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Research article

On a shape derivative formula for star-shaped domains using Minkowski deformation

Abdesslam Boulkhemair[1,*], **Abdelkrim Chakib**[2] **and Azeddine Sadik**[1,2]

[1] Laboratoire de Mathématiques Jean Leray, UMR6629 CNRS, UFR Sciences et Techniques, 2 rue de la Houssinière, BP92208, 44322 Nantes, France

[2] Applied Mathematics Team, Faculty of Sciences and Techniques, Sultan Moulay Slimane University, Beni Mellal, Morocco

* **Correspondence:** Email: boulkhemair-a@univ-nantes.fr.

Abstract: We consider the shape derivative formula for a volume cost functional studied in previous papers where we used the Minkowski deformation and support functions in the convex setting. In this work, we extend it to some non-convex domains, namely the star-shaped ones. The formula happens to be also an extension of a well-known one in the geometric Brunn-Minkowski theory of convex bodies. At the end, we illustrate the formula by applying it to some model shape optimization problem.

Keywords: shape derivative; volume functional; convex domain; star-shaped domain; support function; gauge function; Minkowski deformation; shape optimization; Brunn-Minkowski theory
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1. Introduction

This paper deals with the generalization of a shape derivative formula for a volume cost functional with respect to a class of convex domains, a formula that we already studied in [2, 3], and our aim is to extend it to non-convex domains. To be precise, consider the shape functional $\mathcal{J}$ defined by

$$\mathcal{J}(\Omega) = \int_{\Omega} f(x)\,dx,$$

where Ω is a bounded open subset of $\mathbb{R}^n$ and f is a fixed function defined in $\mathbb{R}^n$.

Using the deformation $(1-\varepsilon)\Omega_0 + \varepsilon\Omega$, $\varepsilon \in [0, 1]$, of Ω_0 and a C^1 function f, A. A. Niftiyev and Y. Gasimov [19] first gave the expression of the shape derivative of $\mathcal{J}$ with respect to the class of convex domains of class C^2 by means of support functions:

Theorem 1.1. *(A. Niftiyev, Y. Gasimov) If Ω_0, Ω are bounded convex domains of class C^2 and the function f is of class C^1, then, the limit*

$$\lim_{\varepsilon\to 0^+} \frac{\mathcal{J}((1-\varepsilon)\Omega_0 + \varepsilon\Omega) - \mathcal{J}(\Omega_0)}{\varepsilon}$$

exists and is equal to

$$\int_{\partial\Omega_0} f(x)\ \left(P_\Omega(\nu_0(x)) - P_{\Omega_0}(\nu_0(x))\right)\, d\sigma(x), \tag{1.1}$$

where $\nu_0(x)$ denotes the outward unit normal vector to $\partial\Omega_0$ at x, and P_{Ω_0}, P_Ω are the support functions of the domains Ω_0, Ω, respectively.

Recently, A. Boulkhemair and A. Chakib [3] extended this formula to the case where f is in the Sobolev space $W^{1,1}_{loc}(\mathbb{R}^n)$. Inspired by the Brunn-Minkowski theory (see, for example, R. Schneider [20]), they also proposed a similar shape derivative formula by considering the Minkowski deformation $\Omega_0 + \varepsilon\Omega$ of Ω_0 :

Theorem 1.2. *(A. Boulkhemair, A. Chakib) If Ω_0, Ω are bounded convex domains of class C^2 and the function f is in the Sobolev space $W^{1,1}_{loc}(\mathbb{R}^n)$, then, the limit*

$$\lim_{\varepsilon\to 0^+} \frac{\mathcal{J}(\Omega_0 + \varepsilon\Omega) - \mathcal{J}(\Omega_0)}{\varepsilon}$$

exists and is equal to

$$\int_{\partial\Omega_0} f(x)\, P_\Omega(\nu_0(x))\, d\sigma(x), \tag{1.2}$$

where $\nu_0(x)$ denotes the outward unit normal vector to $\partial\Omega_0$ at x, and P_Ω is the support function of the domain Ω.

In fact, this formula holds true even for bounded convex domains, see [2].

If one compares (1.1) and (1.2), one can easily remark that, unlike the first formula, the second one does not depend on the support function of Ω_0. This suggests that (1.2) should hold true for non-convex Ω_0, which would be very interesting for applications in shape optimization. Unfortunately, up to now, we have not been able to treat the case of general non-convex domains. In this paper, we shall extend formula (1.2) to the case where Ω_0 is a star-shaped domain of class C^2. In fact, we were naturally led to star-shapedness because one can use parts of the proof in [2, 3] based on gauge functions. Note that by using such a method, this is the best result one can obtain since the star-shaped domains are exactly the sub-level sets of non negative continuous homogeneous functions. Thus, the case of non star-shaped domains is an open question and clearly needs other methods to study it. Anyhow, we shall return to this problem in a future work.

Another important motivation for this work came from the fact that, when $f = 1$, (1.2) is a well-known formula in the Brunn-Minkowski theory of convex bodies, see [20] for example. Indeed, when Ω_0 and Ω are bounded convex domains in $\mathbb{R}^n$, we know from that theory that, if t is a non negative real number, one can write

$$V(\overline{\Omega}_0 + t\,\overline{\Omega}) = \sum_{j=0}^{n} \binom{n}{j} t^j V_j(\overline{\Omega}_0, \overline{\Omega}),$$

where V denotes the volume functional, that is, the n-dimensional Lebesgue measure, the $\binom{n}{j}$ are the usual binomial coefficients, and the coefficients $V_j(\overline{\Omega}_0, \overline{\Omega})$ are what one calls *mixed volumes* of $\overline{\Omega}_0$ and $\overline{\Omega}$ and are significant in convex geometry, see [16,20,22] for example. Let us first remark that the first mixed volume $V_0(\overline{\Omega}_0, \overline{\Omega})$ is simply the volume $V(\overline{\Omega}_0)$. Next, it is known since a long time that

$$\frac{V(\overline{\Omega}_0 + t\,\overline{\Omega}) - V(\overline{\Omega}_0)}{t} = \sum_{j=1}^{n} \binom{n}{j} t^{j-1} V_j(\overline{\Omega}_0, \overline{\Omega}) \longrightarrow n\, V_1(\overline{\Omega}_0, \overline{\Omega}) = \int_{\partial\Omega_0} P_\Omega(\nu_0(x))\, d\sigma(x) \tag{1.3}$$

as $t \to 0^+$. Thus, (1.2) is an extension of the above formula to the case where f is not necessarily 1 and the aim of this work is to extend (1.2) and (1.3) to the case where Ω_0 is not necessarily convex. Another remark is that formula (1.3) is known to be a basic ingredient for solving the classical Minkowski problem in convex geometry, see [20] for example. Moreover, this idea has been used by some authors to solve Minkowski type problems associated with geometric functionals other than the volume one. We quote, for example, [10,11,17,18]. Using our result, one should likely be able to do a similar work using the functional studied in the present paper.

Originally, even if it is a theoretical one, this work was also motivated by numerical approximations in shape optimization problems, since it is indeed the most difficult aspect of this subject. We refer to [1], for example, for explanations about the issues that arise when implementing numerically the minimization of a shape integral functional, via some gradient method, by using the usual expression of the shape derivative by means of vector fields. Briefly, the reason is that, when using vector fields, at each iteration we have to extend the vector field (obtained only on the boundary) to all the domain or to re-mesh, and both approaches are expensive. On the other hand, when we use support functions, at each iteration, we get not only a set of boundary points but also a support function which, by taking its sub-differential at the origin, gives the next domain. This is why we are interested in the above formulas that is, expressions that use support functions instead of vector fields. In the last section, we give an idea on how to apply these formulas to the computation of the shape derivative of a simple shape optimization problem by means of an algorithm based on the gradient method. Anyway, these formulas are actually applied and implemented in recent papers [5–7].

Concerning the method of proof, we first assume that the deformation domain Ω is strongly convex, which allows us to construct some parameterization of the perturbed domain $\Omega_0 + \varepsilon\Omega$ by means of some C^1-diffeomorphism defined on Ω_0. The construction is based on some analytical and geometric properties of gauge and support functions of star-shaped domains, and reduces the problem to the usual computation of the shape derivative using vector fields. The case of a general convex Ω is then treated by using an approximation of Ω by a sequence of strongly convex domains and is based on some crucial analytical and geometric lemmas.

In fact, we have followed the idea of proof of [3]. However, even if the general scheme is the same, our proofs are far from being a straightforward consequence of the work in [3], essentially because the theory of star-shaped sets is not as well established as that of convex sets. For example, the construction of the C^1-diffeomorphism that parameterizes the perturbed domains relies on a tricky argument using the convolution of two hypersurfaces. Let us also quote the result on the continuity of the gauge function with respect to star-shaped domains by means of the Hausdorff distance (Proposition 4.1), a result that is new to our knowledge.

The outline of the paper is as follows. In Section 2, we recall some facts about star-shaped domains and give their proofs. The main results are stated in Section 3 where we also prove consequences of Formula (1.2) to the situation where the function f depends also on domains, which is customary in shape optimization problems. The fourth section is devoted to the proof of the main results using several lemmas. Finally, in Section 5, in order to illustrate these results, we give an application to a model shape optimization problem and an algorithm for solving this type of problem based on the gradient method.

2. Preliminaries on star-shaped domains

There are several definitions of what is called a star-shaped set in the literature. Here, we shall use the following one:

Definition 2.1. *An open subset (or a domain) Ω of $\mathbb{R}^n$ is said to be star-shaped with respect to some $x_0 \in \Omega$, if for all $x \in \overline{\Omega}$, Ω contains the segment* $[x_0, x[= \{(1-t)x_0 + tx\,;\, 0 \leq t < 1\}$.

