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Research article

Analytic solution for the lightning current induced mutually coupled resistive filament wire model

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Abstract: When a lightning current flows between the lightning entry and exit points of a structure, the lightning current density varies in different parts of the structure depending on the shape of the structure and material variance. The structure can be discretized into parallel wires, called filament wires, running parallel to the current direction. Furthermore, using the filament wire method, we can calculate the current distribution among the wires. For a structure that has a low resistance material such as aluminum, current distribution can be calculated by considering self-inductance of the wire and mutual-inductance between wires but resistance is not considered. However, in modern aircraft, composite materials are used for parts of the structure because of their strength and weight. These composite materials have high resistance compared to metal, and resistance cannot be ignored. Thus, to solve a system of ordinary differential equations for a filament model, inhomogeneous structure, aperture, and resistance of each wire must be considered to obtain the correct current distribution of each part of the structure. However, the numerical solution of the filament wire model does not reveal the region of convergence and the accuracy of the given mathematical model. It also has high time complexity. This paper presents the analytic solution and stability condition for the mutually coupled resistive filament wire model using eigenvalues of given filament wire matrix model. The stability condition is rigorously calculated and the solution is also consistent with the numerical model.

Keywords: lightning current; differential equation; filament wire model; matrix equation; systems of ODE
Mathematics Subject Classification: 15A06, 15A24

1. Introduction

Lightning is one of the most powerful natural phenomena. When lightning hits a flying object, such as an aircraft, a rotorcraft, a missile, or a rocket, the lightning current flows through the object and could damage the object. Moreover, the electric and magnetic fields induced by the lightning current, which flows through the internal ground and surface of the object, induce current and voltage in the equipment or cable inside the object. The total lightning current that flows through the object is not affected by the resistivity or inductance of the object because lightning is considered an ideal current source. However, the current waveform and/or intensity in each part of the object can be different because of the various materials and shape that make up the object. To evaluate the effect of the electromagnetic field induced by the lightning currents, determining the current distribution flowing through the structure is necessary. The formula of the standard lightning current waveforms is defined in the SAE ARP-5412 document in the form of an exponential sum [1]. There are many methods for calculating current distribution by excited lightning current, such as the numerical solution of the Laplace's equation, Finite Difference Time Domain (FDTD) Method, Finite Element Method (FEM), and the filament wire method [2–4]. However, solving the Laplace's equation, MoM, FDTD, and FEM for big objects such as aircraft requires a lot of computation time and memory [5]. Calculating the current distribution and induced voltage of wires is essential in the process of designing lightning protection. A lot of trial and error is made in the design process because installation location of the equipment and cables must be adjusted according to the induced current and voltage. Thus simulation of induced current and voltage must be accurate and fast to facilitate the design process. This fast simulation is useful for predicting and verifying results in pre-tests prior to full vehicle lightning tests and debugging. The filament wire method and its analytical solution provide a fast solution through a simplified object model. The filament wire method assumes the length of the conductor is infinite and cross section doesn't change. The actual aircraft has finite length and complex shape, so we can apply this method to each section of the aircraft and combine the results. Moreover, the lightning current waveform formula also needs to be applied by changing the naturally occurring lightning into an idealized waveform because the actual lightning waveform is very diverse, complex, and noisy. Because of these assumption, the exact model cannot be obtained. However, it is worthwhile to be able to predict the interaction between aircraft and lightning at the pre-design stage where the detailed designs are not determined and aircraft lightning test cannot be held. Although the filament wire method uses a simplified model, it considers both the shape of the object and its internal grounding structure and the materials. For conventional aircraft, we only consider self-inductance and mutual-inductance, because the structure consists of only metal. However, modern aircrafts use various composite materials. Thus, to solve a system of ordinary differential equations for a filament model, inhomogeneous structure, aperture, and resistance of each wire must be considered to obtain the correct current density of each part of the structure. To numerically solve the systems of first-order ODE of the filament wire, the time domain or the frequency domain must be sampled at desired intervals. If the excited waveform is long in the time domain or has a high frequency component in the frequency domain, sampling problems may occur. Low sampling frequency will cause aliasing, and high sampling frequency requires considerable time to calculate the whole waveform across the time domain. In particular, the induced voltage waveform is normally a short waveform with a high frequency compared to excited current waveform, rendering sampling frequency difficult to determine. However, the analytical method can provide exact solutions without sampling. The standard lightning current formula generally consists of many parts. To analyze the lightning induced effect, we chose the current component A, which is the most significant current component of a lightning stroke, as input. The approach presented in this paper is independent of input. Current component A can be expressed by the quad-exponential formula [1].

$$I = I_0\left(e^{-\alpha_{I_0}t} - e^{-\beta_{I_0}t}\right)\left(1 - e^{-\gamma_{I_0}t}\right)^2. \tag{1.1}$$

This formula implies that the initial condition of the excited lightning currents is zero. Thus, the initial conditions for the currents in each filament wire are also zero. In addition, the derivative of this kind of quadruple exponential equation is zero at time zero, thus the initial condition for the voltage is also zero. Therefore, (1.1) can be rewritten as

$$I = \sum_{j=1}^{N_I} I_j e^{-\tau_j t} \tag{1.2}$$

where the N_I is the number of terms of (1.2). The excited lightning current flows through the structure, as shown in Figure 1a. The filament wire method discretizes the structure into several wires called filament wires, composed of the same material as the structure, as shown in Figure 1b. In this paper, the filament was created for the scenario shown in Figure 1, but other scenarios can be applied depending on the actual location of the lightning strike and the geometry of the aperture attached to the aircraft. Depending on the geometry of the internal conductors where the current is induced, the current around the aperture, and the geometry of the aperture, a more dominant scenario can be applied. In some situations, a final solution can be obtained by combining the solutions by superposition. The sum of the circumference of the filaments should be equal to the cross-section length of the structure to simulate the surface current of the original structure. Thus, the radius of the filament is determined by the section length l_k, as shown in Figure 2a.

$$r_k = \frac{l_k}{2\pi}. \tag{1.3}$$

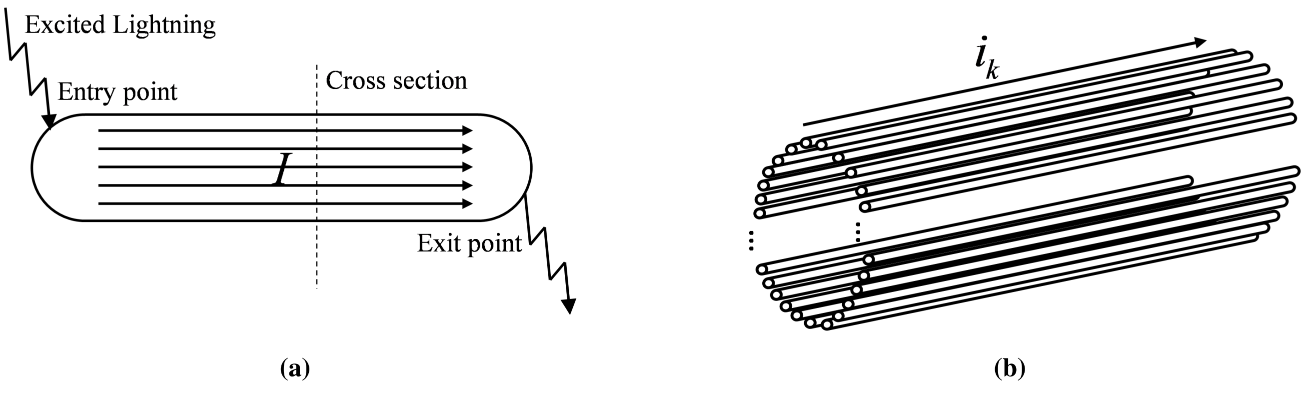

Figure 1. (a) Lightning current flowing through the structure; (b) Discretized structure by filament.

