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Research article

Fixed point results with applications to nonlinear fractional differential equations

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Abstract: The aim of this paper is to define a Berinde type (ρ, μ)-ϑ contraction and establish some fixed point results for self mappings in the setting of complete metric spaces. We derive new fixed point results, which extend and improve some results in the literature. We also supply a non trivial example to support the obtained results. Finally, we investigate the existence of solutions for the nonlinear fractional differential equation.

Keywords: nonlinear fractional differential equation; Berinde type (ρ, μ)-ϑ contractions; complete metric spaces; fixed point
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1. Introduction

The most fascinating and vital theory in the growth of nonlinear analysis is the fixed point theory. In this extent, the Banach fixed point theorem [1] was pioneer result for investigators over the past hundred years. This theorem plays a significant and essential role in the existence and uniqueness of solutions to different nonlinear integrals, functional equations and differential equations. In 2008, Berinde [2, 3] gave the notions of almost and generalized almost contractions and obtained a fixed point result as an extension and generalization of the Banach fixed point theorem. Samet et al. [4] initiated the notions of ρ-admissible mappings and (ρ, ψ)-contractions to prove certain fixed point results for such contractive mappings in the framework of complete metric spaces. Later on, Salimi et al. [5] generalized the notion of ρ-admissible mappings by introducing the idea of twisted (ρ, μ)-admissible mappings. They also defined the concept of (ρ, μ, ψ)-contractions and obtained some fixed point theorems in this context of metric spaces.

On the other hand, Jleli et al. [6] gave a new family of contractions, named Θ-contractions, and established a result associated with these contractions in the framework of complete metric spaces.

Later on, Ahmad et al. [7] defined generalized Θ-contractions to obtain additional generalized fixed point results. Recently, Abbas et al. [8] proved some fixed point theorems for a Suzuki type multivalued $(\Theta, \Re)$-contraction in the context of a complete metric space equipped with a binary relation. They employed their results to prove the existence of solutions of nonlinear fractional differential equations.

In this article, we combined the concepts of generalized almost contractions, twisted (ρ, μ)-admissible mappings, and Θ-contractions to define Berinde type (ρ, μ)-Θ contractions and obtain some fixed point results for self mappings in complete metric spaces. We derive new fixed point results, which enhance and upgrade some results in the literature. We also supply a non trivial example to support the obtained results. Finally, we discuss the solutions for the nonlinear fractional differential equations.

2. Preliminaries

In 1922, Banach [1] established a theorem that is, known as Banach's contraction principle (or Banach's fixed point theorem), which is one of the decisive results of nonlinear analysis, which states that if $\mathcal{L}$ is self mapping on a complete metric space $(\mathcal{U}, \varrho)$ such that, for some $\eth \in [0, 1)$,

$$\varrho(\mathcal{L}(\varkappa), \mathcal{L}(\varpi)) \leq \eth\varrho(\varkappa, \varpi) \tag{2.1}$$

for all $\varkappa, \varpi \in \mathcal{U}$, then there exists a unique point $\varkappa^* \in \mathcal{U}$ such that $\mathcal{L}\varkappa^* = \varkappa^*$. Because of its significance and importance, many researchers have proved lots of fascinating enhancements, development, and generalizations of Banach's contraction principle.

Berinde [2, 3] introduced the following notions of almost contraction and generalized almost contractions and proved a fixed point theorem.

Definition 1. ([2]) Let $(\mathcal{U}, \varrho)$ be a metric space. A mapping $\mathcal{L} : (\mathcal{U}, \varrho) \to (\mathcal{U}, \varrho)$ is said to be an almost contraction if there exists a constant $\eth \in [0, 1)$ and some $L \geq 0$ such that

$$\varrho(\mathcal{L}(\varkappa), \mathcal{L}(\varpi)) \leq \eth\varrho(\mathcal{L}(\varkappa), \mathcal{L}(\varpi)) + L\varrho(\varpi, \mathcal{L}(\varkappa)) \tag{2.2}$$

for all $\varkappa, \varpi \in \mathcal{U}$.

Definition 2. ([3]) Let $(\mathcal{U}, \varrho)$ be a metric space. A mapping $\mathcal{L} : (\mathcal{U}, \varrho) \to (\mathcal{U}, \varrho)$ is called a generalized almost contraction if there exists a constant $\eth \in [0, 1)$ and some $L \geq 0$ such that

$$\varrho(\mathcal{L}(\varkappa), \mathcal{L}(\varpi)) \leq \eth\varrho(\mathcal{L}(\varkappa), \mathcal{L}(\varpi))$$

$$+L \min\{\varrho(\varkappa, \mathcal{L}(\varkappa)), \varrho(\varpi, \mathcal{L}(\varpi)), \varrho(\varkappa, \mathcal{L}(\varpi)), \varrho(\varpi, \mathcal{L}(\varkappa))\} \tag{2.3}$$

for all $\varkappa, \varpi \in \mathcal{U}$.

Theorem 1. *([3]) Let $(\mathcal{U}, \varrho)$ be a complete metric space and $\mathcal{L} : (\mathcal{U}, \varrho) \to (\mathcal{U}, \varrho)$ is a generalized almost contraction, then $\mathcal{L}$ has a unique fixed point.*

In 2015, Jleli [6] presented new class of contractions named Θ-contractions. He presented a new result associated with Θ-contractions in the framework of complete metric spaces.

Definition 3. Let $(\mathcal{U},\varrho)$ be a complete metric space. A mapping $\mathcal{L}:(\mathcal{U},\varrho)\to(\mathcal{U},\varrho)$ is said to be a Θ-contraction if there exists $\daleth\in(0,1)$ such that $\varrho(\mathcal{L}(\varkappa),\mathcal{L}(\varpi))>0$ implies

$$\Theta(\varrho(\mathcal{L}(\varkappa),\mathcal{L}(\varpi)))\leq[\Theta(\varrho(\varkappa,\varpi))]^{\daleth} \tag{2.4}$$

for all $\varkappa,\varpi\in\mathcal{U}$, where $\Theta:(0,+\infty)\to(1,+\infty)$ is a mapping satisfying the following conditions:

(Θ_1) $\Theta(\varkappa)<\Theta(\varpi)$ for $0<\varkappa<\varpi$;
(Θ_2) for $\{\varkappa_r\}\subseteq(0,+\infty)$, $\lim_{r\to\infty}\varkappa_r=0\Leftrightarrow\lim_{r\to\infty}\Theta(\varkappa_r)=1$;
(Θ_3) there exists $0<h<1$ and $l\in(0,+\infty]$ such that $\lim_{\varkappa\to0^+}\frac{\Theta(\varkappa)-1}{\varkappa^h}=l$.

