AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(8): 19313–19333.
DOI: 10.3934/math.2023985
Received: 15 March 2023
Revised: 20 May 2023
Accepted: 04 June 2023
Published: 08 June 2023

Research article

Existence results for nonlinear multi-term impulsive fractional q-integro-difference equations with nonlocal boundary conditions

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Abstract: This paper is concerned with the existence of solutions for a nonlinear multi-term impulsive fractional q-integro-difference equation with nonlocal boundary conditions. The appropriated fixed point theorems are applied to accomplish the existence and uniqueness results for the given problem. We demonstrate the application of the obtained results with the aid of examples.

Keywords: fractional q-derivative operator; Riemann-Liouville fractional q-integral operator; impulse; nonlocal boundary conditions; existence of a solution
Mathematics Subject Classification: 34A08, 39A13, 34B15, 34B37

1. Introduction

Impulsive systems are found to be of great help in the study of the phenomena exhibiting abrupt changes, such as the passage of a solid body from a given fluid density to another fluid density [1], remittent vibrators [2], instantaneous disturbances in cellular neural networks [3], shock changes in stock exchange [4], etc. Evolutionary hereditary processes undergoing abrupt or sudden changes are modeled with the aid of impulsive fractional differential equations [5, 6]. Such processes naturally occur in bio-medical sciences [7, 8], thermo-elasticity [9], heat conduction [10], plasma physics [11], and engineering problems [12].

The subject of fractional calculus has been extensively studied during the last two decades in view of its vast applications in science and technology. Examples include chaotic synchronization [13], immune systems [14], neural networks [15], fractional diffusion [16], ecology [17], etc. For the fundamental concepts of fractional calculus , we refer the reader to the text [18]. In particular, there has been shown a great interest in investigating the existence, uniqueness and stability of solutions for initial and boundary value problems. One can find an uptodate account of these problems in the book [19], while a variety of recent results involving different kinds of fractional derivatives can be found in the articles [20–26]. The natural extension of fractional differential equations to fractional q-difference equations also received significant attraction. For details and examples of nonlinear fractional q-difference equations subject to different kinds of boundary conditions involving q-derivatives and q-integrals, for instance, the articles [27–33] and the references cited therein.

Let us now review some recent results on impulsive fractional and fractional q-difference equations. In [34], the authors studied a Caputo-Hadamard type fractional impulsive hybrid system with nonlinear fractional integral conditions. Impulsive fractional q-integro-difference equations with separated boundary conditions were investigated in [35]. Some stability results for abstract fractional differential equations with non-instantaneous impulses can be found in [36]. The authors in [37] discussed the existence of solutions for an impulsive fractional q-difference equation with nonlocal condition. Some existence results for an impulsive fractional q-difference equation with antiperiodic boundary conditions were proved in [38]. In [39], the authors obtained exact solutions for linear Riemann-Liouville fractional differential equations with impulses.

Motivated by the recent development on impulsive fractional q-difference equations, in this paper, we introduce and study a nonlinear multi-term impulsive nonlocal boundary value problem involving Caputo type fractional q-derivative operators of different orders and the Riemann-Liouville fractional q-integral operator. In precise terms, we discuss the existence of solutions for the following problem:

$$\begin{cases} \varpi\, {}^cD_q^{\alpha}u(t) + (1-\varpi)\, {}^cD_q^{\beta}u(t) = af(t,u(t)) + bI_q^{\delta}g(t,u(t)),\ t \in \mathcal{J} = [0,1], t \neq t_\sigma, \\ \Delta u(t_\sigma) = \mathcal{I}_\sigma(u(t_\sigma)),\ \ \Delta u^{'}(t_\sigma) = \tilde{\mathcal{I}}_\sigma(u(t_\sigma)),\ \sigma = 1,2,3.....,p, \\ u(0) + u'(0) = 0, \quad u(1) + u'(\eta) = 0,\ \ \eta \in (t_m, t_{m+1}), 0 \le m \le p,\ \eta \neq t_\sigma, \end{cases} \tag{1.1}$$

where ${}^cD_q^{\omega}$ denotes the Caputo fractional q-derivative of order ω, ($\omega = \alpha,\ \beta$) and I_q^{δ} denotes the Riemann-Liouville fractional q-integral of order δ, $0 < \delta < 1,\ 0 < q < 1,\ 1 < \alpha < 2$ and $0 < \beta < 1$ such that $\alpha - \beta > 1,\ 0 < \varpi \le 1,\ a, b \in \mathbb{R}^+$ and $f, g : \mathcal{J} \times \mathbb{R} \to \mathbb{R}$ are continuous functions. Further, $\mathcal{I}_\sigma, \tilde{\mathcal{I}}_\sigma : \mathbb{R} \to \mathbb{R}$ are also continuous functions, $\Delta u(t_\sigma) = u(t_\sigma^+) - u(t_\sigma^-)$, $\Delta u^{'}(t_\sigma) = u^{'}(t_\sigma^+) - u^{'}(t_\sigma^-)$, where $u(t_\sigma^+)$ and $u(t_\sigma^-)$ represent the right and left-hand limits of $u(t)$, respectively, at $t = t_\sigma$ ($\sigma = 1,2,3,...,p$) and $0 = t_0 < t_1 < t_2 < ... < t_\sigma < ... < t_p < t_{p+1} = 1$, $\mathcal{J}^{'} = \mathcal{J} - \{t_1, t_2, t_3, ..., t_\sigma\}$.

Here, we emphasize that the problem (1.1) is novel in the sense that it consists of a multi-term fractional q-integro-difference equation involving Caputo type fractional q-derivative and q-Riemann-Liouville integral operators subject to impulsive conditions and nonlocal boundary data. We make use of the fixed-point approach to discuss the existence and uniqueness of solutions for the problem at hand. First of all, we prove an auxiliary lemma dealing with the linear variant of the problem (1.1), which enables us to convert the given nonlinear problem into a fixed-point problem. Then we apply the Schaefer's fixed point theorem to establish an existence result for the problem (1.1), while the uniqueness of its solutions is obtained via Banach's contraction mapping principle. The work accomplished in this paper is not only new in the given configuration but it also accounts for some new results as special cases, for details, see Section 5.

The structure of the rest of the paper is as follows. In section 2, we recall some basic definitions from fractional q-calculus and prove a fundamental lemma which plays a key role in the analysis of the problem at hand. Main results concerning the existence and uniqueness of solutions for the problem (1.1) are proved in Section 3, while examples illustrating these results are discussed in Section 4.

2. Preliminaries

Let $\mathcal{J}_0 = [0, t_1], \mathcal{J}_1 = (t_1, t_2], \mathcal{J}_2 = (t_2, t_3], ..., \mathcal{J}_\sigma = (t_\sigma, t_{\sigma+1}]$, $\sigma = 1, 2, 3, ..., p$ with $t_{p+1} = 1$ and introduce the space: $PC(\mathcal{J}, \mathbb{R}) = \{u : \mathcal{J} \to \mathbb{R} | u \in C(\mathcal{J}_\sigma),\ \sigma = 0, 1, ..., p$ and $u(t_\sigma^+)$ exist, for $\sigma = 1, ..., p\}$ endowed with the norm $\|u\| = \sup_{t\in\mathcal{J}} |u(t)|$. Obviously, $PC(\mathcal{J}, \mathbb{R})$ is a Banach space.

Let us first collect some necessary concepts and definitions from q-fractional calculus [40, 41].

We define a q-real number as

$$[a]_q = \frac{1 - q^a}{1 - q},\ a \in \mathbb{R},\ q \in \mathbb{R}^+ \setminus \{1\}.$$

The q-analogue of the Pochhammer symbol (q-shifted factorial) is defined by

$$(a; q)_0 = 1,\ \ (a; q)_k = \prod_{i=0}^{k-1} (1 - aq^i),\ k \in \mathbb{N} \cup \{\infty\}.$$

The q-analogue of the exponent $(x - y)^k$ is given by

$$(x - y)^{(0)} = 1,\ \ (x - y)^{(k)} = \prod_{j=0}^{k-1} (x - yq^j),\ k \in \mathbb{N},\ x, y \in \mathbb{R}.$$

The q-gamma function $\Gamma_q(y)$ is defined as

$$\Gamma_q(y) = \frac{(1 - q)^{(y-1)}}{(1 - q)^{y-1}},$$

where $y \in \mathbb{R} \setminus \{0, -1, -2, ...\}$. Observe that $\Gamma_q(y + 1) = [y]_q\Gamma_q(y)$.

Definition 2.1. *Let f be a function defined on $[0, b]$, $b > 0$ and $a \in (0, b)$ is an arbitrary fixed point. The fractional q-integral of the Riemann-Liouville type is defined by*

$$(I^\beta_{q,a} f)(t) = \int_a^t \frac{(t - qs)^{(\beta-1)}}{\Gamma_q(\beta)} f(s) d_q(s),\ \ \beta > 0,$$

provided the integral exists. Moreover, $(I^\gamma_{q,a} I^\beta_{q,a} f)(t) = (I^{\beta+\gamma}_{q,a} f)(t);\ \gamma, \beta \in \mathbb{R}^+,\ a \in (0, t)$.

We define the q-derivative of a function f as follows:

$$(D_q f)(t) = \frac{f(t) - f(qt)}{t - qt},\ t \neq 0,\ \ (D_q f)(0) = \lim_{t\to 0} (D_q f)(t).$$

Furthermore, $D^0_q f = f,\ \ D^n_q f = D_q(D^{n-1}_q f),\ \ n = 1, 2, 3, ...$

Definition 2.2. *([41]) The fractional q-derivative of the Riemann-Liouville type of order β is defined as*

$$(D^{\beta}_{q,a}f)(t)=\begin{cases}(I^{-\beta}_{q,a}f)(t), & \beta<0,\\ f(x), & \beta=0,\\ (D^{\lceil\beta\rceil}_{q}I^{\lceil\beta\rceil-\beta}_{q,a}f)(t), & \beta>0,\end{cases}\tag{2.1}$$

where $\lceil\beta\rceil$ is the smallest integer greater than or equal to β.

In passing, we remark that

(i) $(D^{\beta}_{q,a}I^{\beta}_{q,a}f)(t)=f(t),\ 0<a<t.$

(ii) $I^{\beta}_{q,a}\left((x-a)^{(\lambda)}\right)=\frac{\Gamma_q(\lambda+1)}{\Gamma_q(\beta+\lambda+1)}(x-a)^{(\beta+\lambda)},\ 0<a<x<b,\beta\in\mathbb{R}^+,\lambda\in(-1,\infty).$

Definition 2.3. *([41]) The fractional q-derivative of the Caputo type of order $\beta\in\mathbb{R}^+$ is defined by*

$$({}^{c}D^{\beta}_{q,a}f)(t)=(I^{\lceil\beta\rceil-\beta}_{q,a}D^{\lceil\beta\rceil}_{q}f)(t).$$

Remark 2.1. *For $0<a<t$ and $\beta\in\mathbb{R}\setminus\mathbb{N}$, the following relations hold [41]:*

(a) $({}^{c}D^{\beta+1}_{q,a}f)(t)=({}^{c}D^{\beta}_{q,a}D_qf)(t);$

(b) $({}^{c}D^{\beta}_{q,a}I^{\beta}_{q,a}f)(t)=f(t);$

(c) $(I^{\beta}_{q,a}\,{}^{c}D^{\beta}_{q,a}f)(t)=f(t)-\sum_{k=0}^{\lceil\beta\rceil-1}\frac{(D^k_qf)(a)}{\Gamma_q(k+1)}t^k(a/t;q)_k;$

In the following lemma, we solve the linear variant of the problem (1.1), which plays a fundamental role in the forthcoming analysis.

