) \geq h - u_x - m_x(\alpha) \overset{(4.61)}{\geq} h - 2m_x(\alpha) \overset{(4.62)}{\geq} h - \gamma. \tag{4.63}$$

Hence, on the event $\big\{\max_{K_1} \varphi \leq \alpha\big\}$,

$$\begin{aligned} \mathbb{P}\big[A^h_{\mathcal{T}_2} | \varphi_{K_1}\big] &\overset{(4.60)}{=} \tilde{\mathbb{P}}\big[A^h_{\mathcal{T}_2}\big((\tilde{\varphi}_x + u_x)_{x \in K_2}\big)\big] \\ &\leq \tilde{\mathbb{P}}\big[A^h_{\mathcal{T}_2}\big((\tilde{\varphi}_x + m_x(\alpha))_{x \in K_2}\big)\big] = \mathbb{P}\big[A^{h-\gamma}_{\mathcal{T}_2} | \varphi_{K_1} = -\alpha\big], \end{aligned} \tag{4.64}$$

where the last equality follows by (4.20), nothing that, on the event $\{\varphi_{|K_1} = -\alpha\}$, we have $u_x = m_x(-\alpha) = -m_x(\alpha)$ for all $x \in K_2$, c.f. (4.61). Applying Lemma 4.3.1 to the right-hand side of (4.64), we immediately obtain that, on the event $\big\{\max_{K_1} \varphi \leq \alpha\big\}$,

$$\mathbb{P}\big[A^h_{\mathcal{T}_2} | \varphi_{K_1}\big] \leq \mathbb{P}\big(A^{h-\gamma}_{\mathcal{T}_2} | \varphi_{|K_1} \geq -\alpha\big) \leq \mathbb{P}\big(A^{h-\gamma}_{\mathcal{T}_2}\big)\big(\mathbb{P}(\varphi_{|K_1} \geq -\alpha)\big)^{-1}. \tag{4.65}$$

□

Bound 2:

$$\mathbb{P}\big(\max_{K_1} \varphi > \alpha\big) \leq \min\left\{\frac{1}{2}, \quad e^{-\left(\frac{\alpha}{\sqrt{2G(0)}} - M(n,L_0)\right)^2}\right\}. \tag{4.66}$$

Proof of Bound 2. By virtue of the BTIS-inequality (see section 2.2), for arbitrary $\emptyset \neq K \subset\subset \mathbb{Z}^d$, we have

$$\mathbb{P}\Big(\max_K \varphi > \alpha\Big) \leq \exp\left\{-\frac{(\alpha - \mathbb{E}[\max_K \varphi])^2}{2G(0)}\right\}, \qquad \text{if } \alpha > \mathbb{E}\big[\max_K \varphi\big], \tag{4.67}$$

and by Proposition 3.1.1 we know that

$$\mathbb{E}\big[\max_K \varphi\big] \leq c\sqrt{\log |K|}. \tag{4.68}$$

In the relevant case $K = K_1$ with $|K_1| = 2^n(3L_0)^d$, we thus obtain

$$\mathbb{E}\big[\max_{K_1}\varphi\big] \leq c_2(2G(0)\log(2^n(3L_0)^d))^{1/2} \stackrel{(4.49)}{=} \sqrt{2G(0)}\cdot M(n,L_0) \tag{4.69}$$

where the first inequality defines the constant c_2 from (4.49). We now require

$$\alpha/\sqrt{2G(0)} \geq \sqrt{\log 2} + M(n,L_0) \tag{4.70}$$

where the factor $\log 2$ is kept for later convenience, thus (4.67) applies and yields

$$\mathbb{P}\big(\max_{K_1}\varphi > \alpha\big) \leq \min\left\{\frac{1}{2},\quad e^{-\left(\frac{\alpha}{\sqrt{2G(0)}}-M(n,L_0)\right)^2}\right\}, \tag{4.71}$$

since $\frac{\alpha-\mathbb{E}[\max_{K_1}\varphi]}{\sqrt{2G(0)}} \geq \frac{\alpha}{\sqrt{2G(0)}} - M(n,L_0) \geq \sqrt{\log 2}$. $\square$

We now insert (4.59) into (4.58), noting that, since φ has the same law as $-\varphi$, we have $\mathbb{P}(\varphi_{|K_1} \geq -\alpha) = 1-\mathbb{P}(\min_{K_1}\varphi < -\alpha) = 1-\mathbb{P}(\max_{K_1}\varphi > \alpha)$, to get

$$\mathbb{P}\big(A^h_{\mathcal{T}}\big) \leq \mathbb{P}\big(A^h_{\mathcal{T}_1}\big)\cdot\mathbb{P}\big(A^{h-\gamma}_{\mathcal{T}_2}\big)\cdot\Big(1-\mathbb{P}\big(\max_{K_1}\varphi>\alpha\big)\Big)^{-1} + \mathbb{P}\Big(\max_{K_1}\varphi>\alpha\Big). \tag{4.72}$$

