Received: 6 October 2021 | Revised: 14 November 2022 | Accepted: 29 December 2022

DOI: 10.1002/asna.20210039

Astronomische Nachrichten

ORIGINAL ARTICLE

Alpha tensor and dynamo excitation in turbulent fluids with anisotropic conductivity fluctuations

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Abstract

A mean-field theory of the electrodynamics of a turbulent fluid is formulated under the assumption that the molecular electric conductivity is correlated with the turbulent velocity fluctuation in the (radial) direction, $\boldsymbol{g}$. It is shown that for such homogeneous fluids a strong turbulence-induced field advection anti-parallel to $\boldsymbol{g}$ arises almost independently of rotation. For rotating fluids, an extra α effect appears with the known symmetries and with the expected maximum at the poles. Fast rotation, however, with Coriolis number exceeding unity suppresses this term. Numerical simulations of forced turbulence using the nirvana code demonstrate that the radial advection velocity, γ, always dominates the α term. We show finally with simplified models that α^2 dynamos are strongly influenced by the radial pumping: for $\gamma < \alpha$ the solutions become oscillatory, while for $\gamma > \alpha$ they become highly exotic if they exist at all. In conclusion, dynamo models for slow and fast solid-body rotation on the basis of finite conductivity–velocity correlations are unlikely to work, at least for $\alpha^2\Omega$ dynamos without strong shear.

KEYWORDS

astrophysical plasma, dynamo theory

1 | INTRODUCTION

If apart from the velocity and magnetic field, for any reason also the electric conductivity in a turbulent fluid fluctuates around a certain value then also the local magnetic diffusivity fluctuates around its average. Krause and Roberts (1973) started to consider the consequences of this constellation with the result that the effective decay time of a large-scale nonuniform magnetic field is changed by reducing the effective eddy diffusivity of the turbulence field.

Moreover, in convection-driven turbulent fields, the always-existing temperature fluctuations should produce magnetic resistivity fluctuations which are correlated with one of the velocity components, for example, the vertical one. In this case, even a turbulent diffusivity-flux vector $\langle \eta' \boldsymbol{u}' \rangle$—with $\eta = 1/\mu_0\sigma$ denoting the magnetic resistivity and $\boldsymbol{u}'$ the velocity fluctuations—occurs. This, in connection with the magnetic background field or electric current, may form new terms in the mean-field induction equation. Pétrélis et al. (2016) suggested a new sort of α effect arising in such systems.

They derived an expression for the diffusivity-current correlation, in which the diffusivity-flux vector, multiplied with the mean magnetic field, $\overline{\boldsymbol{B}}$, appears so that a new α effect could be possible in spite of the assumed homogeneity of the turbulence field. However, there are two possibilities for the relation between the electromotive force and the mean magnetic field: the latter can be (i) parallel to the electromotive force or (ii) perpendicular to the electromotive force. Only in the first case, one formally speaks of an α effect, which may lead to self-excitation of large-scale magnetic fields, while in the second case the expression describes a turbulent diamagnetism (also called "topological pumping") which is known to hamper dynamo instability. If the correlation $\langle \eta' \boldsymbol{u}' \rangle$ exclusively defines a preferred direction $\boldsymbol{g}$ the resulting turbulent electromotive force is perpendicular to the mean magnetic field and an alpha-effect is not obtained.

Later on, quasi-linear SOCA calculations applicable to rotating forced turbulence and/or magneto-convection indeed confirmed the existence of an α effect in the presence of global rotation. Without rotation, the conductivity fluctuations lead to a reduction of the eddy diffusivity and—if correlated with one of the velocity components—to a new but rather strong diamagnetic pumping effect (Rüdiger et al. 2020). In that work, rotating magneto-convection was numerically used to derive the radial turbulent electric current flux $\left\langle u_r' \mathrm{curl} \boldsymbol{B}' \right\rangle$—where r is the radial coordinate—which serves as a proxy of the turbulent diffusivity-current vector $\left\langle \eta' \mathrm{curl} \boldsymbol{B}' \right\rangle$ if η' and u_r' are correlated or anti-correlated. The flux vector always exists for rotating convection under the influence of an azimuthal magnetic background field. The result is a well-defined diamagnetic pumping and, with rotation, an α effect, which is anti-symmetric with respect to the equator.

However, convection only exists if the fluid is stratified in the radial direction, $\boldsymbol{g}$. The main difference caused by the fluctuating-conductivity concept is the occurrence of an α effect in fully uniform fluids in which an anisotropy exists rather than any form of stratification. This makes the idea a promising one for a dynamo theory of planetary magnetism.

In the present paper, therefore, the existence of the α effect in absolutely homogeneous fluids is shown by numerical simulations of forced rotating turbulence. We shall demonstrate that the α effect indeed occurs, if the global rotation is not too slow or too fast but that it is, however, always accompanied by a dominating diamagnetic pumping term, γ. Even without rotation (and only slightly suppressed in its presence) a strong radial advection term occurs by which the horizontal field (i.e., perpendicular to $\boldsymbol{g}$) is lifted to either of the radial boundary layers, depending on the sign of the effect.

We note that a large-scale α^2 dynamo can in principle operate for very *weak* α effect if only the region is big enough, or—with other words—if it hosts a sufficiently large number of eddies. In our final Section, the consequences of this puzzling situation are shown by the presentation of a sequence of mean-field α^2 dynamo models with stronger and stronger magnetic pumping term (i.e., turbulence-induced diamagnetism). We shall show that such dynamos can only operate as long as the α term (in form of a pattern velocity) exceeds the pumping velocity. This condition is unfortunately not met—at least, according to the results of the derived electrodynamics, which is based on the correlations with conductivity fluctuations.

2 | THE EQUATIONS

The basic equation of the problem is the induction equation

$$\frac{\partial \boldsymbol{B}}{\partial t} = \mathrm{curl}(\boldsymbol{u} \times \boldsymbol{B} - \eta\ \mathrm{curl} \boldsymbol{B}), \tag{1}$$

with the continuity condition $\mathrm{div}\, \boldsymbol{B} = 0$. Moreover, we assume $\mathrm{div} \boldsymbol{u} = 0$ as the condition for an incompressible fluid for the analytic derivations, while for the numerical experiments, this constraint is relaxed. Here, $\boldsymbol{u}$ is the fluid velocity, $\boldsymbol{B}$ is the magnetic field vector and η the (molecular) magnetic diffusivity. We consider a turbulent fluid with $\boldsymbol{u} = \boldsymbol{u} + \boldsymbol{u}'$ and with a fluctuating magnetic diffusivity $\eta = \overline{\eta} + \eta'$. For the expectation values of the perturbations we shall use the notations $u_{\mathrm{rms}} = \left\langle \boldsymbol{u}'^2 \right\rangle^{1/2}$ and $\eta_{\mathrm{rms}} = \left\langle \eta'^2 \right\rangle^{1/2}$. Large-scale observables (i.e., mean values) are marked with overbars, while brackets are used for the correlations of fluctuations. Low or high values of the magnetic Reynolds number

$$\mathrm{Rm} = u_{\mathrm{rms}} \frac{\ell}{\eta} \tag{2}$$

(for Strouhal number $\simeq 1$, and with ℓ the correlation length) distinguish between the regimes of low/high conductivity. Within the realm of the electrodynamics with finite fluctuations, the high-conductivity limit $\overline{\eta} \to 0$ may not be allowed.

If the fluctuations $\boldsymbol{u}'$ and η' exist and are correlated, then the turbulence-originated diffusivity flux

$$\boldsymbol{U} = \left\langle \eta' \boldsymbol{u}' \right\rangle \tag{3}$$

forms a vector, which is polar by definition. The existence of the radial component of this vector is obvious for thermal convection, where both the radial velocity and the electric conductivity are due to temperature fluctuations. The correlation (3) can be understood as transport of magnetic diffusivity in a certain direction. If, for example, the correlation between η' and $u_{r'}$ is positive then resistivity is transported upwards—balanced by a downward radial velocity $\nabla(-\eta)$ which "pumps" the horizontal field downwards in the direction where the magnetic decay is maximum (the "diamagnetic effect" of turbulent origin).

