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А. Маляренко[1], *д-р. наук, проф.*
Ю. С. Мішура[2], *д-р. наук, проф.*
Є. А. О. Рудик[3], *студ.*

Наближення дробових інтегралів гельдерових функцій

[1] Факультет математики та фізики, університет Мелардален, 721 23 Вестерос, Швеція
e-mail: anatoliy.malyarenko@mdu.se
[2] Кафедра теорії ймовірностей, статистики та актуарної математики, Київський національний університет імені Тараса Шевченка, 64/13, вул.Володимирська, 01601 Київ, Україна
e-mail: yuliyamishura@knu.ua
[3] Кафедра теорії ймовірностей, статистики та актуарної математики, Київський національний університет імені Тараса Шевченка, 64/13, вул.Володимирська, 01601 Київ, Україна
e-mail: rudykyao@knu.ua

A. Malyarenko[1], *PhD, Prof.*
Yu. S. Mishura[2], *Dr of Sci., Prof.*
Y. A. O. Rudyk,[3] *Stud.*

Approximation of fractional integrals of Hölder functions

[1]Division of Mathematics and Physics, Mälardalen University, 721 23 Västerås, Sweden
e-mail: anatoliy.malyarenko@mdu.se
[2]Department of Probability Theory, Statistics and Actuarial Mathematics, Taras Shevchenko National University of Kyiv, 64/13, Volodymyrska Street, 01601 Kyiv, Ukraine
e-mail: yuliyamishura@knu.ua
[3] Department of Probability Theory, Statistics and Actuarial Mathematics, Taras Shevchenko National University of Kyiv, 64/13, Volodymyrska Street, 01601 Kyiv, Ukraine
e-mail: rudykyao@knu.ua

Статтю присвячено дослідженню швидкості збіжності інтегральних сум двох типів до дробового інтегралу. В першій теоремі доведено гельдерівську властивість дробових інтегралів від функцій з різних інтегральних просторів. Потім ми оцінюємо швидкість збіжності інтегральних сум, побудованих за гельдерівськими функціями, до відповідних дробових інтегралів. Отримані результати проілюстровано декількома чисельними прикладами.

Ключові слова: дробовий інтеграл, інтегральні суми, наближення, гельдеровість.

The paper is devoted to the rate of convergence of integral sums of two different types to fractional integrals. The first theorem proves the Hölder property of fractional integrals of functions from various integral spaces. Then we estimate the rate of convergence of the integral sums of two types corresponding to the Hölder functions, to the respective fractional integrals. We illustrate the obtained results by several figures.

Key Words: fractional integral, integral sums, approximation, Hölder property.

1 Introduction

Fractional integrals as the basic elements of fractional calculus are widely used for solving fractional differential equations, both nonrandom and stochastic, and for construction of various fractional processes. At the same time, unfortunately, it is impossible to calculate fractional integrals explicitly even for many elementary functions. Naturally, the problem of approximation of fractional integrals appears. In this research we discover Hölder properties of fractional integrals and then the rate at which integral sums approximate fractional integrals. To solve this problem, we use the properties of fractional integrals presented in the classical book [1] and discussed in [2]. At the moment, the approximation of fractional integrals is of great interest, we note only the works [3]–[6].

2 Main definitions

Definition 1. Let $[a,b]$ be a finite interval. The function $f\colon [a,b]\to\mathbb{R}$ satisfies the Hölder condition of order λ on $[a,b]$ if there exists such $C>0$ that

$$|f(x_1)-f(x_2)|\leqslant C|x_1-x_2|^{\lambda} \qquad (1)$$

for any $x_1,x_2\in[a,b]$, where λ is the Hölder exponent. Denote Hölder constant

$$C_\lambda=\sup_{x_1,x_2\in[a,b],x_1\neq x_2}\frac{|f(x_1)-f(x_2)|}{|x_1-x_2|}.$$

It is clear that the function f is continuous on $[a,b]$ if it satisfies the Hölder condition of any positive order on this interval.