Consistent with Jleli et al. [6], we represent the class of all mappings $\Theta:(0,+\infty)\to(1,+\infty)$ by Δ_Θ satisfying (2.4) and (Θ_1)–(Θ_3). For more details in this direction, we refer the readers to [7–19].

On the other hand, Samet et al. [4] initiated the notion of ρ-admissible mappings in 2012.

Definition 4. ([4]) Let $\mathcal{L}:\mathcal{U}\to\mathcal{U}$ and $\rho:\mathcal{U}\times\mathcal{U}\to[0,+\infty)$. Then $\mathcal{L}$ is said to be ρ-admissible if the following condition

$$\rho(\varkappa,\varpi)\geq1\quad\Longrightarrow\quad\rho(\mathcal{L}\varkappa,\mathcal{L}\varpi)\geq1$$

for all $\varkappa,\varpi\in\mathcal{U}$, holds.

In this manner, Salimi et al. [5] gave the idea of twisted (ρ,μ)-admissible mappings.

Definition 5. ([5]) Let $\mathcal{L}:\mathcal{U}\to\mathcal{U}$ and $\rho,\mu:\mathcal{U}\times\mathcal{U}\to[0,+\infty)$. Then $\mathcal{L}$ is said to be twisted (ρ,μ)-admissible if

$$\begin{cases}\rho(\varkappa,\varpi)\geq1\\ \mu(\varkappa,\varpi)\geq1\end{cases}\quad\Longrightarrow\quad\begin{cases}\rho(\mathcal{L}\varkappa,\mathcal{L}\varpi)\geq1\\ \mu(\mathcal{L}\varkappa,\mathcal{L}\varpi)\geq1\end{cases}$$

for all $\varkappa,\varpi\in\mathcal{U}$.

3. Results

We define the notion of a Berinde type (ρ,μ)-Θ contraction in this way.

Definition 6. Let $(\mathcal{U},\varrho)$ be a metric space. A mapping $\mathcal{L}:(\mathcal{U},\varrho)\to(\mathcal{U},\varrho)$ is said to be a Berinde type (ρ,μ)-Θ contraction if there exist $\daleth\in(0,1)$, $\Theta\in\Delta_\Theta$, $L\geq0$ and $\rho,\mu:\mathcal{U}\times\mathcal{U}\to[0,+\infty)$ such that

$$\rho(\varkappa,\varpi)\mu(\varkappa,\varpi)\Theta\left(\varrho(\mathcal{L}(\varkappa),\mathcal{L}(\varpi))\right)\leq\left[\Theta\left(\varrho(\varkappa,\varpi)\right)\right]^{\daleth}$$
$$+L\min\left\{\varrho(\varkappa,\mathcal{L}(\varkappa)),\varrho(\varpi,\mathcal{L}(\varpi)),\varrho(\varkappa,\mathcal{L}(\varpi)),\varrho(\varpi,\mathcal{L}(\varkappa))\right\} \tag{3.1}$$

for all $\varkappa,\varpi\in\mathcal{U}$ and $\varrho(\mathcal{L}(\varkappa),\mathcal{L}(\varpi))>0$.

Theorem 2. *Let $(\mathcal{U},\varrho)$ be a complete metric space and $\mathcal{L}:(\mathcal{U},\varrho)\to(\mathcal{U},\varrho)$ be a Berinde type (ρ,μ)-Θ contraction. If these assertions are satisfied:*

(a) $\mathcal{L}$ is twisted (ρ,μ)-admissible;
(b) there exists $\varkappa_0\in\mathcal{U}$ such that $\rho(\varkappa_0,\mathcal{L}(\varkappa_0))\geq1$ and $\mu(\varkappa_0,\mathcal{L}(\varkappa_0))\geq1$;
(c) $\mathcal{L}$ is continuous.

Then there exists $\varkappa^*\in\mathcal{U}$ such that $\mathcal{L}\varkappa^*=\varkappa^*$.

Proof. Let $\varkappa_0 \in \mathcal{U}$ such that $\rho(\varkappa_0, \mathcal{L}(\varkappa_0)) \geq 1$ and $\mu(\varkappa_0, \mathcal{L}(\varkappa_0)) \geq 1$. Define $\{\varkappa_r\}$ in $\mathcal{U}$ by $\varkappa_{r+1} = \mathcal{L}(\varkappa_r)$, for all $r \in \mathbb{N}$. If $\varkappa_{r+1} = \varkappa_r$ for some $r \in \mathbb{N}$, then $\varkappa^* = \varkappa_r$ is fixed point of $\mathcal{L}$. Now we suppose that $\varkappa_{r+1} \neq \varkappa_r$, for all $r \in \mathbb{N}$. As $\mathcal{L}$ is twisted (ρ, μ)-admissible, we obtain $\rho(\varkappa_0, \varkappa_1) = \rho(\varkappa_0, \mathcal{L}(\varkappa_0)) \geq 1 \Longrightarrow \rho(\varkappa_1, \varkappa_2) = \rho(\mathcal{L}(\varkappa_0), \mathcal{L}(\varkappa_1)) \geq 1$ and $\mu(\varkappa_1, \varkappa_2) = \mu(\mathcal{L}(\varkappa_0), \mathcal{L}(\varkappa_1)) \geq 1$. By mathematical induction, we obtain $\rho(\varkappa_r, \varkappa_{r+1}) \geq 1$ and $\mu(\varkappa_r, \varkappa_{r+1}) \geq 1$, for all $r \in \mathbb{N}$. From (3.1), we have

$$\begin{aligned}
1 &< \Theta\left(\varrho(\varkappa_r, \varkappa_{r+1})\right) = \Theta\left(\varrho(\mathcal{L}(\varkappa_{r-1}), \mathcal{L}(\varkappa_r))\right) \\
&\leq \Theta\left(\rho(\varkappa_{r-1}, \varkappa_r)\mu(\varkappa_{r-1}, \varkappa_r)\varrho(\mathcal{L}(\varkappa_{r-1}), \mathcal{L}(\varkappa_r))\right) \\
&\leq [\Theta(\varrho(\varkappa_{r-1}, \varkappa_r))]^{\supset} \\
&\quad + L \min\{\varrho(\varkappa_{r-1}, \mathcal{L}(\varkappa_{r-1})), \varrho(\varkappa_r, \mathcal{L}(\varkappa_r)), \varrho(\varkappa_{r-1}, \mathcal{L}(\varkappa_r)), \varrho(\varkappa_r, \mathcal{L}(\varkappa_{r-1}))\} \\
&= [\Theta(\varrho(\varkappa_{r-1}, \varkappa_r))]^{\supset} \\
&\quad + L \min\{\varrho(\varkappa_{r-1}, \varkappa_r), \varrho(\varkappa_r, \varkappa_{r+1}), \varrho(\varkappa_{r-1}, \varkappa_{r+1}), \varrho(\varkappa_r, \varkappa_r)\} \\
&= [\Theta(\varrho(\varkappa_{r-1}, \varkappa_r))]^{\supset}.
\end{aligned}$$

