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Строго консистентне оцінювання всіх параметрів моделі Васічека за дискретними спостереженнями

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Strongly consistent estimation of all parameters in the Vasicek model by discrete observations

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Побудовано оцінки для всіх трьох невідомих параметрів у моделі Васічека на основі дискретних спостережень та доведено їхню строгу консистентність. Теоретичні результати проілюстровано за допомогою моделювання.

Ключові слова: стохастичне диференціальне рівняння, модель відсоткової ставки, низькочастотні дані, ерогодична теорема.

We construct estimators of all three unknown parameters in the Vasicek interest rate model based on discrete observations and prove their strong consistency. Theoretical results are illustrated by simulations.

Key words: stochastic differential equation, interest rate model, low-frequency data, ergodic theorem.

1 Introduction

In 1977 O. Vasicek [1] proposed a mathematical model for the interest rate, which was later named after him. It is described by the following stochastic differential equation

$$dX_t = (\alpha - \beta X_t)\, dt + \gamma dW_t,$$

where α, β, γ are constants and W is a standard Brownian motion. This model is widely used in the economics and in the mathematical finance; however, it also has applications in various other fields. Usually, it is assumed that $\beta > 0$. In this case the process X has ergodic properties and the property of mean reversion, which is important for financial applications. We will discuss them below. From the financial point of view, the model parameters have the following interpretation: β corresponds to the speed of recovery, the ratio α/β is the long-term average interest rate, and γ represents the stochastic volatility.

For practical applications of this model, parameter estimation methods are required. If continuous-time observations for a whole trajectory of the process $X = \{X_t, t \in [0, T]\}$ are available, then usually the diffusion parameter γ is assumed to be known, since it can be evaluated almost surely with the help of realized quadratic variations. The drift parameters α and β are estimated using the maximum likelihood method or the least squares method. Both approaches lead to the same estimators, namely

$$\widehat{\alpha} = \frac{(X_T - X_0)\int_0^T X_t^2\,dt - \int_0^T X_t\,dX_t \int_0^T X_t\,dt}{T\int_0^T X_t^2\,dt - \left(\int_0^T X_t\,dt\right)^2}, \tag{1}$$

$$\widehat{\beta} = \frac{(X_T - X_0)\int_0^T X_t\,dt - T\int_0^T X_t\,dX_t}{T\int_0^T X_t^2\,dt - \left(\int_0^T X_t\,dt\right)^2}, \tag{2}$$

see Example 1.35 in the monograph [2].

In practice, a trajectory of the process can be observed only at some discrete time instants, for example $t_k = kh$, $k = 0, 1, 2, \ldots$, with a fixed step h. If h is close to zero, then it can be assumed that the process is observed continuously, so we can use the above-mentioned approach to parameter estimation. If h is considerably large (the case of low-frequency data), then we cannot assume that the observations are continuous. Moreover, the coefficient γ cannot be calculated using realized quadratic variations, and integrals in formulas (1) and (2) cannot be well-approximated by integral sums. So, a problem of simultaneously estimation of three parameters α, β, γ by the discrete low-frequency data arises. This paper is devoted to construction of strongly consistent estimators of these three parameters.

2 Problem formulation

Let the process X be a solution to the following stochastic differential equation

$$dX_t = (\alpha - \beta X_t)dt + \gamma dW_t, \quad X_0 = x_0 \in \mathbb{R}, \tag{3}$$

where $W = \{W_t, t \geqslant 0\}$ is a Wiener process. The parameters $\alpha \in \mathbb{R}$, $\beta > 0$ and $\gamma > 0$ are assumed to be fixed, but unknown.

It is well known, that the equation (3) has a unique solution, which can be found explicitly:

$$X_t = \frac{\alpha}{\beta} + e^{-\beta t}\left(x_0 - \frac{\alpha}{\beta}\right) + \gamma \int_0^t e^{-\beta(t-s)} dW_s,$$

see, e. g., [3, Eq. (9.7)].

