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Research article

A relation theoretic *m*-metric fixed point algorithm and related applications

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Abstract: In this article, we introduce the concept of generalized rational type F-contractions on relation theoretic m-metric spaces (denoted as F_R^m-contractions, where R is a binary relation) and some related fixed point theorems are provided. Then, we achieve some fixed point results for cyclic rational type F_R^m- generalized contraction mappings. Moreover, we state some illustrative numerically examples to show our results are true and meaningful. As an application, we discuss a positive definite solution of a nonlinear matrix equation of the form $\Lambda = S + \sum_{i=1}^{\mu} Q_i^* \Xi(\Lambda) Q_i$.

Keywords: relation theoretic m-metric; generalized rational type F_R^m-contractions; matrix equation
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1. Introduction and preliminaries

The most papular Banach contraction mapping principle (BCMP) [1] is the largest powerful fundamental fixed point result. This principle has a lot of applications in pure and applied mathematics (see [2–4]). In the past few decades, many authors extended and generalized the (BCMP) in several ways (see [5–10]). Ran and Reurings [11] obtained positive definite solutions of matrix equations using the aid of the Banach contraction principle in partially ordered sets. Nieto and Rodriguez-Lopez [12] also used partially ordered spaces and fixed point theorems to find solutions of some differential equations. Very recently, Wardowski [13] furnished the idea of an F-contraction, which is an extension of the (BCMP). Furthermore, common fixed point theorems for rational F_R-contractive pairs of mappings with applications are announced in [14] as an extension of F-contractions in relation theoretic metric spaces. On the other hand, Matthews [15] introduced the notion of a partial metric space as a part of the study of semantics of dataflow network, and for more results in this direction see ([16–20]). One of the latest extensions of a metric space and a partial metric space is initiated through the concept of a m-metric space [21], and some researchers work in this direction (see more [22–30]). In our article, we utilize two last notions to give an interesting type of generalized F_R^m-contractions in the frame of relation theoretic m-metric spaces and to prove some fixed point results.

Generally saying that, we generalize and extend some recent results in [31]. We also extend the earlier mentioned results in the setting of relation theoretic m-metric spaces, that contain only the last two conditions imposed on the Wardowski function F in the first section. Furthermore, the consequences of our main results improve and generalize some corresponding theorems appearing in the literature.

Our article consists four sections. In the first section, we recall some fundamental definitions and theorems concerning m-metric spaces and different types of F-contractions. In the second section, we define the notion of generalized F_R^m-contractions of rational type and generalized F_R^m-contractions of cyclic type. In the third section, we use the whole Wardowski function in the setting of F_R^m-contractions of rational type as consequences of main results in section II. Using these ideas, we prove some new fixed point results in the frame of relation theoretic m-metric spaces and we present some examples to show that our obtained results are meaningful. In section IV, we present an application and we ensure the existence of a solution of a class of nonlinear matrix equations.

Throughout this article, $\mathbb{N}$ indicates a set of all natural numbers, $\mathbb{R}$ indicates set of real numbers and $\mathbb{R}^+$ indicates set of positive real numbers, respectively. We also denote $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Henceforth, U will denote a non-empty set. Given a self mapping $\gamma : U \to U$. A Picard sequence based on an arbitrary ζ_0 in U is given by $\zeta_\mu = \gamma\left(\zeta_{\mu-1}\right) = \gamma^\mu(\zeta_0)$ for all μ in $\mathbb{N}$, where γ^μ denotes the μ^{th}-iteration of γ.

In 2013, the notion of a m-metric space was introduced by Asadi et al. [21]. They also extended the well known Banach contraction fixed point theorem from partial metric spaces to m-metric spaces. We start recalling some definitions and properties:

Definition 1.1. *[21]* Let $U \neq \emptyset$. The function $m : U \times U \to \mathbb{R}^+$ is a m-metric on the set U if for all $\zeta, \mathfrak{I}, \aleph \in U$,

(i) $\zeta = \mathfrak{I} \Longleftrightarrow m(\zeta,\zeta) = m(\mathfrak{I},\mathfrak{I}) = m(\zeta,\mathfrak{I})$ (T_0-separation axiom);
(ii) $m_{\zeta\mathfrak{I}} \leq m(\zeta,\mathfrak{I})$ (minimum self distance axiom);
(iii) $m(\zeta,\mathfrak{I}) = m(\mathfrak{I},\zeta)$ (symmetry);
(iv) $m(\zeta,\mathfrak{I}) - m_{\zeta\mathfrak{I}} \leq (m(\zeta,\aleph) - m_{\zeta\aleph}) + (m(\aleph,\mathfrak{I}) - m_{\aleph\mathfrak{I}})$ (modified triangle inequality),

where

$$m_{\zeta\mathfrak{I}} = \min\{m(\zeta,\zeta), m(\mathfrak{I},\mathfrak{I})\},$$

$$M_{\zeta\Im} = \max\{m(\zeta,\zeta), m(\Im,\Im)\}.$$

Here, the pair (U,m) is called a m-metric space.

On among the classical examples of m-metric spaces is the pair (ζ,m) where $U = \{\zeta,\Im,\aleph\}$ and $m(\zeta,\zeta) = 1$, $m(\Im,\Im) = 9$, $m(\aleph,\aleph) = 5$. Other examples of m-metric spaces may be found, for instance in [21]. Clearly, each partial metric is a m-metric space, but the converse does not hold (see [32–34]).

Every m-metric m on U generates a T_0 topology τ_m (say) on U which has a base of collection of m-open balls

$$\{B_m(\zeta,\epsilon) : \zeta \in U, \epsilon > 0\},$$

where

$$B_m(\zeta,\epsilon) = \{\Im \in U : m(\zeta,\Im) < m_{\zeta\Im} + \epsilon\} \text{for all } \zeta \in U, \varepsilon > 0.$$

If m is a m-metric space on U, then the functions $m^w, m^s : U \times U \to \mathbb{R}^+$ given by:

$$m^w(\zeta,\Im) = m(\zeta,\Im) - 2m_{\zeta\Im} + M_{\zeta\Im},$$

$$m^s = \begin{cases} m(\zeta,\Im) - m_{\zeta\Im}, \text{ if } \zeta \neq \Im \\ 0, \text{ if } \zeta = \Im. \end{cases}$$

define ordinary metrics on U. It is easy to see that m^w and m^s are equivalent metrics on U.

Definition 1.2. According to [21],

(i) a sequence $\{\zeta_\mu\}$ in a m-metric space (U,m) converges with respect to τ_m to ζ if and only if

$$\lim_{\mu\to\infty}\left(m\left(\zeta_\mu,\zeta\right) - m_{\zeta_\mu\zeta}\right) = 0,$$

(ii) a sequence $\{\zeta_\mu\}$ in a m-metric space (U,m) is called m-Cauchy if $\lim_{\mu,\nu\to\infty}\left(m\left(\zeta_\mu,\zeta_\nu\right) - m_{\zeta_\mu\zeta_\nu}\right)$ and $\lim_{\mu,\nu\to\infty}\left(M_{\zeta_\mu,\zeta_\nu} - m_{\zeta_\mu\zeta_\nu}\right)$ exist and are finite,

(iii) (U,m) is said to be complete if every m-Cauchy sequence $\left\{\zeta_\mu\right\}$ in U is m-convergent to ζ with respect to τ_m in U such that

$$\lim_{\mu\to\infty}\left(m\left(\zeta_\mu,\zeta\right) - m_{\zeta_\mu\zeta}\right) = 0, \text{ and } \lim_{\mu\to\infty}\left(M_{\zeta_\mu,\zeta} - m_{\zeta_\mu\zeta}\right) = 0,$$

(iv) $\{\zeta_\mu\}$ is a Cauchy sequence in (U,m) if and only if it is a Cauchy sequence in the metric space (U,m^w),

(v) (U,m) is complete if and only if (U,m^w) is complete.

Consider a function $F : (0,\infty) \to R$ so that:

(F_1) $F(\zeta) < F(\Im)$ for all $\zeta < \Im$,

(F_2) for each sequence $\left\{\varpi_\mu\right\} \subseteq (0,\infty)$, $\lim_{\mu\to\infty}\varpi_\mu = 0$ iff $\lim_{\mu\to\infty} F\left(\varpi_\mu\right) = -\infty$,

(F_3) there exists $p \in (0,1)$ such that $\lim_{\varpi_\mu\to 0^+}\varpi^p F(\varpi) = 0$.

According to [13], denote by $\nabla(F)$ the collection of functions $F:(0,\infty)\to R$ satisfying (F_2) and (F_3). Take also

$$\Pi(F)=\{F\in\nabla_F : F\ ,\ \text{verifies}\,(F_1)\}\,.$$

Example 1.1. *[13]* The following below functions belong to $\Pi(F)$:

(1) $F(s)=\ln s$,
(2) $F(s)=s+\ln s$,
(3) $F(s)=\ln\left(s^2+s\right)$,
(4) $F(s)=-\frac{1}{\sqrt{s}}$,
for all $s>0$.

Example 1.2. The following functions are not strictly increasing and belong to $\nabla(F)$:

(1) $F(s)=100\ln\left(\frac{s}{2}+\sin s\right)$,
(2) $F(s)=\sin s+\ln s$,
(3) $F(s)=\sin s-\frac{1}{\sqrt{s}}$,

for all $s>0$.

Let γ be a self-mapping on a *mm*-space U. The following are some valuable notations that are useful for the rest.

(i) $(\gamma)_{Fix}$ is the set of all fixed points of γ,
(ii) $\Theta(\Psi,S)=\{\zeta\in U:(\zeta,\gamma(\zeta))\in R\}$,
(iii) $\digamma(\zeta,\Im,\nabla)$ is thefashion of all paths in ∇ from ζ to $\Im$.

Altun et al. [35] gave two fixed point results for multivalued F-contractions on *mm*-spaces. We ensure the existence of fixed point results for generalized F_R^m-contractions by using the concept given in [35] to the metric space setup. The motivation of this study is to solve nonlinear matrix equations. First, inspired by Altun et al. [31] and Wardowski [13], we give the following concepts.