Figure 2a shows the cross section of the structure. The cross section of the structure is discretized into a finite number of sections of length l_k. Each discretized section forms a single filament wire. The discretized section need not all be the same size, nor are the each filament's radius. Each filament wire has its own resistance, self-inductance, and mutual inductance with other filaments. The mutual and self-inductance, as well as the total resistance of the filament wires connected in parallel, are all equal to the structure's resistance. The resistance of each filament can be obtained by the material resistivity ρ_k of each section and the filament wire length L [6]. Because actual aircraft has complex shape, we can divide the aircraft into several quasi-uniform sections. And combine all the solutions for each section to get the current distribution between entry and exit point. The self-inductance of each filament wire is determined only by its length and radius [7].

$$S_k = \frac{\mu_0 L}{2\pi}\left[\ln\frac{2L}{r} - 1\right] \tag{1.4}$$

where $\mu_0 = 4\pi \cdot 10^{-7}$ [H/m] is the magnetic permeability of free space. When current flow in one wire produces a magnetic flux around other wires, mutual coupling occurs between wires because of the mutual-inductance. The amount of coupling is determined by the length of the wire L, the distance between wires d, and the current direction [8]. The mutual inductance between two parallel wires is

$$M = \frac{\mu_0 L}{2\pi}\left[\ln\left(\frac{L}{d} + \sqrt{1+\frac{L^2}{d^2}}\right) - \sqrt{1+\frac{d^2}{L^2}} + \frac{d}{L}\right]. \tag{1.5}$$

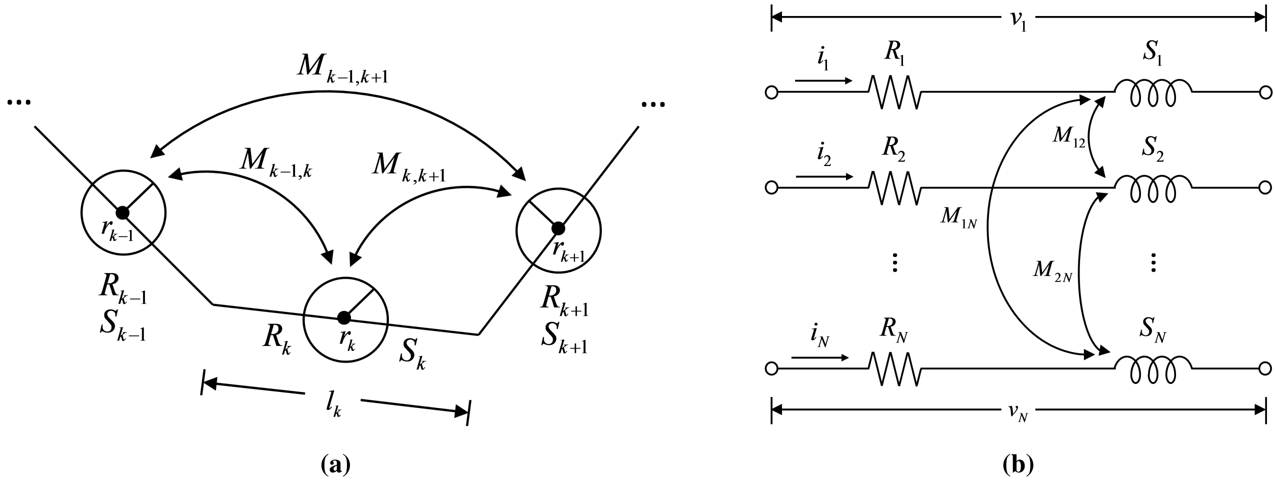

Figure 2. (a) Cross section of the filament wire bundle; (b) Circuit model of the filament wire bundle.

By principle of reciprocity, the induced flux of wire m on wire n, is equal to the induced flux of wire m on wire n. This means,

$$M_{mn} = M_{nm}. \tag{1.6}$$

For a real-world case, the mutual inductance matrix is always positive-definite [9]. However, this filament wire model simplifies the actual aircraft skin and internal structure, and is modeled as a filament wire structure that is parallel and has a finite length, so although it is very rare, a positive-definiteness can be broken. It is important not to create a filament wire model with this condition because it affects the solution and convergence. Nonetheless, even with negative eigenvalues, there may be some solutions. This is covered in more detail in Case 4.

Figure 2b shows the circuit model of the filament wire bundle. As previously mentioned, each filament wire has resistance, self-inductance, and mutual inductance. If we assume that these values do not change with frequency, based on these variables, the system of first order differential equation can be established as follows:

$$\begin{aligned}
v_1 &= R_1 i_1 + S_1 \frac{\mathrm{di}_1}{dt} + M_{12}\frac{di_2}{dt} + M_{13}\frac{di_3}{dt} + \cdots + M_{1N}\frac{di_N}{dt},\\
v_2 &= R_2 i_2 + M_{21}\frac{di_1}{dt} + S_2\frac{di_2}{dt} + M_{23}\frac{di_3}{dt} + \cdots + M_{2N}\frac{di_N}{dt},\\
&\vdots\\
v_N &= R_N i_N + M_{N1}\frac{di_1}{dt} + M_{N2}\frac{di_2}{dt} + M_{N3}\frac{di_3}{dt} + \cdots + S_N\frac{di_N}{dt}.
\end{aligned} \tag{1.7}$$

The voltage drops of each filament wire are $v_1, v_2, \cdots, v_N$. Furthermore, we assumed that the structure is infinitely long and no current flows in a direction perpendicular to the lightning current. This means the cross section of the structure is equipotential surface and all voltage drops are same as (1.8).

$$v := v_1 = v_2 = \cdots = v_N. \tag{1.8}$$

In addition, for convenience, we assumed that self-inductance notation is similar to mutual inductance.

$$M_{kk} := S_k. \tag{1.9}$$

Using matrix notation, (1.7) becomes

$$\mathbf{e}v = \begin{bmatrix} v \\ v \\ \vdots \\ v \end{bmatrix} = \begin{bmatrix} R_1 & 0 & \cdots & 0 \\ 0 & R_2 & & 0 \\ \vdots & & \ddots & 0 \\ 0 & 0 & 0 & R_N \end{bmatrix}\begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_N \end{bmatrix} + \begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1N} \\ M_{21} & M_{22} & & M_{2N} \\ \vdots & & \ddots & \\ M_{N1} & M_{N2} & & M_{NN} \end{bmatrix} \frac{d}{dt}\begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_N \end{bmatrix} \tag{1.10}$$

where $\mathbf{e}$ is a column vector of 1's. This equation can be expressed as

$$\mathbf{v} = \mathbf{Ri} + \mathbf{M}\frac{d}{dt}\mathbf{i}. \tag{1.11}$$

We note that the Matrix $\mathbf{R}$ is a real diagonal matrix and $\mathbf{M}$ is real-symmetric. Both $\mathbf{R}$ and $\mathbf{M}$ are Hermitian matrices. Theoretically, the self-inductance is greater than induced mutual inductance. That is,

$$M_{kk} > M_{kl},\ k \in \{1, 2, \cdots, N\},\ l \in \{1, 2, \cdots, N\},\ l \neq k. \tag{1.12}$$

The excited input lightning current is defined as the sum of the exponential function as in Eqs (1.1) and (1.2). Furthermore, (1.11) is a first order differential equation, where each filament current i_k can be expressed as a form of exponential function $i = C\exp(-\tau t)$. That is, (1.11) can be rewritten as follows for a specific τ:

$$\mathbf{v} = (\mathbf{R} - \tau\mathbf{M})\,\mathbf{i} \tag{1.13}$$

$$\begin{cases} \mathbf{v} &= \mathbf{v}_p &+ \mathbf{v}_h \\ \mathbf{i} &= \mathbf{i}_p &+ \mathbf{i}_h \end{cases} \tag{1.14}$$

where subscript p denotes the particular solution and subscript h denotes the homogeneous solution.

