Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(8): 19334–19352.
DOI: 10.3934/math.2023986
Received: 20 April 2023
Revised: 25 May 2023
Accepted: 01 June 2023
Published: 08 June 2023

Research article

Iterative algorithm for solving monotone inclusion and fixed point problem of a finite family of demimetric mappings

Anjali[1,*], **Seema Mehra**[1], **Renu Chugh**[2], **Salma Haque**[3] **and Nabil Mlaiki**[3,*]

[1] Department of Mathematics, Maharshi Dayanand University, Rohtak 124001, India
[2] Department of Mathematics, Gurugram University, Gurugram 122413, India
[3] Department of Mathematics and Sciences, Prince Sultan University, Riyadh 11586, Saudi Arabia

* **Correspondence:** Email: anjaliahuja3108@gmail.com, nmlaiki2012@gmail.com.

Abstract: The goal of this study is to develop a novel iterative algorithm for approximating the solutions of the monotone inclusion problem and fixed point problem of a finite family of demimetric mappings in the context of a real Hilbert space. The proposed algorithm is based on the inertial extrapolation step strategy and combines forward-backward and Tseng's methods. We introduce a demimetric operator with respect to M-norm, where M is a linear, self-adjoint, positive and bounded operator. The algorithm also includes a new step for solving the fixed point problem of demimetric operators with respect to the M-norm. We study the strong convergence behavior of our algorithm. Furthermore, we demonstrate the numerical efficiency of our algorithm with the help of an example. The result given in this paper extends and generalizes various existing results in the literature.

Keywords: fixed point problem; monotone inclusion problem; strong convergence; forward-backward splitting algorithm
Mathematics Subject Classification: 47H05, 47H10, 47J25, 65K15

1. Introduction

Monotone inclusion problem (MIP) is the problem to identify a zero of the sum of two operators and it is described as

$$\text{Finding } \zeta \in H \text{ such that } 0 \in (A + B)\zeta, \tag{1.1}$$

where $A : H \to H$ is an M-cocoercive operator, M is a linear, bounded operator on a real Hilbert space H and $B : H \to 2^H$ is a maximal monotone operator. Monotone inclusion problem incorporates various problems such as convex optimization, machine learning, statistical regression, signal and image processing, see [1–17]. Lions and Mercier [18] defined the forward-backward splitting algorithm which is one of the most effective method for solving the problem (1.1) and it is defined by

$$\zeta_{n+1} = (I + \lambda_n A)^{-1}(I - \lambda_n B)\zeta_n \text{ for all } n \in \mathbb{N}, \tag{1.2}$$

where A is monotone, B is $1/L$-cocoercive operator and $\lambda_n \in (0, 2/L)$ is a step size parameter. But the algorithm defined in (1.2) converges weakly to a solution of the monotone inclusion problem. Later, Tseng [15] improved this forward-backward splitting algorithm and also proved weak convergence of it. After that, Gibali and Thong [19] proposed a modified version of Tseng's splitting algorithm and proved strong convergence of the proposed algorithm. To speed up the convergence rate of the algorithms, Moudafi and Oliny [20] introduced the following algorithm which includes the inertial parameter.

$$\begin{cases} \varphi_n = \zeta_n + \epsilon_n(\zeta_n - \zeta_{n-1}), \\ \zeta_{n+1} = (I + \lambda_n A)^{-1}(I - \lambda_n B)\zeta_n \text{ for all } n \in \mathbb{N}, \end{cases} \tag{1.3}$$

where $\epsilon_n \in [0, 1)$ is inertial parameter. They proved the weak convergence of above algorithm (1.3) assuming the conditions $\sum_{n=1}^{\infty} \epsilon_n \|\zeta_n - \zeta_{n-1}\|^2 < \infty$ and $\lambda_n < 2/L$, where L is Lipschitz constant of operator B. Later on, the following preconditioning algorithm was defined by Lorenz and Pock [10] for solving the monotone inclusion problem and proved weak convergence of it.

$$\begin{cases} \varphi_n = \zeta_n + \epsilon_n(\zeta_n - \zeta_{n-1}), \\ \zeta_{n+1} = (I + \lambda_n M^{-1}A)^{-1}(I - \lambda_n M^{-1}B)\varphi_n. \end{cases} \tag{1.4}$$

It is clear that the algorithm (1.4) reduces to forward-backward algorithm (1.2) if we take $\epsilon_n = 0$ and $M = I$. Dixit et al. [4], in 2021, introduced accelerated preconditioning forward-backward normal S-iteration and proved weak convergence under few assumptions in a real Hilbert space H.

$$\begin{cases} \varphi_n = \zeta_n + \epsilon_n(\zeta_n - \zeta_{n-1}), \\ \zeta_{n+1} = J_{\lambda,M}^{A,B}((1 - \gamma_n)\varphi_n + \gamma_n J_{\lambda,M}^{A,B}(\varphi_n)) \text{ for all } n \in \mathbb{N}, \end{cases} \tag{1.5}$$

where $J_{\lambda,M}^{A,B} = (I + \lambda M^{-1}A)^{-1}(I - \lambda M^{-1}B), \gamma_n \in (0, 1), \epsilon_n \in [0, 1)$ and $\lambda \in [0, 1)$. Next, Altiparmak and Karahan [21] proposed a new preconditioning forward-backward splitting algorithm and proved strong convergence of it.

$$\begin{cases} \varphi_n = \zeta_n + \epsilon_n(\zeta_n - \zeta_{n-1}), \\ \upsilon_n = J_{\lambda,M}^{A,B}((1 - \beta_n)\varphi_n + \beta_n J_{\lambda,M}^{A,B}(\varphi_n)), \\ \zeta_{n+1} = (1 - \gamma_n)J_{\lambda,M}^{A,B}(\upsilon_n) + \gamma_n f(\upsilon_n) \text{ for all } n \in \mathbb{N}, \end{cases} \tag{1.6}$$

where $\epsilon_n \subset [0, \theta]$ with $\theta \in [0, 1)$ and $\beta_n, \gamma_n \in (0, 1)$ and $f : H \to H$ is a k- contraction mapping with respect to M-norm. Recently, in 2021, the same authors [22] proved strong convergence of a modified preconditioning algorithm for solving the monotone inclusion problem in the following manner:

$$\begin{cases} \varphi_n = \zeta_n + \epsilon_n(\zeta_n - \zeta_{n-1}), \\ \upsilon_n = (1 - \alpha_n)\varphi_n + \alpha_n J_{\lambda,M}^{A,B}(\varphi_n), \\ \kappa_n = J_{\lambda,M}^{A,B}((1 - \beta_n)\upsilon_n + \beta_n J_{\lambda,M}^{A,B}(\upsilon_n)), \\ \zeta_{n+1} = (1 - \gamma_n)J_{\lambda,M}^{A,B}(\kappa_n) + \gamma_n f(\kappa_n) \text{ for all } n \in \mathbb{N}, \end{cases} \tag{1.7}$$

where $J_{\lambda,M}^{A,B} = (I + \lambda M^{-1}B)^{-1}(I - \lambda M^{-1}A), \alpha_n, \beta_n, \gamma_n \in (0,1), \epsilon_n \subset [0,\theta]$ with $\theta \in [0,1)$ and $h : H \to H$ is k-contraction with respect to M-norm.

On the other hand, the theory of fixed points has been an appealing topic of research. Several researchers have worked in this direction, see [23–28].

We have considered the Monotone inclusion problem and fixed point problem of a finite family of demimetric mappings in the context of Hilbert space in this paper as we defined the demimetric operator in the notion of inner product with respect to M-norm, where M is the linear, self-adjoint, positive and bounded operator. In the setting of a Hilbert space, that is, in the complete inner product space, the existence of a solution to the monotone inclusion problem is guaranteed under certain conditions. The authors defined contraction, nonexpansive, quasi-nonexpansive operators with respect to M-norm in these algorithms and proved weak and strong convergence of those algorithms under suitable assumptions. Our main contributions to this research are as follows:

- We introduce an operator namely, a demimetric operator with respect to M-norm;
- We define a new algorithm which is the combination of forward-backward and Tseng method for solving the monotone inclusion problem together with a new step for solving the fixed point problem of the finite family of demimetric operators;
- We also utilize the inertial extrapolation step strategy due to Polyak [29] to enhance the convergence rate of the proposed algorithm.

2. Preliminaries

In this section, we list some definitions and lemmas which contribute significantly to our main result. Throughout the study, suppose H is a real Hilbert space, Q is a nonempty, closed and convex subset of H, M is a linear, self-adjoint, positive and bounded operator on H and $Fix(U)$ denotes the set of all fixed points of the mapping U.

Definition 2.1. *[30]* Assume that $A : H \to 2^H$ is a set-valued operator. It is called monotone if

$$\langle \zeta - \varphi, \upsilon - \kappa \rangle \geq 0 \text{ for all } \zeta, \varphi \in H, \upsilon \in A\zeta \text{ and } \kappa \in A\varphi.$$

The operator A is called maximal monotone if the graph of the operator A is not properly contained in the graph of any other monotone operator.

Definition 2.2. *[31]* Assume that Q is nonempty subset of H and $\zeta \in H$. If for any $\upsilon \in H$, there exists a unique point $\varphi \in Q$ such that $\|\varphi - \zeta\| \leq \|\upsilon - \zeta\|$ for all $\upsilon \in H$ then φ is called metric projection of ζ onto Q. It is symbolized by $\varphi = P_Q\zeta$. If $P_Q\zeta$ exists and it can be uniquely obtained for all $\zeta \in H$, then $P_Q : H \to Q$ is called metric projection operator. The operator P_Q is nonexpansive and it satisfies the following inequality

$$\langle \zeta - P_Q\zeta, \psi - P_Q\zeta \rangle \leq 0 \text{ for all } \psi \in Q.$$

Definition 2.3. *[32]* Assume that M is a bounded and linear operator on H. The operator M is said to be self-adjoint if $M^* = M$ where M^* denotes the adjoint of M. If $\langle M(\zeta), \zeta \rangle > 0$ for all $\zeta (\neq 0) \in H$, then M is called positive definite operator. The M-inner product is given by $\langle \zeta, \psi \rangle_M = \langle \zeta, M(\psi) \rangle$ for all $\zeta, \psi \in H$ and the corresponding M-norm is also given by $\|\zeta\|_M^2 = \langle \zeta, M(\zeta) \rangle$ for all $\zeta \in H$ by using the self-adjoint, linear and bounded operator M.

