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Research article

Analytical solutions to the coupled fractional neutron diffusion equations with delayed neutrons system using Laplace transform method

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Abstract: The neutron diffusion equation (NDE) is one of the most important partial differential equations (PDEs), to describe the neutron behavior in nuclear reactors and many physical phenomena. In this paper, we reformulate this problem via Caputo fractional derivative with integer-order initial conditions, whose physical meanings, in this case, are very evident by describing the whole-time domain of physical processing. The main aim of this work is to present the analytical exact solutions to the fractional neutron diffusion equation (F-NDE) with one delayed neutrons group using the Laplace transform (LT) in the sense of the Caputo operator. Moreover, the poles and residues of this problem are discussed and determined. To show the accuracy, efficiency, and applicability of our proposed technique, some numerical comparisons and graphical results for neutron flux simulations are given and tested at different values of time t and order α which includes the exact solutions (when $\alpha = 1$). Finally, Mathematica software (Version 12) was used in this work to calculate the numerical quantities.

Keywords: diffusion equation; kinetic exact solution; Laplace transform; Caputo factional operator
Mathematics Subject Classification: 35R11, 44A10, 82D75

1. Introduction

In the past decades, various contributions have been made to both theory and applications in the branch of partial fractional differential equations (P-FDEs) as it has shown its strength in how it has expressed vast amounts of physical models which helps in explaining vast amounts of natural phenomena such as dynamical models, fluid dynamics, population dynamical systems, random walks, electrical circuits, control, wave propagation phenomena, pantograph system, Schrodinger equation, continuum and statistical mechanics, dispersive models, oscillation of earthquake, heat transfer model, bacteria population model and NDEs [1–26].

For example, Oldham and Spanier [1] studied the theory of fractional calculus and its applications to differentiation and integration at arbitrary order; Miller and Ross [2] presented the basic concepts of fractional calculus as well as FDEs; Podlubny [3] presented the solutions of some FDEs with important applications; Kilbas et al. [4] studied the theory and some applications of FDEs; Mainardi [5] presented mathematical models of fractional calculus and waves in linear viscoelasticity; Almeida et al. [6] introduced the variable-order fractional calculus of variations; Sun et al. [7] introduced a collection of real world applications of fractional calculus in science and engineering; Al-Zhour [8] presented a new computational formulae of fundamental fractional exponential matrix with some electrical applications; Aguilar [9] described the behavioral characteristics of a cap-resistor, memcapacitor, and a memristor from the response obtained of RC and RL electrical circuits by FDEs; Gomez-Aguilar et al. [10] investigated analytical and numerical solutions of electrical circuits described by fractional derivatives; Nigmatullin and Baleanu [11] derived the fractional Newtonian equations of motion with memory; Hasan et al. [12] analyzed a fractional population dynamical system using the Atangana-baleanu fractional framework to reproduce the kernel technique; Magin [13] studied fractional calculus in bioengineering; Manna and Merle [14] studied asymptotic dynamics of short waves as an example of nonlinear dispersive models; El-Ajou and Al-Zhour [15] introduced a vector series solution for a class of hyperbolic system of Caputo-time-F-PDEs with variable coefficients; El-Ajou et al. [16] investigated the construction and prediction of solitary pattern solutions for nonlinear time-fractional dispersive PDEs; A. El-Ajou et al. [17] presented an analytical solutions for a class of linear non-homogenous higher order matrix F-PDEs; El-Ajou et al. [18] presented analytical and numerical solutions of the fractional multi-pantograph system; Moa'ath et al. [19] made an interesting comparison between conformable and Caputo derivative series of the solutions for nonlinear time fractional Schrodinger PDEs; El-Ajou et al. [20] created the series solutions of nonlinear conformable fractional KdV-burgers PDEs with some applications; Shqair et al. [21] used the residual power series method to solve multi-energy groups of NDEs; Eriqat et al. [22] introduced the Laplace-residual power series technique for solving linear and nonlinear neutral fractional pantograph equations, El-Ajou et al. [23] presented the smooth expansion to solve high-order linear conformable fractional PDEs, Fisher [24] studied the fractional wave of advantageous genes; Merdan [25] solved the time-fractional reaction-diffusion equation with modified Riemann-Liouville derivative; and El-Sayed et al. [26] solved the generalized reaction-diffusion model for bacterial colony.

The reasons why fractional topics have become famous are [8–11]:

(i) It can extend the time domain in models.

(ii) It can explicate as memory and the physical meaning of the fractional order, which is used extremely for most physics, control models and engineering.