Also the magnetic field will fluctuate, hence $\boldsymbol{B} = \overline{\boldsymbol{B}} + \boldsymbol{B}'$. The magnetic fluctuation $\boldsymbol{B}'$ fulfills a nonlinear induction equation, which follows from (1). The turbulence-originated electromotive force $\boldsymbol{\mathcal{E}} = \langle \boldsymbol{u}' \times \boldsymbol{B}' \rangle$ and the diffusivity-current correlation $\boldsymbol{\mathcal{J}} = -\langle \eta' \mathrm{curl} \boldsymbol{B}' \rangle$ enter the induction equation for large-scale magnetic field via

$$\frac{\partial \overline{\boldsymbol{B}}}{\partial t} = \mathrm{curl}(\boldsymbol{\mathcal{E}} + \boldsymbol{\mathcal{J}} - \overline{\eta}\ \mathrm{curl}\overline{\boldsymbol{B}}). \tag{4}$$

Under the assumption that the large-scale field, $\overline{\boldsymbol{B}}$, varies sufficiently slowly in space and time, the electromotive force can be written as

$$\boldsymbol{\mathcal{E}} = \alpha \circ \overline{\boldsymbol{B}} - \eta_{\mathrm{t}} \mathrm{curl} \overline{\boldsymbol{B}}, \tag{5}$$

where the tensor α and the coefficient η_{t} represent the α effect and the turbulent magnetic diffusivity (Krause & Rädler 1980), respectively, and where '$\circ$' denotes a tensor multiplication. The tensorial structure of η_{t} under the presence of magnetic field and rotation has been discussed later by Kitchatinov et al. (1994). As in Rüdiger et al. (2020), the spectral vector of the correlation (3) may be written as

$$\hat{U}_i = u_1(k, \omega) \left(g_i - \frac{(\boldsymbol{g} \cdot \boldsymbol{k}) k_i}{k^2} \right). \tag{6}$$

The vector $\boldsymbol{g}$ gives the unit vector of the direction in which the correlation between velocity and diffusivity is non-vanishing. The expression (6) must be odd in $\boldsymbol{g}$ and its real part must be even in the wave number $\boldsymbol{k}$. The quantity u_1 reflects the correlation of the velocity component $\boldsymbol{g} \cdot \boldsymbol{u}'$ with η' where ω is the Fourier frequency of the spectrum. As it should, the transformation $\boldsymbol{g} \to -\boldsymbol{g}$ only changes the sign of $\boldsymbol{U}$.

3 | THE DIFFUSIVITY-CURRENT CORRELATION

It has been shown earlier that a relation

$$\boldsymbol{\mathcal{J}} = -\gamma\ \boldsymbol{g} \times \overline{\boldsymbol{B}} \tag{7}$$

between the diffusivity-current correlation, $\boldsymbol{\mathcal{J}}$, and the large-scale magnetic field, $\overline{\boldsymbol{B}}$, results with

$$\gamma = \frac{1}{3} \int \int \frac{\overline{\eta} k^4 u_1}{\omega^2 + \overline{\eta}^2 k^4}\ \mathrm{d}\boldsymbol{k}\, \mathrm{d}\omega, \tag{8}$$

representing a turbulent advection of the magnetic background field where $\boldsymbol{u}_{\mathrm{adv}} = -\gamma \boldsymbol{g}$ is the advection velocity (Rüdiger et al. 2020). We find a coefficient γ of the same sign as the diffusivity flux (3). For positive u_1 (i.e., for positive correlation of η' and u_r'), the advection velocity, $\boldsymbol{u}_{\mathrm{adv}}$, points downward if $\boldsymbol{g}$ is the radial unit vector. Anti-correlated η' and $u_{r'}$ lead to an upward turbulent transport of the mean magnetic field. This means that the field is always attracted by the islands of lower resistivity—or, equivalently, of higher electric conductivity. As a consequence, the large-scale magnetic field favors the direction toward longer diffusive decay times. The advection velocity is opposite to the diffusivity flux (3). The integral expression for γ of Equation (8) scales linearly with Rm until it saturates for large magnetic Reynolds numbers.

Let $\hat{V}$ be the spectral function of the two-point autocorrelation function $V(\xi, \tau) = \langle \eta'(\boldsymbol{x}, t) \eta'(\boldsymbol{x} + \xi, t + \tau) \rangle$ of the diffusivity fluctuations. For the diffusivity-current correlation $\boldsymbol{\mathcal{J}}$ the term with $\hat{V}$ leads to

$$\boldsymbol{\mathcal{J}} = \ldots + \frac{2}{3} \int \int \frac{k^2 \hat{V}}{-\mathrm{i}\omega + \overline{\eta} k^2}\ \mathrm{d}\boldsymbol{k}\, \mathrm{d}\omega\ \ \mathrm{curl} \overline{\boldsymbol{B}}, \tag{9}$$

which provides an extra contribution to the magnetic field dissipation. The question is whether this term reduces or enhances the eddy diffusivity η_{t} representing turbulence without η-fluctuations. The small-scale diffusivity fluctuations obviously lead to a *reduction* of the large-scale eddy diffusivity η_{t} which, however, is only weak as it runs with the small value $(\eta_{\mathrm{rms}}/\overline{\eta})$ in second order (Krause & Roberts 1973; Rüdiger et al. 2020). The actual value of the turbulence dissipation will not have relevance for the results of the present paper.

Our assumed background turbulence is homogeneous but anisotropic, where the anisotropy is only implicit. If the turbulence rotates, an additional pseudo-scalar $\boldsymbol{g} \cdot \boldsymbol{\Omega}$ appears with which a relation

$$\begin{aligned} \boldsymbol{\mathcal{J}} = -\gamma\, \boldsymbol{g} \times \overline{\boldsymbol{B}} - \alpha_1 [(\boldsymbol{g} \cdot \overline{\boldsymbol{B}}) \boldsymbol{\Omega} \ \ + (\boldsymbol{g} \cdot \boldsymbol{\Omega}) \overline{\boldsymbol{B}}] \\ - \alpha_2 (\overline{\boldsymbol{B}} \cdot \boldsymbol{\Omega}) \boldsymbol{g} \end{aligned} \tag{10}$$

can be formulated—with yet unknown coefficients α_1 and α_2 for the diffusivity-current correlation, $\boldsymbol{\mathcal{J}}$, in presence of a large-scale magnetic field and rotation. For the above expression, γ is again given by Equation (8). Relation (10) formally describes the existence of an α tensor, which connects the correlation $\boldsymbol{\mathcal{J}}$ with the large-scale magnetic field $\overline{\boldsymbol{B}}$. This connection exists despite the turbulence model being assumed as strictly homogeneous (so that the standard α tensor cannot appear). The α effect according to (10) is highly anisotropic, the middle term with the coefficient α_1 provides the rotation-induced standard α expression. While the diamagnetic term with γ also exists for $\Omega = 0$, the α terms need global rotation. We shall show below that, independently of the sign of the correlations $\langle \eta' u_r' \rangle$, the values of α_1 and γ are always of opposite sign.

The dimensionless ratio

$$\hat{\gamma} = \frac{\gamma}{\alpha_1 \Omega} \tag{11}$$

of the pumping velocity γ and the rotation-induced α effect indicates the ratio of anti-symmetric and symmetric elements in the complete α tensor. Simulating electromotive forces for models of rotating magnetoconvection, Ossendrijver et al. (2001); Ossendrijver et al. (2002) found $\hat{\gamma} \simeq 1$ where both α and γ were about 10% of the rms value of the convective velocity. Also Käpylä et al. (2009) reached typical values of order unity in their numerical models of turbulent magnetoconvection. Additionally, with their extensive numerical simulations, Gressel et al. (2008) derived $\hat{\gamma} = O(1)$ for interstellar turbulence driven by collective supernova explosions. All these examples summarize the results of α effect calculations from the relation between the electromotive force $\boldsymbol{\mathcal{E}}$ and the mean magnetic field $\overline{\boldsymbol{B}}$, which only appears if the turbulence is nonuniform. On the other hand, we shall demonstrate in the following that for *homogeneous* models with fluctuating conductivities, the corresponding ratio (11) reaches values even exceeding unity—with severe consequences for associated dynamo models.