Definition 2.4. *[4]* Assume that Q is a nonempty subset of H, M is a positive definite operator on H and $U : Q \to H$ is an operator. Then U is called

(i) M-cocoercive operator if

$$\|U\zeta - U\psi\|^2_{M^{-1}} \leq \langle \zeta - \psi, U\zeta - U\psi\rangle, \text{ for all } \zeta, \psi \in H,$$

(ii) nonexpansive operator with respect to M -norm if

$$\|U\zeta - U\psi\|_M \leq \|\zeta - \psi\|_M \text{ for all } \zeta, \psi \in H,$$

(iii) quasi-nonexpansive operator with respect to M -norm if

$$\|U\zeta - U\psi\|_M \leq \|\zeta - \psi\|_M \text{ for all } \zeta \in H, \psi \in Fix(U),$$

(iv) h-contraction with respect to M-norm if there exists $h \in [0, 1)$ such that

$$\|U\zeta - U\psi\|_M \leq h\|\zeta - \psi\|_M \text{ for all } \zeta, \psi \in H.$$

Lemma 2.1. *[30] For $\zeta, \psi, \omega, \iota \in H$ and $\alpha, \beta, \gamma \in [0, 1]$, where $\alpha + \beta + \gamma = 1$, we have*

(i) $\|\zeta + \psi\|^2 \leq \|\zeta\|^2 + 2\langle \psi, \zeta + \psi\rangle$,
(ii) $\|\beta\zeta + (1 - \beta)\psi\|^2 = \beta\|\zeta\|^2 + (1 - \beta)\|\psi\|^2 - \beta(1 - \beta)\|\zeta - \psi\|^2$,
(iii) $\|\alpha\zeta + \beta\psi + \gamma\omega\|^2 = \alpha\|\zeta\|^2 + \beta\|\psi\|^2 + \gamma\|\omega\|^2 - \alpha\beta\|\zeta - \psi\|^2 - \alpha\gamma\|\zeta - \omega\|^2 - \beta\gamma\|\psi - \omega\|^2$,
(iv) $2\langle \zeta - \psi, \omega - \iota\rangle = \|\zeta - \iota\|^2 + \|\psi - \omega\|^2 - \|\zeta - \omega\|^2 - \|\psi - \iota\|^2$,
(v) $\|\zeta \pm \psi\|^2 = \|\zeta\|^2 \pm 2\langle \zeta, \psi\rangle + \|\psi\|^2$.

Lemma 2.2. *[33] Let $U : Q \to H$ be the nonexpansive operator with $Fix(U) \neq \phi$. Then the mapping $I - U$ is demiclosed at origin, that is, for any sequence $\{\zeta_n\} \in H$ such that $\zeta_n \to \zeta \in H$ and $\|\zeta_n - U\zeta_n\| \to 0$ as $n \to \infty$, we have $\zeta \in Fix(U)$.*

Lemma 2.3. *[34] Assume that $S : Q \to H$ is ξ-demimetric operator with $\xi \in (-\infty, 1)$. Then, $Fix(S)$ is closed and convex.*

Lemma 2.4. *[4] Suppose that A is M-cocoercive operator on H, $B : H \to 2^H$ is maximal monotone operator. Then, $J^{A,B}_{\lambda,M} = (I+\lambda M^{-1}B)^{-1}(I-\lambda M^{-1}A)$ is nonexpansive with respect to M-norm for $\lambda \in (0, 1]$.*

Lemma 2.5. *[4] Suppose that A is M-cocoercive operator on H, $B : H \to 2^H$ is a maximal monotone operator and λ is a non-negative real number. Then, $\zeta \in H$ is a solution of monotone inclusion problem (1.1) if and only if $(I + \lambda M^{-1}B)^{-1}(I - \lambda M^{-1}A)(\zeta) = \zeta$.*

Lemma 2.6. *[35] Suppose that $\{t_n\} \subset [0, \infty)$ is a sequence of real numbers. Let*

$$s_{n+1} \leq (1 - p_n)s_n + p_n t_n \textit{ for all } n \in \mathbb{N},$$

where $\{p_n\} \subset [0, 1]$ and $\{t_n\} \subset (-\infty, \infty)$ satisfying the following assumptions:

(1) $\sum_{n=1}^{\infty} p_n = \infty$,
(2) $\limsup_{n\to\infty} t_n \leq 0$.

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.7. *[36] Assume that $\{\zeta_n\} \subseteq [0, \infty)$ such that there exists a subsequence $\{\zeta_{n_j}\}$ of $\{\zeta_n\}$ such that $\zeta_{n_j} < \zeta_{n_{j+1}}$. Then, there exists a nondecreasing sequence $\{n_k\}$ of natural numbers satisfying $\lim_{k\to\infty} n_k = \infty$, $\zeta_{n_k} \leq \zeta_{n_{k+1}}$ and $\zeta_k \leq \zeta_{n_{k+1}}$ for all $k \in \mathbb{N}$. Also, n_k is the greatest number n in the set $\{1, 2, ..., k\}$ satisfying $\zeta_n < \zeta_{n+1}$.*

3. Main result

In this part, we will prove the strong convergence of the following algorithm for finding a common solution of monotone inclusion and fixed point problem of a finite family of demimetric operators in the setting of a real Hilbert space H. Assume that A is M-cocoercive operator on H, $B : H \to 2^H$ is a maximally monotone operator. Suppose S_i is a finite family of ξ-demimetric operators with $\xi \in (-\infty, 1)$ such that $I - S_i$ is demiclosed at origin for all $i = 0, 1, 2, ...N-1$, $h : H \to H$ is a contraction mapping with respect to M-norm with constant $k \in (0, 1]$.

Algorithm 3.1. Let $\zeta_0, \zeta_1 \in H$. Compute $\{\varphi_n\}, \{\upsilon_n\}, \{\kappa_n\}$ and $\{\zeta_n\}$ using

$$\begin{cases} \varphi_n = \zeta_n + \epsilon_n(\zeta_n - \zeta_{n-1}), \\ \upsilon_n = (1-\alpha_n)\varphi_n + \alpha_n J^{A,B}_{\lambda,M}(\varphi_n), \\ \kappa_n = J^{A,B}_{\lambda,M}((1-\beta_n)\upsilon_n + \beta_n J^{A,B}_{\lambda,M}(\upsilon_n)), \\ \zeta_{n+1} = \gamma_n h(\zeta_n) + (1-\gamma_n-\delta_n)J^{A,B}_{\lambda,M}(\kappa_n) + \delta_n S_n \kappa_n, \end{cases} \tag{3.1}$$

where $S_n = \frac{1}{N}\sum_{i=0}^{N-1}(1-q_n)I + q_n S_i$

Definition 3.1. A mapping $U : Q \to H$, where Q is closed, convex and nonempty subset of H is called ξ-demimetric with respect to M-norm, where $\xi \in (-\infty, 1)$ if $Fix(U) \neq \phi$ such that

$$\langle \zeta - \zeta^*, (I-U)\zeta \rangle_M \geq \frac{1}{2}(1-\xi)\|(I-U)\zeta\|^2_M, \text{ for all } \zeta \in Q, \zeta^* \in Fix(U).$$

Example 3.1. Let $H = \mathbb{R}^3, Q = H$ and $M(\zeta) = (5\zeta_1, 4\zeta_2, 5\zeta_3)$ for all $\zeta = (\zeta_1, \zeta_2, \zeta_3) \in H$. The mapping $U : H \to H$ defined by $U(\zeta) = -2\zeta$ for all $\zeta = (\zeta_1, \zeta_2, \zeta_3)$ is $\frac{1}{3}$-demimetric mapping with respect to M-norm.

Now, we prove a result for ξ-demimetric operator with respect to M-norm which is motivated by Lemma (2.2) of [37].

Lemma 3.1. *Suppose $S : Q \to H$ is ξ-demimetric operator with respect to M-norm, where $\xi \in (-\infty, 1)$ and $Fix(S)$ is nonempty. Let $P = (1-\gamma)I + \gamma S$, where $\gamma \in (-\infty, \infty)$ with $\gamma \in (0, 1-\xi]$, then $P : Q \to H$ is a quasi-nonexpansive operator.*

Proof. It is direct that $Fix(S) = Fix(P)$. Now, since $S : Q \to H$ is demimetric operator, we have for any $\zeta \in Q$ and $y \in Fix(P)$,

$$\begin{aligned} \langle \zeta - y, \zeta - P\zeta \rangle_M &= \langle \zeta - y, \gamma(\zeta - S\zeta) \rangle_M \\ &= \gamma \langle \zeta - y, \zeta - S\zeta \rangle_M \\ &\geq \frac{\gamma}{2}(1-\xi)\|(I-S)\zeta\|^2_M \\ &= \frac{1-\xi}{2\gamma}\|\zeta - P\zeta\|^2_M \\ &\geq \frac{\gamma}{2\gamma}\|\zeta - P\zeta\|^2_M \\ &= \frac{1}{2}\|\zeta - P\zeta\|^2_M. \end{aligned}$$

This implies that P is 0-demimetric mapping. Also, for $\zeta \in Q$, $y \in Fix(P)$ and using Lemma (2.1) , we deduce

$$\begin{aligned} \frac{1}{2}\|\zeta - P\zeta\|_M^2 &\leq \langle \zeta - y, \zeta - P\zeta \rangle_M \\ \iff \|\zeta - P\zeta\|_M^2 &\leq 2\langle \zeta - y, \zeta - P\zeta \rangle_M \\ \iff \|\zeta - P\zeta\|_M^2 \leq \|\zeta - P\zeta\|_M^2 &+ \|\zeta - y\|_M^2 - \|P\zeta - y\|_M^2 \\ \iff \|P\zeta - y\|_M^2 &\leq \|\zeta - y\|_M^2 \\ \iff \|P\zeta - y\|_M &\leq \|\zeta - y\|_M. \end{aligned}$$

Hence, P is a quasi-nonexpansive operator.