The Caputo fractional derivative of one-dimensional flux $\varphi(x,t)$ of order $\alpha > 0$ is defined by:

$$\mathfrak{D}_t^{\alpha}\,\varphi(x,t) = J_t^{m-\alpha}\mathfrak{D}_t^{m}\,\varphi(x,t)\,, m-1<\alpha<m, \qquad m \in \mathbb{N}, \qquad x \in I, t>0,$$

where I is the interval and J_t^{β} the time Riemann-Liouville fractional integral operator of order $\beta > 0$, given by:

$$J_t^{\beta}\ \varphi(x,t) = \begin{cases} \dfrac{1}{\Gamma(\beta)}\displaystyle\int_0^t (t-\tau)^{\beta-1}\,\varphi(x,t)\,d\tau\,, & \beta>0\,, t>\tau\geq 0, \\ \varphi(x,t)\,, & \beta = 0. \end{cases}$$

Some useful and important properties of fractional operators can be summarized for $\mu > -1$, $c \in \mathbb{R}$, $m-1<\alpha\leq m$, $t \geq 0$, and $\omega > 0$ as [15–26]:

(i) $\mathfrak{D}_t^{\alpha} c = 0.$

(ii) $\mathfrak{D}_t^{\alpha} t^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+1-\beta)} t^{\mu-\alpha}.$

(iii) $\mathfrak{D}_t^{\alpha} J_t^{\alpha}\,\varphi(x,t) = \varphi(x,t).$

(iv) $\mathcal{L}[\mathfrak{D}_t^{\alpha}\,\varphi(x,t)] = \omega^{\alpha}\,\phi(x,\omega) - \sum_{k=0}^{m-1}\omega^{\alpha-k-1}\,\partial_t^{k}\,\varphi(x,0),\ 0<\alpha<1$

where $\phi(x,\omega) = \mathcal{L}[\varphi(x,t)]$ is the LT of $\varphi(x,t)$.

Nuclear reactors guarantee the demands of human needs for energy. For over 7 decades, since 1951, when the first small experimental breeder reactor, EBR-1, was made in the Argonne National Laboratory, nuclear reactor theory has been of very high interest in the community of science all over the world.

The main books of nuclear reactor theory work cover nuclear reactor theory such as nuclear reactor physics by Stacey [27], nuclear reactor analysis by Duderstadt and Hamilton [28], and introduction to nuclear engineering by Lamarsh [29]. Because nuclear reactors are not homogeneous systems (they have many components such as fuel, cladding, coolant, and control rods), dealing with these parts is a difficult task that requires a complete team working system.

The most significant element governing nuclear reactors is neutron behavior, where the spread of neutrons throughout the reactor illustrates the nuclear reaction, which is typically fission. When a heavy nucleus, such as uranium (235), splits into two new nuclei known as fragments, energy is released; however, this energy release must be calculated because it depends on the reactor's neutron count, which must remain constant. This situation is known mathematically as the steady state neutron diffusion equation.

Neutron behavior in nuclear reactors is governed by the NDE (which is a special case of the neutron transport equation after applying Fick's law) [27–29], which captured many authors' interest in solving with various techniques. Such as: Hemispherical symmetry, NDE in cylindrical symmetry, reflected spherical reactor NDEs and a two-group reflected cylindrical reactor were solved using the homotopy perturbation method [30–33]; multi-energy groups of the NDEs and symmetric fractional Helmholtz cylindrically equation were solved using the residual power series technique [21,34]; and modified differential transform technique was used to find the analytical solution of NDE in reflected reactors employing [35].

The continuity equation represents the NDE from high to low regions of neutron flux, i.e.,

$$\frac{1}{V}\frac{\partial\varphi(x,t)}{\partial t} = s(x,t) - \Sigma_a\varphi(x,t) - \nabla.J(x,t), \tag{1.1}$$

where V is neutron velocity, $\sum_a$ is the macroscopic absorption cross-section. Eq (1.1) relates the neutron flux $\varphi(x,t)$ to the neutron current density $J(x,t)$. While $s(x,t)$ represents the source of neutrons. The neutron flux $\varphi(x,t)$ is equivalent to the neutron intensity in the case of a unidirectional neutron beam. According to Fick's law, the current density $J(x,t)$ is proportional to the negative gradient of the neutron flux $\varphi(x,t)$, i.e.,

$$J(x,t) = -D\nabla\varphi(x,t), \tag{1.2}$$

where D is the coefficient of neutron diffusion. In the case of a homogeneous system and after applying Fick's law, Eq (1.1) can be written as:

$$\frac{1}{V}\frac{\partial\varphi(x,t)}{\partial t} = s(x,t) - \Sigma_a\varphi(x,t) + D\nabla^2\varphi(x,t). \tag{1.3}$$

The concept of criticality in nuclear reactors (i.e., safe number of neutron generation in every generation in nuclear reactors) must be observed carefully to sustain the safety of nuclear reactors.

The mathematical expression of the criticality is maintaining the multiplication factor to one, which means that the neutron number must always be constant in the reactor. The criticality condition makes the NDE in Eq (1.3), a time independent equation and it can be re-written as:

$$s(x) - \Sigma_a(x)\varphi(r) + D\nabla^2\varphi(x) = 0, \tag{1.4}$$

where,

$$s(x) = v\Sigma_f\varphi(x), \tag{1.5}$$

where Σ_f is the fission macroscopic cross-sections, v is the neutron number per fission. In general, the microscopic cross-section σ represents the probability of the reaction while the relation between the macroscopic cross-section Σ and microscopic cross-section σ is $\Sigma = \sigma N$. Eq (1.4), represents the time independent NDE, in the reactor which is represented as:

$$\nabla^2\varphi(x) + B^2\varphi(x) = 0, \tag{1.6}$$

where the nuclear reactor concept called the nuclear buckling B^2 is defined as:

$$B^2 = \frac{v\Sigma_f - \left(\Sigma_f + \Sigma_\gamma\right)}{D}. \tag{1.7}$$

In general, the first boundary condition is:

$$\varphi(0) = I, J(0) = 0, \tag{1.8}$$

which represents constant flux, and zero neutron current density at the center of the reactor.
The second boundary condition is at the reactor edges. That is:

$$\varphi\,(a) = 0, \varphi\,(\infty) = 0, \tag{1.9}$$

where the flux must be zero at infinity (or in the case of finite reactors at its edge, i.e., $x = a$) [27–29].