Let $h > 0$ be fixed. Assume that the following values are observed:

$$X_0, X_h, X_{2h}, \dots, X_{nh}.$$

The aim of this paper is to construct strongly consistent estimators of all three unknown parameters α, β and γ.

3 Convergence of statistics

For parameter estimation we will use the following three statistics:

$$\xi_n := \frac{1}{n}\sum_{k=0}^{n-1} X_{kh}, \tag{4}$$

$$\eta_n := \frac{1}{n}\sum_{k=0}^{n-1} X_{kh}^2, \tag{5}$$

$$\zeta_n := \frac{1}{n}\sum_{k=0}^{n-1} X_{kh}X_{(k+1)h}. \tag{6}$$

In this section we will investigate their asymptotic behavior.

Put

$$Z_t = \gamma \int_0^t e^{-\beta(t-s)} dW_s,$$
$$Y_t = \gamma \int_{-\infty}^t e^{-\beta(t-s)} dW_s.$$

Then

$$X_t = \frac{\alpha}{\beta} + e^{-\beta t}\left(x_0 - \frac{\alpha}{\beta}\right) + Z_t, \tag{7}$$

$$Z_t = Y_t - e^{-\beta t}Y_0. \tag{8}$$

The process $\{Y_t, t \geqslant 0\}$ is a two-sided Ornstein–Uhlenbeck process, restricted to the positive half-axis. It is known, that it is stationary and ergodic, see, for example, [4]. It follows from ergodicity that for any measurable function $g\colon \mathbb{R}^2 \to \mathbb{R}$, such that $\mathsf{E}|g(Y_0, Y_h)| < \infty$, the following convergence holds:

$$\frac{1}{n}\sum_{k=0}^{n-1} g\left(Y_{kh}, Y_{(k+1)h}\right) \to g(Y_0, Y_h) \tag{9}$$

a.s., as $n \to \infty$.

Let us start with establishing several auxiliary results.

Lemma 1. *The following convergences hold:*

$$\frac{1}{n}\sum_{k=0}^{n-1} Y_{kh} \to 0,$$
$$\frac{1}{n}\sum_{k=0}^{n-1} Y_{kh}^2 \to \frac{\gamma^2}{2\beta},$$
$$\frac{1}{n}\sum_{k=0}^{n-1} Y_{kh}X_{(k+1)h} \to \frac{\gamma^2}{2\beta}e^{-\beta h},$$

a.s., as $n \to \infty$.

Proof. 1. Using (9) with the function $g(x, y) = x$, we get

$$\frac{1}{n}\sum_{k=0}^{n-1} Y_{kh} \to \mathsf{E}Y_0 = \gamma \mathsf{E}\int_{-\infty}^0 e^{\beta s} dW_s = 0,$$

a. s., as $n \to \infty$.

2. Applying (9) with the function $g(x, y) = x^2$ and Itô isometry, we obtain

$$\frac{1}{n}\sum_{k=0}^{n-1} Y_{kh}^2 \to \mathsf{E}Y_0^2 = \gamma^2 \mathsf{E}\left(\int_{-\infty}^0 e^{\beta s} dW_s\right)^2$$
$$= \gamma^2 \int_{-\infty}^0 e^{2\beta s} ds = \frac{\gamma^2}{2\beta}.$$

3. It follows from (9) with $g(x,y)=xy$ and from the properties of the Itô integral that

$$\frac{1}{n}\sum_{k=0}^{n-1} Y_{kh}X_{(k+1)h} \to \mathsf{E}Y_0Y_h = \gamma^2 \mathsf{E}\left(\int_{-\infty}^{0} e^{\beta s}dW_s \cdot \int_{-\infty}^{h} e^{\beta(s-h)}dW_s\right) = \gamma^2 e^{-\beta h}\int_{-\infty}^{0} e^{2\beta s}ds = \frac{\gamma^2}{2\beta}e^{-\beta h}. \qquad \square$$

Now let us prove similar convergences for the process Z.