Lemma 3.2. *The mapping S_n defined by $S_n = \frac{1}{N}\sum_{i=0}^{N-1}(1-q_n)I + q_nS_i$ is quasi-nonexpansive.*

Proof. Let $\zeta^* \in \Omega$.

Consider

$$\begin{aligned} \|S_n\zeta - \zeta^*\|_M &= \left\| \frac{1}{N}\sum_{i=0}^{N-1}((1-q_n)I + q_nS_i)\zeta - \zeta^* \right\|_M \\ &\leq \frac{1}{N}\sum_{i=0}^{N-1}\|((1-q_n)I + q_nS_i)\zeta - \zeta^*\|_M \\ &\leq \frac{1}{N}\sum_{i=0}^{N-1}\|\zeta - \zeta^*\|_M \\ &= \|\zeta - \zeta^*\|_M. \end{aligned}$$

Hence, S_n is quasi-nonexpansive.

Lemma 3.3. *Assume that $\{\zeta_n\}$ is bounded sequence of real numbers and $\zeta^* \in \Omega = (A+B)^{-1}(0) \cap \bigcap_{i=0}^{N-1} Fix(S_i)$. If $lim_{n\to\infty}\|J_{\lambda,M}^{A,B}\zeta_n - \zeta_n\|_M = 0$ and $lim_{n\to\infty}\|\kappa_n - S_n\kappa_n\|_M = 0$. Then $\limsup_{n\to\infty}\langle h(\zeta^*) - \zeta^*, \zeta_{n+1} - \zeta^*\rangle_M \leq 0$.*

Proof. Since the sequence $\{\zeta_n\}$ is bounded, so there exists a subsequence $\{\zeta_{n_k}\}$ of $\{\zeta_n\}$ such that $\zeta_{n_k} \rightharpoonup \zeta$ Consider

$$\begin{aligned} \limsup_{n\to\infty}\langle h(\zeta^*) - \zeta^*, \zeta_{n+1} - \zeta^*\rangle_M &= \limsup_{k\to\infty}\langle h(\zeta^*) - \zeta^*, \zeta_{n_{k+1}} - \zeta^*\rangle_M \\ &= \langle h(\zeta^*) - \zeta^*, \zeta - \zeta^*\rangle_M. \end{aligned} \tag{3.2}$$

Using the condition $\|J_{\lambda,M}^{A,B}\zeta_n - \zeta_n\|_M \to 0$ as $n \to \infty$ and by using Lemma (2.5), we deduce that $\zeta \in (A+B)^{-1}(0)$. Also, since $lim_{n\to\infty}\|\zeta_n - \kappa_n\| = 0$ and $\zeta_{n_k} \rightharpoonup \zeta$, so we obtain $\kappa_{n_k} \rightharpoonup \zeta$. By using the condition $lim_{n\to\infty}\|\kappa_n - S_n\kappa_n\| = 0$ and using Lemma (2.2), we obtain that $\zeta \in Fix(S_i)$. Thus $\zeta \in \Omega$.

From Eq (3.2) and by the property of metric projection, we have

$$\limsup_{n\to\infty}\langle h(\zeta^*)-\zeta^*,\zeta_{n+1}-\zeta^*\rangle_M \le \langle h(\zeta^*)-\zeta^*,\zeta-\zeta^*\rangle \\ \le 0.$$

Hence the lemma is proved.

Theorem 3.1. *Suppose that the solution set* $\Omega = (A+B)^{-1}(0) \cap \bigcap_{i=0}^{N-1} Fix(S_i)$ *is non-empty and the sequence* $\{\zeta_n\}$ *is generated by algorithm (3.1), where* $\{\epsilon_n\}\subset[0,\theta]$ *with* $\theta\in[0,1), \{\alpha_n\},\{\beta_n\},\{\gamma_n\},\{\delta_n\}\in(0,1)$ *such that the following conditions hold:*

(i) $0<a\le\alpha_n\le b<1$ *for some* $a,b\in\mathbb{R}$,
(ii) $0<c\le\beta_n\le d<1$ *for some* $c,d\in\mathbb{R}$,
(iii) $lim_{n\to\infty}\gamma_n=0, \sum_{n=1}^{\infty}\gamma_n=\infty$,
(iv) $\sum_{n=1}^{\infty}\epsilon_n\|\zeta_n-\zeta_{n-1}\|<\infty$,
(v) *for any* $n\in\mathbb{N}, 0<a^*<\liminf_{n\to\infty}\delta_n\le\limsup_{n\to\infty}\delta_n<b^*<1-\gamma_n$, *where* $a^*,b^*\in\mathbb{R}^+$.

Then the sequence $\{\zeta_n\}$ *converges strongly to a point* $\zeta^*\in\Omega=P_\Omega h(\zeta^*)$.

Proof. First we prove that the sequence $\{\zeta_n\}$ is bounded. Let $\zeta^*\in\Omega$.
Consider

$$\begin{aligned}\|\varphi_n-\zeta^*\|_M &= \|\zeta_n+\epsilon_n(\zeta_n-\zeta_{n-1})-\zeta^*\|_M\\ &\le \|\zeta_n-\zeta^*\|_M+\|\epsilon_n(\zeta_n-\zeta_{n-1})\|_M\\ &= \|\zeta_n-\zeta^*\|_M+\epsilon_n\|\zeta_n-\zeta_{n-1}\|_M.\end{aligned} \tag{3.3}$$

Since, $J_{\lambda,M}^{A,B}$ is nonexpansive with respect to M-norm, so using nonexpansiveness of $J_{\lambda,M}^{A,B}$, we have

$$\begin{aligned}\|\upsilon_n-\zeta^*\|_M &= \|(1-\alpha_n)\varphi_n+\alpha_n J_{\lambda,M}^{A,B}(\varphi_n)-\zeta^*\|_M\\ &\le (1-\alpha_n)\|\varphi_n-\zeta^*\|_M+\alpha_n\|J_{\lambda,M}^{A,B}(\varphi_n)-\zeta^*\|_M\\ &\le (1-\alpha_n)\|\varphi_n-\zeta^*\|_M+\alpha_n\|\varphi_n-\zeta^*\|_M\\ &= \|\varphi_n-\zeta^*\|_M,\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}\|\kappa_n-\zeta^*\|_M &= \|J_{\lambda,M}^{A,B}((1-\beta_n)\upsilon_n+\beta_n J_{\lambda,M}^{A,B}(\upsilon_n))-\zeta^*\|_M\\ &\le \|(1-\beta_n)\upsilon_n+\beta_n J_{\lambda,M}^{A,B}(\upsilon_n)-\zeta^*\|_M\\ &\le (1-\beta_n)\|\upsilon_n-\zeta^*\|_M+\beta_n\|J_{\lambda,M}^{A,B}(\upsilon_n)-\zeta^*\|_M\\ &\le (1-\beta_n)\|\upsilon_n-\zeta^*\|_M+\beta_n\|\upsilon_n-\zeta^*\|_M\\ &= \|\upsilon_n-\zeta^*\|_M.\end{aligned} \tag{3.5}$$

From Eqs (3.1)–(3.5) and using the fact that h is k-contraction and S_n is quasi-nonexpansive with respect to M-norm, we have

$$\|\zeta_{n+1}-\zeta^*\|_M = \|\gamma_n h(\zeta_n)+(1-\gamma_n-\delta_n)J_{\lambda,M}^{A,B}(\kappa_n)+\delta_n S_n\kappa_n-\zeta^*\|_M$$

$$
\begin{aligned}
&\le \gamma_n\|h(\zeta_n)-\zeta^*\|_M+(1-\gamma_n-\delta_n)\|J^{A,B}_{\lambda,M}(\kappa_n)-\zeta^*\|_M+\delta_n\|S_n\kappa_n-\zeta^*\|_M\\
&\le \gamma_n\|h(\zeta_n)-\zeta^*\|_M+(1-\gamma_n-\delta_n)\|\kappa_n-\zeta^*\|_M+\delta_n\|\kappa_n-\zeta^*\|_M\\
&= \gamma_n\|h(\zeta_n)-\zeta^*\|_M+(1-\gamma_n)\|\kappa_n-\zeta^*\|_M\\
&\le \gamma_n\|h(\zeta_n)-h(\zeta^*)\|_M+\gamma_n\|h(\zeta^*)-\zeta^*\|_M+(1-\gamma_n)\|\upsilon_n-\zeta^*\|_M\\
&\le \gamma_n\|h(\zeta_n)-h(\zeta^*)\|_M+\gamma_n\|h(\zeta^*)-\zeta^*\|_M+(1-\gamma_n)\|\varphi_n-\zeta^*\|_M\\
&\le \gamma_n\|h(\zeta_n)-h(\zeta^*)\|_M+\gamma_n\|h(\zeta^*)-\zeta^*\|_M+(1-\gamma_n)[\|\zeta_n-\zeta^*\|_M+\epsilon_n\|\zeta_n-\zeta_{n-1}\|_M]\\
&\le \gamma_n k\|\zeta_n-\zeta^*\|_M+\gamma_n\|h(\zeta^*)-\zeta^*\|_M+(1-\gamma_n)\|\zeta_n-\zeta^*\|_M+\epsilon_n(1-\gamma_n)\|\zeta_n-\zeta_{n-1}\|_M\\
&= (1-\gamma_n+\gamma_n k)\|\zeta_n-\zeta^*\|_M+\gamma_n\|h(\zeta^*)-\zeta^*\|_M+\epsilon_n(1-\gamma_n)\|\zeta_n-\zeta_{n-1}\|_M\\
&\le [1-\gamma_n(1-k)]\|\zeta_n-\zeta^*\|_M+\gamma_n\|h(\zeta^*)-\zeta^*\|_M+\epsilon_n\|\zeta_n-\zeta_{n-1}\|_M\\
&= [1-\gamma_n(1-k)]\|\zeta_n-\zeta^*\|_M+\gamma_n\|h(\zeta^*)-\zeta^*\|_M+\frac{\epsilon_n}{\gamma_n}\gamma_n\|\zeta_n-\zeta_{n-1}\|_M. \qquad (3.6)
\end{aligned}
$$

