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"PRICING AND HEDGING OF A PORTFOLIO OF OPTIONS IN THE PRESENCE OF STOCHASTIC VOLATILITY"

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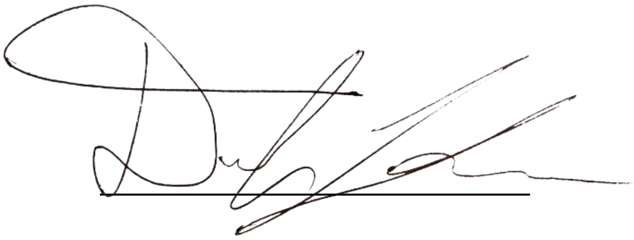

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Abstract

Financial markets have become an important component of people's life. All sorts of media now provide us with a daily coverage on financial news from all markets around the world. At the same time, not only large institutions but also more and more small private investors are taking an active part in financial trading. In particular, the internet arrival has led to an unprecedented increase in small investors direct trading. Needless to say, the rapid expansion of financial markets calls upon products and systems designed to help investors to manage their financial risks. Banks have developed a lot of strategies to control risks induced by market fluctuations. Mathematics and statistics have emerged as the leading disciplines to address fundamental questions in finance as asset pricing model and hedging strategies. In this thesis some hedging strategies in different frameworks would be analyzed, indeed the author will assume the position of an issuer of options, i.e. the position of a bank. The bank is assumed to have the objective to reduce its risk as much as possible, that is, the bank is assumed to make money primarily on the price the bank charge for the transaction, and not on the return of the position it takes on the market when dealing with its clients. Thus the bank wants to hedge the short positions in the contracts. To this end, as said, different dynamic hedging strategies will be considered[1]. In this thesis, the hedging strategies are performed on a weekly basis; a set of Call options and with different strikes and maturities on the S&P 500 (SPX) index would be used to form an initial portfolio. The Hedging strategies are performed throughout 5 months, from 30 January 2015 to 16 June 2015. With this work the author especially aim to answer the following questions:

"Which one out of the dynamic-hedging strategies is most efficient in reducing the risk of an options Portfolio on the S&P 500 Index? How things change by considering realistic features such as stochastic volatility, leverage effect, volatility clustering, volatility smiles and non-normal distribution of returns? How much is reduced, in terms of error, the pricing and hedging considering more advanced features? Does the higher complexity "pays" in terms of reducing the errors?"

The thesis has mainly two parts, in the first one the hedging strategies are implemented in a Black&Scholes framework, in the second part some of the Black&Scholes assumption are relaxed and the hedging strategies are performed in a widely used stochastic volatility model:

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[1] A Delta-hedging strategy, a Delta-Gamma hedging strategy and a strategy with Delta-Gamma-Vega will be considered in this work.

the Heston model. In the second part, some issues appear: that of the accuracy of the Black&Scholes predicted changes in price, which is based on assumptions such as Ito Process, frictionless markets, constant volatility and distributional assumption. However in reality observed prices from financial markets are not normal distributed, trading in real markets is not done continuously, or even highly frequently, and trading of instruments is not cost free. All these observations are in contrast to the standard assumptions of pricing and hedging methods in classical option theories such as the famous Black&Scholes pricing formula, for that reason a stochastic volatility models is used. All contracts considered in the thesis are Vanilla option of European[2] style. For a general understanding, the basic objects, ideas and results of the classical Black-Scholes framework will be presented in Chapter 2, and in the results, the BS model will be used as benchmark. Moreover the author will also use an extension of the Black&Scholes model allowing volatility to vary by means of a rolling method, considered as "hybrid-solution" between stochastic volatility models and fixed volatility models. As introduction for Chapter 3, the attention will be drawn to an historical event, the 1987 crash, which will lead to the assumption of stochastic volatility. In Chapter 3 the Heston model will be discussed. Furthermore, the semi-closed form solution for European options will be presented and the Heston parameters will be analyzed in greater details. The understanding of the parameter influence is important for accurate calibration of the model that in this work is performed using both a global stochastic optimizer and a local gradient optimizer in order to increase the trade-off precision/computational time. Results and the methodology used for the empirical application are presented in Chapter 4. In particular for each framework, statistical and economic performances in terms of in-sample and out-of- sample pricing errors, P/L, of each strategy are calculated. Finally, the conclusions are given in Chapter 5, together with suggestions of further extension of the subjects. All the calculations were performed using the MATLAB® software R.2014a for Windows. The appendix contains most of the MATLAB® Codes used for the calculations throughout the thesis.

" This thesis will proceed by first investigating and building a simple option pricing and hedging framework under Black&Scholes assumptions. Later those assumptions will be released and more realistic market conditions will be used."

[2] A European option can only be exercised on the last day. An American option can be exercised any time between its inception and the end date. A hybrid, the Bermudian Option can be exercised on a set number of days between inception and expiration.

Chapter 1

1.1 What is Hedging?

"Hedging is a technique not aimed in making money but in reducing potential losses."

When financial institutions decide to implement hedging strategies, they are protecting themselves against a negative event. This doesn't prevent a negative event from happening, but if it happens and the hedge is done properly, the impact of the event, in term of losses, is reduced. In a practical way hedging means strategically using instruments in the market to offset the risk of any adverse price movements; in other words banks hedge one investment by making another. For the buyers[3], hedging techniques generally involve the use of financial instruments known as derivatives, the two most common of which are options and futures. With these instruments you can develop trading strategies where a loss in one investment is offset by a gain in a derivative. On the other hand for the sellers of options i.e. banks, hedging techniques usually involve the use of cash[4] or in some cases others derivatives.

1.2 Why Banks use hedging?

"Companies attempt to hedge price changes because those fluctuations are marginal risks to the central business in which they operate."

Options constitute a substantial part of the Global equity derivatives market, which has an estimated value of $630 trillion[5]. The utter size of the market is an incentive for considering the possibilities of hedging options, i.e. the issuers of the option-contracts needs to protect their position against excessive risk. But why banks have these large portfolios of short options? From which are they generated? Banks have become provider of options because there is such a demand of options asked from clients that use options in the most exotic ways. "Why there is such a demand of options?" The increasing demand in options in past years is the engine of a boost in offering and as a consequences the creation of huge portfolios of short options that have to be managed by banks. Factors that push

[3] Buyers of options.
[4] Underlying asset.
[5] Value of December 2014. Source: Bank of international Settlements, Statistic release, April 2015.

buyers of options to increase their demand are the classic advantages that option trading offers such as:

- Leverage:

Options give the buyer the right to buy a number of shares of the underlying instrument from the option seller. The amount of shares (or futures contracts) to buy is determined by the number of option contracts, multiplied by the contract multiplier. The contract multiplier[6] is different for most classes of options and is determined by each exchange. In the US, the contract size for options on shares is 100. This means that every 1 option contract gives buyer the right to buy 100 shares from the option seller. In this thesis, since the author will consider the S&P 500 index, the contract size would be 100.

Example 1.1 *if you buy 10 IBM option contracts, it means that you have the right to buy 1,000 IBM shares at expiration if the price is right.*

This also means that the price of the option is also multiplied by the contract multiplier.

> ***Example 1.2*** *say in the above you purchased 10 options contracts that were quoted in the marketplace for 15c, then you would actually pay the seller $150. If you go out and buy 5 IBM share options for 15c that have a Strike Price of $25, then you will:*
>
> - *Pay the option seller $75*
> - *If you decide to exercise your right and buy the shares, you will have to buy 500 (5 x 100) (100 being the contract size) shares at the exercise price of $25, which will cost you $12,500.*
>
> *In this case, your initial investment of $75 has given you $12,500 exposure in the underlying security. Option trading is very attractive for the small investor as it gives him the opportunity to trade a very large exposure whilst only outlaying a small amount of capital.*

- Insurance:

Another reason investors may use options is for portfolio insurance. Option contracts can give the risk averse investor a method to protect his downside risk in the event of a stock market crash.

[6] Also called "contract size".

- Limited risk and unlimited profit potential

Since options have an asymmetric payoff, buyers have a potential unlimited profit potential and a limited loss, indeed in case the option is not exercised the only loss for the buyer, whatever is the price of the underlying, is only the premium paid to the seller (the option price).

From the other side the sellers, as consequences, have a potential unlimited loss and a limited profit; for this reason sellers after sold the option they can't just sit down and wait the maturity and hope that the option will close out-the-money[7], they need to offset a potential infinite risk. In order to offset this market risk, sellers use Hedging strategies.

1.3 Different Hedging strategies

In order to hedge a position, the banks could use different techniques. The choice could depend on the nature of the derivative used, or coming from a personal strategy.

1.3.1 Naked position

One strategy open to the financial institution is to do nothing. This is referred as naked position. It is a good strategy if the option at the maturity closes out-of-money. A naked position could lead to a significant loss if the option is exercised.

1.3.2 Covered Position

As an alternative to a naked position the financial institution can adopt a covered position. This involves to buy the quantity of shares considered in the contract. (Considering a Call option). In this case the strategy work well if the option is exercised but could lead to a significant loss if the option is not exercised.

1.3.3 Stop loss strategy

[7] An option is said in-the-money if it is exercisable, at-the-money if the strike is equal to the underlying and out-of-money if is not exercisable.

One interesting hedging procedure that is sometimes proposed involves a stop-loss strategy. To illustrate the basic idea, consider an institution that has written a call option with strike price K to buy one unit of a stock. The hedging procedure involves buying one unit of the stock as soon as its price rises above K and selling it as soon as its price is less than K.
The objective is to hold a naked position whenever the stock is less than K and a covered position whenever the stock price is greater than K. The procedure is designed to ensure that at time T (time to maturity) the bank owns the stock if the option closes in the money and does not old own the stock if the option closes out of money. This strategy would work perfectly in absence of transaction costs and bid ask spread.

1.3.4 Dynamic and static Hedging

Most traders use more sophisticated hedging schemes than those mentioned so far.Dynamic Hedging involves adjusting a hedge as the underlying moves often several times a day. But why do we have dynamic and static hedging?" Most of the answer depends on the nature of the derivative, indeed traded instruments or positions can generally be broken down into two types:

- linear
- non-linear[8]

The former includes spot positions, forward positions and futures. Their payoffs or market values are either linear or almost linear functions of their underlying. Non-linear instruments include vanilla options, exotic derivatives and bonds with embedded options. Their payoffs or market values are non-linear functions of their underlying asset.

[8] A nonlinear derivative with respect to a parameter is one that presents a second derivative (or partial derivative with respect to that parameter) different from 0.

Figure 1.a, 1.b, 1.c and 1.d illustrate linear and nonlinear derivative[9] .Source: Nassim Taleb, Dynamic Hedging, 1997, Wiley.

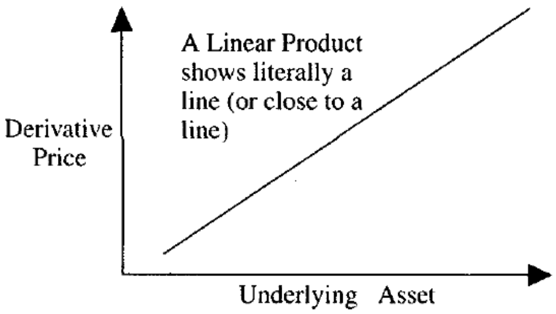


Fig. 1.a Linearity.

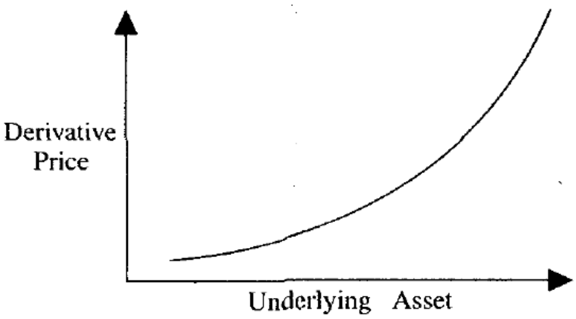


Fig. 1.b nonlinear derivative: Convex security.

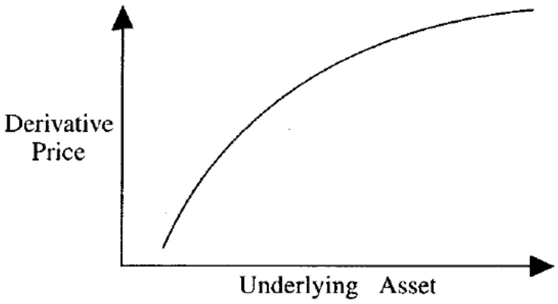


Fig. 1.c nonlinear derivative: Concave security.

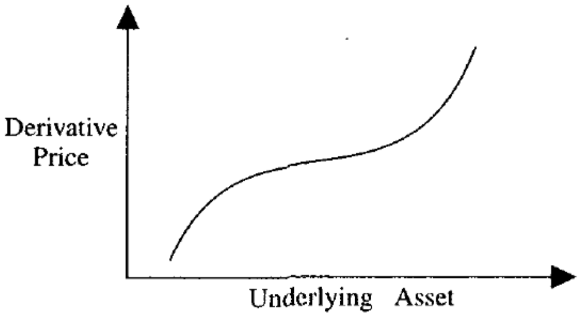


Fig. 1.d mixed nonlinear derivative.

"The best way to look at derivatives is to separate them into two broad categories: linear and nonlinear derivatives. A linear derivative is easy to hedge and lock in completely, whereas a nonlinear one will present serious instability and require the use of the dynamic hedging."

Source: Nassim Taleb, Dynamic Hedging, 1997

Banks transact in both instruments with clients. They prefer to sell non-linear instruments, such as options, because these are more difficult for clients to evaluate, which means they can make larger earnings.

After selling to multiple clients, banks are left holding large short positions. To hedge those positions, they would like to purchase offsetting long options, it makes little sense to buy them from other derivatives dealers or financial institutions, who are in the same boat with their own large short options positions. The solution is to dynamically hedge the short options positions.

[9] In the appendix there is a test in order to recognize if a derivative is linear or nonlinear.

The general idea of dynamic hedging is to construct a portfolio consisting of the instrument that is to be hedged (i.e. short options), and the hedge (that could be the spot but also a combination of the same options), that is locally neutral with respect to one or several measures[10]. As measure, traders normally use the so-called Greeks. The Greeks measures different dimension to the risk in an option position and the aim of a trader is to manage the Greeks so that all risks are acceptable. The Greeks are Delta, Gamma, Rho, Theta and Vega[11]; we could have different strategies coming from different combination of the Greeks. In this thesis a delta-neutral, delta-gamma-neutral, and delta-gamma-vega-neutral strategy for each model would be considered. As consequence this lead to the evaluation of 9 strategies, since the author will consider the pure Black&Scholes model, the extension of the Black&Scholes model and the Heston model. The difference between the Black&Scholes and the Extended Black&Scholes is the calculation of the volatility, this would be better explained in Chapter 2, section 2.5. The strategies for the Dynamic Hedging are:

> "A Static hedge is a position that is taken when the contract is purchased, and which is maintained throughout the lifetime of the contract. On the contrary dynamic hedging requires dynamics adjustments."

- Black&Scholes model (Fixed volatility):
    - Delta Hedging
    - Delta-Gamma Hedging
    - Delta-Gamma-Vega Hedging
- Extended Black&Scholes (Rolling window volatility):
    - Delta Hedging
    - Delta-Gamma Hedging
    - Delta-Gamma-Vega Hedging
- Heston model (Stochastic volatility):
    - Delta Hedging
    - Delta-Gamma Hedging
    - Delta-Gamma-Vega Hedging

[10] Locally neutral because we are talking about nonlinear function, so we need measures that linearize locally the function, for this the Greeks require the computation of derivatives.
[11] Vega is considered a Greek letter also if is not into the Greek Alphabet.

Chapter 2 | Black&Scholes Framework

Introduction

In the Black&Scholes framework we assume that the bank has a liquid market which is free of arbitrage, where the Stock, a risk free asset are traded, at its disposal. The bank may thus use any of these assets to hedge the options.

A clear distinction is made between those contracts that are traded on the market, and those contracts that are not. However, in order for the bank to be able to hedge the options dynamically, it needs to calculate the movements of the price of the options throughout its lifetime, obtain a close form solution for the price, and then obtaining the Greeks; for this we need a whole framework where the first piece is to introduce the basis of option pricing, that are explained in subsection 2.4. In this part of the thesis the price of the options will be calculated using risk-neutral valuation. The motivation of using risk-neutral evaluation are explained in the Focus "Why using risk neutral evaluation" at the end of section 2.4. In subsection 2.1 the Vanilla options are introduced. Subsection 2.3 states how the assets that are traded in the market are priced. The Greeks are introduce in subsection 2.6. Limitations of this model are discussed in Section 2.8 as link for the introduction of stochastic volatility models discussed in Chapter 3.

2.1 Vanilla options[12]

As said in the introduction, the contracts considered in this thesis are Vanilla options. A vanilla option is a normal call or put option that has standardized terms and no special or unusual features. It is generally traded on an exchange, that is, the contract function Φ only depends on the price of the Stock at the day of expiration that is:

$$\Phi = \Phi(S(T))$$

Which are denoted respectively by S and T.

---

[12] Plain vanilla is an adjective describing the simplest version of something, without any optional extras, basic or ordinary. In analogy with the default ice cream flavor vanilla, which became widely and cheaply available with the development of artificial vanillin flavor. Some financial instruments like put options or call options are often described as plain vanilla options. The opposite of plain vanilla options are exotic options.

2.1.1 Call option

The holder of a Call option written on the Stock S (t), with the strike price K and the expiration date T, has an option[13] to buy the Stock on the day of expiration to the fixed price K. The contract function Φ(S (T)) of a Call option is equal to:

$$Max[S(T) - K, 0]^{+}$$

The terminal payoff of a long Call option with a strike price *x* can be found in figure 2a.

2.1.2 Put option

The holder of a Put option written on the Stock *S* (*t*), with the strike price *K* and the expiration date *T*, has an option to sell the Stock on the day of expiration to the fixed price *K*. The contract function Φ(*S* (*T*)) of a Put option is equal to:

$$Max[K - S(T), 0]^{+}$$

In figure 2b, the terminal payoff of a long Put option with a strike price *x* is on display. In this thesis we will consider a portfolio composed by Calls and Puts.

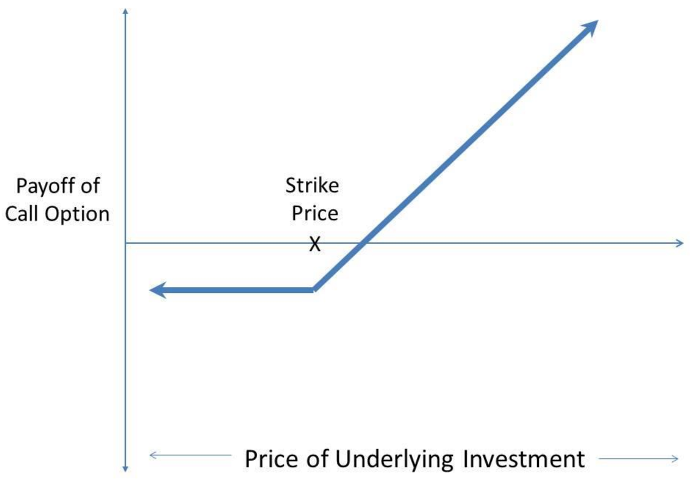


Figure 2.a: Payoff of a long call option strike x

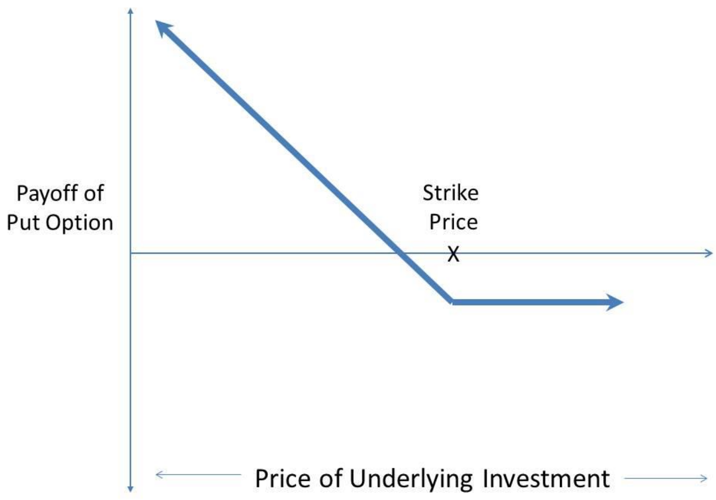


Figure 2.b: Payoff of a long put option strike x

[13] Not the obligation.

2.2 Moneyness of an Option

Moneyness of an option indicates whether an option is worth exercising or not. Moneyness of an option at any given time depends on where the price of the underlying asset is at that point of time relative to the strike price. The following three terms are used to define the moneyness of an option. An option is ITM (in-the-money) if on exercising the option, it would produce a cash inflow for the buyer.

Thus, call options are ITM at time t when the value of the price of the underlying exceeds the strike price, $S_t > K$. On the other hand, put options are ITM when the price of the underlying is lower than the strike price, $S_t < K$. An OTM (out-of-the-money) option is an opposite of an ITM option. A holder will not exercise the option when it is OTM. A call option is OTM when its strike price is greater than the price of the underlying and a put option is OTM when the price of the underlying is greater than the option's strike price. An ATM (at-the-money) is one in which the price of the underlying is equal to the strike price. It is at the stage where with any movement in the price of the underlying, the option will either become ITM or OTM. The moneyness for call and put options is defined by:

$$M_t = \frac{S_t}{K}$$

Moneyness	Call	Put
< 0.91	Deep OTM	Deep ITM
0.91 – 0.97	OTM	ITM
0.97 – 1.00	ATM^-	ATM^-
1.00 – 1.03	ATM^+	ATM^+
1.03 – 1.09	ITM	OTM
>1.09	Deep ITM	Deep OTM

Table 1: Moneyness of Call and Put options.

In this work, the author will consider a portfolio composed with different initial moneyness of options. Since the initial Underlying price is the same, it is sufficient to say that the author will consider a portfolio with different strikes.

2.3 The prices of the contracts that are traded in the market

The prices for the assets that are traded in the marked are given in this section.

2.3.1 Risk free asset

The price process of the risk free asset $B(t)$ is given by:

$$dB(t) = rB(t)dt \tag{1}$$

Which is shorthand for:

$$B(t) = \int_0^t rB(s)ds \tag{2}$$

Solving this equation gives:

$$B(t) = B(0)e^{rT} \tag{3}$$

2.3.2 Stock

The price process of the Stock $S(t)$ is assumed to follow the Geometric Brownian Motion:

$$dS_t = rS_t dt + \sigma S_t dW_t^Q \tag{4}$$

Where the drift r and the volatility σ are constants, and where $W(t)$ is a Wiener process, which has the following properties:

- $W_0 = 0$.
- The process $W(t)$ has independent increments, if $r < s \leq t < u$ then $W_u - W_t$ and $W_s - W_r$ are independent stochastic variables.
- For $s < t$ the stochastic variable $W_t - W_s$ has the Gaussian distribution $N(0, \sqrt{t-s})$
- W Has continuous trajectories.

Without loss of generality, the drift of the Stock has been chosen to be equal to the risk free interest rate r; solving the stochastic differential equation (4) using Itô's Lemma gives:

$$S_t = S_0 e^{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t^Q)} \tag{5}$$

2.4 Pricing the derivatives: Black & Scholes formula derivation

The prices of the options are calculated using risk-neutral valuation. For a European Call option, the potential cash-flows at time t as shown in figure 2.a. in subsection 2.1.1 occur if $S_t > K$

- Receive a stock worth S_t with probability $\Pr(S_t > K)$
- Pay K with probability $\Pr(S_t > K)$

So, the expected value of a European call option is:

$$E(Call\ payoff) = Pr(S_t > K)\ [E(S_t|S_t > K) - K]$$

In order to find the price we have to discount back the expected value following the classical financial assumption that the price of an option is the discounted expected value of the payoff.

$$PV_0[E(Call\ payoff)] = e^{-rT} Pr(S_t > K)\ [E(S_t|S_t > K) - K]$$

For a European put option, the potential cash-flows at time t with occur if $S_t < K$, so

- Receive K with probability $\Pr(S_t < K)$
- Pay (buy a stock worth) S_t with probability $\Pr(S_t < K)$

$$E(Put\ payoff) = \Pr(S_t < \mathrm{K})\ [\mathrm{K} - \mathrm{E}(S_t|S_t < \mathrm{K})]$$

$$PV_0[(Put\ payoff)] = e^{-rT} \Pr(S_t < \mathrm{K})\ [\mathrm{K} - \mathrm{E}(S_t|S_t < \mathrm{K})$$

In order to develop the Black-Scholes formula, we need to know the following quantities:

- $\Pr(S_t > K)$
- $E(S_t|S_t > K)$

Applying Ito's Lemma to the function LnS_t where S_t is driven by the diffusion in equation (4). Then LnS_t follows the SDE:

$$d\ lnS_t = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t$$

Where $\mu = r$ Integrating from 0 to t, we have

$$\int_0^t d\ lnS_u = \int_0^t \left(\mu - \frac{1}{2}\sigma^2\right) du + \int_0^t dW_u$$

So that

$$\ln S_t - \ln S_0 = \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t$$

Since we assume that $W_0 = 0$. Hence the solution to the SDE is equation (5):

$$S_t = S_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t)}$$

Let A be normally distributed random variable for the stock return:

$$S_t = S_0 e^{At} \; where \; A \sim N(\mu, \sigma^2)$$

It follows that:

$$\frac{S_t}{S_0} \sim LN(m = \left(\mu - \frac{1}{2}\sigma^2\right)t, v = \sigma\sqrt{t})$$

These parameters are chosen such that $\mathrm{E}(\frac{\mathrm{St}}{\mathrm{S0}}) = e^{\mu t}$ where μ is the capital gains rate. We can see that this is true since $\mathrm{E}\left(\frac{\mathrm{St}}{\mathrm{S0}}\right) = e^{m + \frac{1}{2}v^2} = e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \frac{1}{2}\sigma^2 t} = e^{\mu t}$

For $t = 1$ the volatility of the stock return equals to the volatility of the $ln\left(\frac{\mathrm{St}}{\mathrm{S0}}\right)$. Otherwise, the volatility of $ln\left(\frac{\mathrm{St}}{\mathrm{S0}}\right)$ must be adjusted for time so $v = \sigma\sqrt{t})$

$$\Pr(S_t < \mathrm{K}) = \Pr\left(\frac{S_t}{S_0} < \frac{K}{S_0}\right)$$

$$= \Pr\left(ln\frac{S_t}{S_0}\right) < \left(ln\frac{K}{S_0}\right)$$

Since $ln\frac{S_t}{S_0} \sim N(m, v^2)$ then $\frac{ln\frac{S_t}{S_0} - m}{v} = Z \sim N(0,1)$ where Z is the standard normal random variable. Therefore:

$$\Pr(S_t < \mathrm{K}) = \Pr\left(Z < \frac{\ln(\frac{K}{S_0}) - m)}{v}\right)$$

$$= \Pr(Z < -d_2)$$

$$= N(-d_2)$$

Where $d_2 = \frac{\ln\left(\frac{S_0}{K}\right)+m}{v} = \frac{\ln\left(\frac{S_0}{K}\right)+\left(\mu-\frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}}$ (6)

Since $\Pr(S_t < \mathrm{K}) = N(-d_2)$ then

$$\Pr(S_t > \mathrm{K}) = N(d_2)$$

We found the first piece, now in order to complete the equation, as said above we need to find $E(S_t|S_t > K)$

To find $E(S_t|S_t > K)$ we use the following formula:

$$E(S_t|S_t < K) = \frac{PE(S_t|S_t < K)}{\Pr(S_t < K)}$$

Where PE is the partial expectation from $S_t = 0$ to $S_t = K$. Note that:

$$PE\left(\frac{S_t}{S_0}\middle|\frac{S_t}{S_0} < \frac{K}{S_0}\right) = E\left(\frac{S_t}{S_0}\right)N\left(\frac{\ln\left(\frac{S_0}{K}\right) - m - v^2}{v}\right)$$

We can calculate $PE\left(S_t\middle|\frac{S_t}{S_0} < \frac{K}{S_0}\right) = S_0\left(PE\left(\frac{S_t}{S_0}\middle|\frac{S_t}{S_0} < \frac{K}{S_0}\right)\right)$.

