1-1-2019

Uniform Stabilization of Navier-Stokes Equations in Lq-based Sobolev and Besov Spaces

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UNIFORM STABILIZATION OF NAVIER-STOKES EQUATIONS IN L^q-BASED SOBOLEV AND BESOV SPACES

by

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A Dissertation
Submitted in Partial Fulfillment of the
Requirements for the Degree of
Doctor of Philosophy

Major: Mathematics

University of Memphis
May 2019

Acknowledgments

Throughout this journey of my doctoral degree, researching and dissertation, I have received invaluable guidance and encouragements from my advisors, Dr. Irena Lasiecka (university distinguished professor & chair) and Dr. Roberto Triggiani (university distinguished professor). Their expertise in the area of control theory of partial differential equations has been irreplaceable in both formulating the research topic and the development of the methodology.

I would like to extend my words to thank my dissertation committee members, Dr. Hongqui Chen and Dr. Bentuo Zheng.

In addition, I would like to thank my parents for giving me the motivation and being sympathetic during the vicissitudes of my fate even if they are ten thousand miles away from me.

Abstract

We consider 2- or 3-dimensional Navier-Stokes equations defined on a bounded domain Ω subject to an external force, assumed to cause instability. We then seek to uniformly stabilize such N-S system, in the vicinity of an unstable equilibrium solution, in L^q-based Sobolev and Besov spaces, by finite dimensional feedback controls. This work is divided in to two parts. In Part I, the finite dimensional feedback controls are localized on an arbitrarily small open interior subdomain ω of Ω. Instead, in Part II seeks tangential boundary feedback stabilizing controls. It provides a solution to the following recognized open problem in the theory of uniform stabilization of d-dimensional Navier-Stokes equations in the vicinity of an unstable equilibrium solution, by means of tangential boundary localized feedback controls: can these stabilizing controls be asserted to be finite dimensional also in the physical dimension $d = 3$? To achieve the desired finite dimensionality result of the feedback tangential boundary controls, it was then necessary to abandon the Hilbert-Sobolev functional setting of past literature and replace it with an appropriate L^q-based/Besov setting with tight parameters related to the physical dimension d, where the compatibility conditions are not recognized. This result is also a new contribution to the area of maximal regularity with inhomogeneous boundary feedback.

Contents

1 Part I: Using finitely many tangential, boundary, localized, feedback controls also in dimension $d = 3$.

Abstract

We consider 2- or 3-dimensional Navier-Stokes equations defined on a bounded domain Ω, with no-slip boundary conditions and subject to an external force, assumed to cause instability. We then seek to uniformly stabilize such N-S system, in the vicinity of an unstable equilibrium solution, in L^q-based Sobolev and Besov spaces, by finite dimensional feedback controls. In this Part I, the feedback controls are localized on an arbitrarily small open interior subdomain ω of Ω. The present treatment much improves and simplifies at both conceptual and computational level, the solution of the present stabilization problem, given in the more restrictive Hilbert space setting in [B-T.1]. Moreover, such treatment, sets the foundation for the authors' subsequent Part II, which solves in the affirmative a presently open problem: whether uniform stabilization is possible by localized tangential boundary feedback controls, which-in addition-are finite dimensional, also for dim $\Omega = 3$.

2 Introduction

2.1 Controlled Dynamic Navier-Stokes Equations

Let, at first, Ω be an open connected bounded domain in $\mathbb{R}^d$, $d = 2, 3$ with sufficiently smooth boundary $\Gamma = \partial\Omega$. More specific requirements will be given below. Let ω be an arbitrarily small open smooth subset of the interior Ω, $\omega \subset \Omega$, of positive measure. Let m denote the characteristic function of ω: $m(\omega) \equiv 1$, $m(\Omega \backslash \omega) \equiv 0$. Consider the following controlled Navier-Stokes Equations with non-slip Dirichlet boundary conditions, where $Q = (0, \infty) \times \Omega$, $\Sigma = (0, \infty) \times \Gamma$:

$$y_t(t,x) - \nu \Delta y(t,x) + (y \cdot \nabla)y + \nabla \pi(t,x) = m(x)u(t,x) + f(x) \quad \text{in } Q \tag{2.1a}$$

$$\text{div } y = 0 \quad \text{in } Q \tag{2.1b}$$

$$y = 0 \quad \text{on } \Sigma \tag{2.1c}$$

$$y(0,x) = y_0(x) \quad \text{in } \Omega \tag{2.1d}$$

Notation: As already done in the literature, for the sake of simplicity, we shall adopt the same notation for function spaces of scalar functions and function spaces of vector valued functions. Thus, for instance, for the vector valued (d-valued) velocity field y or external force f , we shall simply write say $y, f \in L^q(\Omega)$ rather than $y, f \in (L^q(\Omega))^d$ or $y, f \in \mathbf{L}^q(\Omega)$. This choice is unlikely to generate confusion. By way of orientation, we state at the outset two main points. For the linearized w-problem (2.13) below, the corresponding well-posedness and global feedback uniform stabilization result, Theorem 3.2, holds in general for $1 < q < \infty$. Instead, the final, main well-posedness and feedback uniform, local stabilization result, Theorem 3.5, for the original nonlinear problem (3.27) or (3.28) will require $q > 3$, see (8.16), in the $d = 3$-case, hence $1 < p < \dfrac{6}{5}$, and $q > 2$, in the $d = 2$-case, hence $1 < p < \frac{4}{3}$; see (2.16). Let $u \in L^p(0, T; L^q(\Omega))$ be the control input and $y = (y_1, \dots, y_d)$ be the corresponding state (velocity) of the system. Let $\nu > 0$ be the viscosity coefficient. The function $v(t, x) = m(x)u(t, x)$ can be viewed as an interior controller with support in $Q_\omega = (0, \infty) \times \omega$. The initial condition y_0 and the body force $f \in L^q(\Omega)$ are given. The scalar function π is the unknown pressure.

2.2 Stationary Navier-Stokes equations

The following result represents our basic starting point.

Theorem 2.1. *Consider the following steady-state Navier-Stokes equations in Ω*

$$-\nu\Delta y_e + (y_e.\nabla)y_e + \nabla\pi_e = f \qquad \text{in } \Omega \tag{2.2a}$$

$$div\ y_e = 0 \qquad \text{in } \Omega \tag{2.2b}$$

$$y_e = 0 \qquad \text{on } \Gamma \tag{2.2c}$$

Let $1 < q < \infty$. For any $f \in L^q(\Omega)$ there exits a solution (not necessarily unique) $(y_e, \pi_e) \in (W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)) \times (W^{1,q}(\Omega)/\mathbb{R})$.

For the Hilbert case $q = 2$, see [C-F, Thm 7.3 p 59] . For the general case $1 < q < \infty$, see [A-R, Thm 5.iii p 58].

Remark 2.1. It is well-known [Lad], [Li], [Te] that the stationary solution is unique when "the data is small enough, or the viscosity is large enough" [Te, p 157; Chapt 2] that is, if the ratio $\|f\|/_{\nu^2}$ is smaller than some constant that depends only on Ω [FT, p 121]. When non-uniqueness occurs, the stationary solutions depend on a finite number of parameters [FT, Theorem 2.1, p 121] asymptotically in the time dependent case.

Remark 2.2. The case where $f(x)$ in (2.1a) is replaced by $\nabla g(x)$ is noted in the literature as arising in certain physical situations, where f is a conservative vector field. The analysis of this relevant case is postponed to Remark 2.4, at the end of Section 2.

2.3 Main goal of the present paper

For a given external force f, if the Reynolds number $\frac{1}{\nu}$ is sufficiently large, then the steady state solution y_e in (2.2) becomes unstable (in a quantitative sense to be made more precise in Section 3.2 below) and turbulence occurs.

The main goal of the present paper is then - at first qualitatively - to feedback stabilize the non-linear N-S model (2.1) subject to rough (non-smooth) initial condition y_0, in the vicinity of an (unstable) equilibrium solution y_e in (2.2). Thus this paper pertains to the general context of "turbulence suppression or attenuation" in fluids. The general topic of turbulence suppression (or attenuation) in fluids has been the object of many studies over the years, mostly in the engineering literature through experimental studies and via numerical simulation and under different geometrical and dynamical settings. The references cited in the present paper by necessity refer mostly to the mathematical literature, and most specifically on the localized interior control setting of problem (2.1). A more precise description thereof is as follows: *establish interior localized exponential stabilization of problem (2.1) near an unstable equilibrium solution by means of a finite dimensional localized, spectral-based feedback control, in the important case of initial conditions y_0 of low regularity*, as technically expressed by y_0 being in suitable L^q/Besov space with tight indices. In particular, local exponential stability for the velocity field y near an equilibrium solution y_e will be achieved in the topology of the Besov space

$$B^{2-2/p}_{q,p}(\Omega), \quad 1 < p < \frac{2q}{2q-1}; \ q > d, \ d = 2, 3. \tag{2.3}$$

In such setting, the compatibility conditions on the boundary of the initial conditions are not recognized. This feature is precisely our key objective within the stabilization problem. The fundamental reason is that such feature will play a critical role in the successor paper [L-P-T] in showing that: local tangential boundary feedback stabilization near an unstable equilibrium solution with finitely many controls is possible also in dimension $d = 3$, thus solving in the affirmative a recognized open problem in the stabilization area. This point will be more appropriately expanded in Section 2.6 below.

The successor [L-P-T] to the present work will extensively review the literature as it pertains to the boundary stabilization case, particularly with tangential control action. Accordingly, below we shall review the present study mostly in comparison with the prior solution of the localized interior stabilization in the Hilbert-based treatment of [B-T.1] and textbook versions thereof.

2.4 Qualitative Orientation

2.4.1 On the local, interior, feedback stabilization problem: Past Literature

We start with an unstable steady state solution y_e, given an external force f, and a sufficiently large Reynold number $\frac{1}{\nu}$, as described in Section 2.3. We then seek a finite-dimensional interior localized feedback control u, such that the corresponding N-S problem is well-posed in a suitable function space setting and its solution y in (2.1) is locally exponentially stable near the equilibrium solution y_e, in a suitably corresponding norm. This problem was originally posed and solved in the Hilbert space setting in [B-T.1, Theorem 2.2 p 1449] by means of a finite dimensional Riccati-based feedback control u, where exponential decay is obtained in the $\mathcal{D}(A^{1/4})$-topology. Here A is the positive self-adjoint Stokes operator in (2.17) with $q = 2$ in the space H with $L^2(\Omega)$-topology. See (2.6) below. A similar exponential decay result, in the same $\mathcal{D}(A^{1/4})$-topology, is given in [B-L-T.3, Thm 5.1, p 42], this time by means of a finite-dimensional, spectral-based feedback control u.

Regarding the solution given in these references, we point out (at present) two defining, linked characteristics of their finite dimensional treatment:

(i) The number of stabilizing (localized) controls for the (complex-valued) nonlinear dynamics (2.1) is $N = \sup\{N_i;\ i = 1, \dots, M\}$, that is, the max of the *algebraic* multiplicity of the M distinct unstable eigenvalues λ_i, see (3.2), of the projected Oseen operator $\mathcal{A}_N^u$ in (3.5).

(ii) in the fully general case, the algebraic (Kalman rank) conditions for controllability under which the finite dimensional feedback control is *explicitly* constructed involve the Grahm-Schmidt orthogonalization of the generalized eigenfunctions of the adjoint $(\mathcal{A}_N^u)^*$, making the test difficult to verify. Only in the case where the restriction $\mathcal{A}_N^u$ in (3.5) of the Oseen operator $\mathcal{A}$ in (2.10) is semisimple (algebraic and geometric multiplicity of the unstable eigenvalues coincide), are the controllability tests given in terms of eigenfunctions of $(\mathcal{A}_N^u)^*$.

2.4.2 Additional goals of the present paper as definite improvements over the literature

We list these main additional goals of the present work aimed at markedly improving both the results and the approach of the basic reference [B-T.1]. They are:

(i) With reference to part (i) in Section 2.4.1, our next goal is to obtain (in the general case of Theorem 5.1) that the number of stabilizing controls needed for the (complex valued version of the) dynamics (2.1) is $K = \sup\{\ell_i;\ i = 1, \dots, M\}$, where ℓ_i is the *geometric* multiplicity of the M distinct unstable eigenvalues λ_i in (3.2). This is a notable reduction in the number of needed controls over the max algebraic multiplicity N in (i) of Section 2.4.1.

(ii) Intimately linked to goal (i) is the next goal to obtain the controllability Kalman rank condition be expressed in terms of only the eigenfunctions - not the generalized eigenfunctions - of the adjoint $(\mathcal{A}_N^u)^*$

(iii) An important additional goal is to simplify and make more transparent the well-posedness and local stabilization arguments for the non-linear problem, in particular through a direct analysis of the nonlinear operator $\mathcal{N}_q$ in (2.11) called B in [B-T.1], not its approximation sequence B_ε as in [B-T.1, Section 4, p1480]. More precisely, unlike [B-T.1], the present paper carries out an analysis of the critical issues based on the maximal regularity property of the linearized feedback operator $\mathbb{A}_{_F} (= \mathbb{A}_{_{F,q}})$ in (7.1). This point also is further expanded in Section 2.6 below.

(iv) A final goal - in line with goal (iii) above - is to obtain corresponding results for the pressure π in (2.1a), as part of the same maximal regularity property of the linearized feedback operator $\mathbb{A}_{_F} (= \mathbb{A}_{_{F,q}})$ in (7.1), not through ad-hoc subsequent argument as in [B-T.1, Theorem 2.3, p1450]

2.5 What is the motivation for seeking interior localized feedback exponential stabilization of problem (2.1) in the topology of the Besov space in (2.3)?

Obtaining the resulting stabilization in a non-Hilbertian setting may be of theoretical interest in itself in line with recent developments in parabolic equations . However, our main motivation for the present study is another. The present paper intends to test L^q/Besov spaces techniques initially in the interior localized feedback stabilization problem (2.1). The true aim is however, to export them with serious additional technical difficulties, to solve the presently recognized **open** problem of the local feedback exponential stabilization of the N-S equations with *finite-dimensional* feedback *tangential boundary* controllers in the case of dimension $d = 3$. In fact, present state-of-the-art has succeeded [L-T.2], [L-T.3] in establishing local exponential stabilization (asymptotic turbulence suppression) by means of *finite-dimensional tangential* feedback boundary control in the Hilbert setting and with no assumptions whatsoever on the Oseen operator in two cases:

(i) when the dimension $d = 2$,

(ii) when the dimension $d = 3$ but the initial condition y_0 in (2.1.d) is compactly supported.

In the general $d = 3$ case, the non-linearity of the N-S problem forces a Hilbert space setting with a high-topology $H^{1/2+\epsilon}(\Omega)$ for the initial conditions, whereby the compatibility conditions on the boundary kick in. These then cannot allow the stabilizing feedback control to be finite-dimensional in general. More precisely, even at the level of the *linearized boundary* problem for $d = 3$, *open loop* exponential stabilization [B-L-T.3, Proposition 3.7.1 Remark 3.7.1], [L-T.2, Proposition 2.5, eq (2.48)] provide a *boundary* control consisting of a finite-dimensional term plus the term $e^{-2\gamma_1 t}((I.C.)|_\Gamma)$, with $\gamma_1 > 0$ preassigned, which spoils the finite-dimensionality, unless the initial condition is compactly supported. These limitations are in subsequent literature. In contrast, the Besov space in (2.3) above, which is "close" to the space $L^q(\Omega), q > 3$, has the key, fundamental advantage of not recognizing the boundary conditions. In fact, the critical role of $L^3(\Omega)$ (for $d = 3$) has been recognized as in the well-posedness of the (uncontrolled) $3 - d$ Navier-Stokes equations [J-S]. That is why this paper is interested in a stabilization result in such a space in $d = 3$, at first in the case of interior localized control.

2.6 Comparison with the prior work [B-T.1], once the present treatment is specialized to the Hilbert setting ($q = 2$)

A comparison with the prior 2004-work [B-T.1] carried out in the Hilbert setting is in order.

Orientation Even when specialized to the Hilbert space setting ($q = 2$), the present treatment offers distinct, notable advantages both conceptual and computational over [B-T.1]. These include not only definitely simpler and more direct arguments but also transparent simplifications in the actual construction of the finitely many stabilizing controllers. Qualitative details are given below. The main conceptual approach and the final results of the present paper are (when specialized to the Hilbert setting) qualitatively in line with those in [B-T.1]: local uniform stabilization of the non-linear y-problem (2.1) near an unstable equilibrium solution y_e by means of finitely many, arbitrarily localized controllers is based on the corresponding result on (global) uniform stabilization of the linearized w-system (2.13). This in turn rests on the space decomposition technique introduced in [RT.1] for parabolic problems (and also for differentiable semigroups): its foundational starting point is the controllability of the finite dimensional unstable projected system w_N in (3.8a). However, in the implementation of these two fundamental phases, linear analysis - in particular, its finite dimensional w_N component - and nonlinear analysis, the present paper provides a much more attractive, more powerful and mature treatment. We mention the most relevant new features. They are:

1. finite dimensional analysis leading, through a much more simplified and more direct approach, to a lower (optimal) number of feedback controls
2. infinite dimensional analysis on the nonlinear effects (the operator $\mathcal{N}_q$ in (2.11)) handled by critical and clean use of maximal regularity of the linearized feedback operator $\mathbb{A}_F (\equiv \mathbb{A}_{F,q})$ in (7.1), rather than by the approximation argument as in [B-T.1]. This refers to both y and π.

These two points are explained below.

1. **Stabilization of the linearized w-problem (2.13).** The key foundational algebraic test for controllability of the finite dimensional w_N-system (3.8a) on the finite dimensional unstable subspace W_N^u is much simplified, sharper and leads, in principle, to checkable conditions and to an implementable procedure to obtain constructively the finite dimensional stabilizing vectors $u_k \in W_N^u,\ p_k \in (W_N^u)^*$. In fact, the present treatment shows that (for the complexified version of the N-S) the required number of feedback stabilizing controllers is $K = \sup\{\ell_i,\ i = 1, \dots, M\}$, the max of the geometric multiplicity ℓ_i of the M distinct unstable eigenvalues λ_i of the Oseen operator $\mathcal{A}$; not the larger $\sup\{N_i,\ i = 1, \dots, M\}$, the max of the algebraic multiplicity of its distinct unstable eigenvalues, as in [B-T.1], [B-L-T.1], [B-L-T.3]. Let alone $N = \displaystyle\sum_{i=1}^{M} N_i$ = dim

W_N^u (dimension of the generalized eigenspace of the unstable eigenvalues) as in the treatment of [B-T.1, assumption K.2 p 123], where, in addition, the simplifying assumption that algebraic and geometric multiplicities coincide for the unstable eigenvalues. Moreover, the entire analysis of the present paper rests only on the (true) eigenvectors corresponding to the unstable eigenvalues of the adjoint operator in (4.1); not only under the Finite Dimensional Spectral Condition (semisimplicity) as in [B-T.1] where W_N^u has a basis of such (true) eigenvectors, but also in the most general case where the projected Oseen operator $\mathcal{A}_N^u$ is in Jordan form, and hence the basis on W_N^u consists instead of all generalized eigenvectors corresponding to the unstable eigenvalues. As first noted in [L-T.2] in the study of *tangential boundary stabilization* of the NS equations, even in the general case possessing only a basis of generalized eigenfunctions arising from the Jordan form, the final test for controllability involves only the true eigenfunctions of the adjoint operator: the algebraic test (5.13), (5.14) for controllability in the general case is exactly the same as the algebraic test (4.18) in the semisimple (diagonalizable) FDSA-case; and only the true eigenfunctions count. This justifies why the number K of (complex valued) stabilizing controllers as in Theorem 3.1 is equal to the supremum of the geometric multiplicity of the unstable eigenvalues, not the supremum of their larger algebraic multiplicity as in past references [B-T.1], [B-L-T.1], [B-L-T.3], [?], as noted above. Moreover, in [B-T.1] the procedure for testing controllability in the general case was much more cumbersome and far less implementable: the original basis of generalized eigenfunctions of the adjoint operator in the general case was transformed into an orthonormal basis of W_N^u via the Schmidt orthogonalization process, and the test for the finite dimensional controllability was then based on such transformed, and thus in principle difficult to check, orthogonalized system: a much more complicated test than the one using just the true eigenfunctions as in (5.13).

2. **Local Stabilization of the nonlinear translated z-equation (2.7) near the origin, hence of the original y-equation (2.1) near an equilibrium solution y_e.** Treatment of the nonlinearity in the present work is much more transparent and direct than the one performed in [B-T.1]. Here the analysis is directly in terms of the nonlinear operator $\mathcal{N}_q$ in (2.11) and makes use of maximal regularity properties of the linearized feedback operator $\mathbb{A}_F (\equiv \mathbb{A}_{F,q})$ in (7.1) (maximal regularity is equivalent to analyticity of the semigroup in the Hilbert setting). In contrast, in [B-T.1] with $q = 2$, an approximation argument of the nonlinear operator $\mathcal{N}$, denoted by B, was used, by introducing a

sequence of approximating operators B_ε thereof [B-T.1, Section 4, p1480]. A critical step in [B-T.1] is that the nonlinearity B (or its approximation) be controlled by the topology of the $A^{3/4}$-power; and this in turn is achieved by using an optimal control approach with $A^{3/4}$-penalization of the solution via Riccati equations. There is no need of this in the present treatment (the analysis of the optimization problem and Riccati equation in a non-Hilbert setting is not the right tool). We likewise note that our present treatment of the passage from the w-linearized problem (3.16) to the fully non-linear z-system (3.20) is also different from the one employed in [B-L-T.1], [L-T.3] which was also direct in terms of the nonlinear operator $\mathcal{N}$. It was however not maximal regularity - based as in the present paper.

3. **Well-posedness of the pressure π for the original y-problem in the feedback form as in (3.22) in the vicinity of the equilibrium pressure π_e in (2.2a).** The well-posedness result Theorem 26.2 on the pressure π of the original y-problem on feedback form as given by (3.26), (3.27) is the L^q/Besov space counterpart of the Hilbert (L^2)-version given by [B-T.1, Theorem 2.3 p1450]. The present proof is much more direct as, again, is based on maximal regularity properties. In contrast, the proof in [B-T.1, p 1484] is based on the approximation of the original problem.

2.7 Helmholtz decomposition

A first difficulty one faces in extending the local exponential stabilization result for the interior localized problem (2.1) from the Hilbert-space setting in [B-T.1], [B-L-T.1] to the L^q setting is the question of the existence of a Helmholtz (Leray) projection for the domain Ω in $\mathbb{R}^d$. More precisely: Given an open set $\Omega \subset \mathbb{R}^d$, the Helmholtz decomposition answers the question as to whether $L^q(\Omega)$ can be decomposed into a direct sum of the solenoidal vector space $L^q_\sigma(\Omega)$ and the space $G^q(\Omega)$ of gradient fields. Here,

$$\begin{aligned} L^q_\sigma(\Omega) &= \overline{\{y \in C_c^\infty(\Omega) : \text{div } y = 0 \text{ in } \Omega\}}^{\|\cdot\|_q} \\ &= \{g \in L^q(\Omega) : \text{div } g = 0;\ g \cdot \nu = 0 \text{ on } \partial\Omega\}, \\ &\qquad \text{for any locally Lipschitz domain } \Omega \subset \mathbb{R}^d, d \geq 2 \text{ [Ga.3}, p \text{ 119]} \\ G^q(\Omega) &= \{y \in L^q(\Omega) : y = \nabla p,\ p \in W^{1,q}_{loc}(\Omega)\} \text{ where } 1 \leq q < \infty. \end{aligned} \tag{2.4}$$

Both of these are closed subspaces of L^q.

Definition 2.1. Let $1 < q < \infty$ and $\Omega \subset \mathbb{R}^n$ be an open set. We say that the Helmholtz decomposition for $L^q(\Omega)$ exists whenever $L^q(\Omega)$ can be decomposed into the direct sum (non-orthogonal)

$$L^q(\Omega) = L^q_\sigma(\Omega) \oplus G^q(\Omega). \tag{2.5}$$

The unique linear, bounded and idempotent (i.e. $P_q^2 = P_q$) projection operator $P_q : L^q(\Omega) \longrightarrow L^q_\sigma(\Omega)$ having $L^q_\sigma(\Omega)$ as its range and $G^q(\Omega)$ as its null space is called the Helmholtz projection. Additional information is given in Appendix A.

This is an important property in order to handle the incompressibility condition $div\ y \equiv 0$. For instance, if such decomposition exists, the Stokes equation (say the linear version of (2.1) with control $u \equiv 0$) can be formulated as an equation in the L^q setting. Here below we collect a subset of known results about Helmholtz decomposition. We refer to [H-S, Section 2.2], in particular to the comprehensive Theorem 2.2.5 in this reference, which collects domains for which the Helmholtz decomposition is known to exist. These include the following cases:

(i) any open set $\Omega \subset \mathbb{R}^d$ for $q = 2$, i.e. with respect to the space $L^2(\Omega)$; more precisely, for $q = 2$, we obtain the well-known orthogonal decomposition (in the standard notation, where ν =unit outward normal vector on Γ) [C-F, Prop 1.9, p 8]

$$L^2(\Omega) = H \oplus H^\perp \tag{2.6a}$$

$$H = \{\phi \in L^2(\Omega) : div\ \phi \equiv 0 \text{ in } \Omega;\ \phi \cdot \nu \equiv 0 \text{ on } \Gamma\} \tag{2.6b}$$

$$H^\perp = \{\psi \in L^2(\Omega) : \psi = \nabla h,\ h \in H^1(\Omega)\}; \tag{2.6c}$$

(ii) a bounded C^1-domain in $\mathbb{R}^d$ [F-M-M], $1 < q < \infty$ [Ga.3, Theorem 1.1 p 107, Theorem 1.2 p 114] for C^2-boundary

(iii) a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ $(d = 3)$ and for $\frac{3}{2} - \epsilon < q < 3 + \epsilon$ sharp range [F-M-M];

(iv) a bounded convex domain $\Omega \subset \mathbb{R}^d, d \geq 2, 1 < q < \infty$ [F-M-M].

On the other hand, on the negative side, it is known that there exist domains $\Omega \subset \mathbb{R}^d$ such that the Helmholtz decomposition does not hold for some $q \neq 2$ [M-S].

Assumption (H-D) Henceforth in this paper, we assume that the bounded domain $\Omega \subset \mathbb{R}^d$ under consideration admits a Helmholtz decomposition for the values of q, $1 < q < \infty$, here considered at first, for the linearized problem (2.13) below. The final result Theorem 3.5 for the non-linear problem (2.1) will require $q > d$, see (9.16), in the case of interest $d = 2, 3$.

2.8 Translated nonlinear Navier-Stokes z-problem: reduction to zero equilibrium

We return to Theorem 2.1 which provides an equilibrium pair $\{y_e, \pi_e\}$. Then, as in [B-T.1], [B-L-T.1], [L-T.2] we translate by $\{y_e, p_e\}$ the original N-S problem (2.1). Thus we introduce new variables

$$z = y - y_e, \quad \chi = \pi - \pi_e \tag{2.7a}$$

and obtain the translated problem

$$z_t - \nu\Delta z + (y_e \cdot \nabla)z + (z \cdot \nabla)y_e + (z \cdot \nabla)z + \nabla\chi = mu \qquad \text{in } Q \tag{2.7b}$$

$$\text{div } z = 0 \qquad \text{in } Q \tag{2.7c}$$

$$z = 0 \qquad \text{on } \Sigma \tag{2.7d}$$

$$z(0, x) = y_0(x) - y_e(x) \qquad \text{on } \Omega \tag{2.7e}$$

We shall accordingly study the local null feedback stabilization of the z-problem (2.7), that is, feedback stabilization in a neighborhood of the origin. As usual, we next apply the projection P_q below (2.5) to the translated N-S problem (2.7) to eliminate the pressure χ. We thus proceed to obtain the corresponding abstract setting for the problem (2.7) as in [B-T.1] except in the L^q-setting rather than in the L^2-setting as in this reference. Note that $P_q z_t = z_t$, since $z \in L^q_\sigma(\Omega)$ in (2.4).

2.9 Abstract nonlinear translated model

First, for $1 < q < \infty$ fixed, the Stokes operator A_q in $L^q_\sigma(\Omega)$ with Dirichlet boundary conditions is defined by [G-G-H.1, p 1404], [H-S, p 1]

$$A_q z = -P_q \Delta z, \quad \mathcal{D}(A_q) = W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega) \cap L^q_\sigma(\Omega). \tag{2.8}$$

The operator A_q has a compact inverse A_q^{-1} on $L^q_\sigma(\Omega)$, hence A_q has a compact resolvent on $L^q_\sigma(\Omega)$. Next, we introduce the first order operator $A_{o,q}$,

$$A_{o,q} z = P_q[(y_e \,.\, \nabla)z + (z \,.\, \nabla)y_e], \quad \mathcal{D}(A_{o,q}) = \mathcal{D}(A_q^{1/2}) \subset L^q_\sigma(\Omega). \tag{2.9}$$

where the $\mathcal{D}(A_q^{1/2})$ is defined explicitly in (2.22) below. Thus, $A_{o,q}A_q^{-1/2}$ is a bounded operator on $L^q_\sigma(\Omega)$, and thus $A_{o,q}$ is bounded on $\mathcal{D}(A_q^{1/2})$

$$\|A_{o,q}f\| = \left\|A_{o,q}A_q^{-1/2}A_q^{-1/2}A_q f\right\| \leq C_q \left\|A_q^{1/2}f\right\|, \quad f \in \mathcal{D}(A_q^{1/2}).$$

This leads to the definition of the Oseen operator

$$\mathcal{A}_q = -(\nu A_q + A_{o,q}), \quad \mathcal{D}(\mathcal{A}_q) = \mathcal{D}(A_q) \subset L^q_\sigma(\Omega). \tag{2.10}$$

Finally, we define the projection of the nonlinear portion of the static operator in (2.7b)

$$\mathcal{N}_q(z) = P_q[(z \cdot \nabla)z], \quad \mathcal{D}(\mathcal{N}_q) = W^{1,q}(\Omega) \cap L^\infty(\Omega) \cap L^q_\sigma(\Omega). \tag{2.11}$$

[As shown in (9.16) in the analysis of the non-linear problem, at the end we shall use $W^{1,q}(\Omega) \subset L^\infty(\Omega)$ for $q > \dim \Omega = 3$ [Kes, Theorem 2.4.4, p74, requiring C^1 boundary.]]

Thus, the Navier-Stokes translated problem (2.7), after application of the Helmholtz projector P_q in Definition 2.1 and use of (2.8)-(2.11), can be rewritten as the following abstract equation in $L^q_\sigma(\Omega)$:

$$\begin{cases} \dfrac{dz}{dt} + \nu A_q z + A_{o,q} z + P_q[(z \cdot \nabla)z] = P_q(mu) \quad \text{or} \quad \dfrac{dz}{dt} - \mathcal{A}_q z + \mathcal{N}_q z = P_q(mu) \text{ in } L^q_\sigma(\Omega) & \text{(2.12a)} \\ z(x,0) = z_0(x) = y_0(x) - y_e \text{ in } L^q_\sigma(\Omega). & \text{(2.12b)} \end{cases}$$

2.10 The linearized problem of the translated model

Next, still for $1 < q < \infty$, we consider the following linearized system of the translated model (2.7) or (2.12):

$$\begin{cases} \dfrac{dw}{dt} + \nu A_q w + A_{o,q} w = P_q(mu) \quad \text{or} \quad \dfrac{dw}{dt} - \mathcal{A}_q w = P_q(mu) \text{ in } L^q_\sigma(\Omega) & \text{(2.13a)} \\ w_0(x) = y_0(x) - y_e \text{ in } L^q_\sigma(\Omega). & \text{(2.13b)} \end{cases}$$

2.11 Some auxiliary results for problem (2.13): analytic semigroup generation, maximal regularity, domains of fractional powers

In this subsection we collect some known results to be used in the sequel.

(a) **Definition of Besov spaces $B^s_{q,p}$ on domains of class C^1 as real interpolation of Sobolev spaces:** Let m be a positive integer, $m \in \mathbb{N}, 0 < s < m, 1 \leq q < \infty, 1 \leq p \leq \infty$, then we define [G-G-H.1, p 1398]

$$B^s_{q,p}(\Omega) = (L^q(\Omega), W^{m,q}(\Omega))_{\frac{s}{m},p} \tag{2.14a}$$

This definition does not depend on $m \in \mathbb{N}$ [Wahl, p xx]. This clearly gives

$$W^{m,q}(\Omega) \subset B^s_{q,p}(\Omega) \subset L^q(\Omega) \quad \text{and} \quad \|y\|_{L^q(\Omega)} \leq C \|y\|_{B^s_{q,p}(\Omega)}. \tag{2.14b}$$

We shall be particularly interested in the following special real interpolation space of the L^q and $W^{2,q}$ spaces $\left(m = 2, s = 2 - \frac{2}{p}\right)$:

$$B^{2-\frac{2}{p}}_{q,p}(\Omega) = \left(L^q(\Omega), W^{2,q}(\Omega)\right)_{1-\frac{1}{p},p}. \tag{2.15}$$

Our interest in (2.15) is due to the following characterization [Amann.2, Thm 3.4], [G-G-H.1, p 1399]: if A_q denotes the Stokes operator introduced in (2.8), then

$$\left(L^q_\sigma(\Omega), \mathcal{D}(A_q)\right)_{1-\frac{1}{p},p} = \left\{g \in B^{2-2/p}_{q,p}(\Omega) : \text{ div } g = 0, \ g|_\Gamma = 0\right\} \quad \text{if } \frac{1}{q} < 2 - \frac{2}{p} < 2 \tag{2.16a}$$

$$\left(L^q_\sigma(\Omega), \mathcal{D}(A_q)\right)_{1-\frac{1}{p},p} = \left\{g \in B^{2-2/p}_{q,p}(\Omega) : \text{ div } g = 0, \ g \cdot \nu|_\Gamma = 0\right\} \equiv \widetilde{B}^{2-2/p}_{q,p}(\Omega) \tag{2.16b}$$

$$\text{if } 0 < 2 - \frac{2}{p} < \frac{1}{q}; \text{ or } 1 < p < \frac{2q}{2q-1}.$$

Notice that, in (2.16b), the condition $g \cdot \nu|_\Gamma = 0$ is an intrinsic condition of the space $L^q_\sigma(\Omega)$ in (2.4), not an extra boundary condition as $g|_\Gamma = 0$ in (2.16a).

Remark 2.3. In the analysis of well-posedness and stabilization of the nonlinear N-S problem (2.1), with control u in feedback form - such as the non linear translated feedback problem (3.20) = (9.1)- we shall need to impose the constrain $q > 3$, see Eq (9.16), to obtain the embedding $W^{1,q} \hookrightarrow L^\infty(\Omega)$ in our case of interest $d = 3$, as already noted below (2.11). What is then the allowable range of the parameter p in such case $q > 3$? The intended goal of the present paper is to obtain the sought-after stabilization result in a function space, such as a $B^{2-2/p}_{q,p}(\Omega)$-space, that does not recognize boundary conditions of the I.C. Thus, we need to avoid the case in (2.16a), as this implies a Dirichlet homogeneous B.C. Instead, we need to fit into the case (2.16b). We shall then impose the condition $2 - \frac{2}{p} < \frac{1}{q} < \frac{1}{3}$ and then obtain that p must satisfy $p < \frac{6}{5}$.

Moreover, analyticity and maximal regularity of the Stokes problem will require $p > 1$. Thus, in conclusion, the allowed range of the parameters p, q under which we shall solve the well-posedness and stabilization problem of the nonlinear N-S feedback system (3.20) = (9.1) for $d = 3$, in the space $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$ which - as intended - does not recognize boundary conditions is: $q > 3,\ 1 < p < \dfrac{6}{5}$. See Theorems 3.3 through 3.5.

(b) **The Stokes and Oseen operators generate a strongly continuous analytic semigroup on $L^q_\sigma(\Omega)$, $1 < q < \infty$.**

Theorem 2.2. *Let $d \geq 2, 1 < q < \infty$ and let Ω be a bounded domain in $\mathbb{R}^d$ of class C^3. Then*

(i) the Stokes operator $-A_q = P_q\Delta$ in (2.8), repeated here as

$$-A_q\psi = P_q\Delta\psi, \quad \psi \in \mathcal{D}(A_q) = W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega) \cap L^q_\sigma(\Omega) \tag{2.17}$$

generates a s.c analytic semigroup $e^{-A_q t}$ on $L^q_\sigma(\Omega)$. See [Gi.1] and the review paper [H-S, Theorem 2.8.5 p 17].

(ii) The Oseen operator $\mathcal{A}_q$ in (2.10)

$$\mathcal{A}_q = -(\nu A_q + A_{o,q}), \quad \mathcal{D}(\mathcal{A}_q) = \mathcal{D}(A_q) \subset L^q_\sigma(\Omega) \tag{2.18}$$

generates a s.c analytic semigroup $e^{\mathcal{A}_q t}$ on $L^q_\sigma(\Omega)$. This follows as $A_{o,q}$ is relatively bounded with respect to $A_q^{1/2}$, defined in (2.22), see below (2.9): thus a standard theorem on perturbation of an analytic semigroup generator applies [Pazy, Corollary 2.4, p 81].