Theoretic relations have been used in many research articles, for examples see [36]. A non-empty subset R of U^2 is said to be a relation on the m-metric space (U,m) if $R=\left\{(\zeta,\Im)\in U^2:\zeta,\Im\in U\right\}$. If $(\zeta,\Im)\in R$, then we say that $\zeta\leq\Im$ (ζ precede $\Im$) under R denoted by $(\zeta,\Im)\in R$, and the inverse of R is denoted as $R^{-1}=\left\{(\zeta,\Im)\in U^2:(\Im,\zeta)\in R\right\}$. Set $S=R\cup R^{-1}\subseteq U^2$. Consequently, we illustrate another relation on U denoted S^* and is given as $(\zeta,\Im)\in S^*\Leftrightarrow(\Im,\zeta)\in S$ and $\Omega\neq\Im$.

Definition 1.3. *[36]* Let $U\neq\emptyset$ and R be a binary relation on U. Then R is transitive if $(\zeta,\xi)\in R$ and $(\xi,\Im)\in R\Rightarrow(\zeta,\Im)\in R$, for all $\zeta,\Im,\xi\in U$.

Definition 1.4. *[36]* Let $U\neq\emptyset$. A sequence $\zeta_\mu\in U$ is called R-preserving, if $\left(\zeta_\mu,\zeta_{\mu+1}\right)\in R$.

Definition 1.5. *[36]* Let $U\neq\emptyset$ and $\gamma:U\to U$. A binary relation R on U is called γ-closed if for any $\zeta,\Im$ in U, we deduce $(\zeta,\Im)\in R\Rightarrow(\gamma(\zeta),\gamma(\Im))\in R$.

2. Main results

We begin with the following definitions.

Definition 2.1. We say that (U,m,R) is regular if for each sequence $\{\zeta_\mu\}$ in U,

$$\left.\begin{array}{r}\left(\zeta_\mu,\zeta_{\mu+1}\right)\in R \text{ for all } \mu\in\mathbb{N}\\ \lim_{\mu\to\infty}\left(m\left(\zeta_\mu,\zeta\right)-m_{\zeta_\mu\zeta}\right)=0,\text{ i.e., } \zeta_\mu\overset{t_m}{\to}\zeta\in R,\end{array}\right\}\Rightarrow\left(\zeta_\mu,\zeta\right)\in R,\text{ for all } \mu\in\mathbb{N}.$$

Definition 2.2. A relation theoretic m-metric space (U,m,R) is said to be R-complete if for an R-preserving m-Cauchy sequence $\{\zeta_\mu\}$ in U, there exists some ζ in U such that

$$\lim_{\mu\to\infty} m\left(\zeta_\mu,\zeta\right)-m_{\zeta_\mu\zeta}=0,\text{ and }\lim_{\mu\to\infty}\left(M_{\zeta_\mu,\zeta}-m_{\zeta_\mu\zeta}\right)=0.$$

Definition 2.3. Let (U,m) be a m-metric space endowed with a binary relation R on U and γ be a self-mapping on U. Then, γ is said to be a F_R^m-contractions, if there exist $F_R^m\in\Pi(\nabla)$ and $\xi>0$, such that

$$\xi+F_R^m\left(m\left(\gamma(\zeta),\gamma(\Im)\right)\right)\leq F_R^m\left(m\left(\zeta,\Im\right)\right) \tag{2.1}$$

for all $\zeta,\Im\in U$ with $(\zeta,\Im)\in S^*$.

Now, we introduce the concept of a generalized rational type F_R^m-contraction.

Definition 2.4. Let (U,m) be a m-metric space endowed with a binary relation R on U. Let $\gamma:U\to U$ be a self-mapping on U. It is called a generalized rational type F_R^m-contraction if there are $F_R^m\in\nabla(F)$ and $\xi>0$ such that

$$\xi+F_R^m\left(m(\gamma(\zeta),\gamma(\Im))\right)\leq F_R^m\left(\max\left\{\begin{array}{c} m(\zeta,\Im),m(\zeta,\gamma(\zeta)),m(\Im,\gamma(\Im)),\\ \frac{m(\zeta,\gamma(\zeta))\left[1+m\left(\Im,\gamma(\Im)\right)\right]}{1+m\left(\zeta,\Im\right)}\end{array}\right\}\right), \tag{2.2}$$

for all $\zeta,\Im\in U$ with $(\zeta,\Im)\in S^*$.

Theorem 2.1. *Let (U,m) be a complete m-metric space with a binary relation R on U and γ be a self-mapping on U such that:*

(*i*) *the class $\Theta(\gamma,R)$ is nonempty;*
(*ii*) *R is γ-closed;*
(*iii*) *the mapping γ is R-continuous;*
(*iv*) *γ is a generalized rational type F_R^m-contraction mapping.*

Then γ possesses a fixed point in U.

Proof. Let $\zeta_0\in\Theta([\gamma,R])$. We define a sequence $\{\zeta_\mu\}$ by $\zeta_{\mu+1}=\gamma\left(\zeta_\mu\right)=\gamma^\mu(\zeta_0)$ for each $\mu\in\mathbb{N}$. If there is μ_0 in $\mathbb{N}$ so that $\gamma\left(\zeta_{\mu_0}\right)=\zeta_{\mu_0}$, then γ has a fixed point ζ_{μ_0} and the proof is complete. Let $\zeta_{\mu+1}\neq\zeta_\mu$ for all μ in $\mathbb{N}$, so $m\left(\zeta_{\mu+1},\zeta_\mu\right)>0$. Since $(\gamma(\zeta_0),\zeta_0)\in S^*$, using γ-closedness of R, we get $\left(\gamma\left(\zeta_{\mu+1}\right),\zeta_\mu\right)\in S^*$. Then using the fact that γ is a generalized rational type F_R^m-contraction mapping, one writes

$$F_R^m m\left(\zeta_{\mu+1},\zeta_\mu\right)=F_R^m\left(m\left(\zeta_{\mu+1},\zeta_\mu\right)\right).$$

$$\leq F_R^m\left(\max\left\{\begin{array}{c} m\left(\zeta_\mu,\zeta_{\mu-1}\right),m\left(\zeta_\mu,\gamma\left(\zeta_\mu\right)\right),m\left(\zeta_{\mu-1},\gamma\left(_{\mu-1}\right)\right),\\ \frac{m\left(\zeta_\mu,\zeta_{\mu+1}\right)\left[1+m\left(\zeta_{\mu-1},\zeta_\mu\right)\right]}{1+m\left(\zeta_{\mu-1},\zeta_\mu\right)}\end{array}\right\}\right)-\xi \tag{2.3}$$

$$\leq F_R^m\left(\max\left\{\begin{array}{c} m\left(\zeta_\mu,\zeta_{\mu-1}\right), m\left(\zeta_\mu,\zeta_{\mu+1}\right), m\left(\zeta_{\mu-1},\zeta_\mu\right), \\ \frac{m\left(\zeta_\mu,\zeta_{\mu+1}\right)\left[1+m\left(\zeta_{\mu-1},\zeta_\mu\right)\right]}{1+m\left(\zeta_{\mu-1},\zeta_\mu\right)} \end{array}\right\}\right) - \xi$$
$$\leq F_R^m\left(\max\left\{m\left(\zeta_\mu,\zeta_{\mu-1}\right), m\left(\zeta_\mu,\zeta_{\mu+1}\right)\right\}\right) - \xi.$$

If $\max\left\{m\left(\zeta_\mu,\zeta_{\mu-1}\right), m\left(\zeta_\mu,\zeta_{\mu+1}\right)\right\} = m\left(\zeta_\mu,\zeta_{\mu+1}\right)$, then from (2.3) , we have

$$F_R^m\left(m\left(\zeta_\mu,\zeta_{\mu+1}\right)\right) \leq F_R^m\left(m\left(\zeta_\mu,\zeta_{\mu+1}\right)\right) - \xi < F_R^m\left(m\left(\zeta_\mu,\zeta_{\mu+1}\right)\right),$$

which is a contradiction. Thus, $\max\left\{m\left(\zeta_\mu,\zeta_{\mu-1}\right), m\left(\zeta_\mu,\zeta_{\mu+1}\right)\right\} = m\left(\zeta_\mu,\zeta_{\mu-1}\right)$ and so from (2.3) , we have

$$F_R^m\left(m\left(\zeta_\mu,\zeta_{\mu+1}\right)\right) \leq F_R^m\left(m\left(\zeta_{\mu-1},\zeta_\mu\right)\right) \text{ for all } \mu \in \mathbb{N}. \tag{2.4}$$

Denote $\delta_\mu = m\left(\zeta_\mu,\zeta_{\mu+1}\right)$. We have $\delta_\mu > 0$ for all $\mu \in \mathbb{N}$ and using (2.4) we deduce that

$$F_R^m\left(\delta_\mu\right) \leq F_R^m\left(\delta_{\mu-1}\right) - \xi \leq F_R^m\left(\delta_{\mu-1}\right) - 2\xi \leq ...F_R^m\left(\delta_{\mu-1}\right) - \mu\xi. \tag{2.5}$$

It implies that $\lim_{\mu\to\infty} F_R^m\left(\delta_\mu\right) = -\infty$, then by (F_2), we have $\lim_{\mu\to\infty}\delta_\mu = 0$. Due to (F_3), there exists $k \in (0,1)$ such that $\lim_{\mu\to\infty}\delta_\mu^k F_R^m\left(\delta_\mu\right) = 0$.