This simplifies as follows:

$$PE(S_t|S_t < K) = PE\left(S_t\middle|\frac{S_t}{S_0} < \frac{K}{S_0}\right)$$

$$= S_0\left(PE\left(\frac{S_t}{S_0}\middle|\frac{S_t}{S_0} < \frac{K}{S_0}\right)\right)$$

$$= S_0\,E\left(\frac{S_t}{S_0}\right)N\left(\frac{\ln\left(\frac{S_0}{K}\right) - m - v^2}{v}\right)$$

$$= S_0 e^{m+\frac{1}{2}v^2} N(\frac{\ln\left(\frac{S_0}{K}\right) - \left(\mu - \frac{1}{2}\sigma^2\right)t - \sigma^2 t}{\sigma\sqrt{t}}$$

$$= S_0 e^{\mu t} N(\frac{\ln\left(\frac{S_0}{K}\right) - \left(\mu + \frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}}$$

$$= S_0 e^{\mu t} N(-d_1)$$

Where $d_1 = \frac{\ln\left(\frac{S_0}{K}\right)+\left(\mu+\frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}}$. Notice that $d_2 = d_1 - \sigma\sqrt{t}$ (6 bis)

Since $E(S_t) = PE(S_t|S_t > K) + PE(S_t|S_t < K)$ then:

$$PE(S_t|S_t > K) = E(S_t) - PE(S_t|S_t < K)$$

$$= S_0 e^{\mu t} - S_0 e^{\mu t} N(-d_1)$$

$$= S_0 e^{\mu t}(1 - N(-d_1))$$

$$= S_0 e^{\mu t} N(d_1)$$

Which leads to the following formulas:

$$E(S_t|S_t < K) = \frac{S_0 e^{\mu t} N(-d_1)}{N(-d_2)}$$

$$E(S_t|S_t > K) = \frac{S_0 e^{\mu t} N(d_1)}{N(d_2)}$$

Substituting in the formulas derived above, we find that for a European call option

$$E(Call\ payoff) = Pr(S_t > K)\ [E(S_t|S_t > K) - K]$$

$$= N(d_2)\left(\frac{S_0 e^{\mu t} N(d_1)}{N(d_2)} - K\right)$$

$$= S_0 e^{\mu t} N(d_1) - KN(d_2)$$

$$PV_0[E(Call\ payoff)] = e^{-rT}[(S_0 e^{\mu t} N(d_1) - KN(d_2)]$$

By evaluating under a risk neutral methodology, so by putting $\mu = r$ we have:

$$e^{-rT}[(S_0 e^{rt} N(d_1) - KN(d_2)]$$

That is equal to the Black-Scholes Formula for a European Call option:

$$= S_0 N(d_1) - e^{-rT} KN(d_2)] \tag{7}$$

Similarly, substituting in the formulas derived above, we find that, for a European Put option:

$$E(Put\ payoff) = \Pr(S_t < \mathrm{K})\ [\mathrm{K} - \mathrm{E}(S_t|S_t < \mathrm{K})]$$

$$= N(-d_2)\left(K - \frac{S_0 e^{\mu t} N(-d_1)}{N(-d_2)}\right)$$

$$= KN(-d_2) - S_0 e^{\mu t} N(-d_1)$$

$$PV_0[(Put\ payoff)] = e^{-rT}[KN(-d_2) - S_0 e^{\mu t} N(-d_1)]$$

By evaluating under a risk neutral methodology, so by putting $\mu = r$ we have:

$$e^{-rT}[KN(-d_2) - S_0 e^{rt} N(-d_1)]$$

That is equal to the Black-Scholes formula for a European Put option:

$$= e^{-rT} KN(-d_2) - S_0\, N(-d_1)] \qquad (8)$$

There are several ways to derive such a formula, we could also have used a replicating strategy argument to derive the formula, by using the replicating strategy argument in continuous time to derive the Black-Scholes partial differential equation. Another common method is the Martingale pricing, by using the Feynman-Kac formula. Martingale pricing theory states that deflated security prices are martingales. Then Ito's Lemma and Girsanov's Theorem imply:

$$\begin{aligned} dY_t &= (\mu - r)Y_t dt + \sigma Y_t dW_t \\ &= (\mu - r)Y_t dt + \sigma Y_t (dW_t^Q - \eta_t dt) \\ &= (\mu - r - \sigma\eta_t)Y_t dt + \sigma Y_t dW_t^Q \end{aligned}$$

Where Q denotes a new probability measure and W_t^Q is a Q-Brownian motion. But we know from martingale pricing that if Q is an equivalent martingale measure then it must be the case that Y_t is a martingale. This then implies that $\eta_t = (\mu - r)/\sigma$ for all t. It also implies that the dynamics of S_t satisfy

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dW_t \\ &= r S_t dt + \sigma S_t dW_t^Q \end{aligned} \qquad (9)$$

Using the new stochastic process for the stock price, we can derive the call price using martingale pricing. In particular, we have

$$Option\ price = E_t^Q[e^{r(T-t)} Payoff(T)] \qquad (10)$$

Where the price of the Call option is:

$$C = E_t^Q[e^{r(T-t)}(S_t - K)] \qquad (11)$$

While the calculations are a little tedious, it is straightforward to solve (11) and obtain (7) as the solution, as before.

Focus: Why using Risk neutral evaluation?

Risk-neutral valuation is a valuable tool for the analysis of derivatives and is commonly used when pricing options. According to Hull, in a risk-neutral world all investors are indifferent to risk and only require the risk-free rate to invest. Investors require no compensation for risk and therefore the expected return can be substituted with the known risk free-rate. By using this principle, all the variables that enter models are without risk preference and the pricing of options becomes easier. This general principle in option pricing is known as *risk-neutral valuation.*

The fascinating and surprising fact is that the solution is valid in a world with risk-averse investors as well. The argument is in a risk-averse world the expected growth rate in the stock price changes and the discount rate that must be used for any payoffs from the derivative changes. These two effects offset each other perfectly and the price is valid in the risk-neutral world as well as in the real world. According to Hull risk-free valuation is correct when the risk free rate is constant.

2.4.1 Call option

The price of the European Call option $C\,(t, s)$ is calculated using Black-Scholes formula, which can is derived in section above, recalling from section (2.4):

$$C = S_0 N(d_1) - e^{-rT} K N(d_2) \tag{7}$$

Where N is the cumulative distribution function for the Gaussian distribution with zero mean and unit variance, and (d_1) and (d_2) defined in equations (6) and (6 bis):
In order to pricing a Call option, the inputs required are underlying stock, strike price, risk free, volatility, time to maturity.[14]

[14] The expected return of the underlying is irrelevant to the pricing of and option.

Example 2.1 *imagine that a financial institution has sold for 300.000€ a European Call Option on 100.000 shares of a non-dividend paying stock. We assume that the stock price is 49, the strike price is 50, the risk free interest rate is 5% per annum, the stock price volatility is 20% per annum, the time to maturity is 20 weeks (0,3846 years). With our notation this means that:*

$S_0 = 49$ €	$K = 50$ €	$r = 0, 05$	$\sigma = 0, 2$	$T = 0, 3846$

The B&S price of the option is about 240.000.

2.4.2 Put option

The price of the European Put option P (t, S) is calculated using Black-Scholes formula[15], recalling equation (8) from section 2.4:

$$P = e^{-rT} K N(-d_2) - S_0\, N(-d_1)] \qquad (8)$$

Where (d_1) and (d_2) is the same as above in equation (6) and (6 bis).

In MATLAB®, the price of a Call and a Put option is calculated by the function "`blsprice`" providing the 5 input discussed above. In MATLAB®, the function appear as:

```
[Call, Put] = blsprice (Price, Strike, Rate, Time, Volatility,[])
```

2.5 Volatility

As said in section 2.4.1, for practical application of the Black-Scholes Theory, one needs to have numerical estimates of all the input parameters. The input data consists of the string S, K, r, T and σ. Out of these five parameters, S, r, T and K can be observed directly, which leaves the problem of obtaining an estimate of the volatility σ, that is the only input needed to calibrate the BS model. The volatility of a stock is a measure of the uncertainty about the returns provided by the stock. Stocks typically have a volatility between 15% and 60%. Two basic approaches for the estimation are discussed in the following sub-section, namely historical volatility or implied volatility.

[15] Could be also evaluated by the Put-Call Parity.

2.5.1 Historical Volatility

To value an European option, an obvious idea is to use historical data in order to estimate σ. Since, in real life, the volatility is not constant over time, one standard practice is to use historical data for a period of the same length as the time to maturity. This approach takes the standard deviation of the underlying's log-returns and times the time length. The log-return of the underlying asset is

$$R_t = log\frac{S_t}{S_{t-1}}$$

Using elementary statistical theory, an estimate of σ is given by:

$$\tilde{\sigma}^H = \sqrt{\frac{Var}{\Delta t}} \qquad (9)$$

Where the sample variance Var is given by:

$$Var = \frac{1}{n-1}\sum_{t=1}^{n}(R_t - \bar{R}_n)^2$$

With $\bar{R}_n = \frac{1}{n}\sum_{t=1}^{n} R_t$ being the sample mean

An argument against the use of historical volatility is that in real life volatility is not constant, but changes over time. It should be an estimate of the volatility for the coming time period, but this approach only yields an estimate for the volatility over the past time period.

2.5.2 Implied Volatility

In this thesis it's defined as the volatility, obtained when equating the option's market value to its Black&Scholes value, given the same strike price and time to maturity. It is extracted numerically due to the fact that the Black&Scholes formula cannot be solved for σ in terms of the other parameters. Given an observed European call option price C_{Market} (market price observable on internet) for a contract with strike K and expiration date T, the implied volatility $\sigma^{implied}$ is defined to be the value of the volatility parameter that must go into the Black-Scholes formula, Equation (7) to match this price:

(10)

$$C_{Black\&Scholes}(t, S, K, T, \sigma^{implied}) = C_{Market}$$

Thus implied volatilities are embedded in option prices, which in turn reflect the future expectations of the market participants. Whereas historical volatilities are backward looking, implied volatilities are forward looking. Traders often quote the implied volatility of an option rather than its price. This is convenient because the implied volatility tends to be less variable than the option price. Note, that implied volatilities can be used to test the Black&Scholes model. If the model is correct (i.e. with a constant volatility) then, if one plots implied volatility as a function of the strike price, one should obtain a horizontal straight line. Contrary to this, it is often observed that options deep OTM or deep ITM are traded at higher implied volatilities than ATM options. But more on this would be explained in the limitations of the Black&Scholes at the end of this Chapter, in section (2.9). For the empirical part in MATLAB® concerning the calculation of the option value, the author will use the annual Historical Volatility of the underlying, i.e. Equation (9) to calibrate the Black&Scholes model and the Historical volatility calculated from 3-months rolling window for the Black&Scholes extension.

2.6 The Greeks

Having the close-form solution for the pricing formula in section (2.4), we are able to "derive" the Greeks, in order to proceed with the hedging strategies. The Greeks is the common name of the set of the derivatives of the price of the instrument, with respect to the Stock and the model parameters. The knowledge of the Greeks provides information about how sensitive the price is to changes in the Stock and the model parameters.

2.6.1 Delta

Delta means the sensitivity of a derivative price to the movement in the underlying asset. A delta is expressed as the first mathematical derivative of the product with respect to the underlying asset. It means that is the hedge ratio[16] of the asset for an infinitely small move. The orthodox definition of delta is:

$$\Delta = \frac{\partial \Pi}{\partial S} \tag{11}$$

[16] Hedge ratio is defined as the number of shares that has to be bought in order to offset the risk.

Where Π is the derivative price Π(S,t) and S is the underlying asset. It is the derivative of the option price to the underlying. In plain English, it would correspond to changes in the option price from infinitely small changes in the underlying asset. Delta is closely related to the Black-Scholes-Merton analysis. They showed that it is possible to set up a riskless portfolio consisting of a position in an option on a stock and a position in the stock expressed in terms of Delta, where the Black&Scholes portfolio is:

- 1: option
+Δ: shares of the stock

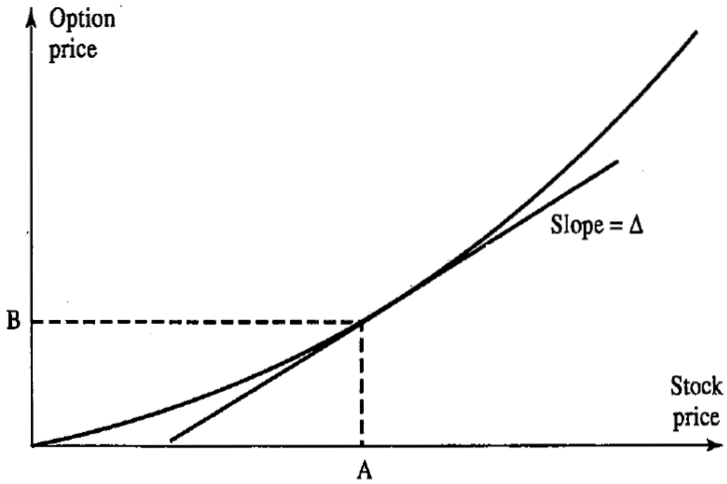


Figure 3: As seen, since the call option follows a nonlinear structure a linear hedge could not possibly be a perfect hedge. Delta try to catch local linearity through derivative, since, as said in the in Chapter 1, Vanilla options are nonlinear products. Source: Dynamic Hedging, Nassim Taleb, Wiley, 1997

Example 2.2 *Suppose that, in figure above, the delta of a call option on a stock is 0.6*[17]*(it can be seen in point A, imagine Δ=0.6), the stock price is 100 and the option price is €10. Imagine an investor who has sold 20 call option contracts, that is, option to buy 2000 shares. The investor's position could be hedged by buying 0.6*2000 = 1200 shares. The gain (loss) on the option position would then tend to be offset by the loss (gain) on the stock position. If the stock price goes up by €1 (producing a gain of 1200€ on the shares purchased), the option price will tend to go up by 0.6*1€=0.6€ (producing a loss of €1200 on the shares purchased), the option price will tend to go down by €0.6 (producing a gain of €1200 on the options written). In this example, the delta of the investor's option position is 0.6 * (-2000) = -1200. In other word, the investor loses 1200ΔS on the short option position when the stock price increases by ΔS. The delta of the stock is 1, so that the long position in 1200 shares has a delta of +1200. The delta of the investor's overall position is, therefore, zero. The delta of the stock position offsets the delta of the option position. It is important to realize that, because delta changes, the investor's position remains delta hedged for only relatively short period of time. The hedge has to be adjusted (i.e. rebalancing). In our example, imagine that after 3 days the stock price might increase to €110, as indicated in the figure, an increase in the stock price leads to an increase in delta. Suppose that delta rises from 0.60 to 0.65. An extra 0.05 * 2000 = 100 shares would then have to be purchased to maintain the hedge, for this we will talk later about gamma and hedging error.*

2.6.1.1 Delta of a European Option

Considering the classic Black-Scholes-Merton formula, for a European Call option on a non-dividend paying stock the delta (Δ) is the first derivative with respect to price, by taking the derivative with respect to S of equation (7):

$$\Delta(Call) = N(d_1) \tag{12}$$

Where d_1 is defined in equation (6 bis). This formula gives the delta of a long position in one Call option[18]. For a European put option on a non-dividend-paying stock, delta is given by the derivative with respect to S of equation (8):

$$\Delta(Put) = N(d_1) - 1 \tag{13}$$

[17] This means that when the stock price changes by a small amount, the option prices changes by about 60% of that amount.

[18] The delta of a short position in one call option is -N(d_1). Using delta hedging for a short position in a European call option involves maintaining a long position of N(d_1) for each option sold. Similarly, using delta hedging for a long position in a European call option involves maintaining a short position of - N(d_1) shares for each option purchased.

In MATLAB® the delta is calculated using the function `blsdelta`:

```
[CallDelta, PutDelta] = blsdelta(Price, Strike, Rate, Time, Volatility)
```

Delta for long Put options is negative, which means that a long position in a put option should been hedged with a long position in the underlying stock, and a short position in a put option should be hedged with a short position in the underlying stock.

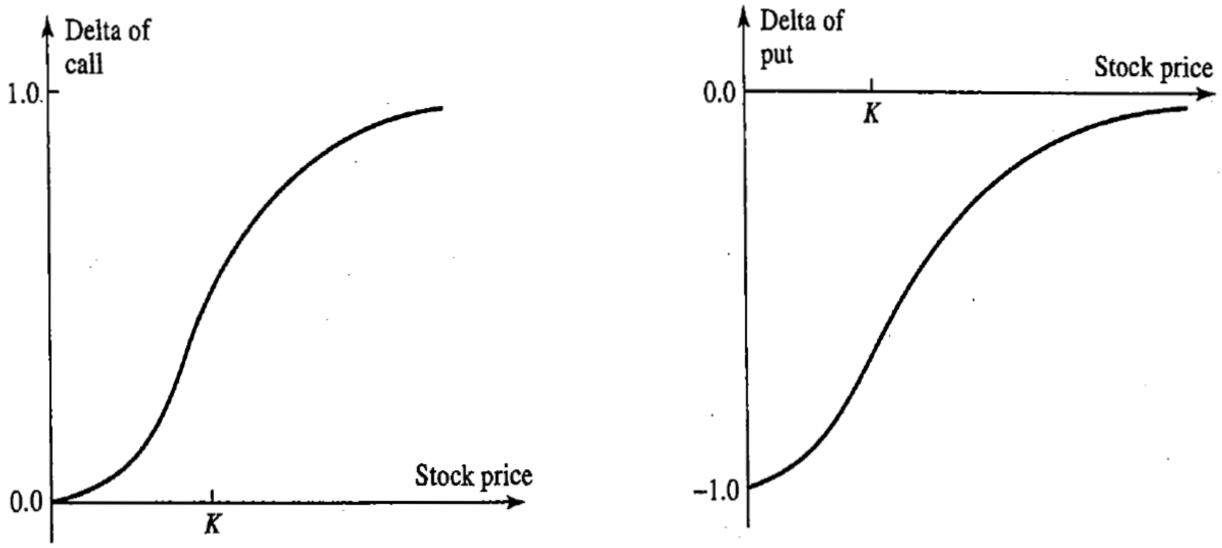


Figure 4: Variation of delta with stock price for (left) a call option and (right) a put option on a non-dividend-paying stock.

2.6.1.2 Delta of a Portfolio

For our purpose, the delta of a Portfolio is needed since the author is considering a hedging on option portfolio dependent on a single asset. In formula we have:

$$\Delta_{Portfolio} = \frac{\partial P}{\partial S} \tag{14}$$

Where P is the value of the portfolio. The delta of the portfolio in this part of the thesis is calculated from the deltas of the individual options in the portfolio. If a portfolio consists of a quantity w_i of option i ($1 < w_i < n$), the delta of the portfolio is given by:

$$\Delta_{Portfolio} = \sum_{i=1}^{n} w_i \Delta_i \tag{15}$$

Where Δ_i is the delta of the i_{th} option. The formula can be used to calculate the position in the underlying asset necessary to make the delta of the portfolio zero. When this position has been taken, the portfolio is referred to as being delta neutral.

Example 2.3 *Suppose a financial institution has the following three position in options on a stock:*

- *A long position in 100,000 call options with strike price €55 and an expiration date in 3 months. The delta of each option is 0.533*
- *A short position in 200,000 call options with strike price €56 and an expiration date in 5 months. The delta of each option is 0.468*
- *A short position in 50,000 put options with strike price €56 and an expiration date in 2 months. The delta of each option is -0.508*

The delta of the whole portfolio is $100.000 * 0.533 - 200.000 * 0.468 - 50.000 * (-0.508) = -14.900$. *This means that the portfolio can be made delta neutral by buying 14,900 shares.*

If a portfolio is delta-neutral, then the value of the portfolio remain constant when the price of the Stock changes. However, the value of the delta changes over time and as the price of the Stock changes. Thus the portfolio is only locally delta neutral. To ensure that the portfolio remains delta neutral over time, it needs to be re-balanced frequently over the lifetime of the instrument, i.e. hedged dynamically. The more often the portfolio is re-balanced, the less the value of the portfolio will change over the lifetime of the instrument (in the limit where the portfolio is continuously re-balanced, the value of the portfolio remains constant throughout the life-time of the instrument). However, in real life there are costs associated with re-balancing the portfolio. There is thus a trade-off between risk-reduction (re-balancing often) and return (re-balance less often), in this thesis, however no transaction costs are considered.

2.6.2 Gamma

The gamma (Γ) of an option on an underlying asset is the rate of change of the option's delta with respect to the price of the underlying asset. Mathematically is the second mathematical derivative of the product with respect to the underlying asset:

$$\Gamma = \frac{\partial^2 \Pi}{\partial S^2} \tag{16}$$

2.6.2.1 Gamma of a European option

For a European call or put option on a non-dividend paying stick the gamma is given by:

$$\Gamma = \frac{N'(d_1)}{S\sigma\sqrt{T}} \tag{17}$$

Where d_1 is defined in equation (6 bis) and N'(x) is the probability density function for a standard normal distribution and is defined as:

$$N'(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \tag{18}$$

The gamma of a long position is always positive. In MATLAB® the gamma is calculated using the function `blsgamma`:

```
Gamma = blsgamma (Price, Strike, Rate, Time, Volatility)
```

Example 2.4 *Consider a Call option on a non-dividend-paying stock where the stock price is 49€, the strike is 50€, r is 5%, T is 20 weeks and volatility is 20%. In this case the option gamma is 0.066. When the stock price changes by ΔS, the delta of the option changes by 0.066ΔS.*

2.6.2.2 The gamma of a Portfolio

The gamma (Γ) of a portfolio of options on an underlying asset is the rate of change of the portfolio's delta with respect to the price of the underlying asset. Mathematically it is the second partial derivative of the portfolio with respect to asset price:

$$\Gamma_{Portfolio} = \frac{\partial^2 P}{\partial S^2} \tag{19}$$

The calculation of the portfolio Gamma is done with the same criteria of the portfolio Delta.. If a portfolio consists of a quantity w_i of option i ($1 < w_i < n$), the Gamma of the portfolio is given by:

(20)

$$\Gamma_{Portfolio} = \sum_{i=1}^{n} w_i \Gamma_i$$

As shown in Figure (5), Gamma changes with time, when we consider a portfolio normally we have inside with different maturities, so a 10 days option is different from a 2 days option in term of gammas all else remaining equal.

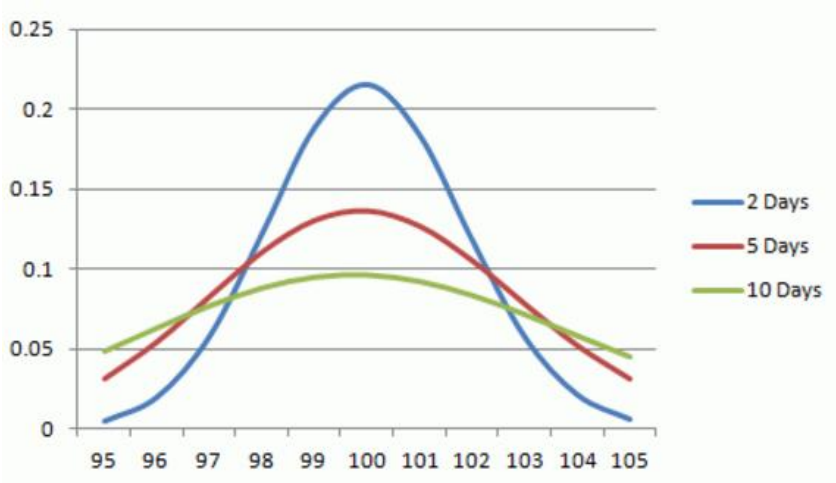


Figure 5: Gamma changes with time. All have a 100 strike. Source: Wilmott.com

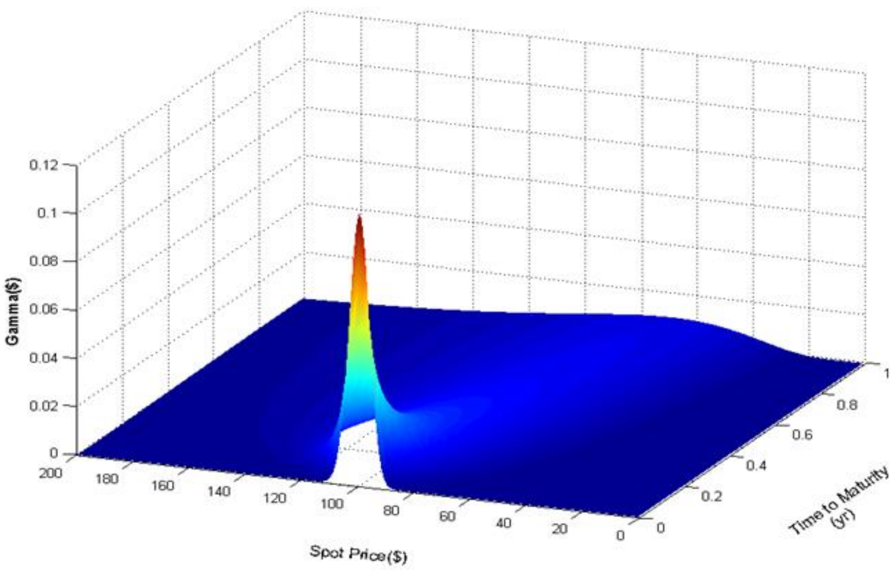


Figure 6: The relationship between Gamma, stock price and time to maturity. Source: Wilmott.com

"Gamma is a proxy for the rebalancing of the portfolio."

Figures above suggest there are important pitfalls for option replication and portfolio stabilization. Often operators hedge their gamma with an option trade that takes care of their immediate need but does not provide long-term stability in the position. If gamma is small, delta changes slowly, and adjustments to keep a portfolio delta neutral to be made only relatively infrequently. However, if the absolute value of gamma is large, delta is highly sensitive to the price of the underlying asset. It is then quite risky to leave a delta-neutral portfolio unchanged for any length of time. Gamma try to adjust the hedging error introduced by nonlinearity of the derivative.[19] Figure (7) illustrates this point.

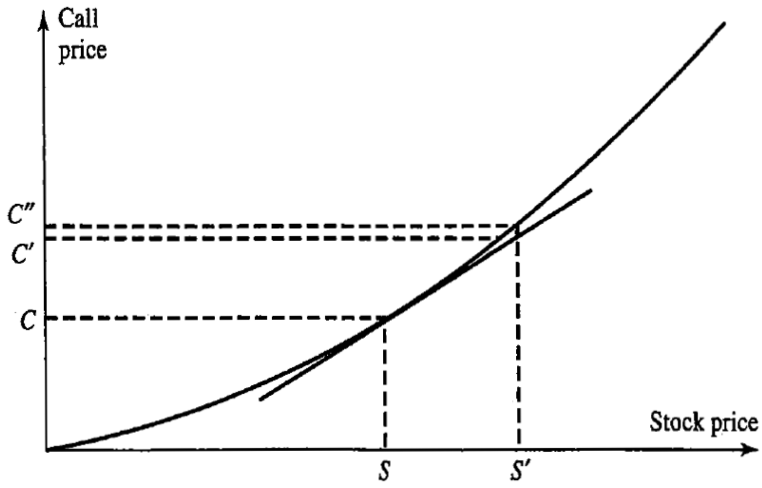


Figure 7: When the stock price move to S to S' delta hedging assumes that the option price moves from C to C', when in reality it moves from C to C''. The difference between C' and C'' leads to a hedging error. The size of the error depends on the curvature of the relationship between the option price and the stock price. Gamma measures this curvature. Source: Dynamic Hedging, Nassim Taleb, Wiley, 1997.

2.6.2.3 Making a Portfolio Gamma Neutral

"Delta neutrality provides protection against relatively small stock price moves between rebalancing. Gamma neutrality provides protection against larger movements in this stick price between hedge rebalancing."

A position in the underlying asset has zero gamma[20] and cannot be used to change the gamma of a portfolio. What is required is a position in an instrument such as an option that is non-linearly dependent on the underlying asset. Suppose that a delta-neutral portfolio has a gamma equal to Γ, and a traded option has a gamma equal to Γ_T. If the number of traded options added to the portfolio is w_T, the gamma of the portfolio is $w_T\Gamma_T + \Gamma$; Hence the position in the traded option necessary to make the portfolio gamma neutral is $-\Gamma/\Gamma_T$. Including the traded option is likely to change the delta of the portfolio, so the position in the underlying asset then has to be changed to maintain delta neutrality. As time passes, gamma neutrality can be maintained only if the

[19] I.e. starting from a delta neutral portfolio, this adjustment in gamma meaning that another adjustment in delta has to be made.

[20] The second derivative of the underlying price is zero.

position in the traded option is adjusted so that it is always equal to $-\Gamma/\Gamma_T$. Making a portfolio gamma neutral as well as delta-neutral can be regarded as a correction for the hedging error illustrated before.

Example 2.5 *Suppose that a portfolio is delta neutral and has a gamma of – 3.000. The delta and gamma of a particular traded call option are 0.62 and 1.50, respectively. The portfolio can be made gamma neutral by including in the portfolio a long position of 3000/1.5 = 2000 in the call option. However, the delta of the portfolio will the change from zero to 2000*0.62 = 1240. Therefore 1240 units of the underlying asset must be sold from the portfolio to keep it both delta neutral and gamma neutral.*

2.6.3. Vega

The Vega is the sensitivity of an option to the changes in the implied volatility for a maturity equal to its stopping time. The Vega of a European option of known maturity it is expressed as:

$$V = \frac{\partial \Pi}{\partial \sigma} \tag{21}$$

Where σ is the implied volatility for the maturity matching that of the option, Π is the price of the derivative security. The best way to ascertain it numerically is by re-pricing the instrument at different levels of volatilities.

2.6.3.1 The Vega of a European option

The Vega for a European call or put option on a non-dividend paying stock is given by:

$$V = S_0\sqrt{T}N'(d_1) \tag{22}$$

Where d_1 is defined as in equation (6 bis). The formula for N'(x) is defined in equation (18). The Vega of a short position in a European option is always negative, this is explained in Example 2.6. Vega changes with time, as with gamma, when we consider a portfolio normally we have inside with different maturities, so a 10 days option is different from a 2 days option in term of gammas all else remaining equal. Figure (8) shows this:

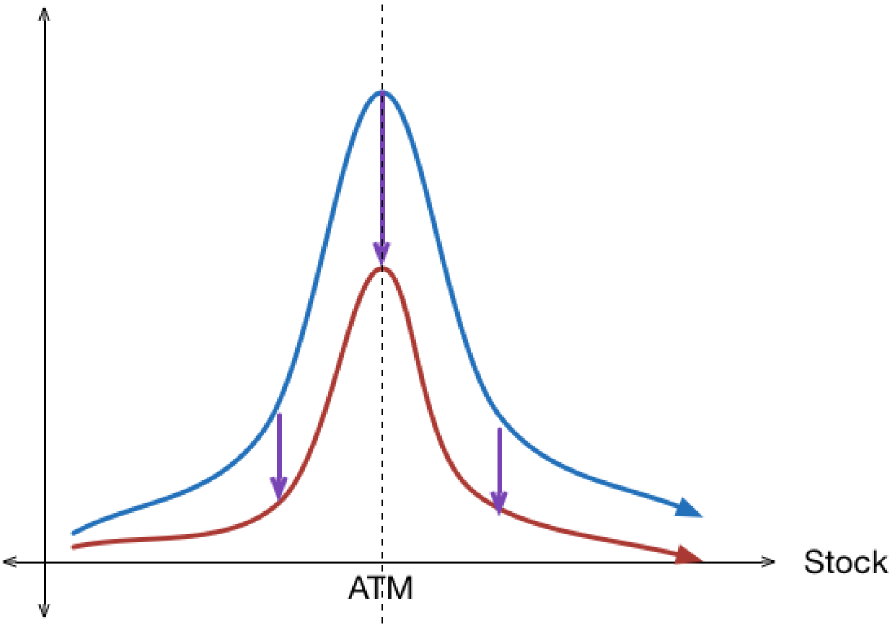


Figure 8: The relationship between Vega and the underlying price and time, for a 10 days option (blue) and 2 days option (red). Source: Wilmott.com

As with the gamma and the theta, it is easy to see that the Vega follows a bell shape, with the maximum reached when the option is at the money, same in the Figure (9) that consider also the passage of time to maturity, adding a third dimension.