Lemma 2.1. *Let $\varrho\in C([0,1],\mathbb{R})$ and $\eta\in(t_m,t_{m+1})$, m is a non-negative integer such that $0\le m\le p$. Then the unique solution of the boundary value problem:*

$$\begin{cases}\varpi\,{}^{c}D^{\alpha}_{q}u(t)+(1-\varpi)\,{}^{c}D^{\beta}_{q}u(t)=\varrho(t),\ \ t\in\mathcal{J},t\neq t_\sigma,\ \sigma=1,2,3.....,p,\\ \Delta u(t_\sigma)=\mathcal{I}_\sigma(u(t_\sigma)),\ \ \Delta u'(t_\sigma)=\tilde{\mathcal{I}}_\sigma(u(t_\sigma)),\ \sigma=1,2,3.....,p,\\ u(0)+u'(0)=0,\quad u(1)+u'(\eta)=0,\ \ \eta\neq t_\sigma,\end{cases}\tag{2.2}$$

is given by

$$u(t)=\begin{cases}\Big(\frac{\varpi-1}{\varpi}\Big)\int_0^t\frac{(t-qs)^{(\alpha-\beta-1)}}{\Gamma_q(\alpha-\beta)}u(s)d_qs+\frac{1}{\varpi}\int_0^t\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\varrho(s)d_qs\\ +(1-t)\,\mathfrak{B},\ \ t\in\mathcal{J}_0;\\ \Big(\frac{\varpi-1}{\varpi}\Big)\int_{t_\sigma}^t\frac{(t-qs)^{(\alpha-\beta-1)}}{\Gamma_q(\alpha-\beta)}u(s)d_qs+\frac{1}{\varpi}\int_{t_\sigma}^t\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\varrho(s)d_qs\\ +\sum_{i=1}^{\sigma}\Big[\Big(\frac{\varpi-1}{\varpi}\Big)\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-\beta-1)}}{\Gamma_q(\alpha-\beta)}u(s)d_qs+\frac{1}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\varrho(s)d_qs\Big]\\ +\sum_{i=1}^{\sigma}(t-t_i)\Big[\Big(\frac{\varpi-1}{\varpi}\Big)\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha-\beta-1)}u(s)d_qs+\frac{1}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\varrho(s)d_qs\Big]\\ +\sum_{i=1}^{\sigma}\mathcal{I}_i(u(t_i))+\sum_{i=1}^{\sigma}(t-t_i)\tilde{\mathcal{I}}_i(u(t_i))+(1-t)\,\mathfrak{B},\ \ t\in\mathcal{J}_\sigma,\ \sigma=1,2,...,p,\end{cases}\tag{2.3}$$

where

$$\begin{aligned}\mathfrak{B} &= \sum_{i=1}^{p+1}\Big[\Big(\frac{\varpi-1}{\varpi}\Big)\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-\beta-1)}}{\Gamma_q(\alpha-\beta)}u(s)d_qs+\frac{1}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\varrho(s)d_qs\Big]\\
&\quad+\sum_{i=1}^{p}(1-t_i)\Big[\Big(\frac{\varpi-1}{\varpi}\Big)\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha-\beta-1)}u(s)d_qs+\frac{1}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\varrho(s)d_qs\Big]\\
&\quad+\Big(\frac{\varpi-1}{\varpi}\Big)\int_{t_m}^{\eta}\frac{(\eta-qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha-\beta-1)}u(s)d_qs+\frac{1}{\varpi}\int_{t_m}^{\eta}\frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\varrho(s)d_qs\\
&\quad+\sum_{i=1}^{m}\Big[\Big(\frac{\varpi-1}{\varpi}\Big)\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha-\beta-1)}u(s)d_qs+\frac{1}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\varrho(s)d_qs\Big]\\
&\quad+\sum_{i=1}^{p}\mathcal{I}_i(u(t_i))+\sum_{i=1}^{p}(1-t_i)\tilde{\mathcal{I}}_i(u(t_i))+\sum_{i=1}^{m}\tilde{\mathcal{I}}_i(u(t_i)),\end{aligned}$$

Proof. Let u be a solution of the q-fractional boundary value problem (2.2). Then, for $t\in\mathcal{J}_0$, we have

$${}^cD_q^{\alpha}u(t)=\Big(\frac{\varpi-1}{\varpi}\Big)\,{}^cD_q^{\beta}u(t)+\frac{1}{\varpi}\varrho(t).$$

Operating the q-integral operator I_q^{α} to both sides of the above equation, we get

$$u(t)=\Big(\frac{\varpi-1}{\varpi}\Big)\int_0^t\frac{(t-qs)^{(\alpha-\beta-1)}}{\Gamma_q(\alpha-\beta)}u(s)d_qs+\frac{1}{\varpi}\int_0^t\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\varrho(s)d_qs-a_1-a_2t,\qquad(2.4)$$

where $a_1,a_2\in\mathbb{R}$ are arbitrary constants. Differentiating (2.4) with respect to t, we obtain

$$u'(t)=\Big(\frac{\varpi-1}{\varpi}\Big)\int_0^t\frac{(t-qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha-\beta-1)}u(s)d_qs+\frac{1}{\varpi}\int_0^t\frac{(t-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\varrho(s)d_qs-a_2.\qquad(2.5)$$

Similarly, for $t\in(t_1,t_2]$, we obtain

$$u(t)=\Big(\frac{\varpi-1}{\varpi}\Big)\int_{t_1}^t\frac{(t-qs)^{(\alpha-\beta-1)}}{\Gamma_q(\alpha-\beta)}u(s)d_qs+\frac{1}{\varpi}\int_{t_1}^t\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\varrho(s)d_qs-b_1-b_2(t-t_1),\qquad(2.6)$$

where $b_1,\ b_2\in\mathbb{R}$ are arbitrary constants.

$$u'(t)=\Big(\frac{\varpi-1}{\varpi}\Big)\int_{t_1}^t\frac{(t-qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha-\beta-1)}u(s)d_qs+\frac{1}{\varpi}\int_{t_1}^t\frac{(t-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\varrho(s)d_qs-b_2.\qquad(2.7)$$

From (2.4)–(2.7), we get

$$u(t_1^-)=\Big(\frac{\varpi-1}{\varpi}\Big)\int_0^{t_1}\frac{(t_1-qs)^{(\alpha-\beta-1)}}{\Gamma_q(\alpha-\beta)}u(s)d_qs+\frac{1}{\varpi}\int_0^{t_1}\frac{(t_1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\varrho(s)d_qs-a_1-a_2t_1,\quad u(t_1^+)=-b_1,$$

$$u'(t_1^-)=\Big(\frac{\varpi-1}{\varpi}\Big)\int_0^{t_1}\frac{(t_1-qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha-\beta-1)}u(s)d_qs+\frac{1}{\varpi}\int_0^{t_1}\frac{(t_1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\varrho(s)d_qs-a_2,\quad u'(t_1^+)=-b_2.$$

In view of the impulsive conditions: $\Delta u(t_1) = \mathcal{I}_1(u(t_1))$ and $\Delta u^{'}(t_1) = \tilde{\mathcal{I}}_1(u(t_1))$, we find that

$$\begin{aligned} -b_1 &= \Big(\frac{\varpi-1}{\varpi}\Big)\int_0^{t_1}\frac{(t_1-qs)^{(\alpha-\beta-1)}}{\Gamma_q(\alpha-\beta)}u(s)d_qs+\frac{1}{\varpi}\int_0^{t_1}\frac{(t_1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\varrho(s)d_qs-a_1-a_2t_1+\mathcal{I}_1(u(t_1)),\\ -b_2 &= \Big(\frac{\varpi-1}{\varpi}\Big)\int_0^{t_1}\frac{(t_1-qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha-\beta-1)}u(s)d_qs+\frac{1}{\varpi}\int_0^{t_1}\frac{(t_1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\varrho(s)d_qs-a_2+\tilde{\mathcal{I}}_1(u(t_1)). \end{aligned}$$

Thus, (2.6) becomes

$$\begin{aligned} u(t) &= \Big(\frac{\varpi-1}{\varpi}\Big)\int_{t_1}^{t}\frac{(t-qs)^{(\alpha-\beta-1)}}{\Gamma_q(\alpha-\beta)}u(s)d_qs+\frac{1}{\varpi}\int_{t_1}^{t}\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\varrho(s)d_qs\\ &\quad+\Big(\frac{\varpi-1}{\varpi}\Big)\int_0^{t_1}\frac{(t_1-qs)^{(\alpha-\beta-1)}}{\Gamma_q(\alpha-\beta)}u(s)d_qs+\frac{1}{\varpi}\int_0^{t_1}\frac{(t_1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\varrho(s)d_qs\\ &\quad+(t-t_1)\Big[\Big(\frac{\varpi-1}{\varpi}\Big)\int_0^{t_1}\frac{(t_1-qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha-\beta-1)}u(s)d_qs+\frac{1}{\varpi}\int_0^{t_1}\frac{(t_1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\varrho(s)d_qs\Big]\\ &\quad+\mathcal{I}_1(u(t_1))+(t-t_1)\tilde{\mathcal{I}}_1(u(t_1))-a_1-a_2t,\quad t\in\mathcal{J}_1. \end{aligned}$$

In general, for $t \in \mathcal{J}_\sigma$, we get

$$\begin{aligned} u(t) &= \Big(\frac{\varpi-1}{\varpi}\Big)\int_{t_\sigma}^{t}\frac{(t-qs)^{(\alpha-\beta-1)}}{\Gamma_q(\alpha-\beta)}u(s)d_qs+\frac{1}{\varpi}\int_{t_\sigma}^{t}\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\varrho(s)d_qs\\ &\quad+\sum_{i=1}^{\sigma}\Big[\Big(\frac{\varpi-1}{\varpi}\Big)\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-\beta-1)}}{\Gamma_q(\alpha-\beta)}u(s)d_qs+\frac{1}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\varrho(s)d_qs\Big]\\ &\quad+\sum_{i=1}^{\sigma}(t_-t_i)\Big[\Big(\frac{\varpi-1}{\varpi}\Big)\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha-\beta-1)}u(s)d_qs+\frac{1}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\varrho(s)d_qs\Big]\\ &\quad+\sum_{i=1}^{\sigma}\mathcal{I}_i(u(t_i))+\sum_{i=1}^{\sigma}(t_-t_i)\tilde{\mathcal{I}}_i(u(t_i))-a_1-a_2t,\quad t\in\mathcal{J}_\sigma. \end{aligned} \tag{2.8}$$