Using that $(1-x)^{-1} \leq 1+2x$ for all $0 \leq x \leq 1/2$ (with $x = \mathbb{P}(\max_{K_1}\varphi > \alpha)$), we finally obtain, for all α satisfying (4.70) and $h' \geq h$,

$$\begin{aligned}\mathbb{P}\big(A^{h'}_{\mathcal{T}}\big) \leq \mathbb{P}\big(A^h_{\mathcal{T}}\big) &\leq \mathbb{P}\big(A^h_{\mathcal{T}_1}\big)\cdot\mathbb{P}\big(A^{h-\gamma}_{\mathcal{T}_2}\big) + 3\cdot\mathbb{P}\Big(\max_{K_1}\varphi>\alpha\Big)\\ &\stackrel{(4.66)}{\leq} \mathbb{P}\big(A^h_{\mathcal{T}_1}\big)\cdot\mathbb{P}\big(A^{h-\gamma}_{\mathcal{T}_2}\big) + 3e^{-\left(\beta-M(n,L_0)\right)^2},\end{aligned} \tag{4.73}$$

where we have set $\beta = \alpha/\sqrt{2G(0)}$. The claim (4.52) now readily follows upon tacking suprema over all $\mathcal{T} \in \Lambda_{n+1,x}$ on both side of (4.73), letting $\beta_n := \beta$, $h_n := h-\gamma \in \mathbb{R}$ (h was arbitrary), $h_{n+1} = h'$, so that requiring $h_{n+1} = h' \geq h_n + \gamma$, by virtute of (4.62), is nothing but (4.51). Noting that (4.70) for $\beta_n = \beta$, we precisely recover (4.50). This concludes the proof of Proposition 4.3.5. $\square$

We now propagate the estimate (4.52) inductively. To this end, we first define, for all $n \geq 0$,

$$\beta_n = (\log 2)^{1/2} + M(n,L_0) + 2^{(n+1)/2}\big(n^{1/2} + K_0^{1/2}\big), \tag{4.74}$$

where $K_0 > 0$ is a certain parameter to be specified below (it will allow us to start the induction). Note in particular that condition (4.50) holds for this choice of $(\beta_n)_{n\geq 0}$. In the next proposition, we inductively derive bounds for $p_n(h_n), n \geq 0$, given any sequence $(h_n)_{n\geq 0}$ satisfying the assumptions of Proposition 4.49, provided the induction can be initiated.

Proposition 4.3.7. *Assume $h_0 \in \mathbb{R}$ and $K_0 \geq 3(1-e^{-1})^{-1} =: B$ are such that*

$$p_0(h_0) \leq e^{-K_0}, \tag{4.75}$$

and let the sequence $(h_n)_{n\geq 0}$ satisfy (4.51) with $(\beta_n)_{n\geq 0}$ as defined in (4.74). Then,

$$p_n(h_n) \leq e^{-(K_0-B)2^n}, \qquad \text{for all } n \geq 0. \tag{4.76}$$

The strategy of the proof is to define a sequence $(K_n)_{n\geq 0}$ such that $K_n \geq K_0 - B$ for all $n \geq 0$ and $p_n(h_n) \leq e^{-K_n 2^n}$ for all $n \geq 0$. This allows us to immediately conclude the proof.

Proof. We define a sequence $(K_n)_{n\geq 0}$ inductively by

$$K_{n+1} = K_n - \log\left(1 + e^{K_n}(3^{2^{-(n+1)}} e^{-2^{-(n+1)}(\beta_n - M(n,L_0))^2})\right), \qquad \text{for all } n \geq 0 \tag{4.77}$$

with β_n give by (4.74). Note that the factor $3^{2^{-(n+1)}} e^{-2^{-(n+1)}(\beta_n - M(n,L_0))^2}$ is the $2^{n+1}-$th root of the remainder term on the right-hand side of (4.52) (i.e. $3e^{-(\beta_n - M(n,L_0))^2}$). Note also that $K_n \leq K_0$ for all $n \geq 0$ since K_n is decreasing. Moreover, we have

- $K_n \geq K_0 - B$ for all $n \geq 0$.

  This is clear for n=0.

  When $n \geq 1$, first note that by the definition of K_n, for all $n \geq 1$,

$$K_n = K_0 - \sum_{m=0}^{n-1} \log\left(1 + e^{K_m}(3^{2^{-(m+1)}} e^{-2^{-(m+1)}(\beta_m - M(m,L_0))^2})\right). \tag{4.78}$$

  Moreover

$$\begin{aligned}(\beta_m - M(m,L_0))^2 &= ((\log 2)^{1/2} + 2^{(m+1)/2}(m^{1/2} + K_0^{1/2}))^2 \\ &\geq \log 2 + 2^{m+1}(m^{1/2} + K_0^{1/2})^2 \geq 2^{m+1}(m + K_0),\end{aligned} \tag{4.79}$$

  for all $m \geq 0$, which, inserted into (4.78), yields

$$\begin{aligned}K_n &\geq K_0 - \sum_{m=0}^{\infty} \log\left(1 + e^{K_m}(3^{2^{-(m+1)}} e^{-2^{-(m+1)}(2^{m+1}(m+K_0))})\right) \\ &= K_0 - \sum_{m=0}^{\infty} \log\left(1 + e^{K_m}(3^{2^{-(m+1)}} e^{-m-K_0})\right) \\ &\geq K_0 - 3\sum_{m=0}^{\infty} e^{-m} = K_0 - 3(1-e^{-1})^{-1} = K_0 - B,\end{aligned} \tag{4.80}$$

where we have used $K_n \leq K_0$ and $\log(1+x) \leq x$ for all $x \geq 0$ in the last inequality.

- $p_n(h_n) \leq e^{-K_n 2^n}$ for all $n \geq 0$.

  We show the result by induction on n.

  The inequality holds for $n = 0$ by assumption (4.75).

  Assume now it holds for some n. By condition (4.52), we find

$$\begin{aligned} p_{n+1}(h_{n+1}) &\leq p_n(h_n)^2 + 3e^{-(\beta_n - M(n,L_0))^2} \\ &\leq (e^{-K_n 2^n})^2 + 3e^{-(\beta_n - M(n,L_0))^2} \\ &\leq \left[e^{-K_n}\left(1 + e^{K_n} 3^{2^{-(n+1)}} e^{-2^{-(n+1)}(\beta_n - M(n,L_0)^2}\right)\right]^{2^{n+1}} \\ &= e^{-K_{n+1}2^{n+1}}. \end{aligned} \tag{4.81}$$

This concludes the proof of the proposition. □

We will now prove the main theorem (Theorem 4.3.4) of this section using Proposition 4.3.7.