2. Solution

2.1. Particular solution

The input function has N_I exponential terms from (1.2), v_p and i_{p_k} also have N_I exponential terms. In addition, we assumed that the voltage drop across the filament wire is same, thus all elements of vector v_p are equal.

$$\begin{cases} v_p &= w_{p_1}\exp(-\tau_1 t) &+ w_{p_2}\exp(-\tau_2 t) &+ \cdots &+ w_{p_{N_I}}\exp(-\tau_{N_I} t) \\ i_{p_k} &= C_{p_{k1}}\exp(-\tau_1 t) &+ C_{p_{k2}}\exp(-\tau_2 t) &+ \cdots &+ C_{p_{kN_I}}\exp(-\tau_{N_I} t) \end{cases} \tag{2.1}$$

where, v_p is the element of the vector $\mathbf{v}_p$. Additionally, the sum of the filament current i_{p_k} is the forcing function I expressed by (1.2).

$$I = \sum_{j=1}^{N_I} I_j \exp(-\tau_j t) = \sum_{k=1}^{N} i_{p_k} = \sum_{k=1}^{N}\sum_{j=1}^{N_I} C_{p_{kj}}\exp(-\tau_j t). \tag{2.2}$$

For a specific τ_j and setting $t = 0$, we can obtain

$$I_j = \sum_{k=1}^{N} C_{p_{kj}}. \tag{2.3}$$

By contrast, for a specific τ_j and setting $t = 0$, (1.10) becomes

$$\begin{bmatrix} w_{p_j} \\ w_{p_j} \\ \vdots \\ w_{p_j} \end{bmatrix} = \left(\begin{bmatrix} R_1 & 0 & \cdots & 0 \\ 0 & R_2 & & 0 \\ \vdots & & \ddots & 0 \\ 0 & 0 & 0 & R_N \end{bmatrix} - \tau_j \begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1N} \\ M_{21} & M_{22} & & M_{2N} \\ \vdots & & \ddots & \\ M_{N1} & M_{N2} & & M_{NN} \end{bmatrix} \right) \begin{bmatrix} C_{p_{1j}} \\ C_{p_{2j}} \\ \vdots \\ C_{p_{Nj}} \end{bmatrix} = (\mathbf{R} - \tau_j\mathbf{M})\mathbf{c}_{p_j}. \tag{2.4}$$

First, we can find w_{p_j} using (2.3).

$$w_{p_j} = \frac{I_j}{\mathbf{e}^{\mathrm{T}}\left(\mathbf{R} - \tau_j\mathbf{M}\right)^{-1}\mathbf{e}} \tag{2.5}$$

where $\mathbf{e}^{\mathrm{T}}$ denotes the transpose of a column vector $\mathbf{e}$, or a row vector of 1's. Furthermore, $\mathbf{e}^{\mathrm{T}}\mathbf{A}\mathbf{e}$ denotes the sum of all elements of a matrix $\mathbf{A}$. Then,

$$\mathbf{c}_{p_j} = (\mathbf{R} - \tau_j\mathbf{M})^{-1}\mathbf{e}w_{p_j}. \tag{2.6}$$

For the case where the denominator part is infinite, or the matrix $\mathbf{R} - \tau_j\mathbf{M}$ is singular, w_{p_j} becomes zero. In this case, $c_{p_k j}$ is kernel of the $(\mathbf{R} - \tau_j\mathbf{M})^{-1}$ and can be uniquely determined using (2.3). This happens if and only if the τ_j is one of the eigenvalues of $\mathbf{RM}^{-1}$. We will discuss it in Lemma 1.

For the case where the denominator part is zero, or the sum of elements of the matrix $(\mathbf{R} - \tau_j\mathbf{M})^{-1}$ is zero, setting the filament currents as $i_{p_j} = c_{p_j}\exp(-\tau_j t) + c'_{p_j} t\exp(-\tau_j t)$ for a corresponding τ_j can solve the case.

2.2. Homogeneous solution

To find a homogeneous solution, the forcing function should be zero and only time dependent.

$$I = \sum_{k=1}^{N} \mathbf{i}_{h_k} = 0 \tag{2.7}$$

$$\begin{cases} v_h = w_{h_1}\exp(-\tau_{h_1}t) + w_{h_2}\exp(-\tau_{h_2}t) + \cdots + w_{h_{(N-1)}}\exp(-\tau_{h_{(N-1)}}t) \\ i_{h_k} = C_{h_{k1}}\exp(-\tau_{h_1}t) + C_{h_{k2}}\exp(-\tau_{h_2}t) + \cdots + C_{h_{k(N-1)}}\exp(-\tau_{h_{(N-1)}}t) \end{cases}. \tag{2.8}$$

In this condition, for a specific τ_{h_j}, (2.4) can be rewritten as follows:

$$\left(\mathbf{R} - \tau_{h_j}\mathbf{M}\right)^{-1} w_{h_j} = \begin{bmatrix} C_{h_{1j}} \\ C_{h_{2j}} \\ \vdots \\ C_{h_{Nj}} \end{bmatrix} = \mathbf{c}_{h_j}. \tag{2.9}$$

Summing every element of (2.9), owing to the homogeneous condition (2.7), the right side becomes zero. To sum the matrix elements, we multiply the left and right side with $\mathbf{e}^{\mathrm{T}}$ and $\mathbf{e}$ respectively, and (2.9) becomes,

$$\mathbf{e}^{\mathrm{T}}\left(\mathbf{R} - \tau_{h_j}\mathbf{M}\right)^{-1}\mathbf{e}w_{h_j} = 0. \tag{2.10}$$

For the case where w_{h_j} is nonzero, and assuming $\tau_{h_j} = \tau_h$,

$$f(\tau_h) := \mathbf{e}^{\mathrm{T}}(\mathbf{R} - \tau_h\mathbf{M})^{-1}\mathbf{e}, \tag{2.11}$$

$$\mathbf{A}(\tau_h) := (\mathbf{R} - \tau_h\mathbf{M}). \tag{2.12}$$

Summing all the elements of the matrix $(\mathbf{R} - \tau_h\mathbf{M})^{-1}$ to find τ_h, we obtain the sum of the rational function of the τ_h. The matrix $(\mathbf{R}-\tau_h\mathbf{M})$ is the first order function of τ_h. To find the inverse of $(\mathbf{R}-\tau_h\mathbf{M})$, we can brutally inverse the matrix. However, the computation complexity of the polynomial matrix increases by more than a cube even for a first-order polynomial matrix as N increases [10, 11]. To reduce the computation complexity, we will use the eigenvalue. The resistivity matrix $\mathbf{R}$ is a diagonal matrix (therefore also symmetric) and consists of only real-positive values. Therefore, it is a positive-definite matrix. The mutual matrix $\mathbf{M}$ is a real-positive symmetric matrix or a Hermitian matrix, in which each diagonal entry is the largest value of each row by (1.12). Therefore, in most real-world cases, $\mathbf{M}$ is invertible. We assumed that $\mathbf{M}$ is an invertible matrix. The positive-definiteness or the sign of eigenvalues of the matrix $\mathbf{M}$ affects the stability of the entire solution of this problem. To find the homogeneous solution of (1.11) with (2.7), let