Thus, we have

$$1 < \Theta\left(\varrho(\varkappa_r, \varkappa_{r+1})\right) \leq [\Theta(\varrho(\varkappa_{r-1}, \varkappa_r))]^{\supset}. \tag{3.2}$$

Therefore

$$1 < \Theta\left(\varrho(\varkappa_r, \varkappa_{r+1})\right) \leq [\Theta(\varrho(\varkappa_{r-1}, \varkappa_r))]^{\supset} \leq [\Theta(\varrho(\varkappa_{r-2}, \varkappa_{r-1}))]^{\supset^2} \leq \ldots \leq [\Theta(\varrho(\varkappa_0, \varkappa_1))]^{\supset^r}.$$

Thus, we have

$$1 < \Theta\left(\varrho(\varkappa_r, \varkappa_{r+1})\right) \leq [\Theta(\varrho(\varkappa_0, \varkappa_1))]^{\supset^r} \tag{3.3}$$

for all $r \in \mathbb{N}$. Since $\Theta \in \Delta_\Theta$, so if we let $r \to \infty$ in (3.3) and by (Θ_2), we get

$$\lim_{r\to\infty} \Theta\left(\varrho(\varkappa_r, \varkappa_{r+1})\right) = 1 \Longleftrightarrow \lim_{r\to\infty} \varrho(\varkappa_r, \varkappa_{r+1}) = 0. \tag{3.4}$$

By (Θ_3), there exist $0 < h < 1$ and $\aleph \in (0, \infty]$ such that

$$\lim_{r\to\infty} \frac{\Theta\left(\varrho(\varkappa_r, \varkappa_{r+1})\right) - 1}{\varrho(\varkappa_r, \varkappa_{r+1})^h} = l.$$

Assume that $\aleph_1 < \infty$, then, let $\aleph_2 = \frac{\aleph_1}{2} > 0$. By the concept of the limit, there exists $r_1 \in \mathbb{N}$ such that

$$\left| \frac{\Theta\left(\varrho(\varkappa_r, \varkappa_{r+1})\right) - 1}{\varrho(\varkappa_r, \varkappa_{r+1})^h} - \aleph_1 \right| \leq \aleph_2$$

for all $r > r_1$. It yields

$$\frac{\Theta\left(\varrho(\varkappa_r, \varkappa_{r+1})\right) - 1}{\varrho(\varkappa_r, \varkappa_{r+1})^h} \geq \aleph_1 - \aleph_2 = \frac{\aleph_1}{2} = \aleph_2$$

for all $r > r_1$. Then

$$r\varrho(\varkappa_r, \varkappa_{r+1})^h \leq \aleph_3 r[\Theta\left(\varrho(\varkappa_r, \varkappa_{r+1})\right) - 1] \tag{3.5}$$

for all $r > r_1$, where $\aleph_3 = \frac{1}{\aleph_2}$. Now we assume that $\aleph_1 = \infty$ and $\aleph_2 > 0$. Therefore, there exists $r_1 \in \mathbb{N}$ such that

$$\aleph_2 \leq \frac{\Theta\left(\varrho(\varkappa_r, \varkappa_{r+1})\right) - 1}{\varrho(\varkappa_r, \varkappa_{r+1})^h}$$

for all $r > r_1$. It yields

$$r\varrho(\varkappa_r, \varkappa_{r+1})^h \leq \aleph_3 r[\Theta\left(\varrho(\varkappa_r, \varkappa_{r+1})\right) - 1]$$

for all $r > r_1$, where $\aleph_3 = \frac{1}{\aleph_2}$. Thus, in all cases, there exist $\aleph_3 > 0$ and $r_1 \in \mathbb{N}$ such that

$$r\varrho(\varkappa_r, \varkappa_{r+1})^h \leq \aleph_3 r[\Theta\left(\varrho(\varkappa_r, \varkappa_{r+1})\right) - 1] \tag{3.6}$$

for all $r > r_1$. Thus, by (3.3) and (3.6), we get

$$r\varrho(\varkappa_r, \varkappa_{r+1})^h \leq \aleph_3 r([(\Theta(\varrho(\varkappa_0, \varkappa_1))]^{\supset^r} - 1).$$

If we let, $r \to \infty$, we get

$$\lim_{r\to\infty} r\varrho(\varkappa_r, \varkappa_{r+1})^h = 0.$$

Thus, there exists $r_2 \in \mathbb{N}$ such that

$$\varrho(\varkappa_r, \varkappa_{r+1}) \leq \frac{1}{r^{1/h}} \tag{3.7}$$

for all $r > r_2$.

For $m, r \in \mathbb{N}$ with $m > r \geq r_1$, we have

$$\begin{aligned}
\varrho(\varkappa_r, \varkappa_m) &\leq \varrho(\varkappa_r, \varkappa_{r+1}) + \varrho(\varkappa_{r+1}, \varkappa_{r+2}) + \varrho(\varkappa_{r+2}, \varkappa_{r+3}) + ... + \varrho(\varkappa_{m-1}, \varkappa_m) \\
&= \sum_{i=r}^{m-1} \varrho(\varkappa_i, \varkappa_{i+1}) \\
&\leq \sum_{i=r}^{\infty} \varrho(\varkappa_i, \varkappa_{i+1}) \\
&\leq \sum_{i=r}^{\infty} \frac{1}{i^{\frac{1}{h}}}.
\end{aligned}$$

If we let, $r \to \infty$ in above inequality and utilizing the fact that the series $\sum_{i=r}^{\infty} \frac{1}{i^{\frac{1}{h}}}$ is convergent, we have $\lim_{r,m\to\infty} \varrho(\varkappa_r, \varkappa_m) = 0$. Hence $\{\varkappa_r\}$ is a Cauchy sequence in $\mathcal{U}$. As $\mathcal{U}$ is complete, there exists $\varkappa^* \in \mathcal{U}$ such that $\{\varkappa_r\} \to \varkappa^*$. As $\mathcal{L}$ is continuous, we get $\mathcal{L}(\varkappa^*) = \lim_{r\to\infty} \mathcal{L}(\varkappa_r) = \lim_{r\to\infty} \varkappa_{r+1} = \varkappa^*$. Thus, $\varkappa^*$ is a fixed point of $\mathcal{L}$. □

Now we omit the continuity condition on $\mathcal{L}$ but use an adjunctive condition on $\mathcal{U}$ and obtain the same conclusion.