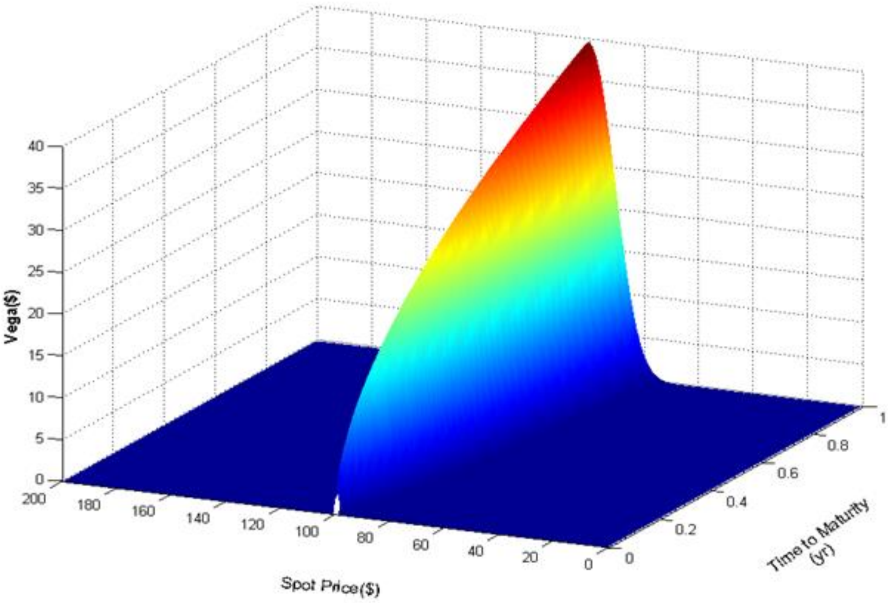
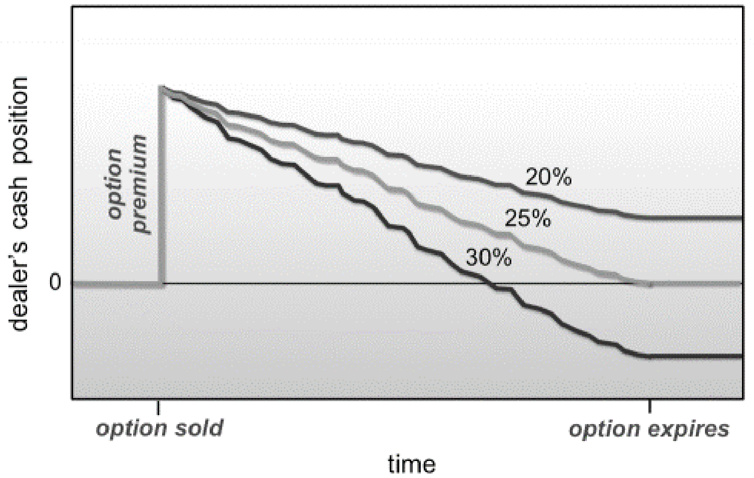


Figure 9: The relationship between Vega, underlying asset and time. Source: The Heston model and its application in Matlab, Wiley, 2013.

Example 2.6 *Imagine the cash position of a derivatives dealer who sells an option and then dynamically hedges it until expiration. The dealer originally prices the option at 25% volatility, but the exhibit considers three volatility scenarios:*

The stock experiences 20% volatility.
The stock experiences 25% volatility.
The stock experiences 30% volatility.



A dealer sells an option priced at 25% volatility and then dynamically hedges the position until expiration. This exhibit considers how the dealer's cash balance evolves over time under three scenarios. Under all scenarios, we assume an initial cash balance of zero. When the option is sold, the dealer receives a premium, so the cash balance jumps. Next, the dealer dynamically hedges the short option, gradually losing cash as he does so. Under the first scenario, the underlying experiences 20% volatility. Dynamic hedging costs less than it would have had the underlying experienced the 25% volatility used to price the option. The dealer ends up with a profit. Under the second scenario, the underlying experiences 25% volatility. This is the volatility at which the option was priced, so the dealer breaks even on the transaction. Finally, under the third scenario, the underlying experiences 30% volatility. This is higher than anticipated, and the dealer ends up with a loss. At a higher volatility, the underlying will fluctuate more, the dealer need to adjust the delta hedge more frequently. He will lose money more rapidly dynamically hedging. The opposite would be true if the underlier's volatility suddenly fell. You could readjust the delta hedge less frequently, and you would lose money more slowly dynamically hedging. When a dealer is dynamically hedging a short options position, he doesn't care whether the underlying goes up or down. Because he is always delta hedged, he is neither long nor short the underlying. He does care whether the underlier's volatility goes up or down. In a very real sense, he is short volatility. This is the same thing as having negative vega (or "short vega"), so the phrases negative vega, short vega and short volatility all mean the same thing. A dealer dynamically hedging a long options position is in the opposite situation. He benefits if volatility increases, so he is long volatility. Synonyms would be long vega or positive vega.

In MATLAB® the gamma is calculated using the function `blsvega`:

```
Vega = blsvega (Price, Strike, Rate, Time, Volatility)
```

2.6.3.2. Vega of a Portfolio

As for delta and gamma, we can express the Vega of a portfolio as:

$$\mathcal{V} = \frac{\partial P}{\partial \sigma} \tag{23}$$

So, for the same criteria, the Vega of the portfolio could be expressed as:

$$\mathcal{V}_{Portfolio} = \sum_{i=1}^{n} w_i \mathcal{V}_i \tag{23 bis}$$

2.6.3.3 Making a Portfolio Vega and Gamma Neutral

"The Vega of at-the-money options is stable to an increase in volatility. Options that are away from the money are convex with respect to volatility for the owner and concave for the seller."

If the absolute value of Vega is high, the portfolio's value is very sensitive to small changes in volatility. If the absolute value of Vega is low, volatility changes have relatively little impact on the value of the portfolio. A position in the underlying asset has zero Vega. However, the Vega of a portfolio can be changed by adding a positon in a traded option. If $\mathcal{V}$ is the Vega of the portfolio and $\mathcal{V}_T$ is the Vega of the traded option, a position of $-\mathcal{V}/\mathcal{V}_T$ in the traded option makes the portfolio instantaneously Vega neutral. Unfortunately, a portfolio that is gamma neutral will not in general be Vega neutral, and vice versa. If a hedger requires a portfolio to be both gamma and Vega neutral at least two traded derivatives dependent on the underlying asset must usually be used.

Example 2.7 *Consider a portfolio that is delta neutral, with a gamma of -5000 and a vega of -8000. The options shown in the table below can be traded. The portfolio can be made vega neutral by including a long position in 4000 of Option 1. This would increase delta to 2400 and require that 2400 unit of the asset be sold to maintain delta neutrality. The gamma of the portfolio would change from -5000 to -3000.*

	Delta	*Gamma*	*Vega*
Portfolio	*0*	*-5000*	*-8000*
Option 1	*0.6*	*0.5*	*2.0*
Option 2	*0.5*	*0.8*	*1.2*

To make the portfolio gamma and vega neutral, both option 1 and option 2 can be used. If w_1 *and* w_1*are the quantities of option 1 and option that are added to the portfolio, we require that*

$$-5000 + 0.5w_1 + 0.8w_2 = 0$$
$$-8000 + 2.0w_1 + 1.2w_2 = 0$$

The solution to these equations is w_1 *= 400 and* w_2 *=6000. The portfolio can therefore be made gamma and vega neutral by including 400 of option 1 and 6000 of option 2. The delta of the portfolio, after the addition of the positons in the two traded options, is 400*0.6+6000*0.5=3240. Hence 3240 unit of the asset would have to be sold to maintain delta neutrality.*

It is important to notice that the number of Greeks that we want to be hedged has to be equal (at least) at the number of instruments in our portfolio. We saw that with a delta neutral portfolio, hedging with a delta-gamma strategy requires another instrument, and hedging for the same delta neutral portfolio, a gamma-vega strategy requires at least adding two instruments. We said also that is possible to construct hedging strategies with the underlying asset, or other options traded in the market, or even a mixed strategy. In this thesis, since the author wants to perform also a delta-gamma-vega strategy he cannot use only the underlying as explained before, he might use a mixed strategy (underlying and options) or trading only options. For the empirical part in MATLAB®, the author decides to construct and hedging all the strategies trading only options, precisely the same options sold at the beginning, leaving the hedging only a matter of "adjusted quantities". For mathematical reasons the minimum number of options (i.e. instruments) in the portfolio has always to be, at least, equal to the number of Greeks considered.

2.7 Minor Greeks

2.7.1 Rho

The rho of a portfolio of options is the rate of change of the value of the product with respect to the interest rate:

$$\rho = \frac{\partial \Pi}{\partial r} \tag{24}$$

It measures the sensitivity of the value of a portfolio to a change in the interest rate when all else remains the same.

2.7.1.1 Rho of a European option

For a European call option on a non-dividend paying stock

$$\rho(Call) = KTe^{-rT}N(d_2) \tag{25}$$

Where d_2 is defined in equation (6). For a European put option:

$$\rho(Put) = -KTe^{-rT}N(-d_2) \tag{26}$$

Example 2.8 *Consider our usual example, a call option on a non-dividend paying stock where the stock price is €49, the strike price is €50, the risk free rate is 5%, the time to maturity is 20 weeks, and the volatility is 20%. In this case the option's rho is:*

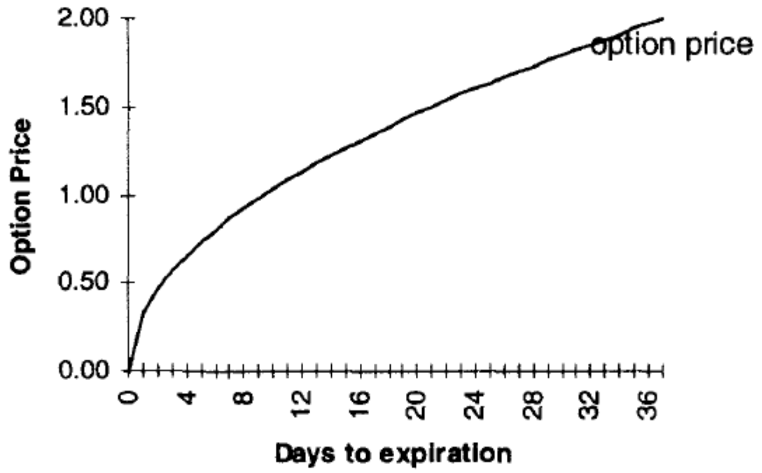
$$\rho(Call) = KTe^{-rT}N(d_2) = 8.91$$

*This means that a 1% (0.01) increase in the risk-free rate (from 5% to 6%) increases the value of the option by approximately 0.01*8.91= 0.0891.*

2.7.2 Theta

"The theta is a loss in time value of an option portfolio that results from the passage of time."

The Theta (Θ) of a portfolio of option is the rate of change of the value of the portfolio with respect to the passage of time with all else remaining the same. Theta is sometimes referred to as the time decay of the portfolio. The pricing is straightforward: the trader can use the difference between the price of an option today and the same price on the next day, keeping everything else constant. One way to look at the representation of theta is that it goes hand in hand with gamma. The alpha (i.e. gamma per theta ratio) will be the same regardless of the number of days to expiration. Figure (10) shows the time decay for an at-the-money and an out-of-money option. Theta corresponds to the re-pricing of a portfolio with one day less to expiration. *However, what if the volatility and other parameters for period t+1 were different than those of period t? Do we offsetting effects or not?*



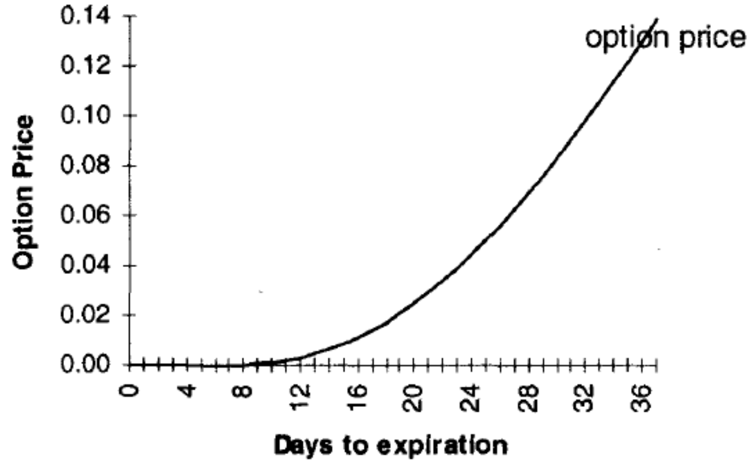


Figure 10: The relationship between the option price and time, for an at-the-money option (above) and an out-of-money option (below) all else remaining equal. Source: Nassim Taleb, Dynamic Hedging, 1997.

2.7.2.1 Theta of a European option

For a European call option on a non-dividend-paying stock, it can be shown from the Black-Scholes formula that:

$$\Theta(Call) = -\frac{S_0 N'(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}N(d_2) \quad (27)$$

Where d_1 and d_2 are defined as in equation (6) and (6 bis) and N'(x) defined in Equation (18). For a European put on the stock we have:

$$\Theta(Put) = \frac{S_0 N'(d_1)\sigma}{2\sqrt{T}} + rKe^{-rT}N(-d_2) \quad (28)$$

Because $N(-d_2) = 1 - N(d_2)$, the theta of a put exceeds the theta of the corresponding Call by rKe^{-rT}. In the se formulas, theta is measured in years.[21]

Example 2.9 *Consider a call option on a non-dividend-paying stock where the stock price is 49€, the strike price is 50€, the risk free is 5%, the time to maturity is 20 weeks (=0.3046 years), and the volatility is 20%. In this case the option theta is:*

$$\Theta(Call) = -\frac{S_0 N'(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}N(d_2) = -4.31$$

The theta is -4.31/365= -0.0118 per calendar day, or -4.31/252 = -0.0171 per trading day.

Theta is usually negative for an option. This because, as time passes with all else remaining the same, the option tends to become less valuable. When the stock price is very low, the theta is close to zero. For an at-the-money call option, theta is large and negative. As the stock price becomes larger, theta tends to $-rKe^{-rT}$. The variation of theta with stock price for a call option on a stock is shown in Figure (11). Theta is not the same type of hedge parameter as delta. There is uncertainty about future stock price, but there is no uncertainty about the passage of time. It does not make any sense to hedge against the passage of time. In spite of this, many traders

[21] Usually when theta is quoted, time is measured in days, so that theta is the change in the portfolio value when 1 day passes with all else remaining the same. We can measure theta either "per calendar day" or "per trading day". To obtain the theta per calendar day, the formula for theta must be divided by 365; to obtain theta per trading day, it must be divided by 252.

regard theta as a useful descriptive statistic for a portfolio. This is because in a delta-neutral portfolio theta is proxy for gamma.[22]

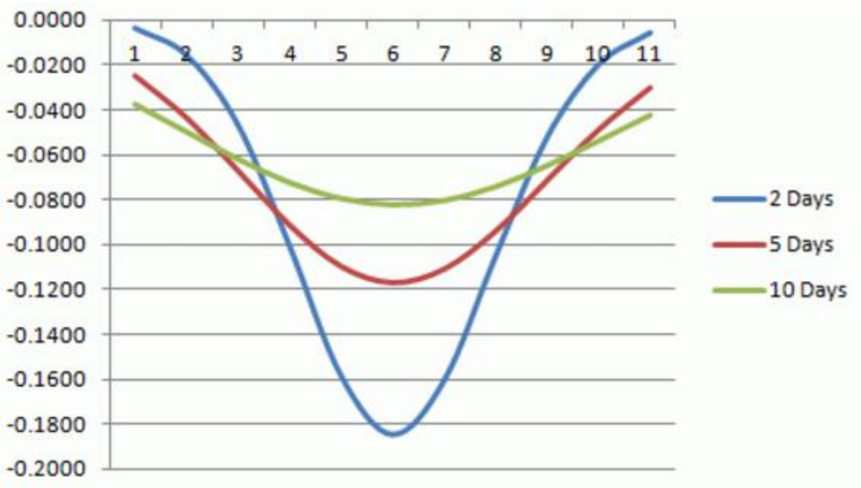


Figure 11: The variation of Theta of a European Call with Stock price. Source: Nassim Taleb, Dynamic Hedging, 1997, Wiley.

Becoming clear the concept of option pricing and the Greeks, we have almost all the instruments to perform a Black&Scholes hedging. In the implementation, in MATLAB®, in order to make a Portfolio neutral among the various strategies, the author will use the function Hedgeopt, introduced at the end of Chapter 3.

2.8 Limitation of Black-Scholes model

The Black&Scholes model has set such an important foundation in financial engineering in the past years and has been really recognized by both academia and practitioners, but it is also well known and accepted that this model is not that accurate in capturing the features in the stock markets in reality. There are several major drawbacks of the Black&Scholes model, mainly because the idealized assumptions do rarely hold in the real world. First of all, the assumption of a normal distribution of log-returns is under critique. Combined factors of extreme events, fat tails, high peak and the volatility clustering effects make the assumption of non-Gaussian distribution more appropriate. Secondly, the volatility smile is simply a violation of the constant volatility assumption. The mentioned drawbacks will be discussed in greater detail in the following sub-sections.

[22] In the appendix the relationship between Gamma, Delta and Theta is reported.

2.8.1 Shortcomings of Gaussian distribution

Economists believed that prices in speculative markets, such as securities markets, behave very much like random walks, which is based on two classic assumptions that are already discussed in the previous sections:

- Price changes are independent random variables
- The changes conform to some probability distribution.

In the study of financial time-series, it is a concept to describe the actual return distribution, where data or the variable turns to cluster around the mean. The two important parameters are the mean μ and the variance σ^2. For a Gaussian distribution the probability density function is given by:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{(x-\mu)^2}{2\sigma^2}}$$

Some notable properties of the Gaussian distribution are the following:

- Symmetry around its mean μ, therefore the skewness of the distribution is 0.
- Both the mode and the median are the same as the mean μ.
- The inflection points (points where the curve changes sign) of the curve occur one standard deviation away from the mean, i.e. at μ- σ and μ+ σ.
- The kurtosis is equal to 3.

Unfortunately, these properties are not suitable in capturing the probability of extreme events in the market. Taking for example the stock market crash of October 1987. Following the standard paradigm, the stock market returns are lognormally distributed with an annualized volatility of 20% (Volatility is usually believed to be between 15% and 60%).

On October 19, 1987, the two month S&P 500 price fell 29 percent. Under the lognormal assumption and according to the calculation from the probability density function, the probability of this event is 10^{-160}, which is virtually impossible. In the history of stock market this is not the only event with little probability that actually did happen. Besides the difficulty in dealing with historical extreme events, empirical research has shown that the actual return distributions in stock market have fatter tails and higher peak than the normal distribution. In Figure (12), the frequency distribution of SPX (SP500) daily log-returns over a 77-year period from 1928 to 2005 is plotted and compared with the normal distribution. Note the - 22,9%

return on October 19, 1987 in Figure (13), which is not directly visible in Figure (12) but the x-axis has been extended to the left to accommodate it.

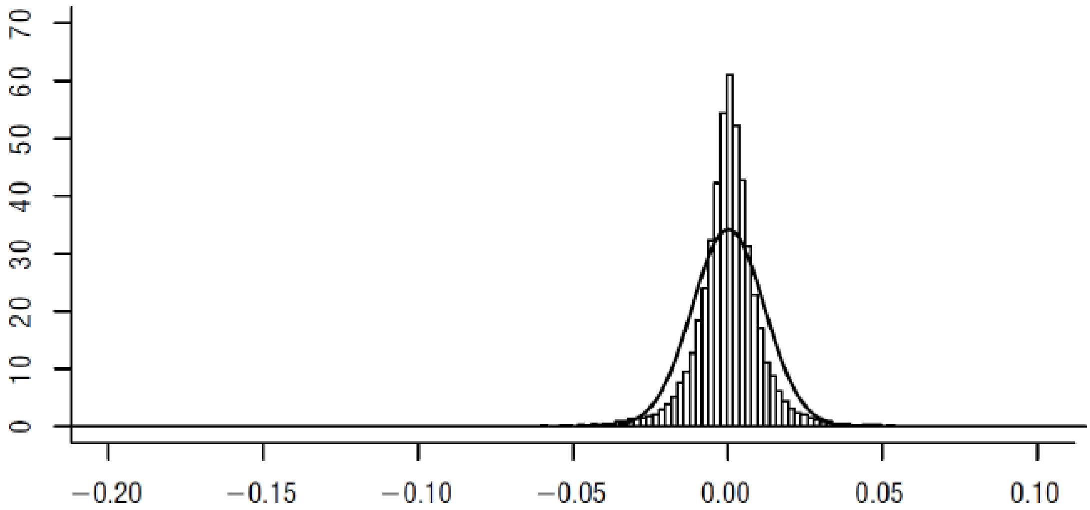


Figure 12: Frequency distribution of 77 years of SPX daily log-returns compared with the normal distribution. Source: The Volatility Surface. A Practioner's Guide. Wiley, 2006.

It is quite obvious that the distribution of log-returns of SPX is highly peaked[23] and fat-tailed relative to the normal distribution.

2.8.2 Clustering and Leverage effect

A clustering effect is often observed in financial time series when stock returns are dependent. In other words, large changes in returns tend to be followed by large movements in returns. The same effect is seen on small changes in assets returns tend to be followed by small changes. This anomaly results in volatility clusters where volatility seem to group together at certain time periods. In the simple BS model the returns are often assumed to be independent and no correlation between the returns. To investigate this assumption one has to graph the correlation between returns and its lags. By analyzing this property it is easy to conclude that asset returns are not correlated. There seem to be no relationship between returns and its lags as the correlation fluctuate around zero and seem to appear randomly. As stated in by previous research[24] the absolute value of returns or squared returns show another interesting property. It seems like the absolute returns do in fact show sign of autocorrelation indicating that returns

[23] Kurtosis affects the height and width of the probability density function. The probability density function is symmetric, but is more or less "peaked" than the normal distribution. A positive kurtosis indicates a high peak, fatter tails and a thin midrange. A positive kurtosis can be interpreted that fewer observations are in the intermediate range and extreme observations occur more often.
[24] Cont, 2001.

are not independent. The absolute returns and its lags seem very dependent, (i.e. significant and slow decaying function of lags). It is this dependence that creates the volatility clustering. This indicates a time varying persistent volatility where today's absolute returns are correlated with past absolute returns. One can observe this trend in Figure (13), where the log-returns of SPX over a 15-year period are plotted.

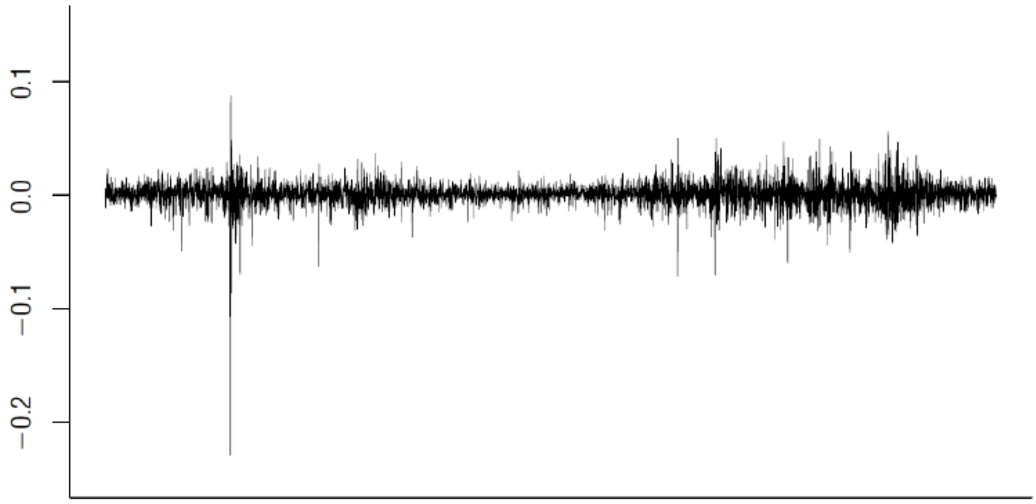


Figure 13: SPX daily log-returns from December 31, 1984 to December 31, 2004. Note the - 22,9% return on October 19, 1987. Source: The Volatility Surface. A Practioner's Guide. Wiley, 2006.

This implies that actually the volatility of the log-returns is auto-correlated. In the model, this is a consequence of mean reversion of volatility, indeed we would see in Chapter 3 that the Heston dynamics for the volatility is expressed with a mean reverting process.

By the way, volatilities have another effect on stocks expressed by a negative correlation, this negative correlation between stock's current prices and their future volatilities, called the *leverage effect* was first noted by Black in 1976[25], who also mentions: *"I have believed for a long time that stock returns are related to volatility changes. When stocks go up, volatility seem to go down; and when stocks go down, volatilities seem to go up."* This could also be explained from intuition. When the return of equity becomes negative, the reactions from the investors will be more volatile, thus the volatility will increase. Otherwise, when the return becomes positive, investors will gain more confidence in the speculative market; therefore the volatility in the near future would decrease. Therefore, this is also an implication that the constant volatility assumption is far away from the reality. On the contrary, we will see the Heston model dealing also with the leverage effect with the rho coefficient.

[25] Black, R. Studies of Stock Price Volatility Changes. Proceedings of the 1976 Meetings of the American Association, Business and Economic Statistic Section, (1976), 177-181.

2.8.3 Volatility Smile and Volatility surface

Recall the definition of the implied volatility from section (2.5.2) as the volatility of the underlying assets which, when substituted into the Black&Scholes formula, gives a theoretical price equal to the market price observed. If the assumption of constant volatility in the Black&Scholes model would hold in the market, the implied volatility of the underlying one could get given an underlying price with different maturities and strikes should be the same. But by considering the price of the Call as given (available on internet), solving the formula for the volatility we wouldn't obtain a line but a curve; even though there are some jumps, rather than a straight line, which means that the volatility should not be a constant value. Indeed as seen in Figure (14), where one can observe a so-called volatility smile, proving that the implied volatility is a function of the strike price, this contradicts the Black&Scholes model assumptions.

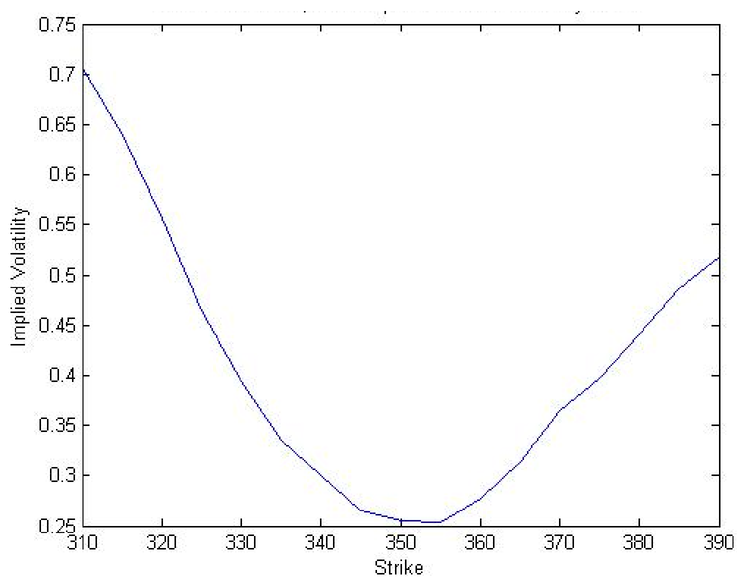


Figure 14: Example of Volatility smile on FX options. Source: Wilmott.com

We have to mention that different underlying assets have different volatility smile graph as can be seen in Figure (15). For Equity and commodities derivatives, whose graph are respectively down and up sloping, we usually call it "volatility smirk/skew" instead of "volatility smile"; while for FX options, the graph is much more familiar with the term "smile" such as in Figure (14). Considering that in this thesis the author uses Equity options, usually they show a negative slope in their implied volatilities.

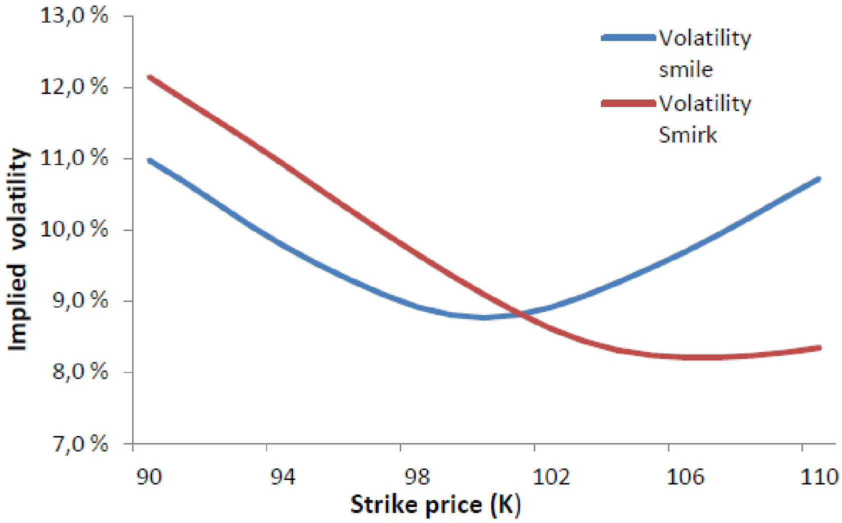


Figure 15: Volatility Smile and Volatility smirk for equity options. Source: Wilmott.com

The smirk is observed when implied volatility is a declining function of strike. As can be seen in Figure (16), for call options, ITM options, having a higher implied volatility, are the most expensive while the OTM are the cheapest relatively speaking.