$$\mathbf{A}(\tau_h) = (\mathbf{R} - \tau_h \mathbf{M}) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{bmatrix}, \tag{2.13}$$

$$\mathbf{B}(\tau_h) := \mathbf{A} - \mathbf{e}\mathbf{A}_{k,*} + \alpha \mathbf{e}_k \mathbf{e}^{\mathrm{T}} = \begin{bmatrix} a_{11} - a_{k1} & a_{12} - a_{k2} & \cdots & a_{1N} - a_{kN} \\ \vdots & \vdots & & \vdots \\ \alpha & \alpha & \ddots & \alpha \\ \vdots & \vdots & & \vdots \\ a_{N1} - a_{k1} & a_{N2} - a_{k2} & \cdots & a_{NN} - a_{kN} \end{bmatrix}, \ (\alpha \neq 0) \tag{2.14}$$

where $\mathbf{A}_{k,*}$ denotes the row vector of the k-th row of $\mathbf{A}$, $\mathbf{e}_k$ denotes a column vector of 0's except in which k-th row is one. Because the left side of (1.11) is the column vector with same values (voltage), if we choose any k-th row of the matrix $\mathbf{A}$, and subtract it to every row of matrix $\mathbf{A}$, the left side becomes a zero vector. Additionally, to meet the homogeneous condition (2.7), we need to insert a nonzero constant row vector to a k-th row of a matrix $\mathbf{B}$. However, the left side remains zero.

$$\mathbf{B} \begin{bmatrix} i_{h_1} \\ \vdots \\ i_{h_N} \end{bmatrix} = \mathbf{B}\mathbf{i}_h = 0. \tag{2.15}$$

Equation (2.15) contains not only the voltage condition (1.8), but also the homogeneous condition or the forcing function condition (2.7). Therefore, for this system to have a solution, (2.15) must have a solution. In summary,

$$\dim(\ker(\mathbf{B})) > 0 \tag{2.16}$$

or,

$$\det(\mathbf{B}) = 0. \tag{2.17}$$

Lemma 1. *The determinant of* $\mathbf{A} - \mathbf{e}\mathbf{A}_{k,*} + \alpha \mathbf{e}_k \mathbf{e}^{\mathrm{T}}$ *is* $\alpha \mathbf{e}^{\mathrm{T}} \mathrm{adj}(\mathbf{A})\mathbf{e}$, $(\alpha \neq 0)$.

Proof. Using matrix determinant lemma [12],

$$\begin{aligned} \det(\mathbf{B}) &= \det(\mathbf{A} - \mathbf{e}\mathbf{A}_{k,*} + \alpha \mathbf{e}_k \mathbf{e}^{\mathrm{T}}) \\ &= \det(1 - \mathbf{A}_{k,*}(\mathbf{A} + \alpha \mathbf{e}_k \mathbf{e}^{\mathrm{T}})^{-1}\mathbf{e})\det(\mathbf{A} + \alpha \mathbf{e}_k \mathbf{e}^{\mathrm{T}}). \end{aligned}$$

Using Sherman–Morrison formula [13],

$$= \left(1 - \mathbf{A}_{k,*}\left(\mathbf{A}^{-1} - \frac{\alpha\mathbf{A}^{-1}\mathbf{e}_k\mathbf{e}^{\mathrm{T}}\mathbf{A}^{-1}}{1+\alpha\mathbf{e}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{e}_k}\right)\mathbf{e}\right)(1+\alpha\mathbf{e}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{e})\det(\mathbf{A})$$
$$= \left(\left(1+\alpha\mathbf{e}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{e}\right) - \mathbf{A}_{k,*}\mathbf{A}^{-1}\left(1+\alpha\mathbf{e}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{e}\right)\mathbf{e} + \mathbf{A}_{k,*}\left(\alpha\mathbf{A}^{-1}\mathbf{e}_k\mathbf{e}^{T}\mathbf{A}^{-1}\right)\mathbf{e}\right)\det(\mathbf{A}).$$

To simplify the aforementioned formula, using $\mathbf{A}_{k,*} = \mathbf{e}_k^{\mathrm{T}}\mathbf{A}$ and other characteristics,

$$\begin{cases} \mathbf{A}_{k,*}\mathbf{A}^{-1} & = & \mathbf{e}_k^{\mathrm{T}} \\ \mathbf{e}_k^{\mathrm{T}}\mathbf{e} & = & 1 \\ \mathbf{e}_k^{\mathrm{T}}\mathbf{e}_k & = & 1 \end{cases}, \tag{2.18}$$

$$\begin{aligned} \det(\mathbf{B}) &= \alpha\mathbf{A}_{k,*}\mathbf{A}^{-1}\mathbf{e}_k\mathbf{e}^{T}\mathbf{A}^{-1}\mathbf{e}\det(\mathbf{A}) \\ &= \alpha\mathbf{e}_k^{\mathrm{T}}\mathbf{e}_k\mathbf{e}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{e}\det(\mathbf{A}) \\ &= \alpha\mathbf{e}^{T}\mathbf{A}^{-1}\mathbf{e}\det(\mathbf{A}) && (2.19) \\ &= \alpha\mathbf{e}^{T}\operatorname{adj}(\mathbf{A})\,\mathbf{e}. && (2.20) \end{aligned}$$

By (2.19), for an invertible matrix $\mathbf{A}$, det($\mathbf{B}$) is the product of constant α, sum of elements of inverse of $\mathbf{A}$, and det($\mathbf{A}$). For a singular matrix $\mathbf{A}$, by (2.20), det($\mathbf{B}$) is the sum of the adjugate of matrix $\mathbf{A}$ multiplied by nonzero constant α. □

Thus, for this system to have a solution, either the sum of the inverse of $\mathbf{R} - \tau_h\mathbf{M}$ should be zero, which is a rediscovery of (2.11); or det($\mathbf{R} - \tau_h\mathbf{M}$) should be zero, which can be useful for the special cases as in (2.5) or the repeated eigenvalue condition.

The method of finding the solution depends on the sign, eigenvectors, and repeatability.

Case 1: The eigenvalues of the matrix $\mathbf{RM}^{-1}$ are all positive and distinctive, $\xi_k \neq 0$.

Because the eigenvalues of the real-symmetric matrix $\mathbf{M}$ are all positive, the matrix $\mathbf{M}$ is positive-definite matrix. Using eigenvalue decomposition, $\mathbf{RM}^{-1}$ can be factorized as

$$\mathbf{QDQ}^{-1} = \mathbf{RM}^{-1} \tag{2.21}$$

where $\mathbf{D}$ is the diagonal matrix and diagonal entries are the eigenvalues of $\mathbf{RM}^{-1}$, with $D_{ii} = \lambda_i$. For convenience, let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$. In addition, $\mathbf{Q}$ is the corresponding eigenvector matrix. Then, (2.11) becomes

$$f(\tau_h) = \mathbf{e}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{Q}(\mathbf{D} - \tau_h\mathbf{I})^{-1}\mathbf{Q}^{-1}\mathbf{e}, \tag{2.22}$$

$$\mathbf{A}^{-1} = \mathbf{M}^{-1}\mathbf{Q}(\mathbf{D} - \tau_h\mathbf{I})^{-1}\mathbf{Q}^{-1}. \tag{2.23}$$