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Definition 2. Let function $\varphi \in L_1(a,b)$. The integrals

$$\left(I_{a+}^{\alpha}\varphi\right)(x) \stackrel{\text{def}}{=} \frac{1}{\Gamma(\alpha)} \int_a^x \frac{\varphi(t)}{(x-t)^{1-\alpha}} dt, \qquad x > a, \tag{2}$$

$$\left(I_{b-}^{\alpha}\varphi\right)(x) \stackrel{\text{def}}{=} \frac{1}{\Gamma(\alpha)} \int_x^b \frac{\varphi(t)}{(t-x)^{1-\alpha}} dt, \qquad x < b, \tag{3}$$

where $\alpha > 0$, are called fractional integrals of the order α, or, more precisely, the Riemann-Liouville fractional integrals. Sometimes they are called left-sided and right-sided fractional integrals, respectively.

3 Hölder properties of fractional integrals

In this section we consider $\alpha \in (0,1)$, and for technical simplicity restrict ourselves with left-sided fractional integrals because right-sided integrals can be considered similarly.

Theorem 3.1.

1) *Let function $f \in L_p[a,b]$ for some $p \in (1,\infty)$. Then for any $\alpha \in (\frac{1}{p}, 1)$:*

$$I_{0+}^{\alpha} f \in C^{\alpha-1/p}[a,b] \quad \textit{and}$$

$$\left|\left(I_{a+}^{\alpha} f\right)(t) - \left(I_{a+}^{\alpha} f\right)(s)\right| \leqslant C(t-s)^{\alpha-1/p},$$

$$\textit{where} \quad C = \frac{\|f\|_{L_p[a,b]}}{\Gamma(\alpha)} *$$

$$* \left(1 + \left(\int_0^{+\infty} \left|(1+y)^{\alpha-1} - y^{\alpha-1}\right|^q dy\right)^{1/q}\right),$$

$a \leqslant s < t \leqslant b$.

2) *Let function $f \in L_\infty[a,b]$. Then for any $\alpha \in (0,1)$*

$$I_{a+}^{\alpha} f \in C^{\alpha}[a,b] \quad \textit{and}$$

$$\left|\left(I_{a+}^{\alpha} f\right)(t) - \left(I_{a+}^{\alpha} f\right)(s)\right| \leqslant C(t-s)^{\alpha},$$

$$\textit{where} \quad C = \frac{3\|f\|_\infty}{\alpha\Gamma(\alpha)}, \quad a \leqslant s < t \leqslant b.$$

Proof.

1) Let $a \leqslant s < t \leqslant b$. We can bound

$$\Delta_{s,t} = \left|\left(I_{a+}^{\alpha} f\right)(t) - \left(I_{a+}^{\alpha} f\right)(s)\right|$$

as follows:

$$\Delta_{s,t} \leqslant \frac{1}{\Gamma(\alpha)} \left(\Im_1(s,t) + \Im_2(s,t)\right),$$

where

$$\Im_1(s,t) = \left|\int_s^t (t-u)^{\alpha-1} f(u) du\right|,$$

and

$$\Im_2(s,t) = \left|\int_0^s \left((t-u)^{\alpha-1} - (s-u)^{\alpha-1}\right) f(u) du\right|.$$

Consider $\Im_1(s,t)$, and let $\frac{1}{q} + \frac{1}{p} = 1$. Then

$$\Im_1(s,t) \leqslant \left(\int_s^t (t-u)^{q(\alpha-1)} du\right)^{1/q} \|f\|_{L_p[a,b]},$$

and

$$\left(\int_s^t (t-u)^{q(\alpha-1)} du\right)^{1/q} = \frac{(t-s)^{\alpha-1+1/q}}{q(\alpha-1)+1}$$

$$= \frac{(t-s)^{\alpha-1/p}}{q\alpha - q/p}.$$

Furthermore,

$$\Im_2(s,t) \leqslant \Im_3(s,t) \|f\|_{L_p[a,b]},$$

where

$$\Im_3(s,t) = \left(\int_0^s \left|(t-u)^{\alpha-1} - (s-u)^{\alpha-1}\right|^q du\right)^{1/q}.$$

Let's evaluate $\Im_3(s,t)$, changing the variables as follows: $s-u = (t-s)y$. Then $t-u = (t-s)(1+y)$, and

$$\Im_3(s,t) =$$

$$= (t-s)^{\alpha-1/p} \left(\int_0^{s/(t-s)} \left|(1+y)^{\alpha-1} - y^{\alpha-1}\right|^q dy\right)^{1/q}$$

$$\leqslant (t-s)^{\alpha-1/p} \left(\int_0^{\infty} \left|(1+y)^{\alpha-1} - y^{\alpha-1}\right|^q dy\right)^{1/q}.$$