From assumptions (*iii*) and (*iv*), we have $\lim_{n\to\infty}\frac{\epsilon_n}{\gamma_n}\|\zeta_n-\zeta_{n-1}\|_M = 0$. So, there exists a positive integer $N_1>0$ such that $\frac{\epsilon_n}{\gamma_n}\|\zeta_n-\zeta_{n-1}\|\le N_1$. By using Eq (3.6), we obtain

$$
\begin{aligned}
\|\zeta_{n+1}-\zeta^*\|_M &\le [1-\gamma_n(1-k)]\|\zeta_n-\zeta^*\|_M+\gamma_n[N_1+\|h(\zeta^*)-\zeta^*\|_M]\\
&\le [1-\gamma_n(1-k)]\|\zeta_n-\zeta^*\|_M+\gamma_n(1-k)\left[\frac{N_1+\|h(\zeta^*)-\zeta^*\|_M}{1-k}\right]\\
&\le max.\left\{\|\zeta_n-\zeta^*\|_M,\frac{N_1+\|h(\zeta^*)-\zeta^*\|_M}{1-k}\right\}.
\end{aligned}
$$

Continuing like this,

$$
\|\zeta_{n+1}-\zeta^*\|_M \le max.\left\{\|\zeta_1-\zeta^*\|_M,\frac{N_1+\|h(\zeta^*)-\zeta^*\|_M}{1-k}\right\}.
$$

Thus the sequence $\{\zeta_n\}$ is bounded and therefore, the sequences $\{\varphi_n\},\{\upsilon_n\},\{\kappa_n\}$ are also bounded. Now, we show that $\zeta_n\to\zeta^*$. Using Lemma (2.1), we obtain

$$
\begin{aligned}
\|\varphi_n-\zeta^*\|_M^2 &= \|\zeta_n+\epsilon_n(\zeta_n-\zeta_{n-1})-\zeta^*\|_M^2\\
&\le \|\zeta_n-\zeta^*\|_M^2+2\epsilon_n\|\zeta_n-\zeta^*\|_M\|\zeta_n-\zeta_{n-1}\|_M+\epsilon_n^2\|\zeta_n-\zeta_{n-1}\|_M^2, \qquad (3.7)
\end{aligned}
$$

$$
\begin{aligned}
\|\upsilon_n-\zeta^*\|_M^2 &= \|(1-\alpha_n)\varphi_n+\alpha_n J^{A,B}_{\lambda,M}(\varphi_n)-\zeta^*\|_M^2\\
&= (1-\alpha_n)\|\varphi_n-\zeta^*\|_M^2+\alpha_n\|J^{A,B}_{\lambda,M}(\varphi_n)-\zeta^*\|_M^2-\alpha_n(1-\alpha_n)\|\varphi_n-J^{A,B}_{\lambda,M}(\varphi_n)\|_M^2\\
&\le (1-\alpha_n)\|\varphi_n-\zeta^*\|_M^2+\alpha_n\|\varphi_n-\zeta^*\|_M^2-\alpha_n(1-\alpha_n)\|\varphi_n-J^{A,B}_{\lambda,M}(\varphi_n)\|_M^2\\
&= \|\varphi_n-\zeta^*\|_M^2-(1-\alpha_n)\alpha_n\|\varphi_n-J^{A,B}_{\lambda,M}(\varphi_n)\|_M^2\\
&\le \|\varphi_n-\zeta^*\|_M^2, \qquad (3.8)
\end{aligned}
$$

and

$$\begin{aligned}
\|\kappa_n - \zeta^*\|_M^2 &= \|J_{\lambda,M}^{A,B}((1-\beta_n)\upsilon_n + \beta_n J_{\lambda,M}^{A,B}(\upsilon_n)) - \zeta^*\|_M^2 \\
&\leq \|(1-\beta_n)\upsilon_n + \beta_n J_{\lambda,M}^{A,B}(\upsilon_n) - \zeta^*\|_M^2 \\
&= \|(1-\beta_n)(\upsilon_n - \zeta^*) + \beta_n (J_{\lambda,M}^{A,B}(\upsilon_n) - \zeta^*)\|_M^2 \\
&= (1-\beta_n)\|\upsilon_n - \zeta^*\|_M^2 + \beta_n \|J_{\lambda,M}^{A,B}(\upsilon_n) - \zeta^*\|_M^2 - (1-\beta_n)\beta_n \|\upsilon_n - J_{\lambda,M}^{A,B}(\upsilon_n)\|_M^2 \\
&\leq (1-\beta_n)\|\upsilon_n - \zeta^*\|_M^2 + \beta_n \|\upsilon_n - \zeta^*\|^2 - (1-\beta_n)\beta_n \|\upsilon_n - J_{\lambda,M}^{A,B}(\upsilon_n)\|_M^2 \\
&\leq \|\upsilon_n - \zeta^*\|_M^2. \qquad (3.9)
\end{aligned}$$

Using Eqs (3.7)–(3.9), Lemma (2.1) and using the fact that $J_{\lambda,M}^{A,B}$ is nonexpansive and S_n is quasi-nonexpansive, we obtain

$$\begin{aligned}
\|\zeta_{n+1} - \zeta^*\|_M^2 &= \|\gamma_n h(\zeta_n) + (1-\gamma_n-\delta_n) J_{\lambda,M}^{A,B}(\kappa_n) + \delta_n S_n \kappa_n - \zeta^*\|_M^2 \\
&= \|\gamma_n (h(\zeta_n) - \zeta^*) + (1-\gamma_n-\delta_n)(J_{\lambda,M}^{A,B}(\kappa_n) - \zeta^*) \\
&+ \delta_n (S_n \kappa_n - \zeta^*)\|_M^2 \\
&= \|\gamma_n (h(\zeta_n) - h(\zeta^*)) + \gamma_n (h(\zeta^*) - \zeta^*) + (1-\gamma_n-\delta_n)(J_{\lambda,M}^{A,B}(\kappa_n) - \zeta^*) \\
&+ \delta_n (S_n \kappa_n - \zeta^*)\|_M^2 \\
&\leq \|\gamma_n (h(\zeta_n) - h(\zeta^*)) + (1-\gamma_n-\delta_n)(J_{\lambda,M}^{A,B}(\kappa_n) - \zeta^*) + \delta_n (S_n \kappa_n - \zeta^*)\|_M^2 \\
&+ 2\langle \gamma_n (h(\zeta^*) - \zeta^*), \zeta_{n+1} - \zeta^* \rangle_M \\
&= \|\gamma_n (h(\zeta_n) - h(\zeta^*)) + (1-\gamma_n-\delta_n)(J_{\lambda,M}^{A,B}(\kappa_n) - \zeta^*) + \delta_n (S_n \kappa_n - \zeta^*)\|_M^2 \\
&+ 2\gamma_n \langle h(\zeta^*) - \zeta^*, \zeta_{n+1} - \zeta^* \rangle_M \\
&\leq \gamma_n \|h(\zeta_n) - h(\zeta^*)\|_M^2 + (1-\gamma_n-\delta_n)\|J_{\lambda,M}^{A,B}(\kappa_n) - \zeta^*\|_M^2 + \delta_n \|S_n \kappa_n - \zeta^*\|_M^2 \\
&+ 2\gamma_n \langle h(\zeta^*) - \zeta^*, \zeta_{n+1} - \zeta^* \rangle_M \\
&\leq \gamma_n \|h(\zeta_n) - h(\zeta^*)\|_M^2 + (1-\gamma_n-\delta_n)\|\kappa_n - \zeta^*\|_M^2 + \delta_n \|\kappa_n - \zeta^*\|_M^2 \\
&+ 2\gamma_n \langle h(\zeta^*) - \zeta^*, \zeta_{n+1} - \zeta^* \rangle_M \\
&= (1-\gamma_n)\|\kappa_n - \zeta^*\|_M^2 + \gamma_n \|h(\zeta_n) - h(\zeta^*)\|_M^2 + 2\gamma_n \langle h(\zeta^*) - \zeta^*, \zeta_{n+1} - \zeta^* \rangle_M \qquad (3.10) \\
&\leq (1-\gamma_n)[\|\zeta_n - \zeta^*\|_M^2 + 2\epsilon_n \|\zeta_n - \zeta^*\|_M \|\zeta_n - \zeta_{n-1}\|_M + \epsilon_n^2 \|\zeta_n - \zeta_{n-1}\|_M^2] \\
&+ \gamma_n \|h(\zeta_n) - h(\zeta^*)\|_M^2 + 2\gamma_n \langle h(\zeta^*) - \zeta^*, \zeta_{n+1} - \zeta^* \rangle_M \\
&\leq (1-\gamma_n)\|\zeta_n - \zeta^*\|_M^2 + 2\epsilon_n (1-\gamma_n)\|\zeta_n - \zeta^*\|_M \|\zeta_n - \zeta_{n-1}\|_M + \epsilon_n^2 (1-\gamma_n)\|\zeta_n - \zeta_{n-1}\|_M^2 \\
&+ \gamma_n k \|\zeta_n - \zeta^*\|_M^2 + 2\gamma_n \langle h(\zeta^*) - \zeta^*, \zeta_{n+1} - \zeta^* \rangle_M \\
&\leq [1 - \gamma_n (1-k)]\|\zeta_n - \zeta^*\|_M^2 + 2\epsilon_n \|\zeta_n - \zeta^*\|_M \|\zeta_n - \zeta_{n-1}\|_M + \epsilon_n^2 \|\zeta_n - \zeta_{n-1}\|_M^2 \\
&+ 2\gamma_n \langle h(\zeta^*) - \zeta^*, \zeta_{n+1} - \zeta^* \rangle_M,
\end{aligned}$$

which implies that

$$\begin{aligned}
\|\zeta_{n+1} - \zeta^*\|_M^2 \leq [1 - \gamma_n (1-k)]\|\zeta_n - \zeta^*\|_M^2 + \gamma_n (1-k)[\frac{2}{1-k}\langle h(\zeta^*) - \zeta^*, \zeta_{n+1} - \zeta^* \rangle \\
+ \frac{2\epsilon_n}{\gamma_n (1-k)}\|\zeta_n - \zeta^*\|_M \|\zeta_n - \zeta_{n-1}\|_M + \frac{\epsilon_n^2}{\gamma_n (1-k)}\|\zeta_n - \zeta_{n-1}\|_M^2]. \qquad (3.11)
\end{aligned}$$