It follows from this definition that the domain Ω is convex if and only if it is star-shaped with respect to each $x_0 \in \Omega$.

In what follows, we shall often work with bounded domains which are star-shaped with respect to 0. The reason for this is that such domains are naturally associated to gauge functions like the convex domains. So, let Ω be a bounded domain which is star-shaped with respect to 0. For each $x \in \mathbb{R}^n$, consider the following set of positive real numbers

$$\{\lambda; \lambda > 0\,,\ x \in \lambda\Omega\},$$

which is always non empty since Ω is a neighborhood of 0. By definition, the gauge function associated to Ω is the real function $J_\Omega : \mathbb{R}^n \to \mathbb{R}_+$ given by

$$J_\Omega(x) = \inf\{\lambda; \lambda > 0\,,\ x \in \lambda\Omega\}.$$

As for convex bodies, the gauge functions characterize the star-shaped domains they are associated to. In the following proposition, we summarize their main properties.

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain which is star-shaped with respect to* 0. *Then, the gauge function J_Ω is a non negative continuous positively homogeneous function of degree* 1. *More precisely, we have the following properties:*

(*i*) $J_\Omega(0) = 0,\ J_\Omega(x) > 0,\ \forall x \neq 0.$

(*ii*) $J_\Omega(tx) = tJ_\Omega(x),\ \forall x \in \mathbb{R}^n,\ \forall t \in \mathbb{R}_+.$

(*iii*) $\Omega = \{x \in \mathbb{R}^n\,;\ J_\Omega(x) < 1\}.$

(*iv*) $\partial\Omega = \{x \in \mathbb{R}^n\,;\ J_\Omega(x) = 1\}.$

(*v*) $J_\Omega : \mathbb{R}^n \to \mathbb{R}$ *is continuous.*

(*vi*) *If Ω' is another domain which is star-shaped with respect to* 0 *and $\Omega' \subset \Omega$, then, $J_\Omega \leq J_{\Omega'}$.*

Proof. (i), (ii) and (vi) are easy consequences of the definition of J_Ω and the fact that Ω is a bounded neighborhood of 0.
(iii): As it follows from the definition, if $J_\Omega(x) < 1$, we have $x \in \lambda\Omega$ for all $\lambda > J_\Omega(x)$, and in particular for $\lambda = 1$. Conversely, if $x \in \Omega$, we have, by definition, only $J_\Omega(x) \leq 1$. However, the fact that Ω is open implies that Ω contains a ball $B(x, r)$ for some $r > 0$. Now, take a λ such that $1 < \lambda < 1 + (r/|x|)$ (note that the case $x = 0$ is obvious). This choice implies $\lambda x \in \Omega$ because $|\lambda x - x| = |x|(\lambda - 1) < r$, hence, $x \in (1/\lambda)\Omega$, and so $J_\Omega(x) \leq 1/\lambda < 1$.
(iv): It follows from (iii) that if $x \in \partial\Omega$, then $J_\Omega(x) \geq 1$. Now, by star-shapedness, we have $tx \in \Omega$, $\forall t \in [0, 1[$ which implies by (iii) that $t\,J_\Omega(x) = J_\Omega(tx) < 1$, $\forall t \in [0, 1[$; hence, $J_\Omega(x) < \lambda$, $\forall \lambda > 1$ which implies $J_\Omega(x) \leq 1$, and so $J_\Omega(x) = 1$. Conversely, if $J_\Omega(x) = 1$, we have $x \notin \Omega$ and, by definition, $x \in \lambda\Omega$ for all $\lambda > 1$, which implies that $tx \in \Omega$, $\forall t \in [0, 1[$. Since $tx \to x$ when $t \to 1^-$, we obtain that $x \in \partial\Omega$.
(v): We know from the classical topology course that a real function f defined in $\mathbb{R}^n$ is continuous if and only if $f^{-1}(I)$ is an open subset of $\mathbb{R}^n$ for any interval I of the form $]a, +\infty[$ or $]-\infty, b[$. Now, using (iii), (iv) and the fact that J_Ω is a positively homogeneous function, one can easily check that

$$J_\Omega^{-1}(]a, +\infty[) = \begin{cases} \mathbb{R}^n, & \text{if } a < 0, \\ \mathbb{R}^n \setminus \{0\}, & \text{if } a = 0, \\ a(\mathbb{R}^n \setminus \overline{\Omega}), & \text{if } a > 0, \end{cases} \qquad \text{and} \qquad J_\Omega^{-1}(]-\infty, b[) = \begin{cases} \emptyset, & \text{if } b \leq 0, \\ b\Omega, & \text{if } b > 0, \end{cases}$$

which shows the continuity of J_Ω. □

It is well known in convex analysis that the gauge function of any convex domain is Lipschitz continuous. This is no longer true for star-shaped domains. Since such a Lipschitz regularity will be needed in the sequel, in fact, we shall work exactly with the star-shaped domains whose gauge functions are Lipschitz continuous. In order to be able to describe geometrically this subfamily of domains, let us give the following definition.

Definition 2.2. *An open set $\Omega \subset \mathbb{R}^n$ is said to be star-shaped with respect to a subset $G \subset \Omega$, if it is star-shaped with respect to any point of G.*

This definition allows us to characterize in a simple manner the star-shaped domains whose gauge functions are Lipschitz continuous. This is done in the following result for which we provide a new and simple proof (see also [8, 12]).

Proposition 2.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain which is star-shaped with respect to 0. Then, its gauge function J_Ω is Lipschitz continuous if and only if Ω is star-shaped with respect to some ball $B(0, r) \subset \Omega$ centered at 0 and with radius $r > 0$. Moreover, when this condition is satisfied, one can take $1/r$ as a Lipschitz constant for J_Ω.*

Proof. Assume first that J_Ω satisfies the inequality $|J_\Omega(y) - J_\Omega(x)| \leq \frac{1}{r}|y - x|$ for all $x, y \in \mathbb{R}^n$ and let us show that Ω is star-shaped with respect to the ball $B(0, r)$. For all $y \in B(0, r)$, $x \in \overline{\Omega}$ and $t \in [0, 1[$, it follows from the assumption that

$$J_\Omega((1 - t)y + tx) \leq J_\Omega(tx) + \frac{1}{r}|(1 - t)y| < t + \frac{1}{r}(1 - t)r = 1,$$

which says exactly that $(1-t)y+tx \in \Omega$ for all $y \in B(0,r)$, $x \in \overline{\Omega}$ and $t \in [0,1[$, that is, Ω is star-shaped with respect to the ball $B(0,r)$.

Conversely, assume that Ω is star-shaped with respect to the ball $B(0,r)$, $r > 0$. For each $x \in \partial\Omega$, consider the convex hull of the set $\overline{B}(0,r) \cup \{x\}$ and denote by Ω_x its interior. Clearly, Ω_x is a convex domain and a subset of Ω. Thus, it follows from Proposition 2.1(vi) that

$$J_\Omega \leq J_{\Omega_x} \leq J_{B(0,r)}.$$

Hence, we can write, for all $y \in \mathbb{R}^n$,

$$J_\Omega(y) \leq J_{\Omega_x}(y) \leq J_{\Omega_x}(x) + J_{\Omega_x}(y-x) \leq J_\Omega(x) + J_{B(0,r)}(y-x) \leq J_\Omega(x) + \frac{1}{r}|y-x|,$$

since J_{Ω_x} is a convex function, $J_\Omega(x) = 1 = J_{\Omega_x}(x)$ and $J_{B(0,r)}(z) = |z|/r$. So, $J_\Omega(y) - J_\Omega(x) \leq \frac{1}{r}|y-x|$ under the assumption $x \in \partial\Omega$. This is also true if $x = 0$ and when $x \neq 0$, it follows from this inequality, since $x/J_\Omega(x)$ is on $\partial\Omega$, that

$$J_\Omega\left(\frac{y}{J_\Omega(x)}\right) - J_\Omega\left(\frac{x}{J_\Omega(x)}\right) \leq \frac{1}{r}\left|\frac{y}{J_\Omega(x)} - \frac{x}{J_\Omega(x)}\right|,$$

which implies by homogeneity that $J_\Omega(y) - J_\Omega(x) \leq \frac{1}{r}|y-x|$ for all $x, y \in \mathbb{R}^n$ and, by symmetry, the Lipschitz continuity of J_Ω. □

We shall also need the following technical results. Note here that the scalar product in $\mathbb{R}^n$ of x by y is denoted in what follows by $\langle x, y\rangle$ or by $x \cdot y$.

Lemma 2.1. *If $\Omega \subset \mathbb{R}^n$ is a bounded domain which is star-shaped with respect to a ball $B(0,r)$, then, the outward unit normal vector $\nu(x)$ to Ω exists for almost every $x \in \partial\Omega$ and is given by*

$$\nu(x) = \frac{\nabla J_\Omega(x)}{|\nabla J_\Omega(x)|}.$$

Proof. First, it follows from Proposition 2.2 that J_Ω is Lipschitz continuous, and from Rademacher's theorem (see [14], for example) that $\nabla J_\Omega(x)$ exists almost everywhere in $\mathbb{R}^n$.