Now our goal is to prove that the latter integral converges. Indeed, it equals

$$\int_0^\infty y^{(\alpha-1)q}\left|\frac{1}{(1+1/y)^{1-\alpha}}-1\right|^q dy$$

On the one hand, it is easy to see that $(\alpha-1)q>-1$. Therefore the integral converges at zero. On the other hand, note that at infinity

$$\begin{aligned}&(1+1/y)^{1-\alpha}\\&=1+\frac{1-\alpha}{y}+\frac{(1-\alpha)(-\alpha)}{2y^2}+o\left(\frac{1}{y^2}\right),\end{aligned}$$

therefore

$$\begin{aligned}&\frac{1}{(1+1/y)^{1-\alpha}}-1\\&=\frac{1}{1+\frac{1-\alpha}{y}-\frac{\alpha(1-\alpha)}{2y^2}+o(\frac{1}{y^2})}-1\\&\sim-\frac{1-\alpha}{y}\quad\text{as}\quad y\to\infty.\end{aligned}$$

Finally,

$$\begin{aligned}&y^{(\alpha-1)q}\left|\frac{1}{(1+1/y)^{1-\alpha}}-1\right|^q\\&\sim(1-\alpha)^q y^{(\alpha-1)q}y^{-q}=(1-\alpha)^q y^{(\alpha-2)q}\\&=(1-\alpha)^q y^{(\alpha-2)p/(p-1)}.\end{aligned}$$

Recall that $\alpha<1$, therefore $\frac{(\alpha-2)p}{p-1}<\alpha-2<-1$, and integral converges at infinity.

2) Recall that now $f\in L_{\infty[a,b]}$, and $\|f\|_\infty=\operatorname{ess\,sup}_{t\in[a,b]}|f|<\infty$. Applying the inequality

$$t^\alpha-s^\alpha\leqslant(t-s)^\alpha,$$

where $s<t$, $0<\alpha<1$, we get that

$$\begin{aligned}&\left|\left(I^\alpha_{a+}f\right)(t)-\left(I^\alpha_{a+}f\right)(s)\right|\\&=\left|\frac{1}{\Gamma(\alpha)}\int_a^t\frac{f(u)}{(t-u)^{1-\alpha}}du\right.\\&\left.-\frac{1}{\Gamma(\alpha)}\int_a^s\frac{f(u)}{(s-u)^{1-\alpha}}du\right|\\&\leqslant\frac{1}{\Gamma(\alpha)}\left(\int_a^s\left|(t-u)^{\alpha-1}-(s-u)^{\alpha-1}\right||f(u)|\,du\right.\\&\left.+\frac{1}{\Gamma(\alpha)}\int_s^t(t-u)^{\alpha-1}|f(u)|\,du\right)\\&\leqslant\frac{\|f\|_\infty}{\Gamma(\alpha)}\left(\int_a^s\left|(t-u)^{\alpha-1}-(s-u)^{\alpha-1}\right|du+\frac{|t-s|^\alpha}{\alpha}\right)\\&=\frac{\|f\|_\infty}{\alpha\Gamma(\alpha)}\,|(t-s)^\alpha-(t-a)^\alpha-(s-s)^\alpha\\&+(s-a)^\alpha+|t-s|^\alpha|\\&\leqslant\frac{3\|f\|_\infty}{\Gamma(\alpha+1)}\cdot(t-s)^\alpha.\end{aligned}$$

