5-2023

G(10,30): A Minor-Minimal Intrinsically Knotted Graph

Johanthan R. Herron

$G(10, 30)$: A MINOR-MINIMAL INTRINSICALLY KNOTTED GRAPH

A Thesis

Submitted to the Graduate Faculty of
The University of South Alabama
in partial fulfillment of the
requirements for the degree of

Masters of Science

in

Mathematics

by
Johnathan Ridley Herron
B.S. Mathematics: University of Montevallo 2021
May 2023

ACKNOWLEDGEMENTS

I would like to thank Dr. Andrei Pavelescu for supervising the research project for which this thesis is the final result. Without his time and patience, I may never have had the opportunity to work in and contribute to such a fascinating subject as graph theory. I would like to express my gratitude to Dr. Elena Pavelescu, Dr. Martin Frank, and Dr. Gayan Abeynanda for their time and consideration as members of my thesis committee. Their input greatly helped in refining this thesis.

I would also like to thank my siblings, Thomas, Jude, and Addy, for their encouragement and support.

I cannot express enough gratitude to my parents, John and Michelle Herron. Without their undivided love, support, and teaching, I never would have come to appreciate the beauty and purity of mathematics.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

ABSTRACT

Herron, Johnathan, Ridley, M.S, University of South Alabama, May 2023. $G(10, 30)$: A Minor-Minimal Intrinsically Knotted Graph. Chair of Committee: Dr. Andrei Pavelescu

In this paper, we shall lay the groundwork for a proof of the minor-minimal intrinsic knotting of the graph $G(10, 30)$. We show that this graph is in fact minor minimal with respect to the property of intrinsic knotting, i.e that no minor of $G(10, 30)$ is intrinsically knotted. Moreover, we discuss the procedure for showing that $G(10, 30)$ itself is intrinsically knotted, and provide a collection of subgraphs that can be used to aid in a proof. In this way, we hope to contribute to the growing list of known minor-minimal intrinsically knotted graphs.

CHAPTER I
INTRODUCTION

One core component of mathematical research is to characterize given mathematical objects–sets, functions, etc.–in terms of their fundamental structure. This is observed, for example, in the characterization of continuous functions in analysis; continuity or discontinuity over a given domain is perhaps the most fundamental feature of a given function, and much effort is spent providing ways to determine if a given function is continuous or not. An analogous effort is being made in the field of graph theory, where such properties as planarity, knotting, and linking are used to characterize a given graph's shape across its numerous embeddings. A graph, being defined by a vertex set and corresponding edge set, is inherently a set theoretic object. However, depictions of the graph can reveal a level of complexity in the graph's shape that is obscured by the notation of set theory. This includes the existence of knots and links, special constructions within the graph that describe the presence and relationship of embeddings of the circle S^1 within the graph. Just as we are often interested in determining if a function is continuous across a domain, we are interested if a given graph expresses knotting or linking across all of its possible embeddings–properties referred to as intrinsic knotting or linking respectively. An intrinsically knotted or linked graph can be thought of as a "permanently knotted or linked" graph–the presence of a knot is a fundamental component of that graph's structure, and cannot be removed without destroying the graph itself (much in the same way that a continuous function can only be made noncontinuous through a change in the underlying topology of the relevant space). However, it should be pointed out that the knot in one embedding may not necessarily be the same knot in every other embedding.

In mathematical research, we are also interested in the most basic examples of an object exhibiting a certain property. The motivating question is thus: What is the smallest or simplest object that has this property? Once that has been determined, it is often easier to characterize the more complicated objects in terms of these primitives. In graph theory, where the construction of graphs is so important to research, this question is exceedingly important. Minor-minimal intrinsically knotted (linked) graphs can be thought of as the most fundamental knotted (linked) graphs, from which all others can be generated. Such graphs lose their knotting (linking) immediately if any edge is deleted or contracted. And all intrinsically knotted (linked) graphs will, in turn, possess an intrinsically knotted (linked) graph as a minor. Thus, compiling a complete list of such minor-minimal intrinsically knotted (linked) graphs would make the characterization of more complicated knotted (linked) graphs significantly easier. For knotlessly (linklessly) embeddable graphs, such specimens could never be found as minors–they are the "forbidden minors". This is observed with intrinsic linking, where its list of minor-minimal intrinsically linked graphs has been fully described [11]. This has not been done for intrinsic knotting.

Intrinsic knotting (linking) in graphs is not only of interest to the researcher of pure mathematics. Graph theory and knot theory together are two fields of mathematics which can be applied directly to more practical endeavors. The best example would be applications to the field of chemistry, where the properties of a molecule are central to its study and use. A molecule can be described by the number of atoms and their various relationships through bonding. This can be

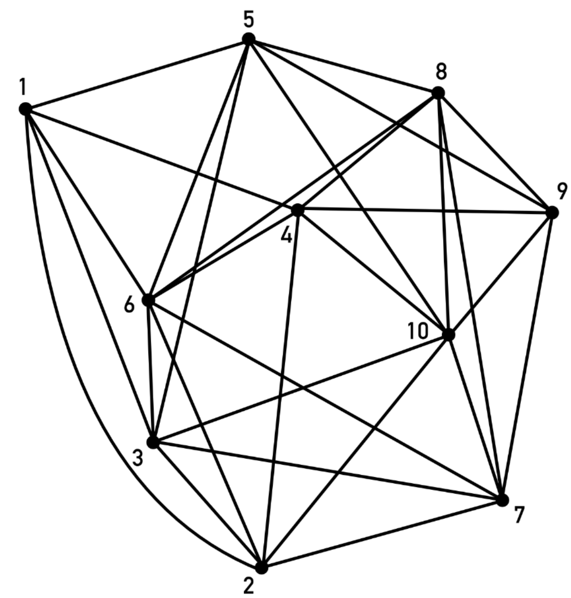

Figure 1. The graph $G(10, 30)$.

readily modeled through the use of vertices and edges–graphs. Changes to a molecule's structure can be modeled by changes in embeddings of the corresponding graph. One example of this is described in Adams' *The Knot Book* where the action of enzymes on a DNA strand can be shown to produce knotting in the strand [1]. Similarly, if a chemist wished to synthesize a molecule with a knotted or linked structure, they could base the molecule upon an intrinsically knotted or intrinsically linked graph, where the presence of the desired configuration is guaranteed [1].

The goal of this thesis is to contribute to the growing body of research around minor-minimal intrinsically knotted graphs. We attempt to prove that the graph of ten vertices and thirty edges, denoted $G(10, 30)$, is among the forbidden minors of knotlessly embeddable graphs. This graph is depicted with our chosen labeling in Figure 1. The edge list, under this labeling, is given by

$$\{1,2\}\,,\{1,3\}\,,\{1,4\}\,,\{1,5\}\,,\{1,6\}\,,$$
$$\{2,3\}\,,\{2,4\}\,,\{2,6\}\,,\{2,7\}\,,\{2,10\}\,,$$
$$\{3,5\}\,,\{3,6\}\,,\{3,7\}\,,\{3,10\}\,,\{4,6\}\,,$$
$$\{4,8\}\,,\{4,9\}\,,\{4,10\}\,,\{5,6\}\,,\{5,8\}\,,$$
$$\{5,9\}\,,\{5,10\}\,,\{6,7\}\,,\{6,8\}\,,\{7,8\}\,,$$
$$\{7,9\}\,,\{7,10\}\,,\{8,9\}\,,\{8,10\}\,,\{9,10\}\,.$$

It should be noted that the vertices of $G(10,30)$ can be broken into eight equivalence classes up to symmetry: $\{1\}\,,\{2,3\}\,,\{4,5\}\,,\{6\}\,,\{7\}\,,\{8\}\,,\{9\}\,,\{10\}\,.$ This indicates that any cycle including the vertex 2 should be mirrored by the corresponding cycle where 2 is replaced by 3, provided we also switch 4 and 5 at the same time. For example, the cycle $(1,3,6,5)$ is mirrored by the cycle $(1,2,6,4)$ under graph symmetry.

This graph was first introduced in a paper by Mattman et al. as an example of an intrinsically knotted graph with μ-invariant 5 [8]. Moreover, it is given as an example of a minimal-order knotted graph with an edge-contraction minor that is linklessly embeddable. By proving that this graph is minor-minimal with respect to the property of intrinsic knotting, and by providing a starting point for the proof of its intrinsic knotting, we are moving one step closer to providing a formal proof that $G(10,30)$ is among the forbidden minors for knotlessly embeddable graphs. This will not only help to prove the assertion in the paper mentioned above, but will also carry us forward in making a complete list of such forbidden minors.

CHAPTER II

REVIEW OF EXISTING BODY OF KNOWLEDGE

We begin by considering the progression of definitions and results from the most foundational definition in graph theory to those theorems utilized directly in our work. To start, we shall consider the formal definition of a graph, and then proceed through the other basic notions of graph theory and knot theory.

Definition 1. *Let V be a set, and $[V]^2 \subset \mathfrak{P}(V)$ a subset of the power set of V containing all two-element subsets of V. A graph $G = (V, E)$ is an ordered pair of sets E and V such that $E \subseteq [V]^2$. The set V is the set of vertices of G, and the set E is the set of edges of G. A graph is said to be complete if $E = [V]^2$. A graph is said to be simple if every pair of vertices is connected by at most one edge (otherwise, it is a nonsimple graph).*

Example 1. *Let $V = \{a, b, c, d, e\}$. Then we find that*

$$[V]^2 = \{\{a,b\}, \{a,c\}, \{a,d\}, \{a,e\}, \{b,c\}, \{b,d\}, \{b,e\}, \{c,d\}, \{c,e\}, \{d,e\}\}$$

is the set of all possible edges between the vertices of the graph on V. The pair $F = (V, E)$ where

$$E = \{\{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{c,d\}, \{d,e\}\}$$

is a graph of five vertices and six edges. The graph $K_5 = (V, [V]^2)$ is the complete graph on five vertices. Both F and K_5 are shown in 2.

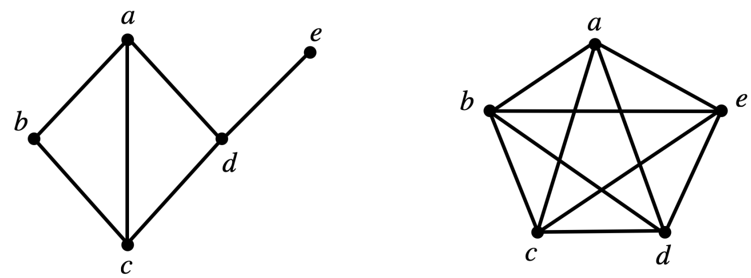

Figure 2. Two graphs on five vertices. F is shown to the left, and K_5 to the right. Both of these graphs are simple.

In addition to Definition 1, we have additional terminology to describe the environment of a graph. Given $G = (V, E)$, two vertices $x, y \in V$ are said to be incident if the edge $\{x, y\} \in E$. Similarly, two edges $\{x, y\}$ and $\{y, z\}$ would be said to be adjacent. The degree of a given vertex x is the number of sets in E to which x belongs. This corresponds graphically to the number of edges incident to x.

A cycle of a graph $G = (V, E)$ is a subgraph $G = (V', E')$ where $V' = \{x_1, x_2, \ldots, x_n\}$ and $E' = \{\{x_1, x_2\}, \{x_2, x_3\}, \ldots, \{x_{n\;1}, x_n\}, \{x_n, x_1\}\}$, where $n \quad 3$. We use the notation $(x_1, x_2, \ldots, x_n)$ to represent a cycle in a graph, and call it an n-cycle. A Hamiltonian cycle is a cycle which touches every vertex in a graph exactly once.

Given a graph G, it is often of interest to consider other graphs produced by acting upon G with various operations. The most basic operations are edge or vertex deletions. In set theoretic terms, this amounts to considering nontrivial subsets of the vertex set V or the edge set E, and examining the graph described by those subsets. In the case of an edge deletion, we simply remove the edge between two vertices. With regards to vertex deletions, we delete a given vertex and all edges which are incident to it. The result of such deletions is a subgraph, defined in Definition 2.