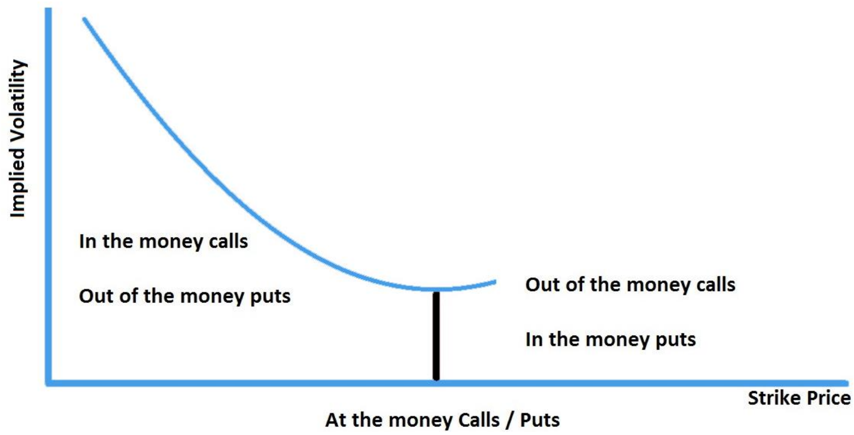


Figure 16: Higher implied volatility for ITM Call options and OTM Put options. Source: Wilmott.com

Therefore the OTM for put options are more expensive than ATM and ITM options[26]. The put-call-parity ensures that the volatility smile/smirk is the same for both puts and calls[27]. There is an economic interpretation to why this is the case. ITM call options have higher implied volatilities because investors use options to leverage their position. Since deep ITM options

[26] Important to keep in mind that ITM for calls is the same as OTM for puts when studying the volatility smile.
[27] Hull, 2011, p. 381.

fluctuate approximate the same as the index, investors can increase their return on investment by using the leverage incorporated in options. For put options deep OTM can be used as a protective insurance for a market turndown. Since deep OTM put options is a “cheap” insurance, investors use these options as downside protection and their implied volatilities increase. This “anomaly” is pronounced and is observed in almost every equity market. The smirk is created when there is a negative correlation between the stock index returns and volatility, not surprisingly, when options data are used to estimate the parameters of the Heston model, the correlation will in most cases turn out to be negative. We might say also that the volatility smile/skew is also a function of time, adding a third dimension we create the volatility surface represented in Figure (17), noticing the negative slope and the effect of time that smooth the volatility surface. It is found that that longer dated maturities tend to have less skew effect[28] compared to shorter-dated options. This effect is referred to as the maturity bias.

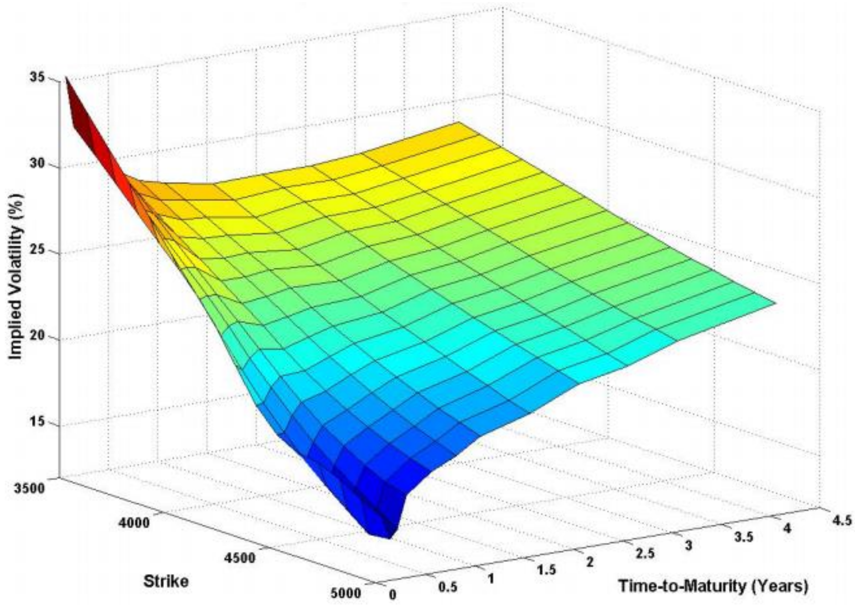


Figure 17: Example of Volatility surface on Equity options. Source: The Heston model and its application in Matlab, Wiley, 2013.

[28] Duque & Lopes, 2000.

Focus: Why is there a Skew in equity options?

For stocks and stock indices the shape of the volatility surface is always changing. There is generally a skew, however, so that for any fixed maturity, T, the implied volatility decreases with the strike, K. It is most pronounced at shorter expirations. There are several explanations for the skew:

- As said in previous sections, stocks do not follow GBM with a fixed volatility. Markets often jump and jumps to the downside tend to be larger and more frequent than jumps to the upside.
- Risk aversion: as markets go down, fear sets in and volatility goes up.
- Supply and demand. Investors like to protect their portfolio by purchasing out-of-the-money puts. This is another form of risk aversion.
- The total value of company assets, i.e. debt + equity, is a more natural candidate to follow GBM. If so, then equity volatility should increase as the equity value decreases. This is known as the explained leverage effect.

"If Black-Scholes model were correct then we should have an implied volatility surface for each type of options. The volatility surface is a function of strike, K, and time-to-maturity, T"

Despite these limitations of idealistic assumptions, which are clearly not suitable to the real market, the Black&Scholes model is still widely used. The main reason is simply its easy analytical tractability, which results in simple formulas for most pricing problems. It is also quite accurate for ATM vanilla options, but one should be careful when using Black&Scholes prices for deep OTM/ITM options or exotic options; in these cases market prices can show huge deviations from the theoretical Black-Scholes prices. However, the content of the following Chapter will digress from the Black-Scholes world to stochastic volatility world. Indeed to overcome the Black&Scholes limits, in the literature, a lot of new models were introduced. These extensions can loosely be grouped into two main approaches: deterministic volatility models and stochastic volatility models. The former allow the volatility to depend uniquely from the price process, the latter volatility is described with another process adding a new source of randomness. An example of stochastic volatility model is the Heston model that would be used as stochastic volatility model for the empirical application and is explained in the next Chapter.

Chapter 3 | Stochastic volatility framework

Introduction

In the previous chapter, precisely in section (2.9) the Black&Scholes framework and its restrictions have been discussed. In this chapter some alternative option pricing models will be mentioned, before the presentation of the widely used Heston model. Since the introduction of the Black&Scholes model, several efforts have been made to construct alternative option pricing models that permit for non-Gaussian return distributions as well as non-constant volatility.

> "Bakshi et al. gives the overall conclusion that the stochastic volatility feature is the most important. Other measurement, such as adding jumps and assumption of stochastic interest rate are not as significant as the assumption of stochastic volatility."

The models of Hull and White (1987), Scott (1987), Wiggins (1987), Chensey and Scott (1989), and Stein and Stein (1991) are among the most significant stochastic volatility models that pre-date Steve Heston's model. The Heston model was not the first stochastic volatility model to be introduced to the problem of pricing options, but it has stood out as the most important and now it is used as a benchmark against which many other stochastic volatility models are compared.

To understand why stochastic-volatility models have become so important, we must go back over an event that shook financial markets: the aforementioned stock market crash of October 1987 and its subsequent impact on mathematical models to price options. The aggravation of smiles and smirk in the implied volatility surface that resulted from the crash brought into question the ability of the Black-Scholes model to provide consisting prices in a new regime of volatility skews, and served to highlight the restrictive assumptions underlying the model. The most tenuous of these assumptions is that of continuously compounded stock returns being normally distributed with constant volatility. An abundance of empirical studies since the 1987 crash have shown that an asset's log-return distribution is non-Gaussian, it is characterized by heavy tails and high peaks. There is also empirical evidence and economic arguments that suggest that equity returns and implied volatility are negatively correlated (i.e. leverage effect). This departure from normality assumptions let the Black-Scholes-Merton model with many problems. In contrast, as shown in the following Sections, the Heston's model can imply a number of different distributions.

Furthermore, the Black&Scholes volatility surfaces generated by Heston's model look like empirical implied volatility surfaces. However, the development of more complex models however comes at the cost of increased intricacy. While the Black-Scholes model only have one unknown parameter, stochastic volatility models typically have between four and fifteen parameters that have to be estimated, by means of a Calibration. The calibration of such models is in general far more problematic than calibrating the model proposed by Black&Scholes[29]. The Heston's parameters are able to include skewness and kurtosis, and produce a smile or skew in implied volatilities extracted from option prices generated by the model. Moreover, the Call price in the Heston model is evaluated in closed form, up to an integral that must be evaluated numerically; these are some reasons why the Heston model has become the most popular stochastic volatility model for pricing equity options. The Heston model is the first to exploit characteristic functions in option pricing, by recognizing that the terminal price density need not be known, only its characteristic function. The prices produced by the model are quite parameter sensitive, hence the calibration of the parameters is as drawback for the model itself. What remains unexamined however is the time consistency, or possibly inconsistency, of these models in terms of parameter variations over time. Large variations in daily parameter estimates would reduce the usefulness of these models as high-frequency recalibrations (i.e. daily/weekly). In this work, the author will perform the calibration every week, since all the hedging and pricing is performed weekly; for the calibration the author will use two type of algorithms to increase the trade-off precision and computer intensity.

"In stochastic volatility models, skewness can be induced by allowing correlation between the processes driving the stock price and the process driving its volatility. Alternatively, skewness can arise by introducing jumps into the stochastic process driving the underlying asset price."

3.1 Heston's Model dynamics

The Heston is represented by the bivariate system of stochastic differential equations (SDEs)

$$
\begin{aligned}
dS_t &= \mu S_t dt + \sqrt{v_t} S_t dW_{1,t} \\
dv_t &= \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} dW_{2,t}
\end{aligned}
\tag{39}
$$

[29] The performance of stochastic volatility models, in terms of pricing and hedging performance, has been investigated in a large number of papers, see for example Bakshi, Cao and Chen 1997, Christofersen, Heston and Jacobs 2009 or Shoutens, Simons and Tistaert 2003.

Where $E^P\left[dW_{1,t}, dW_{2,t}\right] = \rho dt$

The first diffusion process is identical to the BS model[30].The main contribution of this model is that the volatility is stochastic. This is achieved by adding another diffusion process that ensures random volatility. The Heston model converges to the BS model when is $v_t$constant[31]. There are some interesting features of this second equation. The first term is similar to the Cox, Ingersoll & Ross' interest rate model, where the interest rates are mean reverting. Heston uses the same concept with mean reverting volatility due to the volatility clustering effect explained in section (2.9.2). The current variance v_t has at some point converged to the long term volatility theta θ. Even though the current variance is high/low today, it must be some underlying dynamic that pulls the volatility to a long run average. The parameter kappa (κ) is the mean reverting parameter and determines how fast the current variance converges to the long run mean. The second term is the volatility of volatility parameter σ and specifies the magnitude of the stochastic shock. It is multiplied by a different Wiener process which allows the volatility of the model to be stochastic. The two Wiener processes are correlated with a parameter Rho ρ, which ensures that volatility and the stock index returns are correlated. By looking at the Heston model dynamics and comparing the number of random sources with the number of the risky traded assets one we might say that the Heston model is an incomplete model, as explained in the focus "Complete and incomplete markets" at the end of this section. Therefore, as consequence, it is not possible to obtain a unique price for any contingent claim using only the underlying asset and a bank account, which is normally the case for complete models such as the BS model. The parameters of the model are:

"The assumption of mean reversion in volatility is consistent with the behavior observed in financial markets. If were not, markets would be characterized by a considerable amount of assets with volatility exploding or going near zero."

- μ the drift of the process for the stock
- $\kappa > 0$ the mean reversion speed for the variance
- $\theta > 0$ the mean reversion level for the variance
- $\sigma > 0$ the volatility of the variance
- $v_0 > 0$ the initial (time zero) level of the variance

[30] The only difference is that variance is a square root process, this ensures that non-negative numbers can enter the process.

[31] Sigma needs to be approximate zero, as a value of zero will disrupt the calculations in the Heston model.

- $\rho \in [-1,1]$ The correlation between the two Brownian motions; and λ the volatility risk parameter. The author define this parameter in the next section and explain why it has set this parameter to zero.

Later in sub-section (3.2) it would be shown that these parameters affect the distribution of the terminal stock price allowing flexibility. The stock price and variance follow the processes in Equation (39) under the historical measure P, also called the physical measure. For pricing purposes, however, as for the Black&Scholes framework, we need the processes for (S_t, v_t) under the risk-neutral measure Q, by applying the Martingale pricing. In the Heston model, this is done by modifying each SDE in Equation (40) separately by an application of Girsanov's theorem. The risk-neutral process for the stock price is:

$$dS_t = rS_t dt + \sqrt{v_t} S_t d\widetilde{W}_{1,t} \tag{40}$$

$$\widetilde{W}_{1,t} = \left(W_{1,t} + \frac{\mu - r}{\sqrt{v_t}} t \right)$$

It is sometimes convenient to express the price process in terms of the log price instead of the price itself. By an application of Ito's lemma, the log price process is:

$$dlnS_t = \left(\mu - \frac{1}{2} \right) dt + \sqrt{v_t} dW_{1,t}$$

The risk-neutral process for the log price is:

$$dlnS_t = \left(r - \frac{1}{2} \right) dt + \sqrt{v_t} d\widetilde{W}_{1,t}{}^{32}$$

The risk-neutral process for the variance is obtained by introducing a function λ (S_t, v_t, t) into the drift of dv_t in Equation (40), as follows:

$$dv_t = [\kappa(\theta - v_t) - \lambda(S_t, v_t, t)dt] + \sigma\sqrt{v_t} d\breve{W}_{2,t} \tag{41}$$

Where:

$$\breve{W}_{2,t} = \left(W_{2,t} + \frac{\lambda(S_t, v_t, t)}{\sigma\sqrt{v_t}} t \right)$$

[32] If the stock pays a continuous dividend yield, q, then in the equations we must replace r by r-q.

The function $\lambda(S, v, t)$ is called the volatility risk premium. As explained in Heston (1993), Breeden's (1979) consumption model yields a premium proportional to the variance, so that $\lambda(S, v, t) = \lambda v_t$, where λ is a constant. Substituting for λv_t in Equation (41), the risk-neutral version of the variance process is:

$$dv_t = \kappa^*(\theta^* - v_t)dt + \sigma\sqrt{v_t}d\breve{W}_{2,t}$$

Where $\kappa^* = \kappa + \lambda$ and $\theta^* = \kappa\theta/(\kappa + \lambda)$ are the risk-neutral parameters of the variance process. Note that, when $\lambda = 0$, we have $\kappa^* = \kappa$ and $\theta^* = \theta$ so that these parameters under the physical and risk-neutral measures are the same. Throughout this thesis, the author set $\lambda = 0$[33]. Indeed, λ is embedded in the risk-neutral parameters κ^* and θ^*. Hence, when we estimate the risk-neutral parameters to price options we do not need to estimate λ[34]. To summarize, the risk-neutral process is:

$$dS_t = rS_t dt + \sqrt{v_t}S_t d\breve{W}_{1,t}$$
$$dv_t = \kappa^*(\theta^* - v_t)dt + \sigma\sqrt{v_t}d\breve{W}_{2,t}$$

The process for the variance has always to be positive, in order to ensure this the following condition mush therefore holds:

$$2\kappa\theta > \sigma^2 \tag{42}$$

This condition ensures that stochastic shocks are not large enough to create negative variance as the mean reverting parameter κ and long term variance θ pulls the volatility back. This condition is known as the Feller condition.

Focus: Complete and incomplete Markets

Complete markets means that every derivative security is attainable or replicable (i.e hedgeable). In particular, this means that every security can be priced uniquely. It is worth mentioning that the Black-Scholes model is a complete model and so completeness follows from the fact that the EMM in Equation (9) is unique: the only possible choice for eta was $\eta_t = (\mu - r)/\sigma$. In particular, this come by the fact that the first theorem of finance: a model

[33] Since investors are always assumed to be risk neutral and the stochastic risk parameter is set to zero.
[34] Indeed in the calibration the author will use five parameters instead of six (Since $\lambda = 0$).

is arbitrage free if exists a unique martingale measure Q. If we assume that the model is arbitrage free, surely exists a measure Q. The second step is looking if this measure Q is unique. The second theorem of finance say that if we don't assume a model to be arbitrage free, a model is complete if and only if the martingale measure Q in unique. In a complete market the price of derivatives is uniquely determined from arbitrage free, and every derivative is replicable with a portfolio of financial instruments. An example of complete and arbitrage free market is the Black&Scholes model. To state that, we call in some rules, that allow to recognize if a market satisfy the arbitrage principle and completeness. Denote by R the number of state variables and with M the number of underlying tradeable in the market:

- The market is arbitrage free if M > R
- The market is complete if M < R
- The market is both arbitrage free and complete if M = R

As Said the Black&Scholes market is both arbitrage free and complete, indeed we have that M = 1 e R = 1. In particular, In the Black-Scholes-Merton model, a contingent claim is dependent on one or more tradable assets. The randomness in the option value is solely due to the randomness of these assets. Taking a look at the Heston model and comparing the number of random sources (two standard Brownian motions) with the number of the risky traded assets (only the underlying spot since volatility is not traded) one can easily see that the model is an incomplete model. Therefore, it is not possible to obtain an unique price for any contingent claim using only the underlying asset and a bank account.

3.2 Effect of the Heston parameters

3.2.1 Effect of Correlation parameter ρ

Rho (ρ), which can be interpreted as the correlation between the log-returns and volatility of the asset, affects the tails of the return distribution. In particular, if $\rho > 0$, then volatility will move upwards as the asset price/return increases. This will enlarge the right tail and restrict the left tail of the distribution. On the other way, if *rho* < 0, then volatility will rise when the asset price (or return) decreases, thus enlarging the left tail and squeezing the right tail of the distribution. Figure (18) shows this effect for different values *rho*. As explained, since this work is considering Equity options, we will expect a negative rho, emphasizing the fact that equity returns and its related volatility are negatively correlated.

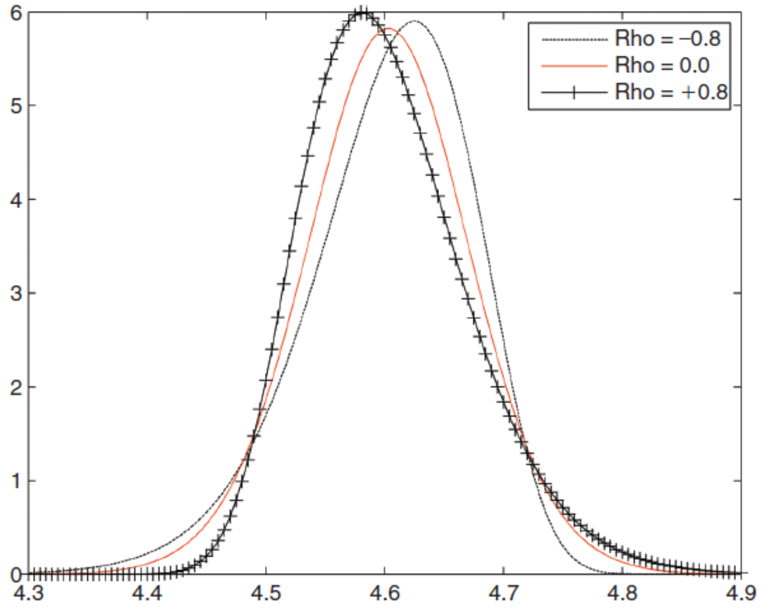


Figure 18: Effect of correlation between Price and Volatility on the return density function. Source: The Heston model and its application in Matlab, Wiley, 2013.

The effect of changing the skewness of the distribution also impacts on the shape of the implied volatility surface. Hence, *rho* also affects this. Figures (19), (20) and (21) show the effect of varying *rho*. As said in the Chapter 2, Section (2.8.3) different underlying assets have different volatility smile graph, we have also said that the model can imply a variety of volatility smile and consequently different volatility surfaces and hence solve another shortcoming of the Black&Scholes model.

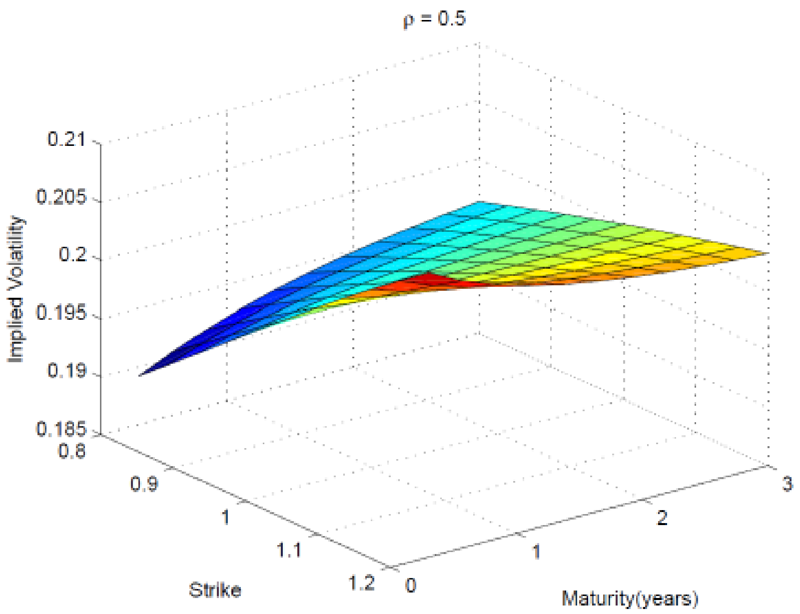


Figure 19: Implied volatility surface, $\rho = 0.5$, $\kappa = 2$, $\theta = 0.04$, $\sigma = 0.1$, $v0 = 0.04$, $r = 1\%$, $S0 = 1$, strikes: 0.8 – 1.2, maturities: 0.5 - 3 years. Source Moodley, N. The Heston Model: A Practical Approach. Faculty of Science, University of the Witwatersrand, Johannesburg, South Africa, 2005.

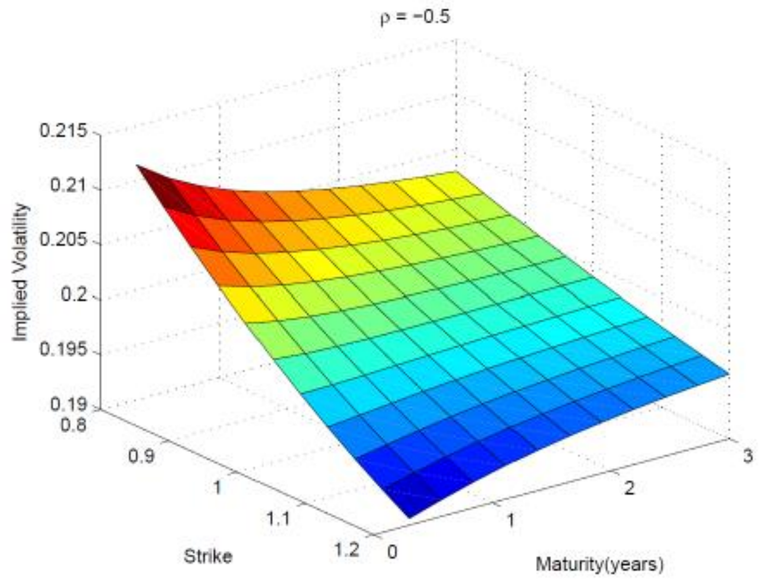


Figure 20: Implied volatility surface, ρ = - 0.5, κ = 2, θ = 0.04, σ = 0.1, v0 = 0.04, r = 1%, S0 = 1, strikes: 0.8 – 1.2, maturities: 0.5 - 3 years. Source Moodley, N. The Heston Model: A Practical Approach. Faculty of Science, University of the Witwatersrand, Johannesburg, South Africa, 2005.

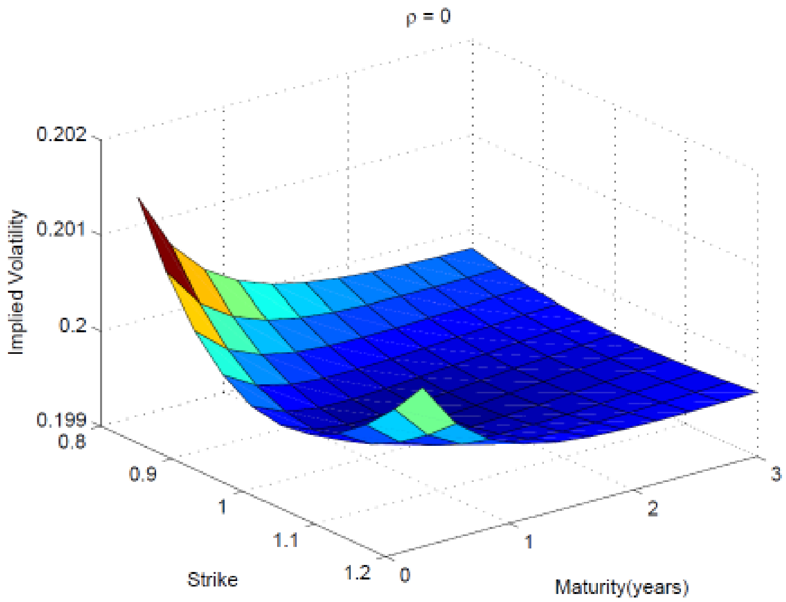


Figure 21: Implied volatility surface, ρ = 0, κ = 2, θ = 0.04, σ = 0.1, v0 = 0.04, r = 1%, S0 = 1, strikes: 0.8 – 1.2, maturities: 0.5 - 3 years. Source Moodley, N. The Heston Model: A Practical Approach. Faculty of Science, University of the Witwatersrand, Johannesburg, South Africa, 2005.

3.2.2 Effect of Volatility of Variance parameter σ

The volatility of variance parameter σ controls the kurtosis of the distribution. When *sigma* is 0 the volatility is deterministic and hence the log-returns will be normally distributed.

Increasing *sigma* will then increase the kurtosis only, creating heavy tails on both sides. Figure (22) shows the effect of varying *sigma*.

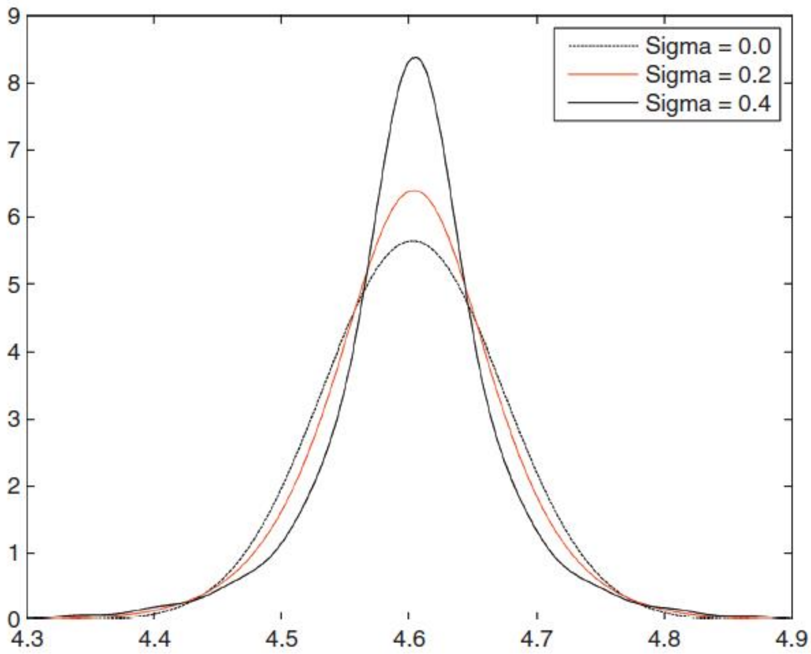


Figure 22: Effect of Volatility of Variance on the return density function. Source: The Heston model and its application in Matlab, Wiley, 2013.

Again, like the parameter rho, the effect of changing the kurtosis of the distribution impacts on the implied volatility. Figures (23), (24) and (25) show how *sigma* affects the curve of the smile/skew.

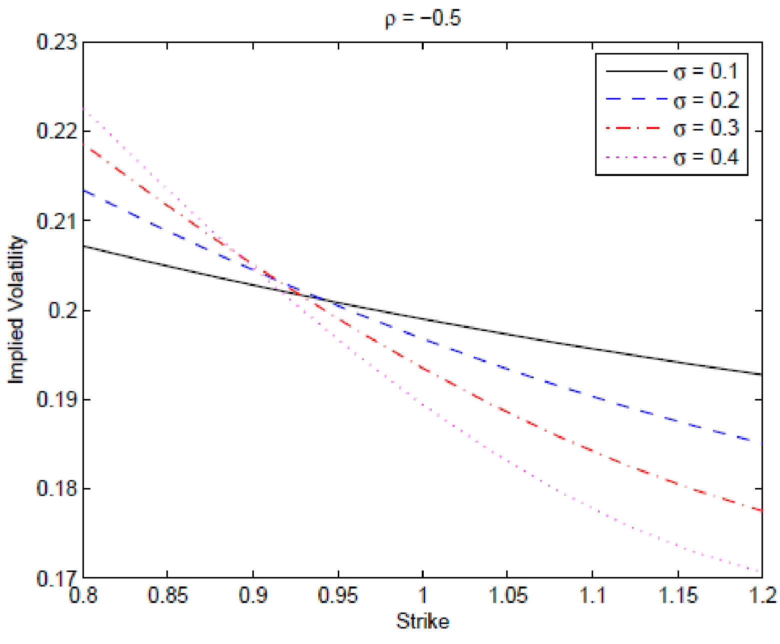


Figure 23: Effect of Volatility of Variance on the Implied Volatility with a negative rho. Source: The Heston model and its application in Matlab, Wiley, 2013

Higher *sigma* makes the skew/smile more prominent. This makes sense relative to the leverage effect. Higher *sigma* means that the volatility is more volatile. This means that the market has a greater chance of extreme movements. So, writers of puts must charge more and those of calls, less, for a given strike.