Lemma 2. *The following convergencies hold:*

$$\frac{1}{n}\sum_{k=0}^{n-1} Z_{kh} \to 0,$$
$$\frac{1}{n}\sum_{k=0}^{n-1} Z_{kh}^2 \to \frac{\gamma^2}{2\beta},$$
$$\frac{1}{n}\sum_{k=0}^{n-1} Z_{kh}Z_{(k+1)h} \to \frac{\gamma^2}{2\beta}e^{-\beta h},$$

a.s., as $n\to\infty$.

Proof. 1. By (8), we get

$$\frac{1}{n}\sum_{k=0}^{n-1} Z_{kh} = \frac{1}{n}\sum_{k=0}^{n-1} Y_{kh} - \frac{1}{n}Y_0\sum_{k=0}^{n-1} e^{-\beta kh} \to 0,$$

a.s., as $n\to\infty$, since the first term tends to zero by Lemma 1, and the sum $\sum_{k=0}^{n-1} e^{-\beta kh}$ in the second term has the finite limit $\frac{1}{1-e^{-\beta h}}$, because it is the sum of a decreasing geometric sequence.

2. Using (8), we may write

$$\frac{1}{n}\sum_{k=0}^{n-1} Z_{kh}^2 = \frac{1}{n}\sum_{k=0}^{n-1}\left(Y_{kh} - e^{-\beta kh}Y_0\right)^2 = \frac{1}{n}\sum_{k=0}^{n-1} Y_{kh}^2 + \frac{1}{n}Y_0^2\sum_{k=0}^{n-1} e^{-2\beta kh} - \frac{2}{n}Y_0\sum_{k=0}^{n-1} Y_{kh}e^{-\beta kh}.$$

In this case, the first term converges to $\frac{\gamma^2}{2\beta}$ by Lemma 1. The second term vanishes as $n\to\infty$, because the sum $\sum_{k=0}^{n-1} e^{-2\beta kh}$ is a sum of decreasing geometric sequence, therefore this sum has the finite limit $\frac{1}{1-e^{-2\beta h}}$. The third term also converges to zero, since it can be bounded by the first two terms due to the Cauchy–Schwarz inequality:

$$\frac{1}{n}\sum_{k=0}^{n-1} Y_{kh}e^{-\beta kh} \leqslant \sqrt{\frac{1}{n}\sum_{k=0}^{n-1} Y_{kh}^2 \cdot \frac{1}{n}\sum_{k=0}^{n-1} e^{-2\beta kh}}.$$

3. In the same way, we obtain the third convergence of Lemma 2. Using (8), we get

$$\frac{1}{n}\sum_{k=0}^{n-1} Z_{kh}Z_{(k+1)h} = \frac{1}{n}\sum_{k=0}^{n-1}\left(Y_{kh} - e^{-\beta kh}Y_0\right) \times \left(Y_{(k+1)h} - e^{-\beta(k+1)h}Y_0\right) = \frac{1}{n}\sum_{k=0}^{n-1} Y_{kh}Y_{(k+1)h} + \frac{1}{n}Y_0^2\sum_{k=0}^{n-1} e^{-2\beta kh-\beta h} - \frac{1}{n}Y_0\sum_{k=0}^{n-1} Y_{(k+1)h}e^{-\beta kh} - \frac{1}{n}Y_0\sum_{k=0}^{n-1} Y_{kh}e^{-\beta(k+1)h}.$$

The first term converges a. s. to $\frac{\gamma^2}{2\beta}e^{-\beta h}$ by Lemma 1. Similarly to the previous part of the proof, we obtain the convergence to zero for the remaining terms. $\square$

Now we are ready to establish convergence of the statistics ξ_n, η_n and ζ_n, defined by (4)–(6).