Proof (of Theorem 4.3.4). We select K_0 appearing in Proposition 4.3.7 as follow:

$$K_0 = \log(2c_0 l_0^{2(d-1)}) + B \qquad \text{(see (4.29) for the definition of } c_0\text{)}. \tag{4.82}$$

Moreover we will solely consider sequences $(h_n)_{n\geq 0}$ with

$$h_0 > 0, \qquad h_{n+1} - h_n = c_1 \beta_n (2l_0^{-(d-2)})^{n+1}, \qquad \text{for all } n \geq 0, \tag{4.83}$$

so that condition (4.51) is satisfied. We recall that β_n is given by (4.74), which now reads

$$\beta_n = (\log 2)^{1/2} + c_2(\log(2^n(3L_0)^d))^{1/2} + 2^{(n+1)/2}(n^{1/2} + (\log(2c_0 l_0^{2(d-1)}) + B)^{1/2}) \tag{4.84}$$

where we have substituted $M(n, L_0)$ from (4.49) and K_0 from (4.82). Note that L_0, l_0 and h_0 are the only parameters which remain to be selected. We observe that the sequence defined in (4.83) has a finite limit $h_\infty = \lim_{n\to\infty} h_n$ for every choice of L_0, l_0 and h_0. Indeed, β_n as given by (4.84) satisfies $\beta_n \leq c(L_0, l_0)2^{n+1}$ for all $n \geq 0$, hence

$$h_\infty = h_0 + c_1 \sum_{n=0}^{\infty} \beta_n (2l_0^{-(d-2)})^{n+1} \leq h_0 + c'(L_0, l_0) \sum_{n=0}^{\infty} (4l_0^{-(d-2)})^{n+1} < \infty, \tag{4.85}$$

since we assumed $l_0 \geq 100$. We set

$$L_0 = 10, \qquad l_0 = 100, \tag{4.86}$$

and now show with Proposition 4.3.7 that there exists $h_0 > 0$ sufficiently large such that

$$p_n(h_n) \leq (2c_0 {l_0}^{2(d-1)})^{-2^n}, \qquad \text{for all } n \geq 0. \tag{4.87}$$

To this end, we note that $p_0(h_0)$ defined in (4.41)

$$\begin{aligned} p_0(h_0) &= \mathbb{P}\big(B_{0,x=0} \overset{\geq h_0}{\longleftrightarrow} \partial^i \widetilde{B}_{0,x=0}\big) \\ &\leq \mathbb{P}\Big(\max_{\widetilde{B}_{0,x=0}} \varphi \geq h_0\Big) \leq \exp\left\{ - \frac{\Big(h_0 - \mathbb{E}\big[\max_{\widetilde{B}_{0,x=0}} \varphi\big]\Big)^2}{2G(0)} \right\}, \end{aligned} \tag{4.88}$$

where the last inequality holds when $h_0 > c$ (e.g using Proposition 3.1.1 to bound $\mathbb{E}\big[\max_{\widetilde{B}_{0,x=0}} \varphi\big]$) thanks to BTIS-inequality (see (2.10)). In particular, since K_0 in (4.82) is completely determined by the choices (4.86), we see that $p_0(h_0) \leq e^{-K_0}$ for all $h_0 \geq c$, i.e. condition (4.75) holds for sufficiently large h_0. by Proposition 4.3.7, setting $h_0 = c$, we obtain

$$p_n(h_n) \overset{(4.76)}{\leq} e^{-(K_0-B)2^n} \overset{(4.82)}{=} (2c_0 l_0^{2(d-1)})^{-2^n} \qquad \text{for all } n \geq 0. \tag{4.89}$$

This result concludes the proof. $\square$

Chapter 5

A generalization of the BLM proof

In both two article [3] and [11] it is remarked that the BLM proof can't be easily gereralized to all $d \geq 3$ since, in $d \geq 4$, a infinite connected set may not be recurrent for the simple random walk on $\mathbb{Z}^d$. In this chapter we try to prove that this is not an issue, since, an infinite cluster over a level h, if it exists than it has positive density by Newman and Schulman theorem, and in particular, using the well-known Wiener's test, it is recurrent for the simple random walk on $\mathbb{Z}^d$.

5.1 Finite energy property for the DGFF

The first step is to prove that Theorem 3.6.2 holds for the level-sets percolation for the DGFF. We consider the measurable map ϕ^h from $\Omega = \mathbb{R}^{\mathbb{Z}^d}$ into $\tilde{\Omega} = \{0,1\}^{\mathbb{Z}^d}$ as defined in (3.26) and the associated pushforward measure $\mathbb{P}^h := \phi^h_*(\mathbb{P})$ on $\tilde{\Omega} = \{0,1\}^{\mathbb{Z}^d}$. We need to show that

1. $\mathbb{P}^h$ is translation invariant;
2. $\mathbb{P}^h$ has the finite energy property;

The first property is a trivial consequence of the translation-invariance property of $\mathbb{P}$. For the second property (that is claimed without proof in [11], Remark 1.6) we give a proof in the following lemma.