(iii)

$$\begin{cases} 0 \in \rho(A_q) = \text{ the resolvent set of the Stokes operator } A_q & (2.19a) \\ A_q^{-1} : L^q_\sigma(\Omega) \longrightarrow L^q_\sigma(\Omega) \text{ is compact} & (2.19b) \end{cases}$$

(iv) The s.c. analytic Stokes semigroup $e^{-A_q t}$ is uniformly stable on $L^q_\sigma(\Omega)$: there exist constants $M \geq 1, \delta > 0$ (possibly depending on q) such that

$$\left\| e^{-A_q t} \right\|_{\mathcal{L}(L^q_\sigma(\Omega))} \leq M e^{-\delta t},\ t > 0. \tag{2.20}$$

(c) **Domains of fractional powers, $\mathcal{D}(A_q^\alpha), 0 < \alpha < 1$ of the Stokes operator A_q on $L^q_\sigma(\Omega), 1 < q < \infty$,**

Theorem 2.3. *For the domains of fractional powers $\mathcal{D}(A_q^\alpha), 0 < \alpha < 1$, of the Stokes operator A_q in (2.8) = (2.17), the following complex interpolation relation holds true [Gi.2] and [H-S, Theorem 2.8.5, p 18]*

$$[\mathcal{D}(A_q), L^q_\sigma(\Omega)]_{1-\alpha} = \mathcal{D}(A_q^\alpha), \ 0 < \alpha < 1, \ 1 < q < \infty; \tag{2.21}$$

in particular

$$[\mathcal{D}(A_q), L^q_\sigma(\Omega)]_{\frac{1}{2}} = \mathcal{D}(A_q^{1/2}) \equiv W_0^{1,q}(\Omega) \cap L^q_\sigma(\Omega). \tag{2.22}$$

Thus, on the space $\mathcal{D}(A_q^{1/2})$, the norms

$$\|\nabla \cdot \|_{L^q(\Omega)} \ \textit{and} \ \| \ \|_{L^q(\Omega)} \tag{2.23}$$

are equivalent via Poincaré inequality.

(d) **The Stokes operator $-A_q$ and the Oseen operator $\mathcal{A}_q, 1 < q < \infty$ generate s.c. analytic semigroups on the Besov space**

$$\Big(L^q_\sigma(\Omega), \mathcal{D}(A_q)\Big)_{1-\frac{1}{p},p} = \Big\{g \in B^{2-2/p}_{q,p}(\Omega) : \ \text{div}\, g = 0, \ g|_\Gamma = 0\Big\} \quad \text{if } \frac{1}{q} < 2 - \frac{2}{p} < 2; \tag{2.24a}$$

$$\Big(L^q_\sigma(\Omega), \mathcal{D}(A_q)\Big)_{1-\frac{1}{p},p} = \Big\{g \in B^{2-2/p}_{q,p}(\Omega) : \ \text{div}\, g = 0, \ g \cdot \nu|_\Gamma = 0\Big\} \equiv \widetilde{B}^{2-2/p}_{q,p}(\Omega) \tag{2.24b}$$

$$\text{if } 0 < 2 - \frac{2}{p} < \frac{1}{q}.$$

Theorem 2.2 states that the Stokes operator $-A_q$ generates a s.c analytic semigroup on the space $L^q_\sigma(\Omega), \ 1 < q < \infty$, hence on the space $\mathcal{D}(A_q)$ in (2.17), with norm $\| \cdot \|_{\mathcal{D}(A_q)} = \|A_q \cdot \|_{L^q_\sigma(\Omega)}$ as $0 \in \rho(A_q)$. Then, one obtains that the Stokes operator $-A_q$ generates a s.c. analytic semigroup on the real interpolation spaces in (2.24). Next, the Oseen operator $\mathcal{A} = -(\nu A_q + A_{o,q})$ likewise generates a s.c. analytic semigroup $e^{\mathcal{A}_q t}$ on $L^q_\sigma(\Omega)$ since $A_{o,q}$ is relatively bounded w.r.t. $A_q^{1/2}$, as $A_{o,q}A_q^{-1/2}$ is bounded on $L^q_\sigma(\Omega)$. Moreover $\mathcal{A}_q$ generates a s.c. analytic semigroup on $\mathcal{D}(\mathcal{A}_q) = \mathcal{D}(A_q)$ (equivalent norms). Hence $\mathcal{A}_q$ generates a s.c. analytic semigroup on the real interpolation space of (2.24). Here below, however, we shall formally state the result only in the case $2-{}^2/_p < {}^1/_q$. i.e. $1 < p < {}^{2q}/_{2q-1}$, in the space $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$, as this does not contain B.C. The objective of the present paper is precisely to obtain stabilization results on spaces that do not recognize B.C.

Theorem 2.4. *Let $1 < q < \infty, 1 < p < {}^{2q}/_{2q-1}$*

(i) The Stokes operator $-A_q$ in (2.17) generates a s.c analytic semigroup $e^{-A_q t}$ on the space $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$ defined in (2.16) = (2.24) which moreover is uniformly stable, as in (2.20),

$$\left\| e^{-A_q t} \right\|_{\mathcal{L}\left(\widetilde{B}^{2-2/p}_{q,p}(\Omega)\right)} \leq M e^{-\delta t}, \quad t > 0. \tag{2.25}$$

(ii) The Oseen operator $\mathcal{A}_q$ in (2.18) generates a s.c. analytic semigroup $e^{\mathcal{A}_q t}$ on the space $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$ in (2.16) = (2.24).

(e) **Space of maximal L^p regularity on $L^q_\sigma(\Omega)$ of the Stokes operator $-A_q$, $1 < p < \infty$, $1 < q < \infty$ up to $T = \infty$.** We return to the dynamic Stokes problem in $\{\varphi(t,x), \pi(t,x)\}$

$$\begin{cases} \varphi_t - \Delta\varphi + \nabla\pi = F & \text{in } (0,T] \times \Omega \equiv Q & (2.26a) \\ div\ \varphi \equiv 0 & \text{in } Q & (2.26b) \\ \varphi|_\Sigma \equiv 0 & \text{in } (0,T] \times \Gamma \equiv \Sigma & (2.26c) \\ \varphi|_{t=0} = \varphi_0 & \text{in } \Omega, & (2.26d) \end{cases}$$

rewritten in abstract form, after applying the Helmholtz projection P_q to (2.26a) and recalling A_q in (2.17) as

$$\varphi' + A_q \varphi = F_\sigma \equiv P_q F, \quad \varphi_0 \in \left(L^q_\sigma(\Omega), \mathcal{D}(A_q)\right)_{1-\frac{1}{p},p} \tag{2.27}$$

Next, we introduce the space of maximal regularity for $\{\varphi, \varphi'\}$ as [H-S, p 2; Theorem 2.8.5.iii, p 17], [G-G-H.1, p 1404-5], with T up to ∞:

$$X^T_{p,q,\sigma} = L^p(0,T;\mathcal{D}(A_q)) \cap W^{1,p}(0,T;L^q_\sigma(\Omega)) \tag{2.28}$$

(recall (2.8) for $\mathcal{D}(A_q)$) and the corresponding space for the pressure as

$$Y^T_{p,q} = L^p(0,T;\widehat{W}^{1,q}(\Omega)), \quad \widehat{W}^{1,q}(\Omega) = W^{1,q}(\Omega)/\mathbb{R}. \tag{2.29}$$

The following embedding, also called trace theorem, holds true [Amann.2, Theorem 4.10.2, p 180, BUC for $T = \infty$], [P-S].

$$X^T_{p,q,\sigma} \subset X^T_{p,q} \equiv L^p(0,T;W^{2,q}(\Omega)) \cap W^{1,p}(0,T;L^q(\Omega)) \hookrightarrow C\Big([0,T]; B^{2-2/p}_{q,p}(\Omega)\Big). \tag{2.30}$$

For a function g such that $div\ g \equiv 0$, $g|_\Gamma = 0$ we have $g \in X^T_{p,q} \iff g \in X^T_{p,q,\sigma}$, by (2.4). The solution of Eq(2.27) is

$$\varphi(t) = e^{-A_q t}\varphi_0 + \int_0^t e^{-A_q(t-s)} F_\sigma(\tau) d\tau. \tag{2.31}$$

The following is the celebrated result on maximal regularity on $L^q_\sigma(\Omega)$ of the Stokes problem due originally to Solonnikov [Sol.2] reported in [H-S, Theorem 2.8.5.(iii) and Theorem 2.10.1 p24 for $\varphi_0 = 0$], [Saa], [G-G-H.1, Proposition 4.1 , p 1405].

Theorem 2.5. *Let $1 < p, q < \infty, T \leq \infty$. With reference to problem (2.26) = (2.27), assume*

$$F_\sigma \in L^p(0,T; L^q_\sigma(\Omega)),\ \ \varphi_0 \in \Big(L^q_\sigma(\Omega), \mathcal{D}(A_q)\Big)_{1-\frac{1}{p},p}. \tag{2.32}$$

Then there exists a unique solution $\varphi \in X^T_{p,q,\sigma}, \pi \in Y^T_{p,q}$ to the dynamic Stokes problem (2.26) or (2.27), continuously on the data: there exist constants C_0, C_1 independent of T, F_σ, φ_0 such that via (2.30)

$$\begin{aligned} C_0 \left\|\varphi\right\|_{C\left([0,T]; B^{2-2/p}_{q,p}(\Omega)\right)} &\leq \|\varphi\|_{X^T_{p,q,\sigma}} + \|\pi\|_{Y^T_{p,q}} \\ &\equiv \left\|\varphi'\right\|_{L^p(0,T;L^q_\sigma(\Omega))} + \|A_q\varphi\|_{L^p(0,T;L^q_\sigma(\Omega))} + \|\pi\|_{Y^T_{p,q}} \\ &\leq C_1 \Big\{ \|F_\sigma\|_{L^p(0,T;L^q_\sigma(\Omega))} + \|\varphi_0\|_{\left(L^q_\sigma(\Omega),\mathcal{D}(A_q)\right)_{1-\frac{1}{p},p}} \Big\}. \end{aligned} \tag{2.33}$$

In particular,

(i) With reference to the variation of parameters formula (2.31) of problem (2.27) arising from the Stokes problem (2.26), we have recalling (2.28): the map

$$F_\sigma \longrightarrow \int_0^t e^{-A_q(t-\tau)} F_\sigma(\tau) d\tau \ \ : \textit{continuous} \tag{2.34}$$

$$L^p(0,T; L^q_\sigma(\Omega)) \longrightarrow X^T_{p,q,\sigma} \equiv L^p(0,T; \mathcal{D}(A_q)) \cap W^{1,p}(0,T; L^q_\sigma(\Omega)) \tag{2.35}$$

(ii) The s.c. analytic semigroup $e^{-A_q t}$ generated by the Stokes operator $-A_q$ (see (2.17)) on the space $\Big(L^q_\sigma(\Omega), \mathcal{D}(A_q)\Big)_{1-\frac{1}{p},p}$ (see statement below (2.24)) satisfies

$$e^{-A_q t}:\ \textit{continuous}\ \ \Big(L^q_\sigma(\Omega), \mathcal{D}(A_q)\Big)_{1-\frac{1}{p},p} \longrightarrow X^T_{p,q,\sigma} \equiv L^p(0,T; \mathcal{D}(A_q)) \cap W^{1,p}(0,T; L^q_\sigma(\Omega)) \tag{2.36a}$$

In particular via (2.24b), for future use, for $1 < q < \infty, 1 < p < \frac{2q}{2q-1}$, the s.c. analytic semigroup $e^{-A_q t}$ on the space $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$, satisfies

$$e^{-A_q t}:\ \textit{continuous}\ \ \widetilde{B}^{2-2/p}_{q,p}(\Omega) \longrightarrow X^T_{p,q,\sigma}. \tag{2.36b}$$

(iii) Moreover, for future use, for $1 < q < \infty, 1 < p < \frac{2q}{2q-1}$*, then (2.33) specializes to*

$$\|\varphi\|_{X^T_{p,q,\sigma}} + \|\pi\|_{Y^T_{p,q}} \le C\Big\{ \|F_\sigma\|_{L^p(0,T;L^q_\sigma(\Omega))} + \|\varphi_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \Big\}. \tag{2.37}$$

(f) **Maximal L^p regularity on $L^q_\sigma(\Omega)$ of the Oseen operator $\mathcal{A}_q$, $1 < p < \infty$, $1 < q < \infty$, up to $T < \infty$.** We next transfer the maximal regularity of the Stokes operator $(-A_q)$ on $L^q_\sigma(\Omega)$-asserted in Theorem 2.5 into the maximal regularity of the Oseen operator $\mathcal{A}_q = -\nu A_q - A_{o,q}$ in (2.18) exactly on the same space $X^T_{p,q,\sigma}$ defined in (2.28), however only up to $T < \infty$.

Thus, consider the dynamic Oseen problem in $\{\psi(t,x), \pi(t,x)\}$ with equilibrium solution y_e, see (2.2):

$$\begin{cases} \psi_t - \Delta\psi + L_e(\psi) + \nabla\pi = F & \text{in } (0,T] \times \Omega \equiv Q \quad (2.38a) \\ div\ \psi \equiv 0 & \text{in } Q \quad (2.38b) \\ \psi|_\Sigma \equiv 0 & \text{in } (0,T] \times \Gamma \equiv \Sigma \quad (2.38c) \\ \psi|_{t=0} = \psi_0 & \text{in } \Omega, \quad (2.38d) \end{cases}$$

$$L_e(\psi) = (y_e.\nabla)\psi + (\psi.\nabla)y_e \tag{2.39}$$

rewritten in abstract form, after applying the Helmholtz projector P_q to (2.38a) and recalling $\mathcal{A}_q$ in (2.18), as

$$\psi_t = \mathcal{A}_q\psi + P_qF = -\nu A_q\psi - A_{o,q}\psi + F_\sigma, \quad \psi_0 \in \big(L^q_\sigma(\Omega), \mathcal{D}(A_q)\big)_{1-\frac{1}{p},p} \tag{2.40}$$

whose solution is

$$\psi(t) = e^{\mathcal{A}_q t}\psi_0 + \int_0^t e^{\mathcal{A}_q(t-\tau)}F_\sigma(\tau)d\tau. \tag{2.41}$$

$$\psi(t) = e^{-\nu A_q t}\psi_0 + \int_0^t e^{-\nu A_q(t-\tau)}F_\sigma(\tau)d\tau - \int_0^t e^{-\nu A_q(t-\tau)}A_{o,q}\psi(\tau)d\tau. \tag{2.42}$$

Theorem 2.6. *Let* $1 < p, q < \infty$, $0 < T < \infty$*. Assume (as in (2.32))*

$$F_\sigma \in L^p\big(0,T; L^q_\sigma(\Omega)\big), \quad \psi_0 \in \big(L^q_\sigma(\Omega), \mathcal{D}(A_q)\big)_{1-\frac{1}{p},p} \tag{2.43}$$

where $\mathcal{D}(A_q) = \mathcal{D}(\mathcal{A}_q)$*, see (2.18). Then there exists a unique solution* $\psi \in X^T_{p,q,\sigma}$*,* $\pi \in Y^T_{p,q}$ *of the dynamic Oseen problem (2.38), continuously on the data: that is, there exist constants* C_0, C_1

independent of F_σ, ψ_0 *such that*

$$
\begin{aligned}
C_0 \|\varphi\|_{C\left([0,T];B^{2-2/p}_{q,p}(\Omega)\right)} &\leq \|\varphi\|_{X^T_{p,q,\sigma}} + \|\pi\|_{Y^T_{p,q}} \\
&\equiv \|\varphi'\|_{L^p(0,T;L^q(\Omega))} + \|A_q\varphi\|_{L^p(0,T;L^q(\Omega))} + \|\pi\|_{Y^T_{p,q}} \qquad (2.44)\\
&\leq C_T\left\{ \|F_\sigma\|_{L^p(0,T;L^q_\sigma(\Omega))} + \|\varphi_0\|_{\left(L^q_\sigma(\Omega),\mathcal{D}(A_q)\right)_{1-\frac{1}{p},p}} \right\} \qquad (2.45)
\end{aligned}
$$

where $T < \infty$*. Equivalently, for* $1 < p, q < \infty$

i. *The map*

$$
\begin{aligned}
&F_\sigma \longrightarrow \int_0^t e^{\mathcal{A}_q(t-\tau)}F_\sigma(\tau)d\tau \ \ : \textit{continuous} \\
&\qquad\qquad L^p(0,T;L^q_\sigma(\Omega)) \longrightarrow L^p\big(0,T;\mathcal{D}(\mathcal{A}_q)=\mathcal{D}(A_q)\big)
\end{aligned} \qquad (2.46)
$$

where then automatically, see (2.40)

$$
L^p(0,T;L^q_\sigma(\Omega)) \longrightarrow W^{1,p}(0,T;L^q_\sigma(\Omega)) \qquad (2.47)
$$

and ultimately

$$
L^p(0,T;L^q_\sigma(\Omega)) \longrightarrow X^T_{p,q,\sigma} \equiv L^p\big(0,T;\mathcal{D}(A_q)\big) \cap W^{1,p}(0,T;L^q_\sigma(\Omega)). \qquad (2.48)
$$

ii. *The s.c. analytic semigroup* $e^{\mathcal{A}_q t}$ *generated by the Oseen operator* $\mathcal{A}_q$ *(see (2.18)) on the space* $\big(L^q_\sigma(\Omega),\mathcal{D}(A_q)\big)_{1-\frac{1}{p},p}$ *satisfies for* $1 < p, q < \infty$

$$
e^{\mathcal{A}_q t} : \ \textit{continuous} \quad \big(L^q_\sigma(\Omega),\mathcal{D}(A_q)\big)_{1-\frac{1}{p},p} \longrightarrow L^p\big(0,T;\mathcal{D}(\mathcal{A}_q)=\mathcal{D}(A_q)\big) \qquad (2.49)
$$

and hence automatically by (2.28)

$$
e^{\mathcal{A}_q t} : \ \textit{continuous} \quad \big(L^q_\sigma(\Omega),\mathcal{D}(A_q)\big)_{1-\frac{1}{p},p} \longrightarrow X^T_{p,q,\sigma}. \qquad (2.50)
$$

In particular, for future use, for $1 < q < \infty, 1 < p < \frac{2q}{2q-1}$*, we have that the s.c. analytic semigroup* $e^{\mathcal{A}_q t}$ *on the space* $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$*, satisfies*

$$
e^{\mathcal{A}_q t} : \ \textit{continuous} \quad \widetilde{B}^{2-2/p}_{q,p}(\Omega) \longrightarrow L^p\big(0,T;\mathcal{D}(\mathcal{A}_q)=\mathcal{D}(A_q)\big), \ T < \infty. \qquad (2.51)
$$

and hence automatically

$$
e^{\mathcal{A}_q t} : \ \textit{continuous} \quad \widetilde{B}^{2-2/p}_{q,p}(\Omega) \longrightarrow X^T_{p,q,\sigma}, \ T < \infty. \qquad (2.52)
$$

A proof is given in Appendix B.

Remark 2.4. The literature reports physical situations where the volumetric force f is actually replaced by $\nabla g(x)$; that is, f is a conservative vector field. Thus, returning to Eq (2.2a) with $f(x)$ replaced now by $\nabla g(x)$ we see that a solution of such stationary problem is $y_e = 0$, $\pi_e = g$, hence $L_e(\cdot) \equiv 0$ by (2.39). Returning to Eq (2.1a) with f replaced by $\nabla g(x)$ and applying to the resulting equation the projection operator P_q, one obtains in this case the projected equation

$$y_t - \nu P_q \Delta y + P_q\big[(y \cdot \nabla)y\big] = P_q(mu) \quad \text{in } Q. \tag{2.53}$$

This, along with the solenoidal and boundary conditions (2.1b), (2.1c), yields the corresponding abstract form recalling also (2.11)

$$y_t + \nu A_q y + \mathcal{N}_q y = P_q(mu) \quad \text{in } L^q_\sigma(\Omega). \tag{2.54}$$

Then y-problem (2.54) is the same as the z-problem (2.12a), except without the Oseen term $A_{o,q}$. The linearized version of problem (2.54) is then

$$\eta_t + \nu A_q \eta = P_q(mu) \quad \text{in } L^q_\sigma(\Omega), \tag{2.55}$$

which is the same as the w-problem (2.13a), except without the Oseen term $A_{o,q}$. The s.c. analytic semigroup $e^{-\nu A_q t}$ driving the linear equation (2.55) is uniformly stable in $L^q_\sigma(\Omega)$, see (2.20), as well as in $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$, see (2.25). Then, in the case of the present Remark, the present paper may be used to enhance at will the uniform stability of the corresponding problem with u given in feedback form as in the RHS of Eq (3.20) as to obtain a decay rate much bigger than the original $\delta > 0$ in (2.20) or (2.25). Thus there is no need to perform the translation $y \longrightarrow z$ of Section 2.8, when f in (2.2a) is replaced by $\nabla g(x)$; i.e. $y_e = 0$ in this case. The important relevance of the present Remark will be pointed out in the follow-out paper [L-P-T] where only finitely many localized tangential boundary feedback controls will be employed to the so far open case dim $\Omega = 3$. The corresponding required "unique continuation property" holds true for the Stokes problem ($y_e = 0$), see [RT.3], [RT.4].

3 Main results

3.1 Orientation

All the main results of this paper, Theorems 3.1 through 3.5, are stated (at first) in the complex state space setting $L^q_\sigma(\Omega) + iL^q_\sigma(\Omega)$. Thus, the finitely many stabilizing feedback vectors p_k, u_k constructed in the subsequent proofs belong to the complex finite dimensional unstable subspace $(W^u_N)^*$ and W^u_N respectively. The question then arises as to transfer back these results into the original real setting. This issue was resolved in [B-T.1]. Here, such translation, taken from [B-T.1], from the results in the complex setting (Theorems 3.1 through 3.5) into corresponding results in the original real setting is given in Section 3.7.

<u>Step 1</u>: First, we will show in Theorem 3.2 that the linearized Navier-Stokes problem $w_t = \mathcal{A}_q w + P_q(mu)$ in (2.13) can be uniformly (exponentially) stabilized in the basic space $L^q_\sigma(\Omega), 1 < q < \infty$ in fact, in the space $\mathcal{D}(A^\theta_q),\ 0 \le \theta \le 1$, or $\big(L^q_\sigma(\Omega), \mathcal{D}(A_q)\big)_{1-\frac{1}{p},p}$, in particular $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$ by means of an explicitly constructed, finite dimensional spectral-based feedback controller mu, localized on ω, whose structure is given in (3.16).

<u>Step 2</u>: Next, we proceed to the non-linear translated Navier-Stokes z-problem (2.12) with a control u having the same structure as the finite-dimensional, spectral based stabilizing control used in the linearized problem (2.13). This strategy leads to the non-linear feedback z-problem (3.20). We then establish two results for problem (3.20):

(i) The first, Theorem 3.3, is that problem (3.20) is locally well-posed, i.e. for small initial data z_0, in the desired space $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$. It will require the constraint $q > 3$, see (9.16), to obtain $W^{1,q}(\Omega) \hookrightarrow L^\infty(\Omega)$ for $d = 3$. In achieving this result, we must factor in what is the deliberate, sought-after goal of the present paper: that is, to obtain (well-posedness and) uniform stabilization of the original non-linear problem (2.1) near an equilibrium solution, in a function space that does not recognize boundary conditions. This is the space $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$ having only the boundary condition $g \cdot \nu|_\Gamma = 0$ inherited from the basic $L^q_\sigma(\Omega)$-space, see (2.16b) and statement below it. In contrast, we deliberately exclude then the space in (2.16a), $p > 2q/_{2q-1}$, having an explicit additional B.C. In conclusion, for the nonlinear problem, we need to work with the space $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$ in (2.16b), and this requires for $d=3$ the range $q>3, 1<p<{}^{2q}/_{2q-1}$, that is $1<p<{}^{6}/_{5}$ where the boundary conditions are not recognized. In this case the space $\widetilde{B}^{2-2/p}_{q,p}(\Omega) = \left(L^q_\sigma(\Omega), \mathcal{D}(A_q)\right)_{1-1/p,p}$ with index $1-{}^{1}/_{p}$ close to zero is "close" to the space $L^q(\Omega)$, for $q>3$. Accordingly, with reference to the feedback z-problem (3.20), we take $z_0 \in \widetilde{B}^{2-2/p}_{q,p}(\Omega)$, $q>3, 1<p<{}^{6}/_{5}$ sufficiently small, and show that (3.20) is well-posed in the function space $X^\infty_{p,q,\sigma}$ in (2.28). To this end, we use critically the maximal regularity result Theorem 8.1. This is Theorem 2.3.ii.

(ii) Second, we address the stabilization problem and show that such Navier-Stokes feedback problem (3.20) is, in fact, locally exponentially stabilizable in a neighborhood of the *zero* equilibrium solution in the state space $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$. This is Theorem 3.4.

Such results, Theorem 3.3 and the Theorem 3.4 for the translated Navier-Stokes z-problem (3.20) in feedback form then at once translate into counterpart results of local well-posedness and local interior stabilization of the *original* y-problem (2.1) in a neighborhood of the equilibrium solution y_e, with an explicit finite dimensional feedback control localized on ω whose structure is given in (3.28b). Thus Theorem 3.5 gives the main result of the present paper.

3.2 Introducing the problem of feedback stabilization of the linearized w-problem (2.13) on the complexified $L^q_\sigma(\Omega)$ space.

Preliminaries: In this subsection we take q fixed, $1<q<\infty$ throughout. Accordingly, to streamline the notation in the preceding setting of Section 1, we shall drop the dependence on q of all relevant quantities and thus write $P, A, A_o, \mathcal{A}$ instead of $P_q, A_q, A_{o,q}, \mathcal{A}_q$. We return to the linearized system (2.13).

Moreover, as in [B-T.1], [B-L-T.1], we shall henceforth let $L^q_\sigma(\Omega)$ denote the complexified space $L^q_\sigma(\Omega) + iL^q_\sigma(\Omega)$, whereby then we consider the extension of the linearized problem (2.13) to such complexified space. Thus, henceforth, w will mean $w+i\tilde{w}$, u will mean $u+i\tilde{u}$, w_0 will mean $w_0+i\tilde{w}_0$:

$$\frac{dw}{dt} + \nu A w + A_0 w = P(mu), \quad \text{or} \quad \frac{dw}{dt} - \mathcal{A}w = P(mu),\ w(0)=w_0 \text{ on } L^q_\sigma(\Omega). \tag{3.1}$$

As noted in Theorem 1.2(iii), the Oseen operator $\mathcal{A}$ has compact resolvent on $L^q_\sigma(\Omega)$. It follows that $\mathcal{A}$ has a discreet point spectrum $\sigma(\mathcal{A}) = \sigma_p(\mathcal{A})$ consisting of isolated eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$, which are repeated according to their (finite) algebraic multiplicity ℓ_j. However, since $\mathcal{A}$ generates a C_0 analytic semigroup on $L^q_\sigma(\Omega)$, its eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$ lie in a triangular sector of a well-known type.

The case of interest in stabilization occurs where $\mathcal{A}$ has a finite number, say N, of eigenvalues $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_N$ on a complex half plane $\{\lambda \in \mathbb{C} : Re\ \lambda \geq 0\}$ which we then order according to their real parts, so that

$$\dots \leq Re\ \lambda_{N+1} < 0 \leq Re\ \lambda_N \leq \dots \leq Re\ \lambda_1, \tag{3.2}$$

each $\lambda_i,\ i = 1, \dots, N$, being an unstable eigenvalue repeated according to its geometric multiplicity ℓ_i. Let M denote the number of distinct unstable eigenvalues λ_j of $\mathcal{A}$, so that ℓ_i is equal to the dimension of the eigenspace corresponding to λ_i. Instead, $N = \sum\limits_{i=1}^{M} N_i$ is the sum of the corresponding algebraic multiplicity N_i of λ_i, where N_i is the dimension of the corresponding generalized eigenspace.

There are results in the literature [J-T] that quantify the number of unstable eigenvalues in terms of the system parameters. Denote by P_N and P_N^* the projections given explicitly by [K-1, p 178], [B-T.1], [B-L-T.1]

$$P_N = -\frac{1}{2\pi i}\int_\Gamma (\lambda I - \mathcal{A})^{-1}\, d\lambda : L^q_\sigma(\Omega) \text{ onto } W^u_N \tag{3.3a}$$

$$P_N^* = -\frac{1}{2\pi i}\int_{\bar{\Gamma}} (\lambda I - \mathcal{A}^*)^{-1}\, d\lambda : (L^q_\sigma(\Omega))^* \text{ onto } (W^u_N)^* \subset L^{q'}_\sigma(\Omega), \tag{3.3b}$$

by (A.2c), where Γ (respectively, its conjugate counterpart $\bar{\Gamma}$) is a smooth closed curve that separates the unstable spectrum from the stable spectrum of $\mathcal{A}$ (respectively, $\mathcal{A}^*$). As in [B-L-T.1, Sect 3.4, p 37], following [RT.1],we decompose the space $L^q_\sigma(\Omega)$ into the sum of two complementary subspaces (not necessarily orthogonal):

$$L^q_\sigma(\Omega) = W^u_N \oplus W^s_N; \quad W^u_N \equiv P_N L^q_\sigma(\Omega); \quad W^s_N \equiv (I - P_N) L^q_\sigma(\Omega); \quad \dim\ W^u_N = N \tag{3.4}$$

where each of the spaces W^u_N and W^s_N (which depend on q, but we suppress such dependence) is invariant under $\mathcal{A}$ $(= \mathcal{A}_q)$, and let

$$\mathcal{A}_N^u = P_N \mathcal{A} = \mathcal{A}|_{W_N^u}; \quad \mathcal{A}_N^s = (I - P_N)\mathcal{A} = \mathcal{A}|_{W_N^s} \tag{3.5}$$

be the restrictions of $\mathcal{A}$ to W_N^u and W_N^s respectively. The original point spectrum (eigenvalues) $\{\lambda_j\}_{j=1}^{\infty}$ of $\mathcal{A}$ is then split into two sets

$$\sigma(\mathcal{A}_N^u) = \{\lambda_j\}_{j=1}^{N}; \quad \sigma(\mathcal{A}_N^s) = \{\lambda_j\}_{j=N+1}^{\infty}, \tag{3.6}$$

and W_N^u is the generalized eigenspace of $\mathcal{A}_N^u$ in (3.1). The system (3.1) on $L_\sigma^q(\Omega)$ can accordingly be decomposed as

$$w = w_N + \zeta_N, \quad w_N = P_N w, \quad \zeta_N = (I - P_N)w. \tag{3.7}$$

After applying P_N and $(I - P_N)$ (which commute with $\mathcal{A}$) on (3.1), we obtain via (3.5)

$$\text{on } W_N^u : w_N' - \mathcal{A}_N^u w_N = P_N P(mu); \; w_N(0) = P_N w_0 \tag{3.8a}$$

$$\text{on } W_N^s : \zeta_N' - \mathcal{A}_N^s \zeta_N = (I - P_N)P(mu); \; \zeta_N(0) = (I - P_N)w_0 \tag{3.8b}$$

respectively.

Main Result: We may now state the main feedback stabilization result of the linearized problem (2.13) (=(3.1)) on the complexified space $L_\sigma^q(\Omega)$). The proof is constructive. How to construct the finitely many stabilizing vectors will be established in the proof.

We anticipate the fact (noted in (4.2) and (4.0)) below that, for $1 < p, q < \infty$:

$$W_N^u = \begin{array}{l}\text{space of generalized} \\ \quad\text{eigenfunctions of } \mathcal{A}_q (= \mathcal{A}_N^u) \\ \quad\text{corresponding to its distinct} \\ \quad\text{unstable eigenvalues}\end{array} \subset \begin{cases} \left(L_\sigma^q(\Omega), \mathcal{D}(A_q)\right)_{1-\frac{1}{p},p} \\ \left[\mathcal{D}(A_q), L_\sigma^q(\Omega)\right]_{1-\alpha} = \mathcal{D}(A_q^\alpha), \; 0 \le \alpha \le 1 \end{cases} \subset L_\sigma^q(\Omega). \tag{3.9}$$

3.3 Uniform (exponential) stabilization of the linear finite-dimensional w_N-problem (3.8a) in the space W_N^u by means of a finite-dimensional, explicit, spectral based feedback control localized on ω.

Theorem 3.1. *Let $\lambda_1, ., \lambda_i, ., \lambda_M$ be the unstable distinct eigenvalues of the Oseen operator $\mathcal{A}(= \mathcal{A}_q)$ (see (2.10)) with geometric multiplicity ℓ_i and set $K = \sup \{\ell_j; j = 1, \dots, M\}$. Let ω be an arbitrarily small open portion of the interior with sufficiently smooth boundary $\partial\omega$. Then: Given $\gamma > 0$ arbitrarily large, one can construct suitable interior vectors $[u_1, \dots, u_K]$ in the smooth subspace W_N^u of $L^q_\sigma(\omega), 1 < q < \infty$, and accordingly obtain a K-dimensional interior controller $u = u_N$ acting on ω, of the form*

$$u = \sum_{k=1}^{K} \mu_k(t) u_k, \quad u_k \in W_N^u \subset L^q_\sigma(\Omega), \quad \mu_k(t) = \text{scalar}, \tag{3.10}$$

such that, once inserted in the finite dimensional projected w_N-system in (3.8), yields the system

$$w_N' - \mathcal{A}_N^u w_N = P_N P\left(m\left(\sum_{k=1}^{K} \mu_k(t) u_k\right)\right). \tag{3.11}$$

whose solution then satisfies the estimate

$$\|w_N(t)\|_{L^q_\sigma(\Omega)} + \|u_N(t)\|_{L^q_\sigma(\omega)} \leq C_\gamma e^{-\gamma t} \|P_N w_0\|_{L^q_\sigma(\Omega)}, \ t \geq 0. \tag{3.12}$$

In (3.12) we may replace the $L^q_\sigma(\Omega)$-norm, $1 < q < \infty$, alternatively either with the $\left(L^q_\sigma(\Omega), \mathcal{D}(A_q)\right)_{1-\frac{1}{p},p}$ norm, $1 < q < \infty$; or else with the $\left[\mathcal{D}(A_q), L^q_\sigma(\Omega)\right]_{1-\alpha} = \mathcal{D}(A_q^\alpha)$-norm, $0 \leq \alpha \leq 1$, $1 < q < \infty$. In particular, we also have

$$\|w_N(t)\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} + \|u_N(t)\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \leq C_\gamma e^{-\gamma t} \|P_N w_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)}, \ t \geq 0, \tag{3.13}$$

in the $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$-norm, $1 < q < \infty$, $p < 2q/2q-1$.
[Estimate (3.13) will be invoked in the nonlinear stabilization proof of Section 10]

Moreover, the above control $u = u_N = \sum_{k=1}^{K} \mu_k(t) u_k$, the terms $u_k \in W_N^u$, in (3.10) can be chosen in feedback form: that is, of the form $\mu_k(t) = (w_N(t), p_k)_\omega$ for suitable vectors $p_k \in (W_N^u)^* \subset L^{q'}_\sigma(\Omega)$ depending on γ. Here and henceforth $(v_1, v_2)_\omega = \int_\omega v_1 \cdot \bar{v}_2 \, d\omega$, $v_1 \in W_N^u \subset L^q_\sigma(\Omega)$, $v_2 \in (W_N^u)^* \subset$

$L^{q'}_\sigma(\Omega)$. *In conclusion,* w_N *in (3.11) satisfying (3.12),(3.13) is the solution of the following equation on* W^u_N *(see (3.8)):*

$$w'_N - \mathcal{A}^u_N w_N = P_N P\left(m\left(\sum_{k=1}^{K}(w_N(t),p_k)_\omega u_k\right)\right),\ u_k \in W^u_N \subset L^q_\sigma(\Omega),\ p_k \in (W^u_N)^* \subset L^{q'}_\sigma(\Omega), \quad (3.14a)$$

rewritten as

$$w'_N = \bar{A}^u w_N, \quad w_N(t) = e^{\bar{A}^u t}P_N w_0, \quad w_N(0) = P_N w_0. \quad (3.14b)$$

A proof of Theorem 3.1 is given in Section 6.

3.4 Global well-posedness and uniform exponential stabilization on the linearized w-problem (3.1) in various $L^q_\sigma(\Omega)$-based spaces, by means of the same feedback control obtained for the w_N-problem in Section 3.3

Again, $1 < q < \infty$ throughout this section.

Theorem 3.2. *With reference to the unstable, possibly repeated, eigenvalues* $\{\lambda\}_{j=1}^N$ *in (3.2),* M *of which are distinct, let* $\varepsilon > 0$ *and set* $\gamma_0 = |Re\ \lambda_{N+1}| - \varepsilon$. *Then the same* K*-dimensional feedback controller*

$$u = u_N = \sum_{k=1}^{K}(w_N(t),p_k)_\omega u_k, \quad u_k \in W^u_N \subset L^q_\sigma(\Omega),\ p_k \in (W^u_N)^* \subset L^{q'}_\sigma(\Omega), \quad (3.15)$$

constructed in Theorem 3.1, (3.14a) and yielding estimate (3.12), (3.13) for the finite-dimensional projected w_N*-system (3.8), once inserted, this time in the full linearized* w*-problem (3.1), yields the linearized feedback dynamics* ($w_N = P_N w$)*:*

$$\frac{dw}{dt} = \mathcal{A}w + P\left(m\left(\sum_{k=1}^{K}(P_N w,p_k)_\omega u_k\right)\right) \equiv \mathbb{A}_{_F} w \quad (3.16)$$

where $\mathbb{A}_{_F}$ *is the generator of a s.c. analytic semigroup in the space* $L^q_\sigma(\Omega)$. *Here,* $\mathcal{A} = \mathcal{A}_q$, $P = P_q$, $\mathbb{A}_{_F} = \mathbb{A}_{_{F,q}}$*Moreover, such dynamics* w *in (3.16) (equivalently, such generator* $\mathbb{A}_{_F}$ *in (3.16)) is uniformly stable in the space* $L^q_\sigma(\Omega)$*:*

$$\left\|e^{\mathbb{A}_{_F}t}w_0\right\|_{L^q_\sigma(\Omega)} = \|w(t;w_0)\|_{L^q_\sigma(\Omega)} \leq C_{\gamma_0}e^{-\gamma_0 t}\|w_0\|_{L^q_\sigma(\Omega)}, \quad t \geq 0 \quad (3.17)$$

or for $0 < \theta < 1$ *and* $\delta > 0$ *arbitrarily small*

$$\left\| A_q^{\theta}\, e^{\mathbb{A}_F t} w_0 \right\|_{L^q_\sigma(\Omega)} = \left\| A_q^{\theta}\, w(t;w_0) \right\|_{L^q_\sigma(\Omega)} \leq \begin{cases} C_{\gamma_0,\theta} e^{-\gamma_0 t} \left\| A_q^{\theta}\, w_0 \right\|_{L^q_\sigma(\Omega)}, \quad t \geq 0,\; w_0 \in \mathcal{D}(A_q^{\theta}) & \text{(3.18a)} \\ C_{\gamma_0,\theta,\delta} e^{-\gamma_0 t} \left\| w_0 \right\|_{L^q_\sigma(\Omega)}, \quad t \geq \delta > 0 & \text{(3.18b)} \end{cases}$$

As in the case of Theorem 3.1, we may replace the $L^q_\sigma(\Omega)$-norm in (3.17), $1 < q < \infty$, with the $\left(L^q_\sigma(\Omega), \mathcal{D}(A_q)\right)_{1-\frac{1}{p},p}$-norm, $1 < p, q < \infty$; in particular, with the $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$-norm

$$\left\| e^{\mathbb{A}_F t} w_0 \right\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} = \left\| w(t;w_0) \right\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \leq C_{\gamma_0} e^{-\gamma_0 t} \left\| w_0 \right\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)}, \; t \geq 0 \qquad 1 < q < \infty,\; 1 < p < \frac{2q}{2q-1}. \tag{3.19}$$

A proof of Theorem 3.2 is given in Section 7.