The time dependence of a related process in determining the core multiplication as a function of the power level of the reactor represents the dynamics of the nuclear reactor.

The time-dependent nuclear reactor system can be represented by the F-NDEs.

The time behavior and the effect of the delayed neutrons in nuclear reactors can also be studied via this model of criticality.

Delayed neutrons are extremely important for reactor time behavior. In the case of fast and thermal reactors, lifetimes of prompt neutrons range from 10^{-4} s and 10^{-7} s.

According to the predictions made by this model, the reactor period is to modicum for effective control of the reactor. Also, the model does not treat the reactor as a point but assumes that the flux shape does not vary in time.

In reactors, the effective lifetime of neutrons plays an important role in delayed neutrons. The effective lifetime is equal to their prompt lifetime plus additional delay time, which is characterized by the β-decay of their antecedent, and considerably longer lifetime than the prompt neutron. Hence, delayed neutrons significantly increase the time constant of a reactor so that effective control is possible. It must be noted that the source term $s(r)$ in Eq (1.3) must be modified to include the effect of delayed neutron effect as:

$$s(x) = (1-\beta)\nu\Sigma_f\,\varphi(x,t) + \lambda c(x,t). \tag{1.10}$$

The neutron point kinetic model for NDE with delayed neutrons can be represented by a coherent system of two equations:

$$\frac{1}{V}\frac{\partial}{\partial t}\varphi(x,t) = D\frac{\partial^2}{\partial x^2}\varphi(x,t) + \left(-\sum_a + (1-\beta)\nu\sum_f\right)\varphi(x,t) + \lambda c(x,t), \tag{1.11}$$

$$\frac{\partial}{\partial t}c(x,t) = \beta\nu\sum_f\varphi(x,t) - \lambda c(x,t), \tag{1.12}$$

where $t>0,\ 0<x<L,\ \varphi(x,t)$is the neutron flux, $c(x,t)$ is the delayed neutron density, V is neutron velocity, $\sum_a$ is the macroscopic absorption cross-section, D is the neutron diffusion coefficient and β, λ, and ν are the fraction of the delayed fission neutrons, the radio-active decay constant, and the average number of neutrons produced per fission respectively.

$$\varphi(0,t) = \varphi(L,t) = 0 \tag{1.13}$$

$$\varphi(x,0) = \varphi_0(x), \qquad c(x,0) = \frac{\beta\nu\sum_f}{\lambda}\varphi_0(x) \tag{1.14}$$

The design and analysis of various operational modes in nuclear reactors is a complicated task. After determining the neutronic conditions of the reactor core, which plays a key role in the analysis-the main objective of such neutronic analysis is to predict and describe the states of a reactor under various conditions which enables determining the optimum configuration (i.e., capable of long-term, self-sustained operation) with minimal human intervention.

The kinetics equations in the diffusion theory of neutrons govern the dynamics of space-time and neutron population. Kinetics equations are classified into point and space kinetic equations. Point kinetic equations describe the variation of neutron density over time and estimate the power response of a reactor, allowing the ability to control the operations of the power plant to avoid the occurrence of any incidents. Now a recent approach is the fractional kinetics model, which reproduces the classical model and enables us to capture the different effects from the usually employed Fick's hypothesis. Also, it is widely used in reactor dynamics because of the simplicity of the resulting equations.

Sure, systems do not obey the normal (Gaussian) diffusion process where the rate at which the particles/objects diffuse either is slower or faster than that of the Gaussian diffusion processes in the system. Such deviation from the Gaussian diffusion is defined as anomalous diffusion.

Anomalous diffusion is characterized by its asymptotic nonlinearity (especially power-law type) and its dependence on the average-squared displacement on time, i.e.,

$$\langle x^2(t)\rangle \sim t^\alpha$$

According to the value of α, normal and anomalous diffusion is classified. When $\alpha = 1$, Gaussian diffusion, and when $\alpha < 1$ or $\alpha > 1$, then anomalous diffusion is studied.

Anomalous diffusion is classified into Sub-diffusion ($\alpha < 1$) which is characterized by the slow movement of particles because of being trapped in the medium for a very long or infinite time. While super diffusion ($\alpha > 1$) is described by the fast movement of particles, because of their non-collision with any particles for a very long time. Many researchers showed how anomalous diffusion is modeled using the technique of F-NDEs [36–37].