4 Approximation rate

Now our goal is to apply obtained formulas to approximate fractional integrals by integral sums, applying, in particular, Theorem 3.1. For technical simplicity, consider $[a,b]=[0,T]$ and create the sequence of uniform partitions π_n of the interval $[0,T]$ consisting of points $t_k=\frac{kT}{n}$, $0\leqslant k\leqslant n$. By $\frac{k_tT}{n}$ we denote the point of partition, which is closest to t being less or equal to t. Without loss of generality, we shall approximate only left-sided integrals, considering sums of two types:

$$S_{1,n}(t)=\frac{1}{\Gamma(\alpha)}\sum_{j=0}^{k_t-1}\left(\frac{k_tT}{n}-\frac{jT}{n}\right)^{\alpha-1}f\left(\frac{jT}{n}\right)\frac{T}{n},$$

and

$$\begin{aligned}S_{2,n}(t)=\frac{1}{\Gamma(\alpha+1)}\sum_{j=0}^{k_t-1}&\left(\left(\frac{k_tT}{n}-\frac{(j+1)T}{n}\right)^\alpha\right.\\&\left.-\left(\frac{k_tT}{n}-\frac{jT}{n}\right)^\alpha\right)f\left(\frac{jT}{n}\right).\end{aligned}$$

To achieve our goal we will evaluate the difference between fractional integrals at the points of partition π_n because the difference between fractional integrals in other points is estimated in Theorem

3.1 and can be reformulated in the following statement. In the point $\frac{kT}{n}$ we have that

$$S_{1,n}\left(\frac{kT}{n}\right)=\frac{1}{\Gamma(\alpha)}\sum_{j=0}^{k-1}\left(\frac{kT}{n}-\frac{jT}{n}\right)^{\alpha-1}f\left(\frac{jT}{n}\right)\frac{T}{n},$$

and

$$S_{2,n}\left(\frac{kT}{n}\right)=\frac{1}{\Gamma(\alpha+1)}\sum_{j=0}^{k-1}\left(\left(\frac{kT}{n}-\frac{(j+1)T}{n}\right)^{\alpha}-\left(\frac{kT}{n}-\frac{jT}{n}\right)^{\alpha}\right)f\left(\frac{jT}{n}\right).$$

Corollary 1. Let function $f\in L_\infty[0,T]$. Then for any $\alpha\in(0,1)$

$$\left|\left(I_{0+}^{\alpha}f\right)(t)-\left(I_{0+}^{\alpha}f\right)\left(\frac{k_tT}{n}\right)\right|\leqslant\frac{3\|f\|_\infty}{\Gamma(\alpha+1)}\cdot\left(\frac{T}{n}\right)^{\alpha}.$$

Proof. That is the particular case of the Theorem 3.1, part 2. Here we have $s=k_tT/n$ and $t-k_tT/n\leqslant\frac{T}{n}$.

It should be noted that a problem of approximation of fractional integrals needs particular discovery because of singularity at the upper (lower) bound of integration. Let's formulate the main result.

Theorem 4.1. *Assume that $f\in C^\beta[0,T]$ with Hölder constant C_β, and $\alpha+\beta>1$. For approximation the fractional integral by the sequences $S_{i,n}, i=1,2$ we have the upper bounds*

$$\max_{k\in[0,n]}\left|\left(I_{0+}^{\alpha}f\right)\left(\frac{kT}{n}\right)-S_{i,n}\left(\frac{kT}{n}\right)\right|\leqslant C_in^{1-\alpha-\beta},$$

where $C_1=\frac{1}{\Gamma(\alpha)}\left(\frac{C_\beta T^{\alpha+\beta}}{\alpha}+\|f\|_{C[0,T]}T^\alpha\right)$, $C_2=\frac{C_\beta T^{\alpha+\beta}}{\Gamma(\alpha+1)}$.

Proof.

(*i*) Denote

$$\Delta_n=\sup_{0\leqslant s\leqslant t\leqslant(s+T/n)\wedge T}|f(s)-f(t)|.$$

Obviously, $\Delta_n\leqslant\left(\frac{T}{n}\right)^\beta\wedge2\|f\|_{C[0,T]}$. Consider separately the last term

$$\Im_{k,n}:=\frac{1}{\Gamma(\alpha)}\left|\int_{(k-1)T/n}^{kT/n}\left(\frac{kT}{n}-s\right)^{\alpha-1}f(s)ds-\left(\frac{T}{n}\right)^{\alpha}f\left(\frac{(k-1)T}{n}\right)\right|,$$

and all other terms:

$$\Im_{j,n}:=\frac{1}{\Gamma(\alpha)}\left|\int_{jT/n}^{(j+1)T/n}\left(\frac{kT}{n}-s\right)^{\alpha-1}f(s)ds-\int_{jT/n}^{(j+1)T/n}\left(\frac{kT}{n}-\frac{jT}{n}\right)^{\alpha-1}f\left(\frac{jT}{n}\right)ds\right|,$$