3.5 Local well-posedness and uniform (exponential) null stabilization of the translated nonlinear z-problem (2.7) or (2.12) by means of a finite dimensional explicit, spectral based feedback control localized on ω

Starting with the present section, the nonlinearity of problem (2.1) will impose for $d = 3$ the requirement $q > 3$, see (9.16) below. As our deliberate goal is to obtain the stabilization result in the space $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$ which does not recognize boundary conditions, then the limitation $p < 2q/2q-1$ of this space applies. In conclusion, our well-posedness and stabilization results will hold under the restriction $q > 3, 1 < p < 6/5$ for $d = 3$, and $q > 2, 1 < p < 4/3$ for $d = 2$.

Theorem 3.3. *For $d = 3$, let $1 < p < 6/5$ and $q > 3$, while for $d = 2$, let $1 < p < 4/3$ and $q > 2$. Consider the nonlinear z-problem (2.12) in the following feedback form.*

$$\frac{dz}{dt} - \mathcal{A}_q z + \mathcal{N}_q z = P_q \left(m \left(\sum_{k=1}^{K} (P_N z, p_k)_\omega u_k \right) \right) \tag{3.20}$$

i.e. subject to a feedback control of the same structure as in the linear w-dynamics (3.16), Here p_k, u_k are the same vectors as constructed in Theorem 3.1, and appearing in (3.14) or (3.16). There exists a positive constant $\rho > 0$ such that, if the initial condition z_0 satisfies

$$\| z_0 \|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} < \rho, \tag{3.21}$$

then problem (3.20) defines a unique solution z in the space (see (2.28), (2.30))

$$z \in X^{\infty}_{p,q,\sigma} \equiv L^p(0,\infty;\mathcal{D}(A_q)) \cap W^{1,p}(0,\infty;L^q_\sigma(\Omega)) \tag{3.22}$$

$$\hookrightarrow C([0,\infty);\widetilde{B}^{2-2/p}_{q,p}(\Omega)) \tag{3.23}$$

where $\mathcal{D}(A_q)$ is topologically $W^{2,q}(\Omega) \cap L^q_\sigma(\Omega)$, see (2.8).

A proof of Theorem 3.3 is given in Section 9.

Theorem 3.4. *In the situation of Theorem 3.3, we have that such solution is uniformly stable on the space $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$: there exist constants $\widetilde{\gamma} > 0, M_{\widetilde{\gamma}} \geq 1$, such that said solution satisfies*

$$\|z(t;z_0)\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \leq M_{\widetilde{\gamma}} e^{-\widetilde{\gamma} t} \|z_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)}, \ t \geq 0. \tag{3.24}$$

A proof of Theorem 3.4 is given in Section 10. It will be critically based on the maximal regularity of the semigroup $e^{\mathbb{A}_F t}$ giving the solution of the feedback w-problem (3.16), $\mathbb{A}_F = \mathbb{A}_{F,q}$. Remark 10.1, at the end of Section 10, will provide insight on the relationship between $\widetilde{\gamma}$ in the nonlinear case in (3.24) and γ_0 in the corresponding linear case in (3.17).

3.6 Local well-posedness and uniform (exponential) stabilization of the original nonlinear y-problem (2.1) in a neighborhood of an equilibrium solution y_e, by means of a finite dimensional explicit, spectral based feedback control localized on ω

The result of this subsection is an immediate corollary of section 3.5.

Theorem 3.5. *Let $1 < p < 6/5, q > 3, d = 3$; and $1 < p < 4/3, q > 2, d = 2$. Consider the original N-S problem (2.1). Let y_e be a given equilibrium solution as guaranteed by Theorem 2.1 for the steady state problem (2.2). For a constant $\rho > 0$, let the initial condition y_0 in (2.1d) be in $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$ and satisfy*

$$\mathcal{V}_\rho \equiv \left\{ y_0 \in \widetilde{B}^{2-2/p}_{q,p}(\Omega) : \|y_0 - y_e\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \leq \rho \right\}, \quad \rho > 0. \tag{3.25}$$

If $\rho > 0$ is sufficiently small, then

(i) for each $y_0 \in \mathcal{V}_\rho$, there exists an interior finite dimensional feedback controller

$$u = F(y - y_e) = \sum_{k=1}^{K} (P_N(y - y_e), p_k)_\omega u_k \tag{3.26}$$

that is, of the same structure as in the translated N-S z-problem (3.20), with the same vectors p_k, u_k in (3.14) or (3.16), such that the closed loop problem corresponding to (2.1)

$$\begin{cases} y_t - \nu\Delta y + (y\cdot\nabla)y + \nabla\pi = m(F(y-y_e)) + f(x) & \text{in } Q \quad (3.27a)\\ div\ y = 0 & \text{in } Q \quad (3.27b)\\ y = 0 & \text{on } \Sigma \quad (3.27c)\\ y|_{t=0} = y_0 & \text{in } \Omega \quad (3.27d) \end{cases}$$

rewritten abstractly after application of the Helmholtz projection P_q as

$$y_t + \nu A_q y + \mathcal{N}_q y = P_q\Big[m\big(F(y-y_e)\big) + f(x)\Big] \tag{3.28a}$$

$$= P_q\left[m\left(\sum_{k=1}^{K}\big(P_N(y-y_e), p_k\big)_\omega u_k\right) + f(x)\right] \tag{3.28b}$$

$$y(0) = y_0 \in \widetilde{B}^{2-2/p}_{q,p}(\Omega) \tag{3.28c}$$

has a unique solution $y \in C\big([0,\infty); \widetilde{B}^{2-2/p}_{q,p}(\Omega)\big)$.

(ii) Moreover, such solution exponentially stabilizes the equilibrium solution y_e in the space $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$: there exist constants $\widetilde{\gamma} > 0$ and $M_{\widetilde{\gamma}} \geq 1$ such that said solution satisfies

$$\|y(t) - y_e\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \leq M_{\widetilde{\gamma}} e^{-\widetilde{\gamma}t}\|y_0 - y_e\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)}, \quad t \geq 0,\ y_0 \in \mathcal{V}_\rho. \tag{3.29}$$

Once the neighborhood $\mathcal{V}_\rho$ is obtained to ensure the well-posedness, then the values of $M_{\widetilde{\gamma}}$ and $\widetilde{\gamma}$ do not depend on $\mathcal{V}_\rho$ and $\widetilde{\gamma}$ can be made arbitrarily large through a suitable selection of the feedback operator F.

See Remark 10.1 comparing $\widetilde{\gamma}$ in (3.29) with γ_0 in (3.17).

3.7 Results on the real space setting

Here we shall complement the results of Theorems 3.1 through 3.5 by giving their version in the real space setting. We shall quote from [B-T.1]. In the complexified setting $L^q_\sigma(\Omega) + iL^q_\sigma(\Omega)$ we have that the complex unstable subspace W^u_N is,

$$W^u_N = W^1_N + iW^2_N \tag{3.30}$$

$$= \text{space of generalized eigenfunctions } \{\phi_j\}_{j=1}^N \text{ of the operator } \mathcal{A}_q (= \mathcal{A}^u_q) \text{ corresponding to its } N \text{ unstable eigenvalues.} \tag{3.31}$$

Set $\phi_j = \phi_j^1 + i\phi_j^2$ with ϕ_j^1, ϕ_j^2 real. Then:

$$W_N^1 = Re\ W_N^u = \text{span}\{\phi_j^1\}_{j=1}^N; \quad W_N^2 = Im\ W_N^u = \text{span}\{\phi_j^2\}_{j=1}^N. \tag{3.32}$$

The stabilizing vectors $p_k, u_k, k = 1, \dots, K$ are complex valued and belong to W_N^u.

The complex-valued uniformly stable linear w-system in (3.16) with K complex valued stabilizing vectors admits the following real-valued uniformly stable counterpart

$$\frac{dw}{dt} = \mathcal{A}_q w + P_q\left(m\left(\sum_{k=1}^{K} \text{Re}\ (w_N(t), p_k)_\omega\ \text{Re}\ u_k - \sum_{k=1}^{K} \text{Im}\ \ (w_N(t), p_k)_\omega\ \text{Im}\ u_k\right)\right) \tag{3.33}$$

with $2K \leq N$ real stabilizing vectors, see [B-T.1, Eq 3.52a, p 1472].If $K = \sup\ \{\ell_i, i = 1, \dots, M\}$ is achieved for a real eigenvalue λ_i (respectively, a complex eigenvalue λ_i), then the *effective* number of stabilizing controllers is $K \leq N$, as the generalized functions are then real, since y_e is real; respectively, $2K \leq N$, for, in this case, the complex conjugate eigenvalue $\bar{\lambda}_j$ contributes an equal number of components in terms of generalized eigenfunctions $\phi_{\bar{\lambda}_j} = \bar{\phi}_{\lambda_j}$. In all cases, the actual (*effective*) upper bound $2K$ is $2K \leq N$. For instance, if all unstable eigenvalues were real and simple then $K = 1$, and only one stabilizing controller is actually needed.

Similarly, the complex-valued locally (near y_e) uniformly stable nonlinear y-system (3.28) with K complex-valued stabilizing vectors admits the following real-valued locally uniformly stable counterpart

$$\frac{dy}{dt} - \nu A_q y + \mathcal{N}_q y = P_q\left(m\left(\sum_{k=1}^{K} \text{Re}\ (y - y_e, p_k)_\omega\ \text{Re}\ u_k - \sum_{k=1}^{K} \text{Im}\ (y - y_e, p_k)_\omega\ \text{Im}\ u_k\right)\right) \tag{3.34}$$

with $2K \leq N$ real stabilizing vectors, see [B-L-T.1, p 43].

4 Algebraic rank condition for the w_N-dynamics in (3.8a) under the (preliminary) Finite-Dimensional Spectral Assumption (FDSA)

Preliminaries: For $i = 1, \dots, M$, we now denote by $\{\varphi_{ij}\}_{j=1}^{\ell_i}$, $\{\varphi^*_{ij}\}_{j=1}^{\ell_i}$ the normalized linearly independent eigenfunctions (on $L^q_\sigma(\Omega)$ and $(L^q_\sigma(\Omega))' = L^{q'}_\sigma(\Omega)$, respectively $1/_q + 1/_{q'} = 1$ invoking property (A.2b) of Appendix A) corresponding to the unstable distinct eigenvalues $\lambda_1, \dots, \lambda_M$ of $\mathcal{A}$ and $\bar{\lambda}_1, \dots, \bar{\lambda}_M$ of $\mathcal{A}^*$, respectively:

$$\mathcal{A}\phi_{ij} = \lambda_i \phi_{ij} \in \mathcal{D}(\mathcal{A}_q) = W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega) \cap L^q_\sigma(\Omega) \in L^q(\Omega) \tag{4.1a}$$

$$\mathcal{A}^*\phi^*_{ij} = \bar{\lambda}_i \phi^*_{ij} \in \mathcal{D}(\mathcal{A}^*_q) = W^{2,q'}(\Omega) \cap W^{1,q'}_0(\Omega) \cap L^{q'}_\sigma(\Omega) \in L^{q'}(\Omega) \tag{4.1b}$$

FDSA: We henceforth assume in this section that for each of the distinct eigenvalues $\lambda_1, \dots, \lambda_M$ of $\mathcal{A}$, algebraic and geometric multiplicity coincide:

$$W^u_{N,i} \equiv P_{N,i} L^q_\sigma(\Omega) = \text{span}\{\varphi_{ij}\}_{j=1}^{\ell_i}; \quad (W^u_{N,i})^* \equiv P^*_{N,i}(L^q_\sigma(\Omega))^* = \text{span}\{\varphi^*_{ij}\}_{j=1}^{\ell_i}; \tag{4.2}$$

Here $P_{N,i}, P^*_{N,i}$ are the projections corresponding to the eigenvalues λ_i and $\bar{\lambda}_i$, respectively. For instance, $P_{N,i}$ is given by an integral such as the one on the RHS of (3.3a), where now Γ is a closed smooth curve encircling the eigenvalue λ_i and no other. Similarly $P^*_{N,i}$. The space $W^u_{N,i}$ = range of $P_{N,i}$ is the algebraic eigenspace of the eigenvalues λ_i, and $\ell_i = \dim W^u_{N,i}$ is the algebraic multiplicity of λ_i, so that $\ell_1 + \ell_2 + \dots + \ell_M = N$. As a consequence of the FSDA, we obtain

$$W^u_N \equiv P_N L^q_\sigma(\Omega) = \text{span}\{\varphi_{ij}\}_{i=1,j=1}^{M \ \ \ell_i}; \quad (W^u_N)^* \equiv P^*_N(L^q_\sigma(\Omega))^* = \text{span}\{\varphi^*_{ij}\}_{i=1,j=1}^{M \ \ \ell_i}; \tag{4.3}$$

Without the FDSA, W^u_N is the span of the generalized eigenfunctions of $\mathcal{A}$, corresponding to its unstable distinct eigenvalues $\{\lambda_j\}_{j=1}^M$; and similarly for $(W^u_N)^*$ (see the subsequent section). In other words, the FDSA says that the restriction $\mathcal{A}^u_N$ in (3.5) is *diagonalizable* or that $\mathcal{A}^u_N$ is *semisimple* on W^u_N in the terminology of [K-1, p 43]. Under the FDSA, any vector $w \in W^u_N$ admits the following unique expansion [K-1, p 12, Eq (2.16)], [B-T.1, p 1453], in terms of the basis $\{\varphi_{ij}\}_{i=1,j=1}^{M \ \ \ell_i}$ in $L^q_\sigma(\Omega)$ and its adjoint basis [K-1, p 12] $\{\varphi^*_{ij}\}_{i=1,j=1}^{M \ \ \ell_i}$ in $(L^q_\sigma(\Omega))^*$:

$$W_N^u \ni w = \sum_{i,j}^{M,\ell_i} (w, \phi_{ij}^*)\phi_{ij}; \quad (\phi_{ij}, \phi_{hk}^*) = \begin{cases} 1 & \text{if } i = h, j = k \\ 0 & \text{otherwise} \end{cases} \tag{4.4}$$

that is, the system consisting of $\{\phi_{ij}\}$ and $\{\phi_{ij}^*\}$, $i = 1, \dots, M,\ j = 1, \dots, \ell_i$, can be chosen to form bi-orthogonal sequences. Here (,) denotes the scalar product between W_N^u and $(W_N^u)^*$ [K-1, p 12]. i.e. ultimately, the duality pairing in Ω between $L_\sigma^q(\Omega)$ and $(L_\sigma^q(\Omega))^*$. Next, we return to the w_N-dynamics in (3.8a), rewritten here for convenience

$$\text{on } W_N^u : w_N' - \mathcal{A}_N^u w_N = P_N P(mu);\ w_N(0) = P_N w_0 \tag{4.5}$$

The term $P_N P(mu)$ expressed in terms of adjoint bases. Next, let $mu \in L^q(\omega)$ where $q > 1$. Here below we compute the RHS of the term $P_N P(mu)$ via the adjoint bases expansion in (4.4), where we notice that $P^* P_N^* \phi_{ij}^* = \phi_{ij}^*$ because $\phi_{ij}^* \in \mathcal{D}(\mathcal{A}^*)$, so that ϕ_{ij}^* is invariant under the projections P^* and P_N^*. With $(f, g)_\omega = \int_\omega f \bar{g}\, d\omega$, we obtain

$$W_N^u \ni P_N P(mu) = \sum_{i,j=1}^{M,\ell_i} (P_N P(mu), \phi_{ij}^*)\phi_{ij} = \sum_{i,j=1}^{M,\ell_i} (mu, \phi_{ij}^*)\phi_{ij} = \sum_{i,j=1}^{M,\ell_i} (u, \phi_{ij}^*)_\omega \phi_{ij}, \tag{4.6}$$

so that the dynamics (4.5) on W_N^u becomes

$$\text{on } W_N^u : w_N' - \mathcal{A}_N^u w_N = \sum_{i,j=1}^{M,\ell_i} (u, \phi_{ij}^*)_\omega \phi_{ij}. \tag{4.7}$$

Selection of the scalar interior control function u in finite dimensional separated form

(**with respect to K coordinates**) Next, we select the control u of the form given in (3.10)

$$u = \sum_{k=1}^{K} \mu_k(t) u_k, \quad u_k \in W_N^u \subset L_\sigma^q(\Omega), \quad \mu_k(t) = \text{scalar} \tag{4.8}$$

so that the term in (4.6) in W_N^u specializes to

$$W_N^u \ni P_N P(mu) = \sum_{i,j=1}^{M,\ell_i} \left\{ \sum_{k=1}^{K} (u_k, \phi_{ij}^*)_\omega \mu_k(t) \right\} \phi_{ij}. \tag{4.9}$$

Substituting (4.9) on the RHS of (4.5), we finally obtain

$$\text{on } W_N^u : w_N' - \mathcal{A}_N^u w_N = \sum_{i,j=1}^{M,\ell_i} \left\{ \sum_{k=1}^{K} (u_k, \phi_{ij}^*)_\omega \mu_k(t) \right\} \phi_{ij}. \tag{4.10}$$

The dynamics (4.10) in coordinate form on W_N^u. Our next goal is to express the finite dimensional dynamics (4.10) on the N-dimensional space W_N^u in a component-wise form. To this end, we introduce the following ordered bases β_i and β of length ℓ_i and N respectively:

$$\begin{aligned} &\beta_i = [\phi_{i1}, \dots, \phi_{i\ell_i}] : \quad \text{basis on } W_{N,i}^u \\ &\beta = \beta_1 \cup \beta_2 \cup \dots \cup \beta_M = [\phi_{11}, \dots, \phi_{1\ell_1}, \phi_{21}, \dots, \phi_{2\ell_2}, \dots, \phi_{M1}, \dots, \phi_{M\ell_M}] : \quad \text{basis on } W_N^u. \end{aligned} \tag{4.11}$$

Thus, we can represent the N-dimensional vector $w_N \in W_N^u$ as column vector $\hat{w}_N = [w_N]_\beta$ as,

$$w_N = \sum_{i,j=1}^{M,\ell_i} w_N^{ij} \phi_{ij}; \text{ and set } \hat{w}_N = \text{col}[w_N^{1,1}, \dots, w_N^{1,\ell_1}, \dots, w_N^{i,1}, \dots, w_N^{i,\ell_i}, \dots, w_N^{M,1}, \dots, w_N^{M,\ell_M}]. \tag{4.12}$$

Remark 4.1. The eigenfunction ϕ_{ij} belongs to $L_\sigma^q(\Omega)$ as well as to $\mathcal{D}(A_q) = \mathcal{D}(\mathcal{A}_q)$. Thus, by real/complex interpolation, see (2.16)/(2.21) they also belong to $\big(L_\sigma^q(\Omega), \mathcal{D}(A_q)\big)_{1-\frac{1}{p},p}$ as well as to $\big[\mathcal{D}(A_q), L_\sigma^q(\Omega)\big]_{1-\alpha} = \mathcal{D}(A_q^\alpha)$, $0 \leq \alpha \leq 1$; in particular, $\phi_{ij} \in \widetilde{B}_{q,p}^{2-2/p}(\Omega)$. See (B.11) or (B.12) in Appendix B. Thus, exponential decay in $\mathbb{C}^N$ of the $\mathbb{C}^N$-vector $\hat{w}_N$ translates at once into exponential decay with the same rate in any of the spaces $L_\sigma^q(\Omega)$, $\big(L_\sigma^q(\Omega), \mathcal{D}(A_q)\big)_{1-\frac{1}{p},p}$, $\mathcal{D}(A_q^\alpha)$, in particular, $\widetilde{B}_{q,p}^{2-2/p}(\Omega)$ for the vector w_N, views as a vector on any one of these spaces. This remark applies to $w_N(t)$ and $u_N(t)$ in Theorem 3.1, Eqts (3.12), (3.13) as well as Theorem 3.2, Eqts (3.17)- (3.19).

Lemma 4.1. *In $\mathbb{C}^N$, with respect to the ordered basis β : $\{\varphi_{ij}\}_{i=1,j=1}^{M \ \ \ell_i}$ of normalized eigenfunctions of $\mathcal{A}_N^u$, we may rewrite system (4.10) = (4.12) = (3.8a) as*

$$(\hat{w}_N)' - \Lambda \hat{w}_N = U \hat{\mu}_K \tag{4.13}$$

where

$$\Lambda = \begin{bmatrix} \lambda_1 I_1 & & & 0 \\ & \lambda_2 I_2 & & \\ & & \ddots & \\ 0 & & & \lambda_M I_M \end{bmatrix} : N \times N, \quad I_i : \ell_i \times \ell_i \ \textit{identity} \tag{4.14}$$

$$U_i = \begin{bmatrix} (u_1, \phi^*_{i1})_\omega & \dots & (u_K, \phi^*_{i1})_\omega \\ (u_1, \phi^*_{i2})_\omega & \dots & (u_K, \phi^*_{i2})_\omega \\ \vdots & \ddots & \vdots \\ (u_1, \phi^*_{i\ell_i})_\omega & \dots & (u_K, \phi^*_{i\ell_i})_\omega \end{bmatrix} : \ell_i \times K; \quad U = \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_M \end{bmatrix} : N \times K; \quad \hat{\mu}_K = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_K \end{bmatrix} : K \times 1; \quad (4.15)$$

where $(f, g)_\omega = \int_\omega f\bar{g}\, d\omega$ *and we take* $K \geq \ell_i,\ i = 1, \dots, M$*. Thus (4.13) gives the dynamics on* W^u_N *as a linear* N*-dimensional ordinary differential equation in coordinate form in* $\mathbb{C}^N$*.*

Proof. Recalling the basis β_i and the definitions of U_i in (4.15), we can rewrite the term in (4.9) with respect to this basis as

$$[P_N P(mu)]_{\beta_i} = U_i \hat{\mu}_K : \ell_i \times 1; \tag{4.16}$$

Then with respect to the basis β in (4.11) and recalling the definition U in (4.15), we can rewrite the term (4.9) with respect to this basis as

$$[P_N P(mu)]_\beta = \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_M \end{bmatrix} \hat{\mu}_K = \begin{bmatrix} U_1\hat{\mu}_K \\ U_2\hat{\mu}_K \\ \vdots \\ U_M\hat{\mu}_K \end{bmatrix} = U\hat{\mu}_K : N \times 1 \tag{4.17}$$

Finally, clearly $\mathcal{A}^u_N$ becomes the diagonal matrix Λ in (4.14) with respect to the basis β, recalling its eigenvalues in (4.1). □

The following is the main result of the present section.

Theorem 4.2. *Assume the FDSA. It is possible to select vectors* $u_1, \dots, u_K \in L^q_\sigma(\omega), q > 1, K = \sup\{\ell_i : i = 1, \dots, M\}$*, such that the matrix* U_i *of size* $\ell_i \times K$ *in (4.15) satisfies*

$$rank[U_i] = full = \ell_i \text{ or } rank \begin{bmatrix} (u_1, \phi^*_{i1})_\omega & \dots & (u_K, \phi^*_{i1})_\omega \\ (u_1, \phi^*_{i2})_\omega & \dots & (u_K, \phi^*_{i2})_\omega \\ \vdots & & \vdots \\ (u_1, \phi^*_{i\ell_i})_\omega & \dots & (u_K, \phi^*_{i\ell_i})_\omega \end{bmatrix} = \ell_i;\ \ell_i \times K \text{ for each } i = 1, \dots, M. \tag{4.18}$$

Proof. Step 1. By selection, see (4.1) and statement preceding it, the set of vectors $\phi^*_{i1}, \dots, \phi^*_{i\ell_i}$ is linearly independent in $L^{q'}_\sigma(\Omega)$, q' is the Hölder conjugate of q, $^1/_q + {}^1/_{q'} = 1$, for each $i = 1, \dots, M$. Next, if the set of vectors $\{\phi^*_{i1}, \dots, \phi^*_{i\ell_i}\}$ were linearly independent in $L^{q'}_\sigma(\Omega)$, $i = 1, \dots, M$, the desired conclusion (4.18) for the matrix U_i to be full rank, would follow for infinitely many choices of the vectors $u_1, \dots, u_K \in L^q_\sigma(\Omega)$.

Claim: The set $\{\phi^*_{i1}, \dots, \phi^*_{i\ell_i}\}$ is linearly independent on $L^{q'}_\sigma(\omega)$, for each $i = 1, \dots, M$.

The proof will critically depend on a unique continuation result [RT.3] see also [B-T.1, Lemma 3.7 p1466]. By contradiction, let us assume that the vectors $\{\phi^*_{i1}, \dots, \phi^*_{i\ell_i}\} \in L^{q'}_\sigma(\Omega)$ are instead linearly dependent, so that

$$\phi^*_{i\ell_i} = \sum_{j=1}^{\ell_i - 1} \alpha_j \phi^*_{i\ell_j} \text{ in } L^{q'}_\sigma(\Omega) \tag{4.19}$$

with constants α_j not all zero. We shall then conclude by [B-T.1, Lemma 3.7] and [RT.3] below, that in fact $\phi^*_{i\ell_i} \equiv 0$ on all of Ω as well, thereby making the system $\{\phi^*_{ij}, j = 1, \dots, \ell_i\}$ linearly dependent on Ω, a contradiction. To this end, define the following function (depending on i) in $L^{q'}_\sigma(\Omega)$

$$\phi^* = \left[\sum_{j=1}^{\ell_i - 1} \alpha_j \phi^*_{i\ell_j} - \phi^*_{i\ell_i} \right] \in L^{q'}_\sigma(\Omega), \quad i = 1, \dots, M. \tag{4.20}$$

As each ϕ^*_{ij} is an eigenvalue of $\mathcal{A}^*$ (or $(\mathcal{A}^u_N)^*$) corresponding to the eigenvalue $\bar{\lambda}_i$, see (4.1), so is the linear combination ϕ^*. This property, along with (4.19) yields that ϕ^* satisfies the following eigenvalue problem for the operator $\mathcal{A}^*$ (or $(\mathcal{A}^u_N)^*$):

$$\mathcal{A}^* \phi^* = \bar{\lambda} \phi^*, \text{ div } \phi^* = 0 \text{ in } \Omega; \quad \phi^* = 0 \text{ in } \omega, \text{ by (4.19)}. \tag{4.21}$$

But the linear combination ϕ^* in (4.20) of the eigenfunctions $\phi^*_{ij} \in \mathcal{D}(\mathcal{A}^*)$ satisfies itself the Dirichlet B.C $\phi^*|_{\partial\Omega} = 0$. Thus the explicit PDE version of problem (4.21) is

$$
\begin{cases}
-\nu\Delta\phi^* - (L_e)^*\phi^* + \nabla p^* = \bar{\lambda}_i \phi^* \text{ in } \Omega; & \text{(4.22a)} \\
\text{div } \phi^* = 0 \text{ in } \Omega; & \text{(4.22b)} \\
\phi^*|_{\partial\Omega} = 0; \quad \phi^* = 0 \text{ in } \omega; & \text{(4.22c)}
\end{cases}
$$

$$
\phi^* \in \mathcal{D}(\mathcal{A}^*); \quad (L_e)^*\phi^* = (y_e.\nabla)\phi^* + (\phi^*.\nabla)^* y_e, \tag{4.23}
$$

where $(f.\nabla)^* y_e$ is a d-vector whose i^{th} component is $\sum_{j=1}^{d}(D_i y_{e_j}) f_j$ [B-L-T.1, p 55].

Step 2. The critical point is now that the over-determined problem (4.22) implies the following unique continuation result.

$$
\phi^* = 0 \text{ in } L^{q'}_\sigma(\Omega); \text{ or by (4.20) } \phi^*_{i\ell_i} = \alpha_1\phi^*_{i1} + \alpha_2\phi^*_{i2} + \cdots + \alpha_{\ell_i-1}\phi^*_{i\ell_i-1} \text{ in } L^{q'}_\sigma(\Omega) \tag{4.24}
$$

i.e. the set $\{\phi^*_{i1}, \ldots, \phi^*_{i\ell_i}\}$ in linearly dependent on $L^{q'}_\sigma(\Omega)$. But this is false, by the very selection of such eigenvectors, see (4.1) and statement preceding it. Thus, the condition (4.24) cannot hold.

The required unique continuation result is established in [B-T.1, Lemma 3.7] or [RT.3]. The original proof is done in the Hilbert setting but we may invoke the same result because ϕ^* has more regularity and integrability than required since ϕ^* is an eigenfunction of $\mathcal{A}^*$. Thus the *claim* is established. In conclusion: it is possible to select, in infinitely many ways, interior functions $u_1, \ldots, u_K \in L^q_\sigma(\Omega)$ such that the algebraic full rank condition (4.18) holds true for each $i = 1, \ldots, M$. □

5 Algebraic rank conditions for the dynamics w_N in (3.8a) in the general case

In the present section we dispense with the FDSA (4.2). More precisely, we shall obtain Theorem 3.1 without assuming the FDSA (4.2). Thus now

$$W_N^u = \text{space of generalized eigenfunctions of } \mathcal{A}_q (= \mathcal{A}_N^u) \\ \text{corresponding to its distinct unstable eigenvalues.} \tag{4.0}$$

Warning: In this section (only) we shall denote by ℓ_i the geometric multiplicity of the eigenvalue λ_i and by N_i its algebraic multiplicity.

Step 1: To treat this computationally more complicated case we shall, essentially invoke the classical result on controllability of a finite-dimensional, time-invariant system $\{\mathbb{A}, \mathbb{B}\}, \mathbb{A} : N \times N, \mathbb{B} : N \times p$ where $\mathbb{A}$ is given in Jordan form J. Let again $\lambda_1, \lambda_2, \dots, \lambda_M$ be the distinct eigenvalues of $\mathbb{A} = J$. Let $\mathbb{A}_i$ denote all the Jordan blocks associated with the eigenvalue λ_i; let ℓ_i be the number of Jordan blocks of $\mathbb{A}$ (i.e the number of linearly independent eigenvectors associated with the eigenvalue λ_i). Let $\mathbb{A}_{ij}$ be j^{th} Jordan block in $\mathbb{A}_i$ corresponding to a Jordan cycle of length N_j^i. That is:

$$\mathbb{A} = \text{diag}\{\mathbb{A}_1, \mathbb{A}_2, \dots, \mathbb{A}_M\}; \quad \mathbb{A}_i = \text{diag}\{\mathbb{A}_{i1}, \mathbb{A}_{i2}, \dots, \mathbb{A}_{i\ell_i}\} \tag{5.1}$$

Partition the matrix $\mathbb{B}$ accordingly:

$$\underset{(N\times N)}{\mathbb{A}} = \begin{bmatrix} \mathbb{A}_1 & & & 0 \\ & \mathbb{A}_2 & & \\ & & \ddots & \\ 0 & & & \mathbb{A}_M \end{bmatrix}; \quad \underset{(N\times p)}{\mathbb{B}} = \begin{bmatrix} \mathbb{B}_1 \\ \mathbb{B}_2 \\ \vdots \\ \mathbb{B}_M \end{bmatrix} \tag{5.2}$$

$$\underset{(N_i\times N_i)}{\mathbb{A}_i} = \begin{bmatrix} \mathbb{A}_{i1} & & & 0 \\ & \mathbb{A}_{i2} & & \\ & & \ddots & \\ 0 & & & \mathbb{A}_{i\ell_i} \end{bmatrix}; \quad \underset{(N_i\times p)}{\mathbb{B}_i} = \begin{bmatrix} \mathbb{B}_{i1} \\ \mathbb{B}_{i2} \\ \vdots \\ \mathbb{B}_{i\ell_i} \end{bmatrix} \tag{5.3}$$

$$\underset{(N^i_j \times N^i_j)}{\mathbb{A}_{ij}} = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \lambda_i & 1 & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix}; \quad \underset{(N^i_j \times p)}{\mathbb{B}_{ij}} = \begin{bmatrix} b_{1ij} \\ b_{2ij} \\ \vdots \\ b_{Lij} \end{bmatrix} \tag{5.4}$$

If E_{λ_i} and K_{λ_i} denote the eigenspace and the generalized eigenspace associated with the eigenvalue λ_i, $i = 1, \dots, M$, then $\dim E_{\lambda_i} = \ell_i = \#$ of Jordan blocks in $\mathbb{A}_i$, $\dim K_{\lambda_i} = N_i$, $N^i_j =$ length of j$^{\text{th}}$-cycle associated with λ_i; $j = 1, \dots, \ell_i$. We have $\dim W^u_N = N = \sum_{i=1}^{M} N_i = \sum_{i=1}^{M} \sum_{j=1}^{\ell_i} N^i_j$. In (5.4), the last row of $\mathbb{B}_{ij}$ is denoted by b_{Lij}. The following result is classical [Chen.2, p 165].

Theorem 5.1. *[L-T.2, Theorem 3.1] The pair $\{J, \mathbb{B}\}$, $J : N \times N$, Jordan form, $\mathbb{B} : N \times p$ is controllable if and only if, for each $i = 1, \dots, M$ (that is for each distinct eigenvalue) the rows of the $\ell_i \times p$ matrix constructed with all "last" rows $b_{Li1}, \dots, b_{Li\ell_i}$*

$$\mathbb{B}^L_i = \begin{bmatrix} \mathbb{B}_{Li1} \\ \mathbb{B}_{Li2} \\ \vdots \\ \mathbb{B}_{Li\ell_i} \end{bmatrix} : \ell_i \times p \tag{5.5}$$

are linearly independent (on the field of complex numbers). [A direct proof uses Hautus criterion for controllability [Chen.2]]

We next apply the above Theorem 5.1 to the w_N-problem (3.8a) and (4.5). To this end, we select a Jordan basis β_i for the operator $(\mathcal{A}^u_N)_i$ on $W^u_{N,i}$ given by

Jordan Basis:

$$\beta_i = \left\{ e^1_1(\lambda_i), e^1_2(\lambda_i), \dots, e^1_{N^i_1}(\lambda_i) \vdots e^2_1(\lambda_i), e^2_2(\lambda_i), \dots, e^2_{N^i_2}(\lambda_i) \vdots \dots \vdots e^{\ell_i}_1(\lambda_i), e^{\ell_i}_2(\lambda_i), \dots, e^{\ell_i}_{N^i_{\ell_i}}(\lambda_i) \right\} \tag{5.6a}$$

Here the first vectors of each cycle: $e^1_1(\lambda_i), e^2_1(\lambda_i), \dots, e^{\ell_i}_1(\lambda_i)$ are eigenvectors of $(\mathcal{A}^u_N)_i$ corresponding to the eigenvalue λ_i, while the remaining vectors in β_i are corresponding generalized eigenvectors. Thus, in the notation (4.1), we have

$$\phi_{i1} = e^1_1(\lambda_i); \ \phi_{i2} = e^2_1(\lambda_i); \dots; \ \phi_{i\ell_i} = e^{\ell_i}_1(\lambda_i) \tag{5.6b}$$

Next, we can choose a bi-orthogonal basis β_i^* of $((\mathcal{A}_N^u)^*)_i$ corresponding to its eigenvalue $\bar{\lambda}_i$ given by

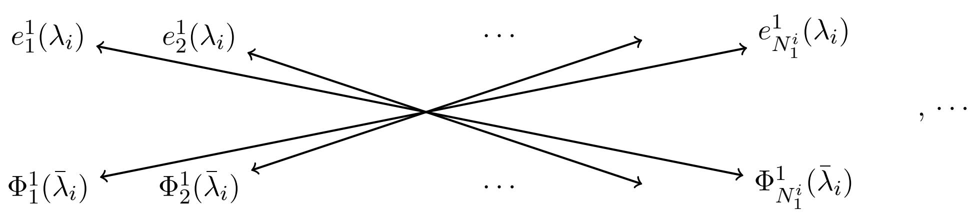
Bi-orthogonal Basis:

$$\beta_i^* = \left\{\Phi_1^1(\bar{\lambda}_i), \Phi_2^1(\bar{\lambda}_i), \dots, \Phi_{N_1^i}^1(\bar{\lambda}_i) \vdots \Phi_1^2(\bar{\lambda}_i), \Phi_2^2(\bar{\lambda}_i), \dots, \Phi_{N_2^i}^2(\bar{\lambda}_i) \vdots \dots \vdots \Phi_1^{\ell_i}(\bar{\lambda}_i), \Phi_2^{\ell_i}(\bar{\lambda}_i), \dots, \Phi_{N_{\ell_i}^i}^{\ell_i}(\bar{\lambda}_i)\right\} \tag{5.7a}$$

Thus, in the notation (4.1), we have

$$\phi_{i1}^* = \Phi_1^1(\bar{\lambda}_i); \ \phi_{i2}^* = \Phi_1^2(\bar{\lambda}_i); \dots; \ \phi_{i\ell_i}^* = \Phi_1^{\ell_i}(\bar{\lambda}_i) \tag{5.7b}$$

In the bi-orthogonality relationship between the vectors in (5.6) and those in (5.7), the first eigenvector $e_1^1(\lambda_i)$ of the first cycle in β_i is associated with the last generalized eigenvector $\Phi_{N_1^i}^1(\bar{\lambda}_i)$ of the first cycle in β_i^*; etc, the last generalized eigenvector $e_{N_1^i}^1(\lambda_i)$ of the first cycle in β_i is associated with the first eigenvector $\Phi_1^1(\bar{\lambda}_i)$ of the first cycle in β_i^*; etc.