Here, both $\mathbf{D}$ and $\tau_h\mathbf{I}$ are the diagonal matrix; therefore they can easily be inverted compared to the original (2.11). Thus, (2.13) can be expressed as

$$f(\tau_h) = \sum_{k=1}^{N}\frac{\xi_k}{\lambda_k - \tau_h} \tag{2.24}$$

where ξ_k is the product of the k-th column sum of $\mathbf{M}^{-1}\mathbf{Q}$ and the k-th row sum of $\mathbf{Q}^{-1}$. Equation (2.24) can be solved directly by multiplying each denominator to obtain N-th order polynomials. However, in the real-world problem, not only are ξ_k and λ_k usually small, but the eigenvalues λ_k might be close to each other. Moreover, multiplying every denominator to form a polynomial function from rational functions causes substantial difference between the coefficient of the highest order term and the constant term. When N becomes hundreds or more, the constant term might not be expressed by ordinary floating-point arithmetic. This causes precision problems when solving higher order polynomial, such as that of hundreds order or more. To solve (2.24) for the positive-definite matrix $\mathbf{M}$, it is essential to know its properties. Because the matrix $\mathbf{R}$ is positive-definite,

$$f(0) = \mathbf{e}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{e} > 0, \tag{2.25}$$

$$\sum_{k=1}^{N} \xi_k = \mathbf{e}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{Q}\cdot\mathbf{Q}^{-1}\mathbf{e} = \mathbf{e}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{e}, \tag{2.26}$$

$$\begin{aligned}\frac{df}{d\tau_h} &= \mathbf{e}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{Q}(\mathbf{D}-\tau_h\mathbf{I})^{-2}\mathbf{Q}^{-1}\mathbf{e}\\ &= \mathbf{e}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{Q}(\mathbf{D}-\tau_h\mathbf{I})^{-1}\mathbf{Q}^{-1}\mathbf{M}\mathbf{M}^{-1}\mathbf{Q}(\mathbf{D}-\tau_h\mathbf{I})^{-1}\mathbf{Q}^{-1}\mathbf{e}\\ &= (\mathbf{A}^{-1}\mathbf{e})^{\mathrm{T}}\mathbf{M}(\mathbf{A}^{-1}\mathbf{e})\\ &= \sum_{k=1}^{N}\frac{\xi_k}{(\lambda_k-\tau_h)^2}.\end{aligned} \tag{2.27}$$

Therefore,

$$\frac{df}{d\tau_h} > 0, \tau_h \neq \lambda_k, \quad k \in \{1, 2, \cdots, N\} \tag{2.28}$$

$$\xi_k > 0, \qquad k \in \{1, 2, \cdots, N\} \tag{2.29}$$

$$\lim_{\tau_h \to \lambda_k^{\mp}} f(\tau_h) = \pm\infty, \quad k \in \{1, 2, \cdots, N\} \tag{2.30}$$

$$\lim_{\tau_h \to \pm\infty} f(\tau_h) = 0^{\mp}. \tag{2.31}$$

The graph of the function $f(\tau_h)$ is shown in Figure 3. The roots of the function $f(\tau_h)$ lie between the eigenvalues of $\mathbf{RM^{-1}}$. The eigenvalues are asymptotes. According to Eqs (2.25), (2.28), and (2.31), no negative root exists and no root greater than the largest eigenvalue for this case exist. Considering the graph between eigenvalues, the graph is continuous on an interval $(\lambda_k, \lambda_{k+1})$, $1 \leq k < N$ and monotonically increasing from $-\infty$ to $+\infty$; therefore, only one root between adjacent eigenvalue exists. In this case, it is good to use the bisection method to find the root rapidly and with desired accuracy [14], as follows:

Step 1 Set $X_{min} = \lambda_k, X_{max} = \lambda_{k+1}$.

Step 2 Calculate $c = (X_{min} + X_{max})/2, f(c)$ and check if convergence is satisfactory.

Step 3-1 If not, and $f(c) < 0$, set $X_{min} = c$ and go to Step 2.

Step 3-2 If not, and $f(c) > 0$, set $X_{max} = c$ and go to Step 2.

Step 3-3 If convergence is satisfactory, c is a root between $(\lambda_k, \lambda_{k+1})$.

In this case, if all eigenvalues are positive and distinctive, and if the numerator part of (2.24) is $\xi_k \neq 0$, $N-1$ roots exist.

Case 2: The eigenvalues of the matrix $\mathbf{RM}^{-1}$ are all positive and distinctive, and some $\xi_k = 0$.

Let $\xi_m = 0, m \in \{1, 2, \cdots, N\}$. In this case, the m-th term of (2.24) vanishes, resulting in less than N asymptotic curves on Figure 3, despite the presence of N eigenvalues. Therefore, it is not possible to find all solutions using the bisection method. To find a root that satisfies (2.16), from (2.24) with $\xi_m = 0, m \in \{1, 2, \cdots, N\}$,

$$f(\tau_h) = \sum_{k=1}^{N} \frac{\xi_k}{\lambda_k - \tau_h} = \frac{\sum\limits_{j=1}^{N} \xi_j \prod\limits_{k=1,\, k\neq j}^{N} (\lambda_k - \tau_h)}{\prod\limits_{k=1}^{N} (\lambda_k - \tau_h)} = \frac{\sum\limits_{j=1}^{N} \xi_j \prod\limits_{k=1,\, k\neq j}^{N} (\lambda_k - \tau_h)}{\det(\mathbf{D} - \tau_h \mathbf{I})}. \tag{2.32}$$

In addition,

$$\begin{aligned} f(\tau_h) = \mathbf{e}^{\mathrm{T}}(\mathbf{R} - \tau_h\mathbf{M})^{-1}\mathbf{e} &= \frac{\mathbf{e}^{\mathrm{T}}\mathrm{adj}\,(\mathbf{R} - \tau_h\mathbf{M})\,\mathbf{e}}{\det(\mathbf{R} - \tau_h\mathbf{M})} \\ &= \frac{\mathbf{e}^{\mathrm{T}}\mathrm{adj}\,(\mathbf{R} - \tau_h\mathbf{M})\,\mathbf{e}}{\det(\mathbf{Q})\det(\mathbf{D} - \tau_h\mathbf{I})\det(\mathbf{Q}^{-1})\det(\mathbf{M})} \\ &= \frac{\mathbf{e}^{\mathrm{T}}\mathrm{adj}\,(\mathbf{R} - \tau_h\mathbf{M})\,\mathbf{e}}{\det(\mathbf{D} - \tau_h\mathbf{I})\det(\mathbf{M})}. \end{aligned} \tag{2.33}$$

Substituting (2.32) into (2.33),

$$\mathbf{e}^{\mathrm{T}}\mathrm{adj}\,(\mathbf{R} - \tau_h\mathbf{M})\,\mathbf{e} = \det(\mathbf{M}) \sum_{j=1}^{N} \xi_j \prod_{k=1,\, k\neq j}^{N} (\lambda_k - \tau_h). \tag{2.34}$$

The matrix $\mathbf{M}$ is invertible; therefore, det($\mathbf{M}$) $\neq 0$. Assuming $\tau_h = \lambda_m$,

$$\mathbf{e}^{\mathrm{T}}\mathrm{adj}\,(\mathbf{R} - \lambda_m\mathbf{M})\,\mathbf{e} = \mathbf{e}^{\mathrm{T}}\mathrm{adj}\,(\mathbf{A}\,(\lambda_m))\,\mathbf{e} = 0 \;\text{ for }\; \xi_m = 0. \tag{2.35}$$

From (2.20), $f(\lambda_m)$ satisfies (2.17). Thus, λ_m is one of the solutions for $\xi_m = 0$. The rest of the solutions can be found with the bisection method used in Case 1.

Case 3: The eigenvalues of the matrix $\mathbf{RM}^{-1}$ are all positive and some eigenvalues are repeated multiple times.