From (2.4) the following is true for all $\mu \in \mathbb{N}$,

$$\delta_\mu^k\left(F_R^m\left(\delta_\mu\right) - F_R^m\left(\delta_0\right)\right) \leq -\delta_\mu^k\mu\tau \leq 0. \tag{2.6}$$

Letting $\mu \to \infty$ in (2.6), we get

$$\lim_{\mu\to\infty}\mu\delta_\mu^k = 0. \tag{2.7}$$

From (2.7), there exists $\mu_1 \in \mathbb{N}$ so that $\mu\delta_n^k \leq 1$ for all $\mu \geq \mu_1$, then we deduce

$$\delta_\mu \leq \frac{1}{\mu^{\frac{1}{k}}} \text{ for all } \mu \geq \mu_1.$$

We claim that $\left\{\zeta_\mu\right\}$ is a m-Cauchy sequence in the m-metric space. Let $\nu,\mu \in \mathbb{N}$ such that $\nu > \mu \geq \mu_1$. Using the triangle inequality of a m-metric space, one writes

$$\begin{aligned} m\left(\zeta_\mu,\zeta_\nu\right) - m_{\zeta_\mu,\zeta_\nu} &\leq m\left(\zeta_\mu,\zeta_{\mu+1}\right) - m_{\zeta_\mu,\zeta_{\mu+1}} + m\left(\zeta_{\mu+1},\zeta_{\mu+2}\right) - m_{\zeta_{\mu+1},\zeta_{\mu+2}} + \\ &\quad ... + m\left(\zeta_{\nu-1},\zeta_\nu\right) - m_{\zeta_{\nu-1},\zeta_\nu} \\ &\leq m\left(\zeta_\mu,\zeta_{\mu+1}\right) + m\left(\zeta_{\mu+1},\zeta_{\mu+2}\right) + \cdots + m\left(\zeta_{\nu-1},\zeta_\nu\right) \\ &\leq \delta_\mu + \delta_{\mu+1} + \cdots + \delta_{\nu-1} \\ &= \sum_{i=\mu}^{\nu-1}\delta_i \leq \sum_{i=\mu}^{\infty}\delta_i \leq \sum_{i=\mu}^{\infty}\frac{1}{i^{\frac{1}{k}}}. \end{aligned}$$

The convergence of the series $\sum_{i=\mu}^{\infty} \frac{1}{i^{\frac{1}{k}}}$ yields that $m\left(\zeta_\mu, \zeta_\nu\right) - m_{\zeta_\mu,\zeta_\nu} \to 0$. Thus, $\left\{\zeta_\mu\right\}$ is a M-Cauchy sequence in (U, m). Since (U, m, R) is R-complete, there exists $\zeta \in U$ such that $\left\{\zeta_\mu\right\}$ converges to ζ with respect to t_κ, that is, $m\left(\zeta_\mu, \zeta\right) - m_{\zeta_\mu,\zeta} \to 0$ as $\mu \to \infty$. Now, the R-continuity of γ implies that

$$\zeta = \lim_{\mu\to\infty} \zeta_{\mu+1} = \lim_{\mu\to\infty} \gamma\left(\zeta_\mu\right) = \gamma\left(\zeta\right).$$

Hence, ζ is a fixed point of γ.

Example 2.1. Let $U = [0, \infty)$ and m be defined by $m(\zeta, \Im) = \min\{\zeta, \Im\}$ for all $\zeta, \Im \in U$. (U, m) is a complete m-metric space. Consider the sequence $\left\{z_\mu\right\} \subseteq U$ given by $z_\mu = \frac{\mu(\mu+1)(2\mu+1)}{6}$ for all $\mu \geq 2$. Set an binary relation on U denoted by R given by $R = \left\{(z_1, z_1), \left(z_{\mu-1}, z_\mu\right) : \mu = 2, 3, ...100\right\}$. Now, give $\gamma : U \to U$ as

$$\gamma(\zeta) = \begin{cases} \zeta, & \text{if } 0 \leq \zeta \leq z_1, \\ z_1, & \text{if } z_1 \leq \zeta \leq z_2, \\ z_\mu + \left(\frac{z_\mu - z_{\mu-1}}{z_{\mu+1} - z_\mu}\right)\left(\zeta - z_\mu\right), & \text{if } z_{\mu+1} \leq \zeta \leq z_\mu \text{ for all } \mu = 2, 3, \cdots, 100. \end{cases}$$

Obviously, R is γ-closed and γ is continuous. Choosing $\zeta = z_\mu$ and $\Im = z_{\mu+1}$ (for $\mu = 1, 2, 3, \cdots, 100$), for first condition of F (which is (F_1)), we have

$$F_R^m\left(m\left(\gamma\left(z_\mu\right), \gamma\left(z_{\mu+1}\right)\right)\right) = F_R^m\left(m\left(z_{\mu-1}, z_\mu\right)\right) = F_R^m\left(z_{\mu-1}\right) = 100 \ln\left(\frac{z_{\mu-1}}{2} - \sin z_{\mu-1}\right),$$

and

$$\begin{aligned} & F_R^m\left(\max\left\{\begin{array}{c} m\left(z_\mu, z_{\mu+1}\right), m\left(z_\mu, \gamma\left(z_\mu\right)\right), m\left(z_{\mu+1}, \gamma\left(z_{\mu+1}\right)\right), \\ \frac{m\left(z_\mu, \gamma\left(z_\mu\right)\right)\left(1 + m\left(z_{\mu+1}, \gamma\left(z_{\mu+1}\right)\right)\right)}{1 + m\left(z_\mu, z_{\mu+1}\right)} \end{array}\right\}\right) \\ = \; & F_R^m\left(\max\left\{\begin{array}{c} m\left(z_\mu, z_{\mu+1}\right), m\left(z_\mu, z_{\mu-1}\right), m\left(z_{\mu+1}, z_\mu\right), \\ \frac{m\left(z_\mu, z_{\mu-1}\right)\left(1 + m\left(z_{\mu+1}, z_\mu\right)\right)}{1 + m\left(z_\mu, z_{\mu+1}\right)} \end{array}\right\}\right). \\ = \; & F_R^m\left(z_\mu\right). \end{aligned}$$

Now, for $\mu = 2, 3, 4, \cdots, 100$, we have

$$\xi + 100 \ln\left(\frac{z_{\mu-1}}{2} + \sin z_{\mu-1}\right) \leq 100 \ln\left(\frac{z_\mu}{2} + \sin z_\mu\right), \tag{2.9}$$

implies that

$$\xi \leq 100 \ln\left(\frac{\frac{z_\mu}{2} + \sin z_\mu}{\frac{z_{\mu-1}}{2} + \sin z_{\mu-1}}\right). \tag{2.10}$$

Let

$$f(\mu) = 100 \ln\left(\frac{\frac{z_\mu}{2} + \sin z_\mu}{\frac{z_{\mu-1}}{2} + \sin z_{\mu-1}}\right). \tag{2.11}$$

In view of Table 1 and Figure 1, since the function $\{f(\mu)\}_{\mu\geq 2}$ is decreasing and discontinuous, the smallest value in (2.11) is 5.02. Therefore, the Eq (2.10) holds for $0 < \xi < 5$. So

$$\xi + F_R^m(m(\gamma(\zeta), \gamma(\mathfrak{I}))) \leq F_R^m\left(\max\left\{\begin{array}{c} m(\zeta, \mathfrak{I}), m(\zeta, \gamma(\zeta)), m(\mathfrak{I}, \gamma(\mathfrak{I})), \\ \frac{m(\zeta,\gamma(\zeta))\left[1+m(\mathfrak{I},\gamma(\mathfrak{I}))\right]}{1+m(\zeta,\mathfrak{I})} \end{array}\right\}\right),$$

for all $\zeta, \mathfrak{I} \in U$ such that $(\zeta, \mathfrak{I}) \in \mathbf{S}^*$ with mm-space. Hence, γ is a generalized rational type F_R^m-contraction mapping with $0 < \xi < 5$. Generally, we can say that γ has infinite $(F.Ps)$.

Table 1. Iterations and $f(\mu)$.

Iter	$f(\mu)$	Iter	$f(\mu)$
$\mu = 2$	13.87	$\mu = 13$	25.21
$\mu = 3$	164.6	$\mu = 14$	23.21
$\mu = 4$	56.16	$\mu = 15$	21.20
$\mu = 5$	63.72	$\mu = 16$	20.20
$\mu = 6$	54.28	$\mu = 17$	18.71
$\mu = 7$	44.23	$\mu = 18$	17.64
$\mu = 8$	36.45	$\mu = 19$	16.53
$\mu = 9$	33.78	...	...
$\mu = 10$	33.78	$\mu = 60$	5.02

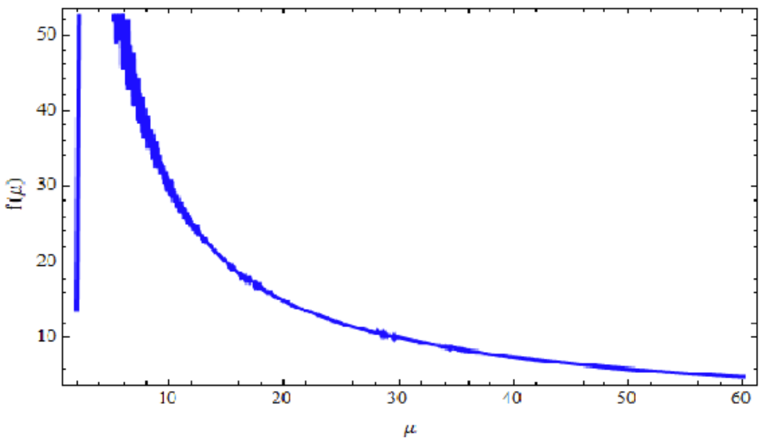

Figure 1. Behaviour of $f(\mu)$, for $\mu \in [2, 60]$.

Theorem 2.2. *Theorem* 2.1 *remains true if the condition* (ii) *is replaced by the following:*

(i) $(ii)'$,
(ii) (X, κ, ∇) *is regular.*

Proof. It is a same argument as Theorem 2.1. Here, the sequence $\left\{\zeta_\mu\right\}$ is m-Cauchy and converges to some Ω in U such that $m\left(\zeta_\mu,\zeta\right)-m_{\zeta_\mu,\zeta}$ as limit $\mu\to\infty$ which implies that

$$\begin{aligned}\lim_{\mu\to\infty} m\left(\zeta_\mu,\zeta\right) &= \lim_{\mu\to\infty} m_{\zeta_\mu,\zeta} = \lim_{\mu\to\infty}\min\left\{m\left(\zeta_\mu\zeta_\mu\right), m\left(\zeta,\zeta\right)\right\} = m\left(\zeta,\zeta\right)\\ &= \lim_{\mu,\nu\to\infty} m\left(\zeta_\mu,\zeta_\nu\right)=0 \text{ and } \lim_{\mu,\nu\to\infty} m_{\zeta_\mu,\zeta_\nu}=0.\end{aligned}$$

As $\left(\zeta_\mu,\zeta_{\mu+1}\right)\in R$, then $\left(\zeta_\mu,\zeta\right)\in R$ for all $\mu\in\mathbb{N}$. Set $\mu=\left\{\mu\in N:\gamma\left(\zeta_\mu\right)=\gamma\left(\zeta\right)\right\}$. We will take two cases depending on μ

C-1. If μ is a finite set, then there exists μ_0 in $\mathbb{N}$, so that $\gamma\left(\zeta_\mu\right)\neq\gamma\left(\zeta\right)$ for every $\mu\geq\mu_0$. In particular, $\left(\zeta_\mu,\zeta\right)\in S^*$ and $\left(\gamma\left(\zeta_\mu\right),\gamma\left(\zeta\right)\right)\in S^*$, then for all $\mu\geq\mu_0$,

$$\xi+F_R^m\left(m\left(\gamma\left(\zeta_\mu\right),\gamma\left(\zeta\right)\right)\right)\leq F\left(m\left(\zeta_\mu,\zeta\right)\right).$$

Since $\lim_{\mu\to\infty} m\left(\zeta_\mu,\zeta\right)=0$ implies that $\lim_{\mu\to\infty} F_R^m\left(m\left(\zeta_\mu,\zeta\right)\right)=-\infty$, one writes $\lim_{\mu\to\infty} F_R^m\left(m\left(\gamma\left(\zeta_\mu\right),\gamma\left(\zeta\right)\right)\right)=-\infty$. Therefore, $\lim_{\mu\to\infty} m\left(\gamma\left(\zeta_\mu\right),\gamma\left(\zeta\right)\right)=0$, which yields that $\gamma\left(\zeta\right)=\zeta$, that is, ζ is a fixed point of γ.