Definition 2. *Let $G = (V, E)$ be a graph. Then some graph $G' = (V', E')$ is said to be a subgraph of G if $E' \subseteq E$ and $V' \subseteq V$. We denote the graph-subgraph relation by $G' \subseteq G$.*

Example 2. *In Figure 2, we find that F is a subgraph of K_5, obtained through the deletion of the following edges:* $\{a, e\}, \{b, d\}, \{d, e\}, \{c, e\}$.

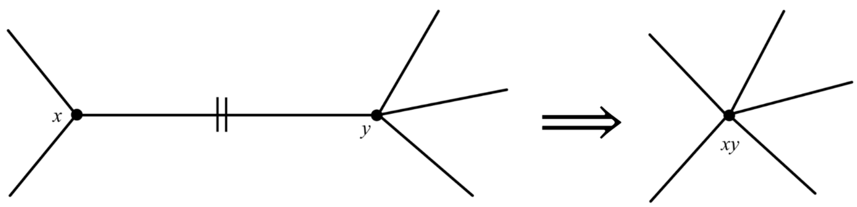

Figure 3. Contracting the edge $\{x, y\}$

Subgraphs are not the only graphs that can be produced from a given graph, however. Let x, y be vertices of G with $\{x, y\} \in G$, such that

$$\{x, a_1\}, \{x, a_2\}, \{x, a_3\}, \ldots$$

are the edges incident to x and

$$\{y, b_1\}, \{y, b_2\}, \{y, b_3\}, \ldots$$

are the edges incident to y. We can contract the edge $\{x, y\}$ by removing it from E and by replacing x and y with a single new vertex xy. Moreover, we replace all edges incident to x and y with the following edges

$$\{xy, a_1\}, \{xy, a_2\}, \{xy, a_3\}, \ldots, \{xy, b_1\}, \{xy, b_2\}, \{xy, b_3\}, \ldots$$

Graphically, this amounts to combining x and y into xy, a new vertex which is incident to all vertices incident to x and y individually. We also remove any double edges that might result from such a contraction; it should be noted, however, that non-simple contractions can be used to obtain a certain desired structure (as we shall observe later on in our research). An example of this is shown in Figure 3. This leads us to the next definition.

Definition 3. *Let G and H be graphs such that H is obtained from G through a sequence of edge deletions and/or contractions, and vertex deletions. Then we call H a minor of G.*

We should point out that there are technically two possible edge contraction operations. One of them, as described above, preserves all edges from the original graph–possibly resulting in a non-simple graph. The other edge contraction would ignore any repeated edges, resulting in a simple graph.

Example 3. *Consider F first given in Example 1. We can apply a (nonsimple) edge contraction on $\{a, b\}$ to produce a minor F' shown in Figure 4. Note that F' is by extension a minor of K_5 through the edge contraction and the same sequence of edge deletions that produced F as a subgraph. Indeed, F itself is a minor as well as a subgraph of K_5. F' is a minor but not a subgraph of K_5.*

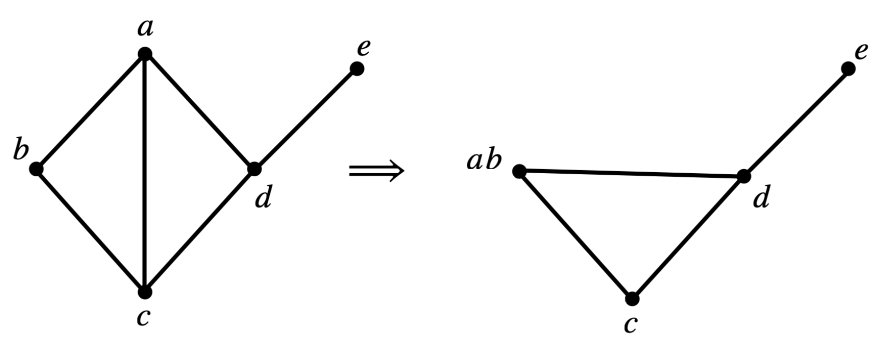

Figure 4. An edge-contraction minor of F.

Definition 4. *A graph G is said to be minor minimal with respect to some property if G possesses the property but no minor H possesses that property.*

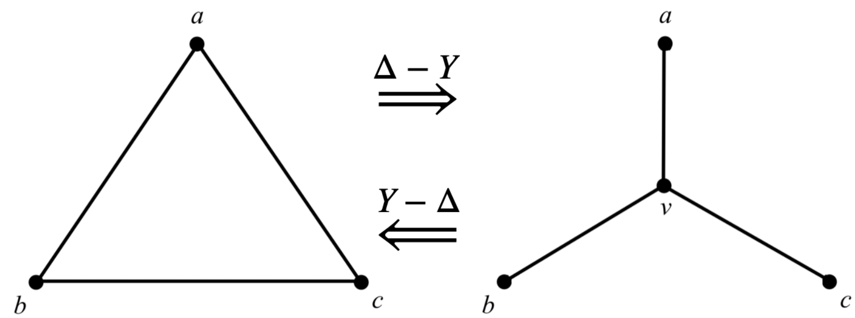

Figure 5. Y and Y transformations.

We shall consider one more operation which is more directly relevant to our work: Y and Y transformations. First, it is necessary to introduce some specific terminology. Let G be a graph such that for vertices $a, b, c \in V$ we have edges $\{a, b\}, \{b, c\}, \{a, c\}$. Then $T = (\{a, b, c\}, \{\{a, b\}, \{b, c\}, \{a, c\}\})$ is a triangle of G. A Y (Delta-Wye) transformation on T is an operation such that a new vertex v is introduced, and the edges $\{a, b\}, \{b, c\}, \{a, c\}$ are all replaced by $\{a, v\}, \{b, v\}, \{c, v\}$. The Y transformation is the inverse of this operation, taking a collection of edges $\{a, v\}, \{b, v\}, \{c, v\}$, deleting the vertex v and forming edges between a, b, and c. Both of these operations are shown in Figure 5.

Definition 5. *Let G and H be two graphs. If H can be obtained from G by a single Y transformation, then we call H a child of G. G is called the parent of H. Graphs related through a finite sequence of Y and Y transformations are called cousins. The family of G is the collection of all graphs obtained through a finite sequence of Y and Y transformations.*

Example 4. *Once again, consider the graph F given in Example 1. We can see that F possesses two triangles, the 3-cycles (a, b, c) and (a, d, c). We can apply a*

Y transformation on (a, b, c) *to produce* F_1*, shown in Figure 6. Note the inclusion of a new vertex* v*. This graph* F_1*, being the result of a single Y transformation on* F*, is a child of* F*.*

Both F *and* F_1 *have a* Y*-configuration in the edges* $\{a, d\}, \{c, d\}$*, and* $\{d, e\}$*. We can apply a* Y *transformation to produce the cousin* F_2*.*

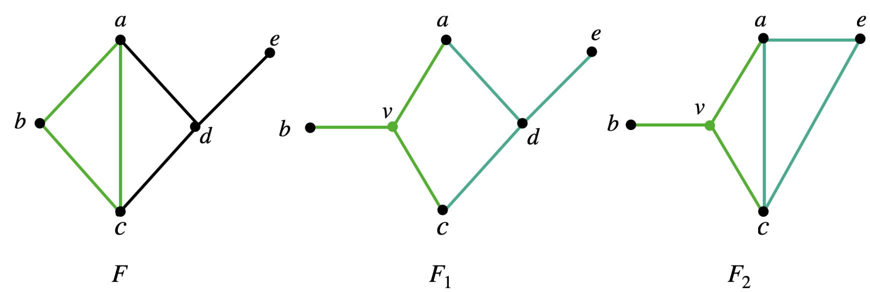

Figure 6. Cousins of F.

Necessary to the study of graph theory is the ability to describe graphs as images rather than simply sets. This is handled mathematically through the notion of the embedding, and specifically the homomorphic embedding, given in Definition 6.

Definition 6. *Let* $G = (V, E)$ *and* $H = (U, F)$ *be two graphs. A function* $: V \longrightarrow U$ *is said to be an embedding of* G *in* H *if for all* $x, y \in V$ *where* $(x) = (y)$*, we find that* $x = y$*. We call a homomorphic embedding if for all* $x, y \in V$ *where* $\{x, y\} \in E$*, we find that* $\{ (x), (y)\} \in F$*.*

From the definition above, and taking to be a homomorphic embedding, we find that the set $(V) = \{ (x) : x \in V\}$ along with the corresponding edge set $(E) = \{\{ (x), (y)\} : x, y \in V, \{x, y\} \in E\}$ is, effectively, the representation of the graph G within H. When we take H to be a space of real numbers $\mathbb{R}^n$ then $(G) = ((V), (E))$ would be a picture of G in this more familiar setting–what we will call an embedding of G in $\mathbb{R}^n$. For our purposes, we are interested only in the cases where $n = 2$ or $n = 3$. These are necessary to begin describing planar, linked, and knotted graphs.

Definition 7. *A graph G is said to be planar if there is an embedding of G in $\mathbb{R}^2$ that contains intersections only at common endpoints.*

Planar graphs can be thought of as the most basic kind of graph which has a defined structure. These are graphs which can be depicted without edges intersecting anywhere other than the endpoints. In terms of structural complexity, they can be thought of as being somewhat simple. The next level of structural complexity can be found in graphs which are, in a sense, almost planar.

Definition 8. *A graph $G = (V, E)$ is said to be t-apex if the graph $G' = (V', E')$ where $V' = V \quad \{v_1, v_2, \ldots, v_t\}$ is a planar graph., where $v_1, \ldots, v_t$ is some collection of elements out of V.*

Now let us consider the linking and knotting of graphs. These notions are more commonly found in the field of knot theory; as such, it is more useful to consider knots and links in terms of the circle S^1.

Definition 9. *A collection of disjoint embeddings of S^1 in $\mathbb{R}^3$ is known as a link.*

Definition 10. *Two embeddings of S^1 are said to be splittably linked if there exists two disjoint embeddings of the 2-sphere, each containing one of the two embeddings of S^1. If no such pair of 2-spheres exist, then the two embeddings are said to be non-splittable.*

To determine if two embeddings of S^1 are linked, it is useful to calculate the linking number of the two embeddings, denoted $lk(C_1, C_2)$, where C_1 and C_2 are the two embeddings. Linking number is calculated by considering all crossings between the S^1 embeddings, setting an orientation to the edges, and applying a value associated with the two possible types of crossings. The two crossings and their corresponding values are defined in Figure 7. This rule assumes that we are

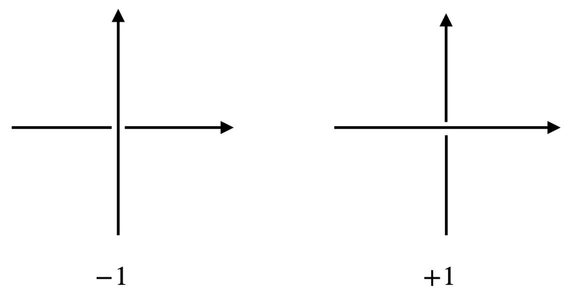

Figure 7. Types of crossings. Orientation of the edges must first be defined.

projecting our links onto $\mathbb{R}^2$. Then, the values are summed, with the result divided by 2. The final value, the linking number, determines whether or not the pair of S^1 are splittably linked or not. Given two embeddings of S^1, C_1 and C_2, we denote their linking with the notation $C_1 \cup C_2$. This provides us with the following useful theorem.

Theorem 1. *Two embeddings of S^1 are non-splittably linked if their linking number is nonzero.*

We now extend the notion of links to graphs.