Lemma 5.1.1. *The probability measure $\mathbb{P}^h$ for the DGFF has the finite energy property.*

Proof. We need to check that for all $x \in \mathbb{Z}^d$ and $\eta = 0$ or 1, if $\tilde{U} \subset \tilde{\Omega}_x$ and $\mathbb{P}^h(\tilde{U} \times \{\mathbb{1}_{\{\varphi_x \geq h\}} = \eta\}) \neq 0$ then $\mathbb{P}^h(\tilde{U} \times \{\mathbb{1}_{\{\varphi_x \geq h\}} = 1 - \eta\}) \neq 0$. From the translation invariance of the DGFF, it is sufficient to show that for all

$\tilde{U} \subset \tilde{\Omega}_x$, setting $U := (\phi^h)^{-1}(\tilde{U})$, holds:

$$\mathbb{P}(U \times \{\varphi_0 \geq h\}) \neq 0 \quad \text{if and only if} \quad \mathbb{P}(U \times \{\varphi_0 < h\}) \neq 0. \tag{5.1}$$

Suppose that $\mathbb{P}(U \times \{\varphi_0 \geq h\}) > 0$. Since the measure $\mathbb{P}$ is tight (for example for the Ulam's theorem), then there exist $L > 0$ such that

$$\mathbb{P}\Big(\Big\{U \cap \big\{\varphi_x \in [h-L, h+L]; |x| = 1\big\}\Big\} \times \big\{\varphi_0 \in [h, h+L]\big\}\Big) > 0. \tag{5.2}$$

Denoting $U_1^L := \big\{\varphi_x \in [h-L, h+L]; |x| = 1\big\} \in \mathcal{F}_{\{x \in \mathbb{Z}^d; |x|=1\}} =: \mathcal{F}_1$ and using the local Markov property of the DGFF, we obtain

$$\begin{aligned}\mathbb{P}\Big(\big\{U \cap U_1^L\big\} \times \big\{\varphi_0 \in [h, h+L]\big\}\Big) &= \mathbb{E}\Big[\mathbb{E}\big[\mathbb{1}_{\{U \cap U_1^L\} \times \{\varphi_0 \in [h,h+L]\}} | \mathcal{F}_1\big]\Big] \\ &= \mathbb{E}\Big[\mathbb{1}_{U_1^L} \cdot \mathbb{E}\big[\mathbb{1}_U | \mathcal{F}_1\big] \cdot \mathbb{E}\big[\mathbb{1}_{\{\varphi_0 \in [h,h+L]\}} | \mathcal{F}_1\big]\Big].\end{aligned} \tag{5.3}$$

Under U_1^L, there exists a constant $\varepsilon_L > 0$ such that

$$\mathbb{E}\big[\mathbb{1}_{\{\varphi_0 \in [h,h+L]\}} | \mathcal{F}_1\big] \leq \varepsilon_L \cdot \mathbb{E}\big[\mathbb{1}_{\{\varphi_0 \in [h-L,h)\}} | \mathcal{F}_1\big], \quad a.s. \,. \tag{5.4}$$

Indeed, using standard results for conditional Gaussian vectors, $\mathbb{E}[\varphi_0 | \mathcal{F}_1]$ is a linear combination of the random variables φ_x for $x \in \mathbb{Z}^d$ such that $|x| = 1$, denoted by $Y := \sum_{x \in \mathbb{Z}^d; |x|=1} \lambda_x \varphi_x$, and more generally, denoting by $q_{\mu,\sigma^2}(x)$ the density of a Gaussian random variable with mean μ and variance σ^2, we have

$$\mathbb{E}\big[\mathbb{1}_{\{\varphi_0 \in [h,h+L]\}} | \mathcal{F}_1\big] = \int_{\mathbb{R}} \mathbb{1}_{[h,h+L]}(x) q_{Y,\sigma^2}(x) dx, \tag{5.5}$$

where $\sigma^2 = \mathbb{E}[(\varphi_0 - Y)^2]$. Finally, noting that under U_1^L, $Y \in [2d(h-L), 2d(h-L)]$, the bound in (5.4) follows from (5.5).

Summing up and repeating the same computation as in (5.3), we obtain

$$\begin{aligned}0 < \mathbb{P}\Big(\big\{U \cap U_1^L\big\} \times \big\{\varphi_0 \in [h, h+L]\big\}\Big) &\leq \varepsilon_L \cdot \mathbb{P}\Big(\big\{U \cap U_1^L\big\} \times \big\{\varphi_0 \in [h-L, h)\big\}\Big) \\ &\leq \varepsilon_L \cdot \mathbb{P}\big(U \times \{\varphi_0 < h\}\big).\end{aligned} \tag{5.6}$$

Since the other implication is similar, this concludes the proof. $\square$

Now, setting $\rho^h := \mathbb{P}\big(|C^h(0)| = \infty\big)$ and noting that $\rho^h > 0$ for all $h < h_*(d)$, we have the following result

Proposition 5.1.2. *For all $h < h_*(d)$, conditioning on the event $\{0 \overset{\geq h}{\longleftrightarrow} \infty\}$,*

$$D\big(C^h(0)\big) = \rho^h, \quad a.s. \,. \tag{5.7}$$

In particular, conditioning on the event $\{0 \overset{\geq h}{\longleftrightarrow} \infty\}$, $D\big(C^h(0)\big) > 0$ almost surely.

We are now ready to give our new proof of the existence of a phase transition for the level-set percolation for the DGFF on $\mathbb{Z}^d$, for all $d \geq 3$.