Equation (3.11) is equivalent to

$$s_{n+1} \leq (1 - p_n)s_n + p_n t_n, \tag{3.12}$$

where, $s_n = \|\zeta_n - \zeta^*\|_M^2$, $p_n = \gamma_n(1-k)$ and $t_n = \frac{2}{1-k}\langle h(\zeta^*) - \zeta^*, \zeta_{n+1} - \zeta^*\rangle + \frac{2\epsilon_n}{\gamma_n(1-k)}\|\zeta_n - \zeta^*\|_M\|\zeta_n - \zeta_{n-1}\|_M + \frac{\epsilon_n^2}{\gamma_n(1-k)}\|\zeta_n - \zeta_{n-1}\|_M^2$

Now, we consider two possible cases on the sequence $\{\|\zeta_n - \zeta^*\|_M\}$.

Case 1. Suppose that there exists some positive integer n_0 such that the sequence $\{\|\zeta_n - \zeta^*\|_M\}$ is nonincreasing sequence for any $n \geq n_0$. Also, the sequence $\{\|\zeta_n - \zeta^*\|_M\}$ is bounded below by zero. So, it is convergent.

By using Lemma (2.1), Eqs (3.7) and (3.9), we have

$$\begin{aligned}
\|\zeta_{n+1} - \zeta^*\|_M^2 &= \|\gamma_n(h(\zeta_n) - \zeta^*) + (1 - \gamma_n - \delta_n)(J_{\lambda,M}^{A,B}(\kappa_n) - \zeta^*) + \delta_n(S_n\kappa_n - \zeta^*)\|_M^2 \\
&\leq \gamma_n\|h(\zeta_n) - \zeta^*\|_M^2 + (1 - \gamma_n - \delta_n)\|J_{\lambda,M}^{A,B}(\kappa_n) - \zeta^*\|_M^2 + \delta_n\|S_n\kappa_n - \zeta^*\|_M^2 \\
&\leq \gamma_n\|h(\zeta_n) - \zeta^*\|_M^2 + (1 - \gamma_n - \delta_n)\|\kappa_n - \zeta^*\|_M^2 + \delta_n\|\kappa_n - \zeta^*\|_M^2 \\
&= \gamma_n\|h(\zeta_n) - \zeta^*\|_M^2 + (1 - \gamma_n)\|\kappa_n - \zeta^*\|_M^2 \\
&\leq \gamma_n\|h(\zeta_n) - \zeta^*\|_M^2 + (1 - \gamma_n)\|\upsilon_n - \zeta^*\|_M^2,
\end{aligned} \tag{3.13}$$

and

$$\begin{aligned}
\|\upsilon_n - \zeta^*\|_M^2 &= \|(1 - \alpha_n)(\varphi_n - \zeta^*) + \alpha_n(J_{\lambda,M}^{A,B}(\varphi_n) - \zeta^*)\|_M^2 \\
&= (1 - \alpha_n)\|\varphi_n - \zeta^*\|_M^2 + \alpha_n\|J_{\lambda,M}^{A,B}(\varphi_n) - \zeta^*\|_M^2 - \alpha_n(1 - \alpha_n)\|J_{\lambda,M}^{A,B}(\varphi_n) - \varphi_n\|_M^2 \\
&= \|\varphi_n - \zeta^*\|_M^2 - \alpha_n(1 - \alpha_n)\|J_{\lambda,M}^{A,B}(\varphi_n) - \varphi_n\|_M^2 \\
&\leq \|\zeta_n - \zeta^*\|_M^2 + 2\epsilon_n\|\zeta_n - \zeta^*\|_M\|\zeta_n - \zeta_{n-1}\|_M + \epsilon_n^2\|\zeta_n - \zeta_{n-1}\|_M^2 \\
&\quad - \alpha_n(1 - \alpha_n)\|J_{\lambda,M}^{A,B}(\varphi_n) - \varphi_n\|_M^2.
\end{aligned} \tag{3.14}$$

From Eqs (3.13) and (3.14), we obtain

$$\begin{aligned}
\|\zeta_{n+1} - \zeta^*\|_M^2 \leq \gamma_n\|h(\zeta_n) - \zeta^*\|_M^2 &+ (1 - \gamma_n)[\|\zeta_n - \zeta^*\|_M^2 + 2\epsilon_n\|\zeta_n - \zeta^*\|_M\|\zeta_n - \zeta_{n-1}\|_M \\
&+ \epsilon_n^2\|\zeta_n - \zeta_{n-1}\|_M^2 - \alpha_n(1 - \alpha_n)\|J_{\lambda,M}^{A,B}(\varphi_n) - \varphi_n\|_M^2],
\end{aligned}$$

which implies that

$$\begin{aligned}
\alpha_n(1 - \alpha_n)\|J_{\lambda,M}^{A,B}(\varphi_n) - \varphi_n\|_M^2 &\leq \gamma_n\|h(\zeta_n) - \zeta^*\|_M^2 + (1 - \gamma_n)\|\zeta_n - \zeta^*\|_M^2 + 2\epsilon_n\|\zeta_n - \zeta^*\|_M\|\zeta_n - \zeta_{n-1}\|_M \\
&\quad + \epsilon_n^2\|\zeta_n - \zeta_{n-1}\|_M^2 - \|\zeta_{n+1} - \zeta^*\|_M^2 \\
&\leq \gamma_n\|h(\zeta_n) - \zeta^*\|_M^2 + \|\zeta_n - \zeta^*\|_M^2 + 2\epsilon_n\|\zeta_n - \zeta^*\|_M\|\zeta_n - \zeta_{n-1}\|_M \\
&\quad + \epsilon_n^2\|\zeta_n - \zeta_{n-1}\|_M^2 - \|\zeta_{n+1} - \zeta^*\|_M^2.
\end{aligned} \tag{3.15}$$

Since, $\sum_{n=1}^{\infty} \epsilon_n\|\zeta_n - \zeta_{n-1}\| < \infty$, so we have

$$\lim_{n\to\infty} \epsilon_n\|\zeta_n - \zeta_{n-1}\|_M = 0. \tag{3.16}$$

Using the condition $\lim_{n\to\infty}\gamma_n = 0$, Eq (3.16) and taking limit $n \to \infty$ in Eq (3.15), we get

$$\lim_{n\to\infty} \|J^{A,B}_{\lambda,M}(\varphi_n) - \varphi_n\|^2_M = 0. \tag{3.17}$$

Now, consider

$$\begin{aligned}\|\upsilon_n - \varphi_n\|_M &= \|(1-\alpha_n)\varphi_n + \alpha_n J^{A,B}_{\lambda,M}(\varphi_n) - \varphi_n\|_M \\ &= \|\alpha_n(J^{A,B}_{\lambda,M}(\varphi_n) - \varphi_n)\|_M \\ &= \alpha_n\|J^{A,B}_{\lambda,M}(\varphi_n) - \varphi_n\|_M.\end{aligned} \tag{3.18}$$

Taking limit $n \to \infty$ in Eq (3.18) and using Eq (3.17), we obtain

$$\lim_{n\to\infty} \|\upsilon_n - \varphi_n\|_M = 0. \tag{3.19}$$

By using triangle inequality, we have

$$\|J^{A,B}_{\lambda,M}(\varphi_n) - \upsilon_n\|_M \le \|J^{A,B}_{\lambda,M}(\varphi_n) - \varphi_n\|_M + \|\varphi_n - \upsilon_n\|_M. \tag{3.20}$$

Taking limit $n \to \infty$ in Eq (3.20) and using Eqs (3.17) and (3.19), we deduce

$$\lim_{n\to\infty} \|J^{A,B}_{\lambda,M}(\varphi_n) - \upsilon_n\|_M = 0. \tag{3.21}$$

Also,

$$\begin{aligned}\|J^{A,B}_{\lambda,M}(\upsilon_n) - \upsilon_n\|_M &= \|J^{A,B}_{\lambda,M}(\upsilon_n) - J^{A,B}_{\lambda,M}(\varphi_n) + J^{A,B}_{\lambda,M}(\varphi_n) - \upsilon_n\|_M \\ &\le \|J^{A,B}_{\lambda,M}(\upsilon_n) - J^{A,B}_{\lambda,M}(\varphi_n)\|_M + \|J^{A,B}_{\lambda,M}(\varphi_n) - \upsilon_n\|_M \\ &\le \|\upsilon_n - \varphi_n\|_M + \|J^{A,B}_{\lambda,M}(\varphi_n) - \upsilon_n\|_M,\end{aligned} \tag{3.22}$$