Next, we have to show in fact that $\nabla J_\Omega(x)$ exists for almost every $x \in \partial\Omega$. To do that, let us remark that, since it is locally bounded (by the Lipschitz constant), ∇J_Ω is locally integrable in $\mathbb{R}^n$. In general, if f is a locally integrable function in $\mathbb{R}^n$ which is also homogeneous of degree 0, we can write, by using polar coordinates, Fubini's theorem and the homogeneity of f,

$$+\infty > \int_{B(0,1)} |f(x)|\,dx = \int_0^1 \int_{\mathbb{S}^{n-1}} |f(\varrho\omega)|\,\varrho^{n-1} d\varrho\, d\omega = \frac{1}{n}\int_{\mathbb{S}^{n-1}} |f(\omega)|\,d\omega.$$

Hence, $\omega \mapsto f(\omega)$ exists a.e., on $\mathbb{S}^{n-1}$ and is even integrable. Consider now the map Ψ defined by $\Psi(0) = 0$ and

$$\Psi(x) = \frac{|x|}{J_\Omega(x)}\,x, \quad x \in \mathbb{R}^n, x \neq 0.$$

One can easily show that this is a bi-Lipschitz homeomorphism from $\mathbb{R}^n$ onto itself and that $\Psi(\mathbb{S}^{n-1}) = \partial\Omega$. By applying the above argument to the function $f = (\nabla J_\Omega) \circ \Psi$ which is locally integrable in $\mathbb{R}^n$

and also homogeneous of degree 0, we obtain that it is defined a.e., on $\mathbb{S}^{n-1}$. Moreover, it follows from the fact that Ψ is bi-Lipschitz continuous that sets of measure 0 in $\mathbb{S}^{n-1}$ correspond to sets of measure 0 in $\partial\Omega$. Hence, ∇J_Ω is defined a.e., on $\partial\Omega$.

The last point is the formula giving the outward unit normal vector $\nu(x)$ at $x \in \partial\Omega$. In fact, the arguments are more or less classical and we indicate them briefly:
(1) If $x \in \partial\Omega$ and $\nabla J_\Omega(x)$ exists, any Lipschitz continuous curve $\gamma : I =]-\epsilon, \epsilon[\to \partial\Omega$ such that $\gamma(0) = x$ and $\gamma'(0)$ exists, satisfies $J_\Omega(\gamma(t)) = 1$, $\forall t \in I$, which implies that $\nabla J_\Omega(x) \cdot \gamma'(0) = 0$, that is, $\nabla J_\Omega(x)$ is normal to tangent vectors to $\partial\Omega$ at x.
(2) At any $x \in \partial\Omega$ such that $\nabla J_\Omega(x)$ exists, we can write, as $t \to 0$,

$$J_\Omega(x + t\nabla J_\Omega(x)) = 1 + t|\nabla J_\Omega(x)|^2 + o(t),$$

which shows that, for small $t > 0$, $x + t\nabla J_\Omega(x)$ is outside $\overline{\Omega}$, that is, $\nabla J_\Omega(x)$ is an outward normal vector to Ω at x. □

Lemma 2.2. *If $\Omega \subset \mathbb{R}^n$ is a bounded domain which is star-shaped with respect to a ball $B(0, r)$, then, we have*

$$\langle \nu(x), x\rangle \geq r\,,$$

for almost every $x \in \partial\Omega$, where $\nu(x)$ is the outward unit normal vector at x.

Proof. It follows from Proposition 2.1 that

$$\Omega = \{x \in \mathbb{R}^n\,;\; J_\Omega(x) = 1\}, \tag{2.1}$$

and from Proposition 2.2 that J_Ω is Lipschitz continuous with a Lipschitz constant equal to $\frac{1}{r}$, that is, for all $x, y \in \mathbb{R}^n$,

$$|J_\Omega(x) - J_\Omega(y)| \leq \frac{1}{r}|x - y|.$$

From this inequality and Lemma 2.1, we deduce that $|\nabla J_\Omega| \leq \frac{1}{r}$ a.e., on $\partial\Omega$ and that the outward unit normal vector is given by

$$\nu(x) = \frac{\nabla J_\Omega(x)}{|\nabla J_\Omega(x)|}$$

for almost every $x \in \partial\Omega$. Therefore, using the homogeneity of J_Ω via Euler relation, we obtain that, for almost every $x \in \partial\Omega$,

$$\langle \nu(x), x\rangle = \frac{1}{|\nabla J_\Omega(x)|}\langle \nabla J_\Omega(x), x\rangle = \frac{J_\Omega(x)}{|\nabla J_\Omega(x)|} = \frac{1}{|\nabla J_\Omega(x)|} \geq r\,.$$

□

As for convex domains, the regularity of a domain which is star-shaped with respect to a ball is that of its gauge function:

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain which is star-shaped with respect to a ball $B(0, r)$, $r > 0$. Then, Ω is of class C^k, $k \geq 1$, if and only if its gauge function J_Ω is of class C^k in $\mathbb{R}^n\backslash\{0\}$.*

The proof of this last result is the same as that given in [3] in the case of convex domains and makes use of the fact that $\langle \nu(x), x\rangle$ does not vanish which is insured by Lemma 2.2 in our case. So, we refer to it.

Finally, the following result will also be needed:

Proposition 2.3. *Let $(\Phi_\varepsilon)_{0\le\varepsilon\le\varepsilon_0}$ be a family of C^1 diffeomorphisms from $\mathbb{R}^n$ onto $\mathbb{R}^n$ such that $\Phi_0(x) = x$ and $(\varepsilon, x) \mapsto \Phi_\varepsilon(x)$ and $(\varepsilon, y) \mapsto \Phi_\varepsilon^{-1}(y)$ are of class C^1 in $[0, \varepsilon_0] \times \mathbb{R}^n$. Then, for all $f \in W^{1,1}_{loc}(\mathbb{R}^n)$, the limit $\lim_{\varepsilon\to 0^+}(f(\Phi_\varepsilon(x)) - f(x))/\varepsilon$ exists in $L^1_{loc}(\mathbb{R}^n)$ and is equal to $\nabla f(x) \cdot \frac{d}{d\varepsilon}\Phi_\varepsilon(x)|_{\varepsilon=0}$.*

For a proof of this lemma, see [15], Chapter 5.

3. Main results

Let us first define the set of admissible domains $\mathcal{U}$ to be the set of bounded open subset of $\mathbb{R}^n$ which are of class C^2 and star-shaped with respect to some ball.

Recall that the support function P_Ω of a bounded convex domain Ω is given by

$$P_\Omega(x) = \sup_{y\in\Omega} x \cdot y = \sup_{y\in\overline{\Omega}} x \cdot y\,,$$

where $x \cdot y$ denotes the standard scalar product of x and y in $\mathbb{R}^n$, a product that we shall also denote sometimes by $\langle x, y\rangle$.

We can now state the first result of this paper which concerns the shape derivative of the volume functional

$$\Omega \mapsto \mathcal{J}(\Omega) = \int_\Omega f(x)dx\,.$$

Theorem 3.1. *Let $\Omega_0 \in \mathcal{U}$, Ω be a bounded convex domain and $f \in W^{1,1}(D)$ where D is a large smooth bounded domain which contains all the sets $\Omega_0 + \varepsilon\Omega$, $\varepsilon \in [0, 1]$. Then, we have*

$$\lim_{\varepsilon\to 0^+} \frac{\mathcal{J}(\Omega_0 + \varepsilon\Omega) - \mathcal{J}(\Omega_0)}{\varepsilon} = \int_{\partial\Omega_0} f(x)P_\Omega(\nu_0(x))d\sigma(x). \tag{3.1}$$

where ν_0 denotes the outward unit normal vector on $\partial\Omega_0$.

The proof of this theorem will be given in the following section. Here, we state and prove a corollary of this result which treats a case that occurs frequently in the applications, that is, the case where the function f itself depends on the parameter ε. Thus, this second result can be also considered as an extension of the first one.

Corollary 3.1. *Let Ω_0, Ω and D be as in Theorem 3.1, let (f_ε), $0 \le \varepsilon \le 1$, be a family of functions in $L^1(D)$ such that $f_0 \in W^{1,1}(D)$ and let h be a function such that $(f_\varepsilon - f_0)/\varepsilon \to h$ in $L^1(D)$ as $\varepsilon \to 0^+$. Let us set $\Omega_\varepsilon = \Omega_0 + \varepsilon\Omega$ and*

$$I(\varepsilon) = \int_{\Omega_\varepsilon} f_\varepsilon(x)dx\,.$$

Then, we have

$$\lim_{\varepsilon\to 0^+} \frac{I(\varepsilon) - I(0)}{\varepsilon} = \int_{\Omega_0} h(x)dx + \int_{\partial\Omega_0} f_0(x)P_\Omega(\nu_0(x))d\sigma(x)\,. \tag{3.2}$$

Proof. We write

$$\frac{I(\varepsilon)-I(0)}{\varepsilon}=\int_{\Omega_\varepsilon}\left(\frac{1}{\varepsilon}(f_\varepsilon-f_0)(x)-h(x)\right)dx+\int_{\Omega_\varepsilon}h(x)dx+\frac{1}{\varepsilon}\left(\int_{\Omega_\varepsilon}f_0(x)dx-\int_{\Omega_0}f_0(x)dx\right),$$

and then study each of the three terms on the right hand side of this equality. It follows from the assumption that

$$\left|\int_{\Omega_\varepsilon}\left(\frac{1}{\varepsilon}(f_\varepsilon-f_0)(x)-h(x)\right)dx\right|\leq\int_D\left|\frac{1}{\varepsilon}(f_\varepsilon-f_0)(x)-h(x)\right|dx\xrightarrow[\varepsilon\to0^+]{}0\,.$$

On the other hand, since the characteristic functions of Ω_ε converge almost everywhere to the characteristic function of Ω_0 when $\varepsilon\to0^+$, it follows from the Lebesgue convergence theorem and from (3.1) that

$$\lim_{\varepsilon\to0^+}\frac{I(\varepsilon)-I(0)}{\varepsilon}=\int_{\Omega_0}h(x)dx+\int_{\partial\Omega_0}f_0(x)P_\Omega(\nu_0(x))d\sigma(x)\,.$$

□

4. Proof of Theorem 3.1

Note first that one can assume that $f\in W^{1,1}_{loc}(\mathbb{R}^n)$ or even $f\in W^{1,1}(\mathbb{R}^n)$. Indeed, one can reduce to this case just by extending the function f to $\mathbb{R}^n$ by means of the usual results on Sobolev spaces.