Definition 11. *Let G be a graph. Then G is said to be intrinsically linked if every embedding of G in $\mathbb{R}^3$ contains a non-splittable link. Otherwise, the graph is said to be linklessly embeddable.*

Definition 12. *A knot is a embedding of S^1, a circle, in $\mathbb{R}^3$. A knot projection is a representation of a knot in $\mathbb{R}^2$. A knot is said to be nontrivial if it cannot bound a 2-cell (disc) in $\mathbb{R}^3$.*

Definition 13. *A graph G is said to be intrinsically knotted if every embedding of G in $\mathbb{R}^3$ contains a nontrivial knot. Otherwise, the graph is said to be knotlessly embeddable.*

In the previous definitions, we developed the language necessary to discuss the theoretical backbone of our work. The notions of intrinsic linking and intrinsic knotting were first confronted in the seminal paper “Knots and Links in Spatial Graphs,” by J. Conway and C. Gordon. In the paper, Conway and Gordon proved that every embedding of the complete graph on seven vertices, K_7, cycles a nontrivial knot, and that every embedding of the complete graph on six vertices, K_6, contained a nontrivial link [3]. Both of these graphs are depicted in Figure 8.

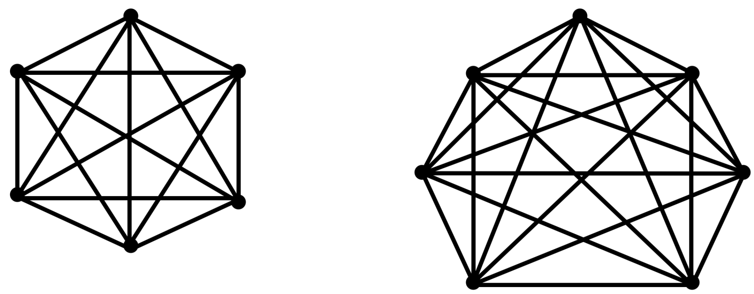

Figure 8. The graphs K_6 and K_7. K_6 is the first graph proven to be intrinsically linked, and K_7 is the first graph to be proven to be intrinsically knotted.

The proof of the intrinsic knotting of K_7 required the use of the arf invariant, which is given by

$$\alpha(K) = \begin{cases} 0, & K \text{ is pass equivalent to the trivial knot} \\ 1, & K \text{ is pass-equivalent to the trefoil knot} \end{cases}$$

for the knot K [1]. Two knots are said to be pass-equivalent if one can be obtained from the other through a sequence of pass moves–which are changes in the projection of a knot made by passing a pair of oppositely-oriented edges through another pair of oppositely-oriented strands. For further discussion of pass moves and pass-equivalence, consider p.222-231 of Adams’ *The Knot Book* [1]. Every non-trivial knot is pass equivalent to the trefoil knot. Conway and Gordon defined a function $\in \mathbb{Z}_2$ to be the sum of all arf invariants of every Hamiltonian cycle of K_7

in an arbitrary embedding, showed to be invariant under changes in edge crossing, and found at least one embedding of K_7 containing the trefoil knot. With one embedding known to be knotted, and so having one instance that $(K_7) \neq 0$, it can be concluded that every embedding is knotted since is invariant. It should be noted that the proof for the intrinsic knotting of K_7 is significantly more involved than the proof for the intrinsic linking of K_6.

With these two examples established, research has moved towards characterizing all intrinsically knotted/linked graphs. In terms of intrinsic linking, this was successful, as P.D. Seymour, Neil Robertson, and Robin Thomas proved in their 1995 paper titled "Linkless Embeddings of Graphs in 3-space" that every intrinsically linked graph must have at least one of seven specific graphs as minors [10, 11]. These graphs are the Petersen family graphs, all of which are obtained through a sequence of Y and Y transformations from K_6. This family is depicted in Figure 9. Prior to this, H. Sachs showed that the Petersen family graphs were all minor minimal intrinsically linked [13]. Between the two results, a class of "forbidden minors" was established for linklessly embeddable graphs. This could be used to easily determine if a given graph was intrinsically linked or not. If a graph had one of the Petersen family as a minor, it would be intrinsically linked. No such list of forbidden minors exist for knotlessly embeddable graphs.

Robertson and Seymour proved that the set of minor-minimal intrinsically knotted graphs must be finite [12]. This was done by proving for every graph H and some infinite set of graphs whose minors are not isomorphic to H, some member of the set is isomorphic to a minor of another member. By taking the set of minor-minimal intrinsically knotted graphs to be infinite–none of whose minors are isomorphic to some graph H–then there must exist a minor of one graph in the set

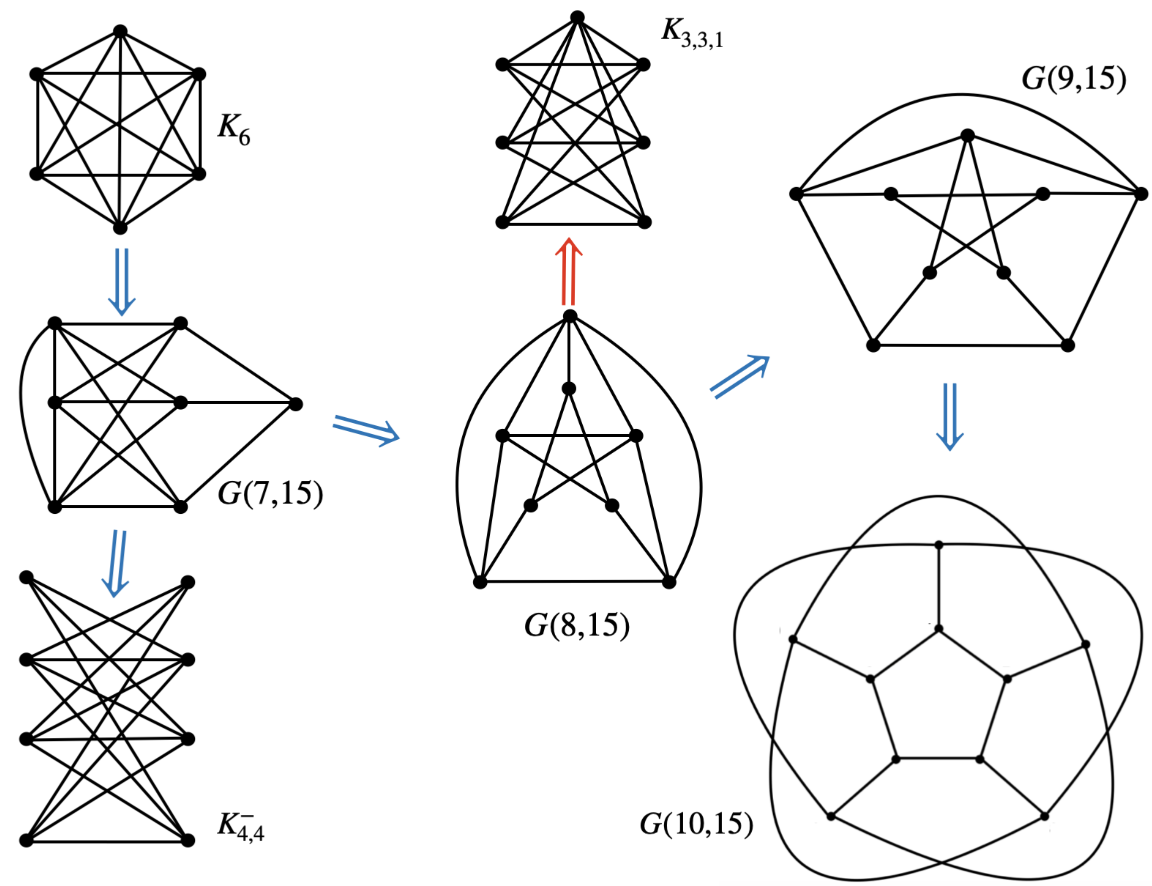

Figure 9. The Petersen family. These seven graphs are the “forbidden minors” for linklessly embeddable graphs.

which is isomorphic to another graph in the set. This is, of course, a contradiction, as no minor of a minor-minimal intrinsically knotted graph is intrinsically knotted, and the property of knotting is preserved under isomorphism. Therefore, the set of forbidden minors for knotlessly embeddable graphs must be finite. Further research into this problem has narrowed down this list; the only minor-minimal intrinsically knotted graphs on eight or fewer vertices are K_7 and $K_{3,3,1,1}$ [2]. Moreover, it has been shown that any intrinsically knotted graph must have at least 21 edges [7].

The work presented here relies heavily upon two results in particular regarding intrinsic knotting. First and foremost, we have the following Lemma proved by J. Foisy [4]:

Lemma 1 (Foisy, 2002, [4]). *Given an embedding of D_4,* $\neq 0$ *if* $lk(C_1, C_3) \neq 0$ *and* $lk(C_2, C_4) \neq 0$.

This lemma, which we shall call "Foisy's Lemma", requires some explanation. The graph D_4 is a graph on four vertices with eight edges, depicted in Figure 10. This graph is comprised of four cycles, formed from the double edges between vertices. The cycle C_1 corresponds to the two edges between vertices 1 and 2. The cycle C_3 sits opposite C_1, being formed by the double edges between 3 and 4. Similarly, C_2 is formed by the vertices 2 and 3, and C_4 by 1 and 4. If the linking number between C_1 and C_3 is nonzero in an embedding, then they are linked. Similarly, if the linking number between C_2 and C_4 is nonzero in a given embedding, they are linked. In the case where both pairs are linked, we have a doubly-linked D_4 graph. By Foisy's Lemma, such a graph guarantees that we have a knotted embedding (the invariant used in Conway and Gordon's paper is nonzero [3]). The success of Foisy's Lemma in proving graphs to be intrinsically knotted can be observed in Foisy's papers "Many More Intrinsically Knotted Graphs," where he proves that the graphs G_{15}, H_{15}, J_{14} and J'_{14} are all intrinsically knotted using the lemma [6].

Foisy's Lemma outlines our primary tool for proving that G(10,30) is intrinsically knotted. We show that for every embedding of $G(10, 30)$ there exists a doubly-linked D_4 subgraph. The linking shall be provided by identifying pairs of intrinsically linked subgraphs within $G(10, 30)$–specifically any instances of a Petersen family graph. Each subgraph shall have a full list of pairs of cycles, at least one of which must be linked in an arbitrary embedding. By going through each pair and assuming linkage, we shall try and demonstrate that, when coupled with another pair of assumed-linked cycles out of the other graph, they will form a doubly-linked D_4 subgraph. Optimally, we shall need a single pair of Petersen family subgraphs of $G(10, 30)$.

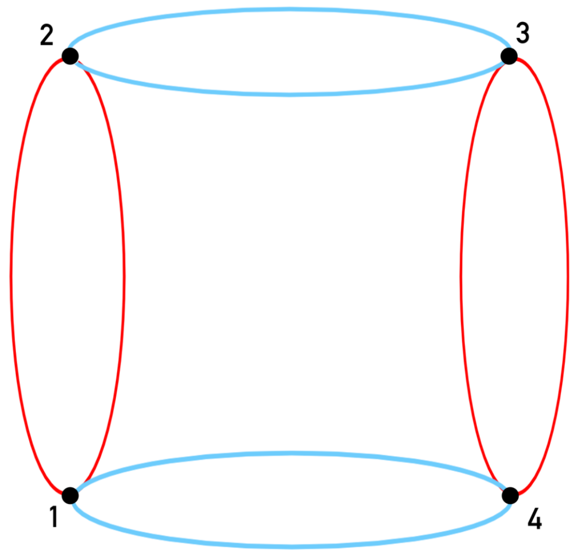

Figure 10. The graph D_4. The linked cycles in a doubly-linked D_4 graph are colored.