Theorem 3. *Let $(\mathcal{U}, \varrho)$ be a complete metric space and $\mathcal{L} : (\mathcal{U}, \varrho) \to (\mathcal{U}, \varrho)$ be a Berinde type (ρ, μ)-Θ contraction. Suppose that the following assertions hold:*

(a) $\mathcal{L}$ is twisted (ρ, μ)-admissible;

(b) there exists $\varkappa_0 \in \mathcal{U}$ such that $\rho(\varkappa_0, \mathcal{L}(\varkappa_0)) \geq 1$ and $\mu(\varkappa_0, \mathcal{L}(\varkappa_0)) \geq 1$;

(c) if $\{\varkappa_r\} \subseteq \mathcal{U}$ such that $\rho(\varkappa_r, \varkappa_{r+1}) \geq 1$ and $\mu(\varkappa_r, \varkappa_{r+1}) \geq 1$, for all r and $\varkappa_r \to \varkappa^* \in \mathcal{U}$ as $r \to \infty$, then $\rho(\varkappa_r, \varkappa^*) \geq 1$ and $\mu(\varkappa_r, \varkappa^*) \geq 1$, for all $r \in \mathbb{N}$.

Then $\mathcal{L}$ has a fixed point $\varkappa^* \in \mathcal{U}$.

Proof. Let $\varkappa_0 \in \mathcal{U}$ such that $\rho(\varkappa_0, \mathcal{L}(\varkappa_0)) \geq 1$ and $\mu(\varkappa_0, \mathcal{L}(\varkappa_0)) \geq 1$. Proceeding in a similar manner to the proof of Theorem 2, we have $\rho(\varkappa_r, \varkappa_{r+1}) \geq 1$ and $\mu(\varkappa_r, \varkappa_{r+1}) \geq 1$ and $\{\varkappa_r\}$ is Cauchy in $\mathcal{U}$ that converges to $\varkappa^*$, i.e.,

$$\lim_{r\to\infty} \varrho(\varkappa_r, \varkappa^*) = 0.$$

By hypothesis (c), we have $\rho(\varkappa_r, \varkappa^*) \geq 1$ and $\mu(\varkappa_r, \varkappa^*) \geq 1$ for all $r \in \mathbb{N}$. On the contrary, we assume that $\mathcal{L}(\varkappa^*) \neq \varkappa^*$, and there exists $r_0 \in \mathbb{N}$ such that $\varkappa_{r+1} \neq \mathcal{L}(\varkappa^*)$, for all $r \geq r_0$, i.e., $\varrho(\mathcal{L}(\varkappa_r), \mathcal{L}(\varkappa^*)) > 0$, for all $r \geq r_0$. By (3.1), (Θ_1) and the triangle inequality, we have

$$\begin{aligned} 1 &< \Theta(\varrho(\varkappa_{r+1}, \mathcal{L}(\varkappa^*))) = \Theta(\varrho(\mathcal{L}(\varkappa_r), \mathcal{L}(\varkappa^*))) \\ &\leq \rho(\varkappa_r, \varkappa^*)\mu(\varkappa_r, \varkappa^*)\Theta(\varrho(\mathcal{L}(\varkappa_r), \mathcal{L}(\varkappa^*))) \\ &\leq [\Theta(\varrho(\varkappa_r, \varkappa^*))]^{\eth} \\ &\quad + L\min\{\varrho(\varkappa_r, \mathcal{L}(\varkappa_r)), \varrho(\varkappa^*, \mathcal{L}(\varkappa^*)), \varrho(\varkappa_r, \mathcal{L}(\varkappa^*)), \varrho(\varkappa^*, \mathcal{L}(\varkappa_r))\} \\ &= [\Theta(\varrho(\varkappa_r, \varkappa^*))]^{\eth} \\ &\quad + L\min\{\varrho(\varkappa_r, \varkappa_{r+1}), \varrho(\varkappa^*, \mathcal{L}(\varkappa^*)), \varrho(\varkappa_r, \mathcal{L}(\varkappa^*)), \varrho(\varkappa^*, \varkappa_{r+1})\}. \end{aligned}$$

If $\varrho(\varkappa^*, \mathcal{L}(\varkappa^*)) > 0$, then from the following fact

$$\lim_{r\to\infty} \varrho(\varkappa_r, \varkappa^*) = 0$$

and Θ is continuous, taking the limit as $r \to \infty$ in above inequality, we get

$$1 < \Theta(\varrho(\varkappa^*, \mathcal{L}(\varkappa^*))) \leq 1,$$

which is a contradiction. Therefore $\varrho(\varkappa^*, \mathcal{L}(\varkappa^*)) = 0$, i.e., $\mathcal{L}(\varkappa^*) = \varkappa^*$ and $\varkappa^*$ is a fixed point of $\mathcal{L}$. □

For the uniqueness of the fixed point, we take the property:
(P) $\rho(\varkappa, \varpi) \geq 1$ and $\mu(\varkappa, \varpi) \geq 1$ for all fixed points $\varkappa, \varpi \in \mathcal{U}$ of $\mathcal{L}$.

Theorem 4. *In addition to the assumptions of Theorem 2, if we take property (P), then we obtain the uniqueness of the fixed point.*

Proof. Let $\varkappa^*, \widehat{\varkappa} \in \mathcal{U}$ be such that $\mathcal{L}(\varkappa^*) = \varkappa^* \neq \widehat{\varkappa} = \mathcal{L}(\widehat{\varkappa})$. Then, from property (P), $\rho(\varkappa^*, \widehat{\varkappa}) \geq 1$ and $\mu(\varkappa^*, \widehat{\varkappa}) \geq 1$. Then

$$\begin{aligned} \Theta(\varrho(\varkappa^*, \widehat{\varkappa})) &= \Theta(\varrho(\mathcal{L}(\varkappa^*), \mathcal{L}(\widehat{\varkappa}))) \\ &\leq \Theta(\rho(\varkappa^*, \widehat{\varkappa})\mu(\varkappa^*, \widehat{\varkappa})\varrho(\mathcal{L}(\varkappa^*), \mathcal{L}(\widehat{\varkappa}))) \\ &\leq [\Theta(\varrho(\varkappa^*, \widehat{\varkappa}))]^{\eth} \\ &\quad + L\min\{\varrho(\varkappa^*, \mathcal{L}(\varkappa^*)), \varrho(\widehat{\varkappa}, \mathcal{L}(\widehat{\varkappa})), \varrho(\varkappa^*, \mathcal{L}(\widehat{\varkappa})), \varrho(\widehat{\varkappa}, \mathcal{L}(\varkappa^*))\} \\ &= [\Theta(\varrho(\varkappa^*, \widehat{\varkappa}))]^{\eth} \\ &\quad + L\min\{\varrho(\varkappa^*, \varkappa^*), \varrho(\widehat{\varkappa}, \widehat{\varkappa}), \varrho(\varkappa^*, \mathcal{L}(\widehat{\varkappa})), \varrho(\widehat{\varkappa}, \mathcal{L}(\varkappa^*))\} \\ &= [\Theta(\varrho(\varkappa^*, \widehat{\varkappa}))]^{\eth} \end{aligned}$$

a contradiction because $\eth \in (0, 1)$. Hence, $\mathcal{L}$ has a unique fixed point in $\mathcal{U}$. □