C-2. If μ is an infinite set, then there exists a subsequence $\left\{\zeta_{\mu_k}\right\}$ of $\left\{\zeta_\mu\right\}$ so that $\zeta_{\mu_k+1}=\gamma\left(\zeta_{\mu_k}\right)=\gamma\left(\zeta\right)$ for $k\in\mathbb{N}$, so $\gamma\left(\zeta_{\mu_k}\right)\to\gamma\left(\zeta\right)$ with respect to t_m as ζ_μ converges ζ, then $\gamma\left(\zeta\right)=\zeta$, i.e., γ has a fixed point. Hence, the proof is complete.

Now, we prove a result of uniqueness.

Theorem 2.3. *Following Theorems 2.1 and 2.2 γ possesses a unique fixed point if $F\left(\zeta,\Im,\nabla\right)\neq\emptyset$, for all $\zeta,\Im\in\left(\gamma\right)_{Fix}$.*

Proof. Let $\zeta,\Im\in\left(\gamma\right)_{Fix}$ such that $\zeta\neq\Im$. Since $F\left(\zeta,\Im,\nabla\right)\neq\emptyset$, there exists a path $\left(\left\{a_0,a_1,...a_\mu\right\}\right)$ of some finite length μ in ∇ from Ω to $\Im$ (with $a_s\neq a_{s+1}$ for all $s\in[0,p-1]$). Then $a_0=\zeta$, $a_k=\Im$, $\left(a_s,a_{s+1}\right)\in S^*$ for every $s\in[0,p-1]$. As $a_s\in\gamma\left(U\right)$, $\gamma\left(a_s\right)=a_s$ for all $s\in[0,p-1]$ we deduce that

$$\begin{aligned}F_R^m\left(m\left(a_s,a_{s+1}\right)\right) &= F_R^m\left(m\left(\gamma\left(a_s\right),\gamma\left(a_{s+1}\right)\right)\right)\\ &\leq F_R^m\left\{\max\left\{\begin{array}{c} m\left(a_s,a_{s+1}\right), m\left(a_s,\gamma\left(a_s\right)\right), m\left(a_{s+1},\gamma\left(a_{s+1}\right)\right),\\ \frac{m\left(a_s,\gamma\left(a_s\right)\right),\left[1+m\left(a_s,\gamma\left(a_{s+1}\right)\right)\right]}{1+m\left(a_s,a_{s+1}\right)}\end{array}\right\}\right\}-\xi\\ &= F_R^m\left\{\max\left\{\begin{array}{c} m\left(a_s,a_{s+1}\right), m\left(a_s,a_s\right), m\left(a_{s+1},a_{s+1}\right),\\ \frac{m\left(a_s,a_s\right)\left[1+m\left(a_s,a_{s+1}\right)\right]}{1+m\left(a_s,a_{s+1}\right)}\end{array}\right\}\right\}-\xi\\ &< F_R^m\left\{\left(m\left(a_s,a_{s+1}\right)\right)\right\}.\end{aligned}$$

It is a contradiction. Hence, γ possesses a unique fixed point.

Now, we say that $\gamma:U\to U$ has the property P if

$$\left(\gamma^\mu\right)_{\text{Fix}}=\left(\gamma\right)_{\text{Fix}} \text{ for each } \mu \text{ is member of } \mathbb{N}.$$

In this theorem, we use above condition having property P.

Theorem 2.4. *Let (U, m) be a complete m-metric space with a binary relation R on U and γ be a self-mapping such that:*

(i) *the class $\Theta(\gamma, R)$ is nonempty,*
(ii) *the binary relation R is γ-closed,*
(iii) *γ is R-continuous,*
(iv) *there are $F_R^m \in \nabla(F)$ and $\xi > 0$ so that*

$$\xi + F_R^m\left(m\left(\gamma(\zeta), \gamma^2(\zeta)\right)\right) \leq F_R^m\left(\max\left\{\begin{array}{c} m(\zeta, \gamma(\zeta)), m\left(\gamma(\zeta), \gamma^2(\zeta)\right), \\ \frac{m(\zeta,\gamma(\zeta))\left[1+m\left(\gamma(\zeta),\gamma^2(\zeta)\right)\right]}{1+m\left(\gamma(\zeta),\gamma^2(\zeta)\right)} \end{array}\right\}\right)$$

for all $\zeta \in U$, with $\left(\gamma(\zeta), \gamma^2(\zeta)\right) \in S^*$.
Then γ has a fixed point. Furthermore, if
(v) $(iv)'$*;*
(vi) *$\zeta \in (\gamma^\mu)_{Fix}$ (for some $\mu \in \mathbb{N}$) which implies that $(\zeta, \gamma(\zeta)) \in R$,*

then γ has a property P.

Proof. Let $\zeta_0 \in \Theta([\gamma, R])$, i.e., $(\zeta_0, \gamma(\zeta_0)) \in \mathbf{R}$,therefore using assumption (ii), we get $\left(\zeta_\mu, \zeta_{\mu+1}\right) \in R$ for each $\mu \in \mathbb{N}$. Denote $\zeta_{\mu+1} = \gamma\left(\zeta_\mu\right) = \gamma^{\mu+1}(\zeta_0)$, for all $\mu \in \mathbb{N}$. If there exists $\mu_0 \in \mathbb{N}$ so that $\gamma\left(\zeta_{\mu_0}\right) = \zeta_{\mu_0}$, then γ has a fixed point ζ_{μ_0} and it completes the proof. Otherwise, assume that $\zeta_{\mu+1} \neq \zeta_\mu$ for every $\mu \in \mathbb{N}$. Then $\left(\zeta_\mu, \zeta_{\mu+1}\right) \in R$ (for all $\mu \in \mathbb{N}$). Continuing this process and using the assumption (iv), we deduce (for all $\mu \in \mathbb{N}$)

$$\begin{aligned} F_R^m\left(m\left(\gamma\left(\zeta_{\mu-1}\right), \gamma^2\left(\zeta_{\mu-1}\right)\right)\right) &\leq F_R^m\left(\max\left\{\begin{array}{c} m\left(\zeta_{\mu-1}, \gamma\left(\zeta_{\mu-1}\right)\right), m\left(\gamma\left(\zeta_{\mu-1}\right), \gamma^2\left(\zeta_{\mu-1}\right)\right), \\ \frac{m\left(\zeta_{\mu-1},\gamma\left(\zeta_{\mu-1}\right)\right)\left[1+m\left(\gamma\left(\zeta_{\mu-1}\right),\gamma^2\left(\zeta_{\mu-1}\right)\right)\right]}{1+m\left(\gamma\left(\zeta_{\mu-1}\right),\gamma^2\left(\zeta_{\mu-1}\right)\right)} \end{array}\right\}\right) \\ &\quad -\xi \\ &= F_R^m\left(\max\left\{\begin{array}{c} m\left(\zeta_{\mu-1}, \zeta_\mu\right), m\left(\zeta_\mu, \zeta_{\mu+1}\right) \\ \frac{m\left(\zeta_{\mu-1},\zeta_\mu\right)\left[1+m\left(\zeta_\mu,\zeta_{\mu+1}\right)\right]}{1+m\left(\zeta_\mu,\zeta_{\mu+1}\right)} \end{array}\right\}\right) \\ &\quad -\xi \\ &\leq F_R^m\left(\max\left\{m\left(\zeta_{\mu-1}, \zeta_\mu\right), m\left(\zeta_\mu, \zeta_{\mu+1}\right)\right\}\right) - \xi. \end{aligned}$$

Assume that $\max\left\{m\left(\zeta_{\mu-1}, \zeta_\mu\right), m\left(\zeta_\mu, \zeta_{\mu+1}\right)\right\} = m\left(\zeta_\mu, \zeta_{\mu+1}\right)$, then we get

$$F_R^m\left(m\left(\zeta_\mu, \zeta_{\mu+1}\right)\right) \leq F_R^m\left(m\left(\zeta_\mu, \zeta_{\mu+1}\right)\right) - \xi < F_R^m\left(m\left(\zeta_\mu, \zeta_{\mu+1}\right)\right).$$

It is a contradiction. Hence, $\max\left\{m\left(\zeta_{\mu-1}, \zeta_\mu\right), m\left(\zeta_\mu, \zeta_{\mu+1}\right)\right\} = m\left(\zeta_{\mu-1}, \zeta_\mu\right)$, and so

$$F_R^m\left(m\left(\zeta_\mu, \zeta_{\mu+1}\right)\right) \leq F_R^m\left(m\left(\zeta_{\mu-1}, \zeta_\mu\right)\right) - \xi \text{ for all } \mu \in \mathbb{N}.$$