To supplement Foisy's Lemma in cases where a D_4 is not readily produced from the pairs of cycles on hand, it is necessary to utilize a result from homology theory. The following statement of the lemma is given in Foisy's paper "A Newly Recognized Intrinsically Knotted Graph" [5]:

Lemma 2 ([5])**.** *Let* ${}_1$, ${}_2$ *and* ${}_3$ *to be simple closed curves in* $\mathbb{R}^3$ *such that* ${}_2 \cap {}_3$ *is an arc, and both* ${}_2 \cap {}_1$ *and* ${}_1 \cap {}_3$ *are empty. Suppose that* $[{}_2]$ *is non-trivial in* $H_1(\mathbb{R}^3 \quad {}_1; \mathbb{Z}_2)$*. Then precisely one of* $[{}_3]$ *and* $[{}_2 + {}_3]$ *is non trivial in* $H_1(\mathbb{R}^3 \quad {}_1; \mathbb{Z}_2)$.

One can understand this lemma from a purely physical perspective. Given three simple closed curves which exhibit nontrivial linking, if two curves intersect over an arc (a common edge between the curves), then the linking must occur on at least one side of this common edge. Consider Figure 11, which shows a pair of cycles: (a, b, d, c) and (e, f, g). The 4-cycle (a, b, d, c) is further made up of two

3-cycles, (a, b, c) and (b, c, d). We may take $_1$ to be the simple closed curve defined by (e, f, g), and $_2$ and $_3$ the simple closed curves defined by (a, b, c) and (b, c, d) respectively. We find that $_2$ and $_3$ intersect over the edge $\{b, c\}$ (an arc), and that $_1 \cap {}_2$ and $_1 \cap {}_3$ are both empty. By Lemma 2, if (a, b, d, c) and (e, f, g) are linked, then either (a, b, c) and (e, f, g) are linked, or (b, c, d) and (e, f, g) are linked.

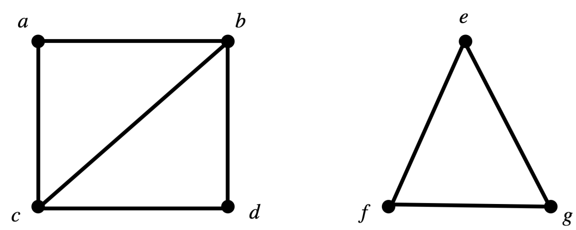

Figure 11. A pair of linked cycles.

Showing $G(10, 30)$ to be minor minimal with respect to the property of intrinsic knotting shall make use of a theorem proven by P. Blain, G. Bowlin, T. Fleming, J. Foisy, and others[1] [2].

Theorem 2 (Blain et al., 2007, [2]). *No 2-apex graph is intrinsically knotted.*

In the special cases where this theorem fails to provide immediate results, we turn to a result from Motwani, Raghunathan, and Saran [9].

Theorem 3 (Motwani et al, 1988, [9]). *Y exchanges preserve intrinsic linking and knotting.*

[1] This result was also proven concurrently by Ozawa and Tsutsumi.

CHAPTER III

SURVEY: PROOF THAT $G(10, 30)$ IS INTRINSICALLY KNOTTED

The first step in proving that $G(10, 30)$ is among the forbidden minors of knotlessly embeddable graphs is showing that it is intrinsically knotted. The proof of $G(10, 30)$'s intrinsic knotting is based entirely upon Foisy's Lemma, with the goal being to identify a doubly-linked D_4 graph in every embedding of $G(10, 30)$. Given that manually checking every embedding for the desired D_4 graph would be impossible, we shall instead take advantage of certain subgraphs $G(10, 30)$ possesses. More specifically, we shall identify a number of subgraphs of $G(10, 30)$ which are members of the Petersen family of graphs, and so are known to be intrinsically linked. For each such subgraph we find, we produce a complete list of all possible linkable cycles. By the definition of intrinsic linking, in any given embedding of the subgraph, and so $G(10, 30)$ itself, at least one pair in this list will be linked.

With a list of Petersen subgraphs and corresponding lists of linkable pairs, we can then develop a body of casework based upon assuming one pair in one subgraph links simultaneously with another pair in another subgraph. With these two pairs of linkable cycles, we can manually show that (through some sequence of edge contractions) a doubly-linked D_4 graph can be produced from the two pairs. By Foisy's Lemma, we find the embedding described by this assumption will be knotted. If we can show every linkable pair in one subgraph generates a doubly-linked D_4 graph with other linkable pairs of another subgraph or subgraphs, then we will effectively cover every embedding in a manageable number of cases.

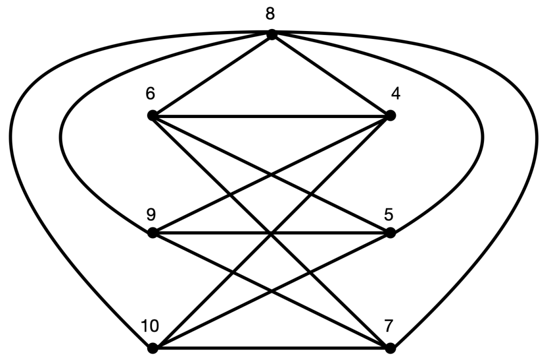

Figure 12. Subgraph A: $K_{3,3,1}$ in $G(10,30)$. $K_{3,3,1}$ is an intrinsically linked graph, as it is a member of the Petersen family.

We begin by identifying as many Petersen subgraphs of $G(10,30)$ as possible. This can be done manually by recognizing the proper vertex-edge arrangements in the edge-list of $G(10,30)$. In this way, we have identified a subgraph of $G(10,30)$ which is a $K_{3,3,1}$ Petersen graph (shown in Figure 12). The graph $K_{3,3,1}$ is obtained through two Y transformations and a Y transformation on K_6, and in every embedding one of the following cycles must be linked:

$$(4,6,8) \cup (5,10,7,9), \quad (5,6,8) \cup (4,9,7,10)$$

$$(4,9,8) \cup (6,5,10,7), \quad (5,9,8) \cup (6,4,10,7)$$

$$(4,10,8) \cup (6,5,9,7), \quad (5,10,8) \cup (4,6,7,9)$$

$$(7,6,8) \cup (4,9,5,10), \quad (7,9,8) \cup (4,6,5,10)$$

$$(7,10,8) \cup (4,6,5,9).$$

Taking this graph as the basis for our proof, we would identify a second Petersen subgraph along with all of its linkable pairs, and for each pair given above demonstrate that every pair in the second Petersen graph will form a D_4 graph from that given pair. Given that $K_{3,3,1}$ has nine linkable pairs, we would expect that there would be nine cases–with at least one subcase for each pair of cycles out of the complementary Petersen subgraph. Fortunately, we can reduce this casework by utilizing the inherent symmetry of $G(10,30)$. Under the graph symmetry, we find that by switching 2 with 3 and 4 with 5, we obtain isomorphic cycles and (as we shall see shortly) even subgraphs. Thus, the pair $(4,6,8) \cup (5,10,7,9)$ is an isomorphic copy of $(5,6,8) \cup (4,9,7,10)$ under the graph symmetry. Similarly, $(4,9,8) \cup (6,5,10,7)$ is mirrored by $(5,9,8) \cup (6,4,10,7)$ and $(4,10,8) \cup (6,5,9,7)$ is mirrored by $(5,10,8) \cup (4,6,7,9)$. Thus, any result found regarding one of these pairs can be readily duplicated for the isomorphic copy, and we can shorten our number of cases from nine to six.

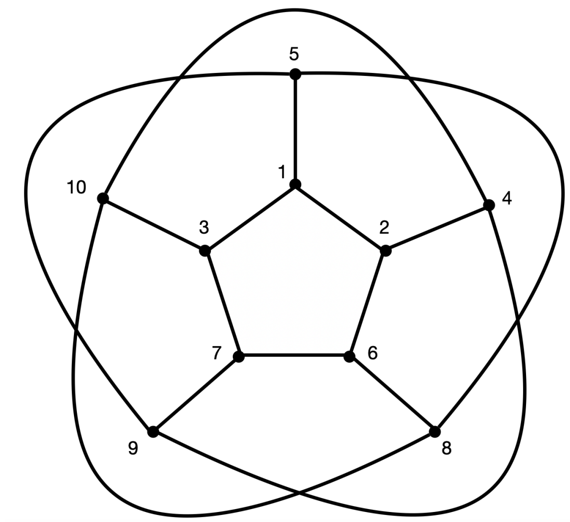

Figure 13. Subgraph $G(10,15)$ 5 in $G(10,30)$.

The $G(10,15)$ subgraph is produced from K_6 through four consecutive Y transformations. In addition to the $K_{3,3,1}$ described above, twenty-four of these Petersen $G(10,15)$ graphs have been identified as subgraphs of $G(10,30)$. Of the twenty-four subgraphs, we have precisely twelve unique graphs and their isomorphic copies under the $\{2,3\}$ and $\{4,5\}$ equivalence. The list of twelve is given in Table 1 at the end of this chapter. To obtain the other twelve subgraphs, simply switch 2 with 3 and 4 with 5 in each edge list. For example, we have identified the $G(10,15)$

subgraph (#5 in Table 1, and so labeled $G(10,15)\quad 5$) shown in Figure 13, which possesses the following linkable pairs of cycles:

$$(5,9,7,6,8)\cup(1,2,4,10,3),\quad (4,9,5,8,10)\cup(1,2,6,7,3)$$

$$(3,10,8,6,7)\cup(1,5,9,4,2),\quad (3,10,4,9,7)\cup(1,5,8,6,2)$$

$$(2,4,9,7,6)\cup(1,5,8,10,3),\quad (2,4,10,8,6)\cup(1,5,9,7,3).$$

Its copy under graph symmetry, $G(10,15)\quad 5^*$, is shown in Figure 14. The list of linkable pairs of cycles in $G(10,15)\quad 5^*$ can be readily found by simply switching 2 with 3 and 4 with 5 in the cycles of $G(10,15)\quad 5$. Thus, we find that the complete list of linkable cycles associated with $G(10,15)\quad 5^*$ would be:

$$(4,9,7,6,8)\cup(1,3,5,10,2),\quad (5,9,4,8,10)\cup(1,3,6,7,2),$$

$$(2,10,8,6,7)\cup(1,4,9,5,3),\quad (2,10,5,9,7)\cup(1,4,8,6,3)$$

$$(3,5,9,7,6)\cup(1,4,8,10,2),\quad (3,5,10,8,6)\cup(1,4,9,7,2).$$

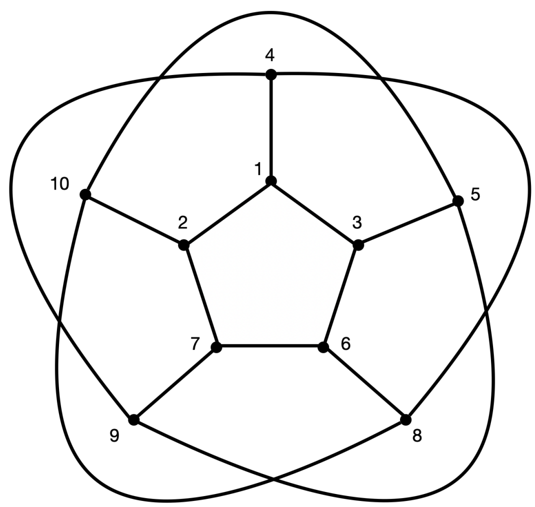

Figure 14. Subgraph $G(10,15)\quad 5^*$ in $G(10,30)$. This graph is obtained by switching 2 with 3 and 4 with 5 in the original subgraph.

Either graph, with six pairs of linkable cycles, will correspond to six cases if used in the proof of $G(10,30)$'s intrinsic knotting. Here we may point out where the difficulty in the proof lies. Let us take the $K_{3,3,1}$ subgraph given in Figure 12, and suppose that for each of its nine pairs of cycles there is a $G(10, 15)$ graph in Table 1 for which every cycle in that $G(10, 15)$ pairs perfectly with a single cycle out of $K_{3,3,1}$. Even taking into account graph symmetry, we would find a total number of thirty-six cases. For each of the six unique cycles of $K_{3,3,1}$, we would test six cycles out of one $G(10, 15)$.