We follow the same idea as [3], that is, we treat first the case where the deformation domain Ω is strongly convex, the general case being obtained by means of an appropriate approximation.

To be able to use gauge functions, we have to assume that Ω_0 and Ω are neighborhoods of 0. However, this is not a restriction of generality. Indeed, assume that Theorem 3.1 (and hence also Corollary 3.1) is proved in this case, then, if Ω_0 and Ω are neighborhoods of 0 and $c_0,c\in\mathbb{R}^n$, we have, by obvious changes of variables,

$$\begin{aligned}&(\mathcal{J}(c_0+\Omega_0+\varepsilon(c+\Omega))-\mathcal{J}(c_0+\Omega_0))/\varepsilon\\=&(\mathcal{J}(c_0+\varepsilon c+\Omega_\varepsilon)-\mathcal{J}(c_0+\Omega_0))/\varepsilon\\=&\frac{1}{\varepsilon}\left(\int_{\Omega_\varepsilon}f(c_0+\varepsilon c+x)\,dx-\int_{\Omega_0}f(c_0+x)\,dx\right).\end{aligned}$$

It follows then from Proposition 2.3 that

$$\frac{f(x+c_0+\varepsilon c)-f(x+c_0)}{\varepsilon}\longrightarrow\nabla f(x+c_0)\cdot c=div(f(x+c_0)c)\quad in\quad L^1_{loc}(\mathbb{R}^n)$$

as $\varepsilon\to0^+$, and from Corollary 3.1 that

$$\lim_{\varepsilon\to0^+}\frac{\mathcal{J}(c_0+\Omega_0+\varepsilon(c+\Omega)-\mathcal{J}(c_0+\Omega_0)}{\varepsilon}=\int_{\partial\Omega_0}f(x+c_0)P_\Omega(\nu_0(x))\,d\sigma(x)+\int_{\Omega_0}div(f(x+c_0)c)dx.$$

Now, it remains to apply the divergence formula to get

$$\lim_{\varepsilon\to0^+}\frac{\mathcal{J}(c_0+\Omega_0+\varepsilon(c+\Omega)-\mathcal{J}(c_0+\Omega_0)}{\varepsilon}$$

$$
\begin{aligned}
&= \int_{\partial\Omega_0} f(x+c_0)P_\Omega(\nu_0(x))\,d\sigma + \int_{\partial\Omega_0} f(x+c_0)\,c\cdot\nu_0(x)\,d\sigma \\
&= \int_{\partial\Omega_0} f(x+c_0)P_{c+\Omega}(\nu_0(x))\,d\sigma \\
&= \int_{\partial(c_0+\Omega_0)} f(x)P_{c+\Omega}(\nu_{c_0+\Omega_0}(x))\,d\sigma,
\end{aligned}
$$

where $\nu_{c_0+\Omega_0}$ is the exterior unit normal vector to $\partial(c_0+\Omega_0)$ at x, which establishes the formula in the case where the domains are not necessarily neighbourhoods of 0.

In what follows Ω_0 is thus assumed to be star-shaped with respect to the ball $B(0,r)$ and Ω is a neighborhood of 0.

4.1. Case where the deformation domain is strongly convex

Assume that Ω is strongly convex, that is, near each point of its boundary, the open set Ω is defined by $\{\varphi < 0\}$ and its boundary $\partial\Omega$ by $\{\varphi = 0\}$, with some C^2 function φ whose Hessian matrix is positive. Such an assumption allows us to do some geometrical construction to show that the domain $\Omega_0+\varepsilon\Omega$ is the deformation of Ω_0 via some diffeomorphism. This reduces the problem to a well known situation of deformations with vector fields, see [15] for example. The construction relies on several lemmas and starts with the following:

Lemma 4.1. *Let Ω_0 and Ω be bounded open subsets of $\mathbb{R}^n$ of class C^2 and assume that Ω is strongly convex. Then, there exists a map $a_0 : \partial\Omega_0 \to \partial\Omega$, such that*
(i) *For all $x \in \partial\Omega_0$, $P_\Omega(\nu_0(x)) = \nu_0(x)\cdot a_0(x)$.*
(ii) *For all $x \in \partial\Omega_0$, $\nu(a_0(x)) = \nu_0(x)$, where $\nu(y)$ denotes the exterior unit normal vector to $\partial\Omega$ at y.*
(iii) *The map $a_0 : \partial\Omega_0 \to \partial\Omega$ is of class C^1.*

The proof of this lemma indeed does not assume a particular geometry for Ω_0 and is the same as that of Lemma 1 of [3], so we refer to it.

Now, we would like to extend a_0 to a map from Ω_0 to Ω and even from $\mathbb{R}^n$ to $\mathbb{R}^n$. This is done by using homogeneity.

Lemma 4.2. *Ω_0 and Ω being as in the preceding lemma, assume moreover that Ω_0 is star-shaped with respect to a ball centered at 0. Then, there exists a map a defined from $\mathbb{R}^n$ to $\mathbb{R}^n$, satisfying the following properties:*
(i) *$a = a_0$ on $\partial\Omega_0$.*
(ii) *$a(\Omega_0) \subset \Omega$ and $a(\mathbb{R}^n \setminus \overline{\Omega}_0) \subset \mathbb{R}^n \setminus \overline{\Omega}$.*
(iii) *a is positively homogeneous of degree 1, Lipschitz continuous on $\mathbb{R}^n$ and of class C^1 in $\mathbb{R}^n \setminus \{0\}$.*

Proof. We define a on $\mathbb{R}^n$ by

$$
a(x) = \begin{cases} 0, & \text{if } x = 0, \\ J_{\Omega_0}(x)\,a_0(x/J_{\Omega_0}(x)), & \text{if } x \neq 0. \end{cases}
$$

Using Propositions 2.1, 2.2, Lemmas 2.3 and 4.1, it is easy to check that a satisfies (i), (ii) and (iii). □

Using the vector field a, let us now consider the map

$$\Phi_\varepsilon(x) = x + \varepsilon a(x), \;\; x \in \mathbb{R}^n, \;\; \varepsilon > 0.$$

Since a is lipschitz continuous on $\mathbb{R}^n$, it is a classical fact (and easy to check) that, if ε is sufficiently small, Φ_ε is a Lipschitz homeomorphism from $\mathbb{R}^n$ onto $\mathbb{R}^n$. Moreover, it follows from the inverse function theorem that Φ_ε is a C^1-diffeomorphism from $\mathbb{R}^n \setminus 0$ onto $\mathbb{R}^n \setminus 0$. See for example [9]. We shall use Φ_ε to parameterize the set $\Omega_0 + \varepsilon\Omega$. In order to be able to do that, we need the following result which estimates the boundary of the Minkowski sum of two subsets of $\mathbb{R}^n$ using the convolution of hypersurfaces.

Lemma 4.3. *Let $A, B \subset \mathbb{R}^n$ be open, bounded and of class C^1. Consider the following set*

$$\partial A \star \partial B := \{x + y \;:\; x \in \partial A, \; y \in \partial B \;\text{ and }\; \nu_A(x) = \nu_B(y)\} \tag{4.1}$$

where ν_A and ν_B are the outward unit normal vectors to ∂A and ∂B respectively. Then, we have

$$\partial(A + B) \subset \partial A \star \partial B.$$

Proof. Recall that the Minkowski sum of two subsets A, B of $\mathbb{R}^n$ can also be written as

$$A + B = \{x \in \mathbb{R}^n \,;\, (-A + x) \cap B \neq \emptyset\}. \tag{4.2}$$

Let $x \in \partial(A + B)$. It follows from (4.2) that $(-A + x) \cap B = \emptyset$ and that $\overline{(-A + x)} \cap \overline{B} \neq \emptyset$; hence, $\partial(-A+x) \cap \partial B \neq \emptyset$. Let $y \in \partial(-A+x) \cap \partial B$. Since $-A+x \subseteq \mathbb{R}^n \setminus B$, the hypersurfaces $\partial(-A+x)$ and ∂B are tangent at y and we have $T_y\, \partial(-A + x) = T_y\, \partial B$ and $\nu_{(-A+x)}(y) = -\nu_B(y)$. Now, $y \in \partial(-A + x) = -\partial A + x$, so that $x \in \partial A + y$ and there exists $a \in \partial A$ such that $x = y + a$. Moreover, since $-A + x$ is the image of A by the diffeomorphism $z \mapsto -z + x$, we also have

$$-\nu_A(a) = \nu_{(-A+x)}(y) = -\nu_B(y)\,,$$

which achieves the proof of the lemma. □

We will also need the following result:

Lemma 4.4. *Let Ω be a bounded and strongly convex domain of class C^2 and let ν denote the outward unit vector field normal to $\partial\Omega$. Then, $\nu : \partial\Omega \mapsto S^{n-1}$ is injective.*