4 | NUMERICAL METHODS

To probe the theoretical predictions we run artificially forced, fully nonlinear numerical simulations with the nirvana MHD code (Ziegler 2004), which solves the equations of compressible magnetohydrodynamics by means of a second-order Godunov approach. In the simulations, the fluctuating component of the magnetic diffusivity is prescribed by $\eta' = c_u u_z$, where the coefficient c_u is used to control the strength of the correlation. We furthermore use $\eta_{\rm rms} = c_u u_{z,{\rm rms}}$ to quantify the amplitude of the fluctuating part of the magnetic diffusivity. The simulation domain is a fully periodic cube with volume L^3. The units of length and time are $[x] = k_1^{-1}$, $[t] = (c_{\rm s} k_1)^{-1}$ where k_1 is the wave number corresponding to the system size and $c_{\rm s}$ is the constant speed of sound. The simulations employ standard non-helical forcing according to Equation (7) of Haugen et al. (2004) and are characterized by the magnetic Reynolds number (2) with $u_{\rm rms}$ volume averaged and $\ell = (k_{\rm f})^{-1}$. The flows under consideration are weakly compressible with Mach number ${\rm Ma} = u_{\rm rms}/c_{\rm s} \approx 0.1$. All simulations have $k_{\rm f} \simeq 4.5$ (using isotropically sampled discrete wave vectors obeying $4 \leq k_{\rm f} \leq 5$) and employ a grid resolution of 80^3. In code units, the molecular diffusivity is fixed at $\overline{\eta} = 0.02$.

5 | THE TURBULENT FLUX OF ELECTRIC CURRENT

Consider a homogeneous and isotropic turbulence that is influenced by uniform magnetic fields and global rotation. Let us write its correlation tensor, $\left\langle u_{i'} \ \mathrm{curl}_j \boldsymbol{B}' \right\rangle$, as

$$\left\langle u_{i'} \mathrm{curl}_j \boldsymbol{B}' \right\rangle = \kappa' \varepsilon_{jik} \overline{\boldsymbol{B}}_k + \kappa_1 \Omega_i \overline{\boldsymbol{B}}_j + \kappa_2 \Omega_j \overline{\boldsymbol{B}}_i + \kappa_3 (\boldsymbol{\Omega} \cdot \overline{\boldsymbol{B}}) \delta_{ij}. \tag{12}$$

The tensor is not a pseudo-tensor and there is no reason that the dimensionless coefficients κ identically vanish. It does not play a known role in the mean-field electrodynamics but it is exploited here as a proxy of the desired diffusivity-current correlation. The correlation vector $\left\langle u_{r'} \ \mathrm{curl} \boldsymbol{B}' \right\rangle$ describes an upward or downward radial flux of electric current in a rotating magnetized turbulence which we shall use below to estimate the diffusion-current correlation $\boldsymbol{\mathcal{J}}$. We note that for $\Omega = 0$ it is $\left\langle \left(\boldsymbol{g} \cdot \boldsymbol{u}' \right) \ \mathrm{curl} \boldsymbol{B}' \right\rangle = \kappa' \boldsymbol{g} \times \ \overline{\boldsymbol{B}}$ for all directions $\boldsymbol{g}$. With $\boldsymbol{g}$ as the radial direction, one finds

$$\left\langle u_{r'} \mathrm{curl}_\theta \boldsymbol{B}' \right\rangle = - \left\langle u_{\theta'} \mathrm{curl}_r \boldsymbol{B}' \right\rangle = -\kappa' \overline{B}_\phi, \tag{13}$$

if the magnetic background field only has an azimuthal component. Based on SOCA calculations, the coefficient κ' is

$$\kappa' = \frac{1}{15} \int_0^\infty \int_0^\infty \frac{\eta k^4 E(k, \omega)}{\omega^2 + \eta^2 k^4} \ \mathrm{d}k \mathrm{d}\omega, \tag{14}$$

with the positive spectral function E of the turbulence intensity,

$$u_{\rm rms}^2 = \int_0^\infty \int_0^\infty E(k, \omega) \ \mathrm{d}k \ \mathrm{d}\omega. \tag{15}$$

As the spectrum $E(k, \omega)$ is positive-definite, the tensor coefficient κ' is positive-definite, too.

Figure 1 gives a numerical representation of the complete tensor (12) in Cartesian coordinates $(r, \theta, \phi) \rightarrow (x, y, z)$, where the rotation vector is $\boldsymbol{\Omega} = \Omega_0(\cos\theta, -\sin\theta, 0)$ and the magnetic field $\overline{\boldsymbol{B}} = (0, 0, B_0)$. The details of the simulations were given in the previous Section. Obviously, the κ_3 coefficient in (12) cannot be determined for this geometry as always $\boldsymbol{\Omega} \perp \boldsymbol{B}$. It is clear from the uppermost and the lowermost curves in the left and the right panel that after (13) the simulation gives $\kappa' > 0$ in accordance to the result (14) of the quasi-linear theory. Only the *xy*-component is anti-symmetric in its indices but the cross correlations *xz* and *yz* are symmetric. The diagonal components *xx*, *yy* and *zz* vanish (not shown) in accordance to the relation (12).

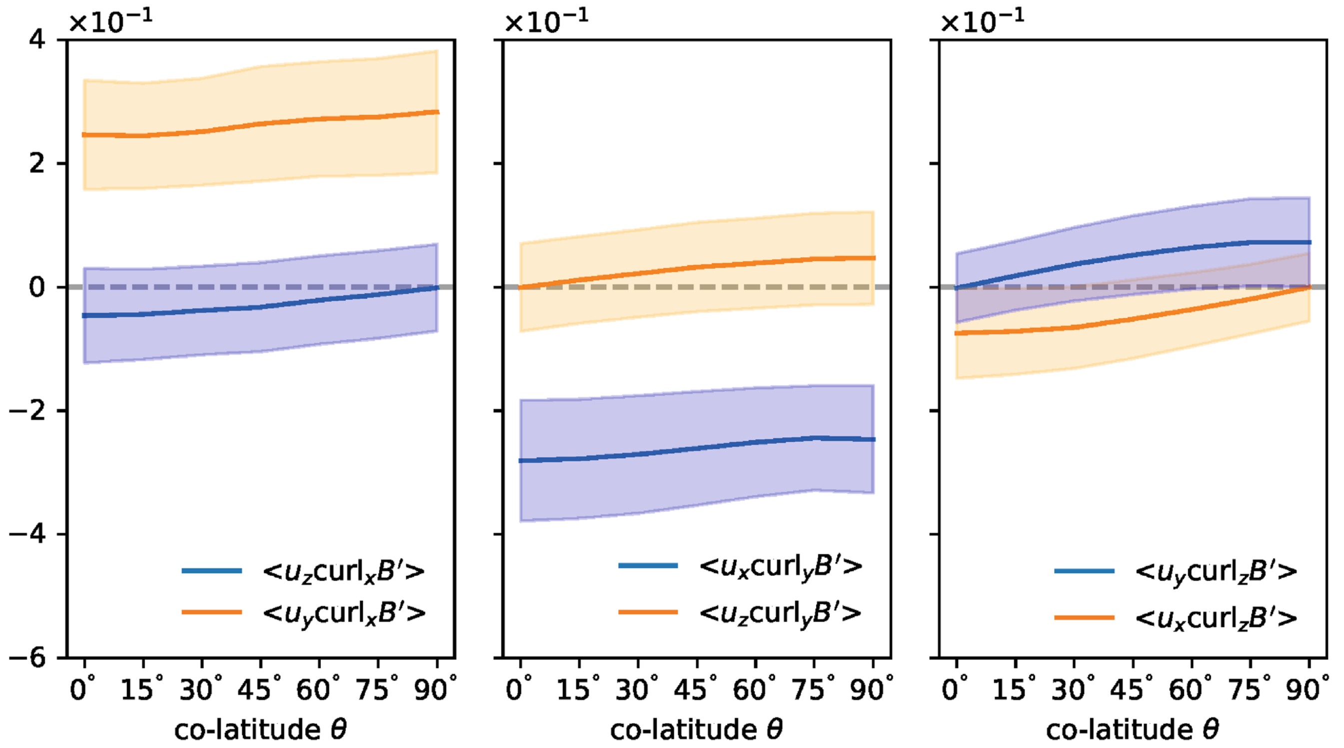

FIGURE 1 The off-diagonal components (expectation value plus temporal variations) of the turbulence-induced electric-current flux tensor $\langle u_i' \mathrm{curl}_j \boldsymbol{B}' \rangle$ normalized with $u_{\mathrm{rms}} k_{\mathrm{f}} B_0$ for various co-latitudes. The plot reflects the symmetry of the tensor except the *xy*-component which is anti-symmetric in accordance with Equation (18). Rm = 100, $B_0 = 2 \times 10^{-8}$, Pm = 1, Ω = 1