We have identified one more Petersen subgraph of $G(10, 30)$ which might prove useful in the proof of $G(10, 30)$'s intrinsic knotting. The graph shown in Figure 15 is a Petersen graph of seven vertices and fifteen edges, produced from K_6 through a single Y transformation. The only possible linkable pairs are made of a 3-cycle linked with a 4-cycle (with the two cycles disjoint). We find that the complete list of all possible linkable pairs is

$$(9, 10, 7) \cup (4, 8, 5, 6), \quad (9, 5, 8) \cup (4, 6, 7, 10),$$
$$(8, 7, 9) \cup (5, 10, 4, 6), \quad (10, 9, 5) \cup (4, 8, 7, 6),$$
$$(8, 5, 10) \cup (4, 9, 7, 6), \quad (8, 7, 10) \cup (4, 9, 5, 6),$$
$$(4, 9, 8) \cup (6, 5, 10, 7), \quad (4, 8, 10) \cup (6, 5, 9, 7),$$
$$(4, 9, 10) \cup (6, 5, 8, 7).$$

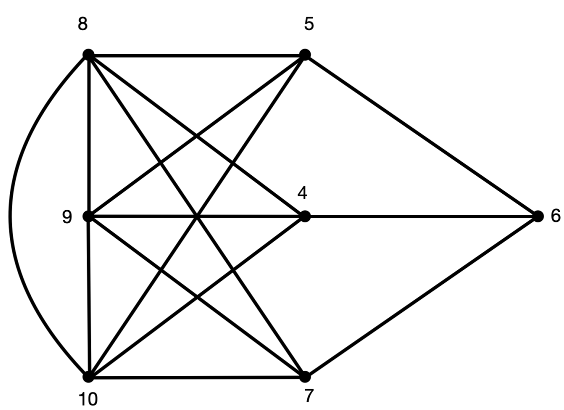

Figure 15. A $G(7, 15)$ subgraph of $G(10, 30)$.

Thus, it would correspond to nine cases in the proof.

With these subgraphs in mind, let us now consider a few examples of how the casework in the proof would go. Let us take $K_{3,3,1}$ as the central graph in the proof; label it subgraph **A**. The goal would be to find for each of its six unique cases a subgraph, the linkable pairs of which would form a doubly-linked D_4 graph. Let us take the first pair given, $(4,6,8) \cup (5,10,7,9)$, and label it **A1**. Let us test $G(10,15) \quad 5$ against this first case, and so consider six subcases for each pair in $G(10,15) \quad 5$–for ease, let us label $G(10,15) \quad 5$ as subgraph B, and its six cycles **B1-B6**.

Case A1. Assume that $(4,6,8) \cup (5,10,7,9)$ is a pair of linked cycles, and consider the six linkable pairs of cycles in subgraph **B1-B6**. We shall consider six subcases–each a pairing of **A1** with a pair of cycles from **B**.

Subcase A1-B1. Suppose that $(5,9,7,6,8) \cup (1,2,4,10,3)$ is a pair of linked cycles in subgraph B, and so in $G(10,30)$. Consider Figure 16

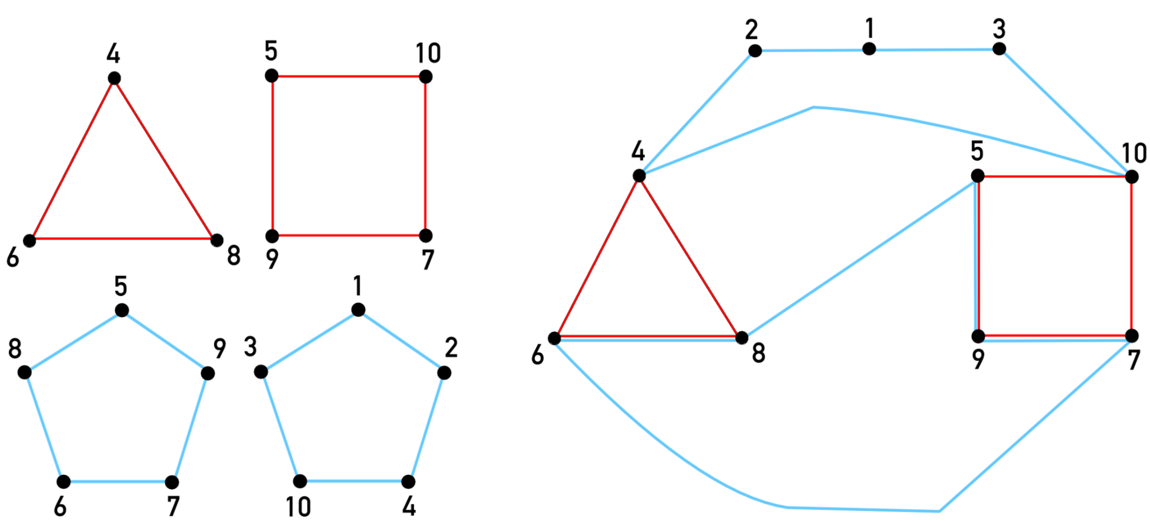

Figure 16. **A1** and **B1**.

If we contract the edges $\{6,8\}, \{5,9\}, \{9,7\}, \{4,2\}, \{2,1\}$, and $\{1,3\}$ in such a way as to leave double edges, we will produce a doubly-linked D_4 graph (two pairs of linked cycles on opposing sides of a D_4). Thus, by Foisy's Lemma, we find that any embedding with **A1** and **B1** linked is knotted.

Subcase A1-B2. Suppose that $(4, 9, 5, 8, 10) \cup (1, 2, 6, 7, 3)$ is a pair of linked cycles in B. Note that $\{4, 8\}$ exists in $G(10, 30)$. Thus, by Lemma 2, we find that if $(1, 2, 6, 7, 3)$ links with $(4, 9, 5, 8, 10)$, it must link with either $(4, 8, 10)$ or $(4, 8, 5, 9)$ (possibly both). Consider Figure 17. We find that we obtain a doubly-linked D_4 graph in either case, and conclude that if **A1** and **B2** are linked simultaneously, then by Foisy's Lemma that embedding of $G(10, 30)$ is knotted.

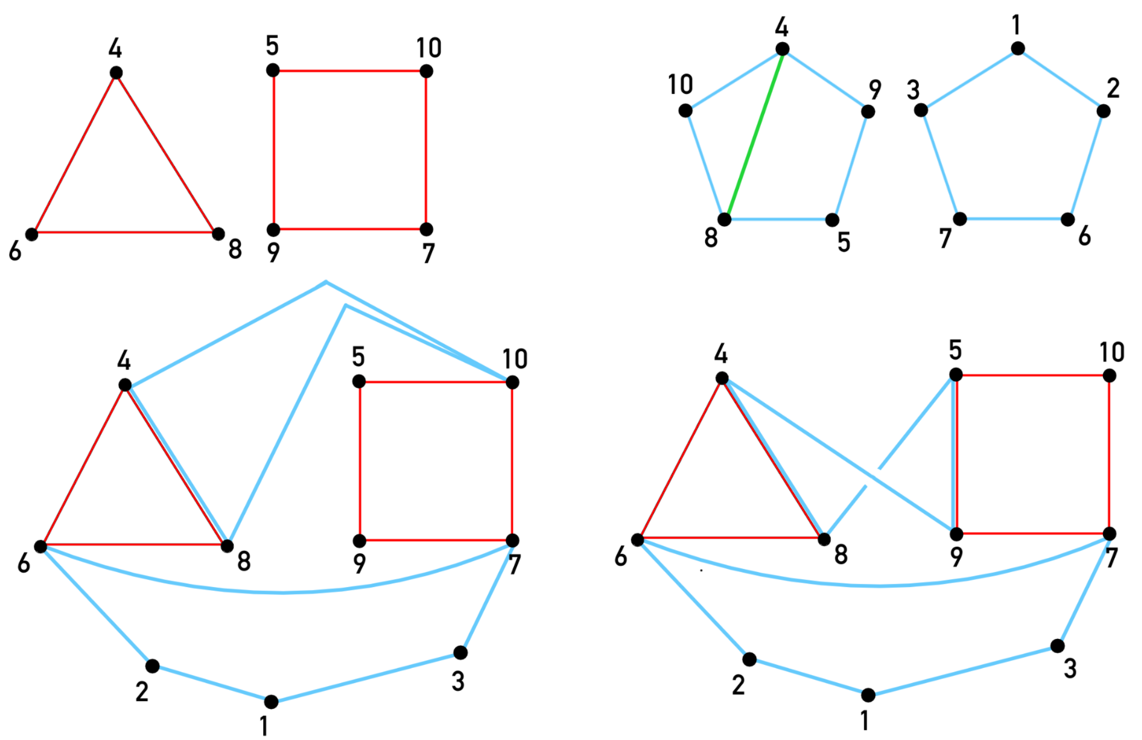

Figure 17. **A1** and **B2**. The green edge corresponds to $\{4, 8\}$; splitting the cycle $(4, 9, 5, 8, 10)$ into $(4, 8, 10)$ and $(4, 8, 9, 5)$ along this diagonal produces two ways of obtaining the doubly-linked D_4. To the left, we have the case for $(4, 8, 10)$, and to the right we have $(4, 8, 9, 5)$.

Subcase A1-B3. Suppose that $(3, 10, 8, 6, 7) \cup (1, 5, 9, 4, 2)$ is the linked pair of cycles in B. Consider that the edge $\{3, 6\}$ is present in $G(10, 30)$. By

Lemma 2, since $(1, 5, 9, 4, 2)$ is linked with $(3, 10, 8, 6, 7)$, we find that it must link with either $(3, 6, 7)$ or $(3, 6, 8, 10)$. Consider Figure 18. We find that in either case we have the desired D_4 graph. Thus, if **A1** and **B3** are the two linked pairs of cycles, $G(10, 30)$ is knotted.

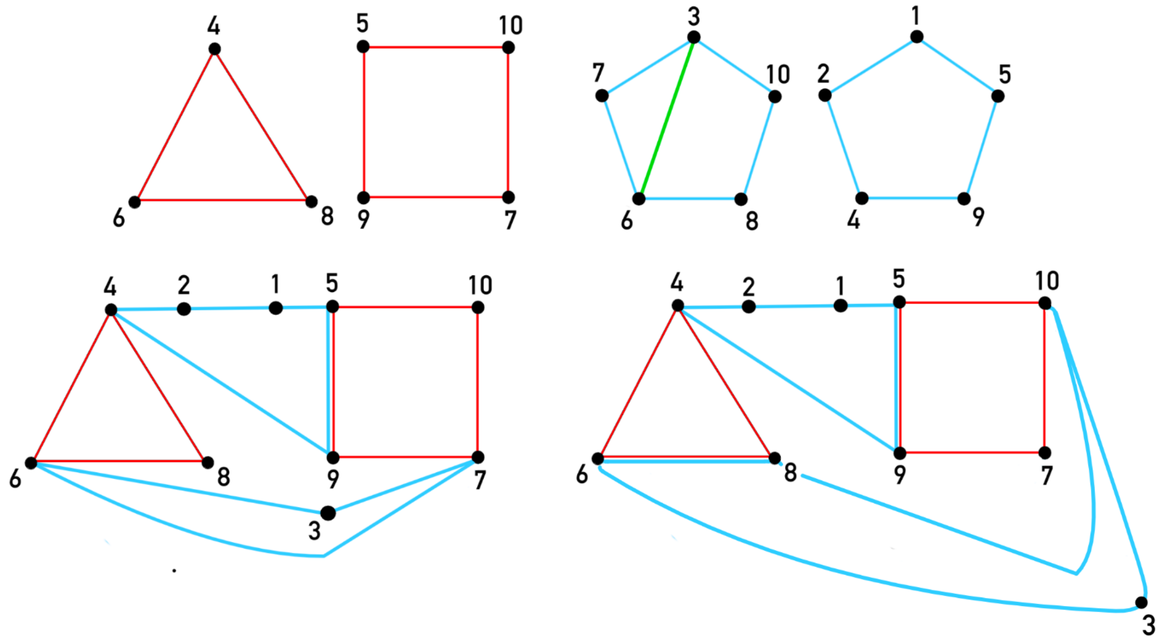

Figure 18. **A1** and **B3**. The green edge corresponds to $\{3, 6\}$; splitting the cycle $(3, 10, 8, 6, 7)$ into $(3, 6, 7)$ and $(3, 6, 8, 10)$ along this diagonal produces two ways of obtaining the doubly-linked D_4. To the left, we have the case for $(3, 6, 7)$, and to the right we have $(3, 6, 8, 10)$.