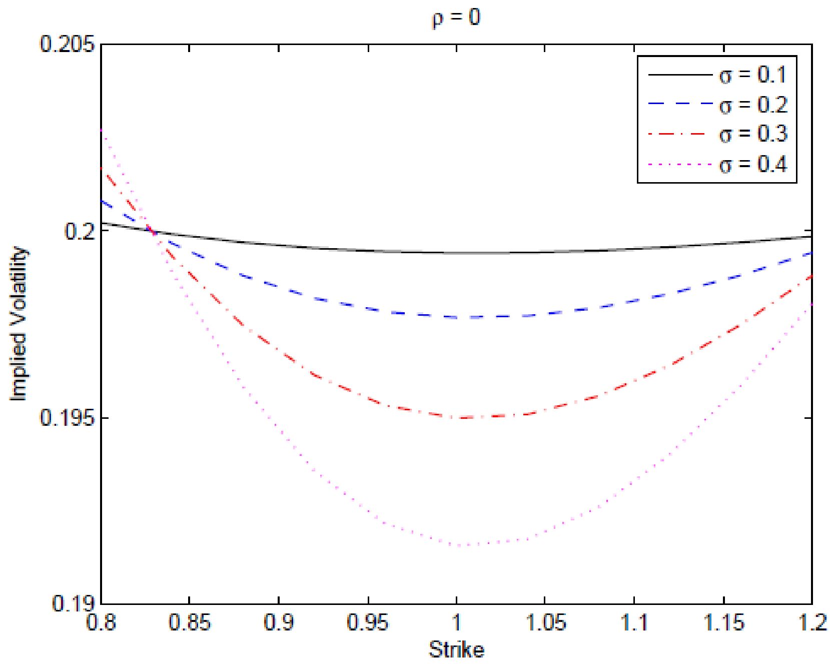


Figure 24: Effect of Volatility of Variance on the Implied Volatility with a zero rho. Source: The Heston model and its application in Matlab, Wiley, 2013

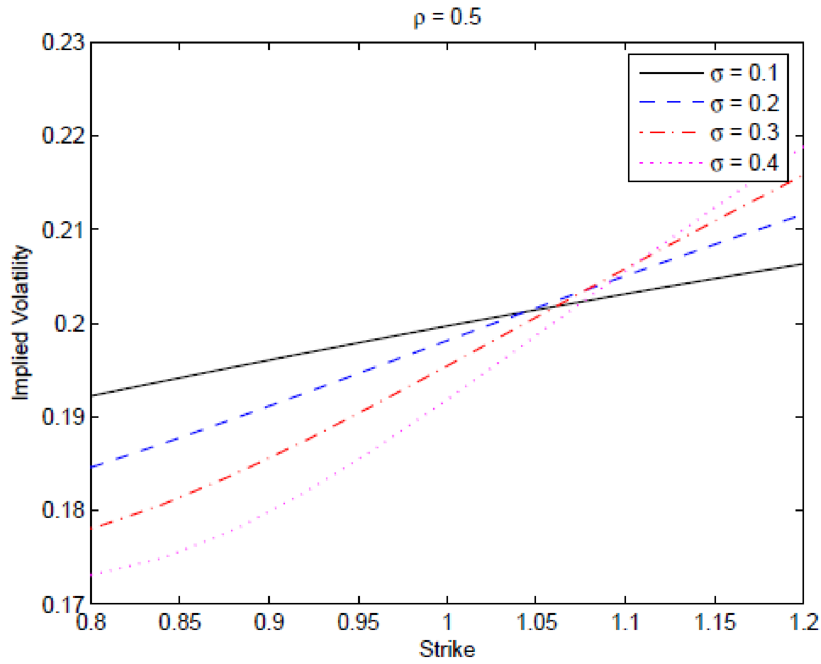


Figure 25: Effect of Volatility of Variance on the Implied Volatility with a positive rho. Source: The Heston model and its application in Matlab, Wiley, 2013

A high positive kurtosis increases the probability of extreme movements in both directions compared to the normal distribution assumed in Black&Scholes. When there is a higher probability of tail events, the option prices in that range are likely to increase (higher implied volatilities) and thereby create a volatility smile. In contrast, if $\sigma \approx 0$ the smile effect disappears and the Heston model converges to the BS model.

3.2.3 Effect of reversion parameter Kappa

Kappa, the mean reversion parameter, can be interpreted as representing the degree of volatility clustering, it defines the how fast the variance process reverting to its long term mean, and it can be found in the real market. This is something that has been observed in the market as shown in Figure (14) in sub-section (2.9.2); large price variations are more likely to be followed by large price variations. The aforementioned features of this model enables it to produce a barrage of distributions. It provides a framework to price a variety of options that is closer to reality.

"The main advantage of the Heston model, however, is the closed-form solution for European Call options, making it more tractable and easier to implement than other stochastic volatility models. In the next section, we derive the general valuation equation and apply it to the Heston model in order to obtain a pricing formula for European calls."

In the following section, it is shown that the call price in the Heston model can be expressed as the sum of two terms that each contains an in-the money probability, but obtained under a separate measure.[35]

3.3 The European Call Price

3.3.1 Risk-neutralized approach with the Heston Model

For stochastic volatility model, a risk-neutralized method, also called an Equivalent Martingale Measure (EMM) or Martingale pricing, is widely used in the pricing of financial derivatives. It is based on the Girsanov's theorem of asset pricing. The basic way is to set up a new model that replaces the drift by the risk-free interest rate, as the author has done for the Black&Scholes framework in Chapter 2, Section (2.4), and transforms the drift in the volatility Equation. The Call price can be expressed by recalling Equation (10):

[35] A result demonstrated by Bakshi and Madan, 2000.

$$Option\ price = E_t^Q[e^{r(T-t)}Payoff(T)]$$

The Call price in the Heston model can be expressed in a manner which resembles the Call price in the Black-Scholes model. Authors sometimes refer to this characterization of the call price as “Black-Scholes–like” or “`a la Black-Scholes.” The time-*t* price of a European call on a non-dividend paying stock with spot price S_t, when the strike is K and the time to maturity is $\tau = T - t$, is the discounted expected value of the payoff under the risk-neutral measure Q.

$$\begin{aligned} C &= e^{-rT}E^Q[(S_T - K)]^+ \\ &= e^{-rT}E^Q[(S_T - K)1_{S_T>K} \\ &= e^{-rT}E^Q[S_T 1_{S_T>K}] - Ke^{-rT}E^Q[1_{S_T>K}] \\ &= S_t P_1 - Ke^{-rT}P_2 \end{aligned} \tag{43}$$

Where 1 is the indicator function. The price of a Put could be recovered by the Put-Call Parity[36]. The last line of is the “Black-Scholes–like” call price formula, with P_1 replacing N (d_1), and P_2 replacing N (d_2) in the Black- Scholes call price in this section. The quantities P_1 and P_2 each represent the probability of the call expiring in-the-money. When the characteristic functions are known, each in-the-money probability P_j for j=1, 2 can be recovered from the characteristic function via the Gil-Pelaez inversion theorem. In order to explain how to recover the distribution function the author need to define the notion of characteristic function and mention, at least, the framework of the Fourier transform and their link between the distribution function.

> “The most difficult part is obtaining P_1 and P_2, by using not-build Matlab function: HestonP, HestonPIntegrand and Hestonf that are available in the appendix inside the HestonCallQuadl.m function”

3.4 Characteristic function, Fourier transform and other headaches

We know that, in this type of framework, and also in reality, we don’t have a close form for the distribution function, but in order to perform Equation (43) for pricing the options, we need a way to recover the distribution. The calculation of the distribution function of random variables

[36] The put-call parity describes the important relationship between European call and put options, which can be used to derive a closed-form expression for the price of a European put option.

is required. A very interesting fact is that even if the random variable of interest does not have an analytical expression, the characteristic function of this random variable always exists, and more, there is a one to one relationship between the probability density and a characteristic function. If the characteristic function is known in closed form, is tractable numerically, or given by empirical data, then we can compute the distribution function by using the Inversion theorem. Let's define at first the characteristic function:

$$\phi_x(u) = E[e^{iux}]$$

of a real valued random variable x is defined for arbitrary real numbers u as the expectation of the complex valued transformation e^{iux}, where $i = \sqrt{-1}$ is the imaginary unit. If $f_x(x)$ is the probability function (PDF) of the random variable then the integral.

$$\phi_x(u) = E\left[e^{iux}\right] = \int_{-\infty}^{\infty} e^{iux} f_x(x) dx$$

Defines the expected value and is by definition the Fourier transform of the density function $f_x(x)$ denoted by $\mathcal{F}[f_x(x)]$. At a given u $\phi_x(u)$, is a single random variable and for $-\infty < u < \infty$ we have a stochastic process. If a characteristic function is absolutely integrable over the real line $-\infty < u < \infty$ then it has an absolutely continuous probability distribution. This is said to be integrable in the Lebesgue sense and belongs to $L^1(R)$.

So, an essential property of characteristic functions is their one to one relationship with distribution functions, and this is fundamental for pricing purposes. Every random variable possesses a unique characteristic function and the characteristic function indeed characterizes the distribution uniquely[37]. The Inversion theorem is the Fundamental Theorem of the Theory of Characteristic Functions since it links the characteristic function back to its probability distribution via an inverse Fourier transform. The inversion algorithms are based on the following particular form of the Gil-Pelaez inversion integral for cumulative distribution function (CDF) $\int_{-\infty}^{X} f_x(x)\, dx$

$$F_X(X) = \mathrm{P}(X \leq x) = \frac{1}{2} - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-iux}\phi_x(u)}{iu} du$$

[37] Waller, 1995.

We see that the recovered distribution function is expressed as an integral in terms of the characteristic function. Since we need to find the probability "In-the-money", we use $F_X^C(X) = \mathrm{P}(X > x) = 1 - F_X(X)$ to obtain the complementary CDF (cCDF). Taking our variables in consideration and by applying some properties of the complex plane that are not discussed here, we can write the the probability of the call expiring in-the-money as:

$$F_X^C(X) = P_j = \Pr(lnS_t > lnK) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left[\frac{e^{-i\phi lnK} f_j(\phi; x, v)}{i\phi}\right] d\phi \qquad (44)$$

Where $f_j(\phi; x, v) = \phi_x(u)$. Heston (1993) postulates that the characteristic functions for the logarithm of the terminal stock price, $x_T = \ln S_T$, are of the log linear form:

$$f_j(\phi; x, v) = \exp(C_j(\tau, \phi) + D_j(\tau, \phi)v_t + i\phi x_i$$

$$D_j(\tau, \phi) = \frac{b_j - \rho\sigma i\phi + d_j}{\sigma^2}\left(\frac{1 - e^{d_j\tau}}{1 - g_j e^{d_j\tau}}\right)$$

$$C_j(\tau, \phi) = ri\phi\tau + \frac{a}{\sigma^2}\left[\,(b_j - \rho\sigma i\phi + d_j)\tau - 2ln\left(\frac{1 - g_j e^{d_j\tau}}{1 - g_j}\right)\right.$$

$$d_j = \sqrt{(\rho\sigma i\phi + b_j)^2 - \sigma^2(2u_j i\phi - \phi^2)}$$

$$g_j = \frac{b_j - \rho\sigma i\phi + d_j}{b_j - \rho\sigma i\phi - d_j}$$

Where

$$u_1 = \frac{1}{2} \quad u_2 = -\frac{1}{2} \quad a = \kappa\theta \quad b_1 = \kappa - \rho\sigma \quad b_2 = \kappa$$

Some authors refer to the Call price as being in ''semi-closed'' form because of the numerical integration required to obtain (P_1) and (P_2). It is important to notice that the Black&Scholes model also requires numerical integration, to obtain (d_1) and (d_2). The difference is that programming languages often have built-in routines for calculating the standard normal cumulative distribution function, whereas the Heston probabilities are not built-in and must be obtained using numerical integration. This integral cannot be evaluated exactly, but can be approximated with reasonable accuracy by using some numerical integration technique, such as Simpson's rule or Gauss Lobatto integration, stuff explained in the next section.

3.5 Numerical Integration: Quadrature Rules

In order to price the Call option, we need to solve Equation (43), as said the tricky part is the evaluation of the numerical integral, i.e. Equation (44). In order to solve the integral, two techniques are briefly discussed together with their implementation in MATLAB®.

3.5.1 Adaptive Simpson's Rule

The MATLAB® function `quad(@fun,a,b)` uses an *Adaptive Simpson's Rule* to numerically integrate a function (@fun) over [a,b]. It produces a result that has an error less than 10^{-6} or a user defined tolerance level which is prescribed by a fourth argument.

3.5.2 Adaptive Gaussian Quadrature Rules

The MATLAB® function `quadl(@fun,a,b)` implements an adaptive *Gauss Lobatto* quadrature rule on the function @fun over the interval [*a; b*]. It's defined as the Gaussian quadrature.

3.5.3 Solving the Heston Integral

In order to evaluate (43) the author need to compute the integral in (44), recalling is:

$$P_j = \Pr(lnS_t > lnK) = \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} Re\left[\frac{e^{-i\phi lnK} f_j(\phi; x, v)}{i\phi}\right] d\phi$$

"Heston (1993) has mentioned that the integrand in equation (44) is a smooth function that decays rapidly and presents no difficulty. Thereby, the author chooses the set of integration between 0 and 100"

For j = 1, 2. This can be done in MATLAB® with the 2 functions, `quad(@fun a,b)` and `quadl(@fun,a,b)` discussed above. A problem arises as Equation (44) is an improper integral and the argument b cannot be specified as 'infinity' i.e., `quadl(@fun,a,b)` evaluates only proper integrals. So, for sufficiently large b, the integral of (44) can be evaluated with the required accuracy. The author has chosen b = 100. MATLAB® code to price a European Call option using Heston's model via *Adaptive Gauss Lobatto* integration/*Simpson's Rule* can be found in the appendix under the HestonCallQuadl.m function. As said, the Put option price could be recovered by the Put-Call parity:

$$Put = Call + Ke^{-rT} - S_t \tag{44bis}$$

3.5.4 Comparison of Quadrature Rules

The purpose of this sub-section is to undestrand which quadrature rule to use[38]. MATLAB® defines the functions, quad and quadl, as low order and high order quadrature rules. One would therefore expect quadl to be superior. Some studies[39] said that for extremely small tolerance levels quadl outperforms quad in:

- Efficiency: as measured by the number of recursive steps required to compute an answer within a certain tolerance level.
- Reliability: as measured by the extent to which the required tolerance is achieved.

For large tolerance levels, quad is more efficient (faster) than quadl, but less reliable. Keeping in mind that we're pricing options, it is require an extremely low tolerance level. Hence quadl would be better suited for our purposes, so in this thesis, the integrand in Equation (44) was calculated using the MATLAB® function quadl, embedded in the function HestonCallQuadl.m.

3.5.5 Integration problems

This sub-section highlights the problem that one's might encounter in solving the integral and is done for completeness. In certain instances, the integrand for P_j

$$Re\left[\frac{e^{-i\phi lnK} f_j(\phi; x, v)}{i\phi}\right]$$

[38] For a complete discussion refer to Gander & Gautschi, 1998.
[39] Gander & Gautschi, 1998.

“It is well-known that the integrand for the call price can sometimes show high oscillation, can dampen very slowly along the integration axis, and can show discontinuities. All of these problems can introduce inaccuracies in numerical integration.”

Is well-behaved in that it poses no difficulties in numerical integration. This corresponds to an integrand that does not oscillate much, that dampens quickly so that a large upper limit in the numerical integration is not required, and that does not contain portions that are excessively steep. In some instances, the integrand is well-behaved and the integration poses no numerical problems. In other cases, however, the integrand is not well-behaved so numerical integration can be problematic. In particular we might have 3 kinds of problems related to the integrand:

- The first problem is that the integrand is not defined at the point $\varphi = 0$, even though the integration range is $[0, \infty)$. This implies that the integration must begin at a very small point close to zero. In order to avoid inaccuracies due to the removal of the origin, the integrand must not be too steep there.

- The second problem is that the integrand may contain discontinuities. To illustrate, Figure (26) plots two integrands for in the range $\varphi \in (0, 10]$. The first integrand has a maturity of $\tau = 3$ years and $\sigma = 0.75$, and the second has $\tau = 1$ year and $\sigma = 0.09$. Both integrands use $\kappa = 10$, $\theta = v_0 = 0.05$, $\rho = -0.9$, $r = 0$, along with spot $S_0 = 100$ and strike $K = 100$. The first integrand (red line) is smooth and shows no particular numerical instability. The second integrand (black line), on the other hand, has discontinuities near the points $\varphi = 1.7$ and $\varphi = 5$, and it is steep near 1.7[40].

[40] Simple modifications of the integrand are made by Albrecher et al. 2007 which is effective at eliminating these discontinuities but are not considered in this thesis.

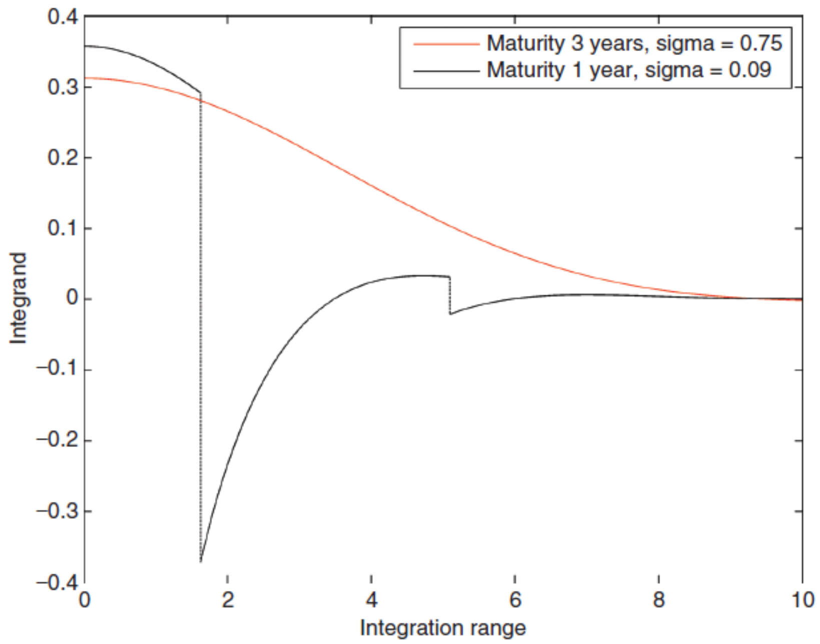


Figure 26: Discontinuity in the Heston integral. Source: The Heston model and its application in Matlab, Wiley, 2013.

- Finally, the third problem that can arise is that of an integrand that oscillates wildly. In Figure (27), the first integrand has a maturity of τ = 1/52 years and uses σ = 0.175, θ = v_0= 0.01, and a spot S_0= 7. The second has τ = 1 year and uses σ = 0.09, θ = v_0= 0.07, and a spot S_0= 10. Both use $\rho = -0.9$, κ = 10, r = 0, and a strike of K = 10. The plots are over the integration range $\varphi \in (0, 100]$. The first integrand (black line) shows high oscillation, which is still not damped at φ = 100. This implies that the numerical integral needs to extend much further beyond φ = 100 to converge.

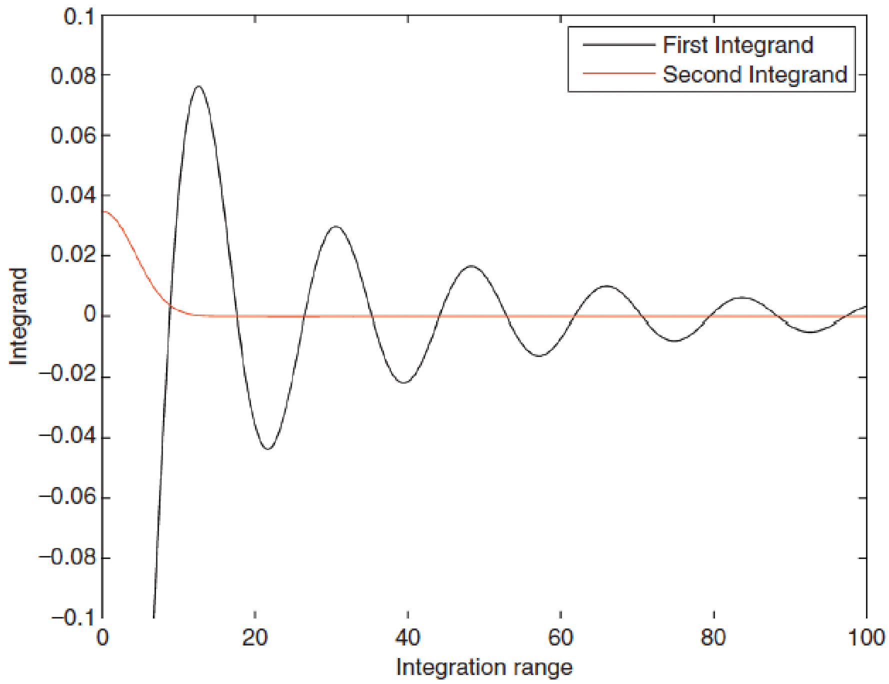


Figure 27: Oscillation in the Heston integral. Source: The Heston model and its application in Matlab, Wiley, 2013.

Moreover, the integrand is very steep near the origin, which requires a very fine grid for the numerical integral. The second integrand (red line) is well-behaved and would pose no numerical difficulties. Indeed, it does not oscillate, is not steep anywhere, and rapidly dampens to zero, starting at around $\varphi = 10$. High oscillation of the integrand is usually associated with short-maturity options. For a complete discussion on the Albrecher Formulation and the Little Heston trap take a look at "The Heston model and its applications in MATLAB®". To illustrate in a whole graph, look at the plot. The integrand using the settings $S = 7$, $K = 10$, and $r = q = 0$, with parameter values $\kappa = 10$, $\theta = v_0 = 0.07$, $\sigma = 0.3$, and $\rho = -0.9$. The plot uses the domain $-50 < \varphi < 50$ over maturities running from 1 week to 3 months. This plot appears in Figure (28). The integrand has a discontinuity at $\varphi = 0$, but this does not show up in the figure. The plot indicates an integrand that has a fair amount of oscillation, especially at short maturities, and that is steep near the origin.

"The ''Little Trap'' formulation of Albrecher et al. (2007) can remedy many of the problems with the numerical integration that arise when the integrand is discontinuous."

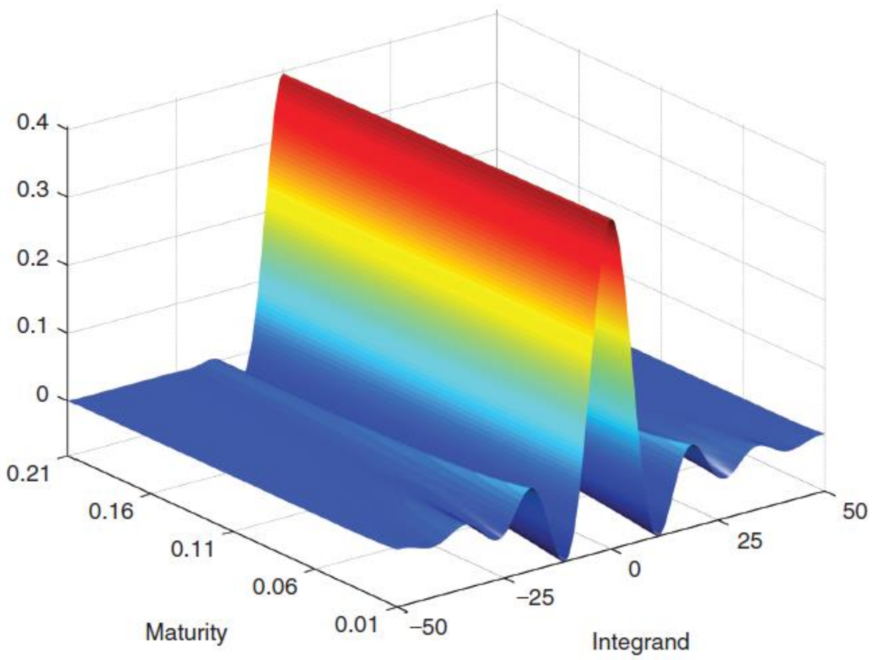


Figure 28: Heston integrand and maturity. Source: The Heston model and its application in Matlab, Wiley, 2013.

3.6 Advantages and Disadvantages of the Heston Model

Both academia and practitioners have recognized the importance of the Heston Model, nevertheless, it is not a model without any drawbacks. A brief summary of its advantages and disadvantages will be presented in this section.

Advantages of the Heston model:

- Semi-closed form solution for European options and thus the model allows a fast calibration to given market data.
- Unlike the Black-Scholes model, the price dynamics in the Heston model allows for non-lognormal probability distribution (high peak, fat tails).
- The model fits the implied volatility surface of option prices in the market, when the maturity is not too small.
- The volatility is mean reverting.
- It takes into account leverage effect, and in addition, it permits the correlation between the asset and the volatility to be changed.

Disadvantages of the Heston model:

- Hard to find proper parameters to calibrate the stochastic model.
- The prices produced by the Heston model are sensitive to the parameters, so the fitness of the model depends on the calibration[41]. In other words, the price to pay for more realistic models is the increased complexity of model calibration.
- It cannot capture the skew at short maturity as the one given by the market[42].

Some drawbacks concerns the importance of calibration in the Heston model. Often, the estimation method for the parameters becomes as crucial as the model itself.

Let’s introduce it in the next Section.

---

[41] Mikhailov & Nogel, 2003.

[42] Mikhailov & Nogel, 2003.

3.7 Calibration

3.7.1 Why we need calibration?

So far we have assumed that the Heston's parameters are given, but it is necessary to specify that in order to pricing the options we need to know the parameters that describe the underlying and the volatility dynamics. So in order to solve a direct problem i.e. pricing the derivatives, we need the inverse of the solution of the problem i.e. the calibration of the parameters.

"Once a model has been chosen for its realistic features, one has to calibrate it. This calibration must be robust and stable and should not be too computer intensive."

In other words, the pricing problem concerns the evaluation of option price given the model parameters, the calibration is interested in the parameters estimation. For both problems, is necessary assume a robust model that represent the evolution of the underlying. In this thesis, calibration is an optimization problem, since we want to estimate the parameters that minimize the distance between two variables, one expressed by the market, the other one by the model.

"The aim of calibration is to find the parameters of the model under the measure Q that better approximate real prices observed under the measure P"

3.7.2 The Calibration problem

The calibration problem has a slight simplification when we price options under a Martingale pricing. The evaluation under an EMM effectively reduces the number of estimated parameters from six to five as explained in Chapter 2, by setting 0 for the parameter that concerns lamba (λ). The following Equation must, therefore, hold:

$$Option\ Value^{P}(\kappa, \theta, \sigma, V_0, \rho, \lambda) = Option\ Value^{Q}(\kappa^*, \theta^*, \sigma, V_0, \rho, 0)$$

The five parameters, needed to be estimated in the Heston model. The change for each parameter will bring a huge impact for the correctness, as said in the drawbacks for the Heston model in the previous section, so the estimation of parameters becomes very important. A variety of methods can be chosen. For instance, one can observe the real market data, and use statistic tool to fit data in the Heston model[43], but recently studies have shown that the implied parameters (i.e. those parameters that produce the correct vanilla option prices) and their time-series estimate counterparts are different[44], so one cannot just use empirical estimates for the parameters.

Monte Carlo simulation is another famous method to do the calibration, by means of a simulation[45]. What the author selected is another common used method, in a way that is called an inverse problem (or the loss function approach). The most popular approach to solving this inverse problem is to minimize the error or discrepancy using a loss function.

"The most popular way to estimate the parameters of the Heston model is with loss functions. This method uses the error between quoted market prices and model prices, or between market and model implied volatilities."

3.7.3 The Choice of Loss Function

This method uses the error between quoted market prices and model prices, or between market and model implied volatilities. The parameter estimates are those values which minimize the value of the loss function, so that the model prices or implied volatilities are as close as possible to their market counterparts. A constrained minimization algorithm must be used in this regard so that the constraints on the parameters:

$$\kappa > 0, \quad \rho \in [-1, +1], \quad \sigma > 0, \quad \theta > 0, \quad V_0 > 0$$

Are respected. To this basis set of constraints, we need to add another one in order to ensure that the process for the variance is positive, as explained in section (3.4) we need the feller condition to be satisfied:

$$2\kappa\theta > \sigma^2$$

Throughout this chapter, the Heston parameters are represented as the vector:

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[43] See Ait-Sahila, Kimmel, 2005.
[44] See Bakshi, Cao & Chen 1997.
[45] See Alexander V.H, 2010.

$$\Omega = (\sigma, \rho, \theta, \kappa, V_0)$$

And their corresponding estimates, as $\tilde{\Omega}$.

There are many possible ways to define a loss function, but they usually fall into one of two categories: loss function based on prices, and those based on implied volatilities.

Suppose we have a set of N_T maturities τ_i ($t = 1, \ldots, N_T$) and a set of N_K strikes K_k ($k = 1, \ldots, N_K$). For each maturity-strike combination (τ_t, K_k), for the Call option, we have a market price C_{tk} and a corresponding model price $C_{tk}(\breve{\Omega})$ generated by the Heston model. Attached to each option is an optional weight w_{tk}.

3.7.3.1 Loss function based on prices

The first category of loss functions are those that minimize the error between quoted and model prices. The error is usually defined as the squared difference between the quoted and model prices, or the absolute value of the difference; relative errors can also be used. For example, parameter estimates obtained using the mean error sum of squares (MSE) loss function are obtained by minimizing:

$$MSE = \frac{1}{N} \sum_{t,k}^{N} w_{tk} \left(C_{tk} - C_{tk}(\breve{\Omega})\right)^2 \tag{45}$$

With respect to $\tilde{\Omega}$, where N is the number of quotes. The relative mean error sum of Squares (RMSE) parameter estimates are obtained with the loss function:

$$RMSE = \frac{1}{N} \sum_{t,k}^{N} w_i \left(\frac{C_{tk} - C_{tk}(\breve{\Omega})}{C_{tk}}\right)^2 \tag{46}$$

One well-known disadvantage of the MSE loss function is that short maturity, deep out-of-the money options with very little value contribute little to the sum in (45). Hence, the optimization will tend to fit long maturity, in-the-money options well, at the detriment of the other options. One remedy is to use in-the-money options only, so that, in (45), Call options are used for strikes less than the spot price, and Put options are used for strikes greater than the spot price. The other remedy is to use the RMSE loss function in (46). The problem with RMSE, however, is that the opposite effect occurs. Indeed, because of the presence of C_{tk} in the denominator, options with low market value will over-contribute to the sum in (46). The over and under-contribution, however, can be mitigated by assigning weights w_{tk} to the individual terms in the objective function, although the choice of the weights is usually subjective and it is discussed later in sub-section (3.8.4).