Example 1. Consider the sequence $\{\varkappa_n\}$ as follows:
$\varkappa_1 = 1,$
$\varkappa_2 = 1 + 2,$
$\cdots$
$\varkappa_n = 1 + 2 + ... + n = \frac{n(n+1)}{2}$, for $n \in \mathbb{N}$.
Let $\mathcal{U} = \{\varkappa_n : n \in \mathbb{N}\}$
and $\varrho(\varkappa, \varpi) = |\varkappa - \varpi|$. Then $(\mathcal{U}, \varrho)$ is a complete metric space.
Define the mapping $\mathcal{L} : \mathcal{U} \to \mathcal{U}$
by

$$\mathcal{L}(\varkappa_1) = \varkappa_1, \quad \mathcal{L}(\varkappa_n) = \varkappa_{n-1}, \quad \text{for all } n > 1.$$

First, we show that $\mathcal{L}$ is not the Banach contraction

$$\lim_{n\to\infty} \frac{\varrho(\mathcal{L}(\varkappa_n), \mathcal{L}(\varkappa_1))}{\varrho(\varkappa_n, \varkappa_1)} = \lim_{n\to\infty} \frac{\varkappa_{n-1} - 1}{\varkappa_n - 1} = 1.$$

Now, if we consider the mapping $\Theta : (0, +\infty) \to (1, +\infty)$ by $\Theta(t) = 2^t$, for $t > 0$. Now, we show that the mapping $\mathcal{L}$ is not the Θ-contraction for $\daleth = \frac{1}{2} \in (0, 1)$, that is, $\Theta(\varrho(\mathcal{L}(\varkappa_n), \mathcal{L}(\varkappa_m))) \geq [\Theta(\varrho(\varkappa_n, \varkappa_m))]^{\frac{1}{2}}$.

Indeed, for $n = 1$ and $m = 4$, we get

$$\Theta(\varrho(\mathcal{L}(\varkappa_1), \mathcal{L}(\varkappa_4))) \geq [\Theta(\varrho(\varkappa_1, \varkappa_4))]^{\frac{1}{2}}$$

that is

$$2^{\varrho(\mathcal{L}(\varkappa_1), \mathcal{L}(\varkappa_4))} \geq 2^{\frac{1}{2}\varrho(\varkappa_1, \varkappa_4)},$$

because

$$2^{2+3} > 2^{\frac{1}{2}(2+3+4)}.$$

Now, we show that the mapping $\mathcal{L}$ is the Berinde type (ρ, μ)-Θ contraction for $\rho, \mu : \mathcal{U} \times \mathcal{U} \to [0, +\infty)$ defined by

$$\rho(\varkappa, \varpi) = \mu(\varkappa, \varpi) = 1$$

for all $\varkappa, \varpi \in \mathcal{U}$ and some $\daleth = \frac{1}{2} \in (0, 1)$ and $L = 2 > 0$.

To see this, we discuss our main result for $(1 < n < m)$. Now since,

$$\begin{aligned} \varrho(\mathcal{L}(\varkappa_m), \mathcal{L}(\varkappa_n)) &= |\mathcal{L}(\varkappa_m) - \mathcal{L}(\varkappa_n)| \\ &= |\varkappa_{m-1} - \varkappa_{n-1}| \\ &= n + (n+1) + ... + (m-1) \end{aligned}$$

$$\begin{aligned} \varrho(\varkappa_m, \varkappa_n) &= |\varkappa_m - \varkappa_n| \\ &= (n+1) + (n+2)... + (m). \end{aligned}$$

Therefore, for $1 < n < m$, we have

$$2^{n+(n+1)+...+(m-1)} < 2^{\frac{1}{2}(n+1)+(n+2)...+(m)} +2\min\left\{\begin{array}{c} m, n, n+(n+1)+...+m, \\ (n+1)+...+(m-2)+(m-1) \end{array}\right\}.$$

Hence, all the conditions of Theorem 4 are satisfied and $\varkappa_1$ is a unique fixed point of mapping $\mathcal{L}$.

The following results are a direct consequence of Theorem 2 and Theorem 3.

Corollary 1. *Let $(\mathcal{U}, \varrho)$ be a complete metric space and $\mathcal{L} : (\mathcal{U}, \varrho) \to (\mathcal{U}, \varrho)$ be twisted (ρ, μ)-admissible mapping such that*

$$\rho(\varkappa, \varpi)\mu(\varkappa, \varpi)\Theta\left(\varrho(\mathcal{L}(\varkappa), \mathcal{L}(\varpi))\right) \leq \left[\Theta\left(\varrho(\varkappa, \varpi)\right)\right]^{\supset}$$

for all $\varkappa, \varpi \in \mathcal{U}$ and $\varrho(\mathcal{L}(\varkappa), \mathcal{L}(\varpi)) > 0$. If these assertions are satisfied:

(a) there exists $\varkappa_0 \in \mathcal{U}$ such that $\rho(\varkappa_0, \mathcal{L}(\varkappa_0)) \geq 1$ and $\mu(\varkappa_0, \mathcal{L}(\varkappa_0)) \geq 1$,

(b) $\mathcal{L}$ is continuous or if $\{\varkappa_r\} \subseteq \mathcal{U}$ such that $\rho(\varkappa_r, \varkappa_{r+1}) \geq 1$ and $\mu(\varkappa_r, \varkappa_{r+1}) \geq 1$ and $\varkappa_r \to \varkappa^* \in \mathcal{U}$ as $r \to \infty$, then $\rho(\varkappa_r, \varkappa^*) \geq 1$ and $\mu(\varkappa_r, \varkappa^*) \geq 1$, $\forall\ r \in \mathbb{N}$,

then there exists $\varkappa^* \in \mathcal{U}$ such that $\varkappa^* = \mathcal{L}\varkappa^*$.