Subcase A1-B4. This case requires some further attention.

Subcase A1-B5. Suppose that $(2, 4, 9, 7, 6) \cup (1, 5, 8, 10, 3)$ is the linked pair of cycles in **B**. We find that the edges $\{2, 7\}$ and $\{5, 10\}$ exist in $G(10, 30)$. We apply Lemma 2 to both cycles, and so must consider four distinct origins for the D_4 graph. Consider Figure 19.

Subcase A1-B6. Suppose that $(2, 4, 10, 8, 6) \cup (1, 5, 9, 7, 3)$ is the linked pair of cycles in **B**. We recognize that the edge $\{2, 10\}$ exists in $G(10, 30)$, and apply Lemma 2 across this diagonal. Consider Figure 20.

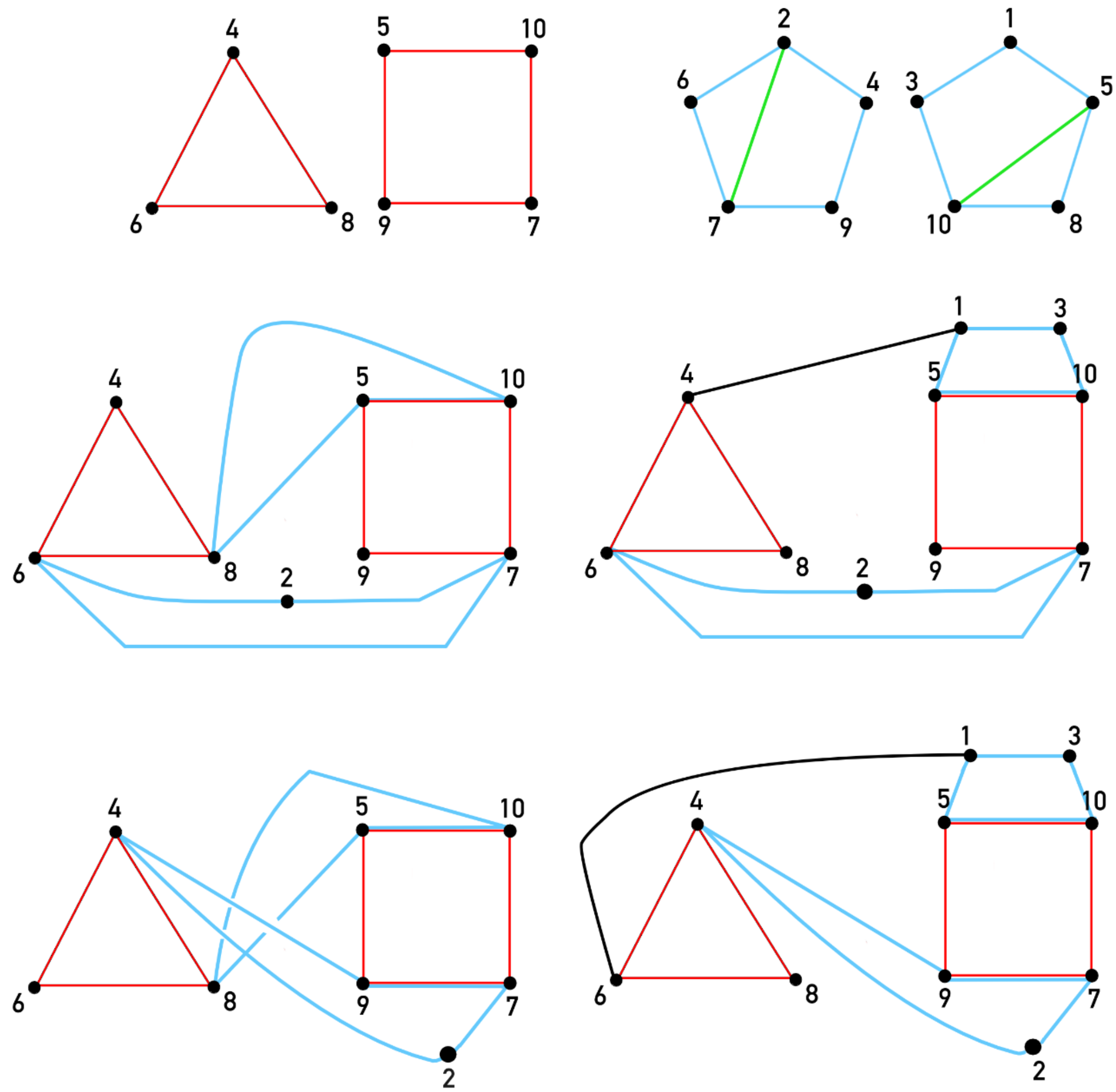

Figure 19. **A1** and **B5**. As with the prior figures, the green edges mark diagonals over which we apply Lemma 2. We are cutting apart $(2,4,9,7,6)$ into $(2,6,7)$ and $(2,4,9,7)$, and $(1,5,8,10,3)$ into $(1,5,10,3)$ and $(5,8,10)$. The top left considers $(5,8,10)$ and $(6,2,7)$, the top right considers $(1,5,10,3)$ and $(6,2,7)$, the bottom left considers $(5,8,10)$ and $(4,2,7,9)$, and the bottom right considers $(1,5,10,3)$ and $(6,2,7,9)$. In the two right cases, we must follow the work up with an edge contraction on $\{1,4\}$ and $\{1,6\}$. In all cases the result is a doubly-linked D_4 graph.

Preferably, every case would yield the desired doubly-linked D_4 graph. We would have shown that every embedding of $G(10,30)$ where $(4,6,8) \cup (5,10,7,9)$ forms a nontrivial link, there is another nontrivial link found somewhere in

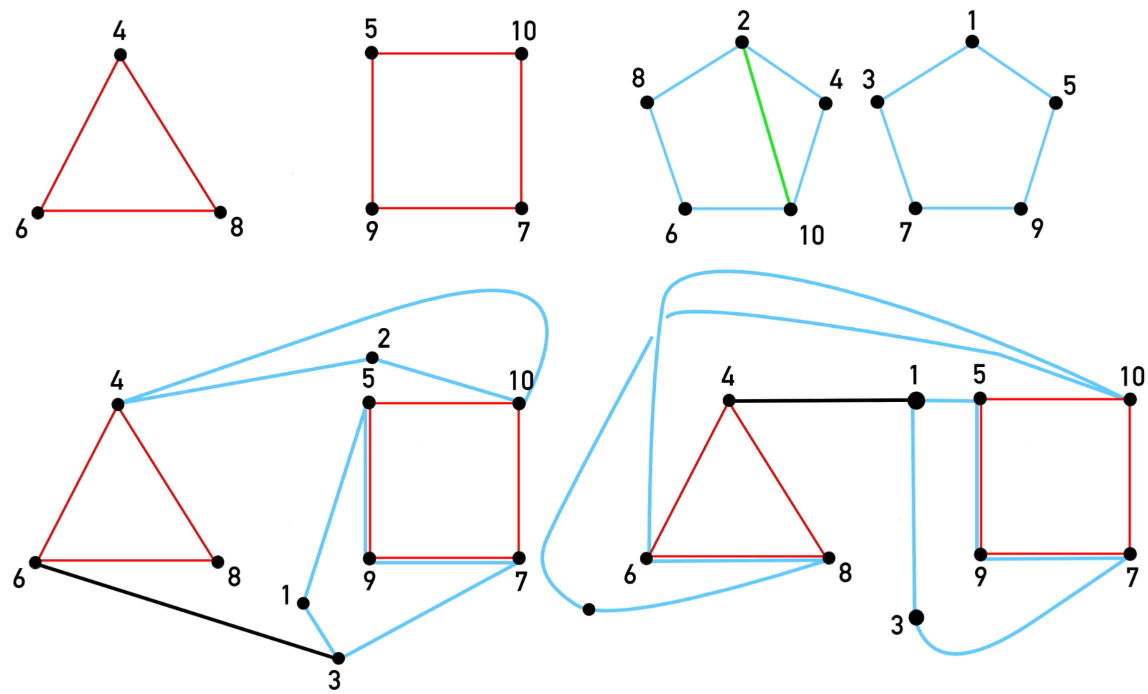

Figure 20. **A1** and **B6**. We break up $(2, 4, 10, 6, 8)$ into the cycles $(2, 4, 10)$ (left) and $(2, 10, 6, 8)$ (right). Both yield the D_4 after contracting an ambient edge ($\{3, 6\}$ and $\{1, 4\}$ respectively).

$G(10, 15)$ 5 that will yield the D_4 graph and produce a knot. Under the graph symmetry, we would then conclude that the same is true if $(5, 6, 8) \cup (4, 10, 9, 7)$ is linked, with the complementary pair coming from $G(10, 15)$ 5^*. However, we find that one case was left unexplored: **A1** and **B4**. This corresponds to assuming that we have an embedding where $(4, 6, 8) \cup (5, 10, 7, 9)$ and $(3, 10, 4, 9, 7) \cup (1, 5, 8, 6, 2)$ are both linked pairs. Under this assumption, however, we find that we have a problem; observe that under Lemma 2 we can break the cycle $(5, 10, 7, 9)$ along the diagonal $\{9, 10\}$ into two 3-cycles $(5, 10, 7)$ and $(7, 9, 10)$. If $(4, 6, 8)$ is linked with $(5, 10, 7, 9)$ as we assume, then the linking must occur through either $(5, 10, 7)$ or $(7, 9, 10)$. It is possible to show that, should the first occur, we will produce a doubly-linked D_4 graph. However, in the case where $(4, 6, 8)$ links with $(7, 9, 10)$, we run into a serious issue. The 3-cycle $(7, 9, 10)$ is present in the cycle $(3, 10, 4, 9, 7)$.

Under Lemma 2, we find that $(3, 10, 4, 9, 7)$ can also be broken along two diagonals: $\{7, 10\}$ and $\{9, 10\}$. The result of this is three different 3-cycles: $(7, 3, 10)$, $(7, 9, 10)$, and $(4, 9, 10)$. Putting this information together, we find that there is a case where we could have $(4, 6, 8) \cup (7, 9, 10)$ a linked pair and $(7, 9, 10) \cup (1, 5, 8, 6, 2)$ a linked pair. There are other cases besides this, and many of those will yield a D_4 graph. However, for the proof we must show every possible arrangement of linking yields this graph. This is impossible in the case of $(4, 6, 8) \cup (7, 9, 10)$ and $(7, 9, 10) \cup (1, 5, 8, 6, 2)$. The doubly-linked D_4, as given in Figure 10, requires that each pair of linked cycles not share vertices. They lie on opposite sides of the graph. Since $(7, 9, 10)$ is linked in what would be both a red and blue cycle, we find that separating the two pairs into the D_4 graph is impossible. This pairing on its own must then be disregarded.

The case of **A1-B4** illustrates the inherent difficulty in a proof of a graph's intrinsic knotting. For a case to work, every one of the subcases must result in a doubly-linked D_4 minor, or else an entirely new set of casework must develop to accommodate the problematic pair of linkable cycles.