The F-NDEs method has been studied by many researchers where fractional of two energy group matrices are used to represent the dynamics of the nuclear reactor in case of an external source [38], while coupled F-NDEs is solved [39], and an exact solution in the case of one-dimensional neutron diffusion kinetic equation with one delayed antecedent concentration in Cartesian coordinates is studied [40]. Various numerical and analytical methods have been used by many researchers to find exact and approximate solutions for P-FDEs [41–44].

Here, the F-NDE with one delayed neutrons group can be written as:

$$\mathfrak{D}_t^\alpha\, \varphi(x,t) = VD\, \frac{\partial^2}{\partial x^2}\varphi(x,t) + \mu\, \varphi(x,t) + \lambda V c(x,t), \tag{1.15}$$

$$\mathfrak{D}_t^\alpha\, c(x,t) = \sigma\, \varphi(x,t) - \lambda\, c(x,t), \tag{1.16}$$

$$\varphi(0,t) = \varphi(L,t) = 0 \tag{1.17}$$

$$\varphi(x,0) = \varphi_0(x), \qquad c(x,t) = \frac{\sigma}{\lambda}\varphi_0(x), \tag{1.18}$$

where $\mu = V\left(-\sum_a + (1-\beta)\nu\sum_f\right)$ and $\sigma = \beta\nu\sum_f$.

2. Materials and methods

In this work, we create the analytical exact and approximate series solutions to the F-NDEs using the LT method via Caputo sense. Also, the poles and residues of this problem are calculated and determined. Finally, some numerical comparisons and graphical results for neutron flux simulations are given and tested at different values of time t and order α, which includes the exact solutions at $\alpha = 1$.

The fractional neutron flux equation in Eq (1.15) can be solved exactly with the use of the LT, as shown below.

Taking the LT to both sides of F-PDEs in Eq (1.15), according to:

$$\mathcal{L}[\mathfrak{D}_t^{\alpha}\, \varphi(x,t)] = \omega^{\alpha}\, \phi(x,\omega) - \omega^{\alpha-1}\, \varphi(x,0)$$

and the initial condition in Eq (1.17), we get:

$$\omega^{\alpha}\, \phi(x,\omega) - \omega^{\alpha-1}\varphi_0(x) = VD\, \frac{\partial^2}{\partial x^2}\phi(x,\omega) + \mu\ \phi(x,\omega) + \lambda V C(x,\omega), \tag{2.1}$$

where $\phi(x,\omega) = \mathcal{L}[\varphi(x,t)]$, $C(x,\omega) = \mathcal{L}[c(x,t)]$.

$$VD\frac{\partial^2}{\partial x^2}\phi(x,\omega) + (\mu - \omega^{\alpha})\,\phi(x,\omega) + \omega^{\alpha-1}\varphi_0(x) + \lambda V C(x,\omega) = 0. \tag{2.2}$$

Again, applying LT on the fractional Precursor concentration in Eq (1.16) and using the initial condition in Eq (1.18), we get:

$$\omega^{\alpha}\, C(x,\omega) - \frac{\sigma\omega^{\alpha-1}}{\lambda}\varphi_0(x) = \sigma\phi(x,\omega) - \lambda C(x,\omega), \tag{2.3}$$

$$(\omega^{\alpha} + \lambda)\, C(x,\omega) = \frac{\sigma\omega^{\alpha-1}}{\lambda}\varphi_0(x) + \sigma\phi(x,\omega), \tag{2.4}$$

$$C(x,\omega) = \frac{\sigma\omega^{\alpha-1}}{\lambda(\omega^{\alpha}+\lambda)}\varphi_0(x) + \frac{\sigma}{\omega^{\alpha}+\lambda}\phi(x,\omega), \tag{2.5}$$

$$C(x,\omega) = \frac{\omega^{\alpha-1}\delta}{\omega^{\alpha}+\lambda}\varphi_0(x) + \frac{\sigma}{\omega^{\alpha}+\lambda}\phi(x,\omega), \tag{2.6}$$

where $\delta = \frac{\sigma}{\lambda}$.

Inserting Eq (2.6) in Eq (2.2), we get:

$$VD\frac{\partial^2}{\partial x^2}\phi(x,\omega) + (\mu - \omega^{\alpha})\,\phi(x,\omega) + \omega^{\alpha-1}\varphi_0(x) + \lambda V\left(\frac{\omega^{\alpha-1}\delta}{\omega^{\alpha}+\lambda}\varphi_0(x) + \frac{\sigma}{\omega^{\alpha}+\lambda}\phi(x,\omega)\right) = 0 \tag{2.7}$$

$$VD\frac{\partial^2}{\partial x^2}\phi(x,\omega) + \left((\mu - \omega^{\alpha}) + \frac{\lambda V\sigma}{\omega^{\alpha}+\lambda}\right)\phi(x,\omega) = -\varphi_0(x)\left(\omega^{\alpha-1} + \frac{\lambda V\omega^{\alpha-1}\delta}{\omega^{\alpha}+\lambda}\right) \tag{2.8}$$