3.7.3.2 Loss function based on Volatilities

> "Therefore, by calibrating these parameters values, we seek to obtain an evolution for the underlying asset that is consistent with the current prices of plain vanilla options."

The second category of loss functions are those that minimize the error between quoted and model implied volatilities. Again, the error is usually defined as the squared difference, absolute difference, or relative difference, between quoted and model implied volatilities. This category of loss function is sensible, since options are often quoted in terms of implied volatility, and since the fit of model is often assessed by comparing quoted and model implied volatilities. Hence, for example, the implied volatility mean error sum of squares (IVMSE) parameter estimates are based on the loss function:

$$IVMSE = \frac{1}{N}\sum_{t,k}^{N} w_{tk}\,(IV_{tk} - IV_{tk}(\breve{\Omega}))^2 \qquad (47)$$

Where IV_{tk} and $IV_{tk}(\tilde{\Omega})$ are the quoted and model implied volatilities, respectively. The main disadvantage of Equation (47) is that it is numerically intensive. The most popular approach to solving this inverse problem is to minimize the error or discrepancy between model prices and market prices, and is the loss function that the author will use in this thesis (i.e. MSE, Equation (45)).

This usually turns out to be a non-linear least-squares optimization problem. More specifically, the squared differences between vanilla option market prices and that of the model are minimized over the parameter space, i.e., evaluating:

$$min_{\breve{\Omega}}\, S(\theta) = min_{\breve{\Omega}} \sum_{i=1}^{N} \frac{1}{N} w_i [C_i^{\breve{\Omega}}(K_i\,, T_i) - C_i^{M}(K_i, T_i)]^2 \qquad (48)$$

“The author want to minimize the error difference between the Heston model prices, and real market price, which can be easily found from Internet.”

Where Ω is the vector of the five parameters values, $C_i^{\Omega}(K_i T_i)$ and $C_i^M(K_i, T_i)$ are the i_{th} option prices from the model and market, respectively, with strike K_i and maturity T_i. N is the number of options used for calibration, and the w_i's are weights. The question now arises as to what market prices to use in this calibration process, as for any given option there is an ask price and a bid price. This might seem an issue but it actually permits flexibility in the calibration. The author will use the mid-price of the option but accept a parameter set, given that:

$$\sum_{i=1}^{N} \frac{1}{N} w_i [C_i^{\Omega}(K_i, T_i) - C_i^M(K_i, T_i)]^2| \leq \sum_{i=1}^{N} \frac{1}{N} w_i [C_i^{\Omega}(bid_i - ask_i)^2 \qquad \text{(48bis)}$$

Where bid_i and ask_i are the bid/ask prices of the i^{th} option. We do not “order” the model to match the mid-prices precisely, but fall, at least on average, within the bid-offer spread. We should bear in mind that the modeling process should produce the required estimates within a certain tolerance level. Accuracy beyond this could be spurious[46]. The minimization mentioned above is not as trivial as it would seem. In general, the loss function $S(\Omega)$ is neither convex nor does it have any particular structure. This poses some complications:

- Finding the minimum of $S(\Omega)$ is not as simple as finding those parameter values that make the gradient of $S(\Omega)$ zero. This means that a gradient based optimization method, sometimes will prove to be futile[47].
- Hence, finding a global minimum is difficult and depends on the optimization method used.
- Unique solutions to (48) need not necessarily exist, in which case only local minima can be found. This has some implications regarding the stationarity of parameter values which are important in these types of models. This is discussed later.

[46] The same procedure is applied for the Put option, but is not reported.
[47] For this reason the author will use two approaches, a Gradient based optimization and a Global stochastic optimization.

The last two points make this an ill-posed problem. Figure (29) plots $S(\Omega)$ as a function of *rho* and sigma. The graph presents a graphical idea of the nature of $S(\Omega)$ as a function of two of its parameters, It is easy to see that gradient based optimizers will struggle to find a global minimum. Notice, also, the number of points that are non-differentiable. This poses a further problem for gradient based optimizers. It is important to remember that $S(\Omega)$ is 5-dimensional and as such could be even nastier in its 'true' form.

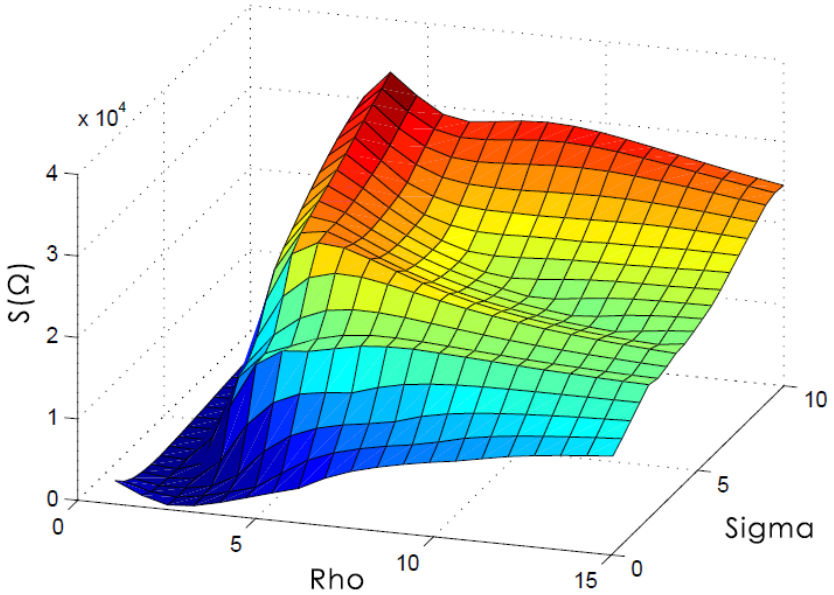

Figure 29: A generic representation of $S(\Omega)$ .Source: Moodley N., The Heston Model: A Practical Approach. Bachelor of Science Honours, Programme in Advanced Mathematics of Finance, University of Witwatersrand (2005).

Further complications arise from the solution, that it does not depends on data and could generate instability in the research of the minimum. We could say that the problem could present a lot of local minima, if they exist.

3.7.4 The choice of Weights

Earlier research[48] has shown that the choice of weighting w_i has a large influence on the error functional, and therefore on the parameter estimates. Two common methods are to either use the bid-ask spread of the options or to choose weights according to the number of options within different maturity categories. Using:

$$w_i = \frac{1}{bid_i - ask_i} \tag{49}$$

[48] Mikhailov and Nögel, 2003.

The bid price represents the highest price a buyer is willing to pay for a security, while the ask price is the lowest offered price a seller is willing to receive for the security. Bid-ask prices are therefore “quotes” that buyer and seller are willing to do a deal, however these recorded prices are not the actual transaction price of the security and may not seem a good proxy for the market price. Nonetheless, the true market price has to be in the bid-ask spread as no one are willing to sell below the bid price or buy higher than the ask price. On the other hand, “last price” represents the last transaction price recorded before the market close. The advantage is that the recorded price represents a fair value of the option at the time of the transaction. However, the time of the last transaction may differ from the “true” market price, especially if the security is not traded actively. Since the recorded price is not time-stamped, it can be affected by a non-synchronous bias. This means that the option price may not be recorded at the same time as the index and therefore be a poor proxy for the true market price. Consequently, by using last price the bid-ask quote may be included in the data set either way. According to previous studies[49], by using the mid-price rather than last price reduces noise in the cross-sectional estimation of the volatility function.

If the bid-ask spread is large, there is a great uncertainty about the true price of the option and we assign it less weight on the sum of (48). Since bid and ask prices may be hard to come by, this method is sometimes difficult to use. An alternative approach is to choose weights so that on each day all maturities have the same influence on the objective function. Moreover, the same weight is assigned to all points of the same maturity. This leads to the weights:

$$w_i = \frac{1}{n_{mat} n_{str}^i}$$

Where n_{mat} denotes the number of maturities, and n_{str}^i denotes the number of strikes with the same maturity as observation “i”. In this thesis, as weights, the author will use the inverse of the bid ask spread, i.e. Equation (49).

3.7.5 The Levenberg-Marquardt algorithm

Defined the function that has to be used, an algorithm is needed in order to find the minima of this function. They exists methods that allow to find local minima in a multidimensional

[49] Dumas et al., (1998).

framework, but it won't be complete because the algorithm does not guarantee a global minimum, since it is considered a gradient optimizer. The most used algorithm for this kind of problems is Levenberg-Marquardt. The calibration aim is to reach the best approximation of market prices, asking the exact calibration would imply that the market obeys the model, and this is an unrealistic assumption. The Levenberg-Marquardt algorithm is iterative, and it needs a starting point, that it would affect the solution in presence of some local minima. As mentioned earlier, most commonly proposed loss functions are non-convex and may exhibit several local (and perhaps global) minima, making standard optimization techniques unqualified. The author will solve the optimization problem using MATLAB® algorithm `lsqnonlin`, which is a local based optimizer. On one hand, as said, we risk getting stuck in a local minima.

3.7.6 Regularization

In addition to the objective function that is minimized a regularization term is added. Regularization can be necessary for two reasons: most commonly proposed error functional may have several global minima[50] and thus the regularization term is needed to get a unique minimum. This method is discussed briefly for completeness[51]. Regularization involves adding a *penalty* function, $p(\Omega)$, to (3.1) such that:

$$\sum_{i=1}^{N} \frac{1}{N} w_i [C_i^{\Omega}(K_i, T_i) - C_i^{M}(K_i, T_i)]^2 + ap(\Omega) \tag{50}$$

Is convex. The "*a*" here, is called the *regularization parameter*. An approach suggested[52] is using $ap(\Omega) = \|\Omega - \Omega_0\|^2$, where Ω_0 is an initial estimate of the parameters. This method is therefore dependent on the choice of the initial parameters. It is, in a sense, a local minimum optimizer. Equation (50) can be minimized in MATLAB® using the function `lsqnonlin`.

3.7.7 MATLAB® `lsqnonlin`

MATLAB® least-squares, non-linear optimizer is the function:

[50] Cont and Hamida 2005.
[51] For a detailed discussion refer to Chiarella, Craddock & El-Hassan 2000.
[52] Mikhailov & Nogel, 2003.

```
Ω =lsqnonlin (fun, x₀, lb, ub)
```

It minimizes the vector-valued function, fun, that in our case is the MSE function, i.e Equation (50), using the vector of initial parameter values, x_0, where the lower and upper bounds of the parameters are specified in vectors l_b and u_b, respectively and Ω are the vector of the parameters as output. The results produced by `lsqnonlin` are dependent on the choice of x_0, the initial estimate. This is, as said, not a global optimizer, but rather, a local one. We have no way of knowing whether the solution is a global/local minimum. The appendix contains the code on using `lsqnonlin` for calibration. By using only `lsqnonlin` the author might encounter the problem explained in the previous section, getting struck in a global minima.

"We must also apply good judgment when selecting starting values. For example, volatility and price are usually negatively correlated, so we may specify for the starting value for the correlation to lie in (−1, 0) instead of (-1, 1)."

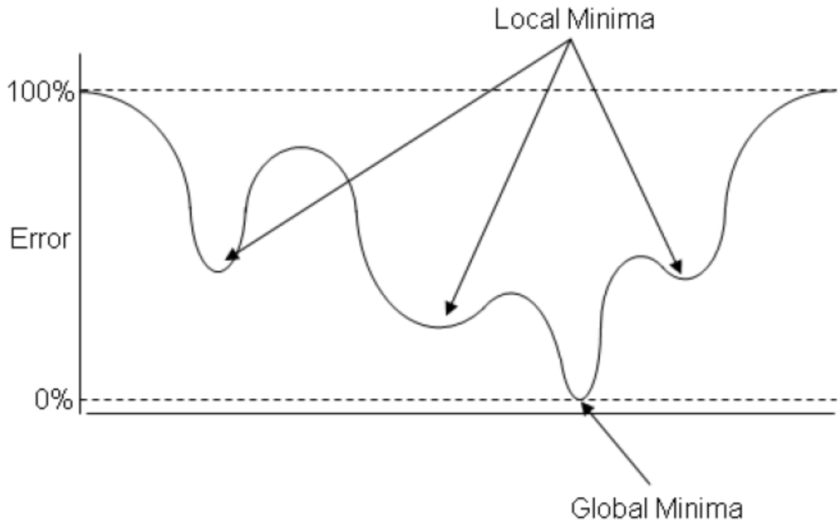

Figure 30: Local al global minima, for each local minima we have a set of 5 parameters that try to minimize the objective function. Source: Pricing and Hedging of an option portfolio in presence of stochastic volatility, David Laguardia, 2015.

To mitigate this problem, we could follow two approaches: take precautions to acquire appropriate results in estimating the model parameters, or using a stochastic global optimizer. The author chooses the latter approach, for this he will introduce and use another algorithm, the Simulated Annealing, that is more precise but also more time consuming.

3.7.8 Simulated Annealing (SA)

The algorithm works in the following way:

- First the objective function is evaluated at the user-specified initial parameter estimates.
- Next a random set of parameters is generated based on point 1 above.
- If the value of the objective function is less than that of point 1, then we 'accept' the parameter set from point 2, else, the parameter set is accepted with probability $\exp\left\{-\frac{\partial f}{T_k}\right\}$, where ∂f is the difference between the objective functions using the parameter sets in points 1 and 2, and T_k is the *temperature* 10 parameter at iteration k, specified by the algorithm.
- This process is iterated, with the *temperature* parameter decreased at each iteration, until a termination condition is met (usually a specified value of the *temperature* parameter).

"It helps to imagine the objective function that we want to minimize, as a geographical terrain. We want to find the deepest valley of this terrain."

The above algorithm not only 'accepts' parameter sets that decrease the objective function, but also that which increases it, consequently this ensures that the algorithm does not get stuck in a local minimum such as the `lsqnonlin` algorithm. SA requires only the value of the objective function for a given set of parameters. It does not require the form of this objective function. This, in a sense, makes the function a 'black-box' to the optimizer. Constraints are encapsulated within the 'black-box'. This means that the objective function should be able to tell the optimizer whether a set of parameters lies within the required parameter space (in our case, whether $2\kappa\theta > \sigma^2$), hence limits the search to a feasible space. Parameters that are determined by the model (eg. the *temperature* parameter) are collectively known as the *annealing scheme*. The annealing scheme broadly determines the efficiency and accuracy of the algorithm. For example, it determines the degree of 'uphill' movement and the rate of decrease of *temperature*, which in turn affects how long the algorithm runs for. It is therefore obvious that the annealing scheme be optimally specified. Such a specification isn't obvious since parameters like *temperature* don't have an explicit/implicit mathematical relationship with the objective function. This sort of specification has therefore become an art form. To reduce the subjective nature of the aforementioned specification, adaptive methods of Simulated Annealing have been developed. The most famous and widely used of these is *Adpative Simulated Annealing*.[53]

[53] For a better explanation of SA algorithm look at The Heston Model: A Practical Approach with Matlab Code Nimalin Moodley, 2005.

3.7.9 The SA algorithm in MATLAB®: Adaptive Simulated Annealing (ASA)

ASA was developed by the theoretical physicist Lester Ingber.[54] ASA is similar to SA except that it uses statistical measures of the algorithm's current performance to modify its control parameters i.e. the annealing scheme. A proof is provided by that Ingber shows that ASA is a global optimizer. He also provides arguments for ASA's computational effeciency and accuracy.

ASA can be implemented in MATLAB® by downloading the function `asamin.m`, written by Shinichi Sakata. Instruction how to use and install `asamin` can be found in the Appendix.

The author reported only the instructions for installing on Windows, since the calculations are performed on a Windows Operating system, for other platform such as Linux or Unix the author suggest to take a look at Moins 2002. `Asamin` is a MATLAB® gateway function to ASA. This means that `asamin` uses the actual C++ code of Ingeber's ASA through MATLAB®. The MATLAB® code for calibrating the model using ASA for one step, can be found in the appendix.

The ASA algorithm is particularly useful when poor starting values are available, such in our case. This is because, even with poor values, the algorithm will converge to the global optimum, albeit at the requirement of many iterations. On the downside, as said, ASA techniques are generally much more time-consuming than for example gradient based optimizers. Therefore, as explained in the following sections, the author will make use of both alternatives. In MATLAB® the `asamin` function appear like this:

```
[Ω, S(Ω),…]= asamin('minimize',@fun,x0,lb,ub,xtype,par1,par2..)
```

where Ω is the output vector of parameters that minimize the function @fun (That in our case should be the HestonCostFunc.m, the MATLAB® form of Equation (50)); S(Ω) is the value of the objective fuction with the parameters Ω; X_0 is the initial vector of parameters, lb and ub are respectively the lower and the upper bound of the parameters, xtype is an option for setting real or imaginary parameters, and par1, par2… are the parameters subject to the xtype command.

[54] The C++ code is open-source and available from www.ingber.com.

3.7.10 Parameters stability

One's might asking if the parameters are time-varying or not. How much are they stable during time and how good they represent the reality also few days after the calibration data. An experiment to validate the stability of parameters is grabbing the data days after the calibration and the data of the option in interest (i.e. Out-of-Sample performances). As expected for days near the calibration data, the parameters represent in a good way the price evolution A lot of studies, such as the aforementioned Mikhailov & Nogel 2003 said that the parameters aren't stationary, and hence the model has to be calibrated each step. In this work, the author follows the spirit of Mikhailov & Nogel, and, as he will present in the following sub-section, he will calibrate the model each step. Thus, he will assume that the parameters variations over time exists. Indeed in the results it'll be shown that the parameters varies over time.

3.7.11 Trade-off between algorithms: `lsqnonlin` or `asamin`?

The trade-off between precision and time consuming is clear, and this lead to an obvious question, that is, which method do we use? There isn't a straight answer to this. The choice of method should be dependent on the amount of information available related to parameter values and the state of the market. If a global minimum was obtained yesterday by using ASA, and the market conditions are quite normal then `lsqnonlin` can quite safely be used. The author expects that today's parameters are in a small neighborhood around yesterday one's. If there has been a crash or dramatic market movement then such an expectation is unreal. In this case ASA would have to be used. So, there should be a marriage of the different methods to create efficiency and not sacrifice accuracy. Since the author doesn't have reliable starting values, the best solution is that he will use the ASA algorithm for the first date to get reliable results. After that he will solve the optimization problem using MATLAB® algorithm `lsqnonlin`. On one hand, after the first calibration, he risks getting stuck in a local minima. On the other hand, unless the market has changed dramatically he does not expect the parameters to change very much. In periods when the stock price fluctuates heavily he will again use the ASA algorithm to obtain reliable parameter estimates. Both the MATLAB® codes needed to calibrate the Heston model for one step, are included in the Appendix.

3.8 Heston Greeks

Having the prices of European Calls and Puts in the Heston model in semi-closed form, it is now possible to differentiate the call or put price and obtain expressions for the Greeks in closed form also. Recalling that the call price, from Equation[55] (43):

$$C = S_t P_1 - K e^{-rT} P_2$$

Where, the in-the-money probability are P_j, recalling Equation (44):

$$P_j = \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} Re\left[\frac{e^{-i\phi lnK} f_j(\phi; x, v)}{i\phi}\right] d\phi$$

Hence, the sensitivity of calls and puts to a parameter or input y usually involves first- and second-order derivatives of the in-the-money probabilities P_j:

$$\frac{\partial P_j}{\partial y} = \frac{1}{\pi} \int_0^{\infty} Re\left[\frac{\partial f_j}{\partial y} * \frac{e^{-i\phi lnK}}{i\phi}\right] d\phi, \qquad \frac{\partial^2 P_j}{\partial y^2} = \frac{1}{\pi} \int_0^{\infty} Re\left[\frac{\partial^2 f_j}{\partial y^2} * \frac{e^{-i\phi lnK}}{i\phi}\right] d\phi \quad (50)$$

In the following subsections, the author use Equation (50) to derive analytic expressions for most of the popular first- and second-order Greeks.

3.8.1 Delta, Gamma, and Vega derivation in the Heston model

Delta, gamma, rho and theta are obtained by differentiating Equation (43) and applying (50) when required. Vega is more arbitrary, since there are several parameters that affect the volatility smile in the Heston model. The delta for the Call is given by, respectively:

$$Delta\ (Call) = \frac{\partial C}{\partial S} = P_1$$

[55] The Put is obtained by Call-Put parity formula.

Gamma is found by differentiating delta. By definition, it is the same for Calls and Puts. Using $\partial f_j/\partial S = i\phi f_j/S_t$ in Equation (50) to obtain $\partial P_1/\partial S$, gamma is expressed as:

$$Gamma(Call, Put) = \frac{\partial^2 C}{\partial S^2} = \frac{\partial P_1}{\partial S} = \frac{1}{\pi S_t}\int_0^\infty Re\,[e^{-i\phi lnK} f_j(\phi; x, v)]d\phi$$

Vega is defined as the derivative of the Call price with respect to the implied volatility. In the Black&Scholes model, the implied volatility is represented by the volatility parameter σ_{BS}, so Vega for the call is readily obtained as $\partial C_{BS}/\partial \sigma_{BS}$, where C_{BS} is the Black-Scholes call price. Recalling from Section (3.2) that in the Heston model, however, the shape of the implied volatility surface is determined by the parameters driving the process for the variance, namely the mean reversion speed κ, the mean reversion level θ, the initial level of the variance v_0, and the correlation ρ. Since v_0 and θ are responsible for the initial and long-term level of the variance, some studies[56] recommends basing Vega on those two parameters. Both parameters represent variance, so to create measures of sensitivity to volatility; we defines two Vegas, one based on $\nu = \sqrt{v_0}$ and the other based on $\omega = \sqrt{\theta}$. The Vegas for the call are, therefore, the derivatives:

$$Vega_1 = \frac{\partial C}{\partial \nu} = \frac{\partial C}{\partial v_0} 2\sqrt{v_0}, \qquad Vega_2 = \frac{\partial C}{\partial \omega} = \frac{\partial C}{\partial \theta} 2\sqrt{\theta}$$

The first Vega is

$$Vega_1 = S_t\,\frac{\partial P_1}{\partial v_0} 2\sqrt{v_0} - Ke^{-rT}\frac{\partial P_2}{\partial v_0} 2\sqrt{v_0} \quad (11.10)$$

Where

$$\frac{\partial P_j}{\partial v_0} = \frac{1}{\pi}\int_0^\infty Re\left[\frac{e^{-i\phi lnK}\, f_j(\phi; x, v)\, D_j(\tau, \phi)}{i\phi}\right] d\phi$$

The second Vega is

$$Vega_2 = S_t\,\frac{\partial P_1}{\partial \theta} 2\sqrt{\theta} - Ke^{-rT}\frac{\partial P_2}{\partial \theta} 2\sqrt{\theta}$$

Where

$$\frac{\partial P_j}{\partial \theta} = \frac{1}{\pi}\int_0^\infty Re\left[\frac{e^{-i\phi lnK}\, f_j(\phi; x, v)\ {}^{\partial C_j}/_{\partial \theta}}{i\phi}\right] d\phi$$

[56] Zhu, 2010.

And

$$\frac{\partial C_j}{\partial \theta} = \frac{\kappa}{\sigma^2}\left[(b_j - \rho\sigma i\phi + d_j)\tau - 2ln\left(\frac{1-g_j e^{d_j\tau}}{1-g_j}\right)\right]$$

By examination of Equation (43), it is easy to verify that $Vega_1$ and $Vega_2$ for the put are the same as those for the call. However in this thesis, for the Delta-Gamma-Vega strategy only Vega1 is considered for the Hedging purposes. Rho and theta are not discussed since the author won't use them.

3.8.2 Approximation of Heston Greeks in MATLAB®

In the previous section, the derivation and analytic expressions for the Greeks from the Heston Call price has be done. However due to the complexity of the formula, the author chose to use finite differences to approximate the derivatives. Applying finite differences to find the Greeks and other option sensitivities also. When this approach is used to calculate Greeks, however, the computation time is increased because the option price must be calculated more than once, multiple times in the case of the second-order Greeks. For the first-order Greeks, the sensitivity of the option price to a parameter or variable can be approximated with first-order central differences.

3.8.2.1 First order Greeks: approximation with first-order central differences

Approximation with first-order central differences include the Greeks Delta and Vega.
Delta for the Heston Call $C(S, v, t)$ is approximated as:

$$Delta(Call) = \Delta_{CallHeston} = \frac{\partial C}{\partial S} \approx \frac{C(S+dS, v, t) - C(S-dS, v, t)}{2dS} \qquad (51)$$

In order to approximate the delta, in MATLAB® the author will use the functions HestonCallDelta.m available in the appendix, based on finite difference in Equation (51) that is based on Equation (13). As another example, to approximate the first Vega the author approximate the derivative with respect to v_0 by a central difference.

$$Vega_1(Call, Put) = V_{1|Heston} = \frac{\partial C}{\partial v} \approx \frac{C(S, v, t; v_0 + dv) - C(S, v, t; v_0 - dv)}{2dv} \tag{52}$$

The MATLAB® form of Equation (52) is the function HestonVega1.m, can be found in the appendix.

3.8.2.2 Second order Greeks: approximation with second-order central differences

For the second-order Greeks, such as gamma (Γ), the second order central differences for a single variable could be used. Hence, for example, gamma is approximated as:

$$Gamma(Call, Put) = \Gamma_{Heston} = \frac{\partial^2 C}{\partial S^2} \approx \frac{C(S + dS, v, t) - 2C(S, v, t) - C(S - dS, v, t)}{(dS)^2} \tag{53}$$

In order to approximate the Gamma in Equation (53), in MATLAB® the author will use the function HestonGamma.m obtainable in the appendix. It is informative to present visual illustrations of the Greeks. Figure (29) presents a plot of gamma from the Heston model.
The mesh with black squares represents gamma with $\rho = -0.9$ and the smooth-colored surface, gamma with $\rho = 0.9$. In section (3.2), it was shown that the correlation parameter introduces skewness in the terminal stock price density, with negative correlation leading to a negative skew, and positive correlation, to a positive skew. This effect is introduced in the Heston Greeks also, and the skewed patterns are discernible also by looking at Figure (31).

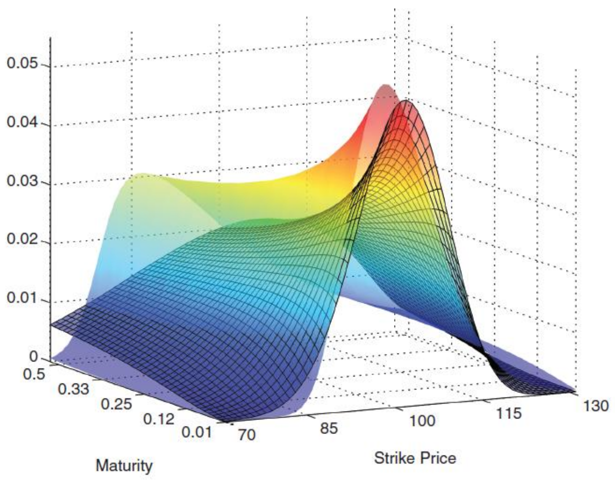

Figure 31: Gamma from Heston model. Source: The Heston model and its application in Matlab, Wiley, 2013.

Having the Greeks the author might proceed, by calculating the Greek of the portfolio[57] with the Equation (15), (20) and (23 bis) explained in Chapter 2, and using the MATLAB® function Hedgeopt, such as in the Black&Scholes framework, in order to calculate the desired hedging strategy. The Hedgeopt function in MATLAB®, appear like this:

```
[PortSens, [], PortHolds] = hedgeopt (Sensitivities, Price,
CurrentHolds, [], [], [], TargetSens, [])
```

The input to be provided, for a basic hedging are a matrix/vector of Sensitivities, depending on the number of the Greeks that one wants to be hedged. A vector of Prices, are the prices of the options; the Current Holdings of the options that the seller have in his portfolio, and the Target Sensitivities that one should achieve. In this work, the author wants a full hedging, so he'll provide a vector of zeros for that variable. To perform the hedgeopt function one should have the previous inputs. A generic example of the variable needed is shown in Figure (32)[58]:

```
disp([Price  Holdings  Sensitivities])

     8.29     10.00      0.59       0.04      53.45
     2.50      5.00     -0.31       0.03      67.00
    12.13      1.00      0.69       0.03      67.00
     3.32      3.00     -0.12      -0.01     -98.08
     7.60      7.00     -0.40  -45926.32      88.18
    11.78      9.00     -0.42 -112143.15     119.19
     4.18      4.00      0.60   45926.32      49.21
     3.42      6.00      0.82  112143.15      41.71
```

Figure 32: Generic representation of inputs needed for using the Hedgeopt function. Source: Financial Derivatives Toolbox™ 5, Mathworks, 2015.

Where the sensitivities matrix (in this exmple 3 sensitivities, as consequence the desired hedging would be a Delta-Gamma-Vega) are calculated by Equations (51), (52) and (53). The Price vector is observable in the market. It is important to notice that Figure (32) is only for a generic step, the table rows represents the Number of options in the portfolio. Since the author needs to perform dynamic hedging, he needs to calculate this for each step in time, he'll use 3-d matrices, where the third dimension is time; a double loop in MATLAB® is needed to perform this kind of problems. This is done for both models (i.e. Heston and BS). As output the most important are the PortHolds, a vector of quantities that will ensure the hedging[59].

[57] Following the same approach in the Black&Scholes model in Chapter 2.
[58] In this case the hedgeopt function calculates the Greeks of the portfolio, since the single Greeks and quantities of each options are provided. Hedgeopt for the portfolio Greeks, use the approach of the author.
[59] That in this thesis would be an array with dimension (NumberofOptions, 1, NumberofStepsintime).