$G(10, 30)$ has been shown using Mathematica to be intrinsically knotted. However, a formal proof of this would follow much in the same way as the examples presented above. The subcase **A1-B4** indicates that $G(10, 15)$ 5 may not be the best choice to pair with the $K_{3,3,1}$. However, with this discussion a path has been laid for future attempts into showing the graph's intrinsic knotting. The subgraphs presented here are given explicitly to aid in this endeavor. On the next page, Table 1 presents the twelve unique $G(10, 15)$ subgraphs, which offer the most promising start to the proof. This is largely due to each graph having only six pairs of linkable cycles, and so representing six individual cases. The goal of the proof should be to minimize the casework as much as possible.

Table 1. A list of unique $G(10,15)$ subgraphs of $G(10,30)$.

Label		**Edge List**
$G(10,15)$	1	$\{1,2\},\{1,4\},\{1,6\},\{2,3\},\{2,7\},\{3,5\},\{3,10\},\{4,9\},$ $\{4,10\},\{5,6\},\{5,9\},\{6,8\},\{7,8\},\{7,9\},\{8,10\}$
$G(10,15)$	2	$\{1,3\},\{1,4\},\{1,5\},\{2,4\},\{2,6\},\{2,10\},\{3,6\},\{3,7\},$ $\{4,9\},\{5,8\},\{5,10\},\{6,8\},\{7,9\},\{7,10\},\{8,9\}$
$G(10,15)$	3	$\{1,2\},\{1,4\},\{1,5\},\{2,7\},\{2,10\},\{3,5\},\{3,6\},\{3,10\},$ $\{4,6\},\{4,9\},\{5,8\},\{6,7\},\{7,8\},\{8,9\},\{9,10\}$
$G(10,15)$	4	$\{1,2\},\{1,3\},\{1,5\},\{2,4\},\{2,7\},\{3,6\},\{3,10\},\{4,6\},$ $\{4,9\},\{5,8\},\{5,9\},\{6,8\},\{7,8\},\{7,10\},\{9,10\}$
$G(10,15)$	5	$\{1,2\},\{1,3\},\{1,5\},\{2,4\},\{2,6\},\{3,7\},\{3,10\},\{4,9\}$ $\{4,10\},\{5,8\},\{5,9\},\{6,7\},\{6,8\},\{7,9\},\{8,10\}$
$G(10,15)$	6	$\{1,2\},\{1,3\},\{1,4\},\{2,6\},\{2,7\},\{3,5\},\{3,10\},\{4,8\},$ $\{4,9\},\{5,6\},\{5,9\},\{6,8\},\{7,9\},\{7,10\},\{8,10\}$
$G(10,15)$	7	$\{1,3\},\{1,4\},\{1,6\},\{2,6\},\{2,7\},\{2,10\},\{3,5\},\{3,7\}$ $\{4,9\},\{4,10\},\{5,8\},\{5,10\},\{6,8\},\{7,9\},\{8,9\}$
$G(10,15)$	8	$\{1,2\},\{1,4\},\{1,5\},\{2,7\},\{2,10\},\{3,5\},\{3,6\},\{3,10\}$ $\{4,6\},\{4,8\},\{5,9\},\{6,7\},\{7,9\},\{8,9\},\{9,10\}$
$G(10,15)$	9	$\{1,2\},\{1,5\},\{1,6\},\{2,3\},\{2,4\},\{3,7\},\{3,10\},\{4,8\}$ $\{4,9\},\{5,9\},\{5,10\},\{6,7\},\{6,8\},\{7,9\},\{8,10\}$
$G(10,15)$	10	$\{1,2\},\{1,4\},\{1,5\},\{2,3\},\{2,7\},\{3,6\},\{3,10\},\{4,8\}$ $\{4,10\},\{5,6\},\{5,9\},\{6,8\},\{7,8\},\{7,9\},\{9,10\}$
$G(10,15)$	11	$\{1,3\},\{1,4\},\{1,5\},\{2,3\},\{2,6\},\{2,10\},\{3,7\},\{4,8\}$ $\{4,10\},\{5,6\},\{5,9\},\{6,8\},\{7,8\},\{7,9\},\{9,10\}$
$G(10,15)$	12	$\{1,2\},\{1,3\},\{1,4\},\{2,6\},\{2,10\},\{3,5\},\{3,7\},\{4,8\}$ $\{4,9\},\{5,8\},\{5,10\},\{6,7\},\{6,8\},\{7,9\},\{9,10\}$

CHAPTER IV

PROOF THAT $G(10,30)$ IS MINOR MINIMAL WITH RESPECT TO INTRINSIC KNOTTING

We now come to the second component of the research–demonstrating that no minor of $G(10,30)$ is intrinsically knotted. Recall that a minor of a graph is a graph formed by the deletion or contraction of an edge or edges of the original graph. It should be noted that a minor of a minor is a minor of the original graph; thus, we may concern ourselves only with the "first generation" of minors–those generated by deleting or contracting each individual edge in $G(10,30)$. By showing that these first-generation minors are not intrinsically knotted, we will show that all subsequent minors in general are not intrinsically knotted.

The first step in this process is to test if each deletion and contraction minor is 2-apex. By Theorem 1, this is sufficient to show that the minor is knotlessly embeddable. In Table 2, we have listed all thirty edges of $G(10,30)$, as well as the results of testing each corresponding deletion or contraction minor for being 2-apex. The values in the "Deletion" and "Contraction" columns correspond to the vertices which, when deleted from the appropriate minor, result in a planar graph. Thus, we find that every single-edge contraction and most of the single-edge deletion minors are 2-apex, and so knotlessly embeddable [2]. Let us consider an example.

Take the edge $\{1,2\}$. By deleting this edge, we obtain the graph $G(10,30)$ $\quad$ $\{1,2\}$, a minor of $G(10,30)$, as depicted on the left in Figure 21. If we then delete the vertices 4 and 7, we will also delete the twelve edges that are

Table 2. Analysis of the single-edge deletion/contraction minors of $G(10,30)$.

Edges	**Deletion**	**Contraction**	Y **Triangle**	**Vertices**	**Figure**
$\{1,2\}$	4,7	5,7	-	-	Figure 21
$\{1,3\}$	5,7	4,7	-	-	
$\{1,4\}$	5,7	7,10	-	-	
$\{1,5\}$	4,7	7,10	-	-	
$\{1,6\}$	7,10	4,7	-	-	
$\{2,3\}$	7,10	4,5	-	-	
$\{2,4\}$	7,10	5,7	-	-	
$\{2,6\}$	5,7	4,5	-	-	
$\{2,7\}$	4,5	4,5	-	-	
$\{2,10\}$	4,5	4,5	-	-	
$\{3,5\}$	7,10	4,7	-	-	
$\{3,6\}$	4,7	4,5	-	-	
$\{3,7\}$	4,5	4,5	-	-	
$\{3,10\}$	4,5	4,5	-	-	
$\{4,6\}$	*	5,7	(5,6,8)	$v,7$	Figure 22, 23
$\{4,8\}$	6,7	1,7	-	-	
$\{4,9\}$	1,7	2,4	-	-	
$\{4,10\}$	*	1,7	(2,3,10)	$v,7$	
$\{5,6\}$	*	4,7	(4,6,8)	$v,7$	
$\{5,8\}$	6,7	1,7	-	-	
$\{5,9\}$	1,7	2,5	-	-	
$\{5,10\}$	*	1,7	(2,3,10)	$v,7$	
$\{6,7\}$	*	4,5	(2,3,7)	$v,10$	
$\{6,8\}$	6,8	6,7	-	-	
$\{7,8\}$	7,8	1, 7	-	-	
$\{7,9\}$	2,4	1,7	-	-	
$\{7,10\}$	*	1,7	*	*	Figure 24
$\{8,9\}$	*	1,7	(2,3,10)	$v,7$	
$\{8,10\}$	1,7	6,7	-	-	
$\{9,10\}$	6,7	1,7	-	-	

incident to 4 and 7:

$\{1,4\}, \{2,4\}, \{4,6\}, \{4,8\}, \{4,9\}, \{4,10\}, \{2,7\}, \{3,7\}, \{6,7\}, \{7,8\}, \{7,9\}$, and $\{7,10\}$. With these twelve edges removed, the resulting minor of $G(10,30)$ $\{1,2\}$ (and by extension $G(10,30)$) can be depicted as a planar graph. One embedding of this minor is given in Figure 21 on the right. Note the absence of edge crossings.

Thus, by Theorem 1 on page 18, we may conclude that the minor $G(10,30)$ $\{1,2\}$ is knotlessly embeddable.

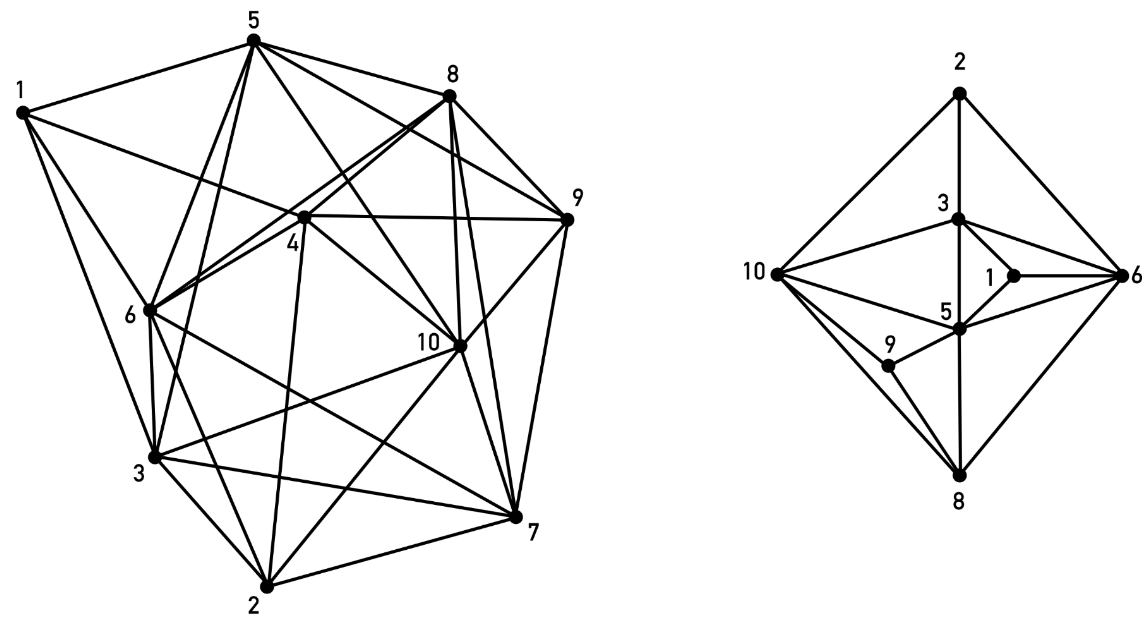

Figure 21. The case of $G(10,30)$ $\{1,2\}$. The minor produced by deleting the edge $\{1,2\}$ is shown on the left. By deleting the vertices 4 and 7, and so all relevant edges, we obtain the planar graph on the right.

However, we find that seven of the thirty deletion minors are not 2-apex: $\{4,6\}$, $\{4,10\}$, $\{5,6\}$, $\{5,10\}$, $\{6,7\}$, $\{7,10\}$, $\{8,9\}$. For these cases, a closer analysis is required. By Theorem 2 on page 18, we know that applying a Y transformation on a graph will preserve intrinsic linking and knotting. For the seven cases, we deleted the necessary edge, and identified all triangles in the resulting graph. We then manually applied a Y transformation on each triangle, until a cousin graph was identified as being 2-apex. Thus, by Theorem 2, the original minor must also be knotlessly embeddable. The results of this analysis are described in the " Y Triangles" and "Vertices" columns of Table 2. The 3-cycle stated in the column is the triangle in the corresponding minor which underwent a single Y transformation to produce a cousin graph, and the vertices given are those that, when deleted give a planar minor of that cousin. Let us consider an example of this process.