For the remaining off-diagonal tensor components, one finds $\kappa_1 = \kappa_2 = \kappa$ with $\kappa < 0$ as

$$\langle u_r' \mathrm{curl}_\phi \boldsymbol{B}' \rangle = \left\langle u_\phi' \mathrm{curl}_r \boldsymbol{B}' \right\rangle = \kappa \Omega \overline{B}_0 \cos\theta < 0 \qquad (16)$$

and

$$\langle u_\theta' \mathrm{curl}_\phi \boldsymbol{B}' \rangle = \left\langle u_\phi' \mathrm{curl}_\theta \boldsymbol{B}' \right\rangle = -\kappa \Omega \overline{B}_0 \sin\theta > 0, \qquad (17)$$

hence for rotating and magnetized (but otherwise isotropic) turbulence, the tensor expression (12) becomes

$$\langle u_i' \ \mathrm{curl}_j \boldsymbol{B}' \rangle = \kappa' \epsilon_{jik} \overline{B}_k + \kappa \left(\Omega_i \overline{B}_j + \Omega_j \overline{B}_i \right) + \kappa_3 (\boldsymbol{\Omega} \cdot \overline{\boldsymbol{B}}) \delta_{ij}. \qquad (18)$$

In a rotating but otherwise isotropic turbulence with an azimuthal background field, the meridional flow fluctuations will always be correlated with the azimuthal electric-current fluctuations. We note that the simulations show that the anti-symmetric (*xy*)-component of the tensor is *always* much larger than the symmetric (*xz*)-component—which, in fact, will have important consequences.

Replace now in the relations (13) and (16) u_r' by η' and the existence of correlations such as $\langle \eta' \mathrm{curl}_\theta \boldsymbol{B}' \rangle$ and $\langle \eta' \mathrm{curl}_\phi \boldsymbol{B}' \rangle$ becomes obvious in (rotating) homogeneous turbulence fields magnetized with an azimuthal background field. Just this finding is formulated by Equation (10). For positive correlation of the η-fluctuation and the radial velocities (i.e., positive U_r), the α_1 in (10) becomes negative and for negative correlations it becomes positive. Note the negative sign in the definitions. In the same relation, the γ results as positive —hence the pumping is downward (i.e., opposite to U_r). We always obtain $\gamma \alpha_1 \leq 0$ for both signs of U_r.

Another basic finding is that the term with κ' always exceeds those with κ, which—in other words—means that, for rotating turbulence, the pumping term (a velocity) will always be larger than the α term (also a velocity). As a consequence, in rotating conducting fluids, the diamagnetic effect may by far exceed the inducting action of the α terms. The remainder of this paper will confirm this suggestion and will show that a dominating turbulent pumping precludes dynamo instability of the α^2-type, that is, in the absence of large-scale shear.

Figure 2 numerically shows the influence of the magnetic Prandtl number on the off-diagonal components of tensor (18). The values are taken for mid-latitudes. The Pm varies by more than one order of magnitude. The numerical values basically grow for growing Prandtl number. Nevertheless, the ratio of the negative

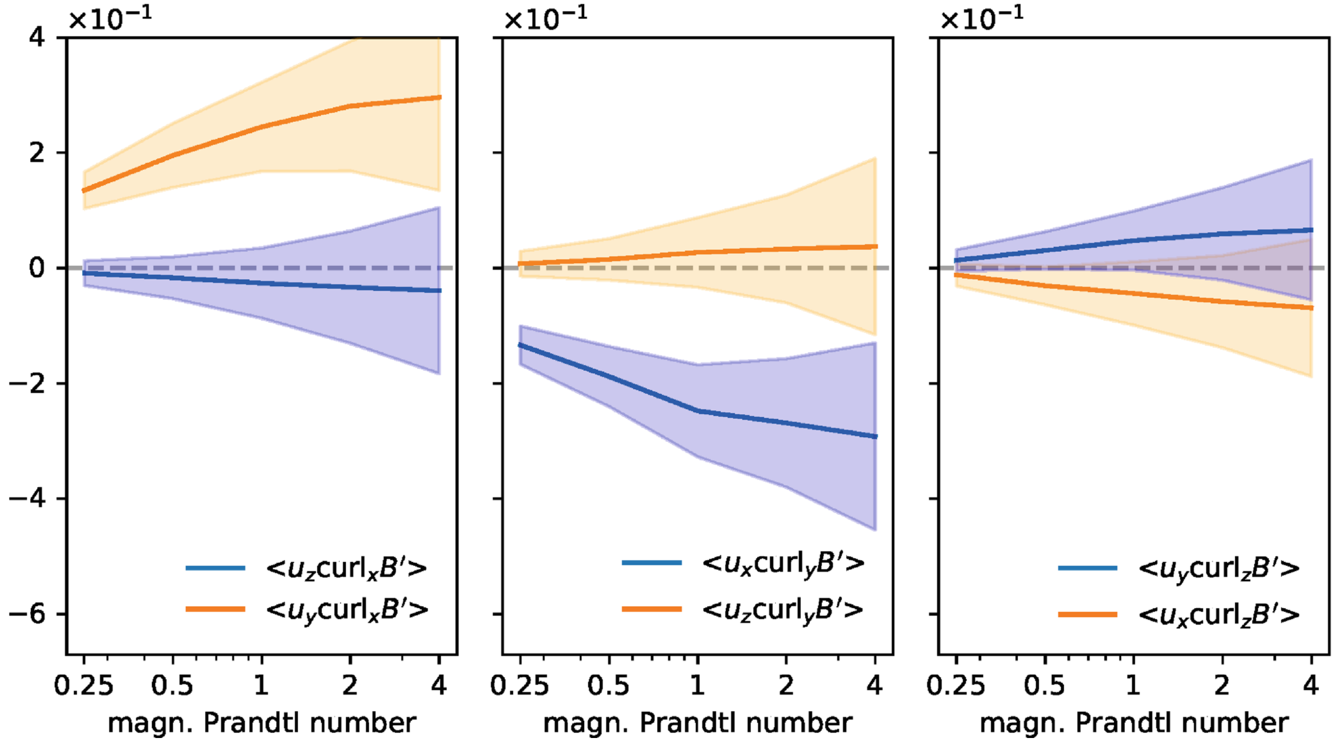

FIGURE 2 Similar to Figure 1 but for $\theta = 45°$ and for various magnetic Prandtl numbers. The blue line in the middle panel ($\langle u_{x'} \mathrm{curl}_y \boldsymbol{B}' \rangle$, leading to the advection term) and the orange line in the right panel ($\langle u_{x'} \mathrm{curl}_z \boldsymbol{B}' \rangle$, leading to the α effect) are of particular relevance. The ratio $\langle u_{x'} \mathrm{curl}_y \boldsymbol{B}' \rangle / \langle u_{x'} \mathrm{curl}_z \boldsymbol{B}' \rangle$ for all Pm always exceeds unity. $\mathrm{Rm} = 11$, $\Omega = 1$

quantities $\langle u_{x'} \mathrm{curl}_y \boldsymbol{B}' \rangle$ and $\langle u_{x'} \mathrm{curl}_z \boldsymbol{B}' \rangle$ remains numerically always much larger than one, also for the important case of $\mathrm{Pm} < 1$.