Proof. Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a C^2 function such that $\Omega = \{x \in \mathbb{R}^n \,;\, \varphi(x) < 0\}$, $\partial\Omega = \{x \in \mathbb{R}^n \,;\, \varphi(x) = 0\}$ and $\nabla\varphi \neq 0$ and $\varphi'' > 0$ in a neighborhood of $\partial\Omega$. We know then that $\nu = \nabla\varphi/|\nabla\varphi|$. Let $x, y \in \partial\Omega$ be such that $\nu(x) = \nu(y)$ and let us show that $x = y$. Assume that $x \neq y$. It follows from Taylor's formula and the positivity of φ'' that

$$\langle\nabla\varphi(x), y - x\rangle < \varphi(y) - \varphi(x) \quad \text{and} \quad \langle\nabla\varphi(y), x - y\rangle < \varphi(x) - \varphi(y).$$

Since $\varphi(x) = \varphi(y) = 0$, multiplying respectively by $\frac{1}{|\nabla\varphi(x)|}$ and $\frac{1}{|\nabla\varphi(y)|}$ yields

$$\langle\nu(x), x - y\rangle > 0 \quad \text{and} \quad \langle\nu(y), x - y\rangle < 0,$$

which gives a contradiction since $\nu(x) = \nu(y)$. So, $x = y$ and the lemma is proved. □

The above lemmas allow us to prove the following crucial one which concerns the parameterization of the perturbed domain Ω_ε by means of Ω_0 and Φ_ε.

Lemma 4.5. *Let $\Omega_0 \in \mathcal{U}$ and Ω be a bounded and strongly convex domain of class C^2 in $\mathbb{R}^n$. Consider the set $\Omega_\varepsilon = \Omega_0 + \varepsilon\Omega$ and the map $\Phi_\varepsilon : x \mapsto x + \varepsilon a(x)$, $\varepsilon > 0$, where a is as in Lemma 4.2. Then, if ε is sufficiently small, we have the following:*

(*i*) $\Phi_\varepsilon(\partial\Omega_0) = \partial\Omega_0 \star \varepsilon\partial\Omega$ *and* $\partial\Omega_\varepsilon \subseteq \partial(\Phi_\varepsilon(\Omega_0))$.

(*ii*) $\Phi_\varepsilon(\Omega_0) = \Omega_\varepsilon$.

Proof. Let ν_0 and ν denote the outward unit normal vectors to Ω_0 and Ω respectively and let $x \in \partial\Omega_0$. According to Lemmas 4.1 and 4.2, $a(x) = a_0(x) \in \partial\Omega$ and $\nu_0(x) = \nu(a(x))$; hence, $\Phi_\varepsilon(x) = x + \varepsilon a(x) \in \partial\Omega_0 \star \varepsilon\partial\Omega$. Conversely, if $z \in \partial\Omega_0 \star \varepsilon\partial\Omega$, there exists $(x, y) \in \partial\Omega_0 \times \partial\Omega$ such that $z = x + \varepsilon y$ and $\nu_0(x) = \nu_{\varepsilon\Omega}(\varepsilon y) = \nu(y)$. Applying once again Lemma 4.2, we have $a(x) \in \partial\Omega$ and $\nu_0(x) = \nu(a(x)) = \nu(y)$. Next, applying Lemma 4.4 yields $a(x) = y$. Therefore, $z = x + \varepsilon a(x) = \Phi_\varepsilon(x) \in \Phi_\varepsilon(\partial\Omega_0)$. Thus, we have proved that $\Phi_\varepsilon(\partial\Omega_0) = \partial\Omega_0 \star \varepsilon\partial\Omega$. Now, according to Lemma 4.3, we have

$$\partial\Omega_\varepsilon = \partial(\Omega_0 + \varepsilon\Omega) \subseteq \partial\Omega_0 \star \partial(\varepsilon\Omega) = \partial\Omega_0 \star \varepsilon\partial\Omega = \Phi_\varepsilon(\partial\Omega_0) = \partial\Phi_\varepsilon(\Omega_0),$$

which achieves the proof of (*i*).

To show (*ii*), note first that $\Phi_\varepsilon(\Omega_0) \subset \Omega_\varepsilon$ is an obvious consequence of Lemma 4.2. To prove the other inclusion, let us first remark that it follows from the homogeneity of Φ_ε that $\Phi_\varepsilon(\Omega_0)$ is also a star-shaped domain with respect to 0 as it can be checked easily. Next, assume that there exists $x \in \Omega_\varepsilon$ such that $x \in \mathbb{R}^n \backslash \Phi_\varepsilon(\Omega_0)$. Then, it follows from Proposition 2.1 that $0 < J_{\Omega_\varepsilon}(x) < 1$ and $J_{\Phi_\varepsilon(\Omega_0)}(x) \geq 1$. Now, consider $x_* = x / J_{\Omega_\varepsilon}(x) \in \partial\Omega_\varepsilon$. Clearly, $J_{\Phi_\varepsilon(\Omega_0)}(x_*) = J_{\Phi_\varepsilon(\Omega_0)}(x) / J_{\Omega_\varepsilon}(x) > 1$, that is, $x_* \notin \partial(\Phi_\varepsilon(\Omega_0))$, which contradicts (*i*). Thus, $\Phi_\varepsilon(\Omega_0) = \Omega_\varepsilon$. □

Lemma 4.5 provides the main tool in the proof of Theorem 3.1 in the case where Ω is strongly convex and of class C^2. Indeed, according to this lemma, $\Omega_\varepsilon = \Phi_\varepsilon(\Omega_0)$ and the problem is reduced to the case of a deformation of Ω_0 by a diffeomorphism or, more precisely, a Lipschitz homeomorphism. According to [21] for example, we have the following shape derivative formula

$$\frac{d}{d\varepsilon}\mathcal{J}(\Omega_0 + \varepsilon\Omega)\bigg|_{\varepsilon=0^+} = \int_{\partial\Omega_0} f(x)\, a(x) \cdot \nu_0(x)\, d\sigma\,,$$

and according to Lemmas 4.1 and 4.2, we have

$$a(x) \cdot \nu_0(x) = a_0(x) \cdot \nu_0(x) = P_\Omega(\nu_0(x)).$$

We obtain therefore the formula

$$\lim_{\varepsilon \to 0^+} \frac{\mathcal{J}(\Omega_0 + \varepsilon\Omega) - \mathcal{J}(\Omega_0)}{\varepsilon} = \int_{\partial\Omega_0} f(x) P_\Omega(\nu_0(x)) d\sigma(x)\,,$$

and this achieves the proof of Theorem 3.1 in the case where Ω is strongly convex.

4.2. The general case

The domain Ω is now assumed to be bounded and (only) convex. We shall approximate it by a sequence of strongly convex ones. To do that, let us recall the following approximation result used in [3].

Lemma 4.6. *Let Ω be a bounded convex domain in $\mathbb{R}^n$. Then, there exists a sequence $(\Omega^k)_{k\in\mathbb{N}}$ of strongly convex smooth open subsets of Ω such that*

$$d^H(\overline{\Omega}^k, \overline{\Omega}) \xrightarrow[k\to\infty]{} 0,$$

where d^H denotes the Hausdorff distance.

Such an approximation is used to prove the following lemma which is an important step in the proof of our theorem.

Lemma 4.7. *Let $\Omega_0 \in \mathcal{U}$, Ω be a bounded convex domain in $\mathbb{R}^n$ and $(\Omega^k)_{k\in\mathbb{N}}$ the sequence given by Lemma 4.6 which approximates Ω. Then, for all $\varepsilon \in [0,1]$ and for all $k \in \mathbb{N}$, we have*

$$d^H(\overline{\Omega}^k_\varepsilon, \overline{\Omega}_\varepsilon) \leq \varepsilon\, d^H(\overline{\Omega}^k, \overline{\Omega}).$$

where $\Omega^k_\varepsilon = \Omega_0 + \varepsilon\Omega^k$ et $\Omega_\varepsilon = \Omega_0 + \varepsilon\Omega$.

Proof. We have $\overline{\Omega}^k_\varepsilon = \overline{\Omega}_0 + \varepsilon\overline{\Omega}^k$ and $\overline{\Omega}_\varepsilon = \overline{\Omega}_0 + \varepsilon\overline{\Omega}$, thus according to [20], Page 64, we have

$$d^H(\overline{\Omega}^k_\varepsilon, \overline{\Omega}_\varepsilon) = d^H(\overline{\Omega}_0 + \varepsilon\overline{\Omega}^k, \overline{\Omega}_0 + \varepsilon\overline{\Omega}) \leq d^H(\overline{\Omega}_0, \overline{\Omega}_0) + d^H(\varepsilon\overline{\Omega}^k, \varepsilon\overline{\Omega}).$$

Since $\Omega^k \subseteq \Omega$, then $d^H(\varepsilon\overline{\Omega}^k, \varepsilon\overline{\Omega}) = \sup_{x\in\varepsilon\overline{\Omega}} d(x, \varepsilon\overline{\Omega}^k) = \varepsilon d^H(\overline{\Omega}^k, \overline{\Omega})$. Thus, $d^H(\overline{\Omega}^k_\varepsilon, \overline{\Omega}_\varepsilon) \leq \varepsilon\, d^H(\overline{\Omega}^k, \overline{\Omega})$. □

We need also the following result.