Differentiating (2.8) with respect to t, we get

$$\begin{aligned} u'(t) &= \Big(\frac{\varpi-1}{\varpi}\Big)\int_{t_\sigma}^{t}\frac{(t-qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha-\beta-1)}u(s)d_qs+\frac{1}{\varpi}\int_{t_\sigma}^{t}\frac{(t-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\varrho(s)d_qs\\ &\quad+\sum_{i=1}^{\sigma}\Big[\Big(\frac{\varpi-1}{\varpi}\Big)\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha-\beta-1)}u(s)d_qs+\frac{1}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\varrho(s)d_qs\Big]\\ &\quad+\sum_{i=1}^{\sigma}\tilde{\mathcal{I}}_i(u(t_i))-a_2. \end{aligned} \tag{2.9}$$

Now, using the condition $u(0)+u'(0)=0$ after finding $u(0)$ and $u'(0)$ from (2.4) and (2.5), respectively, we obtain $a_1 + a_2 = 0$. On the other hand, from (2.8) and (2.9), we have

$$u(1) = \sum_{i=1}^{p+1}\Big[\Big(\frac{\varpi-1}{\varpi}\Big)\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-\beta-1)}}{\Gamma_q(\alpha-\beta)}u(s)d_qs+\frac{1}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\varrho(s)d_qs\Big]$$

$$+\sum_{i=1}^{p}(1-t_i)\Big[\Big(\frac{\varpi-1}{\varpi}\Big)\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha-\beta-1)}u(s)d_qs+\frac{1}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\varrho(s)d_qs\Big]$$
$$+\sum_{i=1}^{p}\mathcal{I}_i(u(t_i))+\sum_{i=1}^{p}(1-t_i)\tilde{\mathcal{I}}_i(u(t_i))-a_1-a_2,$$

$$\begin{aligned} u'(\eta) &= \Big(\frac{\varpi-1}{\varpi}\Big)\int_{t_m}^{\eta}\frac{(\eta-qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha-\beta-1)}u(s)d_qs+\frac{1}{\varpi}\int_{t_m}^{\eta}\frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\varrho(s)d_qs \\ &\quad +\sum_{i=1}^{m}\Big[\Big(\frac{\varpi-1}{\varpi}\Big)\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha-\beta-1)}u(s)d_qs+\frac{1}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\varrho(s)d_qs\Big] \\ &\quad +\sum_{i=1}^{m}\tilde{\mathcal{I}}_i(u(t_i))-a_2. \end{aligned}$$

Making use of the condition: $u(1)+u'(\eta)=0$ and $a_1+a_2=0$, we obtain

$$\begin{aligned} a_1 &= -\sum_{i=1}^{p+1}\Big[\Big(\frac{\varpi-1}{\varpi}\Big)\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-\beta-1)}}{\Gamma_q(\alpha-\beta)}u(s)d_qs+\frac{1}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\varrho(s)d_qs\Big] \\ &\quad -\sum_{i=1}^{p}(1-t_i)\Big[\Big(\frac{\varpi-1}{\varpi}\Big)\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha-\beta-1)}u(s)d_qs+\frac{1}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\varrho(s)d_qs\Big] \\ &\quad -\Big[\Big(\frac{\varpi-1}{\varpi}\Big)\int_{t_m}^{\eta}\frac{(\eta-qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha-\beta-1)}u(s)d_qs+\frac{1}{\varpi}\int_{t_m}^{\eta}\frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\varrho(s)d_qs\Big] \\ &\quad -\sum_{i=1}^{m}\Big[\Big(\frac{\varpi-1}{\varpi}\Big)\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha-\beta-1)}u(s)d_qs+\frac{1}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\varrho(s)d_qs\Big] \\ &\quad -\sum_{i=1}^{p}\mathcal{I}_i(u(t_i))-\sum_{i=1}^{p}(1-t_i)\tilde{\mathcal{I}}_i(u(t_i))-\sum_{i=1}^{m}\tilde{\mathcal{I}}_i(u(t_i)), \end{aligned}$$

$$\begin{aligned} a_2 &= \sum_{i=1}^{p+1}\Big[\Big(\frac{\varpi-1}{\varpi}\Big)\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-\beta-1)}}{\Gamma_q(\alpha-\beta)}u(s)d_qs+\frac{1}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\varrho(s)d_qs\Big] \\ &\quad +\sum_{i=1}^{p}(1-t_i)\Big[\Big(\frac{\varpi-1}{\varpi}\Big)\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha-\beta-1)}u(s)d_qs+\frac{1}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\varrho(s)d_qs\Big] \\ &\quad +\Big[\Big(\frac{\varpi-1}{\varpi}\Big)\int_{t_m}^{\eta}\frac{(\eta-qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha-\beta-1)}u(s)d_qs+\frac{1}{\varpi}\int_{t_m}^{\eta}\frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\varrho(s)d_qs\Big] \\ &\quad +\sum_{i=1}^{m}\Big[\Big(\frac{\varpi-1}{\varpi}\Big)\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha-\beta-1)}u(s)d_qs+\frac{1}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\varrho(s)d_qs\Big] \\ &\quad +\sum_{i=1}^{p}\mathcal{I}_i(u(t_i))+\sum_{i=1}^{p}(1-t_i)\tilde{\mathcal{I}}_i(u(t_i))+\sum_{i=1}^{m}\tilde{\mathcal{I}}_i(u(t_i)). \end{aligned}$$

Inserting the above values of a_1 and a_2 in (2.4) and (2.8), we obtain (2.3). Conversely, if (2.3) has a solution u, then it can easily be verified that that u satisfies the problem (2.2).

3. Main results

By Lemma 2.1, we can transform the problem (1.1) into a fixed point problem: $u = Fu$, where $F : PC(\mathcal{J},\mathbb{R}) \to PC(\mathcal{J},\mathbb{R})$ is defined by

$$\begin{aligned}
(Fu)(t) &= \Big(\frac{\varpi-1}{\varpi}\Big)\int_{t_\sigma}^{t}\frac{(t-qs)^{(\alpha-\beta-1)}}{\Gamma_q(\alpha-\beta)}u(s)d_qs + \frac{a}{\varpi}\int_{t_\sigma}^{t}\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}f(s,u(s))d_qs \\
&\quad+\frac{b}{\varpi}\int_{t_\sigma}^{t}\frac{(t-qs)^{(\alpha+\delta-1)}}{\Gamma_q(\alpha+\delta)}g(s,u(s))d_qs + \sum_{i=1}^{\sigma}\Big[\Big(\frac{\varpi-1}{\varpi}\Big)\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-\beta-1)}}{\Gamma_q(\alpha-\beta)}u(s)d_qs \\
&\quad+\frac{a}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}f(s,u(s))d_qs + \frac{b}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha+\delta-1)}}{\Gamma_q(\alpha+\delta)}g(s,u(s))d_qs\Big] \\
&\quad+\sum_{i=1}^{\sigma}(t-t_i)\Big[\Big(\frac{\varpi-1}{\varpi}\Big)\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha-\beta-1)}u(s)d_qs + \frac{a}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}f(s,u(s))d_qs \\
&\quad+\frac{b}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha+\delta-2)}}{\Gamma_q(\alpha+\delta-1)}g(s,u(s))d_qs\Big] + \sum_{i=1}^{\sigma}\mathcal{I}_i(u(t_i)) + \sum_{i=1}^{\sigma}(t-t_i)\tilde{\mathcal{I}}_i(u(t_i)) \\
&\quad+(1-t)\Big\{\sum_{i=1}^{p+1}\Big[\Big(\frac{\varpi-1}{\varpi}\Big)\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-\beta-1)}}{\Gamma_q(\alpha-\beta)}u(s)d_qs + \frac{a}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}f(s,u(s))d_qs \\
&\quad+\frac{b}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha+\delta-1)}}{\Gamma_q(\alpha+\delta)}g(s,u(s))d_qs\Big] + \sum_{i=1}^{p}(1-t_i)\Big[\Big(\frac{\varpi-1}{\varpi}\Big)\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha-\beta-1)}u(s)d_qs \\
&\quad+\frac{a}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}f(s,u(s))d_qs + \frac{b}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha+\delta-2)}}{\Gamma_q(\alpha+\delta-1)}g(s,u(s))d_qs\Big] \\
&\quad+\Big(\frac{\varpi-1}{\varpi}\Big)\int_{t_m}^{\eta}\frac{(\eta-qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha-\beta-1)}u(s)d_qs + \frac{a}{\varpi}\int_{t_m}^{\eta}\frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}f(s,u(s))d_qs \\
&\quad+\frac{b}{\varpi}\int_{t_m}^{\eta}\frac{(\eta-qs)^{(\alpha+\delta-2)}}{\Gamma_q(\alpha+\delta-1)}g(s,u(s))d_qs + \sum_{i=1}^{m}\Big[\Big(\frac{\varpi-1}{\varpi}\Big)\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha-\beta-1)}u(s)d_qs \\
&\quad+\frac{a}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}f(s,u(s))d_qs + \frac{b}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha+\delta-2)}}{\Gamma_q(\alpha+\delta-1)}g(s,u(s))d_qs\Big] \\
&\quad+\sum_{i=1}^{p}\mathcal{I}_i(u(t_i)) + \sum_{i=1}^{p}(1-t_i)\tilde{\mathcal{I}}_i(u(t_i)) + \sum_{i=1}^{m}\tilde{\mathcal{I}}_i(u(t_i))\Big\}, \qquad (3.1)
\end{aligned}$$

$$\begin{aligned}
(Fu)'(t) &= \Big(\frac{\varpi-1}{\varpi}\Big)\int_{t_\sigma}^{t}\frac{(t-qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha-\beta-1)}u(s)d_qs + \frac{a}{\varpi}\int_{t_\sigma}^{t}\frac{(t-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}f(s,u(s))d_qs \\
&\quad+\frac{b}{\varpi}\int_{t_\sigma}^{t}\frac{(t-qs)^{(\alpha+\delta-2)}}{\Gamma_q(\alpha+\delta-1)}g(s,u(s))d_qs + \sum_{i=1}^{\sigma}\Big[\Big(\frac{\varpi-1}{\varpi}\Big)\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha-\beta-1)}u(s)d_qs \\
&\quad+\frac{a}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}f(s,u(s))d_qs + \frac{b}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha+\delta-2)}}{\Gamma_q(\alpha+\delta-1)}g(s,u(s))d_qs\Big] + \sum_{i=1}^{\sigma}\tilde{\mathcal{I}}_i(u(t_i))
\end{aligned}$$

$$
\begin{aligned}
&-\Big\{\sum_{i=1}^{p+1}\Big[\Big(\frac{\varpi-1}{\varpi}\Big)\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-\beta-1)}}{\Gamma_q(\alpha-\beta)}u(s)d_qs+\frac{a}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}f(s,u(s))d_qs\\
&+\frac{b}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha+\delta-1)}}{\Gamma_q(\alpha+\delta)}g(s,u(s))d_qs\Big]+\sum_{i=1}^{p}(1-t_i)\Big[\Big(\frac{\varpi-1}{\varpi}\Big)\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha-\beta-1)}u(s)d_qs\\
&+\frac{a}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}f(s,u(s))d_qs+\frac{b}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha+\delta-2)}}{\Gamma_q(\alpha+\delta-1)}g(s,u(s))d_qs\Big]\\
&+\Big(\frac{\varpi-1}{\varpi}\Big)\int_{t_m}^{\eta}\frac{(\eta-qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha-\beta-1)}u(s)d_qs+\frac{a}{\varpi}\int_{t_m}^{\eta}\frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}f(s,u(s))d_qs\\
&+\frac{b}{\varpi}\int_{t_m}^{\eta}\frac{(\eta-qs)^{(\alpha+\delta-2)}}{\Gamma_q(\alpha+\delta-1)}g(s,u(s))d_qs+\sum_{i=1}^{m}\Big[\Big(\frac{\varpi-1}{\varpi}\Big)\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha-\beta-1)}u(s)d_qs\\
&+\frac{a}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}f(s,u(s))d_qs+\frac{b}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha+\delta-2)}}{\Gamma_q(\alpha+\delta-1)}g(s,u(s))d_qs\Big]\\
&+\sum_{i=1}^{p}\mathcal{I}_i(u(t_i))+\sum_{i=1}^{p}(1-t_i)\tilde{\mathcal{I}}_i(u(t_i))+\sum_{i=1}^{m}\tilde{\mathcal{I}}_i(u(t_i))\Big\}. \qquad (3.2)
\end{aligned}
$$