The following numerical simulations in a Cartesian box with the vertical (radial) vector $\boldsymbol{g} = (1{,}0{,}0)$ have been done with a *negative* correlation between diffusivity fluctuation and vertical velocity, hence $U_x < 0$. Again, the applied magnetic field is azimuthally directed, $\overline{\boldsymbol{B}} = (0, 0, B_0)$. We find

$$\gamma = -\frac{\langle \eta' \mathrm{curl}_y \boldsymbol{B}' \rangle}{B_0}, \qquad \alpha_1 \Omega = \frac{\langle \eta' \mathrm{curl}_z \boldsymbol{B}' \rangle}{B_0}. \tag{19}$$

Figure 3 displays the three components of the diffusivity-current vector as function of the co-latitude θ. As it should, its radial component vanishes (left panel). It is also understandable that the advection term $\langle \eta' \mathrm{curl}_y \boldsymbol{B}' \rangle$ is positive and does hardly depend on the latitude. According to the first relation in (19), $\gamma < 0$—so that the advection velocity $\boldsymbol{u}_{\mathrm{adv}}$ is directed upwards (i.e., opposite to U_x).

Contrary to this, the z-component of the correlation vector vanishes at the equator—as it is expected for a rotation-induced α-term. Its maximum is obtained at the poles. According to the second definition (19), one finds a positive α_1. Note that the negative sign of the product $\gamma\alpha_1$ is independent of the sign of the correlation of η' and u_x'.

The simulated components of the correlation vector $\langle \eta' \mathrm{curl} \boldsymbol{B} \rangle$ for fixed rotation rate have been given in Figure 3. For a characteristic value $\eta_{\mathrm{rms}}/\overline{\eta} = 0.1$ of the diffusivity fluctuations, the rotation frequency is varied in Figure 4 to obtain the characteristic numbers at the northern pole. Obviously, the maximal correlation appears for rotation $\Omega \simeq 1$ and will be suppressed by faster rotation.

For the ratio (11) we generally obtain a value of about five. The normalized α effect is

$$C_\alpha = \frac{\alpha_1 \Omega L}{\overline{\eta} + \eta_{\mathrm{t}}} \simeq \frac{3\alpha_1 \Omega}{u_{\mathrm{rms}}} \frac{L}{\ell}, \tag{20}$$

with L as the box length in code units. The characteristic turnover time of the turbulence is $\tau_{\mathrm{corr}} \simeq \ell / u_{\mathrm{rms}} \simeq 2$ in the simulation (also in code units), where $u_{\mathrm{rms}} \simeq 0.11$ is set by the amplitude of the forcing. It is $\eta_{\mathrm{t}}/\overline{\eta} \simeq 0.3 u_{\mathrm{rms}}^2 \tau_{\mathrm{corr}}/\overline{\eta} \simeq 0.3$. According to Figure 3 and Equation (19), we have $\alpha_1 \Omega / u_{\mathrm{rms}} \simeq 5 \cdot 10^{-3}$ so that

$$C_\alpha \simeq 1.5 \cdot 10^{-2} \frac{L}{\ell}. \tag{21}$$

The ratio L/ℓ gives the number of cells along the vertical direction, which obviously must exceed 70 to reach C_α of order unity. This is one of the arguments that it would not be easy to simulate such a dynamo in a box.

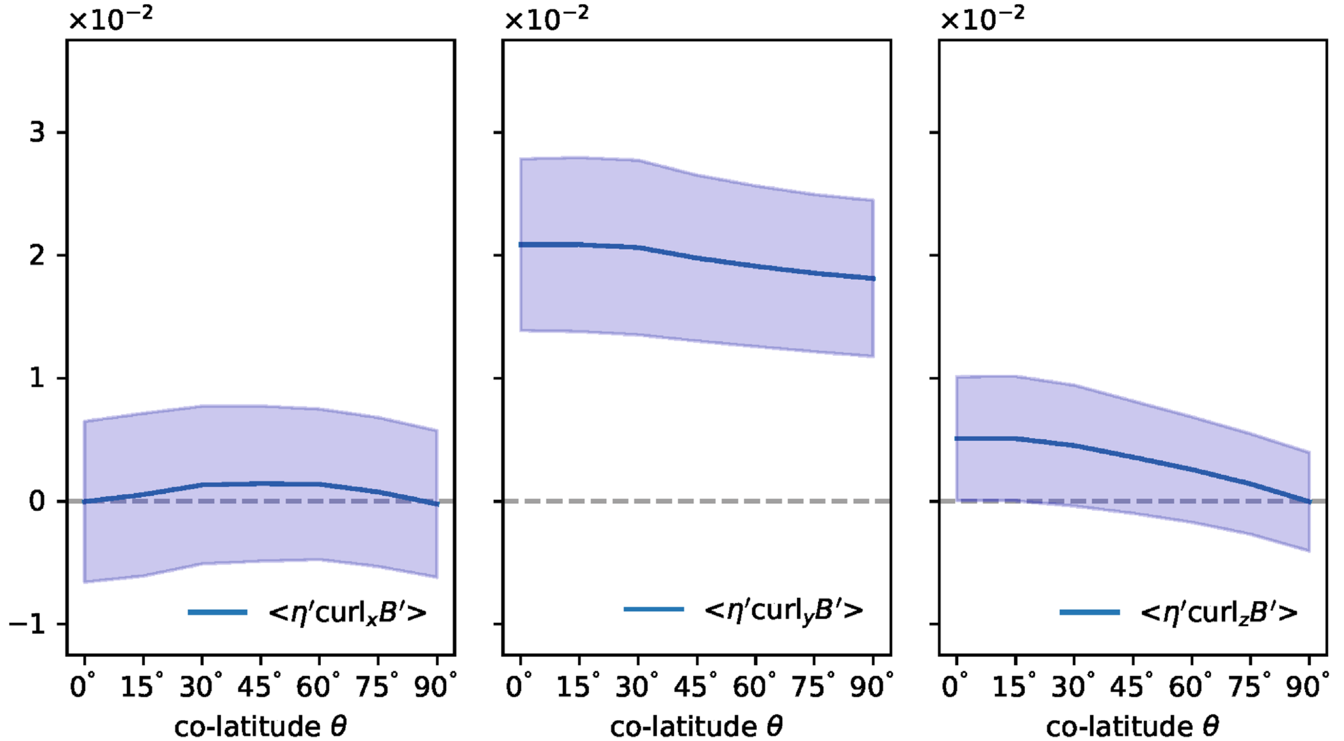

FIGURE 3 The three components of the diffusivity-current vector $\langle \eta' \mathrm{curl} \boldsymbol{B}' \rangle / (u_{\mathrm{rms}} B_0)$ versus co-latitude. $\mathrm{Rm} = 11$. $\eta_{\mathrm{rms}}/\overline{\eta} = 0.1$, $B_0 = 2 \times 10^{-8}$, $\mathrm{Pm} = 1$, $\Omega = 1$