Chapter 4 | Methodology and Results

4.1 Data description

The initial data set consists of 24 Call options[60] on the S&P 500 Index during the time period from January 2014 to June 2015. For all options the author extracts information about maturity, strike price, current index level, Bid and Ask price. In order to calibrate the models, the author applies some traditional filter rules on the option data. For the options chosen sample, only a selective number of strike prices are recorded. Extremely high or low values are omitted. By doing so, all the options chosen are around at-the-money (moneyness range from 0.9 to 1.20). The reason is that very deep in-the-money or vert out-of-the-money option prices are deterministic and do not follow the pricing model. Furthermore, options with a volume of less than 10 contracts or with no open interest are considered as irrelevant. Moreover, options with an expiration date lower than 10 days have been excluded. Options with an expiration date longer than 2 year are rejected since they are less sensitive to volatility. Finally, options with a very large bid-ask spread are excluded. All in all the set of the options consists of contracts with 19 different strikes (ranging from 1705 to 2300) and 7 different maturities where, on the first date, the option maturities range from 2 months to 1.5 years; Table (1 bis) reports the selected options for this thesis. The information is collected from Thomson Reuters Datastream 5.1.
Datastream is one of the world's largest databases for financial and economic information.
It collects data from a number of other information providers and contains more than two million financial instruments, securities and indicators for over 175 countries in 60 markets[61].
In order to recover the desired options, the author at first, downloaded the dead list of call options on the SPX, by searching "LOPTSPXDC[62]" in the Datastream Navigator in the static request menu in Excel.
Then, he recovered each Mnemonics and downloaded each option contract time series by means of a Time-series request.

---

[60] The contract size of this options is 100.
[61] www.datastream.com.
[62] Stand for List of options on the SPX, Dead Calls

Name	Strike	Quantity	Bid	Ask	Volume	Time to maturity	Und.
CALL SPX MAR15 1705	1705	-10	344,4	346,1	>10	50	SPX
CALL SPX MAR15 1715	1715	-11	334,4	336,1	>10	50	SPX
CALL SPX MAR15 1845	1845	-15	204,5	206,2	>10	50	SPX
CALL SPX MAR15 1870	1870	-4	179,5	181,1	>10	69	SPX
CALL SPX MAR15 2000	2000	-7	52,9	54,2	>10	257	SPX
CALL SPX MAR15 2100	2100	-13	1	1,3	>10	257	SPX
CALL SPX APR15 2000	2000	-19	69,2	71,4	>10	83	SPX
CALL SPX APR15 2020	2020	-21	54,7	56,8	>10	82	SPX
CALL SPX APR15 2070	2070	-4	24,3	26,1	>10	82	SPX
CALL SPX APR15 2075	2075	-7	21,9	23,3	>10	83	SPX
CALL SPX APR15 2250	2250	-19	0,2	0,25	>10	83	SPX
CALL SPX MAY15 1900	1900	-11	160,2	164	>10	82	SPX
CALL SPX MAY15 2000	2000	-6	80,8	83,3	>10	82	SPX
CALL SPX MAY15 2050	2050	-7	47,6	49,8	>10	82	SPX
CALL SPX MAY15 2100	2100	-8	21,9	24	>10	82	SPX
CALL SPX MAY15 2150	2150	-12	6,9	8	>10	82	SPX
CALL SPX MAY50 2200	2200	-13	1,45	2,2	>10	82	SPX
CALL SPX JUN15 1850	1850	-15	212,6	214,6	>10	374	SPX
CALL SPX JUN15 1900	1900	-10	169,5	172	>10	374	SPX
CALL SPX JUN15 1975	1975	-8	110,9	112,7	>10	374	SPX
CALL SPX JUN15 2100	2100	-18	33,4	34,9	>10	374	SPX
CALL SPX JUN15 2175	2175	-12	8,9	9,7	>10	367	SPX
CALL SPX JUN15 2200	2200	-5	4,8	5,6	>10	374	SPX
CALL SPX JUN15 2300	2300	-6	0,25	0,35	>10	374	SPX

Table 1bis: Option selected for the initial portfolio.

4.2 Approximate other variables

In this analysis, as explained in Chapter 3 the author assumes the mid-price is the best representative for the true market price which is the average of the bid and ask price[63]. To calculate model prices in practice is clear that the Call price is not only a function of the parameters (Ω), but also of the strike price K, the time to maturity T, the risk-free interest rate r[64].The strike price and the time to maturity is uniquely specified by the contract in question, only the risk-free interest rate has to be determined. According to Hull[65] it is natural to assume

[63] Following Dumas et al., 1998.
[64] No dividends are used in this work.
[65] Hull, 2011.

Treasury bills and Treasury bonds as the correct benchmark for risk free rates. In contrast, traders regard the LIBOR interbank rate as their opportunity cost of capital and usually use LIBOR rates as short-term risk-free rates. The author follows the same procedure as practitioners and use the interbank rate as the risk-free rate. Since results are in dollar the author choose the U.S. LIBOR as interest rate. In theory, one should match the maturity of the risk free rate with the remaining days to expiration for options. However, to approximate this, the author chooses to interpolate the maturity from the 1-3-6-12 months LIBOR with using the Cubic Hermite spline interpolation. In MATLAB® this is done using the interp1 function:

```
K = interp1(x, y, z,'pchip')
```

Where:

x = vector available deadlines for the yield curve; y = observed rate for the available deadlines; z = scalar/vector for deadlines where interpolation is needed; k = scalar/vector of interpolated risk-rate.; ‘pchip’ = option that specify the calculation method.

4.3 Calibration specifics for the Black&Scholes model and the Extended Black&Scholes

The only parameter to be taken care of in order to calibrate the Black&Scholes model is the volatility. As said in sub-section (2.5.2) the author will use 1 year historical volatility calculated from the underlying returns, Equation (9) with annualized adjustment is used. The one year historical volatility is 0,1158 (Calculated from 30 January 2014 to 30 January 2015). Figure (32 bis) is a check that the result is right. For the Extended Black&Scholes the author will use 3-months rolling window volatility. Every step in time, the time-window is the same; this allows volatility “to adjust” to more nearest levels and in some way volatility is no more a fixed value, a raw form of varying volatility. As you can see in Figure 1, volatility tends to average near 15% (the average that many models and academics use for stock market volatility). Although most periods generally fall within a band of 10% to 20% volatility.

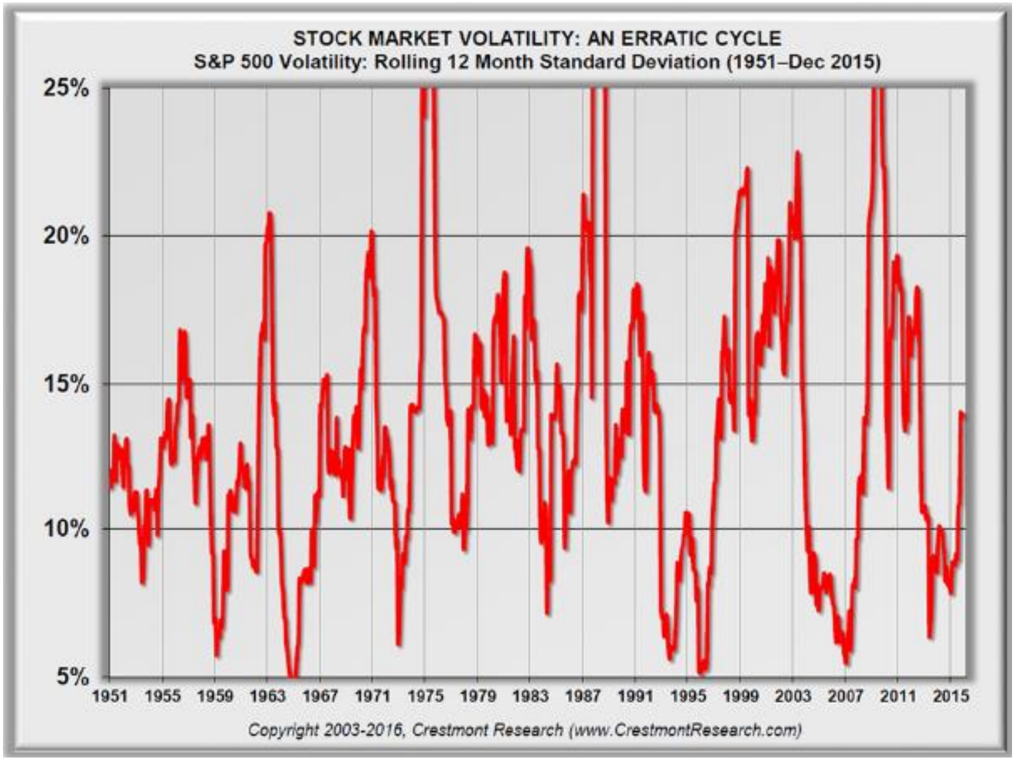

Figure 32 bis: 12-months rolling standard deviation for the S&P 500 Index. Source: Crestmont Research.

4.4 Calibration specifics for the Heston model

The importance of good calibration is vital to get robust results for a complex model like Heston. The following rules were used in the calibration process:

1. Using the `asamin` algorithm, explained in sub-section (3.7.9) for the first date for the 5 parameters. Since the following `lsqnonlin` calibrations are fast, the author decides not to fix initial parameters (such as other studies[66]) to facilitate the complexity of the optimization problem.
2. Use the `lsqnonlin` algorithm, explained in sub-section (3.7.7) from the second to the last time-step.
3. Max iterations of 20.000 for the `lsqnonlin` algorithm.
4. Used previous time-step calculated parameters as an initial guess the next time-step.
5. When the result clearly indicated a wrongly specified parameter set the calibration that day was recalibrated with a new initial guess provided by `asamin`.

[66] Buehler, 2008 fixed kappa and sigma. Concerning kappa, different values of kappa does not alter the pricing performances significantly (kappa=0.01 and kappa=5 are almost the same). Concerning V_0, Buehler proxy it from the 2 months ATM implied volatility.

6. Since the hedging is considered on a weekly basis, and as a consequence the whole time-frame of the thesis is weekly, each step in calibration and pricing, is considered also weekly.

4.5 Calibration, Results and Benchmark of parameters for Heston model

The parameter estimates from the step one and two estimations are shown in Table (2):
The author calibrates the model with the `asamin` algorithm for the first date, and then using the vector of parameters found as initial parameter for feeding the `lsqnonlin` algorithm for the second step. From step 2° to step n (i.e. the final step), `lsqnonlin` was used:

Time	Algorithm	κ	θ	σ	ρ	v_0	S(Ω)	Computation	Date
Step 1	`asamin`	6,10	0,0268	0,56	-0,87	0,13	34,55	108 minutes	13/01/2016
Step 2	`lsqnonlin`	6,09	0,0267	0,5672	-0,78	0,13	33,67	21 seconds	13/01/2016

Table 2: Estimated parameters for the Call options in the Heston model in different steps with different algorithms.

Where "Computation" is the running time of the calibration process[67]. The reason that `asamin` runs longer is because it searches the entire parameter space for a global minimum, unlike `lsqnonlin` which settles down quite quickly into a local minimum. Figures (33)

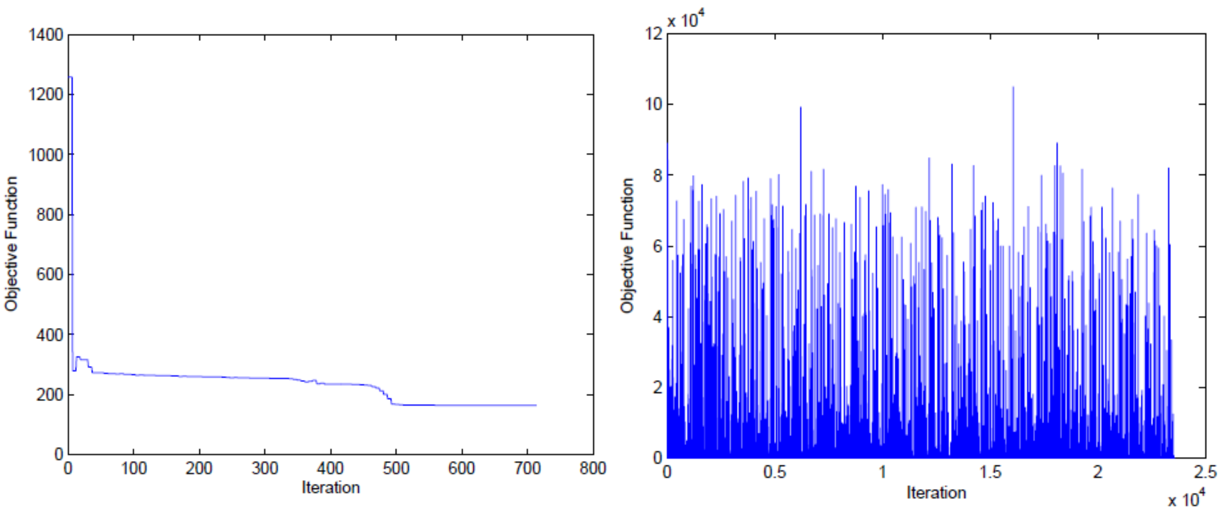

Figure 33: Generic example of minimization of the objective function (i.e. MSE) using lsqnonlin (left) and ASA (right) Source: Fast Calibration in the Heston Model, Stefan Gerhold, 2012.

[67] The Computation time is measured by the "tic" function in Matlab in seconds.

illustrates this point. Notice that `lsqnonlin` drops down very quickly to a local minimum whereas `asamin` keeps on searching for other minima and hence oscillates. As explained in sub-section (3.8.3.2), for the Lsqnonlin, author will require that the difference between model and market prices falls on average within the observed bid-ask spreads. Therefore, he will consider the following set of acceptable solutions. Table (3) reports the pricing for the second time-step to proof Equation (48 bis):

Option Name	Market price (Mid price)	Price (Heston)	Difference (abs)	Within Bid – Ask spread?
CALL SPX MAR15 1705	351	351,81	0,23%	Yes
CALL SPX MAR15 1715	341,35	342,073	0,21%	Yes
CALL SPX MAR15 1845	216	217,21	0,36%	Yes
CALL SPX MAR15 1870	192,25	190,99	0,65%	No
CALL SPX MAR15 2000	79,25	78,41	1,05%	Yes
CALL SPX MAR15 2100	17,6	18,24	3,67%	Yes
CALL SPX APR15 2000	91,65	91,84	0,21%	Yes
CALL SPX APR15 2020	77,21	77,36	0,22%	Yes
CALL SPX APR15 2070	45,01	45,29	0,65%	Yes
CALL SPX APR15 2075	42,25	42,50	0,61%	Yes
CALL SPX APR15 2250	0,975	0,8297	14%	Yes
CALL SPX MAY15 1900	180	180,50	0,28%	Yes
CALL SPX MAY15 2000	101,32	100,89	0,40%	Yes
CALL SPX MAY15 2050	67,65	67,51	0,2%	Yes
CALL SPX MAY15 2100	39,80	39,91	0,3%	Yes
CALL SPX MAY15 2150	19,45	19,77	1,65%	Yes
CALL SPX MAY50 2200	7,65	7,17	6,20%	Yes
CALL SPX JUN15 1850	229,75	230,02	0,12%	Yes
CALL SPX JUN15 1900	188,05	187,55	0,26%	Yes
CALL SPX JUN15 1975	129,95	128	1,03%	No
CALL SPX JUN15 2100	51,25	51,67	0,83%	Yes
CALL SPX JUN15 2175	21,15	21,89	3,53%	Yes
CALL SPX JUN15 2200	14,75	15,12	2,5%	Yes
CALL SPX JUN15 2300	2,7	2,19	18,84%	Yes

Table 3: The model predicted values and its comparison with the market prices for Call options for the second step where lsqnonlin was used.

As the Table (3) shows, the calibrated Heston model provides a good match for most traded options. 22 out of 24 options have a predicted value that falls within the observed bid-ask spread. In addition, when evaluated in terms of our acceptance criterion, the model's average distance from the mid-market price is roughly 2%, which is lower than the average deviation in the bid-ask spreads. Concerning `lsqnonlin`, the so called Feller condition states that if then the variance will never become negative. In order to incorporate this condition in our calibration the author forms a new variable $F = 2\theta\kappa - \sigma^2$. We can then force F to be positive in the optimization routine. Hence a positive bound for F was set, a transformation according to $\kappa = (F + \sigma^2)/2\theta$ can then be sent to the pricer, and after the parameters are obtained, κ is reconstructed from F. This can be seen in the calibration code (in the variable "*Solution*") in the appendix when the `lsqnonlin` calibration is performed.

4.6 Stationarity of the Heston parameters

Table (4) is a summary of the different parameter estimates throughout the period and it seems that the Heston model does not have stationary parameters. For some parameters the standard deviation is larger than the parameter itself. This was also evident in the article by Kim & Kim (2004). The non-stationarity of the parameters clearly indicates that the market changes and the model have to absorb new information by changing its parameters. This is the contrary of what the theory suggests: the model parameters should be slow moving and change little throughout time. Other articles such as Kim & Kim (2004) found evidence that parameters have large standard deviations. They argue that stability of the interdependence between the parameters is far more important than focusing on the standard deviations.

κ	θ	σ	ρ	v_0
3,8210 (3,9004)	0,0547 (0,061)	0,3855 (0,1516)	-0,9312 (0,0667)	0,01523 (0,0075)

Table 4: Mean and standard deviation (in parenthesis) of the parameter estimates for the Heston model for the Call options.

Fortunately in this case the index doesn't move a lot during our sample and making the model parameters less volatile. By prohibiting the parameters to adjust can negatively bias the results. The instability of parameters suggests that out-of-sample pricing can be somewhat mispriced since the parameters are unstable. Moving on to the validity of the parameters, several interesting characteristics can be observed:

- The spot volatilities (i.e. 0,3855) lie in the range of 28-39 %, which is in line such as previous workes than for example Christoffersen, Heston and Jacobs (2009).
- The correlation between return and volatility is negative for all loss functions, which indicates that the Heston model is able to generate the observed smirk shape in volatility skew.

- The estimated long-run mean of the stochastic variance process (i.e. 0,2338), is also in accordance with earlier studies, with an average long-run mean volatility in the interval 22-29 % (note that the long-run mean volatility is defined as $\sqrt{\theta}$).
- The mean reversion parameter κ, the values vary between 1 and 6,30. These values coincide with the estimates obtained by Christoffersen, Heston and Jacobs (2009) as well as with Bakshi, Cao and Zen (1997).

All-in-all, the parameter estimates of the Heston model are in line with the expectations as well as the results of earlier empirical studies[68]. In the next section, the author will presents the results of in-sample and out-of-sample pricing errors.

4.7 Statistical and empirical Performances

Do option pricing models which incorporate the volatility smile perform better than BS empirically using option prices from the S&P 500 Index?

A variety of statistic measures can be selected to check the accuracy of the Heston model. In this thesis, the author and employ 2 yardsticks[69] to compare empirical performances of the option pricing models: In-sample and Out-of-sample pricing errors.

According to the authors, in-sample and out-of-sample errors reflect a model's static performance. To evaluate the pricing errors to compare the performances of the models, the author will use[70]:

$$MPE = \sum_{i=1}^{n} \frac{1}{n} \left(\frac{C_{Model} - C_{Market}}{C_{Market}} \right)$$

$$MAPE = \sum_{i=1}^{n} \frac{1}{n} \left| \left(\frac{C_{Model} - C_{Market}}{C_{Market}} \right) \right|$$

$$MAE = \sum_{i=1}^{n} \frac{1}{n} \left| (C_{Model} - C_{Market}) \right|$$

$$MSE = \sum_{i=1}^{n} \frac{1}{n} (C_{Model} - C_{Market})^2$$

[68] A t-Sahalia & Kimmel, 2005, Gatheral, 2006 and Bakshi et al., 1997.
[69] Following the approach of Bakshi, Cao & Chen, 1997.
[70] The measurements of Kim & Kim, 2004.

Where Model stands for both Heston and Black&Scholes. C_{Model} Is the call price estimated by the model and C_{Market} is the observed market price of the option. To measure the magnitude of the pricing errors, the author uses mean absolute errors (MAE) and mean absolute percentage errors (MAPE). Mean percentage errors (MPE) indicate the direction of the pricing errors while mean squared errors, (MSE) measures the volatility of errors. This analysis will be based on these 4 measurements, although the author will mainly deal with MAPE and MPE because the relative comparison and direction of pricing error is important above all else.

"The relative price error is measured as the difference between market price and model price divided by market price. Hence a negative value means the model underestimate the price."

4.7.1 In-sample performance

The in-sample performance of each model is evaluated by comparing market prices with model prices computed by the estimated parameters of the current time. As mentioned earlier, the weekly re-estimation of parameters is admittedly potentially inconsistent with constant or slow-changing parameters used to compute option prices. On the other hand, such estimation is useful for indicating market outlook on a daily basis. Table (5) reports the In-sample performance[71]:

	BS model	BS extended	Heston Model
MPE	-0,15%	0,097%	0,05%
MAPE	0,30%	0,317%	0,08%
MAE	4,3789	4,178	1,0696
MSE	40,09	32,69	2,5265

Table 5: In-Sample pricing errors for Call options.

Table (5) reports the pricing error for the three models. Overall, the Heston model outperforms the Black&Scholes and the Extended BS model since the MAPE for the Black&Scholes model is 0,3% while for the Heston model is 0,08%. The Extended BS performed similar to the benchmark. Looking at the MPE, notice that the BS model underprices options, since the author has considered both OTM and ITM money, this is a sign of the volatility smile; for this reason the author sorts prices considering Moneyness and Time to maturity. Table (6) sorts the pricing errors according to days to maturity using intervals of less than 20 days (short maturity),

[71] MPE and MAPE have been multiplied by 10.000.

between 20 and 40 days (medium maturity) and above 40 days (long maturity). When sorting errors for maturity (and moneyness later) the lack of observations may be distorted for certain maturity or moneyness categories. BS and Heston have larger pricing errors for short maturities compared to medium and long term maturities in terms of MAPE. This confirms the maturity bias which has been explained theoretically in Chapter 3, where the volatility smile is less prominent for longer expiration dates.

Maturity 20 < T			
	BS model	BS extended	Heston Model
MPE	0,11%	0,24%	0,18%
MAPE	1,43%	0,42%	0,29%
MAE	1,9507	1,6433	1,1272
MSE	4,8726	4,8890	2,5308

Maturity 20 < T < 40			
	BS model	BS extended	Heston Model
MPE	-0,17%	0,46%	0,24%
MAPE	1,06%	0,95%	0,34%
MAE	3,7293	3,34	1,1917
MSE	22,794	18,7371	3,0017

Maturity 40 > T			
	BS model	BS extended	Heston Model
MPE	-1,19%	0,34%	0,10%
MAPE	0,75%	1,69%	0,25%
MAE	6,6509	7,4216	0,9580
MSE	75,11	80,53	2,2406

Table 6: Table reports in-sample pricing errors sorted by days to maturity for the Call options. T represents the remaining trading days to expiration of the option.

To see the degree of moneyness biased errors for in-sample pricing, Table (7) sorts the pricing errors for moneyness. The intervals of moneyness categories expands from:

- Less than 0.90: deep OTM, DOTM
- 0.90 - 0.97: OTM
- 0.97 - 1.00, ATM $^{-}$
- 1.00 - 1.03, ATM $^{+}$
- 1.03 - 1.10: ITM
- Above 1.10: deep ITM, DIT

Deep Out of Money: DOTM			
	BS model	BS extended	Heston Model
MPE	-0,66%	16,3%	0,54%
MAPE	3,22%	16,4%	0,89%
MAE	0,9745	2,87	0,2738
MSE	2,2384	17,95	0,1099

Out of Money: OTM			
	BS model	BS extended	Heston Model
MPE	0,45%	27%	0,38%
MAPE	9,25%	28%	0,29%
MAE	3,83	6,17	0,7851
MSE	21,7	58,46	1,04

At the Money: ATM			
	BS model	BS extended	Heston Model
MPE	4%	1,51%	0,45%
MAPE	4,8%	4,2%	0,05%
MAE	4,94	4,261	1,10
MSE	46,71	29,35	2,72

In the Money: ITM			
	BS model	BS extended	Heston Model
MPE	-6,1%	-3,05%	0,22%
MAPE	7,11%	3,38%	0,93%
MAE	7,75	4,011	0,99
MSE	36,02	37,04	1,61

Deep Out of Money: DITM			
	BS model	BS extended	Heston Model
MPE	-0,31%	-0,03%	0,66%
MAPE	0,42%	0,92%	0,36%
MAE	3,24	2,507	1,78
MSE	20,87	11,217	4,88

Table 7: Table reports in-sample pricing errors sorted by moneyness for the Call options.

Table (7) reports the pricing error of both Black-Scholes model and Heston model for each moneyness category. Also in this case, the Heston model outperforms the Black&Scholes model since for each moneyness class by looking at the MAPE.

Notice that the magnitude of improvement is notable, this can be noted by the MSE. The Heston model performs best on at-the-money options with a smallest MAPE. Besides, looking at the MPE, author found that the Black&Scholes model tends to undervalue the deep out-of-the-money options, in-the-money options and deep in-the-money options. This result supports the existence of volatility smile under the Black-Scholes model. On the contrary, the Heston model is liable to overvalue the deep out-of-the-money options, out-of-the-money options and also the in-the-money options, deep in-the-money options. It suggests that the Heston model does not generates a sneer smile respect to the BS model and that the parameters are able to capture it. To see the degree of moneyness biased errors for in-sample pricing, Table (8), for completeness, sorts the pricing errors for both maturity and moneyness.

Time		Short Maturity T < 20					Medium Maturity 20 < T < 40					Long Maturity T > 40				
Moneyness		DOTM	OTM	ATM	ITM	DITM	DOTM	OTM	ATM	ITM	DITM	DOTM	OTM	ATM	ITM	DITM
MPE	BS	-5,49%	2,26%	-0,02%	-0,01%	0,01%	15,33%	1,14%	-0,11%	-0,15%	-0,02%	-0,68%	0,05%	0,23%	-0,24%	-0,11%
	BS extended	-10,7%	1,89%	0,053%	-0,005%	0,01%	20,84%	2,99%	0,08%	-0,07%	0,01%	3,88%	0,74%	0,03%	-0,13%	-0,06%
	Heston	6,38%	0,72%	0,02%	0,01%	0,01%	-2,72%	0,14%	0,01%	0,95%	0,04%	-0,17%	0,03%	0,01%	0,00%	0,02%
MAPE	BS	7,85%	2,36%	0,07%	0,06%	0,01%	16,57%	1,27%	0,14%	0,16%	0,03%	0,83%	0,20%	0,23%	0,16%	0,11%
	BS extended	10,74%	1,97%	0,12%	0,048%	0,01%	20,84%	2,99%	0,15%	0,08%	0,03%	3,88%	0,7%	0,15%	0,13%	0,07%
	Heston	27,53%	0,82%	0,04%	0,04%	0,01%	5,65%	0,17%	0,03%	0,04%	0,04%	0,19%	0,04%	0,03%	0,02%	0,02%
MAE	BS	0,0589	1,4461	2,1501	1,6164	1,5637	0,4224	4,4845	4,4439	4,3492	2,3559	1,4925	4,4391	8,04	12,8989	6,1574
	BS extended	0,0537	1,1379	2,4010	1,3376	1,5890	0,4428	5,1933	3,7691	2,473	2,4909	4,34	8,899	6,02	6,665	3,809
	Heston	0,2065	0,5009	1,2896	1,1207	1,5477	0,144	0,5968	0,8223	0,9485	2,671	0,3434	0,9822	1,1535	0,9357	1,25
MSE	BS	0,0042	4,8899	7,2559	3,7504	3,6553	0,2651	24,2971	30,311	32,7429	8,7341	3,877	26,9068	98,7982	200,5654	53,8192
	BS extended	0,0043	3,3794	8,623	3,1382	3,7611	0,3560	31,80	21,65	15,714	9,766	27,94	95,17	50,70	71,80	23,10
	Heston	0,1842	0,4532	3,2129	2,5379	3,678	0,0273	0,9454	1,1333	1,27	9,4245	0,1554	2,9125	3,4355	1,2414	2,1494

Table 8: Table reports in-sample pricing errors for Call options sorted by days to maturity and moneyness.

4.7.2 Out-of-sample performance

The analysis also evaluates model's parameter stability over time by analyzing the out-of-sample valuation errors for the next week.

To conduct the 1 time-step ahead (i.e. week) out-of-sample analysis, the author for the Heston model uses the estimated structural parameters from the previous time to price the options for the current time. For the BS extension the author uses the previous level of rolling volatility. The pricing errors of the models are then compared to the benchmark BS, that remains unaltered, since the only parameters needed for the calibration (i.e. volatility) is fixed.

If the results show that a model is not able to outperform the benchmark, the author will conclude that the model is not appropriate to forecast option prices 1 time-step ahead.

Then, the author follows the same procedure for 3 week ahead out-of-sample pricing where the estimated parameters are used to forecast 3 time-step ahead[72].

4.7.2.1 One week ahead results

For the Heston model, the pricing errors worsen when shifting from in-sample pricing to 1 time-step ahead out-of-sample pricing. Looking at both Table (5) and (9), MAPE increases for the Heston model as expected. The MAPE pass from 0,08% to 0,21%. The BS MAPE remains roughly unchanged (0,30% vs 0,32%). The Heston model performs still better respect to BS, but as explained in Chapter 3, in the Heston model drawbacks, is high-sensitive to its parameters.

	BS model	BS extended	Heston Model
MPE	-0,11%	0,15%	0,12%
MAPE	0,32%	0,34%	0,21%
MAE	3,9570	4,1670	2,5365
MSE	30,1787	31,95	11,05

Table 9: 1 day Out-of-Sample pricing errors for Call options.

The Table below, sorts the errors according to maturity for 1 time-step ahead pricing errors.

[72] To further check the robustness of the models one can assess if the model parameters are stable through longer time periods and their ability to predict option prices.

Maturity 20 > T			
	BS model	BS extended	Heston Model
MPE	0,11%	0,27%	0,13%
MAPE	0,43%	0,45%	0,46%
MAE	1,6507	1,7539	1,7701
MSE	4,8726	5,5585	5,6397

Maturity 20 < T < 40			
	BS model	BS extended	Heston Model
MPE	-0,05%	0,74%	0,47%
MAPE	1,07%	1,08%	0,78%
MAE	3,50	3,5187	2,55
MSE	18,4578	20,73	11,5736

Maturity 40 < T			
	BS model	BS extended	Heston Model
MPE	-1,1%	0,64%	0,62%
MAPE	1,83%	1,96%	0,94%
MAE	6,0648	6,493	3,1218
MSE	57,93	60,223	14,9360

Table 10: Table reports 1 day ahead out-of-sample pricing errors for Call options sorted by days to maturity. T represents the remaining trading days to expiration of the option.