(5.8)

Figure 1: Relation between the generalized eigenvectors of $\mathcal{A}_N^u$ and $(\mathcal{A}_N^u)^*$

Thus, if $f \in W_{N,i}^u$, the following expression holds true:

$$\begin{aligned} f = (f, \Phi_{N_1^i}^1(\bar{\lambda}_i))e_1^1(\lambda_i) + \dots + (f, \Phi_1^1(\bar{\lambda}_i))e_{N_1^i}^1(\lambda_i) \\ + \dots + (f, \Phi_{N_{\ell_i}^i}^{\ell_i}(\bar{\lambda}_i))e_1^{\ell_i}(\lambda_i) \quad + \dots + (f, \Phi_1^{\ell_i}(\bar{\lambda}_i))e_{N_{\ell_i}^i}^{\ell_i}(\lambda_i). \end{aligned} \tag{5.9}$$

This expansion is the counterpart of $\sum_{j=1}^{\ell_i}(w, \phi_{ij}^*)\phi_{ij} \in W_{N,i}^u$ in (4.4) under the FDSA. Next, we apply (5.9) to $f = P_N P(mu)$. More specifically, we shall write the vector representation of $P_N P(mu)$ with respect to the basis β_i in (5.6a), and moreover, in line with Theorem 5.1, we shall explicitly note only the coordinates corresponding to the vectors $e^1_{N^i_1}(\lambda_i), e^2_{N^i_2}(\lambda_i), \dots, e^{\ell_i}_{N^i_{\ell_i}}(\lambda_i)$, each being the last vector of each cycle in (5.6a).

$$[P_N P(mu)]_{\beta_i} = \begin{bmatrix} \times\times\times \\ \Big(u, \Phi^1_1(\bar\lambda_i)\Big)_\omega \\ \cdots\cdots\cdots\cdots \\ \times\times\times \\ \Big(u, \Phi^2_1(\bar\lambda_i)\Big)_\omega \\ \cdots\cdots\cdots\cdots \\ \times\times\times \\ \Big(u, \Phi^{\ell_i}_1(\bar\lambda_i)\Big)_\omega \end{bmatrix} \begin{matrix} \\ \leftarrow \text{ last row of the } 1^{st} \text{ cycle} \\ \\ \\ \leftarrow \text{ last row of the } 2^{nd} \text{ cycle} \\ \\ \\ \leftarrow \text{ last row of the } \ell_i^{th} \text{ cycle} \end{matrix} \tag{5.10}$$

The symbol $\times\times\times$ corresponds to terms which we do not care about. In fact, to exemplify, since $P^* P^*_N \Phi^1_1(\bar\lambda_i) = \Phi^1_1(\bar\lambda_i)$ see above (4.6)

$$\Big(P_N P(mu), \Phi^1_1(\bar\lambda_i)\Big)_\Omega = \big(mu, \Phi^1_1(\bar\lambda_i)\big)_\Omega = \big(u, \Phi^1_1(\bar\lambda_i)\big)_\omega \tag{5.11}$$

This is the relevant counterpart of expansion $P_N P(mu) = \sum_{i,j=1}^{M,\ell_i} (u, \Phi^*_{ij})_\omega \Phi_{ij}$ in (4.6) under the FDSA. Notice that (5.10) involves only the eigenvectors $\Phi^1_1(\bar\lambda_i), \Phi^2_1(\bar\lambda_i), \dots, \Phi^{\ell_i}_1(\bar\lambda_i)$ of $(\mathcal{A}^u_N)^*$ corresponding to the eigenvalue $\bar\lambda_i$. Next, recalling (3.10): $u = \sum_{k=1}^{K} \mu_k(t) u_k$, we obtain that the corresponding counterpart of (4.15) is

$$U_i = \begin{bmatrix} \times & \times & & \times \\ (u_1, \Phi^1_1)_\omega & (u_2, \Phi^1_1)_\omega & \dots & (u_K, \Phi^1_1)_\omega \\ \cdots & \cdots & \cdots & \cdots \\ \times & \times & & \times \\ (u_1, \Phi^2_1)_\omega & (u_2, \Phi^2_1)_\omega & \dots & (u_K, \Phi^2_1)_\omega \\ \cdots & \cdots & \cdots & \cdots \\ \times & \times & & \times \\ (u_1, \Phi^{\ell_i}_1)_\omega & (u_2, \Phi^{\ell_i}_1)_\omega & \dots & (u_K, \Phi^{\ell_i}_1)_\omega \end{bmatrix} \begin{matrix} \\ \leftarrow \text{ row } b_{Li1}(u) \\ \\ \\ \leftarrow \text{ row } b_{Li2}(u) \\ \\ \\ \leftarrow \text{ row } b_{Li\ell_i}(u) \end{matrix} \tag{5.12}$$

Again, the relevant rows exhibited in (5.12) correspond to the last rows of each Jordan sub-block $\{\mathbb{A}_{i1}, \mathbb{A}_{i2}, \dots, \mathbb{A}_{i\ell_i}\}$ in (5.3). In (5.12) we have displayed only such relevant rows: $b_{Li1}, b_{Li2}, \dots, b_{Li\ell_i}$ According to Theorem 5.1, the test for controllability as applied to the system (4.5), i.e to the pair $\{\mathcal{A}_N^u, B\}, B = \text{col}\,[B_1, B_2, \dots, B_M]$, is

$$\text{rank} \begin{bmatrix} \text{row } b_{Li1} \text{ of } B_i \\ \text{row } b_{Li2} \text{ of } B_i \\ \vdots \\ \text{row } b_{Li\ell_i} \text{ of } B_i \end{bmatrix} = \text{rank} \begin{bmatrix} \big(u_1, \Phi_1^1(\bar{\lambda}_i)\big)_\omega & \dots & \big(u_K, \Phi_1^1(\bar{\lambda}_i)\big)_\omega \\ \big(u_1, \Phi_1^2(\bar{\lambda}_i)\big)_\omega & \dots & \big(u_K, \Phi_1^2(\bar{\lambda}_i)\big)_\omega \\ \vdots & & \vdots \\ \big(u_1, \Phi_1^{\ell_i}(\bar{\lambda}_i)\big)_\omega & \dots & \big(u_K, \Phi_1^{\ell_i}(\bar{\lambda}_i)\big)_\omega \end{bmatrix} = \ell_i \tag{5.13}$$

$i = 1, \dots, M$. But this is exactly the test obtained in (4.18) via the identification in (5.7b):

$$\phi_{i1}^* = \Phi_1^1(\bar{\lambda}_i), \phi_{i2}^* = \Phi_1^2(\bar{\lambda}_i), \dots, \phi_{i\ell_i}^* = \Phi_1^{\ell_i}(\bar{\lambda}_i) \tag{5.14}$$

involving only eigenvectors, not generalized eigenvectors. Thus the remainder of the proof in section 3 past (4.18) applies and shows Theorem **??** without the FDSA. We have

Theorem 5.2. *With reference to U_i in (5.12), it is possible to select interior vectors $u_1, \dots, u_K \in W_N^u \subset L_\sigma^q(\Omega)$, $K = \sup\,\{\ell_i : i = 1, \dots, M\}$, such that the algebraic conditions (5.13) hold true, $i = 1, \dots, M$.*

We close this section by writing down the counterpart of the expansion (4.10) for the w_N-dynamics in terms of the basis $\beta = \beta_1 \cup \beta_2 \cup \dots \cup \beta_M$, see (4.11), (5.6a), (5.9) of the generalized eigenvectors in the present general case.

$$\begin{aligned} \text{on} \quad & W_N^u : w_N' - \mathcal{A}_N^u w_N \\ & = \sum_{i=1}^M \left\{ \sum_{k=1}^K \Big[(u_k, \Phi_{N_1^i}^1(\bar{\lambda}_i))_\omega \mu_k(t) \Big] e_1^1(\lambda_i) + \dots + \dots + \sum_{k=1}^K \Big[(u_k, \Phi_1^1(\bar{\lambda}_i))_\omega \mu_k(t) \Big] e_{N_1^i}^1(\lambda_i) \right. \\ & + \dots + \dots \\ & \left. + \sum_{k=1}^K \Big[(u_k, \Phi_{N_{\ell_i}^i}^{\ell_i}(\bar{\lambda}_i))_\omega \mu_k(t) \Big] e_1^{\ell_i}(\lambda_i) + \dots + \dots + \sum_{k=1}^K \Big[(u_k, \Phi_1^{\ell_i}(\bar{\lambda}_i))_\omega \mu_k(t) \Big] e_{N_{\ell_i}^i}^{\ell_i}(\lambda_i) \right\} \end{aligned} \tag{5.15}$$

6 Proof of Theorem 3.1: arbitrary decay rate of the w_N-dynamics (4.5) or (5.15) (or (4.13) under the FDSA) by a suitable finite-dimensional interior localized feedback control u

We are now in a position to obtain Theorem 3.1, which we restate for convenience. Let $1 < q < \infty$.

Theorem 6.1. *Let $\lambda_1, \dots, \lambda_M$ be the unstable distinct eigenvalues of $\mathcal{A}$ and let ω be an arbitrarily small open portion of the interior with smooth boundary $\partial\omega$. By virtue of Theorem 5.2, pick interior vectors $[u_1, \dots, u_K]$ in $W_N^u \subset L^q_\sigma(\Omega)$ such that the rank conditions (5.13) hold true, with $K = \sup\{\ell_i : \; i = 1, \dots, M\}$ (respectively, Theorem 4.2 and the (same) rank conditions (4.18) under FDSA).*

Then: Given $\gamma > 0$ arbitrarily large, there exists a K-dimensional interior controller $u = u_N$ acting on ω, of the form given by (4.8), with the vectors u_k given by Theorem 5.2 via the rank conditions (5.13), such that, once inserted in (5.15) yield the estimate

$$\|w_N(t)\|_{L^q_\sigma(\Omega)} + \|u_N(t)\|_{L^q_\sigma(\omega)} \le C_\gamma e^{-\gamma t} \|P_N w_0\|_{L^q_\sigma(\Omega)}, \; t \ge 0, \tag{6.1a}$$

where the $L^q_\sigma(\Omega)$-norm in (6.1a) may be replaced by the $\left(L^q_\sigma(\Omega), \mathcal{D}(A_q)\right)_{1-\frac{1}{p},p}$- norm , $1 < p, q < \infty$; in particular the $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$-norm, $1 < q < \infty, \; 1 < p < \dfrac{2q}{2q-1}$:

$$\|w_N(t)\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} + \|u_N(t)\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \le C_\gamma e^{-\gamma t} \|P_N w_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)}, \; t \ge 0. \tag{6.1b}$$

*Here, w_N is the solution of (5.15) (respectively (4.10) under the FDSA) , i.e., (4.5) corresponding to the control $u = u_N$ in (4.8). Moreover, such controller $u = u_N$ can be chosen in feedback form: that is, with reference to the explicit expression (**??**) for u, of the form $\mu_k(t) = (w_N(t), p_k)_\omega$ for suitable vectors $p_k \in (W_N^u)^* \subset L^{q'}_\sigma(\Omega)$ depending on γ. In conclusion, w_N in (6.1) is the solution of the equation on W_N^u (see (4.5)) specialized as (5.15)*

$$w_N' - \mathcal{A}_N^u w_N = P_N P\left(m\left(\sum_{k=1}^{K} (w_N(t), p_k)_\omega u_k\right)\right), \; u_k \in W_N^u \subset L^q_\sigma(\Omega), \; p_k \in (W_N^u)^* \subset L^{q'}_\sigma(\Omega) \tag{6.2}$$

rewritten as

$$w_N' = \bar{A}^u w_N, \quad w_N(t) = e^{\bar{A}^u t} P_N w_0, \quad w_N(0) = P_N w_0 \tag{6.3}$$

Proof. Step 1: Following [L-T.2] the proof consists in testing controllability of the linear, finite-dimensional system (4.5), in short, the pair

$$\{J, B\}, \; B = U : \; N \times K, K = \sup \{\ell_i; i = 1 \ldots, M\} \tag{6.4}$$

$U = [U_1, \ldots, U_M]^{\text{tr}}$, U_i given by (5.12) (or by (4.15) under FDSA). J is the Jordan form of $\mathcal{A}_N^u$ with respect to the Jordan basis $\beta = \beta_1 \cup \cdots \cup \beta_M$, β_i being given by (5.6a). But the rank conditions (5.13) precisely asserts such controllability property of the pair $\{\mathcal{A}_N^u = J, B\}$, in light of Theorem 5.1.

Step 2: Having established the controllability criterion for the pair $\{\mathcal{A}_N^u = J, B\}$ then by the well-known Popov's criterion in finite-dimensional theory, there exists a real feedback matrix $Q = K \times N$, such that the spectrum of the matrix $(J + BQ) = (J + UQ)$ may be arbitrarily preassigned; in particular, to lie in the left half-plane $\{\lambda : \text{Re}\ \lambda < -\gamma < -\text{Re}\ \lambda_{N+1}\}$, as desired. The resulting closed-loop system

$$(\hat{w}_N') - J\hat{w}_N' = U u_N, \tag{6.5}$$

is obtained with $\mathbb{C}^N$-vector $u_N = Q\hat{w}_N$, Q being the $K \times N$ matrix with row vectors $[\hat{p}_1, \ldots, \hat{p}_K]$, $\mu_N^k = (\hat{w}_N, \hat{p}_k)$ in the $\mathbb{C}^N$-inner product and hence decays with exponential rate

$$|\hat{w}_N(t)|_{\mathbb{C}^N} \leq C_\gamma e^{-\gamma t} \, |\hat{w}_N(0)|_{\mathbb{C}^N} \, , \; t \geq 0 \tag{6.6}$$

But the N-dimensional vector $w_N \in W_N^u \subset L^q_\sigma(\Omega)$ is represented by the $\mathbb{C}^N$-vector $\hat{w}_N = [w_N]_\beta$, where in the general case of Section 5, β is a Jordan basis of generalized eigenfunctions of $\mathcal{A}_q (= \mathcal{A}_N^u)$ corresponding to its M distinct unstable eigenvalues. Such basis is given by $\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_M$, where a representative β_i is given in (5.6a). The whole basis can be read off from (5.15). In the special case of Section 4 where the FDSA holds, the basis β in W_N^u is given by the eigenfunctions of the $\mathcal{A}_N^u$ corresponding to its M distinct eigenvalues, see (4.11). But such eigenfunctions/generalized eigenfunctions are in $\mathcal{D}(\mathcal{A}_q)$, hence smooth. Thus, the exponential decay in (6.6) of the coordinate vector $\hat{w}_N$ in $\mathbb{C}^N$ translates in same exponential decay of the vector $w_N(t) \in W_N^u$ not only in the $L^q_\sigma(\Omega)$-norm but also in the $\mathcal{D}(\mathcal{A}_q) = \mathcal{D}(A_q)$-norm, hence in the $\big(L^q_\sigma(\Omega), \mathcal{D}(A_q)\big)_{1-\frac{1}{p},p}$-norm, in particular in the $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$-norm. See also Remark 4.1. Thus, returning from $\mathbb{C}^N \times \mathbb{C}^N$ back to $W_N^u \times (W_N^u)^*$, there exist suitable $p_1, \ldots, p_K \in (W_N^u)^* \subset L^{q'}_\sigma(\Omega)$, such that $\mu_N^k = (w_k, p_k)$, whereby the closed-loop system

(6.2) corresponds precisely to (5.15) via $P_N P(mu)$ written in terms of the Jordan basis of eigenvectors β in (5.6a).

Thus not only we obtain in view of (6.2), (6.3) and (6.6)

$$\|w_N(t)\|_{L^q_\sigma(\Omega)} = \left\| e^{\bar{A}^u t} P_N w_0 \right\|_{L^q_\sigma(\Omega)} \leq C_\gamma e^{-\gamma t} \|P_N w_0\|_{L^q_\sigma(\Omega)}, \ t \geq 0, \tag{6.7}$$

but also, say $1 < q < \infty, 1 < p < \dfrac{2q}{2q-1}$

$$\|w_N(t)\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} = \left\| e^{\bar{A}^u t} P_N w_0 \right\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \leq C_\gamma e^{-\gamma t} \|P_N w_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)}, \ t \geq 0. \tag{6.8}$$

Hence with $u_N = Q w_N$, we obtain not only

$$\|w_N(t)\|_{L^q_\sigma(\Omega)} + \|u_N(t)\|_{L^q_\sigma(\omega)} = \|w_N(t)\|_{L^q_\sigma(\Omega)} + \|Q w_N(t)\|_{L^q_\sigma(\Omega)} \tag{6.9}$$

$$\leq \big(|Q| + 1 \big) \left\| e^{\bar{A}^u t} P_N w_0 \right\|_{L^q_\sigma(\Omega)} \leq C_\gamma e^{-\gamma t} \|P_N w_0\|_{L^q_\sigma(\Omega)} \tag{6.10}$$

but also, say

$$\|w_N(t)\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} + \|u_N(t)\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \leq C_\gamma e^{-\gamma t} \|P_N w_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)}, \ t \geq 0. \tag{6.11}$$

□

Remark 6.1. Under the FDSA, checking controllability of the system (4.13) is easier. To this end, we can pursue, as usual, two strategies.

A first strategy invokes the well-known Kalman controllability criterion by constructing the $N \times KN$ Kalman controllability matrix

$$\mathcal{K} = [B, \Lambda B, \Lambda^2 B, \dots, \Lambda^{N-1} B] = \begin{bmatrix} B_1 & J_1 B_1 & \dots & J_1^{N-1} B_1 \\ B_2 & J_2 B_2 & \dots & J_2^{N-1} B_2 \\ \dots & \dots & \dots & \dots \\ B_M & J_M B_M & \dots & J_M^{N-1} B_M \end{bmatrix}, \tag{6.12}$$

$$B = \text{col}\ [B_1, B_2, \dots, B_M], \quad B_i = U_i : \ell_i \times \ell_i \tag{6.13}$$

of size $N \times KN$, $N = \dim W^u_N$, $J_i = \lambda_i I_i : \ell_i \times \ell_i$, $B_i = U_i : \ell_i \times \ell_i$, and requiring that it be full rank.

$$\text{rank } \mathcal{K} = \text{full} = N. \tag{6.14}$$

In view of generalized Vandermond determinants, we then have

$$\text{rank } \mathcal{K} = N \quad \text{if and only if rank } U_i = \ell_i \text{ (full) } i = 1, \ldots, M, \tag{6.15}$$

precisely as guaranteed by (4.18). A second strategy invokes the Hautus controllability criterion:

$$rank[\Lambda - \lambda_i I, B] = rank[\Lambda - \lambda_i I, U] = N \text{ (full)} \tag{6.16}$$

for all unstable eigenvalues $\lambda_i, 1, \ldots, M$, yielding again the condition that rank $[U_i] = \ell_i, 1, \ldots, M$, as generated by (4.18)

7 Proof of Theorem 3.2: Feedback stabilization of the original linearized w-Oseen system (2.13) by a finite dimensional feedback controller

The main result on the feedback stabilization of the linearized w-system (2.13) = (3.1) by a finite dimensional controller is Theorem 3.2, here reformulated in part for convenience in the context of the development of the present proof. Throughout this section $1 < q < \infty$.

Theorem 7.1. *Let the Oseen operator $\mathcal{A}$ have N possibly repeated unstable eigenvalues $\{\lambda_j\}_{j=1}^N$ of which M are distinct. Let $\varepsilon > 0$ and set $\gamma_0 = |Re\ \lambda_{N+1}| - \varepsilon$. Consider the setting of Theorem 6.1 so that, in particular, the feedback finite-dimensional control $u = u_N$ is given by $u = u_N = \sum_{k=1}^{K}(w_N(t), p_k)u_k$ and satisfies estimates (6.1) with $\gamma > 0$ arbitrarily large, for vectors $p_1, \dots, p_k \in (W_N^u)^* \subset L^{q'}_\sigma(\Omega)$ and vectors $u_1, \dots, u_k \in W_N^u \subset L^q_\sigma(\Omega)$ given by Theorem 6.1. Thus, the linearized problem (3.1) specializes to (3.16)*

$$\frac{dw}{dt} = \mathcal{A}w + P\left(m\left(\sum_{k=1}^{K}(w_N(t), p_k)_\omega u_k\right)\right) \equiv \mathbb{A}_{_F} w \tag{7.1}$$

Here $\mathbb{A}_{_F} = \mathbb{A}_{_{F,q}}$ is the generator of a s.c. analytic semigroup on either the space $L^q_\sigma(\Omega)$, $1 < q < \infty$, or on the space $\big(L^q_\sigma(\Omega), \mathcal{D}(A_q)\big)_{1-\frac{1}{p},p}$, $1 < p, q < \infty$, in particular on the space $\widetilde{B}^{2-2/p}_{q,p}(\Omega), 1 < q, 1 < p < 2q/2q-1$. Moreover, such dynamics w (equivalently, generator $\mathbb{A}_{_F}$) in (7.1) is uniformly stable in each of these spaces, say

$$\left\| e^{\mathbb{A}_{_F} t} w_0 \right\|_{L^q_\sigma(\Omega)} = \| w(t, w_0) \|_{L^q_\sigma(\Omega)} \le C_{\gamma_0} e^{-\gamma_0 t} \| w_0 \|_{L^q_\sigma(\Omega)}, \quad t \ge 0. \tag{7.2}$$

or say

$$\left\| e^{\mathbb{A}_{_F} t} w_0 \right\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} = \| w(t, w_0) \|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \le C_{\gamma_0} e^{-\gamma_0 t} \| w_0 \|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)}, \quad t \ge 0. \tag{7.3}$$

Proof. Step 1: According to Theorem 6.1, the finite-dimensional system w_N in (3.8a) = (4.5) is uniformly stabilized by the finite dimensional feedback controller $u = u_N$ given in the RHS of (6.2) = RHS of (7.1) with an arbitrary preassigned decay rate $\gamma > 0$, as given, either in the $L^q_\sigma(\Omega)$-norm, or in the $\big(L^q_\sigma(\Omega), \mathcal{D}(A_q)\big)_{1-\frac{1}{p},p}$-norm in (6.1a), or in particular, in the $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$-norm as in (6.1b).

Step 2: Next, we examine the impact of such constructive feedback control u_N on the ζ_N-dynamics (3.8b), whose explicit solution can be given by a variation of parameter formula,

$$\zeta_N(t) = e^{\mathcal{A}_N^s t}\zeta(0) + \int_0^t e^{\mathcal{A}_N^s(t-r)}(I-P_N)P(mu_N(r))dr. \tag{7.4}$$

in the notation $\mathcal{A}_N^s = (I-P_N)\mathcal{A}$, $\mathcal{A} = \mathcal{A}_q$, of (3.5). We now recall from Section 1.10 (d) that the Oseen operator $\mathcal{A}_q$ generates a s.c. analytic semigroup not only on $L^q_\sigma(\Omega)$ but also on $\left(L^q_\sigma(\Omega), \mathcal{D}(A_q)\right)_{1-\frac{1}{p},p}$, in particular on $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$. Hence the feedback operator $\mathbb{A}_F = \mathbb{A}_{F,q}$ similarly generates a s.c. analytic semigroup on these spaces, being a bounded perturbation of the Oseen operator $\mathcal{A} = \mathcal{A}_q$. So we can estimate (7.4) in the norm of either of these spaces. Furthermore, the (point) spectrum of the generator $\mathcal{A}_N^s$ on W_N^s satisfies $\sup\{Re\ \sigma(\mathcal{A}_N^s)\} < -|\lambda_{N+1}| < -\gamma_0$ by assumption. We shall carry our the supplemental computations explicitly in the space $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$ for the case of greatest interest in the nonlinear analysis of sections 9, 10. In the norm of $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$, we obtain from (7.4) since the operators $(I-P_N), P$ are bounded

$$\|\zeta_N(t)\| \le \left\|e^{\mathcal{A}_N^s t}\zeta(0)\right\| + C\int_0^t \left\|e^{\mathcal{A}_N^s(t-\tau)}\right\| \|u_N(\tau)\|\, d\tau \tag{7.5}$$

$$\|\zeta_N(t)\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \le Ce^{-\gamma_0 t}\|\zeta(0)\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} + C\int_0^t e^{-\gamma_0(t-r)}e^{-\gamma r}dr\, \|P_N w_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)}. \tag{7.6}$$

recalling estimate (3.13) or (6.11) for $\|u_N\|$ in the $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$-norm. Since we may choose $\gamma > \gamma_0$ by Theorem 3.1 (or Theorem 6.1), we then obtain

$$\|\zeta_N(t)\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \le C\left[e^{-\gamma_0 t} + e^{-\gamma_0 t}\frac{1-e^{-(\gamma-\gamma_0)t}}{\gamma-\gamma_0}\right]\|w_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \tag{7.7}$$

$$\le Ce^{-\gamma_0 t}\|w_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)}, \quad \forall t > 0 \tag{7.8}$$

Then, estimate (7.8) for $\zeta_N(t)$ along with estimate (3.13) = (6.11) for $w_N(t)$ with $\gamma > \gamma_0$ yields the desired estimate (7.3) for $w = w_N + \zeta_N$ in the $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$-norm:

$$\|w(t)\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \le \|\zeta_N(t)\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} + \|w_N(t)\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \tag{7.9}$$

$$\le \left[\widetilde{C}_{\gamma_0}e^{-\gamma_0 t} + C_\gamma e^{-\gamma t}\right]\|w_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \tag{7.10}$$

$$\le C_{\gamma_0}e^{-\gamma_0 t}\|w_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \tag{7.11}$$

and (7.3) is proved. Similar computations from (7.4) to (7.8) apply in the $L^q_\sigma(\Omega)$-norm for $\zeta_N(t)$, as the Oseen operator generates a s.c. analytic semigroup on $L^q_\sigma(\Omega)$ from Section 1.10 (d). This, coupled with estimate (3.12) for $w_N(t)$, yields estimate (7.2) for the $w = w_N + \zeta_N$ with $L^q_\sigma(\Omega)$-norm. Theorem 7.1 is established. □

Remark 7.1. Computations such as those in [B-T.1, p 1473] using the analyticity of the Oseen semigroup $e^{\mathcal{A}_q t}$ show the alternative estimates (3.18a-b) of Theorem 3.2.

8 Maximal L^p regularity on $L^q_\sigma(\Omega)$ and for $T = \infty$ of the s.c. analytic semigroup $e^{\mathbb{A}_{F,q}t}$ yielding uniform decay of the linearized w-problem (3.1), once specialized as in (7.1) of Theorem 3.2 = Theorem 7.1.

In this section, we return to the w-feedback problem (7.1), $w_t = \mathbb{A}_{F,q}w$, where p_k, u_k are the vectors claimed and constructed in Theorem 3.1, or Theorem 3.2 (Theorem 7.1) and Remark 4.1. As stated in Theorem 7.1, problem (7.1) defines a s.c. analytic, uniformly stable semigroup $e^{\mathbb{A}_{F,q}t}$ as in (7.2):

$$\left\| e^{\mathbb{A}_{F,q}t} \right\|_{\mathcal{L}(\cdot)} \le M_{\gamma_0} e^{-\gamma_0 t}, \quad t \ge 0 \tag{8.1}$$

where $(\cdot)$ denotes the space $L^q_\sigma(\Omega)$ or else $\left(L^q_\sigma(\Omega), \mathcal{D}(A_q)\right)_{1-\frac{1}{p},p}$, in particular $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$. Define the "good" bounded operator

$$Gw = m\left(\sum_{k=1}^{K} (P_N w, p_k)_\omega u_k\right), \; u_k \in W^u_N \subset L^q_\sigma(\Omega), \; p_k \in (W^u_N)^* \subset L^{q'}_\sigma(\Omega), \tag{8.2}$$

By Theorem 2.6, the Oseen operator $\mathcal{A}_q$ enjoys maximal L^p regularity on $L^q_\sigma(\Omega)$ up to $T < \infty$, see (2.48) as well as (2.50), (2.52). Then the same property holds true up to $T < \infty$ for $\mathbb{A}_{F,q} = \mathcal{A}_q + G$, as G is a bounded operator [Dore], [K-W.2], [Weis]. We now seek to establish maximal L^p regularity up to $T = \infty$ of $\mathbb{A}_{F,q}$, i.e. of the following problem

$$\begin{cases} w_t - \Delta w + L_e(w) + \nabla \pi = Gw + F & \text{in } (0,T] \times \Omega \equiv Q \quad (8.3a) \\ div\ w \equiv 0 & \text{in } Q \quad (8.3b) \\ w|_\Sigma \equiv 0 & \text{in } (0,T] \times \Gamma \equiv \Sigma \quad (8.3c) \\ w|_{t=0} = w_0 & \text{in } \Omega, \quad (8.3d) \end{cases}$$

L_e defined in (2.39) rewritten abstractly, upon application of the Helmholtz projection P_q to (8.3a) and $F_\sigma = P_q F$, as

$$w_t = \mathbb{A}_{F,q}w + P_qF = \mathcal{A}_q w + P_q G w + P_q F \tag{8.4}$$

$$= -\nu A_q w - A_{o,q} w + P_q G w + P_q F \tag{8.5}$$

Wlog, we take $\nu = 1$ henceforth. Here we have appended a subscript "q" to the generator $\mathbb{A}_F$ defined in (7.1) which we rewrite as $\mathbb{A}_{F,q}$. With $F_\sigma = P_q F$ its solution on $L^q_\sigma(\Omega)$ is

$$w(t) = e^{\mathbb{A}_{F,q}t} w_0 + \int_0^t e^{\mathbb{A}_{F,q}(t-\tau)} F_\sigma(\tau) d\tau \tag{8.6}$$

$$= e^{-A_q t} w_0 + \int_0^t e^{-A_q(t-\tau)} (P_q G - A_{o,q}) w(\tau) d\tau + \int_0^t e^{-A_q(t-\tau)} F_\sigma(\tau) d\tau. \tag{8.7}$$

As the present section is preparatory for the subsequent sections 9 and 10, the case of greatest interest here is then for $w_0 \in \widetilde{B}^{2-2/p}_{q,p}(\Omega)$, i.e. $1 < q, 1 < p < 2q/_{2q-1}$. Nevertheless we shall treat the general case $1 < p, q < \infty$.

Theorem 8.1. *As in (2.43) of Theorem 2.6, but now with $T = \infty$, assume*

$$F_\sigma \in L^p(0, \infty; L^q_\sigma(\Omega)), \quad w_0 \in \Big(L^q_\sigma(\Omega), \mathcal{D}(A_q)\Big)_{1-1/p,p}. \tag{8.8}$$

Then there exists a unique solution of problem (8.3) = (8.4) = (8.5).

$$\begin{cases} w \in X^\infty_{p,q,\sigma} = L^p(0,\infty;\mathcal{D}(A_q)) \cap W^{1,p}(0,\infty;L^q_\sigma(\Omega)), \text{ equivalently} & (8.9a)\\ w \in X^\infty_{p,q} = L^p(0,\infty;W^{2,q}(\Omega)) \cap W^{1,p}(0,\infty;L^q_\sigma(\Omega)) \hookrightarrow C\big(0,\infty;B^{2-2/p}_{q,p}(\Omega)\big) & (8.9b) \end{cases}$$

(recall [Amann.1, Theorem 4.10.2, p180 in BUC for $T = \infty$] already noted in (2.30)) continuously on the data: there exist constants C_0, C_1 such that

$$C_0 \|w\|_{C\left(0,\infty;B^{2-2/p}_{q,p}(\Omega)\right)} \le \|w\|_{X^\infty_{p,q,\sigma}} + \|\pi\|_{Y^\infty_{p,q}}$$

$$\equiv \|w'\|_{L^p(0,\infty;L^q(\Omega))} + \|A_q w\|_{L^p(0,\infty;L^q(\Omega))} + \|\pi\|_{Y^\infty_{p,q}} \tag{8.10a}$$

$$\le C_1 \left\{ \|F_\sigma\|_{L^p(0,\infty;L^q_\sigma(\Omega))} + \|w_0\|_{\left(L^q_\sigma(\Omega),\mathcal{D}(A_q)\right)_{1-\frac{1}{p},p}} \right\}. \tag{8.10b}$$

Thus for $1 < q, 1 < p < \dfrac{2q}{2q-1}$, then the I.C. w_0 is in $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$. Equivalently,

(i) The map

$$\begin{array}{c} F_\sigma \longrightarrow \displaystyle\int_0^t e^{\mathbb{A}_{F,q}(t-\tau)} F_\sigma(\tau) d\tau \ \text{ : continuous} \\ L^p(0,\infty;L^q_\sigma(\Omega)) \longrightarrow L^p(0,\infty;\mathcal{D}(\mathbb{A}_{F,q}) = \mathcal{D}(\mathcal{A}_q) = \mathcal{D}(A_q)), \end{array} \tag{8.11}$$

whereby then automatically

$$L^p(0,\infty;L^q_\sigma(\Omega)) \longrightarrow W^{1,p}(0,\infty;L^q_\sigma(\Omega)) \tag{8.12}$$

and ultimately

$$L^p(0,\infty;L^q_\sigma(\Omega)) \longrightarrow X^\infty_{p,q,\sigma} = L^p(0,\infty;\mathcal{D}(\mathbb{A}_{F,q})) \cap W^{1,p}(0,\infty;L^q_\sigma(\Omega)) \tag{8.13}$$

(ii) The s.c. analytic semigroup $e^{\mathbb{A}_{F,q}t}$ on the space $\left(L^q_\sigma(\Omega), \mathcal{D}(A_q)\right)_{1-\frac{1}{p},p}$, $1 < p < \infty$, as asserted in Theorem 7.1, in particular on the space $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$, $1 < q$, $1 < p < \dfrac{2q}{2q-1}$, satisfies

$$\begin{aligned} e^{\mathbb{A}_{F,q}t}: \ \textit{continuous} \quad & \left(L^q_\sigma(\Omega), \mathcal{D}(A_q)\right)_{1-\frac{1}{p},p} \longrightarrow X^\infty_{p,q,\sigma} \quad (\textit{equivalently} \ \longrightarrow X^\infty_{p,q}) \\ \textit{in particular} \quad & \widetilde{B}^{2-2/p}_{q,p}(\Omega) \longrightarrow X^\infty_{p,q,\sigma} \quad (\textit{equivalently} \ \longrightarrow X^\infty_{p,q}) \end{aligned} \tag{8.14}$$

Proof. Part i

Orientation The proof is a suitable modification of the proof of Theorem 2.6, that is, of the maximal regularity of the Oseen operator $\mathcal{A}_q$ on $L^q_\sigma(\Omega)$, given in Appendix B. Namely, Step 1 = (B.3) of that proof now exploits the uniform stability of $e^{\mathbb{A}_{F,q}t}$ in (7.2)=(8.1) which was not available for the Oseen semigroup $e^{\mathcal{A}_q t}$ in Appendix B. Hence the convolution argument in (B.8) may now be applied up to $T = \infty$, see below (8.16). Next, Step 2 of the proof in (B.13)-(B.20) in Appendix B applies also in the present proof, for $T \leq \infty$, to include $T = \infty$, as the term $-A_{o,q}$ in (B.13) is replaced in the present proof by $(P_q G - A_{o,q})$, with $P_q G$ bounded.

Step 1: With reference to (8.6) with $w_0 = 0$, we first establish the inequality

$$\int_0^\infty \|w(t)\|^p_{L^q_\sigma(\Omega)} \, dt \leq C \int_0^\infty \|F_\sigma(t)\|^p_{L^q_\sigma(\Omega)} \, dt \tag{8.15}$$

Indeed, from (8.6), in the $L^q_\sigma(\Omega)$-norm, recalling (8.1)

$$\begin{aligned} \|w(t)\| &\leq \int_0^t \left\| e^{\mathbb{A}_{F,q}(t-\tau)} \right\| \|F_\sigma(\tau)\| \, d\tau \\ &\leq M_{\gamma_0} \int_0^t e^{-\gamma_0(t-\tau)} \|F_\sigma(\tau)\| \, d\tau \in L^p(0,\infty) \end{aligned} \tag{8.16}$$

being the convolution of a $L^1(0,\infty)$-function with an $L^p(0,\infty)$-function (Young's Theorem) [Sa]. Then (8.15) is proved.