$$0\leqslant j\leqslant k-2.$$

For the last term we have an upper bound

$$\Im_{k,n}\leqslant\frac{1}{\Gamma(\alpha)}\int_{(k-1)T/n}^{kT/n}\left(\frac{kT}{n}-s\right)^{\alpha-1}\left|f(s)-f\left(\frac{(k-1)T}{n}\right)\right|ds+\frac{1}{\Gamma(\alpha)}*$$

$$*\int_{(k-1)T/n}^{kT/n}\left[\left(\frac{kT}{n}-s\right)^{\alpha-1}-\left(\frac{T}{n}\right)^{\alpha-1}\right]ds\sup_{t\in[0,T]}|f(s)|$$

$$\leqslant\frac{1}{\Gamma(\alpha)}\left(\frac{1}{\alpha}\Delta_n\left(\frac{T}{n}\right)^{\alpha}+\|f\|_{C[0,T]}\left(-1+\frac{1}{\alpha}\right)\left(\frac{T}{n}\right)^{\alpha}\right)\tag{4}$$

$$\leqslant\frac{1}{\Gamma(\alpha)}\|f\|_{C[0,T]}\left(\frac{3}{\alpha}-1\right)\left(\frac{T}{n}\right)^{\alpha}.$$

Now, let $0\leqslant j\leqslant k-2$. Taking into account that

$$\left(\frac{kT}{n}-s\right)^{\alpha-1}-\left(\frac{kT}{n}-\frac{jT}{n}\right)^{\alpha-1}=(1-\alpha)\left(\frac{kT}{n}-\theta_s\right)^{\alpha-2}(s-\frac{jT}{n})\leqslant(1-\alpha)\left(\frac{kT}{n}-s\right)^{\alpha-2}\frac{T}{n},$$

where $\theta_s\in[\frac{jT}{n},s]$, we get the upper bound for the difference of integrals:

$$\left|\int_{jT/n}^{(j+1)T/n}\left(\frac{kT}{n}-s\right)^{\alpha-1}ds-\frac{T}{n}\left(\frac{kT}{n}-\frac{jT}{n}\right)^{\alpha-1}\right|\leqslant(1-\alpha)\frac{T}{n}\int_{jT/n}^{(j+1)T/n}\left(\frac{kT}{n}-s\right)^{\alpha-2}ds,$$

and from now on we can proceed similarly to (4):

$$\Im_{j,n} \leqslant \frac{1}{\Gamma(\alpha)}\Delta_n \int\limits_{jT/n}^{(j+1)T/n} \left(\frac{kT}{n}-s\right)^{\alpha-1} ds$$

$$+\frac{1}{\Gamma(\alpha)}\|f\|_{C[0,T]} \left| \int\limits_{jT/n}^{(j+1)T/n} \left(\frac{kT}{n}-s\right)^{\alpha-1} ds \right.$$

$$\left. -\frac{T}{n}\left(\frac{kT}{n}-\frac{jT}{n}\right)^{\alpha-1} \right|$$

$$\leqslant \frac{1}{\Gamma(\alpha)}\left(\frac{T}{n}\|f\|_{C[0,T]}(1-\alpha) *\right.$$

$$\left. * \int\limits_{jT/n}^{(j+1)T/n} \left(\frac{kT}{n}-s\right)^{\alpha-2} ds + \frac{1}{\alpha}\Delta_n\left(\frac{T}{n}\right)^{\alpha}\right)$$

$$\leqslant \frac{1}{\Gamma(\alpha)}\left(\frac{T}{n}\|f\|_{C[0,T]}(1-\alpha) *\right.$$

$$\left. * \int\limits_{jT/n}^{(j+1)T/n} \left(\frac{kT}{n}-s\right)^{\alpha-2} ds + \frac{C_\beta}{\alpha}\left(\frac{T}{n}\right)^{\alpha+\beta}\right).$$

Taking a sum of integrals over $0 \leqslant j \leqslant k-2$, we get the upper bound

$$\int\limits_{0}^{(k-1)T/n} \left(\frac{kT}{n}-s\right)^{\alpha-2} ds \leqslant \frac{1}{1-\alpha}\left(\frac{T}{n}\right)^{\alpha-1}.$$