$$\frac{\partial^2}{\partial x^2}\phi(x,\omega)+\frac{1}{VD}\left(\frac{(\mu-\omega^\alpha)(\omega^\alpha+\lambda)+\lambda V\sigma}{\omega^\alpha+\lambda}\right)\phi(x,\omega) \\ =\frac{-\varphi_0(x)}{VD}\left(\frac{\omega^{\alpha-1}(\omega^\alpha+\lambda)+\lambda V\delta\omega^{\alpha-1}}{\omega^\alpha+\lambda}\right) \tag{2.9}$$

$$\frac{\partial^2}{\partial x^2}\phi(x,\omega)-\frac{1}{VD}\left(\frac{\omega^{2\alpha}-(\mu-\lambda)\omega^\alpha-\lambda(\mu+V\sigma)}{\omega^\alpha+\lambda}\right)\phi(x,\omega) \\ =\frac{-\varphi_0(x)}{VD}\left(\frac{\omega^{2\alpha-1}+\lambda\omega^{\alpha-1}(1+V\delta)}{\omega^\alpha+\lambda}\right). \tag{2.10}$$

Which is equivalent to the known differential equation,

$$\frac{d^2}{dx^2}\phi(x,\omega)-f^2(\omega^\alpha)\phi(x,\omega)=-g(\omega^\alpha),$$

where:

$$f(\omega^\alpha)=\frac{1}{\sqrt{VD}}\sqrt{\frac{\omega^{2\alpha}-(\mu-\lambda)\omega^\alpha-\lambda(\mu+V\sigma)}{\omega^\alpha+\lambda}}, \tag{2.11}$$

$$g(\omega^\alpha)=\frac{\varphi_0(x)}{VD}\left(\frac{\omega^{2\alpha-1}+\lambda\omega^{\alpha-1}(1+V\delta)}{\omega^\alpha+\lambda}\right). \tag{2.12}$$

The general exact solution is:

$$\phi(x,\omega)=h_1(\omega)\cosh(f(\omega^\alpha)x)+h_2(\omega)\sinh(f(\omega^\alpha)x)+\frac{g(\omega^\alpha)}{f^2(\omega^\alpha)}. \tag{2.13}$$

After applying the LT, we obtain the following two forms for $h_1(\omega)$ and $h_2(\omega)$:

$$h_1(\omega)=-\frac{g(\omega^\alpha)}{f^2(\omega^\alpha)},\ h_2(\omega)=-\frac{g(\omega^\alpha)\big(1-\cosh(f(\omega^\alpha)L)\big)}{f^2(\omega)\sinh(f(\omega^\alpha)L)}.$$

Now, inserting the functions $h_1(\omega)$, $h_2(\omega), g(\omega)$, and $f(\omega)$ into Eq (2.13), we have:

$$\phi(x,\omega)=\varphi_0(x)\left[\frac{2(\omega^{2\alpha-1}+\lambda\omega^{\alpha-1}(1+V\delta))\sinh\left(\frac{f(\omega^\alpha)x}{2}\right)\sinh\left(f(\omega^\alpha)\left(\frac{L-x}{2}\right)\right)}{(\omega^{2\alpha}-(\mu-\lambda)\omega^\alpha-\lambda(\mu+V\sigma))\cosh\left(\frac{f(\omega^\alpha)L}{2}\right)}\right] \tag{2.14}$$

Now, we first need to determine the poles and residues of Eq (2.14) to be able to the inverse LT.

2.1. *Pole calculations of* $\phi(x,s)$

We determine the poles from the zeros of the denominator in Eq (2.14). Specifically, we appoint them from the two factors, $(\omega^{2\alpha}-(\mu-\lambda)\omega^{\alpha}-\lambda(\mu+V\sigma))$ and $\cosh(f(\omega^{\alpha})L/2)$. We determine the first set of poles by the setting:

$$\omega^{2\alpha}-(\mu-\lambda)\omega^{\alpha}-\lambda(\mu+V\sigma)=0. \tag{2.15}$$

Using the definition of $f(\omega^{\alpha})$ in Eq (2.11) to determine the first order poles of Eq (2.15) yields $f^{2}(\omega^{\alpha})=0$, consequently $f(\omega_1^{\alpha})=f(\omega_2^{\alpha})=0$, where:

$$\omega_1=\left(\frac{1}{2}\left[\mu-\lambda+\sqrt{(\mu-\lambda)^2+4\lambda(\mu+V\sigma)}\right]\right)^{1/\alpha}, \tag{2.16}$$

$$\omega_2=\left(\frac{1}{2}\left[\mu-\lambda-\sqrt{(\mu-\lambda)^2+4\lambda(\mu+V\sigma)}\right]\right)^{1/\alpha}. \tag{2.17}$$

To find the second set of poles, we must set the second factor of the denominator in Eq (2.14) as $\cosh\left(\frac{f(\omega^{\alpha})L}{2}\right)=0.$
Consequently,

$$f(\omega^{\alpha})=-i(2n+1)\frac{\pi}{L}, n=0,1,2,.. \tag{2.18}$$

Substituting $f(\omega^{\alpha})$ from Eq (2.18) into Eq (2.11), we get:

$$-(\omega^{\alpha}+\lambda)VD(2n+1)^2\frac{\pi^2}{L^2}=\omega^{2\alpha}-(\mu-\lambda)\omega^{\alpha}-\lambda(\mu+V\sigma), \tag{2.19}$$

$$\omega^{2\alpha}-\left(\mu-\lambda-VD(2n+1)^2\frac{\pi^2}{L^2}\right)\omega^{\alpha}-\lambda\left(\mu+V\sigma-VD(2n+1)^2\frac{\pi^2}{L^2}\right)=0 \tag{2.20}$$