Lemma 3.1. *The operator $F: PC(\mathcal{J},\mathbb{R})\to PC(\mathcal{J},\mathbb{R})$ defined by (3.1) is completely continuous.*

Proof. Observe that continuity of F follows from that of $f, g, \mathcal{I}_\sigma$ and $\tilde{\mathcal{I}}_\sigma$. Let $B_\rho=\{u\in PC(\mathcal{J},\mathbb{R}): \|u\|\le\rho\}\subset PC(\mathcal{J},\mathbb{R})$. Then, there exist positive constants $Q_i>0$ $(i=1,2,3,4)$ such that $|f(t,u)|\le Q_1$, $|g(t,u)|\le Q_2$, $|\mathcal{I}_\sigma(u)|\le Q_3$ and $|\tilde{\mathcal{I}}_\sigma(u)|\le Q_4, \forall u\in B_\rho$. Thus, $\forall u\in B_\rho$, we have

$$
\begin{aligned}
&|(Fu)(t)|\\
\le\ &\frac{|\varpi-1|}{\varpi}\int_{t_\sigma}^{t}\frac{(t-qs)^{(\alpha-\beta-1)}}{\Gamma_q(\alpha-\beta)}|u(s)|d_qs+\frac{|a|}{\varpi}\int_{t_\sigma}^{t}\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}|f(s,u(s))|d_qs\\
&+\frac{|b|}{\varpi}\int_{t_\sigma}^{t}\frac{(t-qs)^{(\alpha+\delta-1)}}{\Gamma_q(\alpha+\delta)}|g(s,u(s))|d_qs+\sum_{i=1}^{\sigma}\Big[\frac{|\varpi-1|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-\beta-1)}}{\Gamma_q(\alpha-\beta)}|u(s)|d_qs\\
&+\frac{|a|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}|f(s,u(s))|d_qs+\frac{|b|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha+\delta-1)}}{\Gamma_q(\alpha+\delta)}|g(s,u(s))|d_qs\Big]\\
&+\sum_{i=1}^{\sigma}|t-t_i|\Big[\frac{|\varpi-1|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha-\beta-1)}|u(s)|d_qs+\frac{|a|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}|f(s,u(s))|d_qs\\
&+\frac{|b|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha+\delta-2)}}{\Gamma_q(\alpha+\delta-1)}|g(s,u(s))|d_qs\Big]+\sum_{i=1}^{\sigma}|\mathcal{I}_i(u(t_i))|+\sum_{i=1}^{\sigma}|t-t_i|\,|\tilde{\mathcal{I}}_i(u(t_i))|\\
&+|1-t|\Big\{\sum_{i=1}^{p+1}\Big[\frac{|\varpi-1|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-\beta-1)}}{\Gamma_q(\alpha-\beta)}|u(s)|d_qs+\frac{|a|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}|f(s,u(s))|d_qs\\
&+\frac{|b|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha+\delta-1)}}{\Gamma_q(\alpha+\delta)}|g(s,u(s))|d_qs\Big]+\sum_{i=1}^{p}|1-t_i|\Big[\frac{|\varpi-1|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha-\beta-1)}|u(s)|d_qs\\
&+\frac{|a|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}|f(s,u(s))|d_qs+\frac{|b|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha+\delta-2)}}{\Gamma_q(\alpha+\delta-1)}|g(s,u(s))|d_qs\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{|\varpi-1|}{\varpi}\int_{t_m}^{\eta}\frac{(\eta-qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha-\beta-1)}|u(s)|d_qs+\frac{|a|}{\varpi}\int_{t_m}^{\eta}\frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}|f(s,u(s))|d_qs\\
&+\frac{|b|}{\varpi}\int_{t_m}^{\eta}\frac{(\eta-qs)^{(\alpha+\delta-2)}}{\Gamma_q(\alpha+\delta-1)}|g(s,u(s))|d_qs+\sum_{i=1}^{m}\Big[\frac{|\varpi-1|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha-\beta-1)}|u(s)|d_qs\\
&+\frac{|a|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}|f(s,u(s))|d_qs+\frac{|b|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha+\delta-2)}}{\Gamma_q(\alpha+\delta-1)}|g(s,u(s))|d_qs\Big]\\
&+\sum_{i=1}^{p}|\mathcal{I}_i(u(t_i))|+\sum_{i=1}^{p}|1-t_i||\tilde{\mathcal{I}}_i(u(t_i))|+\sum_{i=1}^{m}|\tilde{\mathcal{I}}_i(u(t_i))|\Big\}\\
\leq\ &\frac{|\varpi-1|}{\varpi}\int_{t_\sigma}^{t}\frac{(t-qs)^{(\alpha-\beta-1)}}{\Gamma_q(\alpha-\beta)}\rho\, d_qs+\frac{|a|}{\varpi}\int_{t_\sigma}^{t}\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}Q_1d_qs\\
&+\frac{|b|}{\varpi}\int_{t_\sigma}^{t}\frac{(t-qs)^{(\alpha+\delta-1)}}{\Gamma_q(\alpha+\delta)}Q_2d_qs+\sum_{i=1}^{p}\Big[\frac{|\varpi-1|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-\beta-1)}}{\Gamma_q(\alpha-\beta)}\rho\, d_qs\\
&+\frac{|a|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}Q_1d_qs+\frac{|b|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha+\delta-1)}}{\Gamma_q(\alpha+\delta)}Q_2d_qs\Big]\\
&+\sum_{i=1}^{p}\Big[\frac{|\varpi-1|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha-\beta-1)}\rho\, d_qs+\frac{|a|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}Q_1d_qs\\
&+\frac{|b|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha+\delta-2)}}{\Gamma_q(\alpha+\delta-1)}Q_2d_qs\Big]\\
&+\sum_{i=1}^{p+1}\Big[\frac{|\varpi-1|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-\beta-1)}}{\Gamma_q(\alpha-\beta)}\rho\, d_qs+\frac{|a|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}Q_1d_qs\\
&+\frac{|b|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha+\delta-1)}}{\Gamma_q(\alpha+\delta)}Q_2d_qs\Big]+\sum_{i=1}^{p}\Big[\frac{|\varpi-1|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha-\beta-1)}\rho\, d_qs\\
&+\frac{|a|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}Q_1d_qs+\frac{|b|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha+\delta-2)}}{\Gamma_q(\alpha+\delta-1)}Q_2d_qs\Big]\\
&+\frac{|\varpi-1|}{\varpi}\int_{t_m}^{\eta}\frac{(\eta-qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha-\beta-1)}\rho\, d_qs+\frac{|a|}{\varpi}\int_{t_m}^{\eta}\frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}Q_1d_qs\\
&+\frac{|b|}{\varpi}\int_{t_m}^{\eta}\frac{(\eta-qs)^{(\alpha+\delta-2)}}{\Gamma_q(\alpha+\delta-1)}Q_2d_qs+\sum_{i=1}^{p}\Big[\frac{|\varpi-1|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha-\beta-1)}\rho\, d_qs\\
&+\frac{|a|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}Q_1d_qs+\frac{|b|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha+\delta-2)}}{\Gamma_q(\alpha+\delta-1)}Q_2d_qs\Big]+p(2Q_3+3Q_4)\\
\leq\ &\Big[\frac{|\varpi-1|}{\varpi}\Big(\frac{2p+2}{\Gamma_q(\alpha-\beta+1)}+\frac{3p+1}{\Gamma_q(\alpha-\beta)}\Big)\rho\ +\frac{|a|}{\varpi}\Big(\frac{2p+2}{\Gamma_q(\alpha+1)}+\frac{3p+1}{\Gamma_q(\alpha)}\Big)Q_1\\
&+\frac{|b|}{\varpi}\Big(\frac{2p+2}{\Gamma_q(\alpha+\delta+1)}+\frac{3p+1}{\Gamma_q(\alpha+\delta)}\Big)Q_2\ +p(2Q_3+3Q_4)\Big]=\Theta,
\end{aligned}
$$

which implies that $\|Fu\|\leq\Theta$. On the other hand, for $t\in\mathcal{J}_\sigma, 0\leq\sigma\leq p$, we have