Finally,

$$\left|\left(I_{0+}^{\alpha}f\right)\left(\frac{kT}{n}\right) - S_{1,n}\left(\frac{kT}{n}\right)\right|$$

$$\leqslant \Im_{k,n} + \sum_{j=0}^{k-2}\Im_{j,n}$$

$$\leqslant \frac{1}{\Gamma(\alpha)}\left(\frac{1}{\alpha}C_\beta\left(\frac{T}{n}\right)^{\beta}\left(\frac{T}{n}\right)^{\alpha}\right.$$

$$\left. +\|f\|_{C[0,T]}\left(\frac{1}{\alpha}-1\right)\left(\frac{T}{n}\right)^{\alpha}\right)$$

$$+\frac{1}{\Gamma(\alpha)}\left(\frac{C_\beta}{\alpha}\left(\frac{T}{n}\right)^{\alpha+\beta}(n-1)\right.$$

$$\left. +\frac{T}{n}\|f\|_{C[0,T]}(1-\alpha)\frac{1}{1-\alpha}\left(\frac{T}{n}\right)^{\alpha-1}\right)$$

$$= \frac{1}{\Gamma(\alpha)}\left(\frac{C_\beta}{\alpha}\left(\frac{T}{n}\right)^{\alpha+\beta} n\right.$$

$$\left. +\|f\|_{C[0,T]}\left(\frac{T}{n}\right)^{\alpha}\left(\frac{1}{\alpha}-1+1\right)\right)$$

$$= \frac{1}{\Gamma(\alpha)}\left(\frac{C_\beta T^{\alpha+\beta}}{\alpha}n^{1-\alpha-\beta} + \|f\|_{C[0,T]}T^{\alpha}n^{-\alpha}\right)$$

Since $1-\beta > 0$, we have $1-\alpha-\beta > -\alpha$, $n^{-\alpha} < n^{1-\alpha-\beta}$ and respectively the upper bound is

$$\frac{1}{\Gamma(\alpha)}\left(\frac{C_\beta T^{\alpha+\beta}}{\alpha} + \|f\|_{C[0,T]}T^{\alpha}\right)n^{1-\alpha-\beta}$$

(ii) These upper bounds are even simpler. Indeed, we again divide the difference

$$\left|\left(I_{0+}^{\alpha}f\right)\left(\frac{kT}{n}\right) - S_{2,n}\left(\frac{kT}{n}\right)\right|$$

into the sum of respective differences on the subsequent intervals of the partition, and in this case for any $0 \leqslant j \leqslant k-1$ we can estimate these differences as follows:

$$\frac{1}{\Gamma(\alpha)}\left| \int\limits_{jT/n}^{(j+1)T/n} \left(\frac{kT}{n}-s\right)^{\alpha-1} f(s)ds \right.$$

$$\left. -\frac{1}{\alpha}\left(\left(\frac{kT}{n}-\frac{(j+1)T}{n}\right)^{\alpha} - \left(\frac{kT}{n}-\frac{jT}{n}\right)^{\alpha}\right) f\left(\frac{jT}{n}\right)\right|$$

$$\leqslant \frac{1}{\Gamma(\alpha)}\left| \int\limits_{jT/n}^{(j+1)T/n} \left(\frac{kT}{n}-s\right)^{\alpha-1} \left|f(s)-f\left(\frac{jT}{n}\right)\right| ds \right.$$

$$\leqslant \frac{1}{\Gamma(\alpha+1)}\Delta_n\left(\frac{T}{n}\right)^{\alpha} \leqslant \frac{C_\beta}{\Gamma(\alpha+1)}\left(\frac{T}{n}\right)^{\alpha+\beta},$$

whence the proof immediately follows.

5 Graphical illustrations of convergence rate

Consider several examples illustrating the convergence of integral sums for various functions and various number of intervals of partition. We always consider interval $[0,1]$. The values of integrals are received with the help of standard functions integral from MATLAB. It is clear that the sums $S_{2,n}$ give the better approximation.