Taking limit $n \to \infty$ in Eq (3.22) and using Eqs (3.19) and (3.21) in Eq (3.22), we have

$$\lim_{n\to\infty} \|J^{A,B}_{\lambda,M}(\upsilon_n) - \upsilon_n\|_M = 0, \tag{3.23}$$

and

$$\begin{aligned}\|\kappa_n - \varphi_n\|_M &= \|\kappa_n - J^{A,B}_{\lambda,M}(\varphi_n) + J^{A,B}_{\lambda,M}(\varphi_n) - \varphi_n\|_M \\ &\le \|\kappa_n - J^{A,B}_{\lambda,M}(\varphi_n)\|_M + \|J^{A,B}_{\lambda,M}(\varphi_n) - \varphi_n\|_M \\ &= \|J^{A,B}_{\lambda,M}((1-\beta_n)\upsilon_n + \beta_n J^{A,B}_{\lambda,M}(\upsilon_n)) - J^{A,B}_{\lambda,M}(\varphi_n)\|_M + \|J^{A,B}_{\lambda,M}(\varphi_n) - \varphi_n\|_M \\ &\le \|(1-\beta_n)\upsilon_n + \beta_n J^{A,B}_{\lambda,M}(\upsilon_n) - \varphi_n\|_M + \|J^{A,B}_{\lambda,M}(\varphi_n) - \varphi_n\|_M \\ &= \|(\upsilon_n - \varphi_n) + \beta_n(J^{A,B}_{\lambda,M}(\upsilon_n) - \upsilon_n)\|_M + \|J^{A,B}_{\lambda,M}(\varphi_n) - \varphi_n\|_M \\ &\le \|\upsilon_n - \varphi_n\|_M + \beta_n\|J^{A,B}_{\lambda,M}(\upsilon_n) - \upsilon_n\|_M + \|J^{A,B}_{\lambda,M}(\varphi_n) - \varphi_n\|_M.\end{aligned} \tag{3.24}$$

Taking limit $n \to \infty$ in Eq (3.24) and using Eqs (3.17), (3.19) and (3.23), we obtain

$$\lim_{n\to\infty} \|\kappa_n - \varphi_n\|_M = 0. \tag{3.25}$$

Again using Eq (3.1) and Lemma (2.1), we deduce that

$$\begin{aligned}\|\zeta_{n+1}-\zeta^*\|_M^2 &= \|\gamma_n(h(\zeta_n)-\zeta^*)+(1-\gamma_n-\delta_n)(J_{\lambda,M}^{A,B}(\kappa_n)-\zeta^*)+\delta_n(S_n\kappa_n-\zeta^*)\|_M^2\\ &\leq \gamma_n\|h(\zeta_n)-\zeta^*\|_M^2+(1-\gamma_n-\delta_n)\|J_{\lambda,M}^{A,B}(\kappa_n)-\zeta^*\|_M^2+\delta_n\|S_n\kappa_n-\zeta^*\|_M^2\\ &-(1-\gamma_n-\delta_n)\delta_n\|J_{\lambda,M}^{A,B}(\kappa_n)-S_n\kappa_n\|_M^2.\end{aligned}$$

Using nonexpansiveness of $J_{\lambda,M}^{A,B}$, quasi-nonexpansiveness of S_n, Eqs (3.7)–(3.9), we have

$$\begin{aligned}(1-\gamma_n-\delta_n)\delta_n\|J_{\lambda,M}^{A,B}(\kappa_n)-S_n\kappa_n\|_M^2 &\leq \gamma_n\|h(\zeta_n)-\zeta^*\|_M^2+\|\zeta_n-\zeta^*\|_M^2+2\epsilon_n\|\zeta_n-\zeta^*\|_M\|\zeta_n-\zeta_{n-1}\|_M\\ &+\epsilon_n^2\|\zeta_n-\zeta_{n-1}\|_M^2-\|\zeta_{n+1}-\zeta^*\|_M^2. \end{aligned}\tag{3.26}$$

Taking limit $n\to\infty$ in Eq (3.26) and using Eq (3.16), condition $\lim_{n\to\infty}\gamma_n=0$, we obtain

$$\lim_{n\to\infty}\|J_{\lambda,M}^{A,B}(\kappa_n)-S_n\kappa_n\|_M=0. \tag{3.27}$$

By using triangle inequality, we have

$$\begin{aligned}\|J_{\lambda,M}^{A,B}(\kappa_n)-\kappa_n\|_M &= \|J_{\lambda,M}^{A,B}(\kappa_n)-J_{\lambda,M}^{A,B}(\varphi_n)+J_{\lambda,M}^{A,B}(\varphi_n)-\varphi_n+\varphi_n-\kappa_n\|_M\\ &\leq \|J_{\lambda,M}^{A,B}(\kappa_n)-J_{\lambda,M}^{A,B}(\varphi_n)\|_M+\|J_{\lambda,M}^{A,B}(\varphi_n)-\varphi_n\|+\|\varphi_n-\kappa_n\|_M\\ &\leq \|\kappa_n-\varphi_n\|_M+\|J_{\lambda,M}^{A,B}(\varphi_n)-\varphi_n\|_M+\|\varphi_n-\kappa_n\|_M.\end{aligned}\tag{3.28}$$

Taking limit $n\to\infty$ in Eq (3.28) and using Eqs (3.17) and (3.25), we get

$$\lim_{n\to\infty}\|J_{\lambda,M}^{A,B}(\kappa_n)-\kappa_n\|_M=0. \tag{3.29}$$

Also,

$$\begin{aligned}\|S_n\kappa_n-\kappa_n\|_M &= \|S_n\kappa_n-J_{\lambda,M}^{A,B}(\kappa_n)+J_{\lambda,M}^{A,B}(\kappa_n)-\kappa_n\|_M\\ &\leq \|S_n\kappa_n-J_{\lambda,M}^{A,B}(\kappa_n)\|_M+\|J_{\lambda,M}^{A,B}(\kappa_n)-\kappa_n\|_M.\end{aligned}\tag{3.30}$$

Taking limit $n\to\infty$ in Eq (3.30) and using Eqs (3.27) and (3.29), we obtain

$$\lim_{n\to\infty}\|S_n\kappa_n-\kappa_n\|_M=0. \tag{3.31}$$

From Eqs (3.19) and (3.25) and using triangle inequality, we obtain

$$\lim_{n\to\infty}\|\kappa_n-\upsilon_n\|_M=0. \tag{3.32}$$

From Eq (3.1),

$$\begin{aligned}\|\varphi_n-\zeta_n\|_M &= \|\zeta_n+\epsilon_n(\zeta_n-\zeta_{n-1})-\zeta_n\|_M\\ &= \|\epsilon_n(\zeta_n-\zeta_{n-1})\|_M.\end{aligned}\tag{3.33}$$

Taking limit $n\to\infty$ in Eq (3.33) and using Eq (3.16), we get

$$\lim_{n\to\infty}\|\varphi_n-\zeta_n\|_M=0. \tag{3.34}$$

Similarly, we can prove

$$\lim_{n\to\infty} \|\kappa_n - \zeta_n\|_M = 0. \tag{3.35}$$

Now, from Eqs (3.31) and (3.35), we can prove

$$\begin{aligned}\|S_n\kappa_n - \zeta_n\|_M &= \|S_n\kappa_n - \kappa_n + \kappa_n - \zeta_n\|_M \\ &\le \|S_n\kappa_n - \kappa_n\|_M + \|\kappa_n - \zeta_n\|_M,\end{aligned}$$

which implies

$$\lim_{n\to\infty} \|S_n\kappa_n - \zeta_n\|_M = 0. \tag{3.36}$$

Again consider

$$\begin{aligned}\|\zeta_{n+1} - \zeta_n\|_M &= \|\zeta_{n+1} - S_n\kappa_n + S_n\kappa_n - \zeta_n\|_M \\ &\le \|\zeta_{n+1} - S_n\kappa_n\|_M + \|S_n\kappa_n - \zeta_n\|_M \\ &= \|\gamma_n h(\zeta_n) + (1-\gamma_n-\delta_n)J^{A,B}_{\lambda,M}(\kappa_n) + \delta_n S_n\kappa_n - S_n\kappa_n\|_M + \|S_n\kappa_n - \zeta_n\|_M \\ &= \|(J^{A,B}_{\lambda,M}(\kappa_n) - S_n\kappa_n) + \gamma_n(h(\zeta_n) - J^{A,B}_{\lambda,M}(\kappa_n)) + \delta_n(S_n\kappa_n - J^{A,B}_{\lambda,M}(\kappa_n))\|_M \\ &+ \|S_n\kappa_n - \zeta_n\|_M.\end{aligned} \tag{3.37}$$

Taking limit $n \to \infty$ in Eq (3.37) and using the condition $\lim_{n\to\infty}\gamma_n = 0$, Eqs (3.27) and (3.36), we obtain

$$\lim_{n\to\infty} \|\zeta_{n+1} - \zeta_n\|_M = 0. \tag{3.38}$$

Also, using Eqs (3.34) and (3.38), we have

$$\lim_{n\to\infty} \|\zeta_{n+1} - \varphi_n\|_M = 0. \tag{3.39}$$

From Eqs (3.17) and (3.34), we can prove

$$\begin{aligned}\|J^{A,B}_{\lambda,M}(\zeta_n) - \zeta_n\|_M &= \|J^{A,B}_{\lambda,M}(\zeta_n) - J^{A,B}_{\lambda,M}(\varphi_n) - (\zeta_n - \varphi_n) + (J^{A,B}_{\lambda,M}(\varphi_n) - \varphi_n)\|_M \\ &\le 2\|\zeta_n - \varphi_n\|_M + \|J^{A,B}_{\lambda,M}(\varphi_n) - \varphi_n\|_M.\end{aligned}$$

Taking limit $n \to \infty$ in above equation, we have

$$\lim_{n\to\infty} \|J^{A,B}_{\lambda,M}(\zeta_n) - \zeta_n\|_M = 0. \tag{3.40}$$

Since, the sequence $\{\zeta_n\}$ is a bounded sequence of real numbers, so using Lemma (3.3), we obtain

$$\limsup_{n\to\infty} \langle h(\zeta^*) - \zeta^*, \zeta_{n+1} - \zeta^*\rangle_M \le 0. \tag{3.41}$$

Thus, by making use of Eq (3.16),we have $\limsup_{n\to\infty} t_n \le 0$. Hence from Lemma (2.6), $\zeta_n \to \zeta^*$ as $n \to \infty$.