Proof. Take $L = 0$ in Theorem 2 and Theorem 3. □

Corollary 2. *Let $(\mathcal{U}, \varrho)$ be a complete metric space and $\mathcal{L} : \mathcal{U} \to \mathcal{U}$ be such that*

$$\rho(\varkappa, \varpi)\Theta\left(\varrho(\mathcal{L}(\varkappa), \mathcal{L}(\varpi))\right) \leq \left[\Theta\left(\varrho(\varkappa, \varpi)\right)\right]^{\supset}$$

$$+L\min\left\{\varrho(\varkappa, \mathcal{L}(\varkappa)), \varrho(\varpi, \mathcal{L}(\varpi)), \varrho(\varkappa, \mathcal{L}(\varpi)), \varrho(\varpi, \mathcal{L}(\varkappa))\right\}$$

If these assertions are satisfied:

(a) $\mathcal{L}$ is ρ-admissible,

(b) there exists $\varkappa_0 \in \mathcal{U}$ such that $\rho(\varkappa_0, \mathcal{L}(\varkappa_0)) \geq 1$,

(c) $\mathcal{L}$ is continuous or if $\{\varkappa_r\} \subseteq \mathcal{U}$ such that $\rho(\varkappa_r, \varkappa_{r+1}) \geq 1$ and $\varkappa_r \to \varkappa^* \in \mathcal{U}$ as $r \to \infty$, then $\rho(\varkappa_r, \varkappa^*) \geq 1$, $\forall\ r \in \mathbb{N}$, then there exists a unique point $\varkappa^* \in \mathcal{U}$ such that $\varkappa^* = \mathcal{L}\varkappa^*$.

Proof. Taking $\mu(\varkappa, \varpi) = 1$, for all $\varkappa, \varpi \in \mathcal{U}$ in Theorem 2 and Theorem 3. □

Corollary 3. *([6]) Let $(\mathcal{U}, \varrho)$ be a complete metric space and $\mathcal{L} : (\mathcal{U}, \varrho) \to (\mathcal{U}, \varrho)$. If there exists $\supset \in (0, 1)$ and $\Theta \in \Delta_\Theta$ such that*

$$\Theta\left(\varrho(\mathcal{L}(\varkappa), \mathcal{L}(\varpi))\right) \leq \left[\Theta\left(\varrho(\varkappa, \varpi)\right)\right]^{\supset},$$

then there exists a unique point $\varkappa^* \in \mathcal{U}$ such that $\varkappa^* = \mathcal{L}\varkappa^*$.

Proof. Take $\rho(\varkappa, \varpi) = \mu(\varkappa, \varpi) = 1$, for all $\varkappa, \varpi \in \mathcal{U}$ and $L = 0$ in Theorem 2 and Theorem 3. □

4. Applications

On the other hand, the fixed point theory is a very strong mathematical tool to establish the existence and uniqueness of nearly all problems modeled by nonlinear relations. Consequently, the existence and uniqueness problems of fractional differential equations are studied by means of the fixed point theory. Recently, the existence of solutions of fractional differential equations have been studied, see [8,21,22].

In the present section, we discuss the existence of a solution of the following nonlinear fractional differential equation:

$$^{C}\varrho^{\eta}(\varkappa(t)) = f(t,\varkappa(t)) \tag{4.1}$$

$(0 < t < 1,\ 1 < \eta \leq 2)$ via

$$\varkappa(0) = 0\,,\ I\varkappa(1) = \varkappa^{/}(0)$$

where $\varkappa \in C([0,1],\mathbb{R})$. Here, $^{C}\varrho^{\eta}$ represents the Caputo fractional derivative of order η defined by

$$^{C}\varrho^{\eta} f(t) = \frac{1}{\Gamma(j-\eta)} \int\limits_{0}^{t} (t-s)^{j-\eta-1} f^{j}(s)\varrho s,$$

$(j-1 < \eta < j,\ j = [\eta]+1)$ and $I^{\eta} f$ represents the Riemann-Liouville fractional integral of order η of a continuous function $f : \mathbb{R}^{+} \to \mathbb{R}$, given by

$$I^{\eta} f(t) = \frac{1}{\Gamma(\eta)} \int\limits_{0}^{t} (t-s)^{\eta-1} f(s)\varrho s, \quad \text{with } \eta > 0.$$

Let $\mathcal{U} = C[0,1]$ be the space of all continuous functions defined on $[0,1]$. The metric ϱ on $\mathcal{U}$ is given by

$$\varrho(\varkappa,\varpi) = \sup_{t\in[0,1]} |\varkappa(t) - \varpi(t)|$$

for all $\varkappa,\varpi \in \mathcal{U}$, then the space $\mathcal{U} = (C[0,1],\varrho)$ is complete metric space.

Theorem 5. *Consider the nonlinear fractional differential Eq (4.1). Let* $\zeta : (-\infty,+\infty)\times(-\infty,+\infty) \to \mathbb{R}$*. Assume that these assertions hold:*

(i) The function $f : [0,1]\times(-\infty,+\infty) \to \mathbb{R}$ is continuous,

(ii) $\forall \varkappa,\varpi \in \mathcal{U}$ and $\pi \in [1,\infty)$ such that

$$|f(t,\varkappa) - f(t,\varpi)| \leq \frac{\Gamma(\eta+1)}{4} e^{-\pi} |\varkappa - \varpi|$$

$\forall\ t \in [0,1]$,

(iii) there exists $\varkappa_0 \in C([0,1],\mathbb{R})$ such that $\zeta(\varkappa_0(t),\mathcal{L}\varkappa_0(t)) > 0$, for all $t \in [0,1]$, where $\mathcal{L} : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ is defined by

$$\mathcal{L}\varkappa(t) = \frac{1}{\Gamma(\eta)} \int\limits_{0}^{t} (t-s)^{\eta-1} f(s,\varkappa(s))\,\varrho s$$

$$+\frac{2t}{\Gamma(\eta)}\int_0^1\left(\int_0^s (s-m)^{\eta-1} f(m,\varkappa(m))\,\varrho m\right)\varrho s$$

for $t \in [0,1]$,

(iv) for each $t \in [0,1]$ and $\varkappa, \varpi \in C([0,1],\mathbb{R})$, $\zeta(\varkappa(t),\varpi(t)) > 0 \Longrightarrow \zeta(\mathcal{L}\varkappa(t),\mathcal{L}\varpi(t)) > 0$,

(v) for $\varkappa_r\} \subseteq C([0,1],\mathbb{R})$ such that $\varkappa_r \to \varkappa$ in $C([0,1],\mathbb{R})$ and $\zeta(\varkappa_r,\varkappa_{r+1}) > 0$, for all $r \in \mathbb{N}$, then $\zeta(\varkappa_r,\varkappa) > 0$, for all $r \in \mathbb{N}$.

Then, (4.1) has at least one solution.