$$
|(Fu)'(t)|
$$

$$
\begin{aligned}
\leq\ & \frac{|\varpi-1|}{\varpi}\int_{t_\sigma}^{t}\frac{(t-qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha-\beta-1)}|u(s)|d_qs+\frac{|a|}{\varpi}\int_{t_\sigma}^{t}\frac{(t-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}|f(s,u(s))|d_qs \\
&+\frac{|b|}{\varpi}\int_{t_\sigma}^{t}\frac{(t-qs)^{(\alpha+\delta-2)}}{\Gamma_q(\alpha+\delta-1)}|g(s,u(s))|d_qs+\sum_{i=1}^{\sigma}\Big[\frac{|\varpi-1|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha-\beta-1)}|u(s)|d_qs \\
&+\frac{|a|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}|f(s,u(s))|d_qs+\frac{|b|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha+\delta-2)}}{\Gamma_q(\alpha+\delta-1)}|g(s,u(s))|d_qs\Big]+\sum_{i=1}^{\sigma}|\tilde{\mathcal{I}}_i(u(t_i))| \\
&+\sum_{i=1}^{p+1}\Big[\frac{|\varpi-1|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-\beta-1)}}{\Gamma_q(\alpha-\beta)}|u(s)|d_qs+\frac{|a|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}|f(s,u(s))|d_qs \\
&+\frac{|b|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha+\delta-1)}}{\Gamma_q(\alpha+\delta)}|g(s,u(s))|d_qs\Big]+\sum_{i=1}^{p}|1-t_i|\Big[\frac{|\varpi-1|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha-\beta-1)}|u(s)|d_qs \\
&+\frac{|a|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}|f(s,u(s))|d_qs+\frac{|b|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha+\delta-2)}}{\Gamma_q(\alpha+\delta-1)}|g(s,u(s))|d_qs\Big] \\
&+\frac{|\varpi-1|}{\varpi}\int_{t_m}^{\eta}\frac{(\eta-qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha-\beta-1)}|u(s)|d_qs+\frac{|a|}{\varpi}\int_{t_m}^{\eta}\frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}|f(s,u(s))|d_qs \\
&+\frac{|b|}{\varpi}\int_{t_m}^{\eta}\frac{(\eta-qs)^{(\alpha+\delta-2)}}{\Gamma_q(\alpha+\delta-1)}|g(s,u(s))|d_qs+\sum_{i=1}^{m}\Big[\frac{|\varpi-1|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha-\beta-1)}|u(s)|d_qs \\
&+\frac{|a|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}|f(s,u(s))|d_qs+\frac{|b|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha+\delta-2)}}{\Gamma_q(\alpha+\delta-1)}|g(s,u(s))|d_qs\Big] \\
&+\sum_{i=1}^{p}|\mathcal{I}_i(u(t_i))|+\sum_{i=1}^{p}|1-t_i|\,|\tilde{\mathcal{I}}_i(u(t_i))|+\sum_{i=1}^{m}|\tilde{\mathcal{I}}_i(u(t_i))| \\
\leq\ & \frac{|\varpi-1|}{\varpi}\int_{t_\sigma}^{t}\frac{(t-qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha-\beta-1)}\rho\, d_qs+\frac{|a|}{\varpi}\int_{t_\sigma}^{t}\frac{(t-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}Q_1d_qs \\
&+\frac{|b|}{\varpi}\int_{t_\sigma}^{t}\frac{(t-qs)^{(\alpha+\delta-2)}}{\Gamma_q(\alpha+\delta-1)}Q_2d_qs+\sum_{i=1}^{p}\Big[\frac{|\varpi-1|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha-\beta-1)}\rho\, d_qs \\
&+\frac{|a|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}Q_1d_qs+\frac{|b|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha+\delta-2)}}{\Gamma_q(\alpha+\delta-1)}Q_2d_qs\Big] \\
&+\sum_{i=1}^{p+1}\Big[\frac{|\varpi-1|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-\beta-1)}}{\Gamma_q(\alpha-\beta)}\rho\, d_qs+\frac{|a|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}Q_1d_qs \\
&+\frac{|b|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha+\delta-1)}}{\Gamma_q(\alpha+\delta)}Q_2d_qs\Big]+\sum_{i=1}^{p}\Big[\frac{|\varpi-1|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha-\beta-1)}\rho\, d_qs \\
&+\frac{|a|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}Q_1d_qs+\frac{|b|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha+\delta-2)}}{\Gamma_q(\alpha+\delta-1)}Q_2d_qs\Big] \\
&+\frac{|\varpi-1|}{\varpi}\int_{t_m}^{\eta}\frac{(\eta-qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha-\beta-1)}\rho\, d_qs+\frac{|a|}{\varpi}\int_{t_m}^{\eta}\frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}Q_1d_qs
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{|b|}{\varpi}\int_{t_m}^{\eta}\frac{(\eta-qs)^{(\alpha+\delta-2)}}{\Gamma_q(\alpha+\delta-1)}Q_2 d_qs+\sum_{i=1}^{p}\Big[\frac{|\varpi-1|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha-\beta-1)}\rho\, d_qs\\
&+\frac{|a|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}Q_1 d_qs+\frac{|b|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha+\delta-2)}}{\Gamma_q(\alpha+\delta-1)}Q_2 d_qs\Big]+p(Q_3+3Q_4)\\
\le\ &\Bigg[\frac{|\varpi-1|}{\varpi}\Big(\frac{3p+2}{\Gamma_q(\alpha-\beta)}+\frac{p+1}{\Gamma_q(\alpha-\beta+1)}\Big)\rho+\frac{|a|}{\varpi}\Big(\frac{3p+2}{\Gamma_q(\alpha)}+\frac{p+1}{\Gamma_q(\alpha+1)}\Big)Q_1\\
&+\frac{|b|}{\varpi}\Big(\frac{3p+2}{\Gamma_q(\alpha+\delta)}+\frac{p+1}{\Gamma_q(\alpha+\delta+1)}\Big)Q_2+p(Q_3+3Q_4)\Bigg]=\tilde{\Theta}.
\end{aligned}
$$

Hence, for $t_1, t_2 \in \mathcal{J}_\sigma, 0 \le \sigma \le p$, we have

$$
|(Fu)(t_2)-(Fu)(t_1)| \le \int_{t_1}^{t_2} |(Fu)'(s)|d_qs \le \tilde{\Theta}(t_2-t_1),
$$

which implies that F is equicontinuous on all $\mathcal{J}_\sigma, \sigma = 0,1,2,...p$. Thus, by the Arzela-Ascoli theorem, the operator $F : PC(\mathcal{J},\mathbb{R}) \to PC(\mathcal{J},\mathbb{R})$ is completely continuous.

Theorem 3.1. *(Schaefer's fixed-point theorem [42]) Let E be a Banach space. Assume that $F : E \to E$ is a completely continuous operator and the set $\Omega = \{u \in E|\ u = \zeta Fu,\ 0 < \zeta < 1\}$ is bounded. Then F has a fixed point in E.*

Theorem 3.2. *Assume that*

($\mathcal{S}_1$) *There exist nonnegative functions $\mu_1(t), \mu_2(t), \phi_1(t), \phi_2(t) \in L(0,1)$ and positive constants Q_i ($i = 3,4$) such that $|f(t,u)| \le \mu_1(t) + \mu_2(t)|u|$, $|g(t,u)| \le \phi_1(t) + \phi_2(t)|u|$, $|\mathcal{I}_\sigma(u)| \le Q_3$, $|\tilde{\mathcal{I}}_\sigma(u)| \le Q_4$ for $t \in [0,1], u \in \mathbb{R}$ and $\sigma = 1,2,...,p$.*

Then the problem (1.1) has at least one solution on $\mathcal{J}$.

Proof. Let us consider the set $\Omega = \{u \in PC(\mathcal{J},\mathbb{R})|\ u = \zeta Fu,\ 0 < \zeta < 1\}$, where the operator $F : PC(\mathcal{J},\mathbb{R}) \to PC(\mathcal{J},\mathbb{R})$ is defined by (3.1) and define a ball $B_\varepsilon = \{u \in PC(\mathcal{J},\mathbb{R}) :\ \|u\| \le \varepsilon\}$. We just need to show that the set Ω is bounded as it has already been proved in Lemma 3.1 that the operator F is completely continuous. Let $u \in \Omega$, then $u = \zeta Fu,\ 0 < \zeta < 1$. For any $t \in \mathcal{J}$, we have

$$
\begin{aligned}
&|u(t)| = \zeta|(Fu)(t)|\\
\le\ &\frac{|\varpi-1|}{\varpi}\int_{t_\sigma}^{t}\frac{(t-qs)^{(\alpha-\beta-1)}}{\Gamma_q(\alpha-\beta)}|u(s)|d_qs+\frac{|a|}{\varpi}\int_{t_\sigma}^{t}\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}|f(s,u(s))|d_qs\\
&+\frac{|b|}{\varpi}\int_{t_\sigma}^{t}\frac{(t-qs)^{(\alpha+\delta-1)}}{\Gamma_q(\alpha+\delta)}|g(s,u(s))|d_qs+\sum_{i=1}^{\sigma}\Big[\frac{|\varpi-1|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-\beta-1)}}{\Gamma_q(\alpha-\beta)}|u(s)|d_qs\\
&+\frac{|a|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}|f(s,u(s))|d_qs+\frac{|b|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha+\delta-1)}}{\Gamma_q(\alpha+\delta)}|g(s,u(s))|d_qs\Big]\\
&+\sum_{i=1}^{\sigma}|t-t_i|\Big[\frac{|\varpi-1|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha-\beta-1)}|u(s)|d_qs+\frac{|a|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}|f(s,u(s))|d_qs\\
&+\frac{|b|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha+\delta-2)}}{\Gamma_q(\alpha+\delta-1)}|g(s,u(s))|d_qs\Big]+\sum_{i=1}^{\sigma}|\mathcal{I}_i(u(t_i))|+\sum_{i=1}^{\sigma}|t-t_i|\,|\tilde{\mathcal{I}}_i(u(t_i))|
\end{aligned}
$$