For short maturity, all BS model actually outperform Heston. However, for both medium and long term categories, Heston outperforms the BS model in terms of MAPE. Table (11) sorts pricing error across maturities. The extended BS performed very similar to the benchmark in terms of MAPE, but it does not undervalue options. The author decide to not report DOTM for distorting results due to not enough data. Notice that the BS model for the OTM options performs better than the Heston. For the remaining moneyness we might conclude that the Heston model also with the previous parameters outperforms the BS model, and most important, as always the Heston model does not underprice ITM and DITM options by looking the MPE.

Out of Money: OTM			
	BS model	BS extended	Heston Model
MPE	3,7%	23%	4,8%
MAPE	4,7%	23%	8,5%
MAE	2,5606	5,871	1,5397
MSE	10,92	56,38	5,7742

At the Money: ATM			
	BS model	BS extended	Heston Model
MPE	-3,1%	2,5%	1,8%
MAPE	5,4%	4,62%	3,6%

MAE	4,500	4,3079	3,5023
MSE	35,06	29,075	17,96

In the Money: ITM			
	BS model	BS extended	Heston Model
MPE	-4,2%	-2,3%	0,7%
MAPE	4,5%	2,8%	2,2%
MAE	5,958	3,119	2,4415
MSE	58,15	21,8429	9,7441

Deep Out of Money: DITM			
	BS model	BS extended	Heston Model
MPE	-0,37%	-0,02%	0,7%
MAPE	1,2%	1,0%	0,9%
MAE	3,4518	2,637	2,3966
MSE	22,57	12,505	8,480

Table 11: Table reports 1 day out-sample pricing errors sorted by monenyness for the Call options.

Table (12) reports the pricing errors sorted by maturity and moneyness. Across all of the maturity categories, the pricing errors decreases for the Heston model as the maturity increases. This indicates the models suffer from a maturity bias. The fact that MAPE decreases across moneyness confirms a moneyness bias.

Time		Short Maturity T < 20					Medium Maturity 20 < T < 40					Long Maturity 40 < T				
Moneyness		DOTM	OTM	ATM	ITM	DITM	DOTM	OTM	ATM	ITM	DITM	DOTM	OTM	ATM	ITM	DITM
MPE	BS	-	1,36%	-0,017%	-0,30%	-0,013%	-	0,26%	-0,061%	-0,014%	0,02%	-	0,082%	-0,20%	0,20%	-0,10%
	Extended BS	-	1,31%	0,067%	-0,001%	0,014%	-	2,22%	0,13%	-0,039%	0,02%	-	0,68%	0,07%	-0,12%	-0,06%
	Heston	-	1,52%	0,008%	0,005%	0,012%	-	0,33%	0,072%	0,028%	0,04%	-	0,14%	0,09%	0,03%	0,03%
MAPE	BS	-	1,39%	0,12%	0,072%	0,014%	-	0,26%	0,21%	0,15%	0,044%	-	0,12%	0,23%	0,20%	0,11%
	Extended BS	-	1,38%	0,13%	0,039%	0,015%	-	2,22%	0,18%	0,052%	0,040%	-	0,68%	0,17%	0,12%	0,07%
	Heston	-	1,73%	0,15%	0,089%	0,055%	-	1,22%	0,13%	0,085%	0,012%	-	0,24%	0,13%	0,72%	0,05%
MAE	BS	-	1,014	2,0549	2,014	1,5892	-	2,535	4,458	4,087	2,92	-	3,34	6,64	10,51	6,228
	Extended BS	-	0,986	2,695	1,332	1,635	-	4,72	4,202	1,322	2,778	-	8,771	5,773	5,555	3,863
	Heston	-	0,6381	2,6587	1,6947	1,7296	-	0,786	3,389	2,336	2,947	-	2,314	4,323	3,13	2,79
MSE	BS	-	2,4647	7,48	6,116	3,7491	-	11,34	26,35	23,40	11,95	-	14,96	66,95	125,32	54,71
	Extended BS	-	3,00	10,60	3,039	3,8966	-	31,63	27,71	2,84	11,04	-	34,24	45,99	47,52	25,44
	Heston	-	0,8571	10,42	5,0986	4,5510	-	2,08	18,31	8,80	11,59	-	9,84	24,06	14,23	10,93

Table 12: Table reports 1 day out-of-sample pricing errors for Call options, sorted by days to maturity and moneyness.

The author then, performed the 3 weeks ahead Out-of-Sample pricing errors. One interesting result is that the out-of-sample pricing error 3 weeks ahead are mostly in favor of the BS model respect to the Heston model. The author does not report them, but he concludes that three weeks are sufficient to change a lot the Heston parameters, and, as theory said in section (3.6), the Heston model is high-dependent from parameters and calibration. We might conclude by saying that Stochastic volatility models performs better in terms of pricing on the S&P 500 for the sample considered if a lot of computational power is available and used in order to allow the model to adapt to the different market situations. Flexible parameters in time allow to a better fit in reality. On the other hand the BS model gives us very close results comparing to the previous works, this could be possible in a low and stable-volatility market, where the volatility for the sample considered remains stable at the origin levels. In a high-sensitive volatility market, the BS model surely would have performed worse by underpricing a lot both OTM and ITM options.

4.8 Economic results

The scope of this section, is to answer the main question of this work:

"Which one out of the dynamic-hedging strategies is most efficient in reducing the risk of an options Portfolio? How things change by considering advanced features such as stochastic volatility, contingency constraints, leverage effect and non-normal distribution of returns?"

After used for each strategy and each step in time the function `hedgeopt` the author has a disposition the quantity that allow the portfolio to be neutral to the considered Greeks. Having a disposition those quantities and the market prices, the author calculated the portfolio value for each time and for each strategy. Then, he could derive the P/L for the 9 strategies[73] are reported in Table (15):

	Delta	Delta-Gamma	Delta-Gamma-Vega
BS	-5278,73 $	-4203,28 $	-5202,11 $
Extended BS	-6146,05 $	-3819,51 $	-3082,95 $
Heston	-8902,11 $	-5743,62 $	530,83 $

Table 15: P/L of the nine hedging strategies.

[73] The price to calculate the P/L was chosen accordingly between the bid or ask, depends on the transaction.

Numbers are mostly negative, indeed the author has chosen a period where the index was up-trending, starting with a short calls portfolio is a normal result. He has chosen this period to highlight losses if a naked position was implemented.

In order to calculate the P/L it was sufficient to see the variation of the portfolio in time, since the hedging is performed in a self-financing way (i.e. no other funds are requested) and every time that an option is near to expiration (i.e. previous time-step before expiration) the author performed the hedge adding a constraint with zero quantity for the "near-expiration" option. Constraints in the `hedgeopt` function are available in the input "Conset":

```
[PortSens, [], PortHolds] = hedgeopt (Sensitivities, Price,
CurrentHolds, [], [], [], TargetSens, Conset)
```

For example the author, having in expiration 6 options in the portfolio, imposed a zero quantity for these options. The other might vary between the Lowerbound and Upperbound. +/- 180 was chosen to be a reasonable range.

```
LowerBounds = [0 0 0 0 0 0 -180 -180 -180 -180 -180 -180 -180 -180 -180 -180 -180
-180 -180 -180 -180 -180 -180 -180];
UpperBounds = [0 0 0 0 0 0 180 180 180 180 180 180 180 180 180 180 180 180 180
180 180 180 180 180];
ConSet = portcons('AssetLims', LowerBounds, UpperBounds);
```

What happened if no Hedging strategy was implemented, but considering a naked position on the portfolio, so by leaving unaltered the quantity sold at the begin of the portfolio?

The portfolio would have incurred in a loss of – 714'735 $

Chapter 5 | Conclusion and Extension

5.1 Conclusions

The author finds that none of the models presented can fully approximate the market in terms of pricing, but stochastic volatility models such as the Heston model can however improve the pricing errors significantly, and as consequence the Hedging performances in some cases.
Calibrating models is essential to obtain good results. In this thesis the author has used MATLAB® to estimate parameters and the author is positive that the parameter set obtained from the calibration is reasonable, indeed this is a results in very low pricing errors; moreover the parameters set are in line with the previous literature. The Heston model may to some extent be calibrated slightly better without the time constraint, using always a stochastic global optimizer, but this is of minor importance.
In spite of model theory, the parameters have large standard deviations making them flexible and adaptable to new market conditions. This seems to be a major concern since models that have large standard deviations in parameters seem to perform better[74]. All of the parameters seem to incorporate the volatility smile and leverage effect which gives confidence in the results. As a last note, one should not spend infinite time for the perfect calibration when the model by itself is imperfect. It is just as vital to understand the assumptions behind the models and how the different parameters affect the output.
The thesis concludes that models that are able to incorporate the volatility as a stochastic variable improve the ability of pricing from the options on the S&P 500 Index, a familiar result in the academia.

5.2 Extensions

The author presents some extension that could be used to improve results:

- Use a Bates model for pricing short-dated options: It is known in the literature, that the Heston model only has a limited ability of generating extreme skews for short maturities, therefore the model error related to this will be large. The Bates model is an

[74] Kim & Kim, 2004.

extension of the Heston model, adding jumps following a compound Poisson process in the stock dynamics. But a Bates model, as drawback requires a higher level of complexity, the parameters to be estimated pass from 5 to 8. An interesting fact could be evaluate a dynamic hedging strategies also for the Bates model, and see by the statistical performance if outperforms the Heston model in terms of pricing errors.

- Correlation of parameters: From the calibration point of view, there may be a set of parameters in the high vol-vol, high speed of mean reverting regime which gives a compatible fit to a set of parameters in the lower vol-vol, lower mean reverting regime. This feature is sometimes characterized by the phenomenon that ‘parameters that are highly correlated’. As extension one could develop a partial “resampling scheme” which groups the highly correlated parameters together in order to improve the convergence to a global optimum.

- Use different mid-price for calibration: Most studies refer to the daily closing price (or mid-point of the bid-ask spread, as used in this thesis) as reflecting the value of the underlying asset on a daily basis. However, recent studies argued that the use of daily closing prices increase the noise level, which tends to overprice options. One therefore could use the volume weighted average price (VWAP) since VWAP has been shown to be statistically more efficient.

- Weighted measure of volatility: when volatilities change, the implied volatility of short-dated options tend to change by more than the implied volatility of long-dated options. The vega of the portfolio is therefore often calculated by changing the volatilities of long-dated option by less than that of short-dated options.

Appendix

A test of local linearity of a derivative security (that is a function of the underlying asset) between prices S_1 and S_2 with $0 < \lambda < 1$, will satisfy the following equality:

$$V(\lambda S_1 + (1-\lambda)S_2) = \lambda V(S_1) + (1-\lambda)V(S_2)$$

It will be convex between S_1 and S_2 if:

$$V(\lambda S_1 + (1-\lambda)S_2) \leq \lambda V(S_1) + (1-\lambda)V(S_2)$$

It will be concave if:

$$V(\lambda S_1 + (1-\lambda)S_2) \geq \lambda V(S_1) + (1-\lambda)V(S_2)$$

Relationship between delta, theta, and gamma.

The value of a portfolio Π on a derivative must satisfy this differential equation:

$$\frac{\partial \Pi}{\partial t} + rS\frac{\partial \Pi}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \Pi}{\partial S^2} = r\Pi$$

Since:

$$\Theta = \frac{\partial \Pi}{\partial t} \qquad \Delta = \frac{\partial \Pi}{\partial S} \qquad \Gamma = \frac{\partial^2 \Pi}{\partial S^2}$$

It follows that:

$$\Theta + rS\Delta + \frac{1}{2}\sigma^2 S^2 \Gamma = r\Pi$$

For a delta neutral portfolio:

$$\Theta + \frac{1}{2}\sigma^2 S^2 \Gamma = r\Pi$$

This shows that, when Θ is large and positive, gamma of a portfolio tends to be large and negative, and vice versa. This show the relationship between $\Delta\Pi$ and ΔS. When gamma is positive, theta tends be negative. The portfolio declines in value if there is no change in S, but increases in value if there is a large positive or negative change in S. When gamma is negative, theta tends to be positive and the reverse is true; the portfolio increases in value if there is no change in S but decreases in value if there is a large positive or negative change in S. As the absolute value of gamma increases the sensitivity of the value of the portfolio to S increases. This is consistent with the way and explain why theta can to some extend be regarded as proxy for gamma in a delta-neutral portfolio.

MATLAB® CODES

European Call using Numerical Integration

The function Call = HestonCallQuadl(kappa,theta,sigma,rho,v0,r,T,s0,K) calculates the value of a European call, equation () with a specific set of parameters. It calls HestonP(kappa,theta,sigma, rho,v0,r,T,s0,K,type), where type = 1,2, which evaluates (44), either using an adaptive *Gauss Lobatto* rule or adaptive *Simpson's Rule*.

This, in turn, calls HestonPIntegrand(phi,kappa,theta,sigma,rho,v0,r,T,s0,K,type), which evaluates the integrand of (). This, in turn calls Hestf(phi,kappa,theta, sigma,rho,v0,r,T,s0,type), which evaluates the '*f*' function in the integrand of (44).

```
function call = HestonCallQuadl(kappa,theta,sigma,rho,v0,r,T,s0,K)
warning off;
call = s0*HestonP(kappa,theta,sigma,rho,v0,r,T,s0,K,1) - ...
K*exp(-r*T)*HestonP(kappa,theta,sigma,rho,v0,r,T,s0,K,2);
function ret = HestonP(kappa,theta,sigma,rho,v0,r,T,s0,K,type)
ret = 0.5 + 1/pi*quadl(@HestonPIntegrand,0,100,[],[],kappa, ...
theta,sigma,rho,v0,r,T,s0,K,type);
function ret = HestonPIntegrand(phi,kappa,theta,sigma,rho, ...
v0,r,T,s0,K,type)
ret = real(exp(-i*phi*log(K)).*Hestf(phi,kappa,theta,sigma, ...
rho,v0,r,T,s0,type)./(i*phi));
function f = Hestf(phi,kappa,theta,sigma,rho,v0,r,T,s0,type);
if type == 1
u = 0.5;
b = kappa - rho*sigma;
else
u = -0.5;
b = kappa;
end
a = kappa*theta; x = log(s0);
d = sqrt((rho*sigma*phi.*i-b).^2-sigma^2*(2*u*phi.*i-phi.^2));
g = (b-rho*sigma*phi*i + d)./(b-rho*sigma*phi*i - d);
C = r*phi.*i*T + a/sigma^2.*((b- rho*sigma*phi*i + d)*T - ...
2*log((1-g.*exp(d*T))./(1-g)));
D = (b-rho*sigma*phi*i + d)./sigma^2.*((1-exp(d*T))./ ...
(1-g.*exp(d*T)));
f = exp(C + D*v0 + i*phi*x);
```

Calibrating Heston Model using `lsqnonlin`

The following contains the MATLAB® code for calibrating the Heston model, for one step in time, using MATLAB® `lsqnonlin`. The script file HestonLsCalibration.m initiates the calibration process. It creates a handle on the function HestonDifferences.m that calculates the differences between the model and market prices within `lsqnonlin`. The 'load OptionData.xlsx'[75] line imports the strikes, maturities, market prices, bid and offers, etc., of the options and underlying from the pre-built excel file. The first parameter of input that MATLAB® sends into HestonDifference is the feller condition. It is done here in this way because it is easier to encapsulate the constraint $2\kappa\theta - \sigma^2 > 0$. Reasonable bounds on the parameters were chosen relative to this.

```
clear;
NoOfIterations = 0;
[data,text]=xlsread('OptionData.xlsx');
%OptionData = format [r-q,T,S0,K,Option Value,bid,offer]
Size = size(data);
NoOfOptions = Size(1);
OptionData=data;
%input sequence in initial vectors [2*kappa*theta - sigma^2,...
% theta,sigma,rho,v0]
x0 = [6.5482 0.0731 2.3012 -0.4176 0.1838];
lb = [0 0 0 -1 0];
ub = [20 1 5 0 1];
options = optimset('MaxFunEvals',20000);
%sets the max no. of iteration to 20000 so that termination
%doesn't take place early.
tic;
Calibration = lsqnonlin(@HestonDifferences,x0,lb,ub);
toc;
Solution = [(Calibration(1)+Calibration(3)^2)/(2*Calibration(2)),
Calibration(2:5)];
```

```
function ret = HestonCallDifferences(input)
NoOfIterations = NoOfIterations + 1;
%counts the no of iterations run to calibrate model
for i = 1:NoOfOptions
PriceDifference(i) = (OptionData(i,5)-HestonCallQuadl( ...
(input(1)+input(3)^2)/(2*input(2)),input(2), ...
input(3),input(4),input(5), ...
OptionData(i,1),OptionData(i,2),OptionData(i,3), ...
OptionData(i,4)))/sqrt((abs(OptionData(i,6)- ...
OptionData(i,7))));
%input matrix = [kappa theta sigma rho v0]
end
ret = PriceDifference';
```

[75] An example of the composition of OptionData.xlsx is presented at the end of this sub-section

As mentioned before, the file OptionData is an Excel file containing the market data for calibration for a specific step in time. For illustrative purposes, OptionData.xlsx contains the following information, and it is used as input with the following order:

30/01/2015							
Name	rate	Term	Spot	Strike	Mid Price	Bid	Ask
CALL SPX MAR15 1705	0,002058	0,13492	1994,99	1705	290,45	290,2	290,7
CALL SPX MAR15 1715	0,002058	0,13492	1994,99	1715	280,95	280,7	281,2
CALL SPX MAR15 1845	0,002058	0,13492	1994,99	1845	162,35	162,1	162,6
CALL SPX MAR15 1870	0,002058	0,13492	1994,99	1870	141,05	140,8	141,3
CALL SPX MAR15 2000	0,002058	0,13492	1994,99	2000	45,95	45,7	46,2
CALL SPX MAR15 2100	0,002058	0,13492	1994,99	2100	7,2	7,1	7,3
CALL SPX APR15 2000	0,002385	0,21429	1994,99	2000	57,55	57,3	57,8
CALL SPX APR15 2020	0,002385	0,21429	1994,99	2020	46,45	46,2	46,7
CALL SPX APR15 2070	0,002385	0,21429	1994,99	2070	23,95	23,7	24,2
CALL SPX APR15 2075	0,002385	0,21429	1994,99	2075	22,15	21,9	22,4
CALL SPX APR15 2250	0,002385	0,21429	1994,99	2250	0,5	0,45	0,55
CALL SPX MAY15 1900	0,002722	0,29365	1994,99	1900	134,55	134,3	134,8
CALL SPX MAY15 2000	0,002722	0,29365	1994,99	2000	67,15	66,9	67,4
CALL SPX MAY15 2050	0,002722	0,29365	1994,99	2050	41,4	40,9	41,9
CALL SPX MAY15 2100	0,002722	0,29365	1994,99	2100	21,85	21,6	22,1
CALL SPX MAY15 2150	0,002722	0,29365	1994,99	2150	9,5	9,4	9,6
CALL SPX MAY15 2200	0,002722	0,29365	1994,99	2200	3,6	3,5	3,7
CALL SPX JUN15 1850	0,003125	0,39286	1994,99	1850	180,95	180,7	181,2
CALL SPX JUN15 1900	0,003125	0,39286	1994,99	1900	142,95	142,7	143,2
CALL SPX JUN15 1975	0,003125	0,39286	1994,99	1975	91,95	91,7	92,2
CALL SPX JUN15 2100	0,003125	0,39286	1994,99	2100	30,15	29,9	30,4
CALL SPX JUN15 2175	0,003125	0,39286	1994,99	2175	10,6	10,5	10,7
CALL SPX JUN15 2200	0,003125	0,39286	1994,99	2200	7	6,9	7,1
CALL SPX JUN15 2300	0,003125	0,39286	1994,99	2300	1,2	1,1	1,3

`Asamin` Installation Instructions for Windows

The installation procedure for Windows is as follows:

- Ensure that there is a C compiler installed on the PC that will be implementing the ASA routine. The author performed the calculations with Visual studio 2010 for Windows. Available compiler for Matlab versions are available on *mathworks.com*
- Download the ASA packages from http://www.ingber.com
- Download the ASAMIN packages from http://ssakata.sdf.org/software/ and place them in their own directory.

- Place the ASA _les asa.c, asa.h and asa user.h in the same directory in which the ASAMIN packages were placed.
- Open the MATLAB® console and change the \current folder" to the directory containing the ASAMIN and relevant ASA _les.
- In the MATLAB® command window, type:

mex asamin.c asa.c -DUSER_ACCEPTANCE_TEST#TRUE -DUSER_ASA_OUT#TRUE -DDBL_MIN#2.2250738585072014e-308.

MATLAB® will then create a MEX file, allowing the user to interface with the C language ASA code via MATLAB. Importantly, the pathname for the directory containing the ASAMIN, relevant ASA and MEX files must be incorporated into every script that calls the asamin function. This can be done through the use of the addpath command in MATLAB®.

Calibrating Heston Model using ASA

The scheme that controls the 'acceptance' of new solutions is so simple that the cost of implementing `asamin` is purely dependent on the computational efficiency of evaluating the objective function. In our case, calculation of the objective function for a given set of parameters entails the evaluation of a large number of options. This makes `asamin` very computationally demanding and time consuming. The following script, CalibrationASAmin, calibrates the model using `asamin`. It uses the function HestonCostFunc.m. This code reported in only for one step in time.

```
clear;
[data,text]=xlsread('OptionData.xlsx');
%OptionData = [r-q,T,S0,K,Option Value,bid,offer]
OptionData=data;
Size = size(data);
NoOfOptions = Size(1);
%input sequence in initial vectors [kappa,theta,sigma,rho,v0]
x0 = [4 0.05 0.30 -0.90 0.15];
lb = [0 0 0 -1 0];
ub = [10 1 5 0 1];
asamin('set','test_in_cost_func',0)
tic;
[fstar, xstar, grad, hessian, state] = asamin('minimize',...
'HestonCostFunc',x0',lb',ub',-1*ones(5,1));
toc;
```

```
function [cost , flag] = HestonCostFunc(input)
%input matrix = [kappa theta sigma rho v0]
NoOfIterations = NoOfIterations + 1;
if (2*input(1)*input(2)<=input(3)^2) %test for constraint
flag = 0; %flag = 0 if contraint is violated, else = 1
cost = 0
else
for i = 1:NoOfOptions
PriceDifference(i) = (OptionData(i,5)-HestonCallQuadl(...
input(1),input(2),input(3),input(4),input(5), ...
OptionData(i,1),OptionData(i,2),OptionData(i,3),...
OptionData(i,4)))/sqrt((abs(OptionData(i,6)- ...
OptionData(i,7))));
end
cost = sum(PriceDifference.^2)
ObjectiveFunc(NoOfIterations) = cost; %stores the path of
flag = 1; %the optimizer
end
```

Greeks in Heston

Delta of the call option: Function HestonCallDelta.m

```
function ret =HestonCallDelta(kappa,theta,sigma,rho,v0,r,T,s0,K)
epsD = abs(s0)*eps^(1/6);
ret=(HestonCallQuad(kappa,theta,sigma,rho,v0,r,T,s0+epsD,K)...
    -HestonCallQuad(kappa,theta,sigma,rho,v0,r,T,s0-epsD,K))/(2*epsD);
if ret>1
    ret=1;
end
if ret<0
    ret=0;
end
end
```

Gamma: Function HestonGamma.m

```
function ret =Hestondelta(kappa,theta,sigma,rho,v0,r,T,s0,K)
epsD = abs(s0)*eps^(1/6);
a= HestonCallQuad(kappa,theta,sigma,rho,v0,r,T,s0+epsD,K);
b= HestonCallQuad(kappa,theta,sigma,rho,v0,r,T,s0-epsD,K);
c= HestonCallQuad(kappa,theta,sigma,rho,v0,r,T,s0,K);
ret=(a-2*c+b)/(epsD^2);
if ret>1
    ret=1;
end
if ret<0
    ret=0;
end
end
```

Vega: Function HestonVega1.m

```
function ret =HestonVega1(kappa,theta,sigma,rho,v0,r,T,s0,K)
epsD = abs(v0)*eps^(1/6);
a=HestonCallQuad(kappa,theta,sigma,rho,v0+epsD,r,T,s0,K);
b=HestonCallQuad(kappa,theta,sigma,rho,v0-epsD,r,T,s0,K);
ret=((a-b)/(2*epsD)*2*sqrt(v0);
end
```

References

[1] Dynamic Hedging: Managing Vanilla and Exotic Options, John Wiley & Son, New York, Taleb Nassim, 1997.

[2] The Valuation of Options and Corporate Liabilities. Journal of Political Economy, 81, 637-654, Black, F., and Scholes, M., 1973.

[3] A closed-form solution for options with stochastic volatility with applications to bond and currency options. Review of Financial Studies 6, 327-343.Heston, 1993.

[4] Options, Futures, And Other Derivatives, Hull, J. C., Pearson Prentice Hall, 8^{th} edition, 2011.

[5] Empirical comparison of alternative stochastic volatility option pricing models: Evidence from Korean KOSPI 200 index options market. Basin Finance Journal, 12, 117-142.Kim, I. J., and Kim, S., 2004.

[6] Fast calibration in Heston model, Dr. Stefan Gerhold, 2012.

[7] Time Consistency in Option Pricing Models, Johan Nykvist, 2009.

[8] The Heston Model:A Practical Approach with Matlab Code, Nimalin Moodley, 2005.

[9] Heston stochastic volatility model implementation, calibration and some extensions, Wilmott, Mikhailov, S. & Nogel, U., 2003.

[10] Implementation of a simulated annealing algorithm for MATLAB, Technical report, Linkoping Institute of Technology, Moins, S., 2002.

[11] Empirical properties of asset returns: stylized facts and statistical issues, Quantitative Finance 1, 223–236. Cont, R., 2001.

[12] A theory of the term structure of interest rates, Econometrica 53, 385–407.Cox, J. C., Ingersoll, J. E. & Ross, S. A., 1985.

[13] Adaptve quadrature - revisited, Technical report, Departement Informatik, ETH Zurich. Gander, W. & Gautschi, W., 1998.

[14] ‘Lecture 1: Stochastic volatility and local volatility’, Case Studies in Financial Modelling Notes, Courant Institute of Mathematical Sciences. Gatheral, J., 2004.

[15] Simulated annealing: Practice versus theory, Journal of Mathematical Computational Modelling 18(11), 29–57. Ingber, A. L., 1993.

[16] Option Pricing Formulae using Fourier Transform: Theory and Application, Martin Schmelzle, 2010.

[17] The Volatility Surface: A Practitioner’s Guide (Wiley Finance Series), John Wiley & Sons, New York. Gatheral, J., 2006.

[18] Note on the inversion theorem, Biometrika **38**(3–4), 481–482. Gil-Pelaez, J., 1951.

[19] Option Valuation under Stochastic Volatility: With Mathematica Code, Finance Press, Newport Beach. Lewis, A., 2000.

[20] Does the Characteristic Function Numerically Distinguish Distributions, The American Statistician 49(2). Waller, L. A., 1995.

[21] Pricing S&P 500 Index Options under Stochastic Volatility with the Indirect Inference Method, Jinghong Shu, 2003.

[22] Empirical Comparison of Alternative Option Pricing Models, Andreas Nguyen, 2013.

[23] "The Importance of the Loss Function in Option Pricing". Journal of Financial Economics, vol. 72, No 2, 291-318. Christoffersen, P., & Jacobs, K., 2004.

[24] Calibration of the Heston Model with Application in Derivative Pricing and Hedging, Chen Bin, 2007.

[25] Option values under stochastic volatility: Theory and empirical estimates. Journal of Financial Economics, 19(2), 351-372. Wiggins, J. B., 1987.

[26] An Analysis of the Heston Stochastic Volatility Model: Implementation and Calibration using Matlab, Ricardo Crisóstomo, 2014

[27] Two Singular Diffusion Problems. Annals of Mathematics 54. Feller, W., 1951.

[28] Valuing a European Option with the Heston Model. Yuan Yang, 2013.

[29] The Black-Scholes and Heston Models for Option Pricing. Ziqun Ye, 2013.

[30] Beyond Black-Scholes: The Stochastic Volatility Option Pricing Model and Empirical Evidence from Thailand. Woraphon Wattanatorn, 2014.

[31] Delta Gamma Hedging and the Black-Scholes Partial Differential Equation (PDE). Sudhakar Raju, 2012.

[32] Maturity and volatility effects on smiles or dying smiling? Duque, Joao e Patricia T. Lopes 2000.

[33] Empirical performance of alternative option pricing models, Journal of Finance. Bakshi, G., Cao, C. & Chen, Z., 1997.

[34] The Shape and Term Structure of the Index Option Smirk: Why Multifactor Stochastic Volatility Models Work so Well. Peter Christoffersen & Heston & Kris Jacobs, 2009.

[35] A Perfect Calibration! Now What? Wim Schoutens Erwin Simons, Jurgen Tistaert, 2003.

[36] The little Heston trap. Wilmott Magazine, January 2007, 83-92. Albrecher, H., Mayer, P., Schoutens, W. and Tistaert, J., 2007.

[37] Stock Return Characteristic, skew laws and the differential pricing of individual equity options, Bakshi, Madam, 2000.

[38] Maximum Likelihood Estimation of Stochastic Volatility Models, Working paper, Princeton University. Ait-Sahalia, Y., Kimmel, R., 2005.

[39] Almost Exact Solution of the Heston Stochastic Volatility Model, International Journal of Theoretical and Applied Finance, 1, pp. 1-43. Alexander, V.H., 2010.

[40] Implied volatility functions: empirical tests. Journal of Finance 53 Dumas, B., Flemming, J., Whaley, R.E., 1998.

[41] Recovering volatility from option prices by evolutionary optimization, Cont, Hamida, 2005.

[42] The calibration of stock option pricing models using inverse problem methodology. Research Paper Series 39, Quantitative Finance Research Centre, University of Technology, Sydney. C. Chiarella, M. Craddock and N. El-Hassan, 2000.

[43] Applications of Fourier Transform to Smile Modeling, 2nd Edition. Springer J.W. Zhu, 2010.

[44] The Heston model and its application in Matlab, Fabrice Douglas Rouah, Wiley, 2013.

[45] Studies of Stock Price Volatility Changes. Proceedings of the Meetings of the American Association, Business and Economic Statistic Section, 177-181. Black, R, 1976.

Internet sources and data providers

[1] Datastream: www.datastream.com (Accessed on 12th January 2015).

[2] Wilmott: www.wilmott.com | the quantitative finance community.