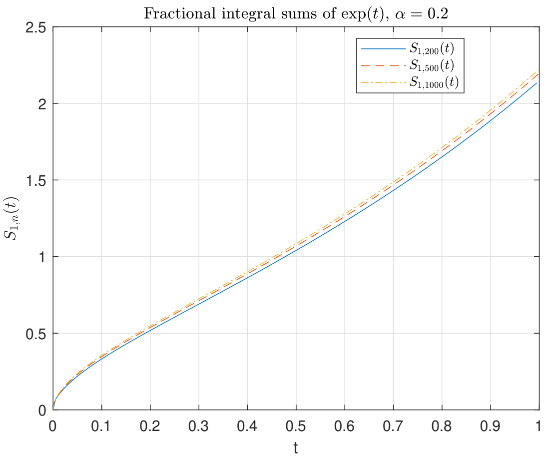


Рис. 1

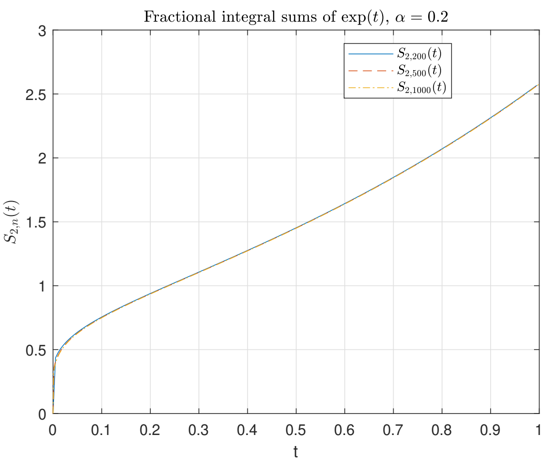


Рис. 2

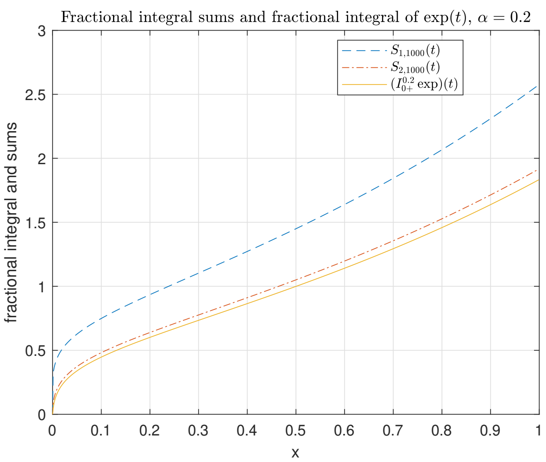


Рис. 3

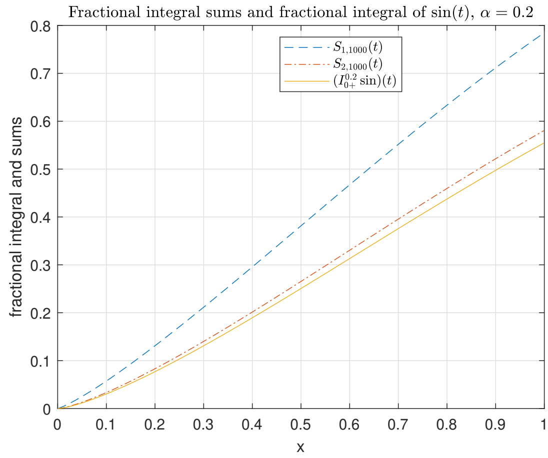
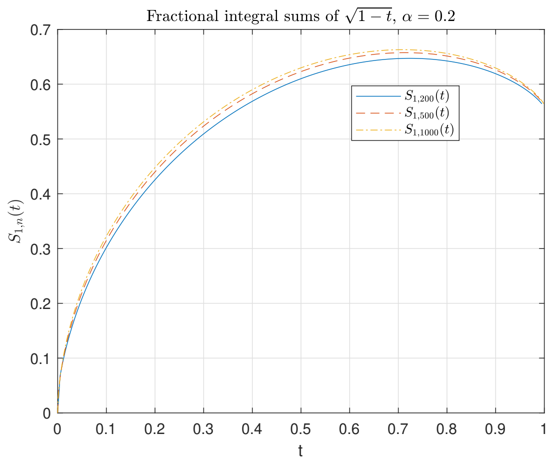