$$
\begin{aligned}
& +|1-t|\Big\{\sum_{i=1}^{p+1}\Big[\frac{|\varpi-1|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-\beta-1)}}{\Gamma_q(\alpha-\beta)}|u(s)|d_qs+\frac{|a|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}|f(s,u(s))|d_qs \\
& +\frac{|b|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha+\delta-1)}}{\Gamma_q(\alpha+\delta)}|g(s,u(s))|d_qs\Big]+\sum_{i=1}^{p}|1-t_i|\Big[\frac{|\varpi-1|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha-\beta-1)}|u(s)|d_qs \\
& +\frac{|a|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}|f(s,u(s))|d_qs+\frac{|b|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha+\delta-2)}}{\Gamma_q(\alpha+\delta-1)}|g(s,u(s))|d_qs\Big] \\
& +\frac{|\varpi-1|}{\varpi}\int_{t_m}^{\eta}\frac{(\eta-qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha-\beta-1)}|u(s)|d_qs+\frac{|a|}{\varpi}\int_{t_m}^{\eta}\frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}|f(s,u(s))|d_qs \\
& +\frac{|b|}{\varpi}\int_{t_m}^{\eta}\frac{(\eta-qs)^{(\alpha+\delta-2)}}{\Gamma_q(\alpha+\delta-1)}|g(s,u(s))|d_qs+\sum_{i=1}^{m}\Big[\frac{|\varpi-1|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha-\beta-1)}|u(s)|d_qs \\
& +\frac{|a|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}|f(s,u(s))|d_qs+\frac{|b|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha+\delta-2)}}{\Gamma_q(\alpha+\delta-1)}|g(s,u(s))|d_qs\Big] \\
& +\sum_{i=1}^{p}|\mathcal{I}_i(u(t_i))|+\sum_{i=1}^{p}|1-t_i||\tilde{\mathcal{I}}_i(u(t_i))|+\sum_{i=1}^{m}|\tilde{\mathcal{I}}_i(u(t_i))|\Big\} \\
\leq\ & \frac{|\varpi-1|}{\varpi}\int_{t_\sigma}^{t}\frac{(t-qs)^{(\alpha-\beta-1)}}{\Gamma_q(\alpha-\beta)}|u(s)|d_qs+\frac{|a|}{\varpi}\int_{t_\sigma}^{t}\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\big(\mu_1(s)+\mu_2(s)\,|u(s)|\big)d_qs \\
& +\frac{|b|}{\varpi}\int_{t_\sigma}^{t}\frac{(t-qs)^{(\alpha+\delta-1)}}{\Gamma_q(\alpha+\delta)}\big(\phi_1(s)+\phi_2(s)\,|u(s)|\big)d_qs+\sum_{i=1}^{p}\Big[\frac{|\varpi-1|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-\beta-1)}}{\Gamma_q(\alpha-\beta)}|u(s)|d_qs \\
& +\frac{|a|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\big(\mu_1(s)+\mu_2(s)\,|u(s)|\big)d_qs \\
& +\frac{|b|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha+\delta-1)}}{\Gamma_q(\alpha+\delta)}\big(\phi_1(s)+\phi_2(s)\,|u(s)|\big)d_qs\Big] \\
& +\sum_{i=1}^{p}\Big[\frac{|\varpi-1|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha-\beta-1)}|u(s)|d_qs+\frac{|a|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\big(\mu_1(s)+\mu_2(s)\,|u(s)|\big)d_qs \\
& +\frac{|b|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha+\delta-2)}}{\Gamma_q(\alpha+\delta-1)}\big(\phi_1(s)+\phi_2(s)\,|u(s)|\big)d_qs\Big]+\sum_{i=1}^{p}|\mathcal{I}_i(u(t_i))|+\sum_{i=1}^{p}\ |\tilde{\mathcal{I}}_i(u(t_i))| \\
& +\sum_{i=1}^{p+1}\Big[\frac{|\varpi-1|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-\beta-1)}}{\Gamma_q(\alpha-\beta)}|u(s)|d_qs+\frac{|a|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\big(\mu_1(s)+\mu_2(s)\,|u(s)|\big)d_qs \\
& +\frac{|b|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha+\delta-1)}}{\Gamma_q(\alpha+\delta)}\big(\phi_1(s)+\phi_2(s)\,|u(s)|\big)d_qs\Big] \\
& +\sum_{i=1}^{p}\Big[\frac{|\varpi-1|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha-\beta-1)}|u(s)|d_qs \\
& +\frac{|a|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\big(\mu_1(s)+\mu_2(s)\,|u(s)|\big)d_qs \\
& +\frac{|b|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha+\delta-2)}}{\Gamma_q(\alpha+\delta-1)}\big(\phi_1(s)+\phi_2(s)\,|u(s)|\big)d_qs\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{|\varpi-1|}{\varpi}\int_{t_m}^{\eta}\frac{(\eta-qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha-\beta-1)}|u(s)|d_qs+\frac{|a|}{\varpi}\int_{t_m}^{\eta}\frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\mu_1(s)+\mu_2(s)\,|u(s)|\Big)d_qs\\
&+\frac{|b|}{\varpi}\int_{t_m}^{\eta}\frac{(\eta-qs)^{(\alpha+\delta-2)}}{\Gamma_q(\alpha+\delta-1)}\Big(\phi_1(s)+\phi_2(s)\,|u(s)|\Big)d_qs\\
&+\sum_{i=1}^{p}\Big[\frac{|\varpi-1|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha-\beta-1)}|u(s)|d_qs\\
&+\frac{|a|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\mu_1(s)+\mu_2(s)\,|u(s)|\Big)d_qs\\
&+\frac{|b|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha+\delta-2)}}{\Gamma_q(\alpha+\delta-1)}\Big(\phi_1(s)+\phi_2(s)\,|u(s)|\Big)d_qs\Big]\\
&+\sum_{i=1}^{p}|\mathcal{I}_i(u(t_i))|+\sum_{i=1}^{p}|\tilde{\mathcal{I}}_i(u(t_i))|+\sum_{i=1}^{p}|\tilde{\mathcal{I}}_i(u(t_i))|\\
\leq\;&\frac{|\varpi-1|}{\varpi}\Big[\frac{2p+2}{\Gamma_q(\alpha-\beta+1)}+\frac{3p+1}{\Gamma_q(\alpha-\beta)}\Big]\|u\|+\frac{|a|}{\varpi}\Big[(2p+2)I_q^{\alpha}\mu_1(1)+(3p+1)I_q^{\alpha-1}\mu_1(1)\Big]\\
&+\frac{|a|}{\varpi}\Big[(2p+2)I_q^{\alpha}\mu_2(1)+(3p+1)I_q^{\alpha-1}\mu_2(1)\Big]\|u\|+\frac{|b|}{\varpi}\Big[(2p+2)I_q^{\alpha+\delta}\phi_1(1)+(3p+1)I_q^{\alpha+\delta-1}\phi_1(1)\Big]\\
&+\frac{|b|}{\varpi}\Big[(2p+2)I_q^{\alpha+\delta}\phi_2(1)+(3p+1)I_q^{\alpha+\delta-1}\phi_2(1)\Big]\|u\|+p(2Q_3+3Q_4),
\end{aligned}
$$

which implies that $\|u\|$ is bounded for any $t\in\mathcal{J}$. So, the set Ω is bounded. Thus, by the conclusion of Theorem 3.1, the operator F has at least one fixed point, which shows that (1.1) has at least one solution on $\mathcal{J}$.

Corollary 3.1. *Assume that functions $f,g,\mathcal{I}_\sigma,\tilde{\mathcal{I}}_\sigma$ ($\sigma=1,2,3,...$) are bounded. Then the nonlinear problem (1.1) has at least one solution $\mathcal{J}$.*

Theorem 3.3. *([42]) Let $F:Y\to Y$ be a contraction on a nonempty closed subset of a Banach space X. Then F has a unique solution.*

Theorem 3.4. *Let $f,g:[0,1]\times\mathbb{R}\to\mathbb{R}$ and $\mathcal{I}_\sigma,\tilde{\mathcal{I}}_\sigma:\mathbb{R}\to\mathbb{R}$ be continuous functions. In addition, the following conditions hold:*

($\mathcal{S}_2$) *There exist positive constants N_i ($i=1,2,3,4$) such that, for each $t\in[0,1]$ and $u,v\in\mathbb{R}$,*

$$|f(t,u)-f(t,v)|\leq N_1|u-v|,\quad |g(t,u)-g(t,v)|\leq N_2|u-v|,$$

$$|\mathcal{I}_\sigma(u)-\mathcal{I}_\sigma(v)|\leq N_3|u-v|,\quad |\tilde{\mathcal{I}}_\sigma(u)-\tilde{\mathcal{I}}_\sigma(v)|\leq N_4|u-v|,$$

for $\sigma=1,2,...,p$.

Then the problem (1.1) has a unique solution on $[0,1]$, provided that

$$
\begin{aligned}
\Xi=\;&\frac{|\varpi-1|}{\varpi}\Big[\frac{2p+2}{\Gamma_q(\alpha-\beta+1)}+\frac{3p+1}{\Gamma_q(\alpha-\beta)}\Big]+\frac{|a|}{\varpi}\Big[\frac{2p+2}{\Gamma_q(\alpha+1)}+\frac{3p+1}{\Gamma_q(\alpha)}\Big]N_1\\
&+\frac{|b|}{\varpi}\Big[\frac{2p+2}{\Gamma_q(\alpha+\delta+1)}+\frac{3p+1}{\Gamma_q(\alpha+\delta)}\Big]N_2+p(2N_3+3N_4)<1.
\end{aligned}\tag{3.3}
$$

Proof. For $u, v \in PC(\mathcal{J}, \mathbb{R})$ and $t \in \mathcal{J}$, we get