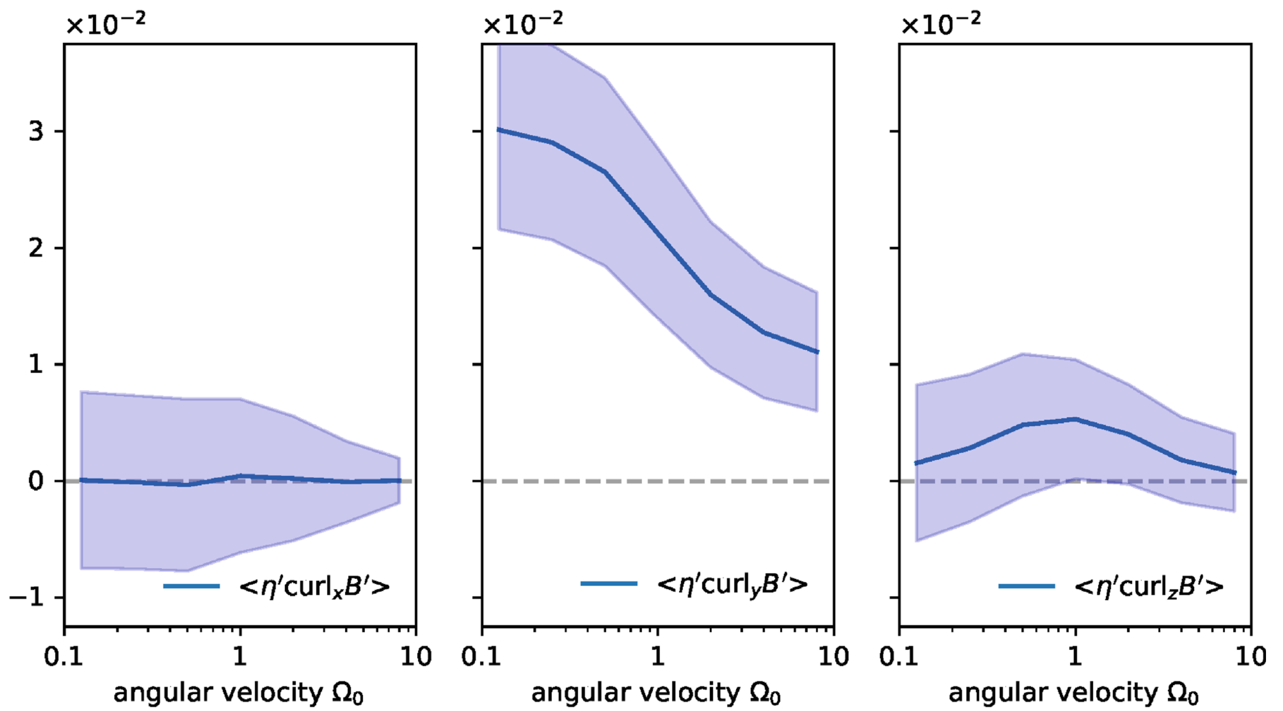

FIGURE 4 Similar to Figure 3 but at the northern pole and for increasing rotation rate Ω

The dependencies of the diffusivity-current vector components on the rotation rate Ω_0 are shown in Figure 4, where for $\Omega_0 = 1$, the rotation period is 2π. As usual, the Strouhal number $\mathrm{St} = u_{\mathrm{rms}} \tau_{\mathrm{corr}} / \ell$ results of order unity. We also note that $\Omega_0 = 1$ describes a rapid rotation with a Coriolis number of $2\tau_{\mathrm{corr}}\Omega \simeq 4.4$ beyond which the α effect is strongly quenched by the rotation (Figure 4, right panel). The maximum correlation exists for $\Omega = 1$; one cannot

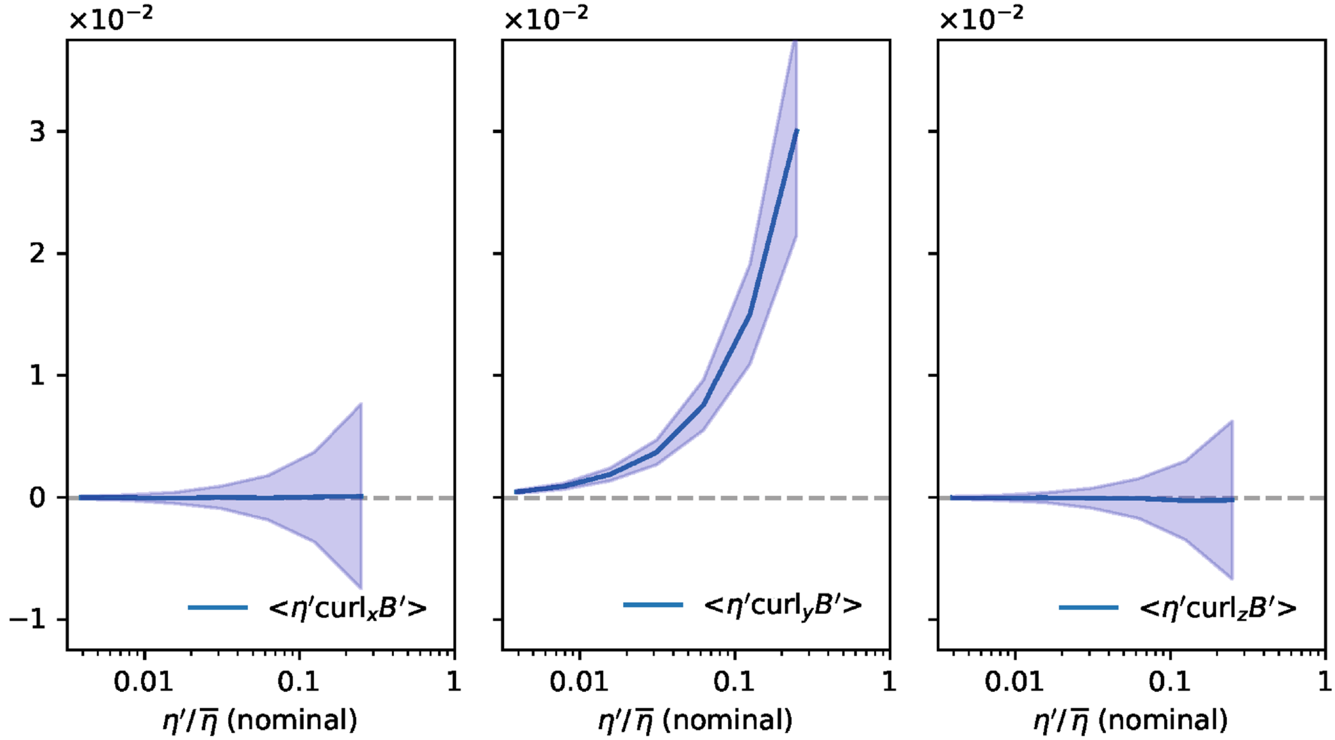

FIGURE 5 The three components of the vector $\langle \eta' \mathrm{curl} \boldsymbol{B}' \rangle / (u_{\mathrm{rms}} B_0)$ for non-rotating turbulence. In accordance with Equation (10) only the y-component (representing the topological pumping) remains finite. $\mathrm{Rm} = 11$, $\mathrm{Pm} = 1$, $\Omega = 0$

increase this value by faster rotation. For $\Omega = 1$ it is $\hat{\gamma} \simeq 5$, and this ratio even *grows* for slower and/or faster rotation. A weak rotational quenching can also be observed in the middle panel, where the advection term is reduced (only) by a factor of three when Ω_0 grows by two orders of magnitude.

Figure 5 refers to non-rotating turbulence with growing ratio of the diffusivity fluctuations, $\eta_{\mathrm{rms}}/\overline{\eta}$. As expected, the curve in the middle panel linearly runs with the normalized diffusivity fluctuation in accordance with the γ expression (8), and it vanishes for $\eta' \to 0$. For non-rotating turbulence, of course, the two remaining components (including the α effect) are identical zero—as shown in the left and the right panel of Figure 5. As it should, the advection term plotted in the middle panel also exists for non-rotating turbulent fluids. We still have to find out how the calculated large values of $\hat{\gamma}$ influence the operation of a global dynamo.

6 | KINEMATIC α^2 DYNAMO MODELS WITH FIELD ADVECTION

It has been shown that for rotating turbulence, the above formulated α effect is always accompanied by a pumping effect in the direction of the component of the flow vector which is correlated with conductivity fluctuations. For all rotation rates, the ratio $\hat{\gamma}$ exceeds unity. We now turn our inquiry to the influence of the turbulent field advection on the operation of an α^2 dynamo. In earlier papers, we already found for disk dynamo models that a too-strong field advection suppresses the field excitation even under the presence of differential rotation (Schultz et al. 1994).

The geometrically simplest model is posed by uniform quantities α and γ existing in a gap between two parallel plates embedded in vacuum. The vertical distance between the boundaries is H. The eddy diffusivity η_0 between the plates is assumed as a free parameter, whose actual value is not important for the result. All quantities are assumed as uniform in the two horizontal directions y and z. Then the condition $\mathrm{div}\, \boldsymbol{B} = 0$ requires that the vertical field B_x does not depend on x hence $B_x = 0$ without lost of generality.