Proof. It is very simple to show that $\varkappa \in \mathcal{U}$ is a solution of (4.1) if and only if $\varkappa \in \mathcal{U}$ is a solution of

$$\begin{aligned}\varkappa(t) &= \frac{1}{\Gamma(\eta)}\int_0^t (t-s)^{\eta-1} f(s,\varkappa(s))\,\varrho s\\ &\quad+\frac{2t}{\Gamma(\eta)}\int_0^1\left(\int_0^s (s-m)^{\eta-1} f(m,\varkappa(m))\,\varrho m\right)\varrho s\end{aligned}$$

for $t \in [0,1]$. Now, let $\varkappa, \varpi \in \mathcal{U}$ such that $\zeta(\varkappa(t),\varpi(t)) > 0$ for all $t \in [0,1]$. By (iii), we have

$$\begin{aligned}|\mathcal{L}\varkappa(t)-\mathcal{L}\varpi(t)| &= \left|\begin{array}{c}\frac{1}{\Gamma(\eta)}\int_0^t (t-s)^{\eta-1} f(s,\varkappa(s))\,\varrho s-\frac{1}{\Gamma(\eta)}\int_0^t (t-s)^{\eta-1} f(s,\varpi(s))\,\varrho s\\ +\frac{2t}{\Gamma(\eta)}\int_0^1\left(\int_0^s (s-m)^{\eta-1} f(m,\varkappa(m))\,\varrho m\right)\varrho s\\ -\frac{2t}{\Gamma(\eta)}\int_0^1\left(\int_0^s (s-m)^{\eta-1} f(m,\varpi(m))\,\varrho m\right)\varrho s\end{array}\right|\\ &\le \frac{1}{\Gamma(\eta)}\int_0^t |t-s|^{\eta-1}\,|f(s,\varkappa(s))-f(s,\varpi(s))|\,\varrho s\\ &\quad+\frac{2t}{\Gamma(\eta)}\int_0^1\left(\int_0^s (s-m)^{\eta-1}\,|f(m,\varkappa(m))-f(m,\varpi(m))|\,\varrho m\right)\varrho s,\end{aligned}$$

which implies that

$$\begin{aligned}|\mathcal{L}\varkappa(t)-\mathcal{L}\varpi(t)| &\le \frac{1}{\Gamma(\eta)}\int_0^t |t-s|^{\eta-1}\,\frac{\Gamma(\eta+1)}{4}e^{-\pi}\,|\varkappa(s)-\varpi(s)|\,\varrho s\\ &\quad+\frac{2}{\Gamma(\eta)}\int_0^1\left(\int_0^s |s-m|^{\eta-1}\,\frac{\Gamma(\eta+1)}{4}e^{-\pi}\,|\varkappa(m)-\varpi(m)|\,\varrho m\right)\varrho s\\ &= e^{-\pi}\frac{\Gamma(\eta+1)}{4\Gamma(\eta)}\int_0^t |t-s|^{\eta-1}\,|\varkappa(s)-\varpi(s)|\,\varrho s\end{aligned}$$

$$
\begin{aligned}
& +2e^{-\pi}\frac{\Gamma(\eta+1)}{4\Gamma(\eta)}\int_0^1\left(\int_0^s |s-m|^{\eta-1}\left|\varkappa(m)-\varpi(m)\right|\varrho m\right)\varrho s \\
\leq\ & e^{-\pi}\frac{\Gamma(\eta+1)}{4\Gamma(\eta)}\varrho(\varkappa,\varpi)\int_0^t |t-s|^{\eta-1}\varrho s \\
& +2e^{-\pi}\frac{\Gamma(\eta+1)}{4\Gamma(\eta)}\varrho(\varkappa,\varpi)\int_0^1\left(\int_0^s |s-m|^{\eta-1}\varrho m\right)\varrho s
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & e^{-\pi}\frac{\Gamma(\eta)\Gamma(\eta+1)}{4\Gamma(\eta)\Gamma(\eta+1)}\varrho(\varkappa,\varpi) \\
& +2e^{-\pi}B(\eta+1,1)\frac{\Gamma(\eta)\Gamma(\eta+1)}{4\Gamma(\eta)\Gamma(\eta+1)}\varrho(\varkappa,\varpi) \\
\leq\ & \frac{e^{-\pi}}{4}\varrho(\varkappa,\varpi)+\frac{e^{-\pi}}{2}\varrho(\varkappa,\varpi),
\end{aligned}
$$

where B is the beta function. From the above inequality, we get

$$\varrho(\mathcal{L}\varkappa,\mathcal{L}\varpi)\leq e^{-\pi}\varrho(\varkappa,\varpi),$$

which implies

$$\sqrt{\varrho(\mathcal{L}\varkappa,\mathcal{L}\varpi)}\leq\sqrt{e^{-\pi}\varrho(\varkappa,\varpi)}.$$

Taking the exponential, we obtain

$$e^{\sqrt{\varrho(\mathcal{L}\varkappa,\mathcal{L}\varpi)}}\leq e^{\sqrt{e^{-\pi}\varrho(\varkappa,\varpi)}},$$

that is

$$e^{\sqrt{\varrho(\mathcal{L}\varkappa,\mathcal{L}\varpi)}}\leq\left(e^{\sqrt{\varrho(\varkappa,\varpi)}}\right)^{\eth},$$

where $\eth=\sqrt{e^{-\pi}}<1$. Consider $\Theta:\mathbb{R}^+\to\mathbb{R}$ defined by $\Theta(u)=e^{\sqrt{u}}$, $\forall\ u>0$, then $\Theta\in\Delta_\Theta$. Additionally, we define $\rho,\mu:\mathcal{U}\times\mathcal{U}\to[0,+\infty)$ by

$$
\begin{aligned}
\rho(\varkappa,\varpi) &= \mu(\varkappa,\varpi) \\
&= \begin{cases} 1 \text{ if } \zeta(\varkappa(t),\varpi(t))>0, t\in[0,1], \\ 0, \text{ otherwise.}\end{cases}
\end{aligned}
$$

and for all $\varkappa,\varpi\in\mathcal{U}$. Thus,

$$\rho(\varkappa,\varpi)\mu(\varkappa,\varpi)\Theta\left(\varrho(\mathcal{L}\varkappa,\mathcal{L}\varpi)\right)\leq\left[\Theta\varrho(\varkappa,\varpi)\right]^{\eth},$$

for all $\varkappa,\varpi\in\mathcal{U}$ and $\varrho(\mathcal{L}\varkappa,\mathcal{L}\varpi)>0$. Now, by using condition (iv), we have

$$\begin{cases}\rho(\varkappa,\varpi)\geq 1\\ \mu(\varkappa,\varpi)\geq 1\end{cases}\implies\zeta(\varkappa(t),\varpi(t))>0$$

implies

$$\zeta(\mathcal{L}(t), \mathcal{L}\varpi(t)) > 0 \Longrightarrow \begin{cases} \rho(\mathcal{L}\varkappa, \mathcal{L}\varpi) \geq 1 \\ \mu(\mathcal{L}\varkappa, \mathcal{L}\varpi) \geq 1 \end{cases}$$

for all $\varkappa, \varpi \in \mathcal{U}$. Hence, $\mathcal{L}$ is a twisted (ρ, μ)-admissible. Additionally, from (iii), there exists $\varkappa_0 \in \mathcal{U}$ such that $\rho(\varkappa_0, \mathcal{L}\varkappa_0) \geq 1$ and $\mu(\varkappa_0, \mathcal{L}\varkappa_0) \geq 1$. Finally, we conclude that the assertion (v) of Theorem 3 is satisfied. Hence, as application of Theorem 3, we obtain $\varkappa^* \in \mathcal{U}$ such that $\varkappa^* = \mathcal{L}\varkappa^*$. Thus, $\varkappa^*$ is a solution of (4.1). □