5.2 The setup

We start this section by introducing some definition and notation. Let $V_n,\ n \in \mathbb{N}$ be a collection of boxes as defined in equation (3.31). We denote by $\Lambda := V_m,$ for some $m \in \mathbb{N}$, a smaller box such that $|V_n| \gg |\Lambda|$. Remember that $C^h(0)$ denote the (random) cluster containing the origin and define the events

$$C^h_{V_n} = \Big\{\varphi \in \Omega \colon C^h(0) \cap \partial^i V_n \neq \emptyset\Big\}, \qquad \text{for all } h \in \mathbb{R} \text{ and } n \in \mathbb{N}. \tag{5.8}$$

Let $\mathcal{S}^h_{V_n}$ be the following collection of subsets of V_n, for all $n \in \mathbb{N}$,

$$\begin{aligned}\mathcal{S}^h_{V_n} = \Big\{K \subseteq V_n \ &: 0 \in K,\ K \text{ is connected and } K \cap \partial^i V_n \neq \emptyset;\ \text{and} \\ &\frac{\rho^h}{2} < \frac{|K \cap V_m|}{m^d} < 2\rho^h,\ \forall m \in \mathbb{N} \text{ s.t. } n/2 \le m \le n \ \Big\}.\end{aligned} \tag{5.9}$$

Note that the family $\mathcal{S}^h_{V_n}$ is a collection of the possible shapes for the connected component of the infinite cluster $C^h(0)$ inside V_n (see Figure 5.1).

Define for a paricular $K \in \mathcal{S}^h_{V_n}$ the event

$$E^h_K = \Big\{\varphi \in \Omega : \varphi_x \ge h, \forall x \in K \text{ and } \varphi_x < h, \forall x \in \partial_{V_n} K\Big\}, \tag{5.10}$$

where $\partial_{V_n} K = \partial K \cap V_n$.

Since in many of the following lemmas the parameter $h \in \mathbb{R}$ will be fixed, we will omit the appendix h on the sets, families and events previously defined, unless absolutely necessary. Moreover, from now till the end, we suppose by contradiction that, for any level h, the event $\{0 \overset{\ge h}{\longleftrightarrow} \infty\}$ occours with positive probability, i.e., $h_*(d) = \infty$.

5.3 The technical lemmas

Before state and prove the four technical lemmas, we note that (from Proposition 5.1.2) we have

$$\lim_{n\to\infty} \frac{|C(0)(\varphi) \cap V_n|}{n^d} = \rho, \quad \text{for a.a. } \varphi \in \{0 \overset{\ge h}{\longleftrightarrow} \infty\}. \tag{5.11}$$

Since $(\Omega, \mathcal{F}, \mathbb{P})$ is a finite measure space, from the Egorov's theorem we know that for all $\delta > 0$, there exists a measurable subset $\Omega_\delta \in \mathcal{F},\ \Omega_\delta \subset \{0 \overset{\ge h}{\longleftrightarrow} \infty\}$, such that

$$\begin{gathered}\mathbb{P}(\{0 \overset{\ge h}{\longleftrightarrow} \infty\} \setminus \Omega_\delta) < \delta \text{ and the limit (5.11) is uniform on } \Omega_\delta, \text{ i.e.,} \\ \lim_{n\to\infty} \sup_{\varphi \in \Omega_\delta} \Big|\frac{|C(0)(\varphi) \cap V_n|}{n^d} - \rho\Big| = 0.\end{gathered} \tag{5.12}$$

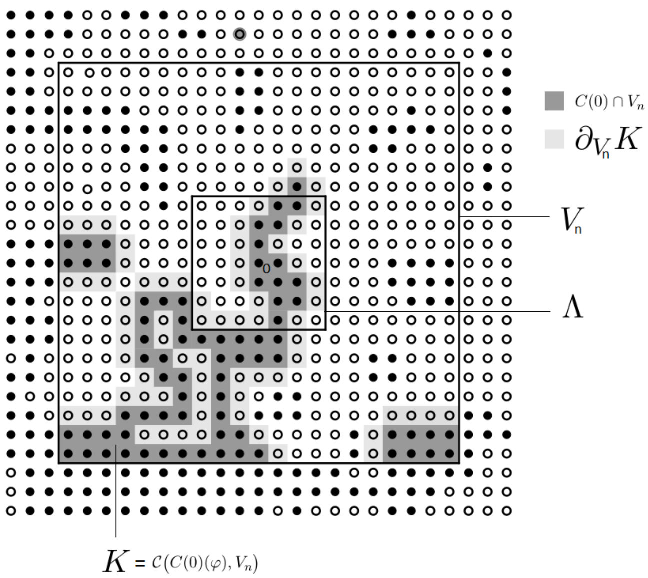

Figure 5.1: In this example we fix a level $h \in \mathbb{R}$ and we paint in black the sites over the level h. Moreover we highlight the sets V_n, Λ, $C(0) \cap V_n$, and $\partial_{V_n} K$. Note that $K = \mathcal{C}\big(C(0)(\varphi), V_n\big)$.

In particular, there exists $n_0 = n_0(\delta) \in \mathbb{N}$ such that for all $n \geq n_0$ we have

$$\frac{\rho}{2} < \frac{|C(0)(\varphi) \cap V_n|}{n^d} < 2\rho, \qquad \text{for all } \varphi \in \Omega_\delta. \tag{5.13}$$

We claim that the bound (5.13) holds also for the connected component in $C(0)(\varphi) \cap V_n$ containing 0, denoted by $\mathcal{C}\big(C(0)(\varphi), V_n\big)$. The proof of this claim still open for the moment.

Remark 5.3.1. We are considering the connected component $\mathcal{C}\big(C(0)(\varphi), V_n\big)$ instead of the whole set $C(0)(\varphi) \cap V_n$ in order to obtain a disjoint union in the left hand side of equation (5.14).

We can now state the first technical lemma.