The equations for this kinematic 1D slab model may be formulated with the normalized quantities

$$C_\alpha = \frac{\alpha H}{\eta_0}, \qquad C_\gamma = \frac{\gamma H}{\eta_0} \tag{22}$$

(let $\Omega = 1$ for simplicity) so that

$$\mathrm{i}\omega B_y - \frac{\mathrm{d}^2 B_y}{\mathrm{d}x^2} = -C_\gamma \frac{\mathrm{d}B_y}{\mathrm{d}x} - C_\alpha \frac{\mathrm{d}B_z}{\mathrm{d}x} \tag{23}$$

and

$$\mathrm{i}\omega B_z - \frac{\mathrm{d}^2 B_z}{\mathrm{d}x^2} = -C_\gamma \frac{\mathrm{d}B_z}{\mathrm{d}x} + C_\alpha \frac{\mathrm{d}B_y}{\mathrm{d}x}, \qquad (24)$$

see Moss et al. (1999); Parker (1979); Rüdiger and Kitchatinov (2006). The real part of the complex frequency ω determines the oscillation frequency (in units of the diffusion rate) of a possible dynamo wave along the vertical direction, while the growth rate of the dynamo is given by the negative value of its imaginary part. We are mainly interested to know the critical $C_{\alpha,0}$ for neutral instability, $\Im(\omega) = 0$. Let us define the ratio

$$\hat{\gamma}_0 = \frac{C_\gamma}{C_{\alpha,0}} \qquad (25)$$

as describing the influence of the pumping effect on the excitation of kinematic α^2 dynamos.

The vacuum boundary conditions $B_y(0) = B_z(0) = B_y(1) = B_z(1) = 0$ are applied. For $\gamma = 0$ the lowest non-trivial eigenvalue of a stationary solution is $C_{\alpha,0} = 2\pi$. The solutions do not depend on the sign of C_γ as they do also not depend on the sign of C_α.

The upper panel of Figure 6 gives the dynamo's growth rates for three values of C_γ as function of C_α. As usual, for sub-critical (super-critical) α the modes are decaying (growing), and we find that the $C_{\alpha,0}$ for neutral instability grows with growing C_γ. All the critical dynamo solutions for non-vanishing γ are oscillating. The lower panel of this figure demonstrates that for these eigensolutions the ratio $\hat{\gamma}_0$ does never exceed unity. The 1D α^2 dynamo, therefore, has no neutral dynamo solution for $C_\gamma > C_\alpha$. A too-strong radial advection effect is not compatible with the operation of α^2 dynamos. The reason for the suppression of the dynamo instability by dominating radial advection is that the field components perpendicular to the advection vector are concentrated inwards (or outwards) so that the dynamo domain is reduced and the critical C_α must grow. This destructive action proves to be even more drastic for α^2 dynamos than for those of $\alpha\Omega$-type (Brandenburg et al. 1992; Moss et al. 1999).

Results for a very special spherical model with α effect and pumping term are plotted in Figure 16.10 in Krause and Rädler (1980). The α effect only exists in an outer hemisphere while the diamagnetic pumping only exists in its inner part. Similar to the above slab model, for growing C_γ also the critical $C_{\alpha,0}$ grows linearly so that the $\hat{\gamma}_0$ never exceeds unity. The mode with the lowest dynamo number is a nonaxisymmetric quadrupole with an azimuthally drifting magnetic field.

Because of its relevance for the concept of conductivity fluctuations, we have designed a simple shell-type dynamo model with an outer turbulence domain filled with α effect independent of the radius, and with uniform radial γ. The α term is anti-symmetric with respect to the equator. The definitions (22) have been used with the replacement $H \to D$ with $D = (1 - r_{\mathrm{in}})R$ and R the radius of the sphere. The inner boundary is a perfect-conducting one while the outer boundary mimics vacuum, so that the Poynting flux is zero. To illustrate the performance of the advection term, examples for the excited magnetic fields are plotted in Figure 7 for a turbulence with outward pumping (top) and inward pumping (bottom). The inner part (or the outer part, in dependence on the sign of γ) of the shell are field-free. Eigensolutions with dipolar symmetry have the same eigenvalue as those with quadrupolar symmetry. The sign of C_γ differs in both models but without consequences for the excitation condition. For both cases, $|\hat{\gamma}| = 0.8$ is prescribed. The radial advection produces non-axisymmetric solutions drifting in the azimuthal direction.

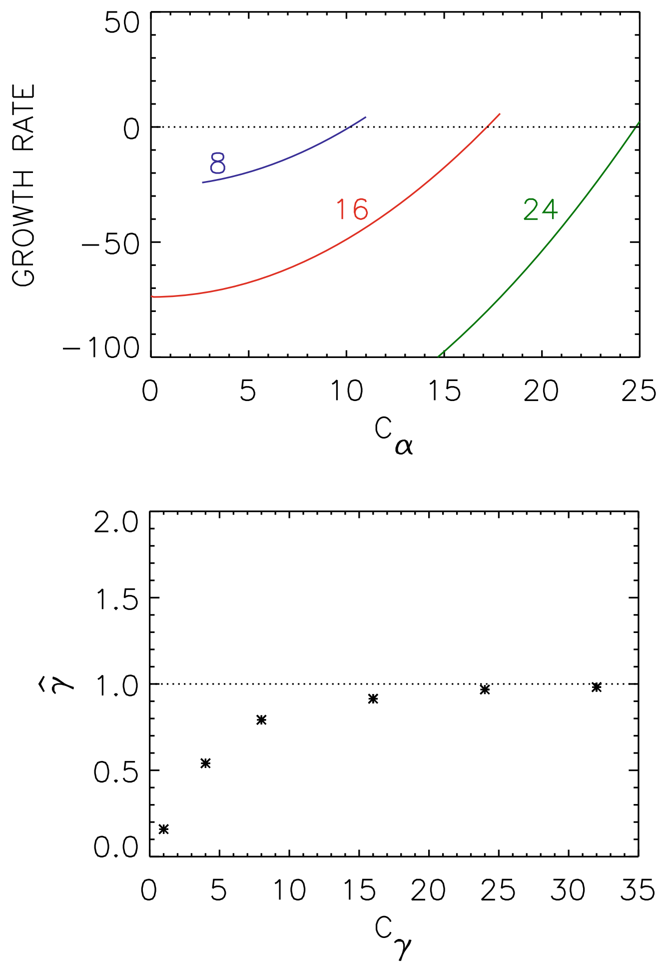

FIGURE 6 Upper panel: Growth rates multiplied with the diffusion time versus C_α for three plane dynamos with $C_\gamma = 8,16,24$. All solutions describe waves traveling in vertical direction. Lower panel: The dimensionless ratio $\hat{\gamma}_0$ versus C_γ for neutral excitation. Kinematic dynamos only exist as long $\hat{\gamma} \leq 1$, the pumping term C_γ suppresses the dynamo action