Lemma 3. *The following convergences hold:*

$$\xi_n \to \frac{\alpha}{\beta}, \tag{10}$$
$$\eta_n \to \frac{\alpha^2}{\beta^2} + \frac{\gamma^2}{2\beta}, \tag{11}$$
$$\zeta_n \to \frac{\alpha^2}{\beta^2} + \frac{\gamma^2}{2\beta}e^{-\beta h} \tag{12}$$

a.s., as $n\to\infty$.

Proof. In order to prove the convergence (10), let us consider the definition of ξ_n and express X in terms of Z using the equality (7). We get

$$\xi_n = \frac{1}{n}\sum_{k=0}^{n-1} X_{kh} = \frac{1}{n}\sum_{k=0}^{n-1}\left(\frac{\alpha}{\beta} + e^{-\beta kh}\left(x_0 - \frac{\alpha}{\beta}\right) + Z_{kh}\right) = \frac{\alpha}{\beta} + \left(x_0 - \frac{\alpha}{\beta}\right)\cdot\frac{1}{n}\sum_{k=0}^{n-1} e^{-\beta kh} + \frac{1}{n}\sum_{k=0}^{n-1} Z_{kh}.$$

By Lemma 2, the last term in this equality converges to 0 a. s. The second term also tends to

0, this fact was proved in the previous lemma (see part 1 of its proof). Thus, $\xi_n \to \frac{\alpha}{\beta}$ a. s., as $n \to \infty$.

The convergences in (11) and (12) can be deduced from Lemma 2 by similar arguments. □

4 Construction of strongly consistent estimators

Lemma 3 gives us an idea for construction of the strongly consistent estimators of α, β, γ. It consists in replacing the sign "$\to$" with "=" in convergences (10)–(12) and solving the obtained system of three equations with respect to α, β and γ. The solutions will be strongly consistent estimators of unknown parameters.

The system of estimating equations has the form

$$\begin{cases} \xi_n = \frac{\hat\alpha}{\hat\beta}, \\ \eta_n = \frac{\hat\alpha^2}{\hat\beta^2} + \frac{\hat\gamma^2}{2\hat\beta}, \\ \zeta_n = \frac{\hat\alpha^2}{\hat\beta^2} + \frac{\hat\gamma^2}{2\hat\beta} e^{-\hat\beta h}. \end{cases}$$

It can be rewritten as

$$\begin{cases} \xi_n \hat\beta = \hat\alpha, \\ \eta_n - \xi_n^2 = \frac{\hat\gamma^2}{2\hat\beta}, \\ \zeta_n - \xi_n^2 = \frac{\hat\gamma^2}{2\hat\beta} e^{-\hat\beta h}. \end{cases}$$

Now, by dividing the second equation by the third one, we get a relation for β. After that, substituting β in the first and the second equations, we immediately obtain α and γ respectively. As a result, we arrive at the following estimators

$$\hat\beta_n = \frac{1}{h} \log^+ \frac{\eta_n - \xi_n^2}{\zeta_n - \xi_n^2}, \qquad \hat\alpha_n = \xi_n \hat\beta_n, \qquad \hat\gamma_n^2 = 2\hat\beta_n(\eta_n - \xi_n^2), \tag{13}$$

where, for the sake of definiteness, we use the following function $\log^+$ instead of log:

$$\log^+ x = \begin{cases} \log x, & x > 1, \\ 0, & x \leqslant 1. \end{cases}$$

The following theorem is the main result of the paper.

Theorem 1. *The following convergences hold a. s. as $n \to \infty$:*

$$\hat\alpha_n \to \alpha, \qquad \hat\beta_n \to \beta, \qquad \hat\gamma_n \to \gamma,$$

i. e., $\hat\alpha_n$, $\hat\beta_n$, $\hat\gamma_n$ are strongly consistent estimators of the parameters α, β, γ respectively.