Step 2: Again for $w_0 = 0$ we obtain from (8.7)

$$A_q w(t) = A_q \int_0^t e^{-A_q(t-\tau)} (P_q G - A_{o,q}) w(\tau) d\tau + A_q \int_0^t e^{-A_q(t-\tau)} F_\sigma(\tau) d\tau \tag{8.17}$$

We shall establish the following inequality

$$\int_0^\infty \|A_q w(t)\|^p_{L^q_\sigma(\Omega)} \, dt \leq C \int_0^\infty \|w(t)\|^p_{L^q_\sigma(\Omega)} \, dt + C \int_0^\infty \|F_\sigma(t)\|^p_{L^q_\sigma(\Omega)} \, dt \tag{8.18}$$

(Compare with (B.4), which holds true for any $T \leq \infty$, including $T = \infty$). In fact, to this end, as in that proof, using the maximal regularity up to $T = \infty$ of the Stokes semigroup, as well as (8.8) for F_σ, we estimate from (8.17)

$$\|A_q w\|_{L^p(0,\infty,L^q_\sigma(\Omega))} \leq C \|F_\sigma\|_{L^p(0,\infty,L^q_\sigma(\Omega))} + C \|[G - A_{o,q}]w\|_{L^p(0,\infty,L^q_\sigma(\Omega))} \tag{8.19}$$

$$\leq C\Big\{ \|F_\sigma\|_{L^p(0,\infty,L^q_\sigma(\Omega))} + C \|w\|_{L^p(0,\infty,L^q_\sigma(\Omega))} \Big\} + C \|A_{o,q} w\|_{L^p(0,\infty,L^q_\sigma(\Omega))}, \tag{8.20}$$

as G is bounded. Using the same interpolation argument leading to (B.20), based on the interpolation inequality (B.11), we obtain from (8.20)

$$\|A_q w\|_{L^p(0,\infty,L^q_\sigma(\Omega))} \leq C \|F_\sigma\|_{L^p(0,\infty,L^q_\sigma(\Omega))} + C \|w\|_{L^p(0,\infty,L^q_\sigma(\Omega))} + \varepsilon C \|A_q w\|_{L^p(0,\infty,L^q_\sigma(\Omega))} + C_\varepsilon \|w\|_{L^p(0,\infty,L^q_\sigma(\Omega))} \tag{8.21}$$

from which we obtain

$$\|A_q w\|_{L^p(0,\infty,L^q_\sigma(\Omega))} \leq \left(\frac{C}{1-\varepsilon C}\right) \|F_\sigma\|_{L^p(0,\infty,L^q_\sigma(\Omega))} + \left(\frac{C + C_\varepsilon}{1-\varepsilon C}\right) \|w\|_{L^p(0,\infty,L^q_\sigma(\Omega))} \tag{8.22}$$

and then estimate in (8.18) in Step 2 is established.

Step 3: Substituting (8.15) in the RHS of (8.18) yields

$$\|A_q w\|_{L^p(0,\infty,L^q_\sigma(\Omega))} \leq C \|F_\sigma\|_{L^p(0,\infty,L^q_\sigma(\Omega))} \tag{8.23}$$

and (8.11) is established via (8.6) with $w_0 = 0$, and $\mathcal{D}(\mathbb{A}_{F,q}) = \mathcal{D}(A_q)$.

Part ii

Let $w_0 \in \big(L^q_\sigma(\Omega), \mathcal{D}(A_q)\big)_{1-\frac{1}{p},p}$, $\big[$in particular $w_0 \in \widetilde{B}^{2-2/p}_{q,p}(\Omega)$ for $1 < q < \infty, 1 < p < 2q/_{2q-1}$ by (2.16b) $\big]$ and consider the s.c. analytic exponentially stable semigroup $e^{\mathbb{A}_{F,q}t}$ in such space, as guaranteed by Theorem 7.1, see (8.1):

$$w(t) = e^{\mathbb{A}_{F,q}t} w_0; \quad w_t = \mathbb{A}_{F,q} w = -A_q w + (P_q G - A_{o,q}) w \tag{8.24}$$

$$w(t) = e^{-A_q t} w_0 + \int_0^t e^{-A_q(t-\tau)} (P_q G - A_{o,q}) w(\tau) d\tau \tag{8.25}$$

$$A_q w(t) = A_q e^{-A_q t} w_0 + A_q \int_0^t e^{-A_q(t-\tau)} (P_q G - A_{o,q}) w(\tau) d\tau \tag{8.26}$$

counterpart of (B.18), that is with $-A_{o,q}$ in (B.18) replaced by $P_qG - A_{o,q}$ now, with P_qG bounded, see (8.2).Thus essentially the same proof leading to (B.24) yields now

$$\begin{aligned} \|\mathbb{A}_{F,q}w\|_{L^P(0,\infty,L^q_\sigma(\Omega))} &= \left\|\mathbb{A}_{F,q}e^{\mathbb{A}_{F,q}t}w_0\right\|_{L^P(0,\infty,L^q_\sigma(\Omega))} \\ &\leq C\,\|w_0\|_{\left(L^q_\sigma(\Omega),\mathcal{D}(A_q)\right)_{1-\frac{1}{p},p}} \end{aligned} \tag{8.27}$$

with $\mathcal{D}(\mathbb{A}_{F,q}) = \mathcal{D}(A_q)$. Then (8.27) proves (8.14). □

9 Proof of Theorem 3.3: Well-posedness on $X^{\infty}_{p,q}$ of the non-linear z-dynamics in feedback form

In this section we return to the translated non-linear z-dynamics (2.12a) and apply to it the feedback control $u = \sum_{k=1}^{K} (P_N z, p_k)_{\omega} u_k$, i.e. of the same structure as the feedback identified on the RHS of the linearized w-dynamics (7.1), which produced the s.c. analytic, uniformly stable feedback semigroup $e^{\mathbb{A}_{F,q}t}$ on $L^q_\sigma(\Omega)$. Here the vectors $p_k \in (W^u_N)^*$, $u_k \in W^u_N$ are precisely those identified in Theorem 6.1 = Theorem 3.1. Thus, returning to (2.12), in this section we consider the following translated feedback non-linear problem

$$\frac{dz}{dt} - \mathcal{A}_q z + \mathcal{N}_q z = P_q\left(m\left(\sum_{k=1}^{K} (z_N, p_k)_{\omega} u_k \right)\right); \quad z_0 = P_N z(0) \tag{9.1}$$

Recalling from Theorem 3.2 = Theorem 7.1, Eq (7.1) the feedback generator $\mathbb{A}_{F,q}$ as well as the bounded operator G in (8.2), we can rewrite (9.1) as

$$z_t = \mathbb{A}_{F,q} z - \mathcal{N}_q z = -(\nu A_q + A_{o,q})z + P_q G z - \mathcal{N}_q z, \quad z(0) = z_0 \tag{9.2}$$

whose variation of parameters formula is

$$z(t) = e^{\mathbb{A}_{F,q}t} z_0 - \int_0^t e^{\mathbb{A}_{F,q}(t-\tau)} \mathcal{N}_q z(\tau) d\tau. \tag{9.3}$$

We already know from (7.3) that for $z_0 \in \widetilde{B}^{2-2/p}_{q,p}(\Omega), 1 < q < \infty,\ 1 < p < 2q/_{2q-1}$ we have

$$\left\| e^{\mathbb{A}_{F,q}t} z_0 \right\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \leq M_{\gamma_0} e^{-\gamma_0 t} \left\| z_0 \right\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)}, \quad t \geq 0 \tag{9.4}$$

with M_{γ_0} possibly depending on p, q. Maximal regularity properties corresponding to the solution operator formula in (9.3) were established in section 8. Accordingly, for $z_0 \in \widetilde{B}^{2-2/p}_{q,p}(\Omega)$ and $f \in X^{\infty}_{p,q,\sigma} \equiv L^p(0,\infty; \mathcal{D}(\mathbb{A}_{F,q})) \cap W^{1,p}(0,\infty; L^q_\sigma(\Omega))$, $\mathcal{D}(\mathbb{A}_{F,q}) = \mathcal{D}(A_q)$, recall (8.11) we define the operator $\mathcal{F}$ by

$$\mathcal{F}(z_0, f)(t) = e^{\mathbb{A}_{F,q}t} z_0 - \int_0^t e^{\mathbb{A}_{F,q}(t-\tau)} \mathcal{N}_q f(\tau) d\tau \tag{9.5}$$

The main result of this section is Theorem 3.3. restated as

Theorem 9.1. *Let $d = 2, 3$, $q > d$ and $1 < p < 2q/_{2q-1}$. There exists a positive constant $r_1 > 0$ (identified in the proof below in (9.24)), such that if*

$$\|z_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} < r_1, \tag{9.6}$$

then the operator $\mathcal{F}$ in (9.5) has a unique fixed point nonlinear semigroup solution on $X^\infty_{p,q,\sigma}$

$$\mathcal{F}(z_0, z) = z, \ \text{ or } z(t) = e^{\mathbb{A}_{F,q}t} z_0 - \int_0^t e^{\mathbb{A}_{F,q}(t-\tau)} \mathcal{N}_q z(\tau) d\tau \tag{9.7}$$

which therefore is the unique solution of problem (9.2) (= (9.1)) in $X^\infty_{p,q,\sigma}$.

The proof of Theorem 3.3 = Theorem 9.1 is accomplished in two steps.

Step 1:

Theorem 9.2. *Let $d = 2, 3$, $q > d$ and $1 < p < {}^{2q}/_{2q-1}$. There exists a positive constant $r_1 > 0$ (identified in the proof below in (9.24)) and a subsequent constant $r > 0$ (identified in the proof below in (9.22)) depending on $r_1 > 0$ and the constant $C > 0$ in (9.20), such that with $\|z_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} < r_1$ as in (9.6), the operator $\mathcal{F}(z_0, f)$ maps the ball $B(0, r)$ in $X^\infty_{p,q,\sigma}$ into itself.* □

Theorem 9.1 will follow then from Theorem 9.2 after establishing that

Step 2:

Theorem 9.3. *Let $d = 2, 3$, $q > 3$ and $1 < p < {}^{2q}/_{2q-1}$. There exists a positive constant $r_1 > 0$, such that if $\|z_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} < r_1$ as in (9.6), then there exists a constant $0 < \rho_0 < 1$, such that the operator $\mathcal{F}(z_0, f)$ defines a contraction in the ball $B(0, \rho_0)$ of $X^\infty_{p,q,\sigma}$* □

The Banach contraction principle then establishes Theorem 9.1, once we prove Theorems 9.2 and 9.3.

Proof of Theorem 9.2. *Step 1*: We start from definition (9.5) of $\mathcal{F}$ and invoke the maximal regularity properties (8.14) for $e^{\mathbb{A}_{F,q}t}$ and (8.13) for $\displaystyle\int_0^t e^{\mathbb{A}_{F,q}(t-\tau)} \mathcal{N}_q f(\tau) d\tau$. We obtain from (9.5)

$$\|\mathcal{F}(z_0, f)(t)\|_{X^\infty_{p,q,\sigma}} \leq \left\| e^{\mathbb{A}_{F,q}t} z_0 \right\|_{X^\infty_{p,q,\sigma}} + \left\| \int_0^t e^{\mathbb{A}_{F,q}(t-\tau)} \mathcal{N}_q f(\tau) d\tau \right\|_{X^\infty_{p,q,\sigma}} \tag{9.8}$$

$$\leq C \Big[\|z_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} + \|\mathcal{N}_q f\|_{L^p(0,\infty; L^q_\sigma(\Omega))} \Big]. \tag{9.9}$$

Step 2: By the definition $\mathcal{N}_q f = P_q[(f.\nabla)f]$ in (2.11), we estimate ignoring $\|P_q\|$ and using, $\sup_{\cdot} \big[|g(\cdot)| \big]^r = [\sup_{\cdot} (|g(\cdot)|)]^r$

$$\|\mathcal{N}_q f\|^p_{L^p(0,\infty;L^q_\sigma(\Omega))} \leq \int_0^\infty \|P_q[(f.\nabla)f]\|^p_{L^q_\sigma(\Omega)}\,dt$$

$$\leq \int_0^\infty \left\{ \int_\Omega |f(t,x)|^q \, |\nabla f(t,x)|^q \, d\Omega \right\}^{p/q} dt \tag{9.10}$$

$$\leq \int_0^\infty \left\{ \left[\sup_\Omega |\nabla f(t,\cdot)|^q \right]^{1/q} \left[\int_\Omega |f(t,x)|^q \, d\Omega \right]^{1/q} \right\}^p dt \tag{9.11}$$

$$\leq \int_0^\infty \|\nabla f(t,\cdot)\|^p_{L^\infty(\Omega)} \, \|f(t,\cdot)\|^p_{L^q_\sigma(\Omega)}\,dt \tag{9.12}$$

$$\leq \sup_{0\leq t\leq\infty} \|f(t,\cdot)\|^p_{L^q_\sigma(\Omega)} \int_0^\infty \|\nabla f(t,\cdot)\|^p_{L^\infty(\Omega)}\,dt \tag{9.13}$$

$$= \|f\|^p_{L^\infty(0,\infty;L^q_\sigma(\Omega))} \, \|\nabla f\|^p_{L^p(0,\infty;L^\infty(\Omega))} \tag{9.14}$$

Step 3: The following embeddings hold true:

(i) [G-G-H.1, Proposition 4.3, p 1406 with $\mu = 0, s = \infty, r = q$] so that the required formula reduces to $1 \geq 1/p$, as desired

$$f \in X^\infty_{p,q,\sigma} \hookrightarrow f \in L^\infty(0,\infty; L^q_\sigma(\Omega)) \tag{9.15a}$$

$$\text{so that, } \|f\|_{L^\infty(0,\infty;L^q_\sigma(\Omega))} \leq C \, \|f\|_{X^\infty_{p,q,\sigma}} \tag{9.15b}$$

(ii) [Kes, Theorem 2.4.4, p 74 requiring C^1-boundary]

$$W^{1,q}(\Omega) \subset L^\infty(\Omega) \text{ for q} > \dim \Omega = d, \ d = 2,3, \tag{9.16}$$

so that, with $p > 1, q > 3$:

$$\|\nabla f\|^p_{L^p(0,\infty;L^\infty(\Omega))} \leq C \, \|\nabla f\|^p_{L^p(0,\infty;W^{1,q}(\Omega))} \leq C \, \|f\|^p_{L^p(0,\infty;W^{2,q}(\Omega))} \tag{9.17}$$

$$\leq C \, \|f\|^p_{X^\infty_{p,q,\sigma}} \tag{9.18}$$

In going from (9.17) to (9.18) we have recalled the definition of $f \in X^\infty_{p,q,\sigma}$ in (2.28), (8.13), as f was taken at the outset on $\mathcal{D}(\mathbb{A}_{F,q}) = \mathcal{D}(\mathcal{A}_q) \subset L^q_\sigma(\Omega)$. Then, the sought-after final estimate of the non-linear term $\mathcal{N}_q f, f \in X^\infty_{p,q,\sigma}$ below (9.4), is obtained from substituting (9.15b) and (9.18) into the RHS of (9.14). We obtain

$$\|\mathcal{N}_q f\|_{L^p(0,\infty;L^q_\sigma(\Omega))} \leq C \, \|f\|^2_{X^\infty_{p,q,\sigma}}, \quad f \in X^\infty_{p,q,\sigma}. \tag{9.19}$$

Returning to (9.8), we finally, obtain by (9.19)

$$\|\mathcal{F}(z_0,f)\|_{X^\infty_{p,q,\sigma}} \le C\Big\{ \|z_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} + \|f\|^2_{X^\infty_{p,q,\sigma}} \Big\}. \tag{9.20}$$

Step 4: We now impose the restrictions on the data on the RHS of (9.20): z_0 is in a ball of radius $r_1 > 0$ in $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$ and f is in a ball of radius $r > 0$ in $X^\infty_{p,q,\sigma}$. We further demand that the final result $\mathcal{F}(z_0,f)$ shall lie in a ball of radius r in $X^\infty_{p,q,\sigma}$. Thus we obtain from (9.20)

$$\|\mathcal{F}(z_0,f)\|_{X^\infty_{p,q,\sigma}} \le C\Big\{ \|z_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} + \|f\|^2_{X^\infty_{p,q,\sigma}} \Big\} \le C(r_1 + r^2) \le r \tag{9.21}$$

This implies

$$Cr^2 - r + Cr_1 \le 0 \quad \text{or} \quad \frac{1-\sqrt{1-4C^2r_1}}{2C} \le r \le \frac{1+\sqrt{1-4C^2r_1}}{2C} \tag{9.22}$$

whereby

$$\Big\{ \text{ range of values of r } \Big\} \longrightarrow \text{ interval } \Big[0, \frac{1}{C}\Big], \text{ as } r_1 \searrow 0 \tag{9.23}$$

a constraint which is guaranteed by taking

$$r_1 \le \frac{1}{4C^2}, \ C \text{ being the constant in (9.20).} \tag{9.24}$$

We have thus established that by taking r_1 as in (9.24) and subsequently r as in (9.22), then the map

$$\mathcal{F}(z_0,f) \text{ takes: } \left\{ \begin{array}{c} \text{ball in } \widetilde{B}^{2-2/p}_{q,p}(\Omega) \\ \text{of radius } r_1 \end{array} \right\} \times \left\{ \begin{array}{c} \text{ball in } X^\infty_{p,q,\sigma} \\ \text{of radius } r \end{array} \right\} \text{ into } \left\{ \begin{array}{c} \text{ball in } X^\infty_{p,q,\sigma} \\ \text{of radius } r \end{array} \right\},$$

$$d < q, \ 1 < p < \frac{2q}{2q-1} \tag{9.25}$$

This establishes Theorem 9.2. □

Proof of Theorem 9.3 Step 1: For f_1, f_2 both in the ball of $X^\infty_{p,q,\sigma}$ of radius r obtained in the proof of Theorem 9.2, we estimate from (9.5):

$$\|\mathcal{F}(z_0,f_1) - \mathcal{F}(z_0,f_2)\|_{X^\infty_{p,q,\sigma}} = \left\| \int_0^t e^{\mathbb{A}_{F,q}(t-\tau)} \big[\mathcal{N}_q f_1(\tau) - \mathcal{N}_q f_2(\tau)\big] d\tau \right\|_{X^\infty_{p,q,\sigma}} \tag{9.26}$$

$$\le \widetilde{m}\, \|\mathcal{N}_q f_1 - \mathcal{N}_q f_2\|_{L^p(0,\infty;L^q_\sigma(\Omega))} \tag{9.27}$$

after invoking the maximal regularity property (8.13).

Step 2: Next recalling $\mathcal{N}_q f = P_q[(f\cdot\nabla)f]$ from (2.11), we estimate the RHS of (9.27). In doing so, we add and subtract $(f_2\cdot\nabla)f_1$, set $A = (f_1\cdot\nabla)f_1 - (f_2\cdot\nabla)f_1$, $B = (f_2\cdot\nabla)f_1 - (f_2\cdot\nabla)f_2$, and use

$$|A+B|^q \leq 2^q\big[\,|A|^q + |B|^q\,\big] \qquad (*).$$

[T-L.1, p 12] We obtain, again ignoring $\|P_q\|$:

$$\|\mathcal{N}_q f_1 - \mathcal{N}_q f_2\|_{L^p(0,\infty;L^q_\sigma(\Omega))} \leq \int_0^\infty \left\{\left[\int_\Omega |(f_1\cdot\nabla)f_1 - (f_2\cdot\nabla)f_2|^q\, d\Omega\right]^{1/q}\right\}^p dt \tag{9.28}$$

$$= \int_0^\infty \left[\int_\Omega |A+B|^q\, d\Omega\right]^{p/q} dt \tag{9.29}$$

$$\leq 2^q \int_0^\infty \left\{\int_\Omega \big[\,|A|^q + |B|^q\,\big] d\Omega\right\}^{p/q} dt \tag{9.30}$$

$$= 2^q \int_0^\infty \left\{\left[\int_\Omega |A|^q\, d\Omega + \int_\Omega |B|^q\, d\Omega\right]^{1/q}\right\}^p dt \tag{9.31}$$

$$= 2^q \int_0^\infty \left\{\left[\, \|A\|^q_{L^q(\Omega)} + \|B\|^q_{L^q(\Omega)}\,\right]^{1/q}\right\}^p dt \tag{9.32}$$

$$(\text{ by } (*)) \qquad \leq 2^q \cdot 2^{1/q} \int_0^\infty \left\{\, \|A\|_{L^q(\Omega)} + \|B\|_{L^q(\Omega)}\,\right\}^p dt \tag{9.33}$$

$$(\text{ by } (*)) \qquad \leq 2^{p+q+1/q} \int_0^\infty \left[\, \|A\|^p_{L^q(\Omega)} + \|B\|^p_{L^q(\Omega)}\,\right] dt \tag{9.34}$$

$$= 2^{p+q+1/q} \int_0^\infty \Big[\, \|((f_1-f_2)\cdot\nabla)f_1\|^p_{L^q(\Omega)} + \|(f_2\cdot\nabla)(f_1-f_2)\|^p_{L^q(\Omega)}\Big] dt \tag{9.35}$$

$$= 2^{p+q+1/q} \int_0^\infty \Big\{\, \|f_1-f_2\|^p_{L^q(\Omega)}\, \|\nabla f_1\|^p_{L^q(\Omega)} + \|f_2\|^p_{L^q_\sigma(\Omega)}\, \|\nabla(f_1-f_2)\|^p_{L^q(\Omega)}\Big\} dt \tag{9.36}$$

Step 3: We now notice that regarding each of the integral term in the RHS of (9.36) we are structurally and topologically as in the RHS of (9.12), except that in (9.36) the gradient terms $\nabla f_1, \nabla(f_1-f_2)$ are penalized in the $L^q_\sigma(\Omega)$-norm which is dominated by the $L^\infty(\Omega)$-norm, as it occurs for the gradient term ∇f in (9.12). Thus we can apply to each integral term on the RHS of (9.36) the same argument as in going from (9.12) to the estimates (9.15b) and (9.18) with $q > \dim \Omega = 3$. We obtain

$$
\begin{aligned}
\|\mathcal{N}_q f_1 - \mathcal{N}_q f_2\|^p_{L^p(0,\infty;L^q_\sigma(\Omega))} &\le \text{RHS of (9.36)} \\
\text{see (9.14)} \qquad &\le C\Big\{ \|f_1 - f_2\|^p_{L^\infty(0,\infty;L^q_\sigma(\Omega))} \|\nabla f_1\|^p_{L^p(0,\infty;L^\infty(\Omega))} \\
&\qquad + \|f_2\|^p_{L^\infty(0,\infty;L^q_\sigma(\Omega))} \|\nabla (f_1 - f_2)\|^p_{L^p(0,\infty;L^\infty(\Omega))} \Big\} \qquad (9.37) \\
\text{see (9.15b) and (9.18)} \qquad &\le C\Big\{ \|f_1 - f_2\|^p_{X^\infty_{p,q,\sigma}} \|f_1\|^p_{X^\infty_{p,q,\sigma}} + \|f_2\|^p_{X^\infty_{p,q,\sigma}} \|f_1 - f_2\|^p_{X^\infty_{p,q,\sigma}} \Big\} \qquad (9.38) \\
&= C\Big\{ \|f_1 - f_2\|^p_{X^\infty_{p,q,\sigma}} \big(\|f_1\|^p_{X^\infty_{p,q,\sigma}} + \|f_2\|^p_{X^\infty_{p,q,\sigma}} \big) \Big\} \qquad (9.39)
\end{aligned}
$$

Finally (9.39) yields

$$
\begin{aligned}
\|\mathcal{N}_q f_1 - \mathcal{N}_q f_2\|_{L^p(0,\infty;L^q_\sigma(\Omega))} &\le C^{1/p} \|f_1 - f_2\|_{X^\infty_{p,q,\sigma}} \Big(\|f_1\|^p_{X^\infty_{p,q,\sigma}} + \|f_2\|^p_{X^\infty_{p,q,\sigma}} \Big)^{1/p} \qquad (9.40) \\
\text{(by } (*)) \qquad &\le 2^{1/p} C^{1/p} \|f_1 - f_2\|_{X^\infty_{p,q,\sigma}} \Big(\|f_1\|_{X^\infty_{p,q,\sigma}} + \|f_2\|_{X^\infty_{p,q,\sigma}} \Big) \qquad (9.41)
\end{aligned}
$$

Step 4: Using estimate (9.41) on the RHS of estimate (9.27) yields

$$
\|\mathcal{F}(z_0, f_1) - \mathcal{F}(z_0, f_2)\|_{X^\infty_{p,q,\sigma}} \le K_p \|f_1 - f_2\|_{X^\infty_{p,q,\sigma}} \Big(\|f_1\|_{X^\infty_{p,q,\sigma}} + \|f_2\|_{X^\infty_{p,q,\sigma}} \Big) \tag{9.42}
$$

$K_p = \widetilde{m} 2^{1/p} C^{1/p}$ ($\widetilde{m}$ as in (9.27), C as in (9.39)). Next, pick f_1, f_2 in the ball of $X^\infty_{p,q,\sigma}$ of radius R:

$$
\|f_1\|_{X^\infty_{p,q,\sigma}}, \|f_2\|_{X^\infty_{p,q,\sigma}} \le R \tag{9.43}
$$

Then

$$
\|\mathcal{F}(z_0, f_1) - \mathcal{F}(z_0, f_2)\|_{X^\infty_{p,q,\sigma}} \le \rho_0 \|f_1 - f_2\|_{X^\infty_{p,q,\sigma}} \tag{9.44}
$$

and $\mathcal{F}(z_0, f)$ is a contraction on the space $X^\infty_{p,q,\sigma}$ as soon as

$$
\rho_0 \equiv 2 K_p R < 1 \text{ or } R < {}^1\!/_{2K_p}, \; K_p = \widetilde{m} 2^{1/p} C^{1/p}. \tag{9.45}
$$

In this case, the map $\mathcal{F}(z_0, f)$ defined in (9.5) has a fixed point z in $X^\infty_{p,q,\sigma}$

$$
\mathcal{F}(z_0, z) = z, \text{ or } z = e^{\mathbb{A}_{F,q} t} z_0 - \int_0^t e^{\mathbb{A}_{F,q}(t-\tau)} \mathcal{N}_q z(\tau) d\tau \tag{9.46}
$$

and such fixed point $z \in X^\infty_{p,q,\sigma}$ is the unique solution of the translated non-linear equation (9.1), or (9.2) with finite dimensional control u in feedback form, as described by the RHS of (9.1). Theorem 9.1 is proved. □

10 Proof of Theorem 3.4. Local exponential decay of the non-linear translated z-dynamics (9.1) with finite dimensional localized feedback control

In this section we return to the feedback problem (9.1) rewritten equivalently as in (9.3)

$$z(t) = e^{\mathbb{A}_{F,q}t} z_0 - \int_0^t e^{\mathbb{A}_{F,q}(t-\tau)} \mathcal{N}_q z(\tau) d\tau. \tag{10.1}$$

For z_0 in a small ball of $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$, Theorem 9.1 provides a unique solution in a ball of $X^{\infty}_{p,q,\sigma}$. We recall from (7.3) = (9.4)

$$\left\| e^{\mathbb{A}_{F,q}t} z_0 \right\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \leq M_{\gamma_0} e^{-\gamma_0 t} \left\| z_0 \right\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)}, \ t \geq 0. \tag{10.2}$$

Our goal now is to show that for z_0 in a small ball of $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$, problem (10.1) satisfies the exponential decay

$$\left\| z(t) \right\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \leq C e^{-at} \left\| z_0 \right\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)}, \ t \geq 0, \ \text{for some constants, } a > 0, C = C_a \geq 1.$$

Step 1: Starting from (10.1) and using (10.2) we estimate

$$\left\| z(t) \right\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \leq M_{\gamma_0} e^{-\gamma_0 t} \left\| z_0 \right\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} + \sup_{0 \leq t \leq \infty} \left\| \int_0^t e^{\mathbb{A}_{F,q}(t-\tau)} \mathcal{N}_q z(\tau) d\tau \right\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \tag{10.3}$$

$$\leq M_{\gamma_0} e^{-\gamma_0 t} \left\| z_0 \right\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} + C_1 \left\| \int_0^t e^{\mathbb{A}_{F,q}(t-\tau)} \mathcal{N}_q z(\tau) d\tau \right\|_{X^{\infty}_{p,q,\sigma}} \tag{10.4}$$

$$\leq M_{\gamma_0} e^{-\gamma_0 t} \left\| z_0 \right\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} + C_2 \left\| \mathcal{N}_q z \right\|_{L^p(0,\infty; L^q_\sigma(\Omega))} \tag{10.5}$$

$$\left\| z(t) \right\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \leq M_{\gamma_0} e^{-\gamma_0 t} \left\| z_0 \right\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} + C_3 \left\| z \right\|^2_{X^{\infty}_{p,q,\sigma}}, \quad C_3 = C_2 C. \tag{10.6}$$

In going from (10.3) to (10.4) we have recalled the embedding $X^{\infty}_{p,q,\sigma} \hookrightarrow L^{\infty}\big(0,\infty; \widetilde{B}^{2-2/p}_{q,p}(\Omega)\big)$ from (2.30). Next, in going from (10.4) to (10.5) we have used the maximal regularity property (8.13). Finally, to go from (10.5) to (10.6) we have invoked estimate (9.19).

Step 2: We shall next establish that

$$\left\| z \right\|_{X^{\infty}_{p,q,\sigma}} \leq M_1 \left\| z_0 \right\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} + K \left\| z \right\|^2_{X^{\infty}_{p,q,\sigma}}, \ \text{hence} \ \left\| z \right\|_{X^{\infty}_{p,q,\sigma}} \left(1 - K \left\| z \right\|_{X^{\infty}_{p,q,\sigma}}\right) \leq M_1 \left\| z_0 \right\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \tag{10.7}$$

In fact, to this end, we take the $X^{\infty}_{p,q,\sigma}$-estimate of equation (10.1). We obtain

$$\|z\|_{X^{\infty}_{p,q,\sigma}} \le \left\| e^{\mathbb{A}_{F,q}t} z_0 \right\|_{X^{\infty}_{p,q,\sigma}} + \left\| \int_0^t e^{\mathbb{A}_{F,q}(t-\tau)} \mathcal{N}_q z(\tau) d\tau \right\|_{X^{\infty}_{p,q,\sigma}} \tag{10.8}$$

from which then (10.7) follows by invoking the maximal regularity property (8.14) on $e^{\mathbb{A}_{F,q}t}$ as well as the maximal regularity estimate (8.13) followed by use of of (9.19), as in going from (10.4) to (10.6)

$$\left\| \int_0^t e^{\mathbb{A}_{F,q}(t-\tau)} \mathcal{N}_q z(\tau) d\tau \right\|_{X^{\infty}_{p,q,\sigma}} \le \widetilde{M} \left\| \mathcal{N}_q z \right\|_{L^p(0,\infty; L^q_\sigma(\Omega))} \tag{10.9}$$

$$\le \widetilde{M} C \left\| z \right\|^2_{X^{\infty}_{p,q,\sigma}}. \tag{10.10}$$

Thus (10.7) is proved with $K = \widetilde{M}C$ where C is the same constant occurring in (9.19), hence in (9.21), (9.22).

Step 3: The well-posedness Theorem 9.1 says that

$$\left\{ \begin{array}{c} \text{If } \|z_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \le r_1 \\ \text{for } r_1 \text{ sufficiently small} \end{array} \right\} \Longrightarrow \left\{ \begin{array}{c} \text{The solution } z \text{ satisfies} \\ \|z\|_{X^{\infty}_{p,q,\sigma}} \le r \end{array} \right\} \tag{10.11}$$

where r satisfies the constraint (9.22) in terms of r_1 and some constant C in (9.19) that occurs for $K = \widetilde{M}C$ in (10.10). We seek to guarantee that we can obtain

$$\begin{cases} \|z\|_{X^{\infty}_{p,q,\sigma}} \le r < \frac{1}{2K} = \frac{1}{2\widetilde{M}C} \left(< \frac{1}{2C} \right) \\ \\ \text{hence } \frac{1}{2} < 1 - K \|z\|_{X^{\infty}_{p,q,\sigma}}, \end{cases} \tag{10.12}$$

where w.l.o.g. we can take the maximal regularity constant $\widetilde{M}$ in (8.13) to satisfy $\widetilde{M} > 1$. Again, the constant C arises from application of estimate (9.19). This is indeed possible by choosing $r_1 > 0$ sufficiently small. In fact, as $r_1 \searrow 0$, (9.23) shows that the interval $r_{min} \le r \le r_{max}$ of corresponding values of r tends to the interval $\left[0, \dfrac{1}{C}\right]$. Thus (10.12) can be achieved as $r_{min} \searrow 0$: $0 < r_{min} < r < \dfrac{1}{2\widetilde{M}C} < \dfrac{1}{2C}$. Next, (10.12) implies that (10.7) holds true and yields then

$$\|z\|_{X^{\infty}_{p,q,\sigma}} \le 2M_1 \|z_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \le 2M_1 r_1. \tag{10.13}$$

Substituting (10.13) in estimate (10.6) then yields with $\widehat{M} = \max\{M_{\gamma_0}, M_1\}$

$$\|z(t)\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \leq M_{\gamma_0} e^{-\gamma_0 t} \|z_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} + 4C_3 M_1^2 \|z_0\|^2_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \tag{10.14}$$

$$= \widehat{M}\left[e^{-\gamma_0 t} + 4\widehat{M} C_3 \|z_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)}\right] \|z_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \tag{10.15}$$

$$\|z(t)\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \leq \widehat{M}\left[e^{-\gamma_0 t} + 4\widehat{M} C_3 r_1\right] \|z_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \tag{10.16}$$

recalling the constant $r_1 > 0$ in (10.11).

Step 4: Now take T sufficiently large and $r_1 > 0$ sufficiently small such that

$$\beta \equiv \widehat{M} e^{-\gamma_0 T} + 4\widehat{M}^2 C_3 r_1 < 1 \tag{10.17}$$

Then (10.15) implies by (10.17)

$$\|z(T)\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \leq \beta \|z_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \quad \text{and hence} \tag{10.18a}$$

$$\|z(nT)\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \leq \beta \|z((n-1)T)\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \leq \beta^n \|z_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)}. \tag{10.18b}$$

Since $\beta < 1$, the semigroup property of the evolution implies that there are constants $M_{\widetilde{\gamma}} \geq 1, \widetilde{\gamma} > 0$ such that

$$\|z(t)\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \leq M_{\widetilde{\gamma}} e^{-\widetilde{\gamma} t} \|z_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)}, \quad t \geq 0 \tag{10.19}$$

This proves Theorem 3.4. □

Remark 10.1. The above computations - (10.17) through (10.19) - can be used to support qualitatively the intuitive expectation that "the larger the decay rate γ_0 in (7.3) = (9.4) = (10.2) of the linearized feedback w-dynamics (7.1), the larger the decay rate $\widetilde{\gamma}$ in (10.19) of the nonlinear feedback z-dynamics (3.20) = (9.1); hence the larger the rate $\widetilde{\gamma}$ in (3.29) of the original y-dynamics in (3.28)".

The following considerations are somewhat qualitative. Let $S(t)$ denote the non-linear semigroup in the space $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$, with infinitesimal generator $\left[\mathbb{A}_{F,q} - \mathcal{N}_q\right]$ describing the feedback z-dynamics (3.20)=(9.1), hence (9.2), as guaranteed by the well posedness Theorem 3.3 = Theorem 9.1. Thus, $z(t;z_0) = S(t)z_0$ on $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$. By (10.17), we can rewrite (10.18a) as:

$$\|S(T)\|_{\mathcal{L}\left(\widetilde{B}^{2-2/p}_{q,p}(\Omega)\right)} \leq \beta < 1. \tag{10.20}$$

It follows from [Bal, p 178] via the semigroup property that

$$-\widetilde{\gamma} \;\text{ is just below }\; \frac{\ln\beta}{T} < 0. \tag{10.21}$$

Pick $r_1 > 0$ in (10.17) so small that $4\widehat{M}^2 C_3 r_1$ is negligible, so that β is just above $\widehat{M}e^{-\gamma_0 T}$, so $\ln\beta$ is just above $\big[\ln\widehat{M} - \gamma_0 T\big]$, hence

$$\frac{\ln\beta}{T} \text{ is just above } (-\gamma_0) + \frac{\ln\widehat{M}}{T}. \tag{10.22}$$

Hence, by (10.21), (10.22),

$$\widetilde{\gamma} \sim \gamma_0 - \frac{\ln\widehat{M}}{T} \tag{10.23}$$

and the larger γ_0, the larger is $\widetilde{\gamma}$, as desired.

11 Well-posedness of the pressure χ for the z-problem (2.7) in feedback form, and of the pressure π for the y-problem (2.1) in feedback form (3.22) in the vicinity of the equilibrium pressure π_e.

The z-problem in feedback form: We return to the translated z problem (2.7), with $Ł_e(z)$ given by (2.39)

$$z_t - \nu\Delta z + L_e(z) + (z\cdot\nabla)z + \nabla\chi = m(Fz) \quad \text{in } Q \tag{11.1a}$$

$$\text{div } z = 0 \quad \text{in } Q \tag{11.1b}$$

$$z = 0 \quad \text{on } \Sigma \tag{11.1c}$$

$$z(0,x) = y_0(x) - y_e(x) \quad \text{on } \Omega \tag{11.1d}$$

with Fz given in the feedback form as in (3.20) = (9.1)

$$m(Fz) = m\left(\sum_{k=1}^{K}\left(z_N, p_k\right)_\omega u_k\right), \quad z_N = P_N z \tag{11.1e}$$

for which Theorem 3.3 = Theorem 9.1 provides a local well-posedness result (3.22), (3.23) for the z variable. We now complement such well-posedness for z with a corresponding local well-posedness result for the pressure χ.