Proposition 4.1. *Let $A, B \subset \mathbb{R}^n$ be two bounded domains which are star-shaped with respect to the ball $B(0,r)$, $r > 0$ and such that $A \subseteq B$. Then, we have*

$$\sup_{S^{n-1}} |J_A - J_B| \leq \frac{1}{r^2} d^H(\overline{A}, \overline{B}). \tag{4.3}$$

Proof. Let $x \in \partial B$. Since $\overline{A} \subset \overline{B}$, there exists $y_x \in \partial A$ such that $d(x, \overline{A}) = |x - y_x|$. According to Proposition 2.2, the gauge functions J_A, J_B are Lipschitz functions with Lipschitz constant $\frac{1}{r}$. Therefore, since $J_A(y_x) = 1 = J_B(x)$ and $\overline{A} \subset \overline{B}$, we can write

$$\begin{aligned}
|J_A(x) - J_B(x)| &\leq |J_A(x) - J_A(y_x)| + |J_A(y_x) - J_B(x)| = |J_A(x) - J_A(y_x)| \\
&\leq r^{-1}|x - y_x| = r^{-1} d(x, \overline{A}) \\
&\leq r^{-1} \sup_{z\in\overline{B}} d(z, \overline{A}) = r^{-1} d^H(\overline{A}, \overline{B}),
\end{aligned}$$

an inequality that holds for $x \in \partial B$. Now, if $x \in S^{n-1}$, we have $\frac{x}{J_B(x)} \in \partial B$, and by using the homogeneity of the gauge functions we obtain

$$|J_A(x) - J_B(x)| = J_B(x)|J_A(\frac{x}{J_B(x)}) - J_B(\frac{x}{J_B(x)})| \leq J_B(x) r^{-1} d^H(\overline{A}, \overline{B}).$$

Since $B(0,r) \subset B$, we have $J_B(x) \leq J_{B(0,r)}(x) = |x|/r = 1/r$, which implies the desired inequality. □

The last lemma is crucial for our proof.

Lemma 4.8. *Let $\Omega_0 \in \mathcal{U}$, Ω be a bounded convex domain and $f \in W^{1,1}_{loc}(\mathbb{R}^n)$, and let $(\Omega^k)_{k\in\mathbb{N}}$ be as in Lemma 4.6. Then, there exists a constant $C > 0$ such that, for all $k \in \mathbb{N}$ and all $\varepsilon \in [0, 1]$, we have*

$$\left| \int_{\Omega_\varepsilon^k} f(x)dx - \int_{\Omega_\varepsilon} f(x)dx \right| \leq C\,\varepsilon\, d^H(\overline{\Omega}^k, \overline{\Omega}),$$

where $\Omega_\varepsilon^k = \Omega_0 + \varepsilon\Omega^k$ et $\Omega_\varepsilon = \Omega_0 + \varepsilon\Omega$.

Proof. We follow the idea of [4]. Let $r > 0$ be such that Ω_0 is star-shaped with respect to $B(0, r)$ and let $B(0, R)$ be a large ball which contains all the sets $\Omega_0 + \varepsilon\Omega$, $\varepsilon \in [0, 1]$. As one can easily check, Ω_ε and Ω_ε^k are star-shaped with respect to $B(0, r)$, and we shall denote by J_ε and J_ε^k respectively their gauge functions. Let us denote by $I_\varepsilon^k(f)$ the difference

$$\int_{\Omega_\varepsilon} f(x)dx - \int_{\Omega_\varepsilon^k} f(x)dx\,.$$

Since $\Omega_\varepsilon = \{J_\varepsilon < 1\}$ and $\Omega_\varepsilon^k = \{J_\varepsilon^k < 1\}$, by using polar coordinates, we can write

$$I_\varepsilon^k(f) = \int_{S^{n-1}} \int_0^{\frac{1}{J_\varepsilon(\omega)}} f(\rho\omega)\rho^{n-1} d\rho d\omega - \int_{S^{n-1}} \int_0^{\frac{1}{J_\varepsilon^k(\omega)}} f(\rho\omega)\rho^{n-1} d\rho d\omega = \int_{S^{n-1}} \int_{\frac{1}{J_\varepsilon^k(\omega)}}^{\frac{1}{J_\varepsilon(\omega)}} f(\rho\omega)\rho^{n-1} d\rho d\omega.$$

This allows us to estimate $I_\varepsilon^k(f)$ as follows:

$$|I_\varepsilon^k(f)| \leq \int_{S^{n-1}} \left| \frac{J_\varepsilon(\omega) - J_\varepsilon^k(\omega)}{J_\varepsilon(\omega) J_\varepsilon^k(\omega)} \right| \sup_{\rho\in[\frac{1}{J_\varepsilon^k(\omega)}, \frac{1}{J_\varepsilon(\omega)}]} |f(\rho\omega)\rho^{n-1}| d\omega.$$

Note that, since $B(0, r) \subset \Omega_\varepsilon^k \subset \Omega_\varepsilon \subset B(0, R)$, we have $J_{B(0,R)} \leq J_\varepsilon \leq J_\varepsilon^k \leq J_{B(0,r)}$. Recall that $J_{B(0,r)}(x) = |x|/r$, $J_{B(0,R)}(x) = |x|/R$. Hence,

$$|I_\varepsilon^k(f)| \leq R^2 \int_{S^{n-1}} |J_\varepsilon(\omega) - J_\varepsilon^k(\omega)| \sup_{\rho\in[r,R]} |f(\rho\omega)\rho^{n-1}| d\omega.$$

Since Ω_ε and Ω_ε^k are star-shaped with respect to $B(0, r)$ and $\Omega_\varepsilon^k \subseteq \Omega_\varepsilon$, it follows from Lemmas 4.1 and 4.7 that

$$\sup_{S^{n-1}} |J_\varepsilon - J_\varepsilon^k| \leq \frac{1}{r^2}\, d^H(\overline{\Omega}_\varepsilon^k, \overline{\Omega}_\varepsilon) \leq \frac{\varepsilon}{r^2}\, d^H(\overline{\Omega}^k, \overline{\Omega}).$$

Therefore,

$$|I_\varepsilon^k(f)| \leq \frac{R^2\varepsilon}{r^2}\, d^H(\overline{\Omega}^k, \overline{\Omega}) \int_{S^{n-1}} \sup_{\rho\in[r,R]} |f(\rho\omega)\rho^{n-1}|\, d\omega.$$

It remains to apply the following classical inequality for functions of one real variable:

$$\|\varphi\|_{L^\infty(I)} \leq \frac{1}{|I|} \int_I |\varphi(t)|\, dt + \int_I |\varphi'(t)|\, dt, \quad \varphi \in W^{1,1}(I), \tag{4.4}$$

where I is a bounded interval and $|I|$ is its length. In fact, this is just a precise version of Sobolev's inequality. Its proof is easy when φ is of class C^1 on $\bar{I}$ and the general case is obtained by a density argument and is left to the reader. Applying (4.4) to the function $\rho \mapsto f(\rho\omega)\,\rho^{n-1}$ yields

$$
\begin{aligned}
|I_{\varepsilon,k}(f)| \leq & \frac{R^2\varepsilon}{r^2} d^H(\overline{\Omega}^k, \overline{\Omega}) \int_{S^{n-1}} \int_r^R \left(|f(\rho\omega)| \Big(\frac{\rho^{n-1}}{R-r} + (n-1)\rho^{n-2} \Big) + |\nabla f(\rho\omega)| \rho^{n-1} \right) d\rho d\omega \\
\leq & \frac{R^2\varepsilon}{r^2} d^H(\overline{\Omega}^k, \overline{\Omega}) \int_{r\leq|x|\leq R} \left(\frac{|f(x)|}{R-r} + \frac{n-1}{|x|} |f(x)| + |\nabla f(x)| \right) dx \\
\leq & \frac{R^2\varepsilon}{r^2} d^H(\overline{\Omega}^k, \overline{\Omega}) \left(\frac{1}{R-r} + \frac{n-1}{r} + 1 \right) \int_{r\leq|x|\leq R} (|f(x)| + |\nabla f(x)|)\, dx \\
\leq & \frac{R^2\varepsilon}{r^2} d^H(\overline{\Omega}^k, \overline{\Omega}) \left(\frac{1}{R-r} + \frac{n-1}{r} + 1 \right) \|f\|_{W^{1,1}(D)},
\end{aligned}
$$

which achieves the proof of the lemma. □

Using the above lemmas, we can now finish the proof of Theorem 3.1. Let $\delta > 0$ be arbitrary. We can write

$$
\begin{aligned}
& \frac{\mathcal{J}(\Omega_0 + \varepsilon\Omega) - \mathcal{J}(\Omega_0)}{\varepsilon} - \int_{\partial\Omega_0} f(x) P_\Omega(\nu_0(x)) d\sigma(x) \\
= & \frac{1}{\varepsilon} \left(\int_{\Omega_\varepsilon} f(x)dx - \int_{\Omega_\varepsilon^k} f(x)dx \right) + \frac{1}{\varepsilon} \left(\int_{\Omega_\varepsilon^k} f(x)dx - \int_{\Omega_0} f(x)dx \right) \\
& - \int_{\partial\Omega_0} f(x) P_{\Omega^k}(\nu_0)(x) d\sigma(x) + \int_{\partial\Omega_0} f(x) \Big(P_{\Omega^k}(\nu_0(x)) - P_\Omega(\nu_0(x)) \Big) d\sigma(x).
\end{aligned}
\tag{4.5}
$$