Case 2. There exists a subsequence $\{\|\zeta_{n_j} - \zeta^*\|_M^2\}$ of $\{\|\zeta_n - \zeta^*\|_M^2\}$ such that $\|\zeta_{n_j} - \zeta^*\|_M^2 \leq \|\zeta_{n_{j+1}} - \zeta^*\|_M^2$ for any $j \in N$. By using Lemma (2.7), we see that there exists a nondecreasing sequence $\{\zeta_{n_k}\}$ of $\mathbb{N}$ such that $\lim_{k\to\infty} n_k = \infty$ and the following inequalities hold for all $k \in \mathbb{N}$.

$$\|\zeta_{n_k} - \zeta^*\|_M^2 \leq \|\zeta_{n_{k+1}} - \zeta^*\|_M^2, \tag{3.42}$$

and

$$\|\zeta_k - \zeta^*\|_M^2 \leq \|\zeta_{n_{k+1}} - \zeta^*\|_M^2. \tag{3.43}$$

Similar to Eq (3.15), we have

$$\begin{aligned}\alpha_{n_k}(1-\alpha_{n_k})\|J_{\lambda,M}^{A,B}(\varphi_{n_k}) - \varphi_{n_k}\|_M^2 \leq \gamma_{n_k}\|h(\zeta_{n_k}) - \zeta^*\|_M^2 + \|\zeta_{n_k} - \zeta^*\|_M^2 + 2\epsilon_{n_k}\|\zeta_{n_k} - \zeta^*\|_M\|\zeta_{n_k} - \zeta_{n_{k-1}}\|_M \\ + \epsilon_{n_k}^2\|\zeta_{n_k} - \zeta_{n_{k-1}}\|_M^2 - \|\zeta_{n_{k+1}} - \zeta^*\|_M^2.\end{aligned} \tag{3.44}$$

Using Eqs (3.16), (3.42) and condition $\lim_{k\to\infty} \gamma_{n_k} = 0$ in Eq (3.44), we obtain

$$\lim_{k\to\infty} \|J_{\lambda,M}^{A,B}(\varphi_{n_k}) - \varphi_{n_k}\|_M = 0. \tag{3.45}$$

Similarly as in Case 1, we have

$$\begin{aligned}\lim_{k\to\infty} \|\upsilon_{n_k} - \varphi_{n_k}\|_M &= 0, \\ \lim_{k\to\infty} \|\varphi_{n_k} - \kappa_{n_k}\|_M &= 0, \\ \lim_{k\to\infty} \|\zeta_{n_{k+1}} - \varphi_{n_k}\|_M &= 0, \\ \lim_{k\to\infty} \|\zeta_{n_{k+1}} - \zeta_{n_k}\|_M &= 0, \\ \lim_{k\to\infty} \|J_{\lambda,M}^{A,B}(\zeta_{n_k}) - \zeta_{n_k}\|_M &= 0.\end{aligned}$$

Using Eq (3.12), we have

$$s_{n_{k+1}} \leq (1 - p_{n_k})s_{n_k} + p_{n_k}t_{n_k}. \tag{3.46}$$

Using Eq (3.42) in Eq (3.46), we obtain

$$\xi_{n_k}s_{n_k} \leq \xi_{n_k}t_{n_k}. \tag{3.47}$$

As $p_{n_k} > 0$, so Eq (3.47) implies $s_{n_k} \leq t_{n_k}$, that is, we have

$$\begin{aligned}\|\zeta_{n_{k+1}} - \zeta^*\|_M^2 \leq \frac{2}{1-k}\langle h(\zeta^*) - \zeta^*, \zeta_{n_{k+1}} - \zeta^*\rangle + \frac{2\epsilon_{n_k}}{\gamma_{n_k}(1-k)}\|\zeta_{n_k} - \zeta^*\|_M\|\zeta_{n_k} - \zeta_{n_{k-1}}\|_M \\ + \frac{\epsilon_{n_k}^2}{\gamma_{n_k}(1-k)}\|\zeta_{n_k} - \zeta_{n_{k-1}}\|_M^2.\end{aligned} \tag{3.48}$$

Similar to Case 1, we obtain $\limsup_{n\to\infty} t_{n_k} \leq 0$. So, we have by using Lemma (2.6),

$$\lim_{k\to\infty} \|\zeta_{n_{k+1}} - \zeta^*\|_M^2 = 0. \tag{3.49}$$

From Eq (3.43), we get $\zeta_k \to \zeta^*$ as $k \to \infty$. This completes the proof.

4. Numerical example

In this section, we provide a numerical example to show the numerical efficiency of the proposed algorithm. We performed all numerical experiments on a Dell computer equipped with a 3.20 GHz Intel (R) Core (TM) i5-3470 CPU and 8 GB of memory. The MATLAB R 2022 a platform was used as the implementation environment.

Let $H = \mathbb{R}^3$. Define the operators $h, A, B, M, S_i : H \to H$ as $h(\zeta) = \frac{\zeta}{2}, A(\zeta) = (5\zeta_1, 4\zeta_2, 5\zeta_3), B = (4\zeta_1, 4\zeta_2, 4\zeta_3), M = A, S_i\zeta = -3\zeta$ for $i = 0, 1$. for all $\zeta = (\zeta_1, \zeta_2, \zeta_3) \in H$. Clearly, M is self-adjoint, bounded and positive operator, A is M-cocoercive operator, B is maximally monotone operator, h is $\frac{1}{2}$-contraction operator with respect to M-norm and S_i is a finite family of $\frac{1}{2}$-demimetric operators. To find the numerical values of ζ_n, we choose $\alpha_n = \frac{1}{2n}, \lambda = 0.5, q_n = 0.1, \beta_n = \frac{1}{2n}, \gamma_n = \frac{1}{n}, \delta_n = \frac{n}{2(n+1)}, \epsilon_n = 0.8$. We compare our algorithm (3.1) with algorithms (1.4) and (1.5) with different choices of ζ_0 and ζ_1. We consider three cases:

Case 1. $\zeta_0 = [20, 20, 20]$ and $\zeta_1 = [10, 10, 10]$.

Case 2. $\zeta_0 = [10, 10, 10]$ and $\zeta_1 = [5, 5, 5]$.

Case 3. $\zeta_0 = [-1, -1, -1]$ and $\zeta_1 = [-2, -2, -2]$.

The values of iteration number (n) for various choices of ζ_0 and ζ_1 are given in the Tables 1–3. Comparison of Algorithm (3.1) by taking $E_n = \|\zeta_n - \zeta_{n-1}\| < 10^{-4}$ with the algorithms Dixit et al. (1.4) and Lorenz and Pock (1.5) for Cases 1–3 are shown in Figures 1–3 respectively.

Table 1. Iteration number (with cpu time) for Algorithms (3.1), (1.4) and (1.5) for case 1.

Case 1	Iteration number(n)	cpu time (in seconds)
Algorithm (3.1)	11	0.015498
Dixit et al. (1.4)	75	0.699281
Lorenz and Pock (1.5)	15	0.019471

Table 2. Iteration number (with cpu time) for Algorithms (3.1), (1.4) and (1.5) for case 2.

Case 2	Iteration number(n)	cpu time (in seconds)
Algorithm (3.1)	10	0.001373
Dixit et al. (1.4)	75	0.003053
Lorenz and Pock (1.5)	14	0.001431

Table 3. Iteration number (with cpu time) for Algorithms (3.1), (1.4) and (1.5) for case 3 .

Case 3	Iteration number(n)	cpu time (in seconds)
Algorithm(3.1)	9	0.001283
Dixit et al. (1.4)	75	0.002931
Lorenz and Pock (1.5)	13	0.001368

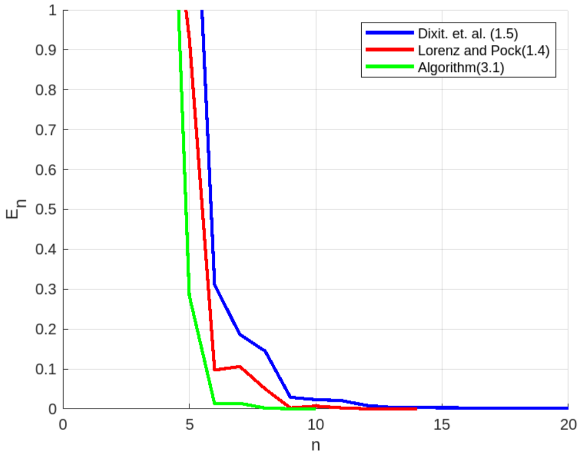

Figure 1. Comparison of Algorithm (3.1) for $\zeta_0 = [20, 20, 20]$ and $\zeta_1 = [10, 10, 10]$.

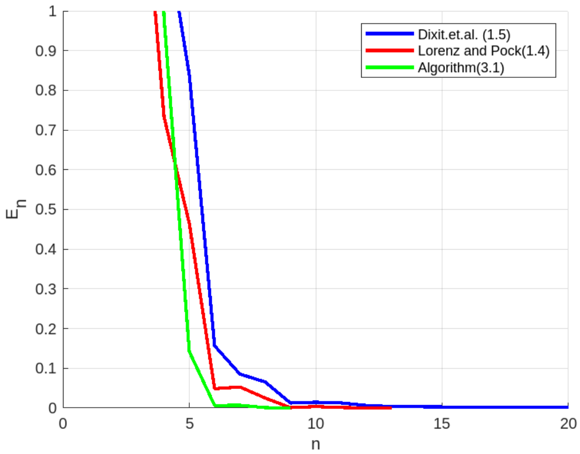

Figure 2. Comparison of Algorithm (3.1) for $\zeta_0 = [10, 10, 10]$ and $\zeta_1 = [5, 5, 5]$.