We assumed that only two eigenvalues are repeated: $\lambda_m = \lambda_{m+1}, 1 \leq m < N$. In this case, the m-th and $(m+1)$-th terms of (2.24) are merged; thus, less than N asymptotic curves are present on Figure 3, despite the presence of N eigenvalues. Therefore, it is not possible to find all solutions using the bisection method. Repeated eigenvalues can occur in a physically symmetric system. Let, $\tau_h = \lambda_m$. Then,

$$\text{rank}\,(\mathbf{A}\,(\lambda_m)) \leq N-2. \tag{2.36}$$

Inequality holds for eigenvalues repeated more than two times. From (2.36), $\text{adj}(\mathbf{A}(\lambda_m)) = 0$ [15]. Thus, λ_m is one of the solutions for the repeated λ_m. If λ_m is repeated p, $1 \leq p \leq N$ times, by rank-nullity theorem [16, 17],

$$\dim\,(\ker\,(\mathbf{A}\,(\lambda_m))) = p, \tag{2.37}$$

$$\text{rank}\,(\mathbf{A}\,(\lambda_m)) = N-p. \tag{2.38}$$

The remaining solutions can be found with the bisection method used in Case 1. In particular, the last root τ_{N-1} can be obtained by Vieta's formula.

From (2.32), to find $f(\tau_h) = 0$,

$$\begin{aligned} f(\tau_h) &= \sum_{j=1}^{N} \xi_j \prod_{k=1,\,k\neq j}^{N} (\lambda_k - \tau_h) \\ &= \left((-1)^N \sum_{j=1}^{N} \xi_j\right)\tau_h^N + \left((-1)^{N-1} \sum_{j=1}^{N}\left(\xi_j \sum_{k=1}^{N} \lambda_k - \xi_j\lambda_j\right)\right)\tau_h^{N-1} \\ &\quad + \cdots + \sum_{j=1}^{N} \frac{\xi_j}{\lambda_j} \prod_{k=1}^{N} \lambda_k, (\tau_h \neq \lambda_k, k \in \{1, 2, \cdots, N\}). \end{aligned} \tag{2.39}$$

The condition $\tau_h \neq \lambda_k$ is equivalent to $\det(\mathbf{D} - \tau_h\mathbf{I}) \neq 0$.

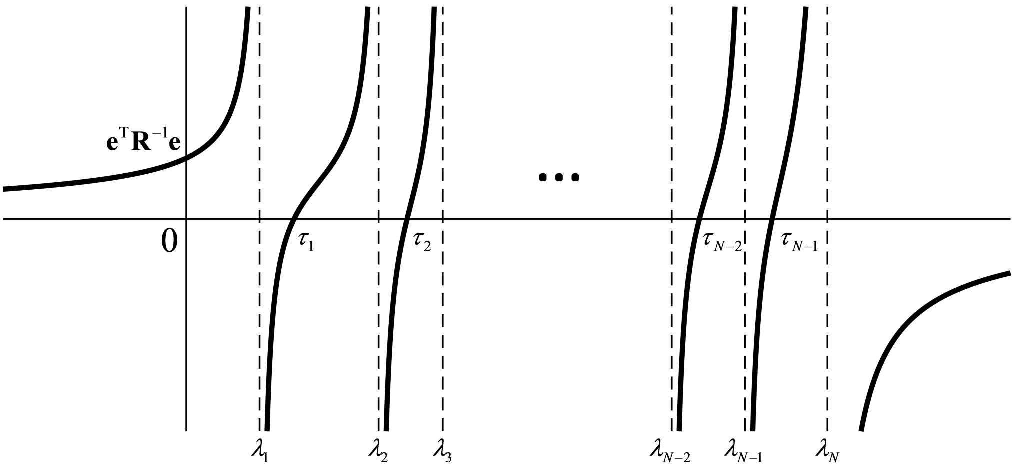

Figure 3. Graph of the function $f(\tau_h)$ with positive-definite matrix $\mathbf{M}$.

Lemma 2. *Let* $\mathbf{QDQ}^{-1} = \mathbf{RM}^{-1}$, *where* $\mathbf{D}$ *is the diagonal matrix in which diagonal entries are the eigenvalues of* $\mathbf{RM}^{-1}$ *and* $\mathbf{Q}$ *is the corresponding eigenvector matrix. If* ξ_k *is the product of the k-th column sum of* $\mathbf{M}^{-1}\mathbf{Q}$ *and the k-th row sum of* $\mathbf{Q}^{-1}$ *, then* $\xi_1\lambda_1 + \xi_2\lambda_2 + \cdots + \xi_N\lambda_N = \mathbf{e}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{RM}^{-1}\mathbf{e}$.

Proof. From (2.21),

$$\mathbf{Q}\mathbf{D}^{-1}\mathbf{Q}^{-1} = \mathbf{M}\mathbf{R}^{-1}. \tag{2.40}$$

Then, (2.22) and (2.24) becomes

$$\sum_{k=1}^{N} \frac{\xi_k}{1/\lambda_k - \tau_h} = \mathbf{e}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{Q}\Big(\mathbf{D}^{-1} - \tau_h\mathbf{I}\Big)^{-1}\mathbf{Q}^{-1}\mathbf{e}. \tag{2.41}$$

Letting $\tau_h = 0$,

$$\sum_{k=1}^{N} \xi_k\lambda_k = \mathbf{e}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{Q}\mathbf{D}\mathbf{Q}^{-1}\mathbf{e} = \mathbf{e}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{R}\mathbf{M}^{-1}\mathbf{e}. \tag{2.42}$$

□

Then, (2.39) can be simplified as

$$\begin{aligned} f(\tau_h) =& (-1)^N\mathbf{e}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{e}\tau_h^N + (-1)^{N-1}\Big(\mathrm{tr}(\mathbf{D})\mathbf{e}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{e} - \mathbf{e}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{R}\mathbf{M}^{-1}\mathbf{e}\Big)\tau_h^{N-1} \\ &+ \cdots + \mathbf{e}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{e}\det(\mathbf{D}), \qquad (\tau_h \neq \lambda_k,\ k \in \{1, 2, \cdots, N\}). \end{aligned} \tag{2.43}$$

Using Vieta's sum of roots formula,

$$\begin{aligned} \tau_{N-1} &= \frac{\mathbf{e}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{e}\,\mathrm{tr}\,(\mathbf{D}) - \mathbf{e}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{R}\mathbf{M}^{-1}\mathbf{e}}{\mathbf{e}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{e}} - \sum_{k=1}^{N-2}\tau_{N-1} && (2.44) \\ &= \mathrm{tr}\,(\mathbf{D}) - \frac{\mathbf{e}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{R}\mathbf{M}^{-1}\mathbf{e}}{\mathbf{e}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{e}} - \sum_{k=1}^{N-2}\tau_{N-1} && (2.45) \end{aligned}$$

where tr($\mathbf{D}$) is the trace of the matrix $\mathbf{D}$ or the sum of eigenvalues of $\mathbf{R}\mathbf{M}^{-1}$. We note that many alternative forms for this expression exist. In addition, using Vieta's product of roots formula,

$$\tau_{N-1} = \frac{\mathbf{e}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{e}\prod_{k=1}^{N}\lambda_k}{\mathbf{e}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{e}\prod_{k=1}^{N-2}\tau_k}. \tag{2.46}$$

Equation (2.46) seems neater than (2.45) and can be calculated by known values rather than matrix multiplications. However, for a large N, multiplying all the eigenvalues or the roots might cause a floating point precision problem.

Case 4: Some eigenvalues are negative.