Theorem 11.1. *Consider the setting of Theorem 3.3 = Theorem 9.1 for problem (10.1a-e). Then the following well-posedness result for the pressure χ holds true, where we recall the spaces $Y^\infty_{p,q}$ for $T=\infty$ and $\widehat{W}^{1,q}(\Omega)$ in (2.29) as well as the steady state pressure π_e from Theorem 2.1:*

$$\|\chi\|_{Y^\infty_{p,q}} \le \widetilde{C}\, \|y_0 - y_e\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \left\{ \|y_0 - y_e\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} + 1\right\}. \tag{11.2}$$

Proof. We first apply the full maximal-regularity (2.33) to the Stokes component of problem (11.1) with $F_q = P_q\big(mF(z) - L_e(z) - (z\cdot\nabla)z\big)$ to obtain

$$\begin{aligned}\|z\|_{X^\infty_{p,q,\sigma}} + \|\chi\|_{Y^\infty_{p,q}} &\le C\Big\{ \|P_q[m(Fz) - (z\cdot\nabla)z - L_e(z)]\|_{L^p\left(0,\infty;L^q_\sigma(\Omega)\right)} + \|z_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)}\Big\}\\ &\le C\Big\{ \|P_q[m(Fz)]\|_{L^p\left(0,\infty;L^q_\sigma(\Omega)\right)} + \|P_q(z\cdot\nabla)z\|_{L^p\left(0,\infty;L^q_\sigma(\Omega)\right)}\\ &\qquad + \|P_q L_e(z)\|_{L^p\left(0,\infty;L^q_\sigma(\Omega)\right)} + \|z_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)}\Big\}.\end{aligned} \tag{11.3}$$

But $P_q[mF(z)] = mF(z)$ as the vectors u_k in the definition of F in (3.26) are $u_k \in W_N^u \subset L^q_\sigma(\Omega)$. Moreover $F \in \mathcal{L}(L^q_\sigma(\Omega))$, we obtain

$$\|P_q[m(Fz)]\|_{L^p\left(0,\infty;L^q_\sigma(\Omega)\right)} \leq C_1 \|z\|_{X^\infty_{p,q,\sigma}}. \tag{11.4}$$

recalling the space $X^\infty_{p,q,\sigma}$ from (2.28). Next, recalling (9.19) for $\mathcal{N}_q z = P_q\big[(z \cdot \nabla)z\big]$, see (2.11), we obtain

$$\|P_q(z \cdot \nabla)z\|_{L^p\left(0,\infty;L^q_\sigma(\Omega)\right)} \leq C_2 \|z\|^2_{X^\infty_{p,q,\sigma}}. \tag{11.5}$$

The equilibrium solution $\{y_e, \pi_e\}$ is given by Theorem 2.1 as satisfying

$$\|y_e\|_{W^{2,q}(\Omega)} + \|\pi_e\|_{\widehat{W}^{q,1}} \leq c\,\|f\|_{L^q(\Omega)}, \quad 1 < q < \infty. \tag{11.6}$$

We next estimate the term $P_q L_e(z) = P_q[(y_e \cdot \nabla)z + (z \cdot \nabla)y_e]$ in (11.3)

$$\|P_q L_e(z)\|_{L^p\left(0,\infty;L^q_\sigma(\Omega)\right)} = \|P_q(y_e.\nabla)z + P_q(z.\nabla)y_e\|_{L^p\left(0,\infty;L^q_\sigma(\Omega)\right)} \tag{11.7}$$

$$\leq \|P_q(y_e.\nabla)z\|_{L^p\left(0,\infty;L^q_\sigma(\Omega)\right)} + \|P_q(z.\nabla)y_e\|_{L^p\left(0,\infty;L^q_\sigma(\Omega)\right)} \tag{11.8}$$

$$\leq \|y_e\|_{L^q(\Omega)} \|\nabla z\|_{L^p\left(0,\infty;L^q_\sigma(\Omega)\right)} + \|z\|_{L^p\left(0,\infty;L^q_\sigma(\Omega)\right)} \|\nabla y_e\|_{L^q(\Omega)} \tag{11.9}$$

$$\leq 2C_2 \|f\|_{L^q(\Omega)} \|z\|_{L^p\left(0,\infty;L^q_\sigma(\Omega)\right)} \tag{11.10}$$

$$\leq C_3 \|z\|_{X^\infty_{p,q,\sigma}} \tag{11.11}$$

with the constant C_3 depending on the $L^q(\Omega)$-norm of the datum f. Setting now $C_4 = C \cdot \{C_1, C_2, C_3\}$ and substituting (11.4), (11.5), (11.11) in (11.3), we obtain

$$\|z\|_{X^\infty_{p,q,\sigma}} + \|\chi\|_{Y^\infty_{p,q}} \leq C_4 \Big\{ \|z\|^2_{X^\infty_{p,q,\sigma}} + 2\|z\|_{X^\infty_{p,q,\sigma}} + \|z_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \Big\} \tag{11.12}$$

Next we drop the term $\|z\|_{X^\infty_{p,q,\sigma}}$ on the left hand side of (11.12) and invoking (10.13) to estimate $\|z\|_{X^\infty_{p,q,\sigma}}$. Thus we obtain

$$\|\chi\|_{Y^\infty_{p,q}} \leq C_5 \Big\{ \|z_0\|^2_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} + 2\|z_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} + \|z_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \Big\} \tag{11.13}$$

$$\leq \widetilde{C} \|z_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \Big\{ \|z_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} + 1 \Big\}, \quad \widetilde{C} = 3C_5 \tag{11.14}$$

and (11.14) proves (11.2), as desired, recalling (2.7e). □

The y-problem in feedback form We return to the original y-problem however in feedback form as in (3.26), (3.27), for which Theorem 2.5(i) proves a local well-posedness result. We now complement such well-posedness result for y with the corresponding local well-posedness result for the pressure π.

Theorem 11.2. *Consider the setting of Theorem 3.5 for the y-problem in (3.27). Then, the following well-posedness result for the pressure π holds true.*

$$\|\pi - \pi_e\|_{Y^T_{p,q}} \leq \|\pi - \pi_e\|_{Y^\infty_{p,q}} \leq \widetilde{C}\, \|y_0 - y_e\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \Big\{ \|y_0 - y_e\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} + 1 \Big\} \tag{11.15}$$

$$\leq \widehat{C} \Big\{ \|y_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} + \|y_e\|_{W^{2,q}(\Omega)} \Big\} \Big\{ \|y_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} + \|y_e\|_{W^{2,q}(\Omega)} + 1 \Big\} \tag{11.16}$$

$$\leq \widehat{C} \Big\{ \|y_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} + \|f\|_{L^q(\Omega)} \Big\} \Big\{ \|y_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} + \|f\|_{L^q(\Omega)} + 1 \Big\} \tag{11.17}$$

$$\|\pi\|_{Y^T_{p,q}} \leq \widehat{C}\, \|y_0 - y_e\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \Big\{ \|y_0 - y_e\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} + 1 \Big\} + cT^{1/p}\, \|\pi_e\|_{\widehat{W}^{1,q}(\Omega)}\,, \quad 0 < T < \infty \tag{11.18}$$

$$\leq \widehat{C} \Big\{ \|y_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} + \|f\|_{L^q(\Omega)} \Big\} \Big\{ \|y_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} + \|f\|_{L^q(\Omega)} + 1 \Big\} + cT^{1/p}\, \|f\|_{L^q(\Omega)}\,, \quad 0 < T < \infty \tag{11.19}$$

Proof. We return to the estimate (11.2) for χ and recall $\chi = \pi - \pi_e$ from (2.7e) to obtain (11.15). We next estimate $y - y_e$ by

$$\|y_0 - y_e\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \leq C \big\{ \|y_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} + \|y_e\|_{W^{2,q}(\Omega)} \big\}. \tag{11.20}$$

which substituted in (11.15) yields (11.16). In turn, (11.16) leads to (11.17) by means of (11.6). □

Appendix A On Helmholtz Decomposition

We return to the Helmholtz decomposition in (2.4), (2.5) and provide additional information. For $M \subset L^q(\Omega),\ 1 < q < \infty$, we denote the annihilator of M by

$$M^{\perp} = \left\{ f \in L^{q'}(\Omega) : \int_{\Omega} fg \; d\Omega = 0, \text{ for all } g \in M \right\} \tag{A.1}$$

where q' is the dual exponent of q : ${}^1/_q + {}^1/_{q'} = 1$.

Proposition A.1. *[H-S, Prop 2.2.2 p6], [Ga.3, Ex. 16 p115], [F-M-M]*
Let $\Omega \subset \mathbb{R}^d$ be an open set and let $1 < q < \infty$.

a) The Helmholtz decomposition exists for $L^q(\Omega)$ if and only if it exists for $L^{q'}(\Omega)$, and we have: (adjoint of P_q) $= P_q^* = P_{q'}$ (in particular P_2 is orthogonal), where P_q is viewed as a bounded operator $L^q(\Omega) \longrightarrow L^q(\Omega)$, and $P_q^* = P_{q'}$ as a bounded operator $L^{q'}(\Omega) \longrightarrow L^{q'}(\Omega)$, ${}^1/_q + {}^1/_{q'} = 1$.

b) Then, with reference to (2.5)

$$\Big[L^q_{\sigma}(\Omega)\Big]^{\perp} = G^{q'}(\Omega) \text{ and } \Big[G^q(\Omega)\Big]^{\perp} = L^{q'}_{\sigma}(\Omega) \tag{A.2a}$$

Remark A.1. Throughout the paper we shall use freely that

$$\big(L^q_{\sigma}(\Omega)\big)' = L^{q'}_{\sigma}(\Omega), \quad \frac{1}{q} + \frac{1}{q'} = 1 \tag{A.2b}$$

Thus can be established as follows. From (2.5) write $L^q_{\sigma}(\Omega)$ as a factor space $L^q_{\sigma}(\Omega) = L^q(\Omega)/G^q(\Omega) \equiv X/M$ so that [T-L.1, p 135].

$$\big(L^q_{\sigma}(\Omega)\big)' = \big(L^q(\Omega)/G^q(\Omega)\big)' = \big(X/M\big)' = M^{\perp} = \Big[G^q(\Omega)\Big]^{\perp} = L^{q'}_{\sigma}(\Omega) \tag{A.2c}$$

In the last step, we have invoked (A.2a), which is also established in [Ga.3, Lemma 2.1, p 116]. Similarly

$$\big(G^q(\Omega)\big)' = \big(L^q(\Omega)/L^q_{\sigma}(\Omega)\big)' = \Big[L^q_{\sigma}(\Omega)\Big]^{\perp} = G^{q'}(\Omega) \tag{A.2d}$$

Appendix B Proof of Theorem 2.6: maximal regularity of the Oseen operator $\mathcal{A}_q$ on $L^q_\sigma(\Omega)$, $1 < p, q < \infty, T < \infty$.

Part I: (2.46). By (2.41) with $\psi_0 = 0$

$$\psi(t) = \int_0^t e^{\mathcal{A}_q(t-\tau)} F_\sigma(\tau) d\tau. \tag{B.1}$$

where by the statement preceding Theorem 2.4

$$\left\| e^{\mathcal{A}_q(t-\tau)} \right\|_{\mathcal{L}(L^q_\sigma(\Omega))} \leq M e^{b(t-\tau)}, \quad 0 \leq \tau \leq t \tag{B.2}$$

for $M \geq 1$, b possibly depending on q.

Step 1: We have the following estimate

$$\int_0^T \|\psi(t)\|^p_{L^q_\sigma(\Omega)} dt \leq C_T \int_0^T \|F_\sigma(t)\|^p_{L^q_\sigma(\Omega)} dt \tag{B.3}$$

where the constant C_T may depend also on p, q, b. This follows at once from the Young's inequality for convolutions [Sa, p26]:

$$\|\psi(t)\|_{L^q_\sigma(\Omega)} \leq M \int_0^t e^{b(t-\tau)} \|F_\sigma(\tau)\|_{L^q_\sigma(\Omega)} d\tau \in L^p(0,T), \; T < \infty,$$

and the convolution of the $L^p(0,T)$-function $\|F_\sigma\|_{L^q_\sigma(\Omega)}$ and the $L^1(0,T)$-function e^{bt} is in $L^p(0,T)$. More elementary, one can use Hölder inequality with $1/p + 1/\tilde{p} = 1$ and obtain an explicit constant.

Step 2: Claim: Here we shall next complement (B.3) with the estimate

$$\int_0^T \|A_q\psi(t)\|^p_{L^q_\sigma(\Omega)} dt \leq C \int_0^T \|\psi(t)\|^p_{L^q_\sigma(\Omega)} dt + C \int_0^T \|F_\sigma(t)\|^p_{L^q_\sigma(\Omega)} dt \tag{B.4}$$

to be shown below. Using (B.3) in (B.4) then yields

$$\int_0^T \|A_q\psi(t)\|^p_{L^q_\sigma(\Omega)} dt \leq C_T \int_0^T \|F_\sigma(t)\|^p_{L^q_\sigma(\Omega)} dt. \tag{B.5}$$

With respect to (2.41) with $\psi_0 = 0$, then (B.5) says

$$F_\sigma \in L^p(0,T; L^q_\sigma(\Omega)) \longrightarrow \psi \in L^p(0,T; \mathcal{D}(A_q) = \mathcal{D}(\mathcal{A}_q)) \tag{B.6}$$

while (2.40) then yields via (B.6)

$$F_\sigma \in L^p(0,T;L^q_\sigma(\Omega)) \longrightarrow \psi_t \in L^p(0,T;L^q_\sigma(\Omega)) \tag{B.7}$$

continuously. Then, (B.6), (B) show is part (i) of Theorem 2.6.

Proof of (B.4): . In this step, with $\psi_0 = 0$, we shall employ the alternative formula, via (2.42) ($\nu = 1$, wlog)

$$\psi(t) = \int_0^t e^{-A_q(t-\tau)}(-A_{o,q})\psi(\tau)d\tau + \int_0^t e^{-A_q(t-\tau)} F_\sigma(\tau)d\tau. \tag{B.8}$$

where by maximal regularity of the Stokes operator $-A_q$ on the space $L^q_\sigma(\Omega)$, as asserted in Theorem 2.5.ii, Eq (2.35), we have in particular

$$F_\sigma \in L^p(0,T;L^q_\sigma(\Omega)) \longrightarrow \int_0^t e^{-A_q(t-\tau)} F_\sigma(\tau)d\tau \in L^p(0,T;\mathcal{D}(A_q)) \quad \text{continuously.} \tag{B.9}$$

Regarding the first integral term in (B.8) we shall employ the (complex) interpolation formula (2.22), and recall from (2.9) that $\mathcal{D}(A_{o,q}) = \mathcal{D}(A_q^{1/2})$:

$$\mathcal{D}(A_{o,q}) = \mathcal{D}(A_q^{1/2}) = [\mathcal{D}(A_q), L^q_\sigma(\Omega)]_{1/2} \tag{B.10}$$

so that the interpolation inequality [Triebel, Theorem p 53, Eq(3)] with $\theta = 1/2$ yields from (B.10)

$$\begin{aligned} \|a\|_{\mathcal{D}(A_{o,q})} = \|a\|_{\mathcal{D}\left(A_q^{1/2}\right)} &\leq C \|a\|^{1/2}_{\mathcal{D}(A_q)} \|a\|^{1/2}_{L^q_\sigma(\Omega)} \\ &\leq \varepsilon \|a\|_{\mathcal{D}(A_q)} + C_\varepsilon \|a\|_{L^q_\sigma(\Omega)} \end{aligned} \tag{B.11}$$

[Since $\mathcal{D}(A_q^{1/2}) = W^{1,q}_0(\Omega) \cap L^q_\sigma(\Omega)$ by (2.22), then for $a \in \mathcal{D}(A_q) = W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega) \cap L^q_\sigma(\Omega)$, see (2.17), we may as well invoke the interpolation inequality for W-spaces. [Adams, Theorem 4.13, p 74]:

$$\|a\|_{W^{1,q}_0(\Omega)} \leq \varepsilon \|a\|_{W^{2,q}(\Omega)} + C_\varepsilon \|a\|_{L^q_\sigma(\Omega)} \Big]$$

We return to (B.8) and obtain

$$A_q\psi(t) = A_q \int_0^t e^{-A_q(t-\tau)}(-A_{o,q})\psi(\tau)d\tau + A_q \int_0^t e^{-A_q(t-\tau)} F_\sigma(\tau)d\tau. \tag{B.12}$$

Hence via the maximal regularity of the uniformly stable Stokes semigroup $e^{-A_q t}$, Eq (2.35), (B.11) yields

$$\|A_q\psi\|_{L^p(0,T;L^q_\sigma(\Omega))} \leq C\Big\{ \|A_{o,q}\psi\|_{L^p(0,T;L^q_\sigma(\Omega))} + \|F_\sigma\|_{L^p(0,T;L^q_\sigma(\Omega))} \Big\} \tag{B.13}$$

$$\text{by (B.11)} \qquad \leq \varepsilon' \|A_q\psi\|_{L^p(0,T;L^q_\sigma(\Omega))} + C_{\varepsilon'} \|\psi\|_{L^p(0,T;L^q_\sigma(\Omega))} + C\|F_\sigma\|_{L^p(0,T;L^q_\sigma(\Omega))} \tag{B.14}$$

$\varepsilon' = \varepsilon C > 0$ arbitrarily small. Hence (B.14) yields

$$\|A_q\psi\|_{L^p(0,T;L^q_\sigma(\Omega))} \leq \frac{C_{\varepsilon'}}{1-\varepsilon'} \|\psi\|_{L^p(0,T;L^q_\sigma(\Omega))} + \frac{C}{1-\varepsilon'} \|F_\sigma\|_{L^p(0,T;L^q_\sigma(\Omega))} \tag{B.15}$$

and estimate (B.4) of Step 2 is established. Part I of Theorem 2.6 is proved.

Part II: (2.49). For simplicity of notation, we shall write the proof on $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$ i.e. for $1 < q, p < {}^{2q}/_{2q-1}$. The proof on $\big(L^q_\sigma(\Omega), \mathcal{D}(A_q)\big)_{1-\frac{1}{p},p}$ in the other case ${}^{2q}/_{2q-1} < p$ is exactly the same.

Step 1: Let $\eta_0 \in \widetilde{B}^{2-2/p}_{q,p}(\Omega)$ and consider the s.c. analytic Oseen semigroup $e^{\mathcal{A}_q t}$ on the space $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$, as asserted by Theorem 1.4.ii (take $\nu = 1$ wlog):

$$\eta(t) = e^{\mathcal{A}_q t}\eta_0, \quad \text{or } \eta_t = \mathcal{A}_q\eta = -A_q\eta - A_{o,q}\eta \tag{B.16}$$

Then we can rewrite η as

$$\eta(t) = e^{-A_q t}\eta_0 + \int_0^t e^{-A_q(t-\tau)}(-A_{o,q})\eta(\tau)\, d\tau \tag{B.17}$$

$$A_q\eta(t) = A_q e^{-A_q t}\eta_0 + A_q\int_0^t e^{-A_q(t-\tau)}(-A_{o,q})\eta(\tau)\, d\tau \tag{B.18}$$

We estimate, recalling the maximal regularity (2.35), (2.36) as well as the uniform decay (2.25) of the Stokes operator.

$$\|A_q\eta\|_{L^p(0,T;L^q(\Omega))} \leq C\|\eta_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} + C\|A_{o,q}\eta\|_{L^p(0,T;L^q_\sigma(\Omega))} \tag{B.19}$$

$$\leq C\|\eta_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} + \varepsilon\widetilde{C}\|A_q\eta\|_{L^p(0,T;L^q_\sigma(\Omega))} + C_\varepsilon\|\eta\|_{L^p(0,T;L^q_\sigma(\Omega))} \tag{B.20}$$

after invoking, in the last step, the interpolation inequality (B.11). Thus (B.20) yields via (2.18)

$$\begin{aligned}\|A_q\eta\|_{L^p(0,T;L^q_\sigma(\Omega))} &= \|\mathcal{A}_q\eta\|_{L^p(0,T;L^q_\sigma(\Omega))} \\ &\leq \frac{C}{1-\varepsilon\widetilde{C}}\|\eta_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} + \frac{C_\varepsilon}{1-\varepsilon\widetilde{C}}\|\eta\|_{L^p(0,T;L^q_\sigma(\Omega))}\end{aligned} \tag{B.21}$$

Step 2: With $\eta_0 \in \widetilde{B}^{2-2/p}_{q,p}(\Omega)$, since $e^{\mathcal{A}_q t}$ generates a s.c (analytic) semigroup on $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$, Theorem 2.4.ii, we have

$$\eta(t) = e^{\mathcal{A}_q t}\eta_0 \in C\Big(0, T; \widetilde{B}^{2-2/p}_{q,p}(\Omega)\Big) \subset L^p\Big(0, T; \widetilde{B}^{2-2/p}_{q,p}(\Omega)\Big) \subset L^p\big(0, T; L^q_\sigma(\Omega)\big) \tag{B.22}$$

continuously, where in the last step, we have recalled that $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$ is the interpolation between $L^q(\Omega)$ and $W^{2,q}(\Omega)$, see (2.16b). (B.22) says explicitly

$$\|\eta\|_{L^p\big(0,T;L^q_\sigma(\Omega)\big)} \leq C\,\|\eta_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \tag{B.23}$$

Step 3: Substituting (B.23) in (B.21) yields

$$\|A_q \eta\|_{L^p\big(0,T;L^q_\sigma(\Omega)\big)} \leq C\,\|\eta_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \tag{B.24}$$

and (2.49) is established, from which (2.50) follows at once. Thus Theorem 2.6 is proved. □

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12 Part II: Using finitely many interior, localized, feedback controls also in dimension $d = 3$.

Abstract

The present dissertation provides a solution to the following recognized open problem in the theory of uniform stabilization of d-dimensional Navier-Stokes equations in the vicinity of an unstable equilibrium solution, by means of tangential boundary localized feedback controls: can these stabilizing controls be asserted to be finite dimensional also in the physical dimension $d = 3$? The result is known for $d = 2$ and also for $d = 3$, however only for compactly supported initial conditions. For physical dimension $d = 3$, the N-S nonlinearity forces a topological level sufficiently high as to dictate compatibility conditions. To achieve the desired finite dimensionality result of the feedback tangential boundary controls, it was then necessary to abandon the Hilbert-Sobolev functional setting of past literature and replace it with an appropriate L^q-based/Besov setting. Eventually, well-posedness of the nonlinear N-S problem as well as its uniform stabilization are obtained in an explicit Besov space with tight parameters related to the physical dimension d, where the compatibility conditions are not recognized. The proof is constructive and is "optimal" also regarding the "minimal" amount of tangential boundary control action needed. The new setting requires the solution of new technical and conceptual issues. These include establishing maximal regularity in the required Besov setting for the overall closed-loop linearized problem with feedback control applied on the boundary. This result is also a new contribution to the area of maximal regularity. The minimal amount of tangential boundary action is linked to the issue of unique continuation properties of over-determined Oseen eigenproblems.

13 Introduction

13.1 Controlled Dynamic Navier-Stokes Equations

Let Ω be an open connected bounded domain in $\mathbb{R}^d$, $d = 2, 3$, with sufficiently smooth boundary $\Gamma = \partial\Omega$, say of class C^2. More specific requirements will be given below. For purposes of illustration, let ω be at first an arbitrary collar (layer) of the boundary Γ in the interior of Ω, $\omega \subset \Omega$ [Fig. 1]. For each point $\xi \in \omega$, we consider the (sufficiently smooth) curve ($d = 2$) or surface ($d = 3$) Γ_ξ, which is the parallel translation of the boundary Γ, passing through $\xi \in \omega$ and lying in ω. Let $\tau(\xi)$ be a unit tangent vector to the oriented curve Γ_ξ at ξ, if $d = 2$; and let $\tau(\xi) = [\tau_1(\xi), \tau_2(\xi)]$ be an orthonormal system of oriented tangent vectors lying on the tangent plane to the surface Γ_ξ at ξ, if $d = 3$, and obtained as isothermal parametrization via a 1-1 conformal mapping of a suitable open set in $\mathbb{R}^2$ with canonical basis $e_1 = \{1, 0\}, e_2 = \{0, 1\}$. See [L-T.2, Appendix] for details and references. We shall in particular allow and study the case where ω is a localized collar based on an arbitrarily small, connected portion $\widetilde{\Gamma}$ of the boundary Γ [Fig. 2]. Let m denote the characteristic function of the collar set ω : $m \equiv 1$ in ω, $m \equiv 0$ in Ω/ω.

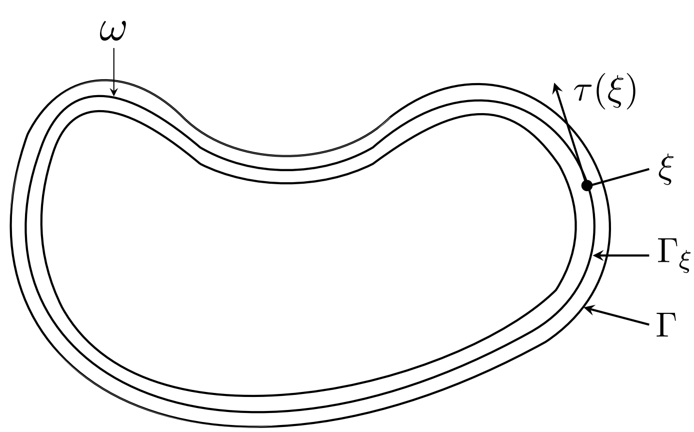


Fig. 1: Internal Collar ω of Full Boundary Γ

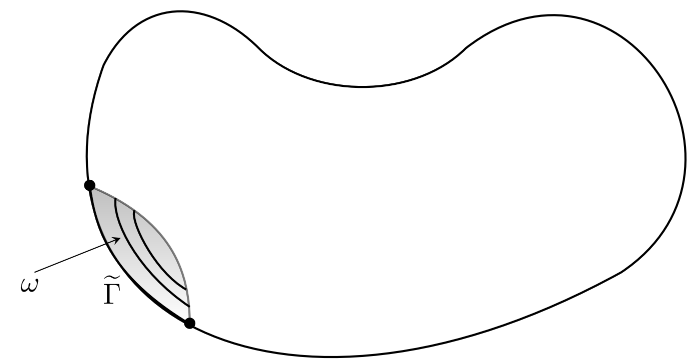


Fig. 2: Internal Localized Collar ω of Subportion $\widetilde{\Gamma}$ of Boundary Γ

We consider the following Navier-Stokes equations perturbed by a force f and subject to the action of a pair u, v of controls, to be described below

$$y_t(t,x) - \nu_o \Delta y(t,x) + (y \cdot \nabla) y + \nabla \pi(t,x) - (m(x)u)\tau = f(x) \qquad \text{in } Q \tag{13.1a}$$

$$\text{div } y = 0 \qquad \text{in } Q \tag{13.1b}$$

$$y = v \qquad \text{on } \Sigma \tag{13.1c}$$

$$y(0,x) = y_0(x) \qquad \text{in } \Omega \tag{13.1d}$$

where $Q = (0,\infty) \times \Omega$, $\Sigma = (0,\infty) \times \Gamma$ and the constant $\nu_o > 0$ is the viscosity coefficient. In (13.1c), v is a d-dimensional *tangential* boundary control $v \cdot \nu \equiv 0$ on Γ, possibly supported on an arbitrarily small connected part $\widetilde{\Gamma}$ of the boundary, where ν is the unit outward normal to Γ. Instead, u is a scalar ($d = 2$) or a two dimensional vector $u = [u^1, u^2]$ ($d = 3$) interior "tangential" control acting in the 'tangential direction' τ (that is, parallel to the boundary) in the small boundary layer ω: $(mu)\tau$, where for $d = 3$ (Fig. 2),

$$(mu)\tau = [(mu^1)\tau_1 + (mu^2)\tau_2] \ \text{ for short}, \quad m \equiv 1 \text{ on } \omega; \quad m \equiv 0 \text{ in } \Omega \backslash \omega. \tag{13.1e}$$

See [L-T.2, Appendix]. The scalar function π is the unknown pressure.

Orientation. The Stabilization Feedback Control Paradigm: purely boundary control action versus arbitrarily short portion of the boundary. We insert the present encompassing orientation at the very outset of our treatment, even though its full content can be documented and understood only after considerable further reading of the present paper. One may wish to refer back to it as reading proceeds.

Case 1: tangential boundary control action on an arbitrarily small portion of the boundary. First, ideally, one would like to establish uniform stabilization of the above problem (13.1) by use of only the boundary control v (thus, with localized interior, tangential-like control $u \equiv 0$), subject to two additional desirable features (regardless, at this stage, of its finite dimensionality):

(i) the boundary control v is applied only on an (arbitrarily) small portion $\widetilde{\Gamma}$ of the boundary Γ, and

(ii) such v acts only tangentially along $\widetilde{\Gamma}$, so that the normal component is not needed (a sort of minimal control action). Tangential actuation is attractive and is described as implementable in the engineering community, by means of jets of air.

Is such idealized purely boundary, tangential control v acting only on a portion $\widetilde{\Gamma}$ of the boundary Γ a possible stabilizing control? ***The answer is in the negative.***

As is known since the studies of boundary feedback stabilization of a parabolic equation with Dirichlet boundary trace in the feedback loop as acting on the Neumann boundary conditions [L-T.4], a critical potential obstruction arises already at the level of the finite-dimensional analysis: more precisely, at the level of enforcing feedback stabilization with large decay rate of the finite-dimensional projected system (4.8a,b) of the linearized w-problem (13.11a) (with $u = 0$). To achieve this requirement, one needs to verify the algebraic Kalman (or Hautus) rank conditions, corresponding to the unstable eigenvalues in (16.2) of the linearized Oseen operator. In the present case, these turn out to be: rank $W_i = \ell_i$, see the matrix W_i in (18.12) or (19.11), with entries restricted only on $\widetilde{\Gamma}$, for each distinct unstable eigenvalue $\lambda_i, i = 1, \dots, M$ in (16.2). In turn, such algebraic controllability conditions are equivalent to the unique continuation property of the Oseen eigenproblem (D.1a,b,c) - (D.2) of Appendix F; [same as the Oseen eigenproblem (6.36a-b-c)- (18.37), with the omission of the interior condition $\varphi \cdot \tau \equiv 0$ on ω]. Such unique continuation property with over-determined conditions $\varphi|_{\widetilde{\Gamma}} \equiv 0,\ \partial_\nu \varphi|_{\widetilde{\Gamma}} \equiv 0$ only on a portion $\widetilde{\Gamma}$ of the boundary Γ is false. In fact, as collected in Appendix F, reference [F-L] provides a simple counterexample to such unique continuation property even for the Stokes problem ($y_e = 0$) on the 2-dimensional half-space $\{(x, y) : x \in \mathbb{R}^+, y \in \mathbb{R}\}$, with over-determination on the infinite boundary $\{x = 0\}$. Such counterexample on the half-space can then be transformed in a counterexample of the unique continuation property on a bounded domain Ω with over-determination on any sub-portion of its boundary $\partial\Omega$. Thus, stabilization (with large decay rate) of the finite dimensional projected system (4.8a,b) - hence of the linearized w-problem (13.11a) - by means only of the boundary control v active only on the small portion $\widetilde{\Gamma}$ of the boundary Γ is not possible [and thus with localized interior, tangent-like control $u = 0$ on ω, the patch supported by $\widetilde{\Gamma}$].

Case 2: the necessity of complementing the localized tangential boundary control v with a corresponding localized interior tangential-like control u. See Fig. 2. If one insists on a boundary control action v active only on a portion of the boundary $\widetilde{\Gamma}$, one then needs to complement such v (as it was introduced in [L-T.2], [L-T.3]) with a localized, interior, tangential-like control u, acting on an arbitrarily small patch ω, supported by $\widetilde{\Gamma}$. This is a sort of minimal extra requirement for keeping v acting only on $\widetilde{\Gamma}$. The role of this additional localized interior, tangential-like control u is to guarantee that the corresponding unique continuation property (6.36a,b,c) of Lemma 18.2, augmented with the interior condition $\varphi \cdot \tau \equiv 0$ on ω, now holds. In short: the unique continuation property (D1a,b,c) $\implies$ (D.2) without the extra condition $\varphi \cdot \tau = 0$ on ω, is false, and this is then "corrected" by falling into the unique continuation of Lemma 18.2 in (6.36a,b,c) augmented with the interior condition $\varphi \cdot \tau \equiv 0$ on ω, which is true. Consequently the correspondingly enlarged controllability matrix in (18.28b) satisfies the required Kalman rank conditions. In this sense, therefore, the results of the present paper (ultimately, Theorem 17.4 yielding null-feedback stabilization in the vicinity of the unstable origin, of the translated z-problem (13.10)) are optimal, in terms of the smallness of the required control action for v and u. Moreover, v is shown here for the first time to be finite dimensional also in the case $d = 3$. This is the key new contribution of the present work (finite dimensionality of u is not an issue, see [L-T.3]).

Case 3: tangential boundary control v on the whole boundary Γ. If, on the other hand, one insists on only exercising tangential boundary control action v - and thus dispensing altogether with the localized, interior, tangential-like control u - then such boundary control action v will have to be applied, as a first step, to the entire boundary Γ. Would then be possible to establish uniform stabilization with only a feedback control v acting tangentially on the entire boundary Γ (regardless of its finite dimensionality)? It seems that a general definitive answer is not known at present. The obstruction is again the unique continuation property of the Oseen eigenproblem (corresponding to the unstable distinct eigenvalues $\lambda_i, i = 1, \dots, M$, in (16.2), with - this time - over-determination $\varphi|_\Gamma \equiv 0$, $\partial_\nu \varphi|_\Gamma \equiv 0$ on the entire boundary Γ: that is Problem 3, implication (D.8) $\Longrightarrow$ (D.9) in Appendix F. Only partial results are known.

a) Such required unique continuation property is true in dimension $d = 2, 3$, if the equilibrium solution $y_e = 0$ (Stokes eigenproblem) or, more generally, y_e is in a sufficiently small ball of the origin in the $W^{1,\infty}$-norm. Several very different proofs are given in [RT.4] and [RT.5]: What is then the implication, if any, on the problem of the present paper? The case $y_e = 0$ is actually physically quite important as it occurs for instance when the forcing function f in (13.1a) or (13.2a) is a conservative vector field $f = \nabla g$ (say an electrostatic field): in which case a solution of problem (1.2a,b,c) is $y_e = 0$ and $\pi = g$, modulo constant.

b) the "good" equilibrium solutions (which yield the unique continuation property with over- determination on the entire boundary Γ) form an open set in, say, the $W^{1,\infty}$ space topology: if y_e is "good" , then there is a full ball in the $W^{1,\infty}$-topology that contains "good" y_e [RT.4], [RT.5]. Of course, with $y_e = 0$, the corresponding Stokes problem (which now replaces the general Oseen problem) is already uniformly stable, with, say a decay rate $-|Re(\lambda_1)|$. A most valuable variation of the problem under investigation is then: enhance the stability of the linearized (uniformly stable) w-problem (1.11a,b,c,d) (with $u = 0$) from the given margin $-|Re(\lambda_1)|$ to an arbitrarily preassigned decay rate $-k^2$, by means of a tangential boundary finite dimensional feedback control of the same form as the operator F in (17.13) as applied to the entire boundary. To this end, it suffices to apply the procedure of the present paper to a finite dimensional projected space spanned by the eigenvectors of the Stoke operator corresponding to its finitely many eigenvalues λ_i with $|Re(\lambda_i)| \leq k^2$. This in turn will provide stability enhancement of the non-linear problem (13.1) in the vicinity of $y_e = 0$ (or small y_e).

c) In the two dimensional case, $d = 2$, there is a genericity result [B-L] about the validity of the unique continuation problem (D.8) $\Longrightarrow$ (D.9) in Appendix F with over-determination on the whole boundary.

d) A variation of the Oseen eigenvalue problem always satisfies a unique continuation property even with over-determination on an arbitrarily small portion $\widetilde{\Gamma}$ of the boundary Γ. This is problem 4 in Appendix D. A proof, along more classical elliptic arguments, is given in [RT.4]. Here however the condition $\partial_\nu \varphi \equiv 0$ is replaced by $\partial_\nu \varphi - p\nu \equiv 0$ on Γ and p on the boundary is unknown in general. Application of this result to the present paper will result in substituting $\partial_\nu \varphi^*_{ij}|_{\widetilde{\Gamma}}$ with $\partial_\nu \varphi^*_{ij} - p_i \nu|_{\widetilde{\Gamma}}$ in the matrix W_i in (18.12) or (19.11) which then, with this modification, becomes full rank, as desired. Thus the stabilizing control will be expressed in terms of the pressure on the boundary, which is typically an unknown.

The stabilization problem by feedback tangential control action. For large Reynolds number $^1/_{\nu_0}$, the presence of the external force f in (13.1a) leads to the existence of equilibrium points $\{y_e, \pi_e\}$ in (13.2) below, which may be unstable, in a quantitative sense to be made more precise in Section 16 below, and may cause turbulence: it is therefore important to be able to suppress turbulence asymptotically in time by selecting a suitable feedback control action. In the present paper, in light with the Orientation, we shall impose a pair $\{u, v\}$ of controls: a tangential boundary control $v(t)$, $v \cdot \nu = 0$ on Γ, acting on the boundary Γ (and preferably in fact supported in an arbitrarily small open connected part $\widetilde{\Gamma}$ of positive measure of the boundary Γ) and an interior localized control u acting on a collar ω supported by Γ [Fig. 1](or, respectively, by $\widetilde{\Gamma}$ [Fig. 2]) in the 'tangential' direction τ, i.e. parallel to the boundary. Thus, the function $m(x)u(t,x)\tau(x)$ can be viewed as an interior tangential-like controller with support in $Q_\omega = (0, \infty) \times \omega$. From the engineering points of view a purely tangential control action is desirable as such control mechanisms can be implemented by using jets of air which are directed in a direction tangential to the motion [ ]. Moreover, in control theory, one generally seeks a 'minimal' control action, and in this spirit we dispense with the 'normal' component.