Proof. The proof follows immediately from the equalities (13) and Lemma 3. □

5 Simulations

In this section we illustrate the quality of the estimators with the help of simulation experiments. We fix the following true values of the parameters: $\alpha = 1$, $\beta = 2$, $\gamma = 1$, and choose the initial value $x_0 = 1$. We generate 1000 trajectories of the process X as a solution to the stochastic differential equation (3), using the Euler–Maruyama approximations with the discretization step 0.001. Then we compute means and standard deviations of the estimates for three values of sampling intervals ($h = 1$, $h = 0.1$ and $h = 0.01$), and for various sample sizes ($n = 100, 500, 1000, 1500, 2000$). The results are reported in Tables 1–3.

We observe that the means of the estimates converge to the true values of the parameters, and their standard deviations tend to zero. Thus, the numerical results confirm theoretical conclusions about the strong consistency of the estimators. It is worth mentioning that the estimators $\hat\gamma_n$ demonstrate much better rate of convergence compared to $\hat\alpha_n$ and $\hat\beta_n$. Note that the estimators of β converge too slowly.

Table 1: Estimation results of the sampling interval $h = 1$

	$n = 100$	$n = 500$	$n = 1000$	$n = 1500$	$n = 2000$
Mean of $\hat\alpha_n$	1.0193	1.0683	1.0393	1.0280	1.0204
Mean of $\hat\beta_n$	1.9983	2.1324	2.0763	2.0555	2.0404
Mean of $\hat\gamma_n$	0.8896	1.0257	1.0168	1.0123	1.0095
S. Dev. of $\hat\alpha_n$	0.6647	0.2278	0.1368	0.1073	0.0897
S. Dev. of $\hat\beta_n$	1.2577	0.4404	0.2634	0.2059	0.1715
S. Dev. of $\hat\gamma_n$	0.4391	0.1073	0.0627	0.0505	0.0421

Table 2: Estimation results of the sampling interval $h = 0.1$

	$n = 100$	$n = 500$	$n = 1000$	$n = 1500$	$n = 2000$
Mean of $\hat{\alpha}_n$	1.5712	1.1057	1.0547	1.0372	1.0265
Mean of $\hat{\beta}_n$	2.9459	2.1681	2.0846	2.0553	2.0410
Mean of $\hat{\gamma}_n$	1.1017	1.0201	1.0107	1.0080	1.0064
S. Dev. of $\hat{\alpha}_n$	0.7631	0.2324	0.1567	0.1301	0.1061
S. Dev. of $\hat{\beta}_n$	1.0773	0.3407	0.2295	0.1836	0.1563
S. Dev. of $\hat{\gamma}_n$	0.0873	0.0364	0.0242	0.0203	0.0178

Table 3: Estimation results of the sampling interval $h = 0.01$

	$n = 100$	$n = 500$	$n = 1000$	$n = 1500$	$n = 2000$
Mean of $\hat{\alpha}_n$	18.3085	2.0973	1.4760	1.3065	1.2248
Mean of $\hat{\beta}_n$	22.0864	3.7253	2.7878	2.5089	2.3657
Mean of $\hat{\gamma}_n$	1.6720	1.1439	1.0772	1.0520	1.0384
S. Dev. of $\hat{\alpha}_n$	24.5680	1.5446	0.8856	0.6552	0.5347
S. Dev. of $\hat{\beta}_n$	22.9114	1.3237	0.6480	0.4836	0.3850
S. Dev. of $\hat{\gamma}_n$	0.2649	0.0603	0.0363	0.0255	0.0210

We also observe that the results for $h = 1$ and $h = 0.1$ for fixed n are quite similar. However, if we compare the results for the same horizon of observations $T = nh$, we will see the clear advantage of more dense partition of the interval $[0, T]$.

At the same time, the results for $h = 0.01$ are substantially worse. Only the estimator of γ is useful in this case. We suppose that this is due to small horizon of observations. As a conclusion, the horizon of observations T is more important for the quality of estimators of α and β, that the sampling interval h.

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