Let $\lambda_1, \lambda_2, \cdots, \lambda_m$, $(m \leq N)$ be negative eigenvalues. As we described in Case 1, with more than one distinctive eigenvalues with $\xi_k \neq 0, (1 \leq k \leq m)$, roots exist between the eigenvalues. If $\xi_k = 0$, as shown in Case 2, a negative eigenvalue itself is a root. If one or more than one of the eigenvalues is repeated, as shown in Case 3, a repeated negative eigenvalue itself is a root. For some special case where only one eigenvalue is negative and $\mathbf{e}^{\mathrm{T}}\mathbf{M}\mathbf{e}>0$, a stable solution with all positive exponents can be obtained by solving the solution presented in this paper. However, as mentioned in introduction, this is a case where the model is incorrect because it is physically impossible.

If all roots $\tau_1, \tau_2, \cdots, \tau_m$, ($m \le N-1$) are found, according to (2.15), the coefficient of current for each root can be obtained by finding the kernel of $\mathbf{B}(\tau_h)$ for each root $\tau_h = \tau_1, \tau_2, \cdots, \tau_m$. If there is no repeated eigenvalue, $N-1$ roots are present because $\dim(\ker(\mathbf{B}(\tau_h))) = 1$ for each root. If there is an eigenvalue that repeats twice, $N-1$ roots are present. If there is an eigenvalue that repeats more than twice, there are less than $N-1$ roots. For the root $\tau_k,\ k \in \{1, 2, \cdots, m\}$.

$$\ker(\mathbf{B}(\tau_k)) = H_k \begin{bmatrix} C_{h_{1k}} \\ \vdots \\ C_{h_{Nk}} \end{bmatrix} \tag{2.47}$$

where H_k is a scaling factor for the kernel.

$$i_{h_k} = \sum_{j=1}^{m} C_{h_{kj}} \exp\left(-\tau_j t\right). \tag{2.48}$$

We note that for the homogeneous conditions (2.7) and (2.15), the column sum of (2.47) is zero. That is, the sum of elements of the kernel is zero. By the initial condition at $t = 0$, and using (2.1) through (2.6),

$$i_k(0) = i_{p_k}(0) + i_{h_k}(0) = \sum_{j=1}^{N_I} C_{p_{kj}} + \sum_{j=1}^{m} C_{h_{kj}} = 0. \tag{2.49}$$

Using matrix form,

$$-\begin{bmatrix} \sum\limits_{j=1}^{N_I} C_{p_{1j}} \\ \vdots \\ \sum\limits_{j=1}^{N_I} C_{p_{Nj}} \end{bmatrix} = \begin{bmatrix} C_{h_{11}} & C_{h_{12}} & \cdots & C_{h_{1m}} \\ C_{h_{21}} & C_{h_{22}} & & \vdots \\ \vdots & & \ddots & \\ C_{h_{N1}} & \cdots & & C_{h_{Nm}} \end{bmatrix} \begin{bmatrix} H_1 \\ H_2 \\ \vdots \\ H_m \end{bmatrix}. \tag{2.50}$$

The coefficient matrix on the right side is N-by-m matrix. If there are no eigenvalues repeated three or more times, it would be N-by-($N-1$) matrix. We previously calculated the left side of (2.50) in Section 2.1. In addition, the coefficient matrix on the right side can be calculated using (2.47). The unknown variable is the scaling factor of the kernel. Although $m(m \le N-1)$ unknowns and N equations are present, the coefficient for the kernels can be easily calculated. Finally, we calculate the voltage as in (2.4).

$$\begin{bmatrix} w_{h_j} \\ w_{h_j} \\ \vdots \\ w_{h_j} \end{bmatrix} = \left(\begin{bmatrix} R_1 & 0 & \cdots & 0 \\ 0 & R_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & R_N \end{bmatrix} - \tau_j \begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1N} \\ M_{21} & M_{22} & & \vdots \\ \vdots & & \ddots & \\ M_{N1} & \cdots & & M_{NN} \end{bmatrix} \right) \begin{bmatrix} H_1 C_{h_{1j}} \\ H_2 C_{h_{2j}} \\ \vdots \\ H_N C_{h_{Nj}} \end{bmatrix}. \tag{2.51}$$

3. Stability

For the system to remain stable, every root must be positive. Otherwise, the system diverges because the roots are the exponential part of the current or the voltage. Although we show from (2.15) that the kernel exists for all roots, some cases may exist where the exponential part diverges and the coefficient is zero. Therefore, no zero column kernel exists. The solution has been investigated through various cases. If the mutual inductance matrix $\mathbf{M}$ is not positive-definite, the system is unstable but it is physically impossible for a correct model.

Lemma 3. *For a positive-definite Hermitian matrix* $\mathbf{R}$ *and Hermitian matrix* $\mathbf{M}$*, the eigenvalues of* $\mathbf{RM}^{-1}$ *and* $\mathbf{M}$ *have the same sign.*

Proof. Let

$$\mathbf{R}^{-1} = \mathbf{C}^*\mathbf{C}. \tag{3.1}$$

In addition, let

$$\left(\mathbf{C}^{-1}\right)^*\mathbf{M}^{-1}\mathbf{C}^{-1} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^* \tag{3.2}$$

where Λ is the eigenvalue of matrix $(\mathbf{C}^{-1})^*\mathbf{M}^{-1}\mathbf{C}^{-1}$. Because $\mathbf{M}$ and $\mathbf{M}^{-1}$ are symmetric, the left side of (3.2) is symmetric. Thus, $\mathbf{U}^* = \mathbf{U}^{-1}$; then,

$$\mathbf{M}^{-1} = \mathbf{C}^*\mathbf{U}\mathbf{\Lambda}\mathbf{U}^*\mathbf{C}. \tag{3.3}$$

Then, (2.21) becomes

$$\mathbf{RM}^{-1} = (\mathbf{C}^*\mathbf{C})^{-1}\mathbf{C}^*\mathbf{U}\mathbf{\Lambda}\mathbf{U}^*\mathbf{C} = \left(\mathbf{C}^{-1}\mathbf{U}\right)\mathbf{\Lambda}\left(\mathbf{C}^{-1}\mathbf{U}\right)^{-1} = \mathbf{QDQ}^{-1}. \tag{3.4}$$

This implies that the eigenvalues of $(\mathbf{C}^{-1})^*\mathbf{M}^{-1}\mathbf{C}^{-1}$ are the same as the eigenvalues of matrix $\mathbf{RM}^{-1}$. In summary,

$$\mathbf{Q} = \mathbf{C}^{-1}\mathbf{U}, \tag{3.5}$$

$$\mathbf{D} = \mathbf{\Lambda}. \tag{3.6}$$

Furthermore, let

$$\mathbf{M}^{-1} = \mathbf{ZXZ}^* \tag{3.7}$$

where $\mathbf{X}$ is diagonal and the eigenvalue matrix of $\mathbf{M}^{-1}$ and $\mathbf{Z}$ is the corresponding eigenvector matrix. Then,

$$\mathbf{X} = \mathbf{Z}^{-1}\mathbf{C}^*\mathbf{UDU}^*\mathbf{C}\left(\mathbf{Z}^{-1}\right)^* = \left(\mathbf{Z}^{-1}\mathbf{C}^*\mathbf{U}\right)\mathbf{D}\left(\mathbf{Z}^{-1}\mathbf{C}^*\mathbf{U}\right)^*. \tag{3.8}$$

Because both $\mathbf{X}$ and $\mathbf{D}$ are diagonal matrices, the eigenvalues of matrix $\mathbf{M}$ and the eigenvalues of matrix $\mathbf{RM}^{-1}$ have the same sign. □

Here, we investigated the stability using the eigenvalues of the matrix $\mathbf{RM}^{-1}$. However, by Lemma 3, the stability depends on only the matrix $\mathbf{M}$, which is the mutual inductance matrix. This also implies if the mutual inductance matrix $\mathbf{M}$ is positive-definite, all eigenvalues of $\mathbf{RM}^{-1}$ are positive. This appears logical because as long as the resistance is positive, only the mutual inductance terms multiplied by the derivatives of the filament current have possibilities of divergence.