Notation: As already done in the literature, for the sake of simplicity, we shall adopt the same notation for function spaces of scalar functions and function spaces of vector valued functions. Thus, for instance, for the vector valued (d-valued) velocity field y or external force f, we shall simply write say $y, f \in L^q(\Omega)$ rather than $y, f \in (L^q(\Omega))^d$ or $y, f \in \mathbf{L}^q(\Omega)$. This choice is unlikely to generate confusion. The initial condition y_0 and the body force $f \in L^q(\Omega)$ are given. The scalar function π is the unknown pressure. By way of orientation, we state at the outset two main points. For the linearized w-problem (13.11) below of the translated non-linear z-problem (13.10), the final well-posedness and global uniform stabilization result, Theorem 17.2, holds in general for $1 < q < \infty$. Instead, the final, main local well-posedness and uniform, local stabilization results, near an equilibrium solution, Theorem A and Theorem B in Section 13.7 (same as Theorem 5.5 (i), (ii) of Section 17.5), for the original nonlinear y-problem (13.1) will require $q > 3$, see (24.16), in the $d = 3$-case, hence $1 < p < {}^{6}/_{5}$, and respectively $q > 2$ in the $d = 2$-case, hence $1 < p < {}^{4}/_{3}$, to satisfy the requirement $p < {}^{2q}/_{2q-1}$ see (13.3), or (15.3b).

13.2 Stationary Navier-Stokes equations

The following result represents our basic starting point.

Theorem 13.1. *Consider the following steady-state Navier-Stokes equations in* Ω

$$-\nu_o \Delta y_e + (y_e.\nabla) y_e + \nabla \pi_e = f \qquad in\ \Omega \tag{13.2a}$$

$$div\ y_e = 0 \qquad in\ \Omega \tag{13.2b}$$

$$y_e = 0 \qquad on\ \Gamma \tag{13.2c}$$

Let $1 < q < \infty$. *For any* $f \in L^q(\Omega)$ *there exits a solution (not necessarily unique)* $(y_e, \pi_e) \in (W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)) \times W^{1,q}(\Omega)$.

For the Hilbert case $q = 2$, see [C-F, Thm 7.3 p 59] . For the general case $1 < q < \infty$, see [A-R, Thm 5.iii p 58].

Remark 13.1. It is well-known [Lad], [Li], [Te] that the stationary solution is unique only when "the data is small enough, or the viscosity is large enough" [Te, p 157; Chapt 2] that is, if the ratio ${}^{\|f\|}/_{\nu_o^2}$ is smaller than some constant that depends only on Ω [FT, p 121]. When non-uniqueness occurs, the stationary solutions depend on a finite number of parameters [FT, Theorem 2.1, p 121] asymptotically, in the time dependent case.

Remark 13.2. As mentioned in the Orientation, the case where $f(x)$ in (13.1a) is replaced by $\nabla g(x)$ is noted in the literature as arising in certain physical situations, where f is a conservative vector field. In this case, a solution of the stationary problem (13.2) is $y_e \equiv 0, \pi_e = g$. The analysis of this relevant case was indicated in the Orientation and is postponed to Remark 15.2, at the end of Section 15.

13.3 Main goal of the present paper: solution of the presently open problem on local uniform stabilization of the N-S equations (13.1), near an unstable equilibrium solution y_e by means of a tangential boundary, localized feedback control v which is finite dimensional also for $d = 3$ (in addition to a corresponding control u).

In the Orientation, supplemented by Appendix F, we have seen that if v is active only on $\widetilde{\Gamma}$, then a corresponding interior, tangential-like control u is needed on a patch supported by $\widetilde{\Gamma}$. We take this fact for guaranteed henceforth. In the literature (to be reviewed below in Section 13.8) on the local uniform feedback stabilization near an equilibrium solution y_e of the N-S equations from the boundary, a presently *open* problem stands prominent: can one achieve such local feedback stabilization of the N-S equations only with *finite-dimensional* feedback *tangential boundary* controller v also in the case $d = 3$?. A second subordinate *open* problem is: does one need also the (finite dimensional) interior tangential-like control u?, if v is applied to the entire boundary Γ? These issues are open at present, as discussed in the Orientation and in Appendix F. When employed, the finite-dimensionality of u is not a problem, see [L-T.2] (and [B-T.1]).

The finite dimensionality of v: Present state-of-the-art has succeeded [L-T.2], [L-T.3] in establishing local exponential stabilization (asymptotic turbulent suppression) near an equilibrium solution y_e by means of a *finite-dimensional tangential* feedback boundary control v, in the Hilbert setting in two cases:

(i) when the dimension $d = 2$,

(ii) when the dimension $d = 3$ but the initial condition y_0 in (13.1d) is compactly supported.

In the general $d = 3$ case, handling of the non-linearity of the N-S problem forces a Hilbert space setting with a high-topology $H^{1/2+\varepsilon}(\Omega)$, $\varepsilon > 0$, whereby the compatibility conditions kick in. These then cannot allow the stabilizing feedback control to be finite-dimensional in general. More precisely, even at the level of establishing (global) uniform stabilization at the $H^{1/2+\varepsilon}(\Omega)$-level of the linearized w-problem (13.11) for $d = 3$, with a Riccati based boundary feedback control v, verification of the preliminary finite cost condition of the optimal control problem is provided by a boundary *open loop* control consisting of a finite-dimensional term plus the term $e^{-\gamma_1 t}((\text{I.C.})|_\Gamma)$, where $\gamma_1 > 0$ is preassigned. This spoils the finite-dimensionality, unless the initial condition is compactly supported. See [B-L-T.3, Proposition 3.7.1 Remark 3.7.1], [L-T.2, Proposition 2.5, eq(2.48)] for such explicit open loop boundary control and [B-L-T.1, Proposition 4.2.2(4.2.4)] for the exponential decay in $H^{1/2+\varepsilon}(\Omega)$.

Thus, the main goal of the present paper is to remove the deficiency noted in (ii) on the tangential stabilizing boundary control v in the case $d = 3$, and thus obtain local uniform feedback stabilization of (13.1) near an unstable equilibrium solution y_e, by means of a stabilizing tangential, localized feedback control v in (13.1d) which is also finite dimensional in the case $d = 3$.

[As explained in the Orientation, it turns out that, in general, we shall also need a finite dimensional interior tangential-like control u supported on the small collar ω of $\widetilde{\Gamma}$, $(mu)\tau$, see [Fig. 2]. The underlying reason - the validity of a unique continuation result for a boundary over-determined Oseen eigenvalue problem - is explained in the Orientation and in Appendix F]. To this end, we need therefore to go beyond the Hilbert setting of [L-T.3] and thus achieve local uniform stabilization near an equilibrium solution y_e in the case $d = 3$ in a space enjoying the following two features: on the one hand, it must accommodate the N-S nonlinearity for $d = 3$; and on the other hand, it must not recognize the boundary conditions, in order not to be subject to compatibility conditions. Thus, the present paper will provide a feedback stabilization pair $\{v, u\}$, in (13.1c) and in (13.1a) respectively, both finite-dimensional also in the case $d = 3$ (in the case of u, this is already known [L-T.2], [B-T.1]) and spectral based, this time however within an L^q/Besov-setting. In particular, local exponential stability for the velocity field y near an unstable equilibrium solution y_e will be achieved for $d = 3$ in the topology of the Besov space

$$\widetilde{B}^{2-2/p}_{q,p}(\Omega), \quad 1 < p < \frac{6}{5}; \quad q > 3, \quad \text{for } \dim \Omega = 3. \tag{13.3}$$

Such space is 'close' to $L^q(\Omega)$, $q > 3$, see (15.3b) below.

See references [J-S], [E-S-S] for the critical scale of the space $L^3(\Omega)$. It was reference [L-T.2] originally in the linearized N-S case, followed by [L-T.3] in the corresponding nonlinear N-S case, that introduced in addition to the feedback tangential boundary control v, such interior, tangential-like control u in (13.1a) localized on ω. [That its feedback stabilizing form was going to be finite dimensional for $d = 2$

and $d = 3$ was never a problem being an internal control].

13.4 Helmholtz decomposition

A first difficulty one faces in extending the local exponential stabilization result near an equilibrium solution y_e with tangential control pair $\{v, u\}$ of the original problem (13.1) from the Hilbert-space setting in [B-T.1], [B-L-T.1] to the L^q/Besov setting is the question of the existence of a Helmholtz (Leray) projection for the domain Ω in $\mathbb{R}^d$. More precisely: Given an open set $\Omega \subset \mathbb{R}^d$, the Helmholtz decomposition answers the question as to whether $L^q(\Omega)$ can be decomposed into a direct sum of the solenoidal vector space $L^q_\sigma(\Omega)$ and the space $G^q(\Omega)$ of gradient fields. Here,

$$\begin{aligned} L^q_\sigma(\Omega) &= \overline{\{y \in C_c^\infty(\Omega) : \text{div } y = 0 \text{ in } \Omega\}}^{\|\cdot\|_q} \\ &= \{g \in L^q(\Omega) : \text{div } g = 0; \; g \cdot \nu = 0 \text{ on } \partial\Omega\}, \\ &\qquad \text{for any locally Lipschitz domain } \Omega \subset \mathbb{R}^d, d \geq 2 \text{ [Ga.3, } p \text{ 119]} \\ G^q(\Omega) &= \{y \in L^q(\Omega) : y = \nabla p, \; p \in W^{1,q}_{loc}(\Omega) \text{ where } 1 \leq q < \infty\}. \end{aligned} \tag{13.4}$$

Both of these are closed subspaces of L^q.

Definition 13.1. Let $1 < q < \infty$ and $\Omega \subset \mathbb{R}^n$ be an open set. We say that the Helmholtz decomposition for $L^q(\Omega)$ exists whenever $L^q(\Omega)$ can be decomposed into the direct sum

$$L^q(\Omega) = L^q_\sigma(\Omega) \oplus G^q(\Omega). \tag{13.5}$$

The unique linear, bounded and idempotent (i.e. $P_q^2 = P_q$) projection operator $P_q : L^q(\Omega) \longrightarrow L^q_\sigma(\Omega)$ having $L^q_\sigma(\Omega)$ as its range and $G^q(\Omega)$ as its null space is called the Helmholtz projection.

This is an important property in order to handle the incompressibility condition $div\ y \equiv 0$. For instance, if such decomposition exists, the Stokes equation (say the linear version of (13.1) with control $u \equiv 0, v \equiv 0$) can be formulated as an equation in the L^q setting. Here below we collect a subset of known results about Helmholtz decomposition. We refer to [H-S, Section 2.2], in particular for the comprehensive Theorem 2.2.5 in this reference, which collects domains for which the Helmholtz decomposition is known to exist. These include the following cases:

(i) any open set $\Omega \subset \mathbb{R}^d$ for $q = 2$, i.e. with respect to the space $L^2(\Omega)$; more precisely, for $q = 2$, we obtain the well-known orthogonal decomposition (in the standard notation, where ν =unit outward normal vector on Γ) [C-F, Prop 1.9, p 8]

$$L^2(\Omega) = H \oplus H^{\perp} \tag{13.6a}$$

$$H = \{\phi \in L^2(\Omega) : \text{div } \phi \equiv 0 \text{ in } \Omega;\ \phi \cdot \nu \equiv 0 \text{ on } \Gamma\} \tag{13.6b}$$

$$H^{\perp} = \{\psi \in L^2(\Omega) : \psi = \nabla h,\ h \in H^1(\Omega)\}; \tag{13.6c}$$

(ii) a bounded C^1-domain in $\mathbb{R}^d$ [F-M-M], $1 < q < \infty$, or [Ga.3, Theorem 1.1 p 107, Theorem 1.2 p 114] for C^2-boundary;

(iii) a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ ($d = 3$) and for $\frac{3}{2} - \epsilon < q < 3 + \epsilon$ sharp range [F-M-M];

(iv) a bounded convex domain $\Omega \subset \mathbb{R}^d, d \geq 2, 1 < q < \infty$ [F-M-M].

On the other hand, on the negative side, it is known that there exist domains $\Omega \subset \mathbb{R}^d$ such that the Helmholtz decomposition does not hold for some $q \neq 2$ [M-S].

Assumption (H-D) Henceforth in this paper, we assume that the bounded domain $\Omega \subset \mathbb{R}^d$ under consideration admits a Helmholtz decomposition for the values of $q,\ 1 < q < \infty$, here considered at first, for the linearized problem (13.11) below. The final results Theorems A and B of Section 13.8 for the non-linear problem (13.1) will require $q > 3$, see (24.16), in the case of interest $d = \dim \Omega = 3$.

Next, for $M \subset L^q(\Omega),\ 1 < q < \infty$, we denote the annihilator of M by

$$M^{\perp} = \left\{ f \in L^{q'}(\Omega) : \int_{\Omega} fg \, d\Omega = 0, \text{ for all } g \in M \right\} \tag{13.7}$$

where q' is the dual exponent of q : ${}^1\!/_q + {}^1\!/_{q'} = 1$.

Proposition 13.2. *[H-S, Prop 2.2.2 p6], [Ga.3, Ex. 16 p115], [F-M-M]*
Let $\Omega \subset \mathbb{R}^d$ be an open set and let $1 < q < \infty$.

a) The Helmholtz decomposition exists for $L^q(\Omega)$ if and only if it exists for $L^{q'}(\Omega)$, and we have: (adjoint of P_q) $= P_q^* = P_{q'}$ (in particular P_2 is orthogonal), where P_q is viewed as a bounded operator $L^q(\Omega) \longrightarrow L^q(\Omega)$, and $P_q^* = P_{q'}$ as a bounded operator $L^{q'}(\Omega) \longrightarrow L^{q'}(\Omega)$, ${}^1\!/_q + {}^1\!/_{q'} = 1$.

b) Then, with reference to (13.5)

$$\left[L^q_\sigma(\Omega)\right]^\perp = G^{q'}(\Omega) \text{ and } \left[G^q(\Omega)\right]^\perp = L^{q'}_\sigma(\Omega). \tag{13.8a}$$

Remark 13.3. Throughout the paper we shall use freely that

$$\left(L^q_\sigma(\Omega)\right)' = L^{q'}_\sigma(\Omega), \quad \frac{1}{q} + \frac{1}{q'} = 1. \tag{13.8b}$$

This can be established as follows. From (13.5) write $L^q_\sigma(\Omega)$ as a factor space $L^q_\sigma(\Omega) = L^q(\Omega)/G^q(\Omega) \equiv X/M$ so that [T-L.1, p 135].

$$\left(L^q_\sigma(\Omega)\right)' = \left(L^q(\Omega)/G^q(\Omega)\right)' = \left(X/M\right)' = M^\perp = \left[G^q(\Omega)\right]^\perp = L^{q'}_\sigma(\Omega). \tag{13.8c}$$

In the last step, we have invoked (13.8a), which is also established in [Ga.3, Lemma 2.1, p 116]. Similarly

$$\left(G^q(\Omega)\right)' = \left(L^q(\Omega)/L^q_\sigma(\Omega)\right)' = \left[L^q_\sigma(\Omega)\right]^\perp = G^{q'}(\Omega). \tag{13.8d}$$

13.5 Translated Nonlinear Navier-Stokes z-Problem: Reduction to zero equilibrium

We return to Theorem 13.1 which provides an equilibrium pair $\{y_e, \pi_e\}$. Then, as in [B-L-T.1], [L-T.2] we translate by $\{y_e, \pi_e\}$ the original N-S problem (13.1). Thus we introduce new variables

$$z = y - y_e, \quad \chi = \pi - \pi_e \tag{13.9}$$

and obtain the translated problem in $\{z, \chi\}$

$$\begin{cases} z_t - \nu_o \Delta z + L_e(z) + (z \cdot \nabla) z + \nabla \chi - (m(x)u)\tau = 0 & \text{in } Q \quad (13.10a) \\ \text{div } z = 0 & \text{in } Q \quad (13.10b) \\ z = v & \text{on } \Sigma \quad (13.10c) \\ z(0,x) = z_0(x) = y_0(x) - y_e(x) & \text{on } \Omega \quad (13.10d) \end{cases}$$

where $v \cdot \nu = 0$ on Σ and the first order Oseen perturbation L_e is given by

$$L_e(z) = (y_e \cdot \nabla) z + (z \cdot \nabla) y_e. \tag{13.10e}$$

We shall accordingly first study the local null feedback stabilization of the z-problem (13.10), that is, feedback stabilization in a neighborhood of the origin.

Our strategy will be to select *constructively* feedback control operators $v = F(z)$ and $u = \widetilde{G}(z)$, with v tangential $v \cdot \nu = 0$ on Γ and supported only on $\widetilde{\Gamma}$, and u supported only on ω, and both F and $\widetilde{G}$ bounded and finite dimensional also for $d = 3$. For $d = 2$ this was achieved in the Hilbert space setting [L-T.3].

13.6 The linearized w-problem of the non-linear translated z-problem (13.10)

The linearization of the non-linear z-problem (13.10) near the equilibrium solution y_e is

$$
\begin{cases}
w_t - \nu_o \Delta w + L_e(w) + \nabla \chi - (m(x)u)\tau = 0 & \text{in } Q & (13.11\text{a}) \\
\text{div } w = 0 & \text{in } Q & (13.11\text{b}) \\
w = v & \text{on } \Sigma & (13.11\text{c}) \\
w(0, x) = w_0(x) & \text{on } \Omega & (13.11\text{d})
\end{cases}
$$

$v \cdot \nu = 0$ on Γ.

13.7 Main contributions of the present paper: for dim $\Omega = d = 2, 3$, local-in-space well-posedness on the space of maximal regularity $X^{\infty}_{p,q,\sigma}$ of the N-S dynamics (13.1) as well as local exponential uniform stabilization near y_e on the space $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$, $q > d$, $1 < p < 2q/_{2q-1}$ by means of a finite dimensional tangential boundary feedback control v, supported on $\widetilde{\Gamma}$ and a feedback finite dimensional tangential-like interior control u, supported on the collar ω of $\widetilde{\Gamma}$

Let us introduce some preliminary notions, with focus on the critical case dim $\Omega = 3$. Let $q > 3$ and $1 < p < 6/_5$ (in order to satisfy the requirement $p < 2q/_{2q-1}$). Recall the Besov space

$$
B^{2-2/p}_{q,p}(\Omega) = \big(L^q(\Omega), W^{2,q}(\Omega)\big)_{1-1/p,p} \tag{13.12}
$$

defined as a real interpolation space, as a specialization of the general formula (15.1a) below for

$m=2, s=2-{}^2/_p$. Consider its subspace

$$\widetilde{B}^{2-2/p}_{q,p}(\Omega)=\Big\{g\in B^{2-2/p}_{q,p}(\Omega):\ \text{div}\ g=0,\ g\cdot\nu|_\Gamma=0\Big\} \tag{13.13a}$$

$$=B^{2-2/p}_{q,p}(\Omega)\cap L^q_\sigma(\Omega),\quad 1<p<\frac{2q}{2q-1} \tag{13.13b}$$

as defined in (15.3b), with $L^q_\sigma(\Omega)$ in (13.4).

Main Theorem A. (On problem (13.1)). Let Ω, dim $\Omega=d=2,3$, be a bounded domain satisfying the Helmholtz decomposition assumption of Definition 13.1. Let Γ be its C^2 boundary and let $\widetilde{\Gamma}$ be an arbitrary small, open connected subset of Γ, of positive measure, supporting the corresponding arbitrary small interior collar ω (Fig. 2). With reference to the N-S dynamics (13.1), consider a given unstable equilibrium solution y_e of problem (13.2), as guaranteed by Theorem 13.1. Let $q>d,\ 1<p<{}^{2q}/_{2q-1}$ and, take the initial condition $y_0\in\mathcal{V}_\rho$, where

$$\mathcal{V}_\rho\equiv\Big\{y_0\in\widetilde{B}^{2-2/p}_{q,p}(\Omega):\|y_0-y_e\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)}\le\rho\Big\},\quad\rho>0. \tag{13.14}$$

Thus $1<p<{}^6/_5$ for $d=3$ and $1<p<{}^4/_3$ for $d=2$. There exists $\rho_0>0$ sufficiently small, such that, if $0<\rho\le\rho_0$, then there exist a tangential boundary feedback controller v and a tangential-like interior feedback controller u, defined respectively by

$$v=F(y-y_e),\ v\cdot\nu|_\Gamma=0;\quad u=\widetilde{G}(y-y_e) \tag{13.15}$$

through bounded operators $F\in\mathcal{L}\big(L^q_\sigma(\Omega),L^q(\widetilde{\Gamma})\big)$ and $\widetilde{G}\in\mathcal{L}\big(L^q_\sigma(\Omega)\big)$, both finite-dimensional, with v supported on $\widetilde{\Gamma}$ and tangential along $\widetilde{\Gamma}$, and u with tangential-like internal action $(mu)\tau=(mu^1)\tau_1+(mu^2)\tau_2$ for $d=3$, supported on a collar ω of $\widetilde{\Gamma}$, such that the corresponding closed loop system (13.1) due to the action of such pair $\{v,u\}$

$$\begin{cases} y_t(t,x)-\nu_o\Delta y(t,x)+(y\cdot\nabla)y+\nabla\pi(t,x)-(m(x)\widetilde{G}(y-y_e))\tau=f(x) & \text{in } Q\\ \text{div}\ y=0 & \text{in } Q\\ y=F(y-y_e) & \text{on } \Sigma\\ y(0,x)=y_0(x) & \text{in } \Omega\end{cases} \tag{13.16}$$

$f(x)$ the external force in (13.1a) or (13.2a),has the following two properties:

(a) the feedback system (13.16) is well-posed as a non-linear s.c. semigroup on the space of maximal regularity, see (11.8a–b)

$$X_{p,q,\sigma}^{\infty} = L^p\big(0,\infty; \mathcal{D}(\mathbb{A}_{_{F,q}})\big) \cap W^{1,p}\big(0,\infty; L^q_\sigma(\Omega)\big) \tag{13.17}$$

$$\subset X_{p,q}^{\infty} = L^p\big(0,\infty; W^{2,q}(\Omega)\big) \cap W^{1,p}\big(0,\infty; L^q_\sigma(\Omega)\big) \tag{13.18}$$

$$\mathcal{D}\big(\mathbb{A}_{_{F,q}}\big) \equiv \big\{\varphi \in W^{2,q}(\Omega) \cap L^q_\sigma(\Omega) : \varphi|_\Gamma = F\varphi\big\}, \tag{13.19}$$

where $\mathbb{A}_{_{F,q}}$ defined in (21.12) or (23.2c), is the generator of the linearized, uniformly stable w-problem in (17.10) in feedback form.

(b) such closed loop system (13.16) is exponentially stable on $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$ with decay rate $\widetilde{\gamma} > 0$:

$$\|y(t) - y_e\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \leq C_{\widetilde{\gamma}} e^{-\widetilde{\gamma}t} \|y_0 - y_e\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)}, \quad t \geq 0, \ y_0 \in \mathcal{V}_\rho, \tag{13.20}$$

for constants $\widetilde{\gamma}, \ C_{\widetilde{\gamma}} \geq 1$, depending on q. □

The bounded finite-dimensional operators $F \in \mathcal{L}\big(L^q_\sigma(\Omega), L^q(\widetilde{\Gamma})\big)$ and $\widetilde{G} \in \mathcal{L}\big(L^q_\sigma(\Omega)\big)$ have the following form:

$$F(y - y_e) = \sum_{k=1}^{K} \big\langle P_N(y - y_e), p_k \big\rangle_{W_N^u} f_k, \quad \text{supported on } \widetilde{\Gamma} \tag{13.21}$$

$$\widetilde{G}(y - y_e) = \sum_{k=1}^{K} \big\langle P_N(y - y_e), q_k \big\rangle_{W_N^u} u_k. \tag{13.22}$$

Here, P_N is the projector, given explicitly in (16.3a), of $L^q_\sigma(\Omega)$ onto W^u_N, $L^q_\sigma(\Omega) = W^u_N \oplus W^s_N$, $W^u_N = P_N L^q_\sigma(\Omega)$ being the finite dimensional subspace of $L^q_\sigma(\Omega)$, spanned by the generalized eigenvectors, see (18.3) corresponding to the unstable eigenvalues, see (16.2), of the Oseen operator. Here, $< \ , \ >_{W^u_N}$ denotes the duality paring between $W^u_N \in L^q_\sigma(\Omega)$ and $(W^u_N)^* \in (L^q_\sigma(\Omega))' = L^{q'}_\sigma(\Omega)$, by (13.8): $< h_1, h_2 >_{W^u_N} = \displaystyle\int_\Omega h_1 h_2 \, d\Omega$. The vectors p_k, q_k in $(W^u_N)^* \subset L^{q'}_\sigma(\Omega)$ as well as the boundary vectors f_k are constructed explicitly in Section 18, in the proof of Theorem 17.1. In particular (see Appendix E, in particular (C.5))

$$f_k \in \mathcal{F} = \text{span}\left\{\frac{\partial \varphi^*_{ij}}{\partial \nu} : \ i = 1, \dots, M; \ j = 1, \dots, \ell_i\right\} \in W^{2-1/q,q}(\Gamma), \ q \geq 2, \tag{13.23}$$

${}^1/_q + {}^1/_{q'} = 1$, where $\varphi^*_{ij} \in W^{3,q}(\Omega)$, see (C.5) in Appendix E, are the eigenfunctions of the adjoint of the Oseen operator, see (18.1), corresponding to the M distinct unstable eigenvalue λ_i, with algebraic multiplicity ℓ_i. Finally, $K \geq \ell_i,\ i = 1, \dots, M$. $\square$

The above main Theorem A for problem (13.1) is an immediate corollary of the following main Theorem B for the translated non-linear z-problem (13.10).

Main Theorem B. (On problem (13.10)) Under the same assumptions and in the same notation of Theorem A, in particular, $q > 3, 1 < p < {}^6/_5$ for dim $\Omega = 3$, consider the following feedback version of the translated non-linear z-problem (13.10), corresponding to the abstract version (24.1)

$$\begin{cases} z_t - \nu_o \Delta z + L_e(z) + (z \cdot \nabla) z + \nabla \chi - \left(m \left(\sum_{k=1}^{K} \left\langle P_N z, q_k \right\rangle_{W_N^u} u_k \right) \right) \cdot \tau = 0 & \text{in } Q \\ \text{div } z = 0 & \text{in } Q \\ z = \sum_{k=1}^{K} \left\langle P_N z, p_k \right\rangle_{W_N^u} f_k & \text{on } \Sigma \\ z(0, x) = z_0(x) = y_0(x) - y_e(x). & \text{on } \Omega \end{cases} \tag{13.24}$$

There exists a positive constant $r_1 > 0$ (identified in (24.24), such that if

$$\|z_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \leq r_1, \tag{13.25}$$

then, the following local well-posedness and uniform feedback stabilization results hold true:

(i) the feedback problem (13.24) problem admits a unique (fixed point nonlinear semigroup) solution z in the space $X^\infty_{p,q,\sigma}$ of maximal regularity.

(ii) Moreover, if the constant $r_1 > 0$ in (13.25) is sufficiently small, then the guaranteed solution z satisfies the exponential decay

$$\|z(t)\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \leq C_{\widetilde{\gamma}} e^{-\widetilde{\gamma} t} \|z_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)}, \quad t \geq 0, \text{ for some constants } \widetilde{\gamma} > 0,\ C_{\widetilde{\gamma}} \geq 1 \tag{13.26}$$

depending on q. $\square$

Remark 25.1 at the end of Section 25 supports qualitatively the intuitive expectation that "the larger the global decay rate $\gamma_0 > 0$ in (17.14) of Theorem 17.2 of the linearized w-problem (13.11) in feedback form as in (17.10), the larger the local decay rate $\widetilde{\gamma}$ in (13.26).

The proof of the well-posedness part in $X^{\infty}_{p,q,\sigma}$ of Theorem B is given (in its concluding arguments) in Section 24, while the exponential decay (13.26) is established (in its concluding arguments) in Section 25. Recalling $z = y - y_e, \chi = \pi - \pi_e$ from (13.9), we see at once that Theorem B implies Theorem A.

13.8 Comparison with the literature

To put the present paper in the context of the literature, we first repeat its main contributions, already discussed in the Orientation of Section 13, supplemented by Appendix F.

Main Contributions of the present paper

1. The main goal of the present paper is to provide an affirmative solution to the recognized open problem as to whether or not uniform stabilization of the Navier-Stokes equations can be achieved by means of a localized 'tangential' feedback controller which, in addition, is *finite dimensional also for* $d = 3$ *in full generality.* To achieve this desired goal, it was necessary to abandon the Hilbert-Sobolev setting of all prior literature on this problem and employ, for the first time, an L^q/Besov space framework, as explained in the Orientation in Section 13. Such stabilizing control consists of a pair $\{v, u\}$ of finite dimensional feedback controls: a localized finite dimensional tangential boundary feedback control v in (13.1d) acting on an arbitrarily small open connected portion $\widetilde{\Gamma}$ of the boundary Γ, and a localized interior finite dimensional feedback control u in (13.1a) acting tangential-like (parallel to the boundary) on an arbitrarily small interior patch ω supported by $\widetilde{\Gamma}$. The interior tangential-like controller u cannot be dispensed with, if one insists in controlling from an arbitrarily small portion $\widetilde{\Gamma}$ of the boundary. This is due to the counter-example in [F-L] to the unique continuation property of the over-determined Oseen eigenproblem in (D.1a-b-c), (D.2) of Appendix F, leading to the implication noted in (D.5). For further details, see the Orientation in Section 13.

2. Thus, the proposed solution has also the additional advantage of *requiring a "minimum" control action or support* $\{\widetilde{\Gamma}, \omega\}$ for the control pair $\{v, u\}$.

3. Moreover, it has another positive feature in that the finite dimensionality of the feedback stabilizing controllers v in (13.1c) and u in (13.1a) is equal to the max of the *geometric* multiplicity not the algebraic multiplicity as in [B-T.1], [B-L-T.1], [B-L-T.2], [B-L-T.3] – of the distinct unstable

eigenvalues of the Oseen operator. This is due to the proof, given originally in [L-T.2], for checking the controllability condition (18.28b) or (19.15b) of the finite dimensional projected system w_N in (16.8). Not only does this proof rest on the geometric rather than the algebraic multiplicity of the unstable eigenvalues, but it also much simplifies the somewhat awkward and unnecessary Gram-Schmidt orthogonalization process of [B-T.1] by employing direct, explicit, sharp tests.

4. Finally, the present work offers a much more attractive and preferable proof over past literature of the ultimate non-linear result: the well-posedness and uniform stabilization of the original (modulo translation) non-linear z-problem (24.1), given in Sections 24 and 25. This new proof now rests on the fundamental preliminary property of maximal regularity of the linearized boundary feedback problem (17.10), or generator $\mathbb{A}_{F,q}$, as stated in Theorem 17.2, and proved in Section 23. Such maximal regularity-based proof is much cleaner and effective over the original proof for the non-linear boundary stabilization result as given in [B-L-T.1]; and even more so over the approximation argument given in the case of localized feedback control given in [B-T.1].

The origin of the studies on the uniform stabilization problem of Navier-Stokes equations. The problem of boundary feedback stabilization of unstable linear classical parabolic equations was investigated extensively in the period, say 1974-1983, see [RT.1], [RT.2], [RT.3], [L-T.4]. The study of uniform stabilization of Navier-Stokes equations apparently initiated with the work of Fursikov [Fur.1], [Fur.2], [Fur.3], first in $2d$, next in $3d$. However, this work used *open-loop boundary controls not closed loop feedback controls*. The nature and dimensionality of the obtained boundary controllers (whether finite or infinite dimensional, whether tangential or otherwise) was not an issue covered by the method of these papers. Fursikov's work was soon followed by paper [B-T.1] which tackled and solved, instead, the (preliminary) problem of uniform stabilization of the Navier-Stokes equations, $d = 2, 3$, by means of a localized interior finite dimensional high-gain, Riccati-based feedback control. All these studies-and the subsequent ones till the present work, some of which are noted below - were carried out in a Hilbert-Sobolev-settings [A marked improvement in both content of results and effectiveness of proofs with spectral based, explicit interior localized controllers on the interior uniform stabilization problem is contained in Part I of the present work. For the first time, its analysis is carried out in an L^q/Besov setting. This study was intended to be a preliminary investigation to test techniques that have been then employed in the more challenging boundary case of the present Part II].

Tangential Boundary feedback stabilization. Paper [B-T.1] opened then the way to a first analysis of the tangential boundary stabilization problem in [B-L-T.1] via a high gain, Riccati-based boundary control, followed by an axiomatic approach, still Riccati-based, in [B-L-T.2], both low and high gain, as well as a complementary, spectral-based approach in [B-L-T.3]. These works required some spectral assumptions of the Oseen eigenvalue problem, equivalent to a unique continuation property for a corresponding overdetermined Oseen eigenproblem. It was only in [L-T.2], [L-T.3] that uniform stabilization with a localized feedback control pair $\{v, u\}$, as described above, was resolved in an "optimal" way regarding the amount of their support $\{\widetilde{\Gamma}, \omega\}$. This setting of [L-T.2], [L-T.3] had the advantage of not requiring any property or assumptions on the distinct unstable eigenvalues of the Oseen operator, as it was the case in prior literature, since the required corresponding unique continuation property can be shown in this context to hold true (Lemma 18.2), due to an extra condition dictated by the employment of the interior localized tangential-like control u. As noted in Section 13.3 in reference [L-T.3], the issue of finite dimensionality of the tangential boundary feedback controller component was resolved positively only for $d = 2$ and for $d = 3$ only in the case of Initial Conditions being compactly supported. The general case for $d = 3$ was left open. It is resolved here in the affirmative.

Additional references; the case of oblique boundary stabilization. Papers [Ray.1], [Ray.2] used low-gain, Riccati-based feedback boundary controllers with a normal component, whose dimensionality is not addressed. Two textbooks have meanwhile appeared. [B.1, Chapter 3] present the earlier results of [B-T.1]. The approach is spectral-based (as in [B-L-T.3]]). The treatment of [B.1] relies on two main assumptions, much stronger than the ones in [B-T.1]. They are [B.1, Assumptions K.1 and K.2 p 123] . The first is the simplifying assumption that the distinct unstable eigenvalues of the Oseen operators be semisimple (geometric = algebraic multiplicity). The second assumes that <u>all</u> N unstable eigenvalues have dual eigenvectors whose normal derivatives are linearly independent as L^2 functions on the whole boundary Γ. This is way too stronger than the conditions (already given in [B-L-T.1]) that require the much weaker property that for each distinct unstable eigenvalue λ_i with geometric multiplicity ℓ_i, only the traces $\partial_\nu \varphi^*_{ij}$ as in (D.5) be linearly independent, $i = 1, 2, \dots, M$; $j = 1, \dots, \ell_i$. We note in passing that in [L-T.4], in the case of say the Laplacian (translated) with Neumann BC on either a rectangle or a disk, it was shown by direct computations that the eigenvectors of the unstable eigenvalues (assumed simple) do have the property that all their Dirichlet traces are Linearly Independent in $L^2(\Gamma)$. Finally, reference [B.2] investigates stabilization with an oblique boundary control-that is one with an additional normal component. The normal component however is not expressed in feedback form. In addition, the strong assumptions K1 and above all K.2 noted above are retained. In both [B.1] and [B.2], the number of controls equals the max algebraic multiplicity of the unstable eigenvalues of the Oseen operator, see [B.1, Eq (3.19)].

14 Abstract models for the non-linear z-problem (13.10) and the linearized w-problem (13.11) in the L^q-setting

We shall next provide abstract models for the translated non-linear z-problem (13.10) and its corresponding linearized w-problem (13.11) in the L^q-setting. This will be the counterpart (extensions) of these introduced in [B-L-T.1] and used in [B-L-T.3] [L-T.3]. The L^q-setting will require a wealth of non-trivial additional results: from the well-posedness and regularity from the boundary of the stationary Oseen problem (that is, the definition of the Dirichlet map D with range in $L^q_\sigma(\Omega) : g \longrightarrow Dg = \psi$ in (14.1) below) to the definition of the adjoint $(\mathcal{A}_q)^* = \mathcal{A}_q^*$ for short, (in the $L^{q'} \longrightarrow L^q$ sense) of the Oseen operator $\mathcal{A}_q$, to the critical meaning of $D^* \mathcal{A}_q^* \varphi,\ \varphi \in \mathcal{D}(\mathcal{A}_q^*)$. These results will be provided below. They will be the perfect counterpart of those obtained in [B-L-T.1], in the Hilbert setting.

14.1 Well-posedness in the L^q-setting of the non-homogeneous stationary Oseen problem: the Dirichlet map D : boundary $\longrightarrow$ interior

Recalling the first order operator $L_e(\psi) = (\psi \cdot \nabla) y_e + (y_e \cdot \nabla)\psi$ from (13.10e) and introducing the differential expression $\mathbb{A}\psi = -\nu_0 \Delta \psi + L_e(\psi)$, we consider the stationary, boundary non-homogeneous Oseen problem on Ω:

$$\begin{cases} \mathbb{A}\psi + \nabla \pi^* = -\nu_o \Delta \psi + L_e(\psi) + \nabla \pi^* = 0 & \text{(14.1a)} \\ \text{div } \psi = 0 \text{ in } \Omega; \quad \psi = g \text{ on } \Gamma,\ g \cdot \nu = 0 \text{ on } \Gamma & \text{(14.1b)} \end{cases}$$

Remark 14.1. Postponing regularity issues to the second part of the present sub-section, our purpose here is to introduce the Dirichlet map $g \longrightarrow \psi$, from the boundary datum to the interior solution of the above Oseen problem, following [B-L-T.1, Chapter 3]

As noted and discussed in [B-L-T.1, Ch 3, Orientation at p. 21; Appendix A.2, pp 99-102], problem (14.1) may not define a unique solution ψ; that is, the operator $g \to \psi$ may have a nontrivial (finite dimensional) null space. To overcome this, one replaces in (14.1) the differential expression $\mathbb{A}\psi = -\nu_o \Delta \psi + L_e(\psi)$ with its translation $k + \mathbb{A}$, for a positive constant k, sufficiently large as to obtain a unique solution ψ. As seen in the subsequent results below, we can take $k = 0$ whenever the Stokes operator is perturbed only by a first order operator such as $\mathbb{A}\psi = -\nu_o \Delta \psi + (a.\nabla)\psi$, div $a = 0$, with a sufficiently regular. Moreover, as documented in [B-L-T.1] in the Hilbert setting $q = 2$ and restated below in the general L^q-setting, the expression $D^*(k)\mathcal{A}_q^*(k)$ does not depend on the translation parameter k. Thus, at the end, also in name of simplicity of notation, we are here justified to admit henceforth that problem (14.1) (with $k = 0$) defines a unique solution ψ. We shall then denote the 'Dirichlet' map $g \longrightarrow \psi$ by $D: \; Dg = \psi$ in the notation of (14.1).