$$
\begin{aligned}
&|(Fu)(t) - (Fv)(t)| \\
\leq\ & \frac{|\varpi - 1|}{\varpi} \int_{t_\sigma}^{t} \frac{(t - qs)^{(\alpha-\beta-1)}}{\Gamma_q(\alpha - \beta)} |u(s) - v(s)| d_q s + \frac{|a|}{\varpi} \int_{t_\sigma}^{t} \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s, u(s)) - f(s, v(s))| d_q s \\
&+ \frac{|b|}{\varpi} \int_{t_\sigma}^{t} \frac{(t - qs)^{(\alpha+\delta-1)}}{\Gamma_q(\alpha + \delta)} |g(s, u(s)) - g(s, v(s))| d_q s + \sum_{i=1}^{\sigma} \Big[\frac{|\varpi - 1|}{\varpi} \int_{t_{i-1}}^{t_i} \frac{(t_i - qs)^{(\alpha-\beta-1)}}{\Gamma_q(\alpha - \beta)} |u(s) - v(s)| d_q s \\
&+ \frac{|a|}{\varpi} \int_{t_{i-1}}^{t_i} \frac{(t_i - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s, u(s)) - f(s, v(s))| d_q s + \frac{|b|}{\varpi} \int_{t_{i-1}}^{t_i} \frac{(t_i - qs)^{(\alpha+\delta-1)}}{\Gamma_q(\alpha + \delta)} |g(s, u(s)) - g(s, v(s))| d_q s \Big] \\
&+ \sum_{i=1}^{\sigma} |t - t_i| \Big[\frac{|\varpi - 1|}{\varpi} \int_{t_{i-1}}^{t_i} \frac{(t_i - qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha - \beta - 1)} |u(s) - v(s)| d_q s \\
&+ \frac{|a|}{\varpi} \int_{t_{i-1}}^{t_i} \frac{(t_i - qs)^{(\alpha-2)}}{\Gamma_q(\alpha - 1)} |f(s, u(s)) - f(s, v(s))| d_q s \\
&+ \frac{|b|}{\varpi} \int_{t_{i-1}}^{t_i} \frac{(t_i - qs)^{(\alpha+\delta-2)}}{\Gamma_q(\alpha + \delta - 1)} |g(s, u(s)) - g(s, v(s))| d_q s \Big] + \sum_{i=1}^{\sigma} |\mathcal{I}_i(u(t_i)) - \mathcal{I}_i(v(t_i))| \\
&+ \sum_{i=1}^{\sigma} |t - t_i|\, |\tilde{\mathcal{I}}_i(u(t_i)) - \tilde{\mathcal{I}}_i(v(t_i))| + |1 - t| \Big\{ \sum_{i=1}^{p+1} \Big[\frac{|\varpi - 1|}{\varpi} \int_{t_{i-1}}^{t_i} \frac{(t_i - qs)^{(\alpha-\beta-1)}}{\Gamma_q(\alpha - \beta)} |u(s) - v(s)| d_q s \\
&+ \frac{|a|}{\varpi} \int_{t_{i-1}}^{t_i} \frac{(t_i - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s, u(s)) - f(s, v(s))| d_q s + \frac{|b|}{\varpi} \int_{t_{i-1}}^{t_i} \frac{(t_i - qs)^{(\alpha+\delta-1)}}{\Gamma_q(\alpha + \delta)} |g(s, u(s)) - g(s, v(s))| d_q s \Big] \\
&+ \sum_{i=1}^{p} |1 - t_i| \Big[\frac{|\varpi - 1|}{\varpi} \int_{t_{i-1}}^{t_i} \frac{(t_i - qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha - \beta - 1)} |u(s) - v(s)| d_q s \\
&+ \frac{|a|}{\varpi} \int_{t_{i-1}}^{t_i} \frac{(t_i - qs)^{(\alpha-2)}}{\Gamma_q(\alpha - 1)} |f(s, u(s)) - f(s, v(s))| d_q s + \frac{|b|}{\varpi} \int_{t_{i-1}}^{t_i} \frac{(t_i - qs)^{(\alpha+\delta-2)}}{\Gamma_q(\alpha + \delta - 1)} |g(s, u(s)) - g(s, v(s))| d_q s \Big] \\
&+ \frac{|\varpi - 1|}{\varpi} \int_{t_m}^{\eta} \frac{(\eta - qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha - \beta - 1)} |u(s) - v(s)| d_q s + \frac{|a|}{\varpi} \int_{t_m}^{\eta} \frac{(\eta - qs)^{(\alpha-2)}}{\Gamma_q(\alpha - 1)} |f(s, u(s)) - f(s, v(s))| d_q s \\
&+ \frac{|b|}{\varpi} \int_{t_m}^{\eta} \frac{(\eta - qs)^{(\alpha+\delta-2)}}{\Gamma_q(\alpha + \delta - 1)} |g(s, u(s)) - g(s, v(s))| d_q s + \sum_{i=1}^{m} \Big[\frac{|\varpi - 1|}{\varpi} \int_{t_{i-1}}^{t_i} \frac{(t_i - qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha - \beta - 1)} |u(s) - v(s)| d_q s \\
&+ \frac{|a|}{\varpi} \int_{t_{i-1}}^{t_i} \frac{(t_i - qs)^{(\alpha-2)}}{\Gamma_q(\alpha - 1)} |f(s, u(s)) - f(s, v(s))| d_q s + \frac{|b|}{\varpi} \int_{t_{i-1}}^{t_i} \frac{(t_i - qs)^{(\alpha+\delta-2)}}{\Gamma_q(\alpha + \delta - 1)} |g(s, u(s)) - g(s, v(s))| d_q s \Big] \\
&+ \sum_{i=1}^{p} |\mathcal{I}_i(u(t_i)) - \mathcal{I}_i(v(t_i))| + \sum_{i=1}^{p} |1 - t_i| |\tilde{\mathcal{I}}_i(u(t_i)) - \tilde{\mathcal{I}}_i(v(t_i))| + \sum_{i=1}^{m} |\tilde{\mathcal{I}}_i(u(t_i)) - \tilde{\mathcal{I}}_i(v(t_i))| \Big\} \\
\leq\ & \frac{|\varpi - 1|}{\varpi} \int_{t_\sigma}^{t} \frac{(t - qs)^{(\alpha-\beta-1)}}{\Gamma_q(\alpha - \beta)} |u(s) - v(s)| d_q s + \frac{|a|}{\varpi} \int_{t_\sigma}^{t} \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} N_1 |u(s) - v(s)| d_q s \\
&+ \frac{|b|}{\varpi} \int_{t_\sigma}^{t} \frac{(t - qs)^{(\alpha+\delta-1)}}{\Gamma_q(\alpha + \delta)} N_2 |u(s) - v(s)| d_q s + \sum_{i=1}^{p} \Big[\frac{|\varpi - 1|}{\varpi} \int_{t_{i-1}}^{t_i} \frac{(t_i - qs)^{(\alpha-\beta-1)}}{\Gamma_q(\alpha - \beta)} |u(s) - v(s)| d_q s \\
&+ \frac{|a|}{\varpi} \int_{t_{i-1}}^{t_i} \frac{(t_i - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} N_1 |u(s) - v(s)| d_q s + \frac{|b|}{\varpi} \int_{t_{i-1}}^{t_i} \frac{(t_i - qs)^{(\alpha+\delta-1)}}{\Gamma_q(\alpha + \delta)} N_2 |u(s) - v(s)| d_q s \Big]
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=1}^{p}\Big[\frac{|\varpi-1|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha-\beta-1)}|u(s)-v(s)|d_qs+\frac{|a|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}N_1|u(s)-v(s)|d_qs\\
&+\frac{|b|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha+\delta-2)}}{\Gamma_q(\alpha+\delta-1)}N_2|u(s)-v(s)|d_qs\Big]+\sum_{i=1}^{p}N_3|u(s)-v(s)|+\sum_{i=1}^{p}N_4|u(s)-v(s)|\\
&+\sum_{i=1}^{p+1}\Big[\frac{|\varpi-1|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-\beta-1)}}{\Gamma_q(\alpha-\beta)}|u(s)-v(s)|d_qs+\frac{|a|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}N_1|u(s)-v(s)|d_qs\\
&+\frac{|b|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha+\delta-1)}}{\Gamma_q(\alpha+\delta)}N_2|u(s)-v(s)|d_qs\Big]\\
&+\sum_{i=1}^{p}\Big[\frac{|\varpi-1|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha-\beta-1)}|u(s)-v(s)|d_qs\\
&+\frac{|a|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}N_1|u(s)-v(s)|d_qs+\frac{|b|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha+\delta-2)}}{\Gamma_q(\alpha+\delta-1)}N_2|u(s)-v(s)|d_qs\Big]\\
&+\frac{|\varpi-1|}{\varpi}\int_{t_m}^{\eta}\frac{(\eta-qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha-\beta-1)}|u(s)-v(s)|d_qs+\frac{|a|}{\varpi}\int_{t_m}^{\eta}\frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}N_1|u(s)-v(s)|d_qs\\
&+\frac{|b|}{\varpi}\int_{t_m}^{\eta}\frac{(\eta-qs)^{(\alpha+\delta-2)}}{\Gamma_q(\alpha+\delta-1)}N_2|u(s)-v(s)|d_qs+\sum_{i=1}^{m}\Big[\frac{|\varpi-1|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-\beta-2)}}{\Gamma_q(\alpha-\beta-1)}|u(s)-v(s)|d_qs\\
&+\frac{|a|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}N_1|u(s)-v(s)|d_qs+\frac{|b|}{\varpi}\int_{t_{i-1}}^{t_i}\frac{(t_i-qs)^{(\alpha+\delta-2)}}{\Gamma_q(\alpha+\delta-1)}N_2|u(s)-v(s)|d_qs\Big]\\
&+\sum_{i=1}^{p}N_3|u(s)-v(s)|+\sum_{i=1}^{p}N_4|u(s)-v(s)|+\sum_{i=1}^{p}N_4|u(s)-v(s)|\\
\leq\ &\Big\{\frac{|\varpi-1|}{\varpi}\Big[\frac{2p+2}{\Gamma_q(\alpha-\beta+1)}+\frac{3p+1}{\Gamma_q(\alpha-\beta)}\Big]+\frac{|a|}{\varpi}\Big[\frac{2p+2}{\Gamma_q(\alpha+1)}+\frac{3p+1}{\Gamma_q(\alpha)}\Big]N_1\\
&+\frac{|b|}{\varpi}\Big[\frac{2p+2}{\Gamma_q(\alpha+\delta+1)}+\frac{3p+1}{\Gamma_q(\alpha+\delta)}\Big]N_2+p(2N_3+3N_4)\Big\}|u(s)-v(s)|.
\end{aligned}
$$

Thus, in view of the condition (3.3), the above inequality takes the form:

$$||Fu-Fv||\leq\Xi||u-v||.$$

As $\Xi<1$ (by the condition (3.3)), therefore F is a contraction. So, by Theorem 3.3, the operator F has a unique fixed point. Hence, the problem (1.1) has a unique solution on $\mathcal{J}$.

4. Examples

In this section, we demonstrate the application of the results obtained in the last section.

Example 4.1. *(Illustration of Theorem 3.1). Consider the following boundary value problem of multi-*

term impulsive fractional q-integro-difference equations:

$$\begin{cases} 0.88\,{}^cD_{0.5}^{1.95}u(t) + (1-0.88)\,{}^cD_{0.5}^{0.02}u(t) = 0.30f(t,u(t)) + 0.90I_{0.5}^{0.25}g(t,u(t)), \quad t \neq t_1 = \dfrac{1}{3}, \\ \Delta u\Big(\dfrac{1}{3}\Big) = \dfrac{4u^2}{1+u^2}, \quad \Delta u'\Big(\dfrac{1}{3}\Big) = \dfrac{3+2u^2}{u^2+1}, \\ u(0)+u'(0) = 0, \quad u(1)+u'(0.4) = 0, \end{cases} \tag{4.1}$$

where $\alpha = 1.95$, $q = 0.5$, $\beta = 0.02$, $\varpi = 0.88$, $\delta = 0.25$, $\eta = 0.4$, $a = 0.30$,$b = 0.90$, $p = 1$ *and*

$$f(t,u(t)) = e^{4t}\cos^2(u(t)) + \sin(t)\,|u|, \quad g(t,u(t)) = \frac{2}{9}(t^3+3) + 3\cos(t)\,|u|.$$

Clearly, $\mu_1(t) = e^{4t}\cos^2(u(t))$, $\mu_2(t) = \sin(t)$, $\phi_1(t) = \frac{2}{9}(t^3+3)$, $\phi_2(t) = 3\cos(t)$, $Q_3 = 4$, $Q_4 = 3$, *and the hypothesis of Theorem 3.1 holds true. Thus, the conclusion of Theorem 3.1 applies to the problem (4.1).*

Example 4.2. *(Illustration of Theorem 3.4). Consider the boundary value problem given by*

$$\begin{cases} 0.88\,{}^cD_{0.5}^{1.95}u(t) + (1-0.88)\,{}^cD_{0.5}^{0.02}u(t) = 0.30f(t,u(t)) + 0.90I_{0.5}^{0.25}g(t,u(t)), \quad t \neq t_1 = \dfrac{1}{3}, \\ \Delta u\Big(\dfrac{1}{3}\Big) = \dfrac{|u(\frac{1}{3})|}{115(|u(\frac{1}{3})|+1)}, \quad \Delta u'\Big(\dfrac{1}{3}\Big) = \dfrac{1}{900}\sin\Big(u\Big(\dfrac{1}{3}\Big)\Big), \\ u(0)+u'(0) = 0, \quad u(1)+u'(0.4) = 0, \end{cases} \tag{4.2}$$

where $\alpha = 1.95$, $q = 0.5$, $\beta = 0.02$, $\varpi = 0.88$, $\delta = 0.25$, $\eta = 0.4$, $a = 0.30$,$b = 0.90$, $p = 1$ *and*

$$f(t,u(t)) = \frac{1}{360+t^2}\Big(\cos t + \frac{|u(t)|}{(|u(t)|+1)} + |u(t)|\Big), \; g(t,u(t)) = \frac{1}{500}\tan^{-1}u(t) + t^4.$$

It is easy to verify that

$$N_1 = \frac{1}{180}, \quad N_2 = \frac{1}{500}, \quad N_3 = \frac{1}{115}, \quad N_4 = \frac{1}{900}.$$

Moreover, $\Xi \approx 0.978211 < 1$ *(*Ξ *is given by (3.3)). Thus, all the assumptions of Theorem 3.4 are satisfied. So, by the conclusion of Theorem 3.4, the problem (4.2) has a unique solution* $[0,1]$.

5. Conclusions

We have investigated a new class of nonlinear nonlocal impulsive boundary value problems of multi-term Caputo fractional q-difference equations involving both usual and Riemann-Liouville fractional q-integral type nonlinearities. The classical fixed point theorems are employed to derive the existence and uniqueness results for the given problem. Our results are indeed new and enrich the related literature on the topic. Moreover, some new results can be recorded as special cases of the present ones by fixing the parameters involved in the governing equation in the problem (1.1). For example, our results reduce to the new ones for the following equations subject to impulsive and nonlocal boundary data:

- ${}^cD_q^{\alpha}u(t) = af(t,u(t)) + bI_q^{\delta}g(t,u(t))$ for $\lambda = 1$;
- $\lambda\,{}^cD_q^{\alpha}u(t) + (1-\lambda)\,{}^cD_q^{\beta}u(t) = f(t,u(t))$ for $a = 1, b = 0$;
- $\lambda\,{}^cD_q^{\alpha}u(t) + (1-\lambda)\,{}^cD_q^{\beta}u(t) = I_q^{\delta}g(t,u(t))$ for $a = 0, b = 1$.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgements

This research work was funded by Institutional Fund Projects under Grant No. (IFPIP: 1444-130-1443). The authors gratefully acknowledge technical and financial support provided by the Ministry of Education and King Abdulaziz University, DSR, Jeddah, Saudi Arabia. The authors also thank the reviewers for their constructive remarks on their work.