4. Example

Figure 4 shows the analytically and numerically calculated result. The analytic results followed the methods and procedures presented in this paper. The numerical results are obtained by solving the following discrete equations.

$$\mathbf{v}_{n+1} = \mathbf{R}\mathbf{i}_n + \mathbf{M}\frac{\mathbf{i}_{n+1} - \mathbf{i}_n}{(\text{sampling period})} \tag{4.1}$$

where the subscript n denotes each sample, not the number of filaments. Both $\mathbf{i}$ and $\mathbf{v}$ have an initial condition of 0 at t= 0, and sampling period was set to 5 ns for convenience. This discrete equation was solved using the conditions that the value of $\mathbf{v}$ is the same for all filaments at the same sample n and that the sum of the current $\mathbf{i}$ of filaments is the same as the excited waveform (1.1).

The red color in Figure 4 denotes the carbon fiber composite (CFC) material, which has thousands times more resistance than typical aluminum. The energy of the excited lightning waveform in this paper is mostly concentrated in the low frequency band below 100 kHz [1]. Therefore, this paper assumes that the conductivity change with frequency characteristics of carbon fiber is negligible. The resistivity of the CFC is $\rho_{\mathrm{CFC}} \approx 2.65 \times 10^{-5}\ \Omega \cdot \mathrm{m}$. In addition, black color in Figure 4 denotes the typical aluminum, resistivity of which is $\rho_{\mathrm{Al}} \approx 2.65 \times 10^{-8}\ \Omega \cdot \mathrm{m}$. The total number of filaments is N=64. Figure 4a shows the physical model of the example. It is a cross section of 1 m long CFC circular cylinder, which has openings on both top sides. In addition, an aluminum plate is located inside along the cylinder. The radius of the CFC cylinder is 1 m, and the width of the aluminum plate is 1 m. The length of the aluminum plate is also 1 m. Figure 4b shows the filament model for Figure 4a. The CFC cylinder is discretized into 54 filaments, and the aluminum plate is discretized into 10 filaments. The radius of the filaments is determined by (1.3). The injected current is current component A, which is defined in SAE ARP-5412B document [1]. The parameters of (1.1) are I_0=1.094 5, α_{I_0}=11 354, β_{I_0}=647 265, γ_{I_0}=5 423 540. Figure 4c shows the calculated filament current. The circle markers denote the numerical results, and the solid lines denote the analytic results using the method presented in this paper. The numbers in the graph represent the index numbers of the filaments. To make the graph easier to read, the currents of all the filaments are not plotted in the graph. The similarity of the lightning waveform to the lightning waveform incident on a conductor inside a resistive surface (CFC) presented in the SAE ARP-5412B document was also verified [1]. Figure 4d shows the calculated filament voltage. We note that by (1.7), every voltage across the filaments is the same. As shown in Figure 4c and 4d, no difference between the numerical and analytic method exists.

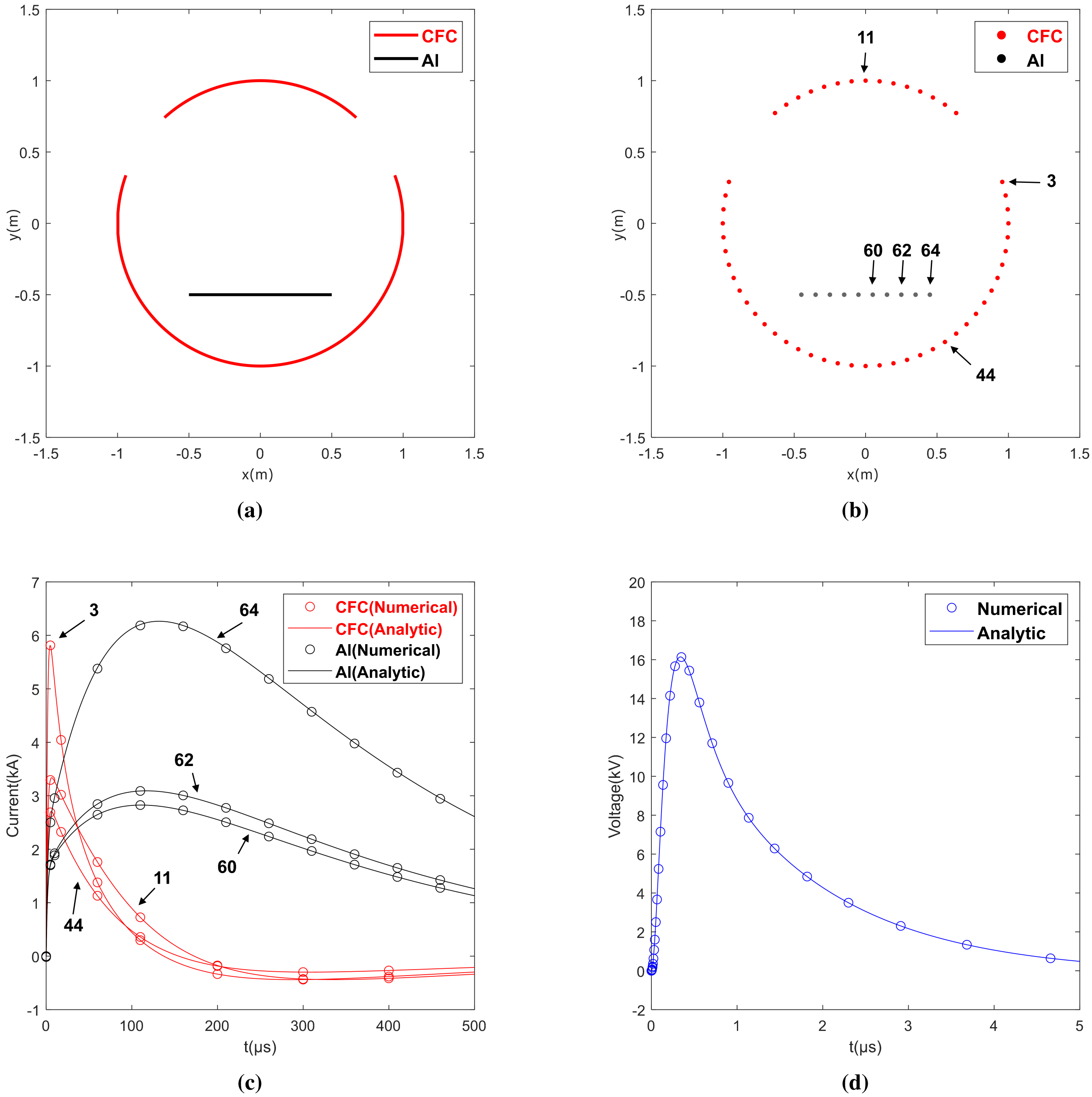

Figure 4. (a) Physical model; (b) Filament model; (c) Filament current waveform; (d) Filament voltage waveform.

5. Conclusions

In this study, we investigated analytic solutions and stability conditions for the filament wire method model. The filament wire method can be mathematically modeled as a system of first-order ordinary differential equations. Using its initial condition and characteristics such as positive-definiteness or Hermitian, we can solve the equation and find the stability condition at which a solution converges. We proved that only the mutual inductance between filament wires has an effect on the stability.

The stability condition is well-matched with the physical meaning, and the results are comparable to waveforms, which are the results of a similar scenario presented in a related document [1, 2, 4].

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors disclosed no conflicts of interest in publishing this paper.

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