This yields that (for all $\mu \in \mathbb{N}$)

$$F_R^m\left(m\left(\zeta_\mu, \zeta_{\mu+1}\right)\right) \leq F_R^m\left(m\left(\zeta_{\mu-1}, \zeta_\mu\right)\right) - \xi \tag{2.1}$$

$$
\begin{aligned}
&\leq \; F_R^m\left(m\left(\zeta_{\mu-2}, \zeta_{\mu-1}\right)\right) - 2\xi \\
&\leq \; \dots \\
&\leq \; F_R^m\left(m\left(\zeta_0, \zeta_1\right)\right) - \mu\xi.
\end{aligned}
$$

By applying limit as μ goes to ∞ in above equation, we deduce $\lim_{\mu\to\infty} F_R^m\left(m\left(\zeta_\mu, \zeta_{\mu+1}\right)\right) = -\infty$. Since $F_R^m \in \nabla(F)$, we deduce that $\lim_{\mu\to\infty} m\left(\zeta_\mu, \zeta_{\mu+1}\right) = 0$. Using (F_3), there is $k \in (0,1)$ so that

$$
\lim_{\mu\to\infty} \left(m\left(\zeta_\mu, \zeta_{\mu+1}\right)\right)^k F_R^m\left(m\left(\zeta_\mu, \zeta_{\mu+1}\right)\right) = 0.
$$

Now, from (2.9), we have

$$
\begin{aligned}
&\left(m\left(\zeta_\mu, \zeta_{\mu+1}\right)\right)^k F_R^m\left(m\left(\zeta_\mu, \zeta_{\mu+1}\right)\right) - \left(m\left(\zeta_\mu, \zeta_{\mu+1}\right)\right)^k F_R^m\left(m\left(\zeta_0, \zeta_1\right)\right) \\
\leq\; & -\left(m\left(\zeta_\mu, \zeta_{\mu+1}\right)\right)^k \mu \leq 0.
\end{aligned} \tag{2.2}
$$

Letting $\mu \to \infty$ in (2.10), we get $\lim_{\mu\to\infty} \left(m\left(\zeta_\mu, \zeta_{\mu+1}\right)\right)^k = 0$. There is μ_1 in $\mathbb{N}$ so that

$$
\mu\left(m\left(\zeta_\mu, \zeta_{\mu+1}\right)\right)^k \leq 1 \text{ for all } \mu \geq \mu_1.
$$

That is,

$$
m\left(\zeta_\mu, \zeta_{\mu+1}\right) \leq \frac{1}{\mu^{\frac{1}{k}}} \text{ for all } \mu \geq \mu_1.
$$

Now, for $\nu > \mu > \mu_1$, we have

$$
m\left(\zeta_\mu, \zeta_\nu\right) - m_{\zeta_\mu,\zeta_\nu} \leq \sum_{i=\mu}^{\nu-1} m\left(\zeta_\mu, \zeta_\nu\right) - m_{\zeta_\mu,\zeta_\nu} \leq \sum_{i=\mu}^{\nu-1} m\left(\zeta_\mu, \zeta_\nu\right) \leq \sum_{i=\mu}^{\nu-1} \frac{1}{i^{\frac{1}{k}}}.
$$

Since the series $\sum_{i=\mu}^{\nu-1} \frac{1}{i^{\frac{1}{k}}}$ is convergent, i.e., $m\left(\zeta_\mu, \zeta_\nu\right) - m_{\zeta_\mu,\zeta_\nu}$ converges to 0, the sequence $\left\{\zeta_\mu\right\}$ is a m-Cauchy sequence. Since (U, m, R) is R-complete and $\left(\zeta_\mu, \zeta_{\mu+1}\right) \in R$ for all $\mu \in \mathbb{N}$, $\left\{\zeta_\mu\right\}$ converges to $\zeta \in U$. Now, using the R-continuity of γ, we deduce that

$$
\zeta = \lim_{\mu\to\infty} \zeta_{\mu+1} = \lim_{\mu\to\infty} \gamma\left(\zeta_\mu\right) = \gamma(\zeta).
$$

Finally, we will prove that $(\gamma^\mu)_{Fix} = (\gamma)_{Fix}$ where $\mu \in \mathbb{N}$. Assume on contrary that $\zeta \in (\gamma^\mu)_{Fix}$ and $\zeta \notin (\gamma)_{Fix}$ for some $\mu \in \mathbb{N}$. Then $m(\zeta, \gamma(\zeta)) > 0$, $(\zeta, \gamma(\zeta)) \in R\left(\text{from condition } (iv)'\right)$. From assumption (ii) we obtain $\left(\gamma^\mu(\zeta), \gamma^{\mu+1}(\zeta)\right) \in R$ for all $\mu \in \mathbb{N}$. Assumption (iv) implies that

$$
\begin{aligned}
F_R^m\left(m\left(\zeta, \gamma(\zeta)\right)\right) &= F_R^m\left(m\left(\gamma\left(\gamma^{\mu-1}(\zeta)\right), \gamma^2\left(\gamma^{\mu-1}(\zeta)\right)\right)\right) \\
&\leq F_R^m\left(m\left(\gamma^{\mu-1}(\zeta)\right), \gamma^\mu(\zeta)\right) - \xi \\
&\leq F_R^m\left(m\left(\gamma^{\mu-2}(\zeta)\right), \gamma^{\mu-1}(\zeta)\right) - 2\xi \\
&\quad \dots \\
&\leq F_R^m\left(m\left(\zeta, \gamma(\zeta)\right)\right) - \mu\xi.
\end{aligned}
$$

Taking $\mu \to \infty$ in above inequality, we obtain $F_R^m(m(\zeta, \gamma(\zeta))) = -\infty$, a contradiction. So, $(\gamma^\mu)_{Fix} = (\gamma)_{Fix}$ for any $\mu \in \mathbb{N}$.

Corollary 2.1. *Let (U, m) be a complete m-metric space with a binary relation R on U and γ be a self-mapping such that:*

(i) *the class $\Theta(\gamma, R)$ is nonempty;*
(ii) *the binary relation R is γ-closed;*
(iii) *γ is R-continuous;*
(iv) *γ is a F_R^m-contraction mapping.*

Then γ possesses a fixed point in U.
Here, we use the definition of F-contractions with the standard conditions $(i - iii)$.

Definition 2.5. Given a *mm*-space (U, m) and a binary relation R on U. Suppose that

$$\varpi = \{(\zeta, \Im) \in S^* : \kappa(\zeta, \Im) > 0\}.$$

We say that a self-mapping $\gamma : U \to U$ is a rational type F_R^m-contraction if there exists $F_R^m \in \Pi(\nabla)$ such that

$$\xi + F_R^m(m(\gamma(\zeta), \gamma(\Im))) \leq F_R^m\left(\max\left\{\begin{array}{c} m(\zeta, \Im), m(\zeta, \gamma(\zeta)), m(\Im, \gamma(\Im)), \\ \frac{m(\zeta,\gamma(\zeta))\left[1+m\left(\Im,\gamma(\Im)\right)\right]}{1+m\left(\zeta,\Im\right)} \end{array}\right\}\right) \tag{3.1}$$

for all $(\zeta, \Im) \in \Xi$.

Theorem 2.5. *Let (U, m) be a complete m-metric space, R be a binary relation on U and γ be a self-mapping on U. Assume that:*

(i) *the class $\Theta(\gamma, R)$ is non-empty;*
(ii) *the binary relation R is γ-closed;*
(iii) *γ is R-continuous;*
(iv) *γ is a rational type F_R^m-contraction mapping.*

Then γ possesses a fixed point in U.

Proof. Let $\zeta_0 \in \Theta([\gamma, R])$, i.e., $([\zeta_0, \gamma(\zeta_0)]) \in R$. We define a sequence $\left\{\zeta_{\mu+1}\right\}$ given as $\zeta_{\mu+1} = \gamma\left(\zeta_\mu\right) = \gamma^{\mu+1}(\zeta_0)$. We have $\left(\zeta_\mu, \zeta_{\mu+1}\right) \in R$ for all μ in $\mathbb{N}$. If there exists μ_0 in $\mathbb{N}$ such that $\gamma\left(\zeta_{\mu_0}\right) = \zeta_{\mu_0}$, then ζ_{μ_0} is a fixed point of γ and the proof is finished. Now, assume that $\zeta_{\mu+1} \neq \zeta_\mu$ for all $\mu \in \mathbb{N}$. Then $\left(\zeta_\mu, \zeta_{\mu+1}\right) \in R$ (for all $\mu \in \mathbb{N}$). Using the condition (iv), we deduce (for all $\mu \in \mathbb{N}$)

$$\begin{aligned} F\left(m\left(\gamma\left(\zeta_{\mu-1}\right), \gamma^2\left(\zeta_{\mu-1}\right)\right)\right) &\leq F_R^m\left(\max\left\{\begin{array}{c} m\left(\zeta_{\mu-1}, \gamma\left(\zeta_{\mu-1}\right)\right), m\left(\gamma\left(\zeta_{\mu-1}\right), \gamma^2\left(\zeta_{\mu-1}\right)\right), \\ \frac{m\left(\zeta_{\mu-1},\gamma\left(\zeta_{\mu-1}\right)\right)\left[1+m\left(\gamma\left(\zeta_{\mu-1}\right),\gamma^2\left(\zeta_{\mu-1}\right)\right)\right]}{1+m\left(\gamma\left(\zeta_{\mu-1}\right),\gamma^2\left(\zeta_{\mu-1}\right)\right)} \end{array}\right\}\right) - \xi \quad (2.3) \\ &= F_R^m\left(\max\left\{\begin{array}{c} m\left(\zeta_{\mu-1}, \zeta_\mu\right), m\left(\zeta_\mu, \zeta_{\mu+1}\right), \\ \frac{m\left(\zeta_{\mu-1},\zeta_\mu\right)\left[1+m\left(\zeta_\mu,\zeta_{\mu+1}\right)\right]}{1+m\left(\zeta_\mu,\zeta_{\mu+1}\right)} \end{array}\right\}\right) - \xi \\ &\leq F_R^m\left(\max\left\{m\left(\zeta_{\mu-1}, \zeta_\mu\right), m\left(\zeta_\mu, \zeta_{\mu+1}\right)\right\}\right) - \xi. \end{aligned}$$

By (F_1), we have $\max\left\{m\left(\zeta_{\mu-1},\zeta_\mu\right),m\left(\zeta_\mu,\zeta_{\mu+1}\right)\right\} = m\left(\zeta_\mu,\zeta_{\mu+1}\right)$, then we get a contradiction. Thus, $\max\left\{m\left(\zeta_\mu,\zeta_{\mu-1}\right),m\left(\zeta_\mu,\zeta_{\mu+1}\right)\right\} = m\left(\zeta_\mu,\zeta_{\mu-1}\right)$ and so from (2.3) we have

$$F_R^m\left(m\left(\zeta_\mu,\zeta_{\mu+1}\right)\right) \leq F_R^m\left(m\left(\zeta_\mu,\zeta_{\mu-1}\right)\right) - \xi \text{ for all } \mu \in \mathbb{N}. \tag{3.3}$$

The proof of Theorem 2.1 is complete.