For the first term in the righthand side of (4.5), according to Lemmas 4.6 and 4.8, there exists $k_0 \in \mathbb{N}$, such for all $k \geq k_0$ and for all $\varepsilon \in]0, 1]$, we have

$$
\frac{1}{\varepsilon} \left| \left(\int_{\Omega_\varepsilon} f(x)dx - \int_{\Omega_\varepsilon^k} f(x)dx \right) \right| \leq \delta. \tag{4.6}
$$

Using the formula $\|P_A - P_B\|_{L^\infty(S^{n-1})} = d^H(A, B)$ for compact convex sets, (see for example, Page 66 of [20]), we can estimate the last term in the righthand side of (4.5) as follows:

$$
\left| \int_{\partial\Omega_0} f \Big(P_{\Omega^k}(\nu_0) - P_\Omega(\nu_0) \Big) d\sigma \right| \leq \|P_{\Omega^k} - P_\Omega\|_{L^\infty(S^{n-1})} \int_{\partial\Omega_0} |f|\, d\sigma = d^H(\overline{\Omega}^k, \overline{\Omega}) \int_{\partial\Omega_0} |f|\, d\sigma,
$$

which implies that it tends to 0 when $k \to \infty$ by virtue of Lemma 4.6. Hence, there exists $k_1 \in \mathbb{N}$, such that, for all $k \geq k_1$, we have

$$
\left| \int_{\partial\Omega_0} f \Big(P_{\Omega^k}(\nu_0) - P_\Omega(\nu_0) \Big) d\sigma \right| \leq \delta. \tag{4.7}
$$

Now, if $k_2 = \max\{k_0, k_1\}$, since Ω^{k_2} is strongly convex, it follows from the first part of the proof that there exists ε_δ such that for all $\varepsilon \leq \varepsilon_\delta$, we have

$$
\left| \frac{1}{\varepsilon} \left(\int_{\Omega_\varepsilon^{k_2}} f(x)dx - \int_{\Omega_0} f(x)dx \right) - \int_{\partial\Omega_0} f(x) P_{\Omega^{k_2}}(\nu_0(x)) d\sigma(x) \right| \leq \delta. \tag{4.8}
$$

By taking $k = k_2$ in (4.5) and using (4.6)–(4.8), we obtain that, for all $\varepsilon \le \varepsilon_\delta$,

$$\left|\frac{1}{\varepsilon}\left(\int_{\Omega_\varepsilon} f(x)dx - \int_{\Omega_0} f(x)dx\right) - \int_{\partial\Omega_0} f(x)P_\Omega(\nu_0(x))d\sigma(x)\right| \le 3\delta\,,$$

which achieves the proof of Theorem 3.1.

5. Application

To illustrate our work, we give an algorithm based on the gradient method to indicate how our formula could be applied to a shape optimization problem and we compute the shape derivative of some functional related to the solution of a partial differential equation. This is done without implementing to keep our paper in a reasonable length. Anyhow, we have successfully implemented such an algorithm in the study of several problems: a Bernoulli type shape optimization problem in [5] and constrained shape optimization ones in [6, 7].

Let us define the set $\mathcal{U}$ of admissible domains by $\Omega \in \mathcal{U} \Longleftrightarrow \Omega$ is a C^3 open subset of $\mathbb{R}^n$ which is star-shaped with respect to some ball of radius r.

If D is an open bounded (convex) and non empty subset of $\mathbb{R}^n$, let us consider the problem

$$(PO)\quad \begin{cases} \text{Find} \quad \Omega_* \in \mathcal{U}(D) \quad \text{such that} \quad \mathcal{J}(\Omega_*) = \inf\limits_{\Omega\in\mathcal{U}(D)} \mathcal{J}(\Omega), \text{where} \quad \mathcal{U}(D) = \{\Omega \in \mathcal{U}\ ;\ \Omega \subset D\}, \\ \mathcal{J}(\Omega) = \displaystyle\int_\Omega (u_\Omega - u_{\mathrm{d}})^2 dx \quad \text{and} \quad u_\Omega \quad \text{is the solution of}(PE) \quad \begin{cases} -\Delta v + v = f, & \text{in} \quad \Omega, \\ \frac{\partial v}{\partial \nu} = g, & \text{on} \quad \partial\Omega, \end{cases} \end{cases}$$

where $u_{\mathrm{d}} \in H^1(D)$, $f \in L^2(D)$ and $g \in H^2(D)$. Let u_0 be the solution of (PE) on $\Omega_0 \in \mathcal{U}(D)$ and u_ε be the solution of (PE) on $\Omega_\varepsilon = \Omega_0 + \varepsilon\Omega$, $\varepsilon \in [0, 1]$. Assuming that Ω is a strongly convex domain, we know from Lemma 4.5 that Ω_ε can be considered as a deformation of the domain Ω_0 by the vector field a, that is $\Omega_\varepsilon = (\mathrm{Id}_{\mathbb{R}^n} + \varepsilon a)(\Omega_0)$ for small enough ε. Therefore, at least when $f \in H^1(D)$, according to [1], we can write

$$\widetilde{u}_\varepsilon = \widetilde{u}_0 + \varepsilon u_0' + \varepsilon v_\varepsilon$$

where $\widetilde{u}_\varepsilon$ and $\widetilde{u}_0$ are respectively extensions of u_ε and u_0 to D, $u_0' \in H^2(\Omega_0) \cap H^1(D)$ is the shape derivative of $\widetilde{u}_0$ with respect to the vector field a and $v_\varepsilon \to 0$ in $L^2(D)$ as $\varepsilon \to 0^+$. It follows from that result that

$$\frac{1}{\varepsilon}[(\widetilde{u}_\varepsilon - u_{\mathrm{d}})^2 - (\widetilde{u}_0 - u_{\mathrm{d}})^2] - 2u_0'(\widetilde{u}_0 - u_{\mathrm{d}}) \xrightarrow[\varepsilon\to 0^+]{} 0 \quad \text{in} \quad L^1(D),$$

which allows one to apply Corollary 3.1 to obtain

$$\lim_{\varepsilon\to 0^+} \frac{\mathcal{J}(\Omega_0 + \varepsilon\Omega) - \mathcal{J}(\Omega_0)}{\varepsilon} = \int_{\Omega_0} 2u_0'(\widetilde{u}_0 - u_{\mathrm{d}})dx + \int_{\partial\Omega_0} (\widetilde{u}_0 - u_{\mathrm{d}})^2 P_\Omega(\nu_0)d\sigma(x).$$

Now, in this expression of the shape derivative of $\mathcal{J}$, even the domain integral can be written as a boundary integral. Indeed, for example, if one follows the same method as [15], one can show that u_0' satisfies the boundary value problem

$$\begin{cases} -\Delta u_0' + u_0' = 0, & \text{in } \Omega_0, \\ \frac{\partial}{\partial \nu_0} u_0' = (\frac{\partial g}{\partial \nu_0} - \frac{\partial^2 u_0}{\partial \nu_0^2})\langle a, \nu_0\rangle + \nabla u_0 \nabla_{\partial\Omega_0}\langle a, \nu_0\rangle, & \text{on } \partial\Omega_0, \end{cases} \tag{5.1}$$

where

$$\frac{\partial^2 u_0}{\partial \nu_0^2} = \sum_{i,j=1}^{n} \frac{\partial^2 u_0}{\partial x_i \partial x_j} \nu_{0,i}\, \nu_{0,j}$$

and $\nabla_{\partial\Omega_0}$ is the tangential gradient (see [15]). Note here that, since Ω_0 is of class C^3, u_0 is in fact in $H^3(\Omega_0)$, $u_0' \in H^2(\Omega_0)$ (see [15]), so that the second derivative $\frac{\partial^2 u_0}{\partial \nu_0^2}$ is well defined on $\partial\Omega_0$. Now, using the solution of the following adjoint state boundary value problem

$$\begin{cases} -\Delta\psi + \psi = -2(u_0 - u_{\mathrm{d}}), & \text{in } \Omega_0, \\ \frac{\partial\psi}{\partial\nu_0} = 0, & \text{on } \partial\Omega_0\,, \end{cases} \tag{5.2}$$

we obtain the following expression for the shape derivative of the functional $\mathcal{J}$ (see [1])

$$\lim_{\varepsilon\to 0^+} \frac{\mathcal{J}(\Omega_0 + \varepsilon\Omega) - \mathcal{J}(\Omega_0)}{\varepsilon} = \int_{\partial\Omega_0} d_{\Omega_0} \langle a, \nu_0\rangle \, d\sigma\,,$$

where

$$d_{\Omega_0} = (u_0 - u_{\mathrm{d}})^2 + \langle \nabla u_0, \nabla\psi\rangle + \psi(u_0 - f) - \frac{\partial(g\psi)}{\partial\nu_0} - \mathcal{H} g\psi.$$

Now, since Ω is strongly convex, it follows from Lemmas 4.1 and 4.2 that

$$\langle a(x), \nu_0(x)\rangle = P_\Omega(\nu_0(x)) \text{ on } \partial\Omega_0, \tag{5.3}$$

so that

$$\lim_{\varepsilon\to 0^+} \frac{\mathcal{J}(\Omega_0 + \varepsilon\Omega) - \mathcal{J}(\Omega_0)}{\varepsilon} = \int_{\partial\Omega_0} d_{\Omega_0} \, P_\Omega \circ \nu_0 \, d\sigma.$$

The last thing we propose is an algorithm to solve the shape optimization problem (PO).

Algorithm.