Lemma 5.3.2. *Let C_{V_n} be the event defined in (5.8). Then for all $n \geq 2n_0$,*

1. *$C_{V_n} \cap \Omega_\delta$ is contained in the disjoint union of the events E_K, i.e.,*

$$C_{V_n} \cap \Omega_\delta \subseteq \bigsqcup_{K \in \mathcal{S}_{V_n}} E_K, \tag{5.14}$$

and if $K, K' \in \mathcal{S}_{V_n}$ and $K \neq K'$, then $E_K \cap E_{K'} = \emptyset$.

2. *$\mathbb{P}(C_{V_n} \cap \Omega_\delta) \geq \mathbb{P}\big(\{0 \overset{\geq h}{\longleftrightarrow} \infty\} \cap \Omega_\delta\big)$ and $\mathbb{P}(E_K) > 0$ for all $K \in \mathcal{S}_{V_n}$.*

Proof. Fix $n \geq 2n_0$. We start by proving that $C_{V_n} \cap \Omega_\delta \subseteq \bigsqcup_{K \in \mathcal{S}_{V_n}} E_K^h$. If $\varphi \in C_{V_n} \cap \Omega_\delta$ then $C(0)(\varphi)$ intersects the inner boundary of V_n, that is $C^h(0) \cap \partial^i V_n \neq \emptyset$. Setting $K := \mathcal{C}\big(C(0)(\varphi), V_n\big)$ then $K \in \mathcal{S}_{V_n}$ since

- $K \subseteq V_n$, $0 \in K$, K is connected and $K \cap \partial^i V_n \neq \emptyset$;
- From the claim below relation (5.13) and the fact that $n \geq 2n_0$ we have

$$\frac{\rho}{2} < \frac{|K \cap V_m|}{m^d} < 2\rho, \qquad \text{for all } m \in \mathbb{N} \text{ s.t. } n/2 \leq m \leq n. \tag{5.15}$$

In particular E_K occurs. The events E_K are disjoint by definition. Part (2) is obvious. □

We now show the key point of our proof, namely, the fact that a set with positive density is recurrent for a symmetric random walk in $\mathbb{Z}^d$.

Lemma 5.3.3. *For some $0 < u \leq 1$ and for all Λ, we can take $V_n = V_n(\Lambda)$ large enough so that*

$$\frac{1}{|\Lambda|} \sum_{x \in \Lambda} f_{\overline{K}}(x) \geq u, \qquad \textit{for all } K \in \mathcal{S}_{V_n} \textit{ and for all } x \in \Lambda, \tag{5.16}$$

where $f_{\overline{K}}(x)$ is defined in (3.22).

Proof. First of all we show that a set $A \subseteq \mathbb{Z}^d$ satisfying

$$\frac{\rho}{2} < \frac{|A \cap V_m|}{m^d} < 2\rho, \qquad \text{for all } m \in \mathbb{N}, \tag{5.17}$$

is recurrent. In order to prove this result we applay Theorem 3.4.1 and we show that $T[A] = +\infty$, where T[A] is defined in (3.21).

From (5.17) we have, for all $l \geq 1$,

$$\begin{aligned} \big|A_l\big| &\overset{(3.20)}{=} \big|A \cap \{y \in \mathbb{Z}^d;\ 2^l < |y|_\infty \leq 2^{l+1}\}\big| = \big|A \cap V_{2^{l+2}}\big| - \big|A \cap V_{2^{l+1}}\big| \\ &> \frac{\rho}{2}\big(2^{l+2}\big)^d - 2\rho\big(2^{l+1}\big)^d = \rho \cdot 2^{(l+1)d}(2^{d-1} - 2) > \rho \cdot 2^{ld}, \end{aligned} \tag{5.18}$$

Similarly, using the other bound in (5.17), we can find the following estimate

$$\big|A_l\big| < \rho 2^{(l+1)d}(2^{d+1} - 2^{-1}) \leq c \cdot 2^{ld}, \qquad \text{for all } l \geq 1. \tag{5.19}$$

Now using estimate (3.18) and (5.18) we obtain

$$\operatorname{cap}(A_l) \geq \frac{\rho \cdot 2^{ld}}{M_l}, \qquad \text{for all } l \geq 1, \tag{5.20}$$

where $M_l = \sup_{z\in A_l}\Big(\sum_{y\in A_l} G(z,y)\Big)$. Note that, using the classical bounds on the Green function (see Theorem 3.2.1), we obtain

$$M_l \le \sup_{z\in A_l}\Big(\sum_{y\in A_l}\frac{c'}{(|y-z|_\infty+1)^{d-2}}\Big), \qquad \text{for all } l\ge 1. \tag{5.21}$$

and taking, for each $l \ge 1$, a box $V_{L(l)}$ such that $|V_{L(l)}| \ge |A_l|$, we have

$$\sup_{z\in A_l}\Big(\sum_{y\in A_l}\frac{c'}{(|y-z|_\infty+1)^{d-2}}\Big) \le \sum_{y\in V_{L(l)}}\frac{c'}{(|y|_\infty+1)^{d-2}}, \qquad \text{for all } l\ge 1, \tag{5.22}$$

since, fixed an element $z \in A_l$, we can easily construct an injective correspondence Γ_l between A_l and $V_{L(l)}$ such that at each element $y \in A_l$, assign an element $\Gamma_l(y) \in V_{L(l)}$, with the property that $|y-z|_\infty \ge |\Gamma_l(y)|_\infty$.