Рис. 4



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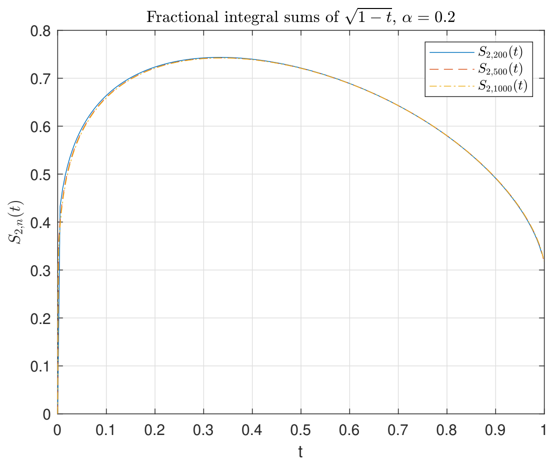

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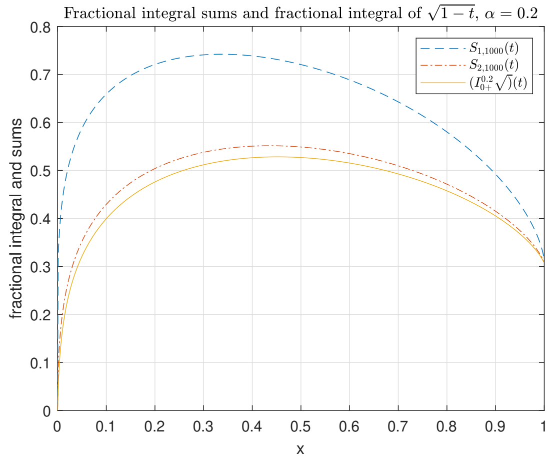

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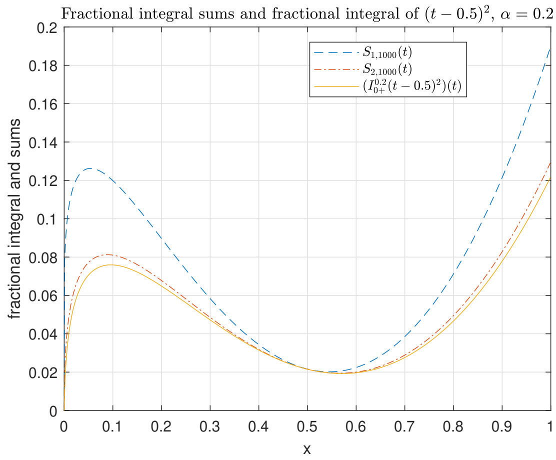

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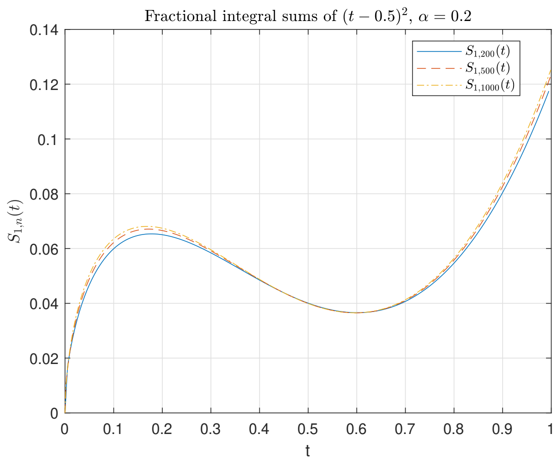

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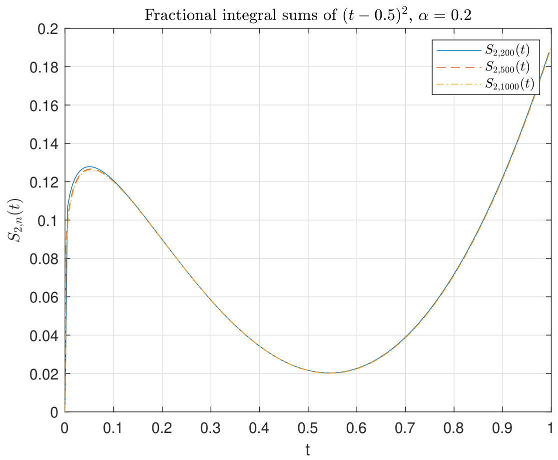

Рис. 9

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