1) Choose $\Omega_0 \in \mathcal{U}(D)$, $\rho \in]0, 1[$ and a precision ε.
2) Solve the state equation:

$$(PE)\begin{cases} -\Delta u_0 + u_0 = f, & \text{in } \Omega_0\,, \\ \dfrac{\partial u_0}{\partial\nu_0} = 0, & \text{on } \partial\Omega_0. \end{cases} \tag{5.4}$$

3) Solve the adjoint state problem

$$(PEA)\begin{cases} -\Delta\psi_0 + \psi_0 = -2(u_0 - u_{\mathrm{d}}), & \text{in } \Omega_0\,, \\ \dfrac{\partial\psi_0}{\partial\nu_0} = 0, & \text{on } \partial\Omega_0. \end{cases} \tag{5.5}$$

4) Calculate $\widehat{p_0}$ solution of

$$\arg\min_{p\in\mathcal{E}} F_0(p) \tag{5.6}$$

where

$$\mathcal{E} = \{\varphi \in C(\overline{D})\ ;\ \varphi \text{ is convex and homogeneous of degree 1 and } \varphi \le P_D\},$$

and

$$F_0(p) = \int_{\partial\Omega_0} \left((u_0 - u_{\mathrm{d}})^2 + \langle \nabla u_0, \nabla\psi\rangle + \psi(u_0 - f) - \frac{\partial(g\psi)}{\partial\nu_0} - \mathcal{H} g\psi \right) p \circ \nu_0 \, d\sigma.$$

5) At step k, if

$$|u_k - u_{\rm d}|_{L^2(\Omega_k)} < \varepsilon,$$

go to 7, where u_k is the solution of the state equation in Ω_k (the domain at step k).

6) Compute

$$\Omega_{k+1} = \left\{ r\theta;\ \theta \in \mathbb{S}^{n-1},\ r \in [0, 1/J_{\Omega_{k+1}}(\theta)[\right\} \tag{5.7}$$

where $J_{\Omega_{k+1}}$ is the gauge function of Ω_{k+1} given by $\Omega_{k+1} := \Omega_k + \rho\,\widehat{\Omega}_k$ with

$$\widehat{\Omega}_k = \partial\widehat{p}_k(0) = \{\ell \in \mathbb{R}^n;\ \widehat{p}_k(x) \geq \langle \ell, x\rangle,\ \forall x \in \mathbb{R}^n\}$$

and go to 2.

7) End.

Let us end this section by making three remarks which clarify some points about the above algorithm: the first one explains the determination of a descent direction for the convergence of this algorithm, the second one is concerned with how to solve problem (5.6) and in the last one we give some details on the computation of Ω_{k+1} at each iteration.

Remark 5.1. *In the above algorithm, the sequence of domains $(\Omega_k)_{k\in\mathbb{N}}$ is constructed in such a way that $(\mathcal{J}(\Omega_k))_{k\in\mathbb{N}}$ is decreasing. Indeed, let $k \in \mathbb{N}^*$, then, for a small $\rho \in]0,1[$, we have*

$$\mathcal{J}(\Omega_{k+1}) - \mathcal{J}(\Omega_k) \;=\; \mathcal{J}(\Omega_k + \rho\widehat{\Omega}_k) - \mathcal{J}(\Omega_k) = \rho\left(\int_{\partial\Omega_k} d_{\Omega_k} P_{\widehat{\Omega}_k} \circ \nu_k d\sigma\right) + O(\rho^2).$$

Now, since $\widehat{p}_k = P_{\widehat{\Omega}_k}$ is a solution of $\arg\min_{p\in\mathcal{E}} F_k(p)$, then

$$F_k(\widehat{p}_k) = \int_{\partial\Omega_k} d_{\Omega_k} P_{\widehat{\Omega}_k} \circ \nu_k d\sigma \leq F_k(0) = 0\,,$$

which guarantees the decrease of the functional $\mathcal{J}$. Thus, $\widehat{\Omega}_k$ defines a descent direction for $\mathcal{J}$.

Remark 5.2. *The problem* (5.6) *admits a solution $\widehat{p} \in \mathcal{E}$ because the functional*

$$p \in \mathcal{E} \longmapsto F_k(p) = \int_{\partial\Omega_k} d_{\Omega_k}\, p \circ \nu_k\, d\sigma$$

is continuous and $\mathcal{E}$ is a compact subset of $C(\overline{D})$. Indeed, the functional F_k being clearly continuous on $C(\overline{D})$, let us show that $\mathcal{E}$ is a compact subset of $C(\overline{D})$. Any $p \in \mathcal{E}$ is the support function of a unique convex bounded open set which is its sub-differential at 0*, that is, $p = P_{\partial p(0)}$ (see for example [20,22]). So, for all $x, y \in \overline{D}$, using the fact that a support function is sub-linear and homogenous of degree* 1 *and $p \leq P_D$, we get*

$$\forall p \in \mathcal{E},\ |p(x) - p(y)| = |P_{\partial p(0)}(x) - P_{\partial p(0)}(y)| \leq \sup_{w\in\mathbb{S}^{n-1}} P_{\partial p(0)}(w)\,\|x-y\| \leq \sup_{w\in\mathbb{S}^{n-1}} P_D(w)\,\|x-y\|,$$

which implies that the family $\mathcal{E}$ is equicontinuous. On the other hand, the fact that $\mathcal{E}$ is a bounded subset of $C(\overline{D})$ is obvious, while the fact that it is closed is easy: because of the homogeneity, the uniform convergence on $\overline{D}$ implies the pointwise convergence in all $\mathbb{R}^n$, which allows to pass to the limit in inequalities. The compactness of $\mathcal{E}$ follows of course by applying Ascoli-Arzela's theorem.

Remark 5.3. *Let $\Omega_0 \in \mathcal{U}(D)$ and let J_{Ω_0} denote its gauge function. In order to determine the domain of the next iteration $\Omega_1 = \Omega_0 + \rho\, \partial\widehat{P_0}\,(0)$ one can consider applying the techniques based on the use of support functions as in [5, 7]. However, support functions do not characterize star-shaped sets unlike gauge functions (see e.g., [12,13]). Because of that, in this work we have proposed an algorithm based on the use of gauge functions, more precisely this concerns the Step* 6 *and the proposed process to achieve this step is as follow: to determine Ω_1, it is numerically sufficient to determine its boundary $\partial\Omega_1$. For this purpose, we recall that $\partial\Omega_1$ can be defined by (see e.g., [12])*

$$\partial\Omega_1 = \left\{\theta/J_{\Omega_1}(\theta);\ \theta \in \mathbb{S}^{n-1}\right\}. \tag{5.8}$$

Next, by homogeneity of the gauge function J_{Ω_1}, we can check that $\partial\Omega_1 = \{w/J_{\Omega_1}(w);\ w \in \partial\Omega_0\}$. We have therefore to compute the gauge function J_{Ω_1} on $\partial\Omega_0$. To do that, let $\delta > 0$ be small enough. According to Lemma 4.6, the convex domain $\partial\widehat{P_0}\,(0)$ can be approximated by a strongly convex subdomain Λ such that

$$|\widehat{P_0} - P_\Lambda|_{\mathbb{S}^{n-1}} = d^H(\partial\widehat{P_0}\,(0)\,, \overline{\Lambda}) \leq \delta. \tag{5.9}$$

Moreover, using (5.9) *and the properties of Hausdorff distance on convex domains (see e.g., [20]), we obtain*

$$d^H(\Omega_1, \Omega_0 + \rho\,\Lambda) = d^H(\Omega_0, \Omega_0) + \rho\, d^H(\partial\widehat{P_0}\,(0)\,, \overline{\Lambda}) \leq \rho\delta,$$

which, combined with Proposition 4.1, gives

$$\sup_{\mathbb{S}^{n-1}} |J_{\Omega_1} - J_{\Omega_0+\rho\,\Lambda}| \leq \frac{1}{r^2}\rho\delta.$$

Hence, we can approximate the functions J_{Ω_1} and $\widehat{P_0}$ by $J_{\Omega_0+\rho\,\Lambda}$ and P_Λ respectively, where $\widehat{P_0}$ is a solution of (5.6)*. Thus, it remains to compute $J_{\Omega_0+\rho\,\Lambda}$ on $\partial\Omega_0$. According to Lemma 1 of [2], the function $(t, x) \mapsto J_t := J_{\Omega_0+t\,\Lambda}(x)$ is smooth at least in $[0, 1] \times (\mathbb{R}^n \setminus \{0\})$ and we have*

$$\frac{d}{dt}J_t = -J_t P_\Lambda(\nabla J_t). \tag{5.10}$$

Using the Taylor expansions of J_t, we have

$$\begin{aligned} J_t &= J_0 + t\frac{d}{dt}\, J_t|_{t=0} + o(t) \\ &= J_0 - tJ_0 P_\Lambda(\nabla J_0) + o(t) \\ &= J_0 - tJ_0 P_\Lambda(\nabla J_{\Omega_0}/|\nabla J_{\Omega_0}|)|\nabla J_{\Omega_0}| + o(t). \end{aligned}$$

Thus, for all $y \in \partial\Omega_0$, using Lemma 2.2 we obtain $J_t(y) = 1 - t\,\frac{P_\Lambda(\nu_{\Omega_0}(y))}{\langle\nu_{\Omega_0}(y), y\rangle} + o(t)$. Finally, $\partial\Omega_1$ can be determined by

$$\partial\Omega_1 = \left\{ y/\left(1 - t\,\frac{P_\Lambda(\nu_{\Omega_0}(y))}{\langle\nu_{\Omega_0}(y), y\rangle}\right)\ ;\ y \in \partial\Omega_0 \right\},$$

where $P_\Lambda(\nu_{\Omega_0})$ and ν_{Ω_0} are known on $\partial\Omega_0$.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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