Corollary 2.2. *Let (U,m) be a complete m-metric space, R be a binary relation on U and γ be a self-mapping on U. Assume that:*

(i) the class $\Theta(\gamma,R)$ is non-empty;
(ii) R is γ-closed;
(iii) γ is R-continuous;
(iv) γ is a F_R^m-contraction mapping.

Then γ possesses a fixed point in U.

Example 2.2. Let $U = [0,1]$ and m be a relation theoretic m-metric defined by

$$m(\zeta,\Im) = \frac{\zeta+\Im}{2} \text{ for all } \zeta,\Im \in U.$$

We define the binary relation

$$(\zeta,\Im) \in S^* \Leftrightarrow m(\zeta,\zeta) = m(\zeta,\Im) \Leftrightarrow \frac{\zeta+\Im}{2}.$$

(U,m) is a complete m-metric space with a binary relation. Define a mapping $\gamma : U \to U$ by

$$\gamma(\zeta) = \begin{cases} \frac{\zeta}{5} & \text{if } \zeta \in [0,1) \\ 0 & \text{if } \zeta = 1. \end{cases}$$

Obviously, $\mathfrak{R}$ is γ-closed, also and γ is $\mathfrak{R}$-continuous. Define $F_{\mathfrak{R}}^m : (0,\infty) \to R$ by

$$F_{\mathfrak{R}}^m(a) = \ln\left(a + a^2\right) \quad \text{for all } \xi \in (0,\infty).$$

Assume that $(\zeta,\Im) \in \Xi = \{(\zeta,\Im) \in S^* : m(\gamma(\zeta),\gamma(\Im)) > 0\}$. Therefore, for all $\zeta,\Im \in U$, with $0 < \zeta < 1, \Im = 1$, we have

$$\begin{aligned} F_{\mathfrak{R}}^m(m(\gamma(\zeta),\gamma(\Im))) &= F_{\mathfrak{R}}^m\left(m\left(\frac{\zeta}{5},0\right)\right) = F_{\mathfrak{R}}^m\left(\frac{\zeta}{10}\right) \\ &= \ln\left(\frac{\zeta}{100}^2 + \frac{\zeta}{10}\right). \end{aligned}$$

Now, consider $Z_A = \max\left\{ \begin{array}{c} m(\zeta,1), m(\zeta,\gamma(\zeta)), m(\Im,\gamma(\Im)), \\ \frac{m(\zeta,\gamma(\zeta))\left[1+m\left(\Im,\gamma\left(\Im\right)\right)\right]}{1+m(\Omega,1)} \end{array} \right\}$

$$F_R^m\left(\max\left\{ \begin{array}{c} m(\zeta,1), m(\zeta,\gamma(\zeta)), m(\Im,\gamma(\Im)), \\ \frac{m(\zeta,\gamma(\zeta))\left[1+m\left(\Im,\gamma\left(\Im\right)\right)\right]}{1+m(\zeta,1)} \end{array} \right\}\right)$$

$$
\begin{aligned}
&= F_R^m\left(\max\left\{\begin{array}{c}\frac{\zeta+1}{2}, \frac{\zeta+\frac{\zeta}{5}}{2}, \frac{1+0}{2}, \\ \frac{\frac{\zeta+\frac{\zeta}{5}}{2}\left[1+\frac{1+0}{2}\right]}{1+\frac{\zeta+1}{2}}\end{array}\right\}\right) \\
&= F_R^m\left(\frac{\zeta+1}{2}\right) \\
&= F_R^m(Z_A) \\
&= \ln\left(\left(\frac{\zeta+1}{2}\right)^2+\frac{\zeta+1}{2}\right).
\end{aligned}
$$

From Table Table.2, γ is a rational type F_R^m-contraction mapping with $\xi = 2$. Moreover, there is $\zeta_0 = 0.1$ in U so that$(\xi_{0,}\gamma(\xi_0)) \in S^*$ and the class $\Theta(\gamma, R)$ is nonempty. Hence, all conditions of Theorem 2.5 hold, and therefore γ has a fixed point.

Table 2. $\xi + F_{\mathfrak{R}}^m(m(\gamma(\zeta), \gamma(\mathfrak{I})))$ and $F_R^m(Z_A)$.

ζ	$\mathfrak{I}$	$\xi + F_{\mathfrak{R}}^m(m(\gamma(\zeta), \gamma(\mathfrak{I})))$	$F_R^m(Z_A)$
0.1	1	−4.595	−0.159
0.2	1	−3.892	−0.040
0.3	1	−3.479	0.069
0.4	1	−3.179	0.174
0.5	1	−2.947	0.272
0.6	1	−2.755	0.364
0.7	1	−2.591	0.452
0.8	1	−2.448	0.536
0.9	1	−2.321	0.616

3. Some fixed point results for cyclic contractions

In [37], Kirk et al. gave the concept of a cyclic contraction, which is the extension of the Banach contraction. It is utilized in the following theorem.

Theorem 3.1. *Suppose that (U, m) is a compete m-metric space, G, H are two nonempty closed subsets of U and $\gamma : U \to U$ verifies the following conditions:*

(i) $\gamma(B) \subseteq D$ *and* $\gamma(D) \subseteq B$;
(ii) *there exists a constant* $k \in (0, 1)$ *such that*

$$m(\gamma(\zeta), \gamma(\mathfrak{I})) \leq km(\zeta, \mathfrak{I}) \text{ for all } \zeta \in B,\ \mathfrak{I} \in D.$$

Then $B \cap D$ is nonempty and there is $\zeta \in B \cap D$ a fixed point of γ.

By Theorems 2.1 and 3.1, we obtain successive fixed point results for cyclic rational type F_R^m-generalized contraction mappings.

Theorem 3.2. *Let (U, m) be a complete m-metric space, G and H be two nonempty closed subsets of U and $\gamma : U \to U$ be an operator. Assume that the successive axioms hold:*

(*i*) $\gamma(G) \subseteq H$ *and* $\gamma(H) \subseteq G$;
(*ii*) *there exist* $F_R^m \in \nabla(F)$ *and* $\xi > 0$ *such that*

$$\xi + F_R^m\left(m(\gamma(\zeta), \gamma(\Im))\right) \le F_R^m\left(\max\left\{\begin{array}{c} m(\zeta, \Im), m(\zeta, \gamma(\zeta)), m(\Im, \gamma(\Im)), \\ \frac{m(\zeta,\gamma(\zeta))\left[1+m(\Im,\gamma(\Im))\right]}{1+m(\zeta,\Im)} \end{array}\right\}\right), \tag{2.10}$$

for all ζ *in G and* $\Im$ *in H.*

Then there is $\zeta \in G \cap H$ *a fixed point of* γ.

Proof. $Z = G \cup H$ is closed, so Z is a closed subspace of U. Therefore, (U, m) is a complete m-metric space. Set a binary relation on Z denoted by R given as

$$R = G \times H.$$

It means that

$$(\zeta, \Im) \in R \Leftrightarrow (\zeta, \Im) \in B \times D \text{ for all } \zeta, \Im \in Z.$$

Set $S = R \cup R^{-1}$ an asymmetric relation. Directly, (U, m, S) is regular. Assume $\{\zeta_\mu\} \in Z$ is any sequence and $\zeta \in Z$ so that

$$\left(\zeta_\mu, \zeta_{\mu+1}\right) \in S \text{ for all } \mu \in \mathbb{N},$$

and

$$\lim_{\mu\to\infty} m\left(\zeta_\mu, \zeta\right) = \lim_{\mu\to\infty} \min\left\{m\left(\zeta_\mu, \zeta_\mu\right), m(\zeta, \zeta)\right\} = m(\zeta, \zeta).$$

Using the definition of **S**, we obtain

$$\left(\zeta_\mu, \zeta_{\mu+1}\right) \in (B \times D) \cup (D \times B) \text{ for all } \mu \in \mathbb{N}. \tag{2.11}$$

Immediately, the product fashion $Z \times Z$ involves a mm-space m given as

$$m((\zeta_1, \Im_1), (\zeta_2, \Im_2)) = \frac{m(\zeta_1, \Im_1) + m(\zeta_2, \Im_2)}{2}.$$

Since (U, m) is a complete m-metric space, we obtain $(Z \times Z, m)$ is complete. Furthermore, $G \times H$ and $H \times G$ are closed in $(Z \times Z, m)$, because G and H are closed in (U, m). Letting $\mu \to \infty$ in (2.11), we have $(\zeta, \zeta) \in (B \times D) \cup (D \times B)$. This implies that $\zeta \in B \cap D$. Furthermore, from Eq (2.11), we have $\zeta_\mu \in B \cup D$. Thus, we get $\left(\zeta_\mu, \zeta\right) \in S$ (for all $\mu \in \mathbb{N}$). Therefore, our assertions hold. Furthermore, since γ is a self-mapping and from condition (*i*), we obtain for all $\zeta, \Im \in U$,

$$\begin{aligned} (\zeta, \Im) \text{ in } G \times H \text{ which implies } (\gamma(\zeta), \gamma(\Im)) &\in H \times G, \\ (\zeta, \Im) \text{ in } H \times G \text{ which implies } (\gamma(\zeta), \gamma(\Im)) &\in G \times H. \end{aligned}$$

The binary relation R is γ-closed. As $B \neq \emptyset$, there exists $\zeta_0 \in B$, such that $\gamma(\zeta_0) \in D$ that is $(\zeta_0, \gamma(\zeta_0)) \in R$. Therefore, all the hypotheses of Theorem 2.2 are satisfied. Hence, $(\gamma)_{Fix} \neq \emptyset$ and also$(\gamma)_{Fix} \subseteq B \cap D$. Finally, as $(\zeta, \Im) \in R$ for all $\zeta, \Im \in G \cap H$, $G \cap H$ is ∇-directed. Hence, the main conditions of Theorem 2.2 are satisfied, so γ has a unique fixed point. It finishes the proof.