5. Conclusions

In this paper, we have defined Berinde type (ρ, μ)-ϑ contractions and obtained some generalized fixed point results. In practice, we discussed the solutions for the nonlinear fractional differential equation. We derived new fixed point results, which upgraded and enhanced some results in the literature. We also supplied a non trivial example to support the obtained results.

Our future work will focus on studying the common fixed points of multivalued mappings and fuzzy mappings for Berinde type (ρ, μ)-Θ contractions in the context of complete metric spaces. Fractional differential inclusions and fractional integral inclusions can be solved as applications of these results.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

References

1. S. Banach, Sur les operations dans les ensembles abstraits et leur applications aux equations integrales, *Fund. Math.*, **3** (1922), 133–181. http://doi.org/10.4064/FM-3-1-133-181
2. V. Berinde, Approximating fixed points of weak contractions using the Picard iteration, *Nonlinear Analysis Forum*, **9** (2004), 43–53.
3. V. Berinde, General constructive fixed point theorem for Ćirić-type almost contractions in metric spaces, *Carpathian J. Math.*, **24** (2008), 10–19.
4. B. Samet, C. Vetro, P. Vetro, Fixed point theorem for α-ψ contractive type mappings, *Nonlinear Anal. Thero.*, **75** (2012), 2154–2165. https://doi.org/10.1016/j.na.2011.10.014

5. P. Salimi, C. Vetro, P. Vetro, Fixed point theorems for twisted (α, β)-ψ-contractive type mappings and applications, *Filomat*, **27** (2013), 605–615. https://doi.org/10.2298/FIL1304605S
6. M. Jleli, B. Samet, A new generalization of the Banach contraction principle, *J. Inequal. Appl.*, **2014** (2014), 38. https://doi.org/10.1186/1029-242X-2014-38
7. J. Ahmad, A. E. Al-Mazrooei, Y. J. Cho, Y. O. Yang, Fixed point results for generalized Θ-contractions, *J. Nonlinear Sci. Appl.*, **10** (2017), 2350–2358. https://doi.org/10.22436/jnsa.010.05.07
8. M. Abbas, H. Iqbal, A. Petrusel, Fixed points for multivalued Suzuki type $(\Theta, \Re)$-contraction mapping with applications, *J. Funct. Space.*, **2019** (2019), 9565804. https://doi.org/10.1155/2019/9565804
9. J. Ahmad, A. S. Al-Rawashdeh, Common fixed points of set mappings endowed with directed graph, *Tbilisi Math. J.*, **11** (2018), 107–123. https://doi.org/10.32513/tbilisi/1538532030
10. J. Ahmad, A. E. Al-Mazrooei, Common fixed point theorems for multivalued mappings on metric spaces with a directed graph, *Bull. Math. Anal. Appl.*, **10** (2018), 26–37.
11. J. Ahmad, A. E. Al-Mazrooei, I. Altun, Generalized Θ-contractive fuzzy mappings, *J. Intell. Fuzzy Syst.*, **35** (2018), 1935–1942. https://doi.org/10.3233/JIFS-171515
12. J. Ahmad, D. Lateef, Fixed point theorems for rational type (α, Θ)-contractions in controlled metric spaces, *J. Nonlinear Sci. Appl.*, **13** (2020), 163–170. https://doi.org/10.22436/jnsa.013.03.05
13. A. E. Al-Mazrooei, J. Ahmad, Multivalued fixed point theorems of generalized (K, J)-contractions with applications, *J. Math. Anal.*, **11** (2020), 113–122.
14. A. Al-Rawashdeh, J. Ahmad, Common fixed point theorems for JS-contractions, *Bull. Math. Anal. Appl.*, **8** (2016), 12–22.
15. Z. Aslam, J. Ahmad, N. Sultana, New common fixed point theorems for cyclic compatible contractions, *J. Math. Anal.*, **8** (2017), 1–12.
16. N. Hussain, A. E. Al-Mazrooei, J. Ahmad, Fixed point results for generalized $(\alpha$-$\eta)$-Θ contractions with applications, *J. Nonlinear Sci. Appl.*, **10** (2017), 4197–4208. https://doi.org/10.22436/JNSA.010.08.15
17. Z. Ma, J. Ahmad, A. E. Al-Mazrooei, D. Lateef, Fixed point results for rational orbitally $(\Theta,\ \delta_b)$-contractions with an application, *J. Funct. Space.*, **2021** (2021), 9946125. https://doi.org/10.1155/2021/9946125
18. D. Baleanu, S. Rezapour, H. Mohammadi, Some existence results on nonlinear fractional differential equations, *Phil. Trans. R. Soc. A*, **371** (2013), 20120144. http://doi.org/10.1098/rsta.2012.0144
19. H. H. Al-Sulami, J. Ahmad, N. Hussain, A. Latif, Relation theoretic $(\Theta,\ R)$ contraction results with applications to nonlinear matrix equations, *Symmetry*, **10** (2018), 767. http://doi.org/10.3390/sym10120767
20. W. Onsod, T. Saleewong, J. Ahmad, A. E. Al-Mazrooei, P. Kumam, Fixed points of a Θ-contraction on metric spaces with a graph, *Commun. Nonlinear Anal.*, **2** (2016), 139–149.
21. J. Zhou, S. Zhang, Y. He, Existence and stability of solution for a nonlinear fractional differential equation, *J. Math. Anal. Appl.*, **498** (2021), 124921. https://doi.org/10.1016/j.jmaa.2020.124921

22. J. Zhou, S. Zhang, Y. He, Existence and stability of solution for nonlinear differential equations with ψ-Hilfer fractional derivative, *Appl. Math. Lett.*, **121** (2021), 107457. https://doi.org/10.1016/j.aml.2021.107457