By (5.19) there exists an odd number $L(l)$ and a constant $\tilde{c} > 0$ such that $L(l) \le \tilde{c}\cdot 2^l$ and $|V_{L(l)}| \ge |A_l|$. In particular, noting that

$$\begin{aligned}\big|\{y\in V_L;\ |y|_\infty = n\}\big| &= |V_{2n+1}| - |V_{2n-1}| \\ &= (2n+1)^d - (2n-1)^d \le c\cdot n^{d-1}, \quad \text{for all } n\ge 1,\end{aligned} \tag{5.23}$$

and using the bound (5.21) and (5.22), we obtain

$$M_l \le c' + c''\sum_{n=1}^{\tilde{c}\cdot 2^l}\frac{n^{d-1}}{(n+1)^{d-2}} \le c' + c''\sum_{n=1}^{\tilde{c}\cdot 2^l} n \le c\cdot 2^{2l}, \qquad \text{for all } l\ge 1. \tag{5.24}$$

Finally from (5.20) and (5.24) we conclude

$$\operatorname{cap}(A_l) \ge c\cdot 2^{l(d-2)}, \qquad \text{for all } l\ge 1, \tag{5.25}$$

and so

$$T[A] \ge \sum_{l=1}^{\infty} c = \infty. \tag{5.26}$$

Now from the proof of the Wiener's test (see Theorem 3.4.1) we know that, for all Λ and $0 < u < 1$, exists $N = N(u,\Lambda) > 0$ such that if a set $A \subset \mathbb{Z}^d$ satisfies $T[A] > N$, then $f_A(x) > u$, for all $x \in \Lambda$.

Finally fixing u, for all Λ, from the divergence of (5.26), we can immediately conclude that there exists $V_n(\Lambda)$ large enough such that

$$T[K] > N, \qquad \text{for all } K \in \mathcal{S}_{V_n}, \tag{5.27}$$

and so

$$f_K(x) > u, \qquad \text{for all } K\in\mathcal{S}_{V_n} \text{ and for all } x\in\Lambda. \tag{5.28}$$

In particular

$$f_{\overline{K}}(x) > u, \qquad \text{for all } K\in\mathcal{S}_{V_n} \text{ and for all } x\in\Lambda. \tag{5.29}$$

□

The following lemma gives us an important lower bound for the function $\mathbb{E}[\varphi_x|E_K]$ for all $x \in \partial K$.

Lemma 5.3.4. *For $h < \infty$ large enough there is a constant $c > 0$ such that for all V_n large enough*

$$\mathbb{E}[\varphi_x|E_K] \geq c, \qquad \text{for all } x \in \partial K, \text{ all } K \in \mathcal{S}_{V_n}. \tag{5.30}$$

Proof. This lemma is proved in [3], Lemma 3, p. 1264. We are not able to follow the last part of the proof where the Ruelle's superstability estimate is applied. □

The following lemma generalizes the previous lower bound for the function $\mathbb{E}[\varphi_x|E_K]$, for all $x \in \mathbb{Z}^d$. The proof of this lemma is based on the result stated in Lemma 1.5.2.

Lemma 5.3.5. *For $h < \infty$ large enough there exists a positive constant $c > 0$ such that for all V_n large enough*

$$\mathbb{E}[\varphi_x|E_K] \geq c \cdot f_{\overline{K}}(x) \qquad \text{for all } x \in \mathbb{Z}^d, \text{ all } K \in \mathcal{S}_{V_n}, \tag{5.31}$$

where $\overline{K} = K \cup \partial K$.

Proof. For all $K \subset\subset \mathbb{Z}^d$, applying Lemma 1.5.5 we have, for all $x \in \mathbb{Z}^d \setminus \overline{K}$,

$$\mathbb{E}[\varphi_x|\mathcal{F}_{\overline{K}}](w) = E_x[\omega_{X_{\tau_{\overline{K}}}}, \tau_{\overline{K}} < \infty], \quad \mathbb{P}\text{-a.s. }, \tag{5.32}$$

where $\mathcal{F}_{\overline{K}} = \sigma(\varphi_x;\ x \in \overline{K})$. Obviously, for all $K \in \mathcal{S}_{V_n}$, $E_K \in \mathcal{F}_{\overline{K}}$ and in particular we have

$$\mathbb{E}[\varphi_x|E_K] = \mathbb{E}[E_x[\varphi_{X_{\tau_{\overline{K}}}}, \tau_{\overline{K}} < \infty]|E_K], \qquad \text{for all } x \in \mathbb{Z}^d \setminus \overline{K}, \tag{5.33}$$

Fixed $h < \infty$ large enough, for all V_n large enough, applying Lemma 5.3.4, we obtain,

$$\begin{aligned}
\mathbb{E}[\varphi_x|E_K] &= \mathbb{E}[E_x[\varphi_{X_{\tau_{\overline{K}}}}, \tau_{\overline{K}} < \infty]|E_K] \\
&= \mathbb{E}[\sum_{k \in \partial K} P_x(X_{\tau_{\overline{K}}} = k, \tau_{\overline{K}} < \infty)\varphi_k|E_K] \\
&= \sum_{k \in \partial K} \mathbb{E}[\varphi_k|E_K] P_x(X_{\tau_{\overline{K}}} = k, \tau_{\overline{K}} < \infty) \\
&\geq c \cdot f_{\overline{K}}(x), \qquad \text{for all } x \in \mathbb{Z}^d \setminus \overline{K},
\end{aligned} \tag{5.34}$$

and in particular, we can conclude that for all V_n large enough,

$$\mathbb{E}[\varphi_x|E_K] \geq c \cdot f_{\overline{K}}(x) \qquad \text{for all } x \in \mathbb{Z}^d, \text{ all } K \in \mathcal{S}_{V_n}. \tag{5.35}$$

□

5.4 The conclusion of the proof

We are now ready to conclude the proof.