Solving Eq (2.20) for ω yields,

$$\omega_3=\left(\frac{1}{2}\left[\mu-\lambda-VD(2n+1)^2\frac{\pi^2}{L^2}+\sqrt{\left(\mu-\lambda-VD(2n+1)^2\frac{\pi^2}{L^2}\right)^2+4\lambda\left(\mu+V\sigma-VD(2n+1)^2\frac{\pi^2}{L^2}\right)}\right]\right)^{1/\alpha},$$

$$\omega_4=\left(\frac{1}{2}\left[\mu-\lambda-VD(2n+1)^2\frac{\pi^2}{L^2}-\sqrt{\left(\mu-\lambda-VD(2n+1)^2\frac{\pi^2}{L^2}\right)^2+4\lambda\left(\mu+V\sigma-VD(2n+1)^2\frac{\pi^2}{L^2}\right)}\right]\right)^{1/\alpha}.$$

The above Poles' values can be real or complex numbers. It depends on the parameter values inside the radical. When presenting the numerical analysis of the results, the values of the poles will be real numbers based on the submitted known physical quantities.

2.2. *Residues calculation of* $\phi(x,s)$

Note that by Eqs (2.11), (2.16) and (2.17), and $f(\omega_1^{\alpha})=f(\omega_2^{\alpha})=0$, it can be easily checked that:

$$Res(\phi(x,\omega)e^{\omega t};\omega_1)=0, \tag{2.21}$$

and similarly,

$$Res(\phi(x,\omega)e^{\omega t};\omega_2)=0, \tag{2.22}$$

Now for $\omega=\omega_3$ and $\omega=\omega_4$, let us reform $\phi(x,\omega)$ by substituting from Eqs (2.11) and (2.18) into Eq (2.14) to get the following formula:

$$\phi(x,\omega)=\frac{2L^2\varphi_0(x)}{(2n+1)^2\pi^2VD}\sin\left[\left(n+\frac{1}{2}\right)\pi\frac{x}{L}\right]\sin\left[\left(n+\frac{1}{2}\right)\pi\left(1-\frac{x}{L}\right)\right]\frac{(\omega^{2\alpha}+\lambda\omega^{\alpha}(1+V\delta))}{\omega(\omega^{\alpha}+\lambda)\cosh\frac{f(\omega^{\alpha})L}{2}} \tag{2.23}$$

Using the last form of $\phi(x,\omega)$ appearing in Eq (2.23) and the value of $f(\omega^{\alpha})$ as in Eq (2.18), we get:

$$Res(\phi(x,\omega)e^{\omega t};\omega_3)=\frac{8\varphi_0(x)}{\alpha\pi}\xi(n,x)e^{\omega_3 t}\frac{(\omega_3^{\alpha}+\lambda)(\omega_3^{\alpha}+\lambda(1+V\delta))}{(\omega_3^{\alpha}+\lambda)^2+\lambda V\sigma}, \tag{2.24}$$

and:

$$Res(\phi(x,\omega)e^{\omega t};\omega_4)=\frac{8\varphi_0(x)}{\alpha\pi}\xi(n,x)e^{\omega_4 t}\frac{(\omega_4^{\alpha}+\lambda)(\omega_4^{\alpha}+\lambda(1+V\delta))}{(\omega_4^{\alpha}+\lambda)^2+\lambda V\sigma}, \tag{2.25}$$

where:

$$\xi(n,x)=\frac{(-1)^n}{(2n+1)}\sin\left[\left(n+\frac{1}{2}\right)\pi\frac{x}{L}\right]\sin\left[\left(n+\frac{1}{2}\right)\pi\left(1-\frac{x}{L}\right)\right] \tag{2.26}$$

Applying the residue theorem method and using Eqs (2.19), (2.22), (2.24) and (2.25), we finally obtain:

$$\varphi(x,t)=\frac{8\varphi_0(x)}{\alpha\pi}\sum_{n=0}^{\infty}\xi(n,x)\left[\varepsilon_1(n)e^{\omega_3(n)t}+\varepsilon_2(n)e^{\omega_4(n)t}\right] \tag{2.27}$$

where:

$$\varepsilon_1(n)=\frac{(\omega_3^{\alpha}+\lambda)(\omega_3^{\alpha}+\lambda(1+V\delta))}{(\omega_3^{\alpha}+\lambda)^2+\lambda V\sigma}, \tag{2.28}$$

and

$$\varepsilon_2(n)=\frac{(\omega_4^{\alpha}+\lambda)(\omega_4^{\alpha}+\lambda(1+V\delta))}{(\omega_4^{\alpha}+\lambda)^2+\lambda V\sigma}. \tag{2.29}$$

3. Numerical results and discussion

To verify the driven theory in Section 2, we solve the time dependent F-NDEs using the LT method, and the results will be tested with the numerical values of the following nuclear reactor cross-section data [40]: $V=1.103497\times10^{7}$, $D=0.96343$, $\lambda=0.08$, $\sigma=0.00014986$, $\mu=191015.738$, $L=22.9$, $\varphi(x,0)=5226.54$.