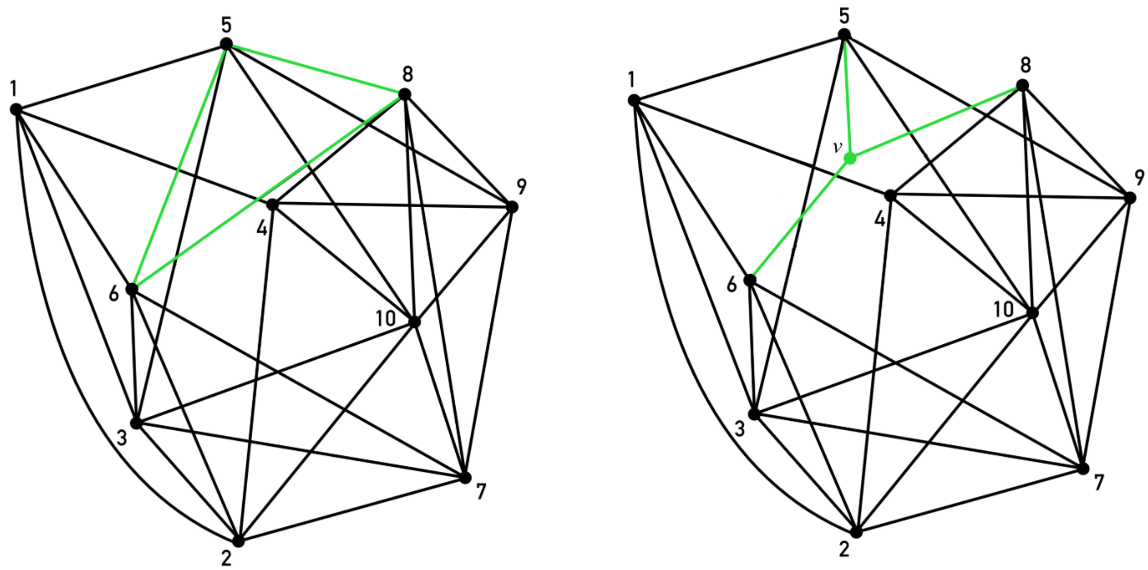

Figure 22. The case of $G(10,30)$ $\{4,6\}$. We have identified the triangle $(5,6,8)$, given in green. Applying a Y transformation on $(5,6,8)$ gives the cousin on the right, which we call H.

Consider the edge $\{4,6\}$. We have found that the minor $G(10,30)$ $\{4,6\}$ is not 2-apex. By manually applying a single Y transformation to each triangle in $G(10,30)$ $\{4,6\}$, we have found a cousin, H, of $G(10,30)$ $\{4,6\}$ which is 2-apex. This is the graph obtained by a Y transformation on the triangle $(5,6,8)$. We delete the edges $\{5,6\}$, $\{5,8\}$ and $\{6,8\}$ in $G(10,30)$ $\{4,6\}$, and replace them with the edges $\{5,v\}$, $\{6,v\}$, and $\{v,8\}$–incident to a new vertex v. Both $G(10,30)$ $\{4,6\}$ and H are show in Figure 22. The graph H is 2-apex, producing a planar graph H' after the deletion of v and 7, shown in Figure 23. Therefore H cannot be intrinsically knotted by Theorem 1, and by extension $G(10,30)$ is not intrinsically knotted by Theorem 2.

Even after applying this process, there still remains a single case which is unresolved–the minor associated with the deletion of the edge $\{7,10\}$. A single

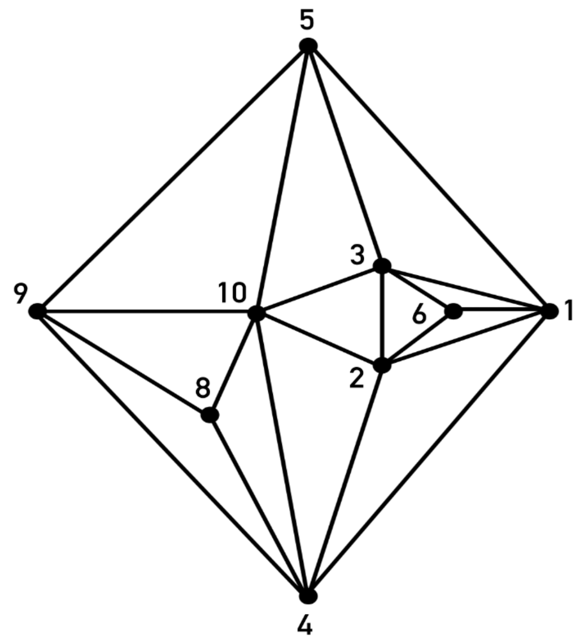

Figure 23. A planar embedding of H'.

Y transformation was applied to every triangle in the deletion minor $G(10, 30) \quad \{7, 10\}$; none of this minor's children were found to be 2-apex. As such, we simply identified a knotless embedding of the minor–a counterexample to the claim that $G(10, 30) \quad \{7, 10\}$ is intrinsically knotted. This knotless embedding was found by deleting the vertices 4 and 7 from the minor, as well as the edge $\{1, 2\}$ to yield a planar graph. We drew underneath this planar graph (meaning behind the plane of the graph) the edge $\{1, 2\}$, which would cross underneath any of the edges in the planar subgraph. After this, the vertex 7 and all relevant edges (all edges incident to 7 in the original graph with the exception of $\{7, 10\}$) were drawn further behind the subgraph, with the edges incident to 7 crossing underneath the planar graphs edges. Finally, we applied the same process to 4, drawing all of the edges incident to 4 underneath the preexisting edges. The result of this process is given in Figure 24.

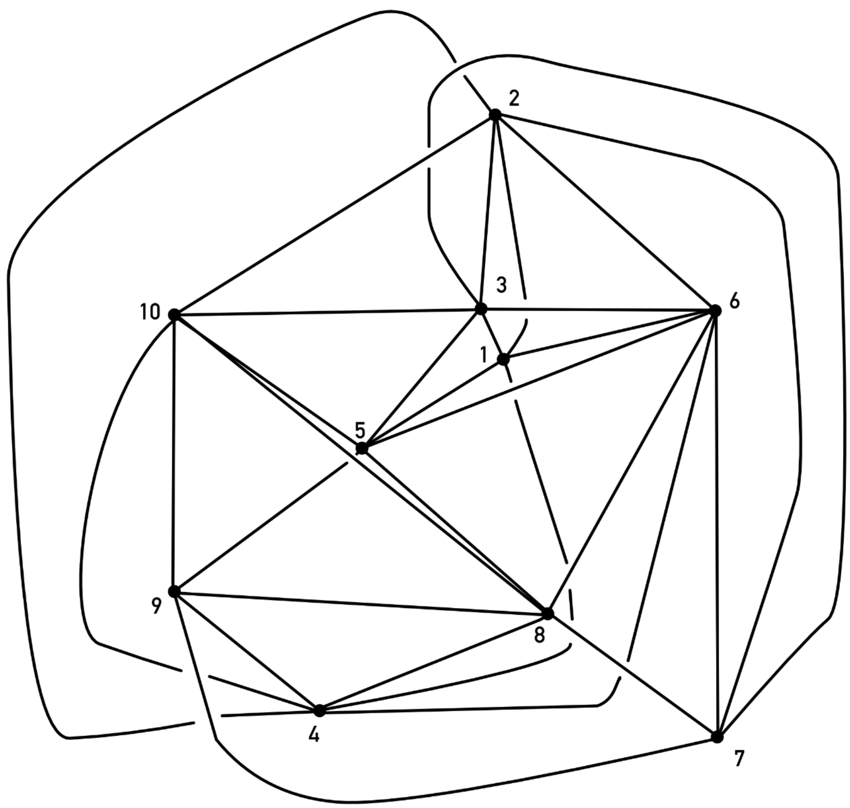

Figure 24. A knotless embedding of $G(10,30)$ $\{7,10\}$.

Thus, every minor of $G(10,30)$ has been found to be knotlessly embeddable, and $G(10,30)$ is minor-minimal with respect to the property of intrinsic knotting.

CHAPTER V

CONCLUSION

From the work demonstrated in Chapters III and IV, we find that a firm foundation has been laid for a formal proof of the minor-minimal intrinsic knotting of $G(10,30)$. While $G(10,30)$ has not been shown to be intrinsically knotted in this thesis, a large number of tools are offered for the completion of that specific result; this includes some new results in addition to the pre-existing theorems presented in Chapter II. In particular, all twenty-four G(10,15) subgraphs of $G(10,30)$ have been identified, which alone presents us with a large bank of subgraphs from which we can develop the casework for the proof. Several examples have been drawn up to demonstrate the general technique, and in the case of **A1-B4** an example is given of the kind of situations which must be avoided for the proof to hold. Furthermore, with the work done in Chapter IV we have effectively completed half of the proof by showing that $G(10,30)$ has no intrinsically knotted minors.

Considering the results presented in this thesis, the next logical step forward is to finish the proof outlined in Chapter III. We believe that we have sufficient tools to tackle this problem, and it is only a matter of time before the full proof can be provided.

REFERENCES

REFERENCES

[1] Adams, C.C., 1994. *The knot book*. American Mathematical Soc..

[2] Blain, P., Bowlin, G., Fleming, T., Foisy, J., Hendricks, J. and Lacombe, J., 2007. Some results on intrinsically knotted graphs. *Journal of Knot Theory and Its Ramifications, 16*(06), pp.749-760.

[3] Conway, J.H. and McA. Gordon, C., 1983. Knots and links in spatial graphs. *Journal of Graph Theory*, 7(4), pp.445-453.

[4] Foisy, J., 2002. Intrinsically knotted graphs. *Journal of Graph Theory, 39*(3), pp.178-187.

[5] Foisy, J., 2003. A newly recognized intrinsically knotted graph. *Journal of Graph Theory, 43*(3), pp.199-209.

[6] Foisy, J., 2007. More intrinsically knotted graphs. *Journal of Graph Theory, 54*(2), pp.115-124.

[7] Johnson, B., Kidwell, M.E. and Michael, T.S., 2010. Intrinsically knotted graphs have at least 21 edges. *Journal of Knot Theory and Its Ramifications, 19*(11), pp.1423-1429.

[8] Mattman, T.W., Naimi, R., Pavelescu, A., and Pavelescu E., 2021. Intrinsically knotted graphs with linklessly embeddable minors. *arXiv preprint arXiv:2111.08859*.

[9] Motwani, R., Raghunathan, A. and Saran, H., 1988, October. Constructive results from graph minors: Linkless embeddings. In *[Proceedings 1988] 29th Annual Symposium on Foundations of Computer Science* (pp. 398-409). IEEE Computer Society.

[10] Robertson, N. Seymour, P.D, and Thomas, R., 1993. Linkless embeddings of graphs in 3-space. *Bulletin of the American Mathematical Society, 28*(1), pp.84-89.

[11] Robertson, N. and Seymour, P.D, 1995. Sachs' linkless embedding conjecture. *Journal of Combinatorial Theory, Series B, 64*, pp.185-227.

[12] Robertson, N. and Seymour, P.D., 2004. Graph minors. XX. Wagner's conjecture. *Journal of Combinatorial Theory, Series B, 92*(2), pp.325-357.

[13] Sachs, H., 1984. On spatial representations of finite graphs. In *Collo. Math. Soc. János Bolyai* (Vol. 37, pp.649-662). North-Holland Amsterdam.

BIOGRAPHICAL SKETCH

BIOGRAPHICAL SKETCH

Johnathan Ridley Herron was born in Charlotte, North Carolina, but was raised primarily in the town of Montevallo, Alabama. After being homeschooled with his two brothers and sister, he attended the University of Montevallo, from which he graduated *summa cum laude* with a Bachelor of Science in Mathematics. Afterwards, he attended the University of South Alabama, graduating from there with a Master of Science in Mathematics.