Proof. Lemma 5.3.5 says that for $\bar{h} < \infty$ large enough there exists a positive constant $c > 0$ such that for all V_n large enough

$$\mathbb{E}[\varphi_x | E_K] \geq c \cdot f_{\overline{K}}(x) \qquad \text{for all } x \in \mathbb{Z}^d, \text{ all } K \in \mathcal{S}_{V_n}. \tag{5.36}$$

Combining Lemma 5.3.3 with (5.36): there is a costant $\tilde{u} > 0$ such that for all Λ, we can choose $V_n = V_n(\Lambda)$ large enough such that

$$\frac{1}{|\Lambda|} \sum_{x \in \Lambda} \mathbb{E}[\varphi_x | E_K] \geq \tilde{u}, \qquad \text{for all } K \in \mathcal{S}_{V_n}. \tag{5.37}$$

By Lemma 5.3.2, denoting $S_\Lambda := \sum_{x \in \Lambda} \varphi_x$ and $U_n := \bigsqcup_{K \in \mathcal{S}_{V_n}} E_K$, we have

$$\mathbb{E}\big[(S_\Lambda)^2\big] \geq \mathbb{E}\big[(S_\Lambda)^2 \mathbb{1}_{U_n}\big] = \sum_{K \in \mathcal{S}_{V_n}} \mathbb{E}\big[(S_\Lambda)^2 \mathbb{1}_{E_K}\big] = \sum_{K \in \mathcal{S}_{V_n}} \mathbb{E}\big[(S_\Lambda)^2 | E_K\big] \mathbb{P}[E_K] \tag{5.38}$$

and by the Cauchy-Schwartz inequality

$$\geq \sum_{K \in \mathcal{S}_{V_n}} \mathbb{E}\big[S_\Lambda | E_K\big]^2 \mathbb{P}[E_K] \geq \sum_{K \in \mathcal{S}_{V_n}} \tilde{u}^2 |\Lambda|^2 \mathbb{P}[E_K], \tag{5.39}$$

where we used (5.37) for the last inequality. Now, by Lemma 5.3.2 again, in particular using that $C_{V_n} \cap \Omega_\delta \subseteq \bigsqcup_{K \in \mathcal{S}_{V_n}} E_K^h$ and $\mathbb{P}(C_{V_n} \cap \Omega_\delta) \geq \mathbb{P}\big(\{0 \overset{\geq h}{\longleftrightarrow} \infty\} \cap \Omega_\delta\big)$, we obtain

$$\geq \tilde{u}^2 |\Lambda|^2 \mathbb{P}(C_{V_n} \cap \Omega_\delta) \geq \tilde{u}^2 |\Lambda|^2 \mathbb{P}\big(\{0 \overset{\geq h}{\longleftrightarrow} \infty\} \cap \Omega_\delta\big). \tag{5.40}$$

Since this chain of inequalities holds for all Λ and $\mathbb{E}\Big[\big((1/|\Lambda|)S_\Lambda\big)^2\Big] \to 0$, for $\Lambda \uparrow \mathbb{Z}^d$, we obtain that $\mathbb{P}\big(\{0 \overset{\geq h}{\longleftrightarrow} \infty\} \cap \Omega_\delta\big) = 0$. Finally taking the limit for $\delta \to 0$, we get $\mathbb{P}\big(\{0 \overset{\geq h}{\longleftrightarrow} \infty\}\big) = 0$, for all $h > \bar{h}$.

This completes the proof. $\square$

Chapter 6

Open problems

We summarize in this final chapter the two open problems that appear in our new proof. We tried to be as independent as possible from the previous chapter with the notation in order to be accessible to lay readers.

Let $(\varphi_x)_{x\in\mathbb{Z}^d}$ be a Discrete Gaussian Free Field on $\mathbb{Z}^d$ and V_n be a box of size $n \in \mathbb{N}$ centered in the origin 0.

Problem 1. We claim that the connected component containing 0 in the cluster $C(0)(\varphi)$ inside V_n, denoted by $\mathcal{C}\big(C(0)(\varphi), V_n\big)$, satisfies for some $\rho > 0$,

$$\frac{\rho}{2} < \frac{|\mathcal{C}\big(C(0)(\varphi), V_n\big)|}{n^d} < 2\rho,$$

for all configurations φ contained in an arbitrary large subset of the configurations such that the origin 0 lies in the infinite cluster over level h. See the claim below equation (5.13) for a more precise statement.

We belive that using an "ergodic argument" the result could be prove.

Problem 2. This is the main problem in our proof. We would like to use a result stated in [3, Lemma 3, p. 1264] but we are not able to follow the last part of the proof where the Ruelle's superstability estimate is applied. The result is intuitive but it seems to be extremely technical to prove. We give a statement: for a box V_n we define

$$\mathcal{S}_{V_n} = \Big\{K \subseteq V_n : 0 \in K,\ K \text{ is connected},\ K \cap \partial^i V_n \neq \emptyset\Big\}.$$

Lemma. *For $h < \infty$ large enough there is a constant $c > 0$ such that for all boxes V_n large enough*

$$\mathbb{E}\Big[\varphi_x\Big|\varphi_x \geq h, \forall x \in K \text{ and } \varphi_x < h, \forall x \in \partial_{V_n} K\Big] \geq c, \qquad \text{for all } x \in \partial K, \text{ all } K \in \mathcal{S}_{V_n},$$

where $\partial_{V_n} K = \partial K \cap V_n$.

Intuitively the result states that the field on the external boundary of a cluster over level h is in mean strictly positive if h is big enough.

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