The neutron flux is an important nuclear reactor notion that defines the number of neutrons moving through a unit area in a unit period. This study demonstrates the neutron flux findings for both anomalous diffusion and sub-diffusion; the super diffusion case is not considered.

Solving non-F-NDE has been used to study neutron behavior in nuclear reactors without taking the effect of delayed neutrons into account [30–33,35]. However, when the effect of the delayed neutron is considered, fractional calculus is introduced, which aids scientists in solving the NDEs [36–40].

Table 1 studies the impact of the number of terms on the neutron, it should be emphasized that Table 1 only examines the non-fractional instance of the flux, which depicts Gaussian diffusion when $\alpha = 1$.

From Table 1, it is shown that the Gaussian flux rapidly converges and becomes stable for the first few calculations $(N = 10)$ although we took up to 400 terms and 10 digits but that is just to clarify the idea and to show that the difference between them is of the order $\Delta\varphi = 1.563 \times 10^{-7}$.

Table 1. Neutron flux simulations at $x = 11.45\ cm,\ t = 1, \alpha = 1$.

N	φ($cm^{-2}\ s^{-1}$)
5	0.1700188336
10	0.1700213179
15	0.1700208490
20	0.1700210249
25	0.1700209455
30	0.1700209742
60	0.1700209838
90	0.1700210039
150	0.1700209568
200	0.1700209595
250	0.1700208786
300	0.1700210515
350	0.1700210453
400	0.1700211616

Table 2 gives the flux at $x = 11.45$ for different time intervals, where the anomalous diffusion, $\alpha < 1$ (sub diffusion), and normal diffusion, $\alpha = 1$, are considered. It must be noted that the results of the normal diffusion case shown in Table 1, are reproduced at $t = 1$. Also, it is clear from Table 2 that the neutron flux decreases as the time increases for both Gaussian and sub-diffusion cases (for all values of α), whereas the flux is expected to vanish as time becomes large and that agrees with nuclear reactor theory [36,37].

Table 2. Neutron flux (φ) at $x = 11.45$ at different values of time (t) and α.

Time (t)	$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 1$
0.0005	0.951223	0.475612	0.317074	0.237806	0.190245
0.0010	0.907964	0.453982	0.302655	0.226991	0.181593
0.0100	0.906962	0.453481	0.302321	0.226740	0.181392
0.0200	0.906369	0.453184	0.302123	0.226592	0.181274

Continued on next page

Time (t)	α = 0.2	α = 0.4	α = 0.6	α = 0.8	α = 1
0.0800	0.902819	0.45141	0.300939	0.225705	0.180564
0.1000	0.901639	0.45082	0.300547	0.22541	0.180328
0.3000	0.889924	0.444962	0.296641	0.222481	0.177985
0.6000	0.872636	0.436318	0.290879	0.218159	0.174527
0.9000	0.855683	0.427842	0.285228	0.213921	0.171137
1.0000	0.850106	0.425053	0.283369	0.212526	0.170021