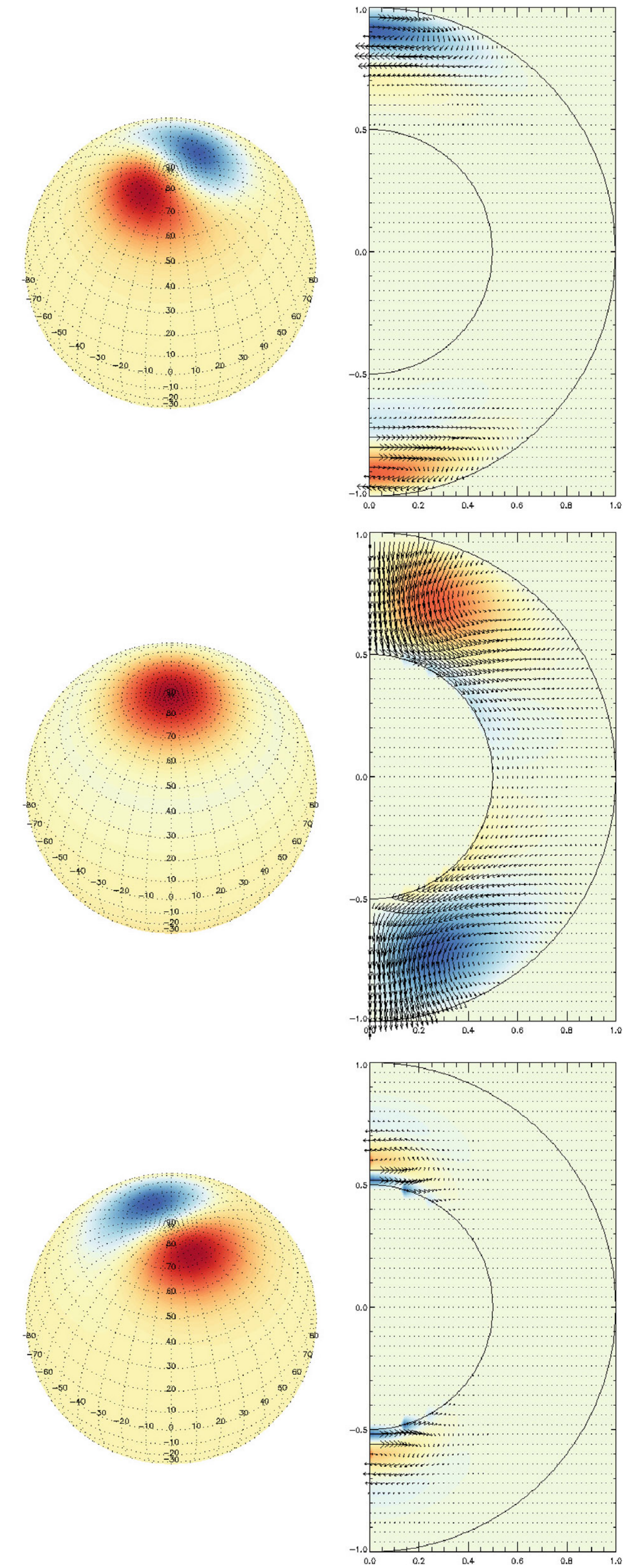

FIGURE 7 Influence of the advection term on α^2 dynamos. The nonaxisymmetric dipolar mode A1 (top) and the quadrupolar mode S1 (bottom) for $|\hat{\gamma}_0| = 0.8$ are excited by the same value of $C_{\alpha,0}$. The kinematic axisymmetric α^2 dynamo for $\gamma = 0$ (middle) is shown for reference. The bottom of the turbulence domain is at a $r = 0.5$, with a perfect-conducting boundary. The models are embedded in vacuum

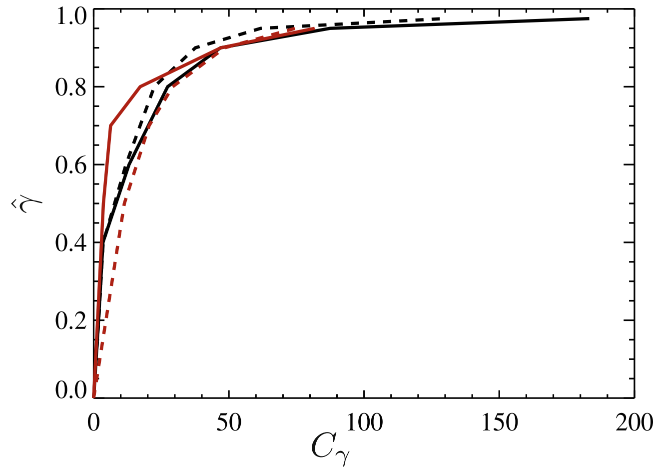

FIGURE 8 The values $\hat{\gamma}_0$ critical for excitation versus C_γ of spherical shell dynamo models. The nonaxisymmetric (dashed lines) solutions possess (slightly) smaller C_α than the axisymmetric solutions (solid line). Dynamo solutions for $\gamma > \alpha_0$ do not exist. $C_\Omega = 0$ (dark), $C_\Omega = 460$ (red). The smallest eigenvalue is $C_{\alpha,0} = 5$ for $\gamma = 0$. $r_{\rm in} = 0.5$

For $\gamma = 0$, the critical eigenvalue for neutral excitation is $C_{\alpha,0} \simeq 5$, independent of the value of $r_{\rm in}$ (see Figure 7, middle panel).

For increasing $\hat{\gamma}$, the horizontal field will be more and more concentrated at the inner or the outer boundary (in dependence on the sign of γ) while the bulk of the shell becomes field-free. The values of C_α necessary for dynamo excitation grow to unrealistic high values (Figure 3). A fluid with values of $\hat{\gamma} > 1$ and without shear cannot maintain large-scale fields via the α^2 mechanism. For the above calculated high value of $\hat{\gamma} \simeq 5$, therefore, kinematic α^2 dynamos are not possible. In other words, the dynamo only works for $C_\alpha \gtrsim Max(5, C_\gamma)$. In case that $\alpha \simeq \hat{\alpha}\Omega$ (which is true for slow rotation), the dynamo only operates as long as the rotation rate exceeds the critical value of $\Omega \simeq \gamma/\hat{\alpha}$. The dynamo decays for $\Omega < \Omega_1$ where Ω_1 denotes the rotation rate where $\hat{\gamma} = 1$. The above-mentioned simulations for solar magneto-convection suggest that indeed $\Omega_1 \simeq \Omega_\odot$. Figure 8 also contains eigenvalues for an $\alpha^2\Omega$ dynamo with the rather flat rotation law $\Omega = \Omega_0/r^{0.3}$. For the normalized rotation rate $C_\Omega = \Omega_0 D/\eta_0$, the value $C_\Omega = 460$ is used. One only finds small deviations from the curves for the α^2 dynamo with $C_\Omega = 0$. For weak field advection, the solutions with the lowest α_0 are axisymmetric and oscillating while for stronger pumping the nonaxisymmetric modes prevail which are drifting in the azimuthal direction.

We note that we only considered the kinematic approximation where any nonlinear feedback of the induced fields onto the turbulence is ignored. In any case, if dynamos ever existed for large values of $\hat{\gamma}$, they must be rather exotic.

7 | CONCLUSIONS

If an anisotropy in a conducting turbulent fluid is defined as one that is (only) in the direction of the conductivity fluctuations, and the velocity fluctuation is correlated, then a turbulent field-advection exists in this direction. It lifts large-scale fields oriented perpendicular to this direction downward or upward, depending on the sign of the correlation.

Our simulations provide the amplitude of this advection term in units of the turbulence velocity. They are on the order of about 10% of the normalized resistivity fluctuation $\eta_{\rm rms}/\overline{\eta}$, while the α effect is generally smaller. Its amplitude grows for growing rotation rate until $\Omega \simeq 1$—declining, however, for faster rotation. On the other hand, the advection term γ is numerically almost uninfluenced by the rotation, in accordance with general expectations. As we have also shown that the Pm-dependence of the results is only weak, one can be sure that in rotating fluids with velocity-correlated conductivity fluctuations, the resulting pumping term γ always exceeds the alpha term velocity $\alpha_1\Omega$.

As demonstrated in Section 6, this constellation has severe consequences for associated dynamo models. There we have considered two dynamo models with different geometries. First, a simplified slab dynamo model with two insulating plates and with a uniform α effect, including a vertical turbulence-induced field advection. This model only yields solutions with neutral stability if the α velocity exceeds the advection velocity. The solution for $\gamma = 0$ is stationary while otherwise it forms a vertical dynamo wave. For $\gamma \geq \alpha$, dynamo solutions no longer exist. The results are very similar for spherical shell dynamos. For growing advection effect the most unstable modes become oscillatory but always the dynamos need $C_\alpha > C_\gamma$, that is, the ratio $\hat{\gamma}$ never exceeds unity. Pure α^2 dynamos on the basis of resistivity fluctuations can thus not work. The same holds for shell dynamos with rather flat rotation laws while the behavior of $\alpha\Omega$ dynamos with large shear is still unknown for the case of strong pumping.

ACKNOWLEDGMENTS

Oliver Gressel thanks Petri Käpylä for helpful correspondence. This work used the nirvana code version 3.3, developed by Udo Ziegler at the Leibniz-Institut für Astrophysik Potsdam (AIP). All direct computations were performed on the Steno node at the Danish Center for Supercomputing (DCSC). Open Access funding enabled and organized by Projekt DEAL.

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How to cite this article: Gressel, O., Rüdiger, G., & Elstner, D. 2023, *Astron. Nachr.*, e20210039. https://doi.org/10.1002/asna.20210039