4. Application

In this section, we illustrate how to guarantee existence of a solution of a matrix type equation. We shall use the following notations. Let $A(\mu)$ be the set of all $\mu \times \mu$ complex matrices, let $H(\mu) \subseteq A(\mu)$ be the family of all $\mu \times \mu$ Hermitian matrices,let $G(\mu) \subseteq A(\mu)$ be the set of all $\mu \times \mu$ positive definite matrices, $H^{+}(\mu) \subseteq F(\mu)$ be the set of all $\mu \times \mu$ positive semidefinite matrices. For Λ in $G(\mu)$, we will also denote $\Lambda \succ 0$. Furthermore, $\Lambda \succeq 0$ means that Λ in $H^{+}(\mu)$. As a different notation for $\Lambda - \Delta \succeq 0$ and $\Lambda - \Delta \succ 0$, we will denote $\Lambda \succeq \Delta$ and$\Lambda \succ \Delta$, respectively. Also, for each Λ, Δ in $A(\mu)$ there is a greatest lower bound and least upper bound, see [38]. In addition, take

$$\|.\| = \text{ denote the spectral norm of matrix } Q \text{ i.e } \|Q\| = \left(\lambda^{+}\left(Q^{*}Q\right)\right)^{\frac{1}{2}},$$

such that

$$\lambda^{+}\left(Q^{*}Q\right) = \text{ is the largest eigenvalue of } Q^{*}Q, \text{ where } Q^{*} \text{ is the conjugate transport of } Q.$$

We use the m-metric induced by the trace norm $\|.\|_{tr}$ given as $\|Q\|_{tr} = \sum_{i=1}^{\mu} \Xi_i(Q)$, where $\Xi_i(Q)$, $i = 1, 2, ..., \mu$ are the singular values of Q in $A(\mu)$. The set $H(\mu)$ endowed with this norm is a complete m-metric space. Moreover, we see that

$$H(\mu) = \text{is a partial ordered set with partial order } \preceq, \text{ where } \Lambda \preceq \Delta \Leftrightarrow \Lambda \succeq \Delta.$$

Consider the following nonlinear matrix equation

$$\Lambda = S + \sum_{i=1}^{\mu} Q_i^{*} \Xi(\Lambda) Q_i, \tag{4.1}$$

where ϑ is a positive definite matrix , $Q_1, Q_2, \cdots, Q_m$ are $\mu \times \mu$ matrices and Ξ is an order persevering continuous map from $H(\mu)$ to $G(\mu)$. Then, $F_R^m \in \nabla(F)$ and $(A(\mu), m)$ is a complete mm-space, where

$$m(\Lambda, \Delta) = \left\|\frac{\Lambda + \Delta}{2}\right\|_{tr} = \frac{1}{2}\left(tr(\Lambda + \Delta)\right). \tag{4.2}$$

In this section, we prove the existence of the positive definite solution to the nonlinear matrix Eq (4.1).

Theorem 4.1. *Assume that there are positive real numbers C and ξ such that:*

(i) *for each Λ, Δ in $H(\mu)$ such that (Λ, Δ) in $\preceq$ with $\sum_{i=1}^{\mu} Q_i^{*}\Xi(\Lambda) Q_i \neq \sum_{i=1}^{\mu} Q_i^{*}\Xi(\Delta) Q_i$,*

$$\left|\frac{tr(\Xi(\Lambda) + \Xi(\Delta))}{2}\right| \leq \frac{\left|\frac{tr(\Lambda+\Delta)}{2}\right|}{C\left(1 + \xi\sqrt{\frac{tr(\Lambda+\Delta)}{2}}\right)^{2}},$$

(ii) *there exists a positive number N for which $\sum_{i=1}^{\mu} Q_i Q_i^{*} < CI_{\mu}$ and $\sum_{i=1}^{\mu} Q_i^{*}\Xi(\Lambda) Q_i \succ 0$.*

Then the matrix Eq (4.1) *has a solution. Furthermore, the iteration*

$$\Lambda_{\mu} = S + \sum_{i=1}^{\mu} Q_i^{*}\Xi\left(\Lambda_{\mu-1}\right) Q_i, \tag{4.2}$$

where Λ_0 *in* $F(\mu)$ *satisfies* $\Lambda_0 \leq \vartheta + \sum_{i=1}^{\mu} Q_i^{*}\Xi\left(S_{\mu-1}\right) Q_i$, *converges in the sense of trace norm* $\|.\|_{tr}$ *to the solution of the matrix Eq* (4.1).

Proof. We define the mapping $\gamma : H(\mu) \to H(\mu)$ and $F_R^m : R^+ \to R$ by

$$\gamma(\Lambda) = S + \sum_{i=1}^{\mu} Q_i^{*}\Xi(\Lambda) Q_i, \text{ for all } \Lambda \in F(\mu),$$

and set

$$H^+(\mu)(\gamma, \preceq) = \{Q \in F(\mu) : Q \preceq \gamma(Q) \text{ or } \gamma(Q) - Q \succeq 0\}.$$

Then, γ is well defined and $\preceq$ is a relation under R, and $\preceq$ on $F(\mu)$ is γ-closed. $F_R^m(a) = -\frac{1}{\sqrt{a}}$ for all $a \in R^+$. Furthermore, a fixed point of γ is a positive solution of (4.1). Now, we want to prove that γ is a F_R^m-contraction mapping with ξ Let $(\Lambda, \Delta) \in \varpi = \{((\Lambda, \Delta) \in R : \Xi(\Lambda) \neq \Xi(\Delta))\}$ which implies that $\Lambda \prec \Delta$. Since Ξ is an order preserving mapping, we deduce that $\Xi(\Lambda) \prec \Xi(\Delta)$. We have

$$\begin{aligned}
\left\|\frac{\gamma(\Lambda) + \gamma((\Delta))}{2}\right\|_{tr} &= \frac{1}{2}\left(tr\left(\gamma(\Lambda) + \gamma(\Delta)\right)\right) \\
&= \sum_{i=1}^{m} \frac{1}{2}\left(tr\left(Q_i Q_i^{*}\left(\gamma(\Lambda) + \gamma(\Delta)\right)\right)\right) \\
&= \frac{1}{2} tr\left(\left(\sum_{i=1}^{m} Q_i Q_i^{*}\right)\gamma(\Lambda) + \gamma(\Delta)\right) \\
&\leq \left\|\sum_{i=1}^{m} E_i E_i^{*}\right\| \frac{1}{2}\left\|\gamma(\Lambda) + \gamma(\Delta)\right\|_1 \\
&\leq \frac{\left\|\sum_{i=1}^{m} E_i E_i^{*}\right\|}{C}\left(\frac{\left\|\frac{\Lambda+\Delta}{2}\right\|}{\left(1 + \xi\sqrt{\left\|\frac{\Lambda+\Delta}{2}\right\|}\right)^2}\right) \\
&< \left(\frac{\left\|\frac{\Lambda+\Delta}{2}\right\|}{\left(1 + \xi\sqrt{\left\|\frac{\Lambda+\Delta}{2}\right\|}\right)^2}\right),
\end{aligned}$$

and so

$$\frac{\left(1 + \xi\sqrt{\left\|\frac{\Lambda+\Delta}{2}\right\|_{tr}}\right)^2}{\left\|\frac{\Lambda+\Delta}{2}\right\|_{tr}} \leq \frac{1}{\left\|\frac{\gamma(\Lambda)+\gamma(\Lambda)}{2}\right\|_{tr}}.$$

This implies that

$$\left(\xi + \frac{1}{\sqrt{\left\|\frac{\Lambda+\Delta}{2}\right\|_{tr}}}\right)^2 \leq \frac{1}{\left\|\frac{\gamma(\Lambda)+\gamma(\Delta)}{2}\right\|_{tr}},$$

and then

$$\xi + \frac{1}{\sqrt{\left\|\frac{\Lambda+\Delta}{2}\right\|_{tr}}} \leq \frac{1}{\sqrt{\left\|\frac{\gamma(\Lambda)+\gamma(\Delta)}{2}\right\|_{tr}}}.$$

Consequently,

$$\xi - \frac{1}{\sqrt{\left\|\frac{\gamma(\Lambda)+\gamma(\Delta)}{2}\right\|_{tr}}} \leq -\frac{1}{\sqrt{\left\|\frac{\Lambda+\Delta}{2}\right\|_{tr}}}.$$

Now, we get

$$\xi + F_R^m\left(\left\|\frac{\gamma(\Lambda) + \gamma(\Delta)}{2}\right\|_{tr}\right) \leq F_R^m\left(\left\|\frac{\Lambda + \Delta}{2}\right\|_{tr}\right).$$

This shows that γ is a F_R^m-contraction. Using $\sum_{i=1}^{\mu} Q_i^* \Xi(\vartheta) Q_i > 0$, we deduce that $\vartheta \leq \gamma(\vartheta)$. This means that ϑ in $H^+(\gamma, \preceq)$. From Corollary 2.2, there exists $\Lambda_0 \in H(\mu)$ such that $\gamma(\Lambda_0) = \Lambda_0$. Hence, the matrix Eq (4.1) has a solution.

Example 4.1. Now, consider the matrix equation

$$\Lambda = S + \sum_{i=1}^{2} Q_i^* \Xi(\Lambda) Q_i,$$

where

$$\vartheta = \begin{pmatrix} 0.1 & 0.01 & 0.01 \\ 0.01 & 0.1 & 0.01 \\ 0.01 & 0.01 & 0.1 \end{pmatrix},$$

$$Q_1 = \begin{pmatrix} 0.2 & 0.01 & 0.01 \\ 0.01 & 0.4 & 0.01 \\ 0.01 & 0.01 & 0.4 \end{pmatrix},$$

$$Q_2 = \begin{pmatrix} 0.6 & 0.01 & 0.01 \\ 0.01 & 0.6 & 0.01 \\ 0.01 & 0.01 & 0.6 \end{pmatrix},$$

and Define $F_R^m : R^+ \to R$ by

$$F_R^m(a) = -\frac{1}{\sqrt{a}},$$

for all $a \in R^+$, and $\Xi : H(\mu) \to H(\mu)$ is given by $\Xi(\Lambda) = \frac{\Lambda}{3}$. Then, all conditions of Corollary 2.2 are satisfied for $N = \frac{6}{10}$ by using the iterative sequence

$$\Lambda_{\mu+1} = S + \sum_{i=1}^{2} Q_i^* \Xi(\Lambda) Q_i,$$

$$\Lambda_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

After some iterations, we get the approximation solution

$$\vartheta_{15} = \begin{pmatrix} 0.0233 & 0.0102 & 0.0912 \\ 0.0102 & 0.0466 & 0.0214 \\ 0.0912 & 0.0532 & 0.0326 \end{pmatrix}.$$

Hence, all the conditions of Theorem 4.1 are satisfied.