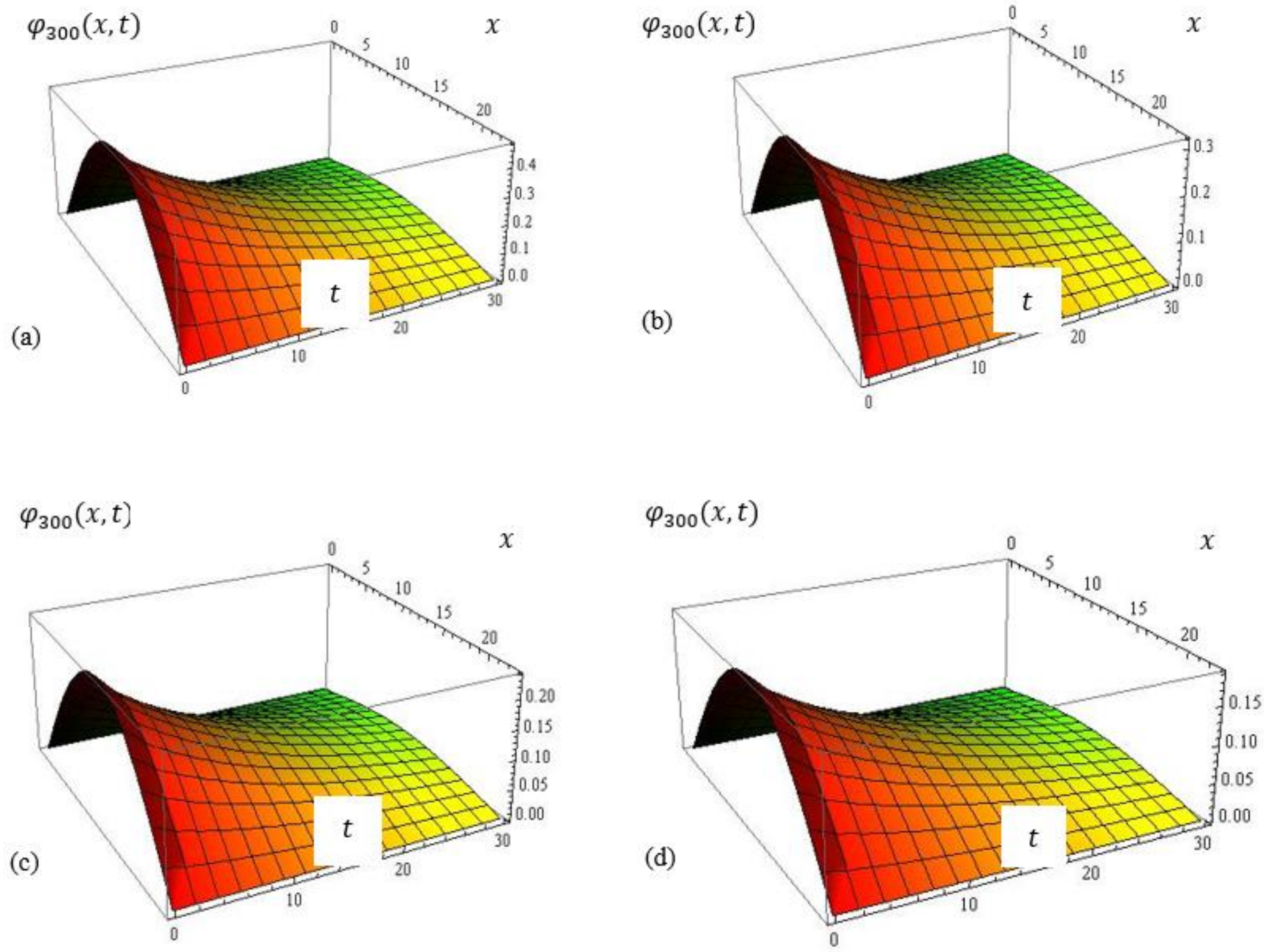

Figure 1. The surface graphs of the 300th approximate solutions, $\varphi_{300}(x,t)$ of the boundary value problem (1.15)–(1.18): (a) when $\alpha = 0.4$, (b) when $\alpha = 0.6$, (c) when $\alpha = 0.8$, (d) when $\alpha = 1.0$.

Then for more clarification, we plotted the profile of the neutron fluxes, $\phi(x,t)$, against α and x for different values of α ($\alpha = 0.4, 0.6,$ and $\alpha = 0.8$) in Figure 1(a)–(c) respectively. The results were compared to the classical result (i.e., $\alpha = 1$) shown in Figure 1(d) [40].

By comparing non-Gaussian flux using the LT method in this work, Figure 1(a)–(c), with Gaussian flux Figure 1(d) and the results are compatible.

It must be noted that the MATHEMATICA package (Version 12) was used to find the needed data to construct the tables and figures to achieve our target.

Finally, as it is observed from the tables and figures, the LT method used in solving the time dependent F-NDEs gives us more details about the neutron flux in non-Gaussian diffusion for different selected times, where the flux values decrease as it is close to the Gaussian case. Furthermore, based on the given values of the parameters, which are the cross-section data taken from truth elements, we would like to indicate the values of $\omega_3(n)$ and $\omega_4(n)$ are negative. Hence, the solution in Eq (2.27) approaches zero in the long-run behavior and appropriate α values. This solution is consistent with the nuclear reactor idea, in which the flux must vanish over an infinite period.

4. Conclusions

We have reformulated the NDEs in the sense of Caputo fractional derivative with integer-order initial conditions, whose physical meanings, in this case, are very evident by describing the whole-time domain of physical processing. Despite some drawbacks in the LT method, some of which can be summarized in the following points:

1) It becomes complicated if the equation coefficients are not constants.
2) It deals with initial conditions and not boundary conditions.
3) It becomes almost impossible to use if the equation is non-linear unless we circumvent the equation form to eliminate nonlinearity.
4) The difficulty in finding the inverse LT directly for the final solution. In this case, we need to calculate the poles and use the residue theorem. Unfortunately, this is often complicated if the poles are complex numbers. Despite all these drawbacks, we have used the LT method for creating a series solutions for the F-NDEs with one delayed neutron group. In addition, the theoretical results were tested numerically at different values of time t for both cases, sub-diffusion $(0 < \alpha < 1)$ and Gaussian diffusion $(\alpha = 1)$. The used method is successful in giving an exact solution for the studied cases (i.e., Gaussian and sub-diffusion) where the neutron flux decreased as the time elapsed for both Gaussian and sub-diffusion cases, whereas the flux, as expected, vanished over a long time.

This work can be considered a step in studying fractional and non-fractional nuclear reactor theory phenomena. How to solve F-NDEs via other methods combined with the LT, such as the Laplace residual power series method, Laplace-ARA method, and other new methods, still require further research.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could appear to influence the work reported in this paper.

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