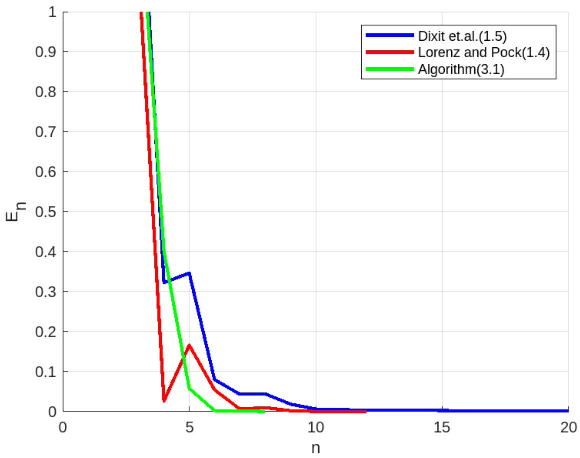

Figure 3. Comparison of Algorithm (3.1) for $\zeta_0 = [-1, -1, -1]$ and $\zeta_1 = [-2, -2, -2]$.

5. Conclusions

We introduced an algorithm for finding the common solution of monotone inclusion and fixed point problem of a finite family of demimetric mappings in a real Hilbert space and showed that our algorithm has strong convergence under some conditions. Moreover, we proved the algorithm has a better rate of convergence by giving a numerical example.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The author Anjali delightedly acknowledge the University Grants Commission (UGC), New Delhi and the authors S. Haque and N. Mlaiki would like to thank the Prince Sultan University for paying the publication fees for this work through TAS LAB.

Conflict of interest

The authors declare no conflicts of interest.

References

1. A. B. Abubakar, K. Muangchoo, A. H. Ibrahim, A. B. Muhammad, L. O. Jolaoso, K. O. Aremu, A new three-term Hestenes-Stiefel type method for nonlinear monotone operator equations and image restoration, *IEEE Access*, **9** (2021), 18262–18277. https://doi.org/10.1109/ACCESS.2021.3053141
2. P. L. Combettes, V. R. Wajs, Signal recovery by proximal forward-backward splitting, *Multiscale Model. Sim.*, **4** (2005), 1168–1200. https://doi.org/10.1137/050626090
3. V. Dadashi, M. Postolache, Forward–backward splitting algorithm for fixed point problems and zeros of the sum of monotone operators, *Arab. J. Math.*, **9** (2020), 89–99. https://doi.org/10.1007/s40065-018-0236-2
4. A. Dixit, D. R. Sahu, P. Gautam, T. Som, J. C. Yao, An accelerated forward-backward splitting algorithm for solving inclusion problems with applications to regression and link prediction problems, *J. Nonlinear Var. Anal.*, **5** (2021), 79–101. https://doi.org/10.23952/jnva.5.2021.1.06
5. A. Hanjing, S. Suantai, A fast image restoration algorithm based on a fixed point and optimization method, *Mathematics*, **8** (2020), 378. https://doi.org/10.3390/math8030378
6. K. Janngam, S. Suantai, An accelerated forward-backward algorithm with applications to image restoration problems, *Thai J. Math.*, **19** (2021), 325–339.
7. D. Kitkuan, K. Muangchoo, Inertial relaxed CQ algorithm with an application to signal processing, *Thai J. Math.*, **18** (2020), 1091–1103.

8. D. Kitkuan, P. Kumam, J. Martínez-Moreno, Generalized Halpern-type forward–backward splitting methods for convex minimization problems with application to image restoration problems, *Optimization*, **69** (2020), 1557–1581. 10.1080/02331934.2019.1646742
9. G. López, V. Martín-Márquez, F. Wang, H. K. Xu, Forward-backward splitting methods for accretive operators in Banach spaces, *Abstract Appl. Anal.*, **2012** (2012), 109236. https://doi.org/10.1155/2012/109236
10. D. A. Lorenz, T. Pock, An inertial forward-backward algorithm for monotone inclusions, *J. Math. Imaging Vis.*, **51** (2015), 311–325. https://doi.org/10.1007/s10851-014-0523-2
11. D. R. Sahu, A. Pitea, M. Verma, A new iteration technique for nonlinear operators as concerns convex programming and feasibility problems, *Numer. Algor.*, **83** (2020), 421–449. https://doi.org/10.1007/s11075-019-00688-9
12. K. Sitthithakerngkiet, J. Deepho, P. Kumam, A hybrid viscosity algorithm via modify the hybrid steepest descent method for solving the split variational inclusion in image reconstruction and fixed point problems, *Appl. Math. Comput.*, **250** (2015), 986–1001. https://doi.org/10.1016/j.amc.2014.10.130
13. S. Sra, S. Nowozin, S. J. Wright, *Optimization for machine learning*, 2012.
14. P. Sunthrayuth, P. Cholamjiak, Iterative methods for solving quasi-variational inclusion and fixed point problem in q-uniformly smooth Banach spaces, *Numer. Algor.*, **78** (2018), 1019–1044. https://doi.org/10.1007/s11075-017-0411-0
15. P. Tseng, A modified forward-backward splitting method for maximal monotone mappings, *SIAM J. Control Optim.*, **38** (2000), 431–446. https://doi.org/10.1137/S0363012998338806
16. G. I. Usurelu, M. Postolache, Algorithm for generalized hybrid operators with numerical analysis and applications, *J. Nonlinear Var. Anal.*, **6** (2022), 255–277.
17. Y. Dong, X. Zhu, An inertial splitting method for monotone inclusions of three operators, *Int. J. Math. Stat. Oper. Res.*, **2** (2022), 43–60. https://DOI:10.47509/IJMSOR.2022.v02i01.04
18. P. L. Lions, B. Mercier, Splitting algorithms for the sum of two nonlinear operators, *SIAM J. Numer. Anal.*, **16** (1979), 964–979. https://doi.org/10.1137/0716071
19. A. Gibali, D. V.Thong, Tseng type method for solving inclusion problems and its applications, *Calcolo*, **55** (2018), 49. https://doi.org/10.1007/s10092-018-0292-1
20. A. Moudafi, M. Oliny, Convergence of a splitting inertial proximal method for monotone operators, *J. Comput. Appl. Math.*, **155** (2003), 447–454. https://doi.org/10.1016/S0377-0427(02)00906-8
21. E. Altiparmak, I. Karahan, A new preconditioning algorithm for finding a zero of the sum of two monotone operators and its application to image restoration problems, *Int. J. Comput. Math.*, **99** (2022), 2482–2498. https://doi.org/10.1080/00207160.2022.2068146
22. E. Altiparmak, I. Karahan, A Modified Preconditioning Algorithm for solving monotone inclusion problem and application to image restoration problem, *U. Politeh. Buch. Ser. A*, **84** (2022), 81–92.
23. W. Shatanawi, T. A. Shatnawi, New fixed point results in controlled metric type spaces based on new contractive conditions, *AIMS Math.*, **8** (2023), 9314–9330. https://doi.org/10.3934/math.2023468

24. A. Z. Rezazgui, A. A. Tallafha, W. Shatanawi, Common fixed point results via $A\nu$-α-contractions with a pair and two pairs of self-mappings in the frame of an extended quasi b-metric space, *AIMS Math.*, **8** (2023), 7225–7241. https://doi.org/10.3934/math.2023363

25. M. Joshi, A. Tomar, T. Abdeljawad, On fixed points, their geometry and application to satellite web coupling problem in S-metric spaces, *AIMS Math.*, **8** (2023), 4407–4441. https://doi.org/10.3934/math.2023220

26. P. Debnath, Z. D. Mitrović, S. Y. Cho, Common fixed points of Kannan, Chatterjea and Reich type pairs of self-maps in a complete metric space, *São Paulo J. Math. Sci.*, **15** (2021), 383–391. https://doi.org/10.1007/s40863-020-00196-y

27. P. Debnath, Banach, Kannan, Chatterjea and Reich-type contractive inequalities for multivalued mappings and their common fixed points, *Math. Method. Appl. Sci.*, **45** (2022), 1587–1596. https://doi.org/10.1002/mma.7875

28. P. Debnath, New common fixed point theorems for Gornicki-type mappings and enriched contractions, *São Paulo J. Math. Sci.*, **16** (2022), 1401–1408. https://doi.org/10.1007/s40863-022-00283-2

29. B. T. Polyak, Some methods of speeding up the convergence of iteration methods, *USSR Comp. Math. Math. Phys.*, **4** (1964), 1–17. https://doi.org/10.1016/0041-5553(64)90137-5

30. H. H. Bauschke, P. L. Combettes, *Convex analysis and monotone operator theory in Hilbert spaces*, New York: Springer, 2011.

31. A. Cegielski, *Iterative methods for fixed point problems in Hilbert spaces*, Springer, 2012.

32. B. V. Limaye, *Functional Analysis*, New Age International, 1996.

33. K. Goebel, W. A. Kirk, *Topics in metric fixed point theory*, Cambridge University Press, 1990. https://doi.org/10.1017/CBO9780511526152

34. S. Takahashi, W. Takahashi, The split common fixed point problem and the shrinking projection method in Banach spaces, *Optimization*, **65** (2016), 281–287. https://doi.org/10.1080/02331934.2015.1020943

35. H. K. Xu, Iterative algorithms for nonlinear operators, *J. London Math. Soc.*, **66** (2002), 240–256. https://doi.org/10.1112/S0024610702003332

36. P. E. Maingé, A hybrid extra gradient-viscosity method for monotone operators and fixed point problems, *SIAM J. Control Optim.*, **47** (2008), 1499–1515. https://doi.org/10.1137/060675319

37. W. Takahashi, C. F. Wen, J. C. Yao, The shrinking projection method for a finite family of demimetric mappings with variational inequality problems in a Hilbert space, *Fixed Point Theor.*, **19** (2018), 407-419. 10.24193/fpt-ro.2018.1.32