5. Conclusions

In this paper, a relation theoretic M-metric fixed point algorithm under rational type F_R^m-contractions (respectively, rational type generalized F_R^m-contractions) is proposed to solve the nonlinear matrix equation $\Lambda = S + \sum_{i=1}^{\mu} Q_i^* \Xi(\Lambda) Q_i$. Some numerical comparison experiments with existing algorithms are presented within given tables and figures. Analogously, this proposed work can be extended to generalized distance spaces, such as symmetric spaces, $m_b m$-spaces rmm-spaces, $rm_b m$-spaces, pm-spaces, $p_b m$-spaces, etc. Some problems of fixed point results could be studied in near future.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest.

References

1. S. Banach, Sur les opérations dans les ensembles abstraits et leurs applications aux équations integrales, *Fund. Math.*, **3** (1922), 133–181. https://doi.org/10.4064/fm-3-1-133-181
2. E. Ameer, H. Aydi, M. Arshad, H. Alsamir, M. S. Noorani, Hybrid multivalued type contraction mappingsin αK-complete partial b-metric Spaces and applications, *Symmetry*, **11** (2019), 86. https://doi.org/10.3390/sym11010086

3. A. Latif, R. F. Subaie, M. O. Alansari, Fixed points of generalized multi-valued contractive mappings in metric type spaces, *J. Nonlinear Var. Anal.*, **6** (2022), 123–138. https://doi.org/10.23952/jnva.6.2022.1.07

4. X. Kang, N. Fang, Some common coupled fixed point results for the mappings with a new contractive condition in a Menger PbM-metric space, *J. Nonlinear Funct. Anal.*, **2023**, (2023), 9. https://doi.org/10.23952/jnfa.2023.9

5. H. Aydi, M. Abbas, C. Vetro, Partial hausdorff metric and Nadler's fixed point theorem on partial metric spaces, *Topol. Appl.*, **159** (2012), 3234–3242. https://doi.org/10.1016/j.topol.2012.06.012

6. I. Beg, A. R. Butt, Common fixed point for generalized set valued contractions satisfying an implicit relation in partially ordered metric spaces, *Math. Commun.*, **15** (2010), 65–76.

7. A. Baklouti, M. Mabrouk, Essential numerical ranges of operators in semi-Hilbertian spaces, *Ann. Funct. Anal.*, **13** (2022), 16. https://doi.org/10.1007/s43034-021-00161-6

8. A. Alam, M. Imdad, Relation-theoretic metrical coincidence theorems, *Filomat*, **31** (2017), 4421–4439. https://doi.org/10.2298/FIL1714421A

9. M. Imdad, Q. H. Khan, W. M. Alfaqih, R. Gubrana, A relation-theoretic (F, R)-contraction principle with applications to matrix equations, *Bul. Math. Anal. Appl.*, **10** (2018), 1–12.

10. S. Reich, A. J. Zaslavski, Convergence of inexact iterates of strict contractions in metric spaces with graphs, *J. Appl. Numer. Optim.*, **4** (2022), 215–220. https://doi.org/10.23952/jano.4.2022.2.07

11. A. C. M. Ran, M. C. B. Reurings, A fixed point theorem in partially ordered sets and some applications to matrix equations, *Proc. Amer. Math. Soc.*, **132** (2004), 1435–1443.

12. J. J. Nieto, R. Rodríguez-López, Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations, *Order*, **22** (2005), 223–239. https://doi.org/10.1007/s11083-005-9018-5

13. D. Wardowski, Fixed points of a new type of contractive mappings in complete metric spaces, *Fixed Point Theory Appl.*, **2012** (2012), 94. https://doi.org/10.1186/1687-1812-2012-94

14. M. B. Zada, M. Sarwar, Common fixed point theorems for rationalF_R-contractive pairs of mappings with applications, *J. Inequal. Appl.*, **2019** (2019), 11. https://doi.org/10.1186/s13660-018-1952-z

15. S. G. Matthews, Partial metric topology, *Ann. N. Y. Acad. Sci.*, **728** (1994), 183–197. https://doi.org/10.1111/j.1749-6632.1994.tb44144.x

16. R. Jain, H. K.Nashine, Z. Kadelburg, Some fixed point results on relational quasi partial metric spaces and application to nonlinear matrix equations, *Symmetry*, **13** (2021), 993. https://doi.org/10.3390/sym13060993

17. A. Baklouti, Quadratic Hom-Lie triple systems, *J. Geom. Phys.*, **121** (2017), 166–175. https://doi.org/10.1016/j.geomphys.2017.06.013

18. C. Vetro, F. Vetro, A homotopy fixed point theorem in 0-complete partial metric space, *Filomat*, **29** (2015), 2037–2048. https://dx.doi.org/10.2298/FIL1509037V

19. N. Hussain, G. Ali, I. Iqbal, B. Samet, The existence of solutions to nonlinear matrix equations via fixed points of multivalued F-contractions, *Mathematics*, **8** (2020), 212. https://doi.org/10.3390/math8020212

20. S. Kumar, S. Luambano, On some fixed point theorems for multivalued F-contractions in partial metricspaces, *Demonstr. Math.*, **54** (2021), 151–161. https://doi.org/10.1515/dema-2021-0012

21. M. Asadi, E. Karapinar, P. Salimi, New extension of p-metric spaces with fixed points results on M-metric spaces, *J. Inequal. Appl.*, **2014** (2014), 18. https://doi.org/10.1186/1029-242X-2014-18

22. A. Ali, H. Işık, H. Aydi, E. Ameer, J. R. Lee, M. Arshad, On multivalued SU-type θ-contractions and related applications, *Open Math.*, **18** (2020), 386–399. https://doi.org/10.1515/math-2020-0139

23. A. Ali, F. Uddin, M. Arshad, M. Rashid, Hybrid fixed point results via generalized dynamic process for F-HRS type contractions with application, *Physica A*, **538** (2020), 122669. https://doi.org/10.1016/j.physa.2019.122669

24. M. Tariq, M. Arshad, E. Ameer, A. Aloqaily, S. S. Aiadi, N. Mlaik, On Relational Weak $\left(F_R^m, \eta\right)$-Contractive Mappings and Their Applicationons, *Symmetry*, **15** (2023), 922. https://doi.org/10.3390/sym15040922

25. M. Tariq, M. Abbas, A. Hussain, M. Arshad, A. Ali, H. Sulami, Fixed points of non-linear set-valued $(\alpha_* \phi_M)$-contraction mappings and related applications, *AIMS Math.*, **7** (2022) , 8861–8878. https://doi.org/10.3934/math.2022494

26. M. Tariq, E. Ameer, A. Ali, H. A. Hammad, F. Jarad, Applying fixed point techniques for obtaining a positive definite solution to nonlinear matrix equations, *AIMS Math.*, **8** (2022), 3842–3859. https://doi.org/10.3934/math.2023191

27. A. Ali, E. Ameer, S. S. Aiadi, M. Tariq, M. Arshad, N. Mlaiki, et al., New extension to fuzzy dynamic system and fuzzy fixed point results with an application, *AIMS Math.*, **8** (2023), 1208–1229. https://doi.org/10.3934/math.2023061

28. H. M. Srivastava, A. Ali, A. Hussain, M. Arshad, H. Al-Sulami, A certain class of θ_L-type non-linear operatorsand some related fixed point results, *J. Nonlinear Var. Anal.*, **6** (2022), 69–87. https://doi.org/10.23952/jnva.6.2022.1.05

29. A. Ali, A. Hussain, M. Arshad, H. A. Sulami, M. Tariq, Certain new development to the orthogonal binary relations, *Symmetry*, **14** (2022), 1954. https://doi.org/10.3390/sym14101954

30. A. Baklouti, J. Schutz, S. Dellagi, A. Chelbi, Selling or leasing used vehicles considering their energetic type, the potential demand for leasing, and the expected maintenance costs, *Energy Rep.*, **8** (2022), 1125–1135. https://doi.org/10.1016/j.egyr.2022.07.074

31. I. Altun, M. Asim, M. Imdad, W. M. Alfaqih, Fixed point results for F_R-generalized contractive mappings in parial metric space, *Math. Slovaca*, **69** (2019), 1413–1424. https://doi.org/10.1515/ms-2017-0318

32. M. Asadi, M. Azhini, E. Karapinar, H. Monfared, Simulation functions over m-metric spaces, *East. Asian Math. J.*, **33** (2017), 559–570. https://doi.org/10.7858/eamj.2017.039

33. H. Monfared, M. Azhini, M. Asadi. Fixed point results on m-metric spaces, *J. Math. Anal.*, **7** (2016), 85–101.

34. E. Karapınar, M. Abbas, S. Farooq, A discussion on the existence of best proximity points that belong to the zero set, *Axioms*, **9** (2020), 19. https://doi.org/10.3390/axioms9010019

35. I. Altun, G. Minak, H. Dag, Multivalued F-contractions on complete metric space, *J. Nonlinear Convex Anal.*, **16** (2015), 659–666.

36. A. Alam, M. Imdad, Relation-theoretic contraction principle, *J. Fixed Point Theory Appl.*, **17** (2015), 693–702. https://doi.org/10.1007/s11784-015-0247-y

37. W. A. Kirk, P. S. Srinivasan, P. Veeramani, Fixed Points for mapping satsifying cyclic contractive conditions, *Fixed Point Theor.*, **4** (2003), 79–89.

38. S. Bose, S. M. Hossein, K. Paul, Positive definite solution of a nonlinear matrix equation, *J. Fixed Point Theory Appl.*, **18** (2016), 627–643. https://doi.org/10.1007/s11784-016-0291-2

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