Conflict of Interest

All authors declare no conflicts of interest in this paper.

References

1. J. Lee, J. Kim, J. Lim, Y. Yun, Y. Kim, J. Ryu, et al., Unsteady flow of shear-thickening fluids around an impulsively moving circular cylinder, *J. Non-Newton. Fluid*, **272** (2019), 104163. https://doi.org/10.1016/j.jnnfm.2019.104163
2. K. F. Chen, Q. Zhang, On the impulse response of a vibrator with a band-limited hysteretic damper, *Appl. Math. Model.*, **35** (2011), 189–201. https://doi.org/10.1016/j.apm.2010.05.017
3. I. M. Stamova, T. Stamov, N. Simeonova, Impulsive control on global exponential stability for cellular neural networks with supremums, *J. Vib. Control*, **19** (2013), 483–490. https://doi.org/10.1177/1077546312441042
4. C. He, Z. Wen, K. Huang, X. Ji, Sudden shock and stock market network structure characteristics: A comparison of past crisis events, *Technol. Forecast. Soc.*, **180** (2022), 121732. https://doi.org/10.1016/j.techfore.2022.121732
5. V. Lakshmikantham, D. D. Bainov, P. S. Simeonov, *Theory of impulsive differential equations*, World Scientific, 1989. https://doi.org/10.1142/0906
6. T. Cardinali, P. Rubbioni, Hereditary evolution processes under impulsive effects, *Mediterr. J. Math.*, **18** (2021), 91. https://doi.org/10.1007/s00009-021-01730-8
7. X. B. Rao, X. P. Zhao, Y. D. Chu, J. G. Zhang, J. S. Gao, The analysis of mode-locking topology in an SIR epidemic dynamics model with impulsive vaccination control: Infinite cascade of Stern-Brocot sum trees, *Chaos Soliton. Fract.*, **139** (2020), 110031. https://doi.org/10.1016/j.chaos.2020.110031
8. R. E. Miron, R. J. Smith, Resistance to protease inhibitors in a model of HIV-1 infection with impulsive drug effects, *Bull. Math. Biol.*, **76** (2014), 59–97. https://doi.org/10.1007/s11538-013-9903-9
9. S. Deswal, S. Choudhary, Impulsive effect on an elastic solid with generalized thermodiffusion, *J. Eng. Math.*, **63** (2009), 79–94. https://doi.org/10.1007/s10665-008-9249-8

10. G. Stamov, T. Stamova, C. Spirova, Impulsive reaction-diffusion delayed models in biology: Integral manifolds approach, *Entropy*, **23** (2021), 1631. https://doi.org/10.3390/e23121631

11. P. Savoini, M. Scholer, M. Fujimoto, Two-dimensional hybrid simulations of impulsive plasma penetration through a tangential discontinuity, *J. Geophys. Res. Space*, **99** (1994), 19377–19391. https://doi.org/10.1029/94JA01512

12. B. Wang, X. Xia, Z. Cheng, L. Liu, H. Fan, An impulsive and switched system based maintenance plan optimization in building energy retrofitting project, *Appl. Math. Model.*, **117** (2023), 479–493. https://doi.org/10.1016/j.apm.2022.12.030

13. Y. Xu, W. Li, Finite-time synchronization of fractional-order complex-valued coupled systems, *Phys. A*, **549** (2020), 123903. https://doi.org/10.1016/j.physa.2019.123903

14. Y. Ding, Z. Wang, H. Ye, Optimal control of a fractional-order HIV-immune system with memory, *IEEE T. Contr. Syst. T.*, **20** (2012), 763–769. https://doi.org/10.1109/TCST.2011.2153203

15. M. S. Ali, G. Narayanan, V. Shekher, A. Alsaedi, B. Ahmad, Global Mittag-Leffler stability analysis of impulsive fractional-order complex-valued BAM neural networks with time varying delays, *Commun. Nonlinear Sci.*, **83** (2020), 105088. https://doi.org/10.1016/j.cnsns.2019.105088

16. X. Zheng, H. Wang, An error estimate of a numerical approximation to a hidden-memory variable-order space-time fractional diffusion equation, *SIAM J. Numer. Anal.*, **58** (2020), 2492–2514. https://doi.org/10.1137/20M132420X

17. M. Javidi, B. Ahmad, Dynamic analysis of time fractional order phytoplankton-toxic phytoplankton-zooplankton system, *Ecol. Model.*, **318** (2015), 8–18. https://doi.org/10.1016/j.ecolmodel.2015.06.016

18. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, 2006.

19. B. Ahmad, S. K. Ntouyas, *Nonlocal nonlinear fractional-order boundary value problems*, Singapore: World Scientific, 2021. https://doi.org/10.1142/12102

20. H. Alsulami, M. Kirane, S. Alhodily, T. Saeed, N. Nyamoradi, Existence and multiplicity of solutions to fractional p-Laplacian systems with concave-convex nonlinearities, *Bull. Math. Sci.*, **10** (2020), 2050007. https://doi.org/10.1142/S1664360720500071

21. A. Ebrahimzadeh, R. Khanduzi, A. Beik, P. Samaneh, D. Baleanu, Research on a collocation approach and three metaheuristic techniques based on MVO, MFO, and WOA for optimal control of fractional differential equation, *J. Vib. Control*, **29** (2023), 661–674. https://doi.org/10.1177/10775463211051447

22. R. Agarwal, S. Hristova, D. O'Regan, Integral presentations of the solution of a boundary value problem for impulsive fractional integro-differential equations with Riemann-Liouville derivatives, *AIMS Math.*, **7** (2022), 2973–2988. https://doi.org/10.3934/math.2022164

23. J. J. Nieto, Fractional Euler numbers and generalized proportional fractional logistic differential equation, *Fract. Calc. Appl. Anal.*, **25** (2022), 876–886. https://doi.org/10.1007/s13540-022-00044-0

24. K. D. Kucche, A. D. Mali, On the nonlinear (k,ψ)-Hilfer fractional differential equations, *Chaos Soliton. Fract.*, **152** (2021), 111335. https://doi.org/10.1016/j.chaos.2021.111335

25. C. Kiataramkul, S. K. Ntouyas, J. Tariboon, Existence results for ψ-Hilfer fractional integro-differential hybrid boundary value problems for differential equations and inclusions, *Adv. Math. Phys.*, **2021** (2021), 9044313. https://doi.org/10.1155/2021/9044313

26. Z. Laadjal, F. Jarad, Existence, uniqueness and stability of solutions for generalized proportional fractional hybrid integro-differential equations with Dirichlet boundary conditions, *AIMS Math.*, **8** (2023), 1172–1194. https://doi.org/10.3934/math.2023059

27. J. R. Graef, L. Kong, Positive solutions for a class of higher order boundary value problems with fractional q-derivatives, *Appl. Math. Comput.*, **218** (2012), 9682–9689. https://doi.org/10.1016/j.amc.2012.03.006

28. B. Ahmad, S. K. Ntouyas, J. Tariboon, *Quantum calculus: New concepts, impulsive IVPs and BVPs, inequalities*, World Scientific, 2016. https://doi.org/10.1142/10075

29. M. Jiang, R. Huang, Existence of solutions for q-fractional differential equations with nonlocal Erdélyi-Kober q-fractional integral condition, *AIMS Math.*, **5** (2020), 6537–6551. https://doi.org/10.3934/math.2020421

30. S. Liang, M. E. Samei, New approach to solutions of a class of singular fractional q-differential problem via quantum calculus, *Adv. Differ. Equ.*, **2020** (2020), 14. https://doi.org/10.1186/s13662-019-2489-2

31. C. Bai, D. Yang, The iterative positive solution for a system of fractional q-difference equations with four-point boundary conditions, *Discrete Dyn. Nat. Soc.*, **2020** (2020), 3970903. https://doi.org/10.1155/2020/3970903

32. A. Wongcharoen, A. Thatsatian, S. K. Ntouyas, J. Tariboon, Nonlinear fractional q-difference equation with fractional Hadamard and quantum integral nonlocal conditions, *J. Funct. Space.*, **2020** (2020), 9831752. https://doi.org/10.1155/2020/9831752

33. A. Alsaedi, H. Al-Hutami, B. Ahmad, R. P. Agarwal, Existence results for a coupled system of nonlinear fractional q-integro-difference equations with q-integral coupled boundary conditions, *Fractals*, **30** (2022), 2240042. https://doi.org/10.1142/S0218348X22400424

34. W. Yukunthorn, B. Ahmad, S. K. Ntouyas, J. Tariboon, On Caputo-Hadamard type fractional impulsive hybrid systems with nonlinear fractional integral conditions, *Nonlinear Anal. Hybri.*, **19** (2016), 77–92. https://doi.org/10.1016/j.nahs.2015.08.001

35. B. Ahmad, S. K. Ntouyas, J. Tariboon, A. Alsaedi, H. H. Alsulami, Impulsive fractional q-integro-difference equations with separated boundary conditions, *Appl. Math. Comput.*, **281** (2016), 199–213. https://doi.org/10.1016/j.amc.2016.01.051

36. S. Abbas, M. Benchohra, A. Alsaedi, Y. Zhou, Some stability concepts for abstract fractional differential equations with not instantaneous impulses, *Fixed Point Theor.*, **18** (2017), 3–16. https://doi.org/10.24193/fpt-ro.2017.1.01

37. D. Vivek, K. Kanagarajan, S. Harikrishnan, Existence of solutions for impulsive fractional q-difference equations with nonlocal condition, *J. Appl. Nonlinear Dyn.*, **6** (2017), 479–486. https://doi.org/10.5890/JAND.2017.12.004

38. M. Zuo, X. Hao, Existence results for impulsive fractional q-difference equation with antiperiodic boundary conditions, *J. Funct. Space.*, **2018** (2018), 3798342. https://doi.org/10.1155/2018/3798342

39. R. P. Agarwal, S. Hristova, D. O'Regan, Exact solutions of linear Riemann-Liouville fractional differential equations with impulses, *Rocky Mountain J. Math.*, **50** (2020), 779–791. https://doi.org/10.1216/rmj.2020.50.779

40. M. H. Annaby, Z. S. Mansour, *q-Fractional calculus and equations*, Berlin: Springer-Verlag, 2012. https://doi.org/10.1007/978-3-642-30898-7

41. P. M. Rajkovic, S. D. Marinkovic, M. S. Stankovic, On q-analogues of Caputo derivative and Mittag-Leffler function, *Fract. Calc. Appl. Anal.*, **10** (2007), 359–373.

42. H. Schaefer, Über die Methode der a priori-Schranken, *Math. Ann.*, **129** (1955), 415–416.