The following two regularity results of the Oseen equation below are critical for our subsequent development. The are the perfect counterpart of the results given in [B-L-T.1] in the Hilbert setting. To state properly the conclusion of uniqueness, they will refer to the Oseen equation with only a first order term, such as

$$\begin{cases} -\nu_o \Delta\psi + (a \cdot \nabla)\psi + \nabla\pi^* = 0 \text{ in } \Omega & (14.2a) \\ \text{div } \psi = 0 \text{ in } \Omega & (14.2b) \\ \psi = g \text{ on } \Gamma & (14.2c) \end{cases}$$

where

$$a \in L^q(\Omega), \; \text{div } a \equiv 0, \; 1 < q < \infty. \tag{14.2d}$$

Theorem 14.1. *[A-R, Thm 15, p 37, where a more general result is given]*
Let

$$a \in L^3(\Omega), \; div \; a \equiv 0; \; g \in W^{1-1/q,q}(\Gamma), \; 2 < q < \infty, \; g \cdot \nu = 0 \; on \; \Gamma. \tag{14.3}$$

Then problem (14.2) has a unique solution

$$(\psi, \pi^*) \in W^{1,q}(\Omega) \times L^q(\Omega)/\mathbb{R} \tag{14.4}$$

continuously: there is a constant $C > 0$ such that

$$\|\psi\|_{W^{1,q}(\Omega)} + \|\pi^*\|_{L^q(\Omega)/\mathbb{R}} \leq C\big(1 + \|a\|_{L^3(\Omega)}\big)^2 \|g\|_{W^{1-1/q,q}(\Gamma)} \quad \Box \tag{14.5}$$

Theorem 14.2. *[A-R, Thm 2, p 6, where a more general result is given]*
Let

$$a \in L^3(\Omega), \; div \; a \equiv 0; \; g \in W^{-1/q,q}(\Gamma), \; 3/2 < q < \infty, \; g \cdot \nu = 0 \; on \; \Gamma. \tag{14.6}$$

Then problem (14.2) has a unique solution

$$(\psi, \pi^*) \in L^q(\Omega) \times W^{-1,q}(\Omega)/\mathbb{R} \tag{14.7}$$

continuously: there is a constant $C_a > 0$ (explicitly depending on the norm of $\|a\|_{L^3(\Omega)}$) such that

$$\|\psi\|_{L^q(\Omega)} + \|\pi^*\|_{W^{-1,q}(\Omega)/\mathbb{R}} \leq C_a \|g\|_{W^{-1/q,q}(\Gamma)} \quad \square \tag{14.8}$$

We note that, in Theorem 14.2, we have also $\Delta\psi \in \left(Y_{r',p'}(\Omega)\right)' =$ dual of $Y_{r,p}(\Omega) = \left\{\varphi \in W_0^{1,r}(\Omega), \text{div } \varphi \in W_0^{1,q}(\Omega)\right\}$, $1 < r, q < \infty$, but we shall not need this result [A-R, p 6].

Returning to our Oseen problem (14.1) of interest, we have $y_e \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$ from Theorem 13.1, hence the embedding $W^{2,q}(\Omega) \hookrightarrow C(\overline{\Omega})$ holds true for $d = 3, q > 3/2$ [Kes, p 79], [Adams, p 97]. Thus, we can apply Theorems 14.1 and 14.2 to problem (14.1) and obtain the following results, where, with $\psi = Dg$, the range of D is in $L^q_\sigma(\Omega)$, since $\text{div}(Dg) \equiv 0$ in Ω, $(Dg) \cdot \nu|_\Gamma = g \cdot \nu|_\Gamma = 0$, see (13.4):

a)

$$\begin{array}{l} g \in W^{1-1/q,q}(\Gamma) \\ g \cdot \nu = 0 \text{ on } \Gamma \end{array} \quad \longrightarrow \quad Dg = \psi \in W^{1,q}(\Omega) \cap L^q_\sigma(\Omega) \tag{14.9}$$

b)

$$\begin{array}{l} g \in W^{-1/q,q}(\Gamma) \\ g \cdot \nu = 0 \text{ on } \Gamma \end{array} \quad \longrightarrow \quad Dg = \psi \in L^q_\sigma(\Omega) \tag{14.10}$$

c) By interpolating in the middle we obtain

$$\begin{array}{l} g \in W^{1/2-1/q,q}(\Gamma) \\ g \cdot \nu = 0 \text{ on } \Gamma \end{array} \quad \longrightarrow \quad Dg = \psi \in W^{1/2,q}(\Omega) \cap L^q_\sigma(\Omega) \tag{14.11}$$

d) More generally

$$\begin{array}{l} g \in W^{\left(1-\frac{1}{q}\right)(1-\theta)-\frac{\theta}{q},q}(\Gamma) \\ g \cdot \nu = 0 \text{ on } \Gamma;\ 0 < \theta < 1 \end{array} \quad \longrightarrow \quad Dg = \psi \in W^{(1-\theta),q}(\Omega) \cap L^q_\sigma(\Omega) \tag{14.12}$$

so that, as $\left(1 - \dfrac{1}{q}\right)(1-\theta) - \dfrac{\theta}{q} = 0$ for $\theta = 1 - \dfrac{1}{q}$, we also obtain

$$g \in U_q \equiv \left\{\widetilde{g} \in L^q(\Gamma),\ \widetilde{g} \cdot \nu = 0 \text{ on } \Gamma\right\} \longrightarrow Dg \in W^{1/q,q}(\Omega) \cap L^q_\sigma(\Omega), \tag{14.13}$$

all continuously. This property will be further complemented by additional information in (15.41) below. In the Hilbert space setting, $q = 2$, we re-obtain the regularity results, that were derived in [B-L-T.1, Theorem A.2.2 p 102], where we recall (1.6a–c)

a)

$$\begin{array}{l} g \in H^{1/2}(\Gamma) \\ g \cdot \nu = 0 \text{ on } \Gamma \end{array} \longrightarrow \quad Dg = \psi \in H^1(\Omega) \cap H \tag{14.14}$$

b)

$$\begin{array}{l} g \in H^{-1/2}(\Gamma) \\ g \cdot \nu = 0 \text{ on } \Gamma \end{array} \longrightarrow \quad Dg = \psi \in L^2(\Omega) \cap H \tag{14.15}$$

c)

$$\begin{array}{c} g \in H^s(\Gamma), \ -1/2 \leq s \leq 1/2 \\ g \cdot \nu = 0 \text{ on } \Gamma \end{array} \longrightarrow \quad Dg = \psi \in H^{s+1/2}(\Omega) \cap H. \tag{14.16}$$

14.2 Abstract model for the non-linear translated z-problem (13.10)

We re-write Eq (13.10a) as $z_t + \mathbb{A}z + (z \cdot \nabla)z + \nabla\chi - (mu)\tau = 0$ recalling the differential expression $\mathbb{A}$ defined above (14.1a), and next subtract $\mathbb{A}\psi = \mathbb{A}Dg = -\nabla\pi^*$ from (14.1a), where presently $g = v$ on $\Gamma, v \cdot \nu = 0$ on Γ. We obtain

$$z_t + \mathbb{A}(z - Dv) + (z \cdot \nabla)z + \nabla(\chi - \pi^*) - (m(x)u)\tau = 0 \quad \text{in } Q \tag{14.17}$$

Next we apply to (14.17) the Helmholtz projector P_q, and obtain [notice that $P_q z_t = z_t$, since $z_t \in L^q_\sigma(\Omega)$ [div$z_t \equiv 0, \ z_t \cdot \nu = v_t \cdot \nu = 0$ on Γ] since $P_q \nabla(\chi - \pi^*) \equiv 0$:

$$z_t + P_q\mathbb{A}(z - Dv) + P_q(z \cdot \nabla z) - P_q((m(x)u)\tau) \equiv 0 \tag{14.18}$$

where via (14.1a), (13.10e)

$$P_q\mathbb{A}f = -\nu_o P_q \Delta f + P_q\big[(y_e \cdot \nabla)f + (f \cdot \nabla)y_e\big]. \tag{14.19}$$

First, for $1 < q < \infty$ fixed, the Stokes operator A_q in $L^q_\sigma(\Omega)$ with Dirichlet boundary conditions is defined by [G-G-H.1, p 1404], [H-S, p 1]

$$A_q z = -P_q \Delta z, \quad \mathcal{D}(A_q) = W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega) \cap L^q_\sigma(\Omega). \tag{14.20}$$

The operator A_q has a compact inverse A_q^{-1} on $L^q_\sigma(\Omega)$, hence A_q has a compact resolvent on $L^q_\sigma(\Omega)$.

Next, we introduce the first order operator $A_{o,q}$, via (13.10e) and (14.20)

$$A_{o,q}z = P_q L_e(z) = P_q[(y_e\,.\,\nabla)z + (z\,.\,\nabla)y_e],\ \mathcal{D}(A_{o,q}) = \mathcal{D}(A_q^{1/2}) = W_0^{1,q}(\Omega)\cap L^q_\sigma(\Omega) \subset L^q_\sigma(\Omega). \quad (14.21)$$

where $A_q^{1/2}$ is defined in (15.9) below. Thus, $A_{o,q}A_q^{-1/2}$ is a bounded operator on $L^q_\sigma(\Omega)$, and thus $A_{o,q}$ is bounded on $\mathcal{D}(A_q^{1/2})$. This leads to the definition of the Oseen operator

$$\mathcal{A}_q = -(\nu_o A_q + A_{o,q}), \quad \mathcal{D}(\mathcal{A}_q) = \mathcal{D}(A_q) \subset L^q_\sigma(\Omega) \quad (14.22)$$

also with compact resolvent. Finally, we define the projection of the nonlinear portion of (13.10a)

$$\mathcal{N}_q(z) = P_q[(z\cdot\nabla)z] \quad (14.23)$$

Thus, after using (14.20)-(14.23) in (14.18), the N-S translated problem (14.18) can rewritten as the following abstract equation on $L^q_\sigma(\Omega), 1 < q < \infty$:

$$z_t - \mathcal{A}_q(z - Dv) + \mathcal{N}_q z - P_q((mu)\tau) = 0,\ \text{on } L^q_\sigma(\Omega),\ v\cdot\nu = 0 \text{ on } \Gamma \quad (14.24)$$

in factor form on $L^q_\sigma(\Omega)$. Next, extending the original Oseen operator $L^q_\sigma(\Omega) \supset \mathcal{D}(\mathcal{A}_q) \longrightarrow L^q_\sigma(\Omega)$ to $\mathcal{A}_q : L^q_\sigma(\Omega) \longrightarrow [\mathcal{D}(\mathcal{A}_q^*)']$, by isomorphism, and retaining the same symbol, we arrive at the definitive abstract model

$$\begin{cases} z_t - \mathcal{A}_q z + \mathcal{N}_q z + \mathcal{A}_q Dv - P_q\big[(mu)\tau\big] = 0 \text{ on } \big[\mathcal{D}(\mathcal{A}_q^*)'\big] \\ z(x,0) = z_0(x) = y_0(x) - y_e \text{ in } L^q_\sigma(\Omega) \end{cases} \quad (14.25)$$

in additive form, on $\big[\mathcal{D}(\mathcal{A}_q^*)'\big]$.

14.3 Abstract model of the linearized w-problem (13.11) of the translated model (13.10)

Still for $1 < q < \infty$, the abstract model (in additive form) of the linearized w-problem in (13.11) is obtained from (14.25) by dropping the nonlinear term

$$\begin{cases} w_t - \mathcal{A}_q w + \mathcal{A}_q Dv - P_q\big[(mu)\tau\big] = 0 \text{ on } \big[\mathcal{D}(\mathcal{A}_q^*)'\big] \\ w(x,0) = w_0(x) = y_0(x) - y_e \text{ in } L^q_\sigma(\Omega). \end{cases} \quad (14.26)$$

14.4 The adjoint operators D^*, $(A_q)^* = A_q^*$ and $(A_{o,q})^* = A_{o,q}^*$, $(\mathcal{A}_q)^* = \mathcal{A}_q^* = -(\nu_o A_q^* + A_{o,q}^*)$, $1 < q < \infty$

(i) Regarding the Helmholtz projection P_q and its adjoint P_q^*, we recall from Proposition 13.2 that $P_q \in \mathcal{L}\big(L^q(\Omega)\big)$, while $P_q^* = P_{q'} \in \mathcal{L}\big(L^{q'}(\Omega)\big)$, ${}^1/_q + {}^1/_{q'} = 1$, by (13.8c)

(ii) Define as in (14.13)

$$U_q = \big\{g \in L^q(\Gamma) : g \cdot \nu = 0 \text{ on } \Gamma\big\}. \tag{14.27}$$

We have seen in (14.13) that

$$D : U_q = \big\{g \in L^q(\Gamma) : g \cdot \nu = 0 \text{ on } \Gamma\big\} \longrightarrow W^{1/_q,q}(\Omega) \cap L^q_\sigma(\Omega), \tag{14.28}$$

so that the dual D^* satisfies

$$D^* : W^{-1/_q,q'} \longrightarrow L^{q'}(\Gamma). \tag{14.29}$$

(iii) The adjoint $A_q^* : L^{q'}_\sigma(\Omega) \supset \mathcal{D}(A_q^*) \longrightarrow L^{q'}_\sigma(\Omega)$, ${}^1/_q + {}^1/_{q'} = 1$ of the Stokes operator A_q in (14.20)

$$\big\langle A_q f_1, f_2\big\rangle_{L^q_\sigma, L^{q'}_\sigma} = \big\langle f_1, A_q^* f_2\big\rangle_{L^q_\sigma, L^{q'}_\sigma}, \quad f_1 \in L^q_\sigma, \ f_2 \in L^{q'}_\sigma \tag{14.30}$$

(duality pairing $L^q_\sigma \longrightarrow L^{q'}_\sigma$) is

$$A_q^* f_2 = -P_{q'} \Delta f_2, \ \mathcal{D}(A_q^*) = W^{2,q'}(\Omega) \cap W_0^{1,q'}(\Omega) \cap L^{q'}_\sigma(\Omega). \tag{14.31}$$

Proof. For $f_1 \in \mathcal{D}(A_q) \subset L^q_\sigma(\Omega) \subset L^q(\Omega)$, so that $A_q f_1 \in L^q_\sigma(\Omega)$ and $f_2 \in \mathcal{D}(A_q^*) \subset L^{q'}_\sigma(\Omega) \subset L^{q'}(\Omega)$ so that $A_q^* f_2 \in L^{q'}_\sigma(\Omega)$, and $P_{q'} f_2 = f_2$, we compute from (14.20): with $P_q^* = P_{q'}$ by Proposition 13.2

$$-\big\langle A_q f_1, f_2\big\rangle_{L^q_\sigma, L^{q'}_\sigma} = \big\langle P_q \Delta f_1, f_2\big\rangle_{L^q_\sigma, L^{q'}_\sigma} \tag{14.32}$$

$$= \big\langle \Delta f_1, P_q^* f_2\big\rangle_{L^q, L^{q'}} = \big\langle \Delta f_1, P_{q'} f_2\big\rangle_{L^q, L^{q'}} = \big\langle \Delta f_1, f_2\big\rangle_{L^q, L^{q'}} \tag{14.33}$$

$$= \int_\Omega f_1 \Delta f_2 \, d\Omega + \int_\Gamma \cancel{\frac{\partial f_1}{\partial \nu} f_2} \, d\Gamma - \int_\Gamma \cancel{f_1 \frac{\partial f_2}{\partial \nu}} \, d\Gamma$$

$$= \big\langle f_1, \Delta f_2\big\rangle_{L^q, L^{q'}} = \big\langle P_q f_1, \Delta f_2\big\rangle_{L^q, L^{q'}}$$

$$= \big\langle f_1, P_{q'} \Delta f_2\big\rangle_{L^q, L^{q'}} = -\big\langle f_1, A_q^* f_2\big\rangle_{L^q, L^{q'}} \tag{14.34}$$

since $f_1 \in W_0^{1,q}(\Omega)$ by (14.20); and so $f_1|_\Gamma = 0$; and since $f_2 \in W_0^{1,q'}(\Omega)$ by (14.31) and so $f_2|_\Gamma = 0$. Moreover, $P_q f_1 = f_1$, since $f_1 \in L^q_\sigma(\Omega)$. Eqt (14.34) proves (14.31). □

(iv) Similarly from $A_{o,q} = P_q L_e : \mathcal{D}(A_{o,q}) = \mathcal{D}(A_q^{1/2}) = W_0^{1,q}(\Omega) \cap L^q_\sigma(\Omega) \longrightarrow L^q_\sigma(\Omega)$, in (14.21), we obtain

$$(A_{o,q})^* = A^*_{o,q} \text{ (for short) } = P_{q'}(L_e)^* : W^{-1,q'}(\Omega) \longrightarrow L^{q'}_\sigma(\Omega) \tag{14.35}$$

where the expression of $(L_e)^*$, which is not needed, is given in [B-L-T.1, p 55], [L-T.2, below (54)], [Fur.1].

(v) As a consequence of (ii), (iii) we have $(\mathcal{A}_q)^* = \mathcal{A}^*_q = -(\nu_0 A^*_q + A^*_{o,q})$, $\mathcal{D}(\mathcal{A}^*_q) = \mathcal{D}(A^*_q)$.

14.5 The operator $D^*\mathcal{A}^*_q$

Theorem 14.3. *Let $1 < q < \infty$. Let $v \in \mathcal{D}(\mathcal{A}^*_q) = \mathcal{D}(A^*_q) = W^{2,q'}(\Omega) \cap W^{1,q'}_0(\Omega) \cap L^{q'}_\sigma(\Omega)$, by (14.31), $\dfrac{1}{q} + \dfrac{1}{q'} = 1$ so that $\left.\dfrac{\partial v}{\partial \nu}\right|_\Gamma \in W^{1-1/q',q'}(\Gamma) \subset L^{q'}(\Gamma)$. Let $g \in L^q(\Gamma), g \cdot \nu = 0$ on Γ. Then*

$$\left\langle D^*\mathcal{A}^*_q v, g\right\rangle_{L^{q'}(\Gamma), L^q(\Gamma)} = \nu_o \left\langle \frac{\partial v}{\partial \nu}, g \right\rangle_{L^{q'}(\Gamma), L^q(\Gamma)} \tag{14.36}$$

where: $q > 3$ for $d = 3$; and $q > 2$ for $d = 2$.

Proof. We shall first prove (14.36) with $g \in W^{1-1/q,q}(\Gamma)$, $g \cdot \nu = 0$ on Γ; and then extend the validity of (14.36) to $g \in L^q(\Gamma)$, $g \cdot \nu = 0$ on Γ by density. By (14.22), $\mathcal{A}_q = -(\nu_o A_q + A_{o,q})$. Accordingly, we consider $D^*A^*_q$ in Step 1 and $D^*A^*_{o,q}$ in Step 2.

Step 1: Let $v \in \mathcal{D}(A^*_q) = W^{2,q'}(\Omega) \cap W^{1,q'}_0(\Omega) \cap L^{q'}_\sigma(\Omega)$, so that $A^*_q v \in L^{q'}_\sigma(\Omega)$, and let initially $g \in W^{1-1/q,q}(\Gamma) \subset L^q(\Gamma), g \cdot \nu = 0$ on Γ, so that $Dg \in W^{1,q}(\Omega) \cap L^q_\sigma(\Omega)$ by (14.9). Our first step is to show

$$-\left\langle D^*A^*_q v, g\right\rangle_{L^{q'}(\Gamma), L^q(\Gamma)} = \int_\Omega v \Delta(Dg)\, d\Omega + \int_\Gamma \frac{\partial v}{\partial \nu} g\, d\Gamma, \tag{14.37}$$

where the integral term under Ω is well-defined as a duality pairing with $v \in W^{1,q'}_0(\Omega)$ and $\Delta(Dg) \in W^{-1,q}(\Omega)$; while the integral term under Γ is well-defined as a duality paring with $\left.\dfrac{\partial v}{\partial \nu}\right|_\Gamma$ in $L^{q'}(\Gamma)$ and $g \in L^q(\Gamma)$.

In fact, we compute - and the computations in (14.38) through (14.40) below actually work even for

$g \in W^{-1/q,q}(\Gamma)$ so that $Dg \in L^q_\sigma(\Omega)$ by (14.10), and hence $P_q Dg = Dg$

$$-\langle D^* A_q^* v, g\rangle_{L^{q'}(\Gamma),L^q(\Gamma)} = -\langle A_q^* v, Dg\rangle_{L^{q'}_\sigma(\Omega),L^q_\sigma(\Omega)} \tag{14.38}$$

$$\text{(by (14.31))} \qquad = \langle P_{q'}\Delta v, Dg\rangle_{L^{q'},L^q} = \langle \Delta v, P_{q'}^* Dg\rangle_{L^{q'},L^q} \tag{14.39}$$

$$= \langle \Delta v, P_q Dg\rangle_{L^{q'},L^q} = \langle \Delta v, Dg\rangle_{L^{q'},L^q} \tag{14.40}$$

where in going from (14.39) to (14.40) we have recalled $P_{q'}^* = P_q$ by Proposition 13.2. Next, we apply Green's theorem in (14.40) and get

$$-\langle D^* A_q^* v, g\rangle_{L^{q'}(\Gamma),L^q(\Gamma)} = \int_\Omega \Delta v Dg\, d\Omega = \int_\Omega v\Delta(Dg)\, d\Omega + \int_\Gamma \frac{\partial v}{\partial \nu} g\, d\Gamma - \int_\Gamma v\frac{\partial Dg}{\partial \nu}\, d\Gamma \tag{14.41}$$

where we have used $Dg|_\Gamma = g$ by definition of D, and $v|_\Gamma = 0$ as $v \in W_0^{1,q'}(\Omega)$. Then (14.41) proves (14.37).

<u>Step 2:</u> Let $v \in \mathcal{D}(A^*_{o,q}) = \mathcal{D}\big((A_q^*)^{1/2}\big) = W_0^{1,q'}(\Omega) \cap L^{q'}_\sigma(\Omega)$ by (14.21) and let $g \in W^{1-1/q,q}(\Gamma)$ and $g \cdot \nu = 0$ on Γ so that $Dg \in W^{1,q}(\Omega) \cap L^q_\sigma(\Omega)$ by (14.9). Recall from Theorem 13.1 that $y_e \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$.

Our second step is to show that

$$\langle D^* A^*_{o,q} v, g\rangle_{L^{q'}(\Gamma),L^q(\Gamma)} = \langle (y_e \cdot \nabla)(Dg) + ((Dg)\cdot\nabla)y_e, v\rangle_{L^q(\Omega),L^{q'}(\Omega)} \tag{14.42}$$

Proof of (14.42).

Step (2a): Let initially $h \in \mathcal{D}(A_{o,q}) = \mathcal{D}(A_q^{\frac{1}{2}}) = W_0^{1,q}(\Omega) \cap L^q_\sigma(\Omega)$ by (14.21). Recalling (14.21) compute

$$\langle A_{o,q} h, v\rangle_{L^q_\sigma(\Omega),L^{q'}_\sigma(\Omega)} = \langle P_q[(y_e\cdot\nabla)h], v\rangle_{L^q_\sigma(\Omega),L^{q'}_\sigma(\Omega)} + \langle P_q[(h\cdot\nabla)y_e], v\rangle_{L^q_\sigma(\Omega),L^{q'}_\sigma(\Omega)} \tag{14.43}$$

$$= \langle [(y_e\cdot\nabla)h], P_q^* v\rangle_{L^q,L^{q'}} + \langle [(h\cdot\nabla)y_e], P_q^* v\rangle_{L^q,L^{q'}} \tag{14.44}$$

$$= \langle [(y_e\cdot\nabla)h], P_{q'} v\rangle_{L^q,L^{q'}} + \langle [(h\cdot\nabla)y_e], P_{q'} v\rangle_{L^q,L^{q'}} \tag{14.45}$$

$$= \langle [(y_e\cdot\nabla)h], v\rangle_{L^q,L^{q'}} + \langle [(h\cdot\nabla)y_e], v\rangle_{L^q,L^{q'}} \tag{14.46}$$

where we have recalled $P_q^* = P_{q'}$ from Proposition 13.2 and $P_{q'} v = v$, as $v \in L^{q'}_\sigma(\Omega)$.

Step (2b): In the next lemma, we show that the terms in (14.46) are well-defined in an appropriate range of q, at any rate for $q > d$, which is our goal, $d = 2, d = 3$.

Lemma 14.4. *With reference to (14.46) we have*

(i)

$$(y_e \cdot \nabla)h \in L^q(\Omega) = W^{0,q}(\Omega) \text{ for } \begin{cases} d = 3, \ q > 3/2 \\ d = 2, \ q > 1 \end{cases} \tag{14.47}$$

(ii)

$$(h \cdot \nabla)y_e \in W^{1,q}(\Omega) \text{ for } \begin{cases} d = 3, \ q > 3 \\ d = 2, \ q > 2. \end{cases} \tag{14.48}$$

Proof. First way: We may use multiplier theory [M-S, Theorem 3, p 252]. We have by Theorem 13.1 on y_e and the assumption on $h \in \mathcal{D}(A_{o,q}) = W_0^{1,q}(\Omega) \cap L^q_\sigma(\Omega)$:

(i)

$$y_e \in W^{2,q}(\Omega), \ |\nabla h| \in L^q(\Omega) = W^{0,q}(\Omega). \tag{14.49}$$

Then [M-S, Theorem 3, p 252 with $m = 2 > \ell = 0$] yields the multiplier space

$$\mathcal{M}\big(W^{2,q} \longrightarrow W^{0,q}\big) = W^{0,q}(\Omega). \tag{14.50}$$

for $mq = 2q > d$ or $q > 3/2$ for $d = 3; q > 1$ for $d = 2$; and part (i) established.

(ii) We start with

$$h \in W_0^{1,q}(\Omega), \ |\nabla y_e| \in W^{1,q}(\Omega). \tag{14.51}$$

Then [M-S, Theorem 3, p 252; with $m = \ell = 1$] yields the multiplier space

$$\mathcal{M}(W^{1,q} \longrightarrow W^{1,q}) = W^{1,q}(\Omega) \tag{14.52}$$

for $mq = 1.q > d$ or $q > 3$ for $d = 3, q > 2$ for $d = 2$ and part (ii) is established.

Second way: We use embedding theory [Kes, p 79]

$$W^{m,q}(\Omega) \hookrightarrow C^k(\overline{\Omega}), \ m > \frac{d}{q}, \ k \text{ integer part of } \left[m - \frac{d}{q}\right] \tag{14.53}$$

Thus

(i)

$$y_e \in W^{2,q}(\Omega) \hookrightarrow y_e \in C^0(\overline{\Omega}) \text{ for } \begin{cases} m = 2, d = 3, q > 3/2, k = 0 \\ m = 2, d = 2, q > 1, k = 0 \end{cases}$$

and since $|\nabla h| \in L^q(\Omega)$, then

$$(y_e \cdot \nabla)h \in L^q(\Omega), d = 3, q > 3/2; \text{ or } d = 2, q > 1 \tag{14.54}$$

and (i) is reproved.

(ii) Similarly, (14.53) gives for $m = 1$

$$h \in W^{1,q}(\Omega) \hookrightarrow h \in C^0(\overline{\Omega}) \text{ for } \begin{cases} d = 3, q > 3, k = 0 \\ d = 2, q > 2, k = 0 \end{cases} \tag{14.55}$$

and since $|\nabla y_e| \in W^{1,q}(\Omega)$, then

$$(h \cdot \nabla)y_e \in W^{1,q}(\Omega), \ d = 3, q > 3; d = 2, q > 2 \tag{14.56}$$

and (ii) is reproved.

Lemma 14.4 is proved. □

Step (2c): Using Lemma 14.4 in (14.46) we see that the two terms are well-defined with $v \in L^{q'}_\sigma(\Omega)$. We rewrite (14.46) as

$$\left\langle h, A^*_{o,q} v \right\rangle_{L^q_\sigma, L^{q'}_\sigma} = \left\langle (y_e \cdot \nabla)h + (h \cdot \nabla)y_e, v \right\rangle_{L^q, L^{q'}}, \tag{14.57}$$

which shows that it can be extended to all $h \in W^{1,q}(\Omega) \cap L^q_\sigma(\Omega)$: the condition $h|_\Gamma = 0$ is not used. With $g \in W^{1-1/q,q}(\Gamma)$, $g \cdot \nu = 0$ on Γ, so that $Dg \in W^{1,q}(\Omega) \cap L^q_\sigma(\Omega)$ by (14.9), we may apply such extended version (14.57) to Dg and obtain

$$\begin{aligned} \left\langle Dg, A^*_{o,q} v \right\rangle &= \left\langle g, D^* A^*_{o,q} v \right\rangle_{L^q(\Gamma), L^{q'}(\Gamma)} \\ &= \left\langle (y_e \cdot \nabla)(Dg) + ((Dg) \cdot \nabla)y_e, v \right\rangle_{L^q(\Omega), L^{q'}(\Omega)} \end{aligned} \tag{14.58}$$

and (14.42) is established.

<u>Step 3:</u> In view of $\mathcal{A}_q = -(\nu_o A_q + A_{o,q})$ by (14.22), we now combine (14.37) of Step 1, with (14.42) of Step 2. Let again $g \in W^{1-1/q,q}(\Gamma)$, $g \cdot \nu = 0$ on Γ, so that $Dg \in W^{1,q}(\Gamma) \cap L^q_\sigma(\Omega)$ by (14.9), and

$v \in \mathcal{D}(A_q^*) = \mathcal{D}(\mathcal{A}_q^*) = W^{2,q'}(\Omega) \cap W_0^{1,q'}(\Omega) \cap L_\sigma^{q'}(\Omega)$ via (14.31). We shall establish the following final relation

$$\left\langle D^* \mathcal{A}_q^* v, g \right\rangle = \nu_0 \int_\Gamma \frac{\partial v}{\partial \nu} g \; d\Gamma = \nu_0 \left\langle \frac{\partial v}{\partial \nu}, g \right\rangle_{L^{q'}(\Gamma), L^q(\Gamma)}. \tag{14.59}$$

In fact, we start from $-\mathcal{A}_q^* = \nu_0 A_q^* + A_{o,q}^*$ via part (iv) of Section 14.4 and next recall (14.37) and (14.42) to obtain

$$-\left\langle D^* \mathcal{A}_q^* v, g \right\rangle_{L^{q'}(\Gamma), L^q(\Gamma)} = \nu_o \left\langle D^* A_q^* v, g \right\rangle_{L^{q'}(\Gamma), L^q(\Gamma)} + \left\langle D^* A_{o,q}^* v, g \right\rangle_{L^{q'}(\Gamma), L^q(\Gamma)} \tag{14.60}$$

$$= -\nu_o \int_\Omega v \Delta(Dg) \; d\Omega - \nu_o \int_\Gamma \frac{\partial v}{\partial \nu} g \; d\Gamma + \left\langle v, (y_e \cdot \nabla)(Dg) + ((Dg) \cdot \nabla) y_e \right\rangle_{L^{q'}(\Omega), L^q(\Omega)} \tag{14.61}$$

$$= \left\langle v, -\nu_o \Delta(Dg) + (y_e \cdot \nabla)(Dg) + ((Dg) \cdot \nabla) y_e \right\rangle_{L^{q'}(\Omega), L^q(\Omega)} - \nu_o \left\langle \frac{\partial v}{\partial \nu}, g \right\rangle_{L^{q'}(\Gamma), L^q(\Gamma)} \tag{14.62}$$

$$= \left\langle v, -\nu_o \Delta(Dg) + L_e(Dg) \right\rangle_{L^{q'}(\Omega), L^q(\Omega)} - \nu_o \left\langle \frac{\partial v}{\partial \nu}, g \right\rangle_{L^{q'}(\Gamma), L^q(\Gamma)}, \tag{14.63}$$

recalling $L_e(\psi = Dg) = (y_e \cdot \nabla)(Dg) + ((Dg) \cdot \nabla) y_e$ by (13.10e). We next invoke the definition $\psi = Dg$ in Eq (14.1a) of the Stationary Oseen Equation (14.1). This way we rewrite (14.63) as

$$\left\langle D^* \mathcal{A}_q^* v, g \right\rangle_{L^{q'}(\Gamma), L^q(\Gamma)} = \left\langle v, \nabla \pi^* \right\rangle_{L^{q'}(\Omega), L^q(\Omega)} + \nu_o \left\langle \frac{\partial v}{\partial \nu}, g \right\rangle_{L^{q'}(\Gamma), L^q(\Gamma)} \tag{14.64}$$

$$= \nu_o \left\langle \frac{\partial v}{\partial \nu}, g \right\rangle_{L^{q'}(\Gamma), L^q(\Gamma)}, \tag{14.65}$$

since

$$\int_\Omega v \cdot \nabla \pi^* = \int_\Gamma \pi^* v \cdot \nu \; d\Gamma - \int_\Omega \pi^* \text{div } v \; d\Omega \equiv 0 \tag{14.66}$$

where $v|_\Gamma = 0$ as $v \in W_0^{1,q'}(\Omega)$ and div $v \equiv 0$ since $v \in L_\sigma^q(\Omega)$, recall (13.4). Thus, (14.65) shows (14.36) so far for $g \in W^{1-1/q,q}(\Gamma), \; g \cdot \nu = 0$ on Γ.

By density, we extend the validity of (14.36) to $g \in L^q(\Gamma), \; g \cdot \nu = 0$ on Γ. Theorem 14.3 is proved. □

Proposition 14.5. *[B-L-T.1, Lemma 3.3.1 p35] Let $\varphi \in C^1(\Omega)$ be a d-function satisfying the following properties:*

(i) $\varphi|_\Gamma = 0$;

(ii) div $\varphi = 0$ *in* $\overline{\Omega}$ *(actually only on an interior strip of* Γ*)*

Then we have that

$$\begin{cases} \textit{the boundary vector } \nabla\varphi \cdot \nu = \frac{\partial \varphi}{\partial \nu} \textit{ is tangential to } \Gamma \\ \\ i.e \ (\nabla\phi \cdot \nu) \cdot \nu \equiv 0 \ \textit{on } \Gamma. \end{cases} \tag{14.67}$$

For $v \in \mathcal{D}(\mathcal{A}_q^*) = W^{2,q'}(\Omega) \cap W_0^{1,q'}\Omega \cap L_\sigma^{q'}(\Omega)$, we have $v|_\Gamma = 0$ and div $v = 0$ in Ω. Thus extending Proposition 14.5 to $v \in W^{2,q'}(\Omega)$, we have $\left.\dfrac{\partial v}{\partial \nu}\right|_\Gamma$ = tangential on Γ. Returning to Theorem 14.3, and recalling that g is tangential. $g \cdot \nu = 0$ on Γ, we then obtain from (14.36) the following

Corollary 14.6. With reference to Theorem 14.3 we have

$$\left\{ \begin{matrix} \text{tangential} \\ \text{component of } D^* \mathcal{A}^* v \end{matrix} \right\} = (D^* \mathcal{A}_q^* v)\tau = \nu_o \frac{\partial v}{\partial \nu}, \ v \in \mathcal{D}(\mathcal{A}_q^*) = W^{2,q'}(\Omega) \cap W_0^{1,q'}\Omega \cap L_\sigma^{q'}(\Omega) \tag{14.68}$$

$q > 3$ for $d = 3$; $q > 2$ for $d = 2$. □

15 Some auxiliary results for the w-linearized problem (14.26): Analytic semigroup generation, Maximal regularity, Domains of fractional powers

In this subsection we collect mostly known results to be used in the sequel.

(a) **Definition of Besov spaces $B^s_{q,p}$ on domains of class C^1 as real interpolation of Sobolev spaces:** Let m be a positive integer, $m \in \mathbb{N}, 0 < s < m, 1 \leq q < \infty, 1 \leq p \leq \infty$, then we define [G-G-H.1, p 1398]

$$B^s_{q,p}(\Omega) = (L^q(\Omega), W^{m,q}(\Omega))_{\frac{s}{m},p} \tag{15.1a}$$

[Wahl, p. xx] This definition does not depend on $m \in \mathbb{N}$. This clearly gives

$$W^{m,q}(\Omega) \subset B^s_{q,p}(\Omega) \subset L^q(\Omega) \quad \text{and} \quad \|y\|_{L^q(\Omega)} \leq C \|y\|_{B^s_{q,p}(\Omega)}. \tag{15.1b}$$

We shall be particularly interested in the following special real interpolation space of the L^q and $W^{2,q}$ spaces $\left(m = 2, s = 2 - \frac{2}{p}\right)$:

$$B^{2-\frac{2}{p}}_{q,p}(\Omega) = \left(L^q(\Omega), W^{2,q}(\Omega)\right)_{1-\frac{1}{p},p}. \tag{15.2}$$

Our interest in (15.2) is due to the following characterization [Amann.2, Thm 3.4], [G-G-H.1, p 1399]: if A_q denotes the Stokes operator introduced in (14.20), then

$$\left(L^q_\sigma(\Omega), \mathcal{D}(A_q)\right)_{1-\frac{1}{p},p} = \left\{g \in B^{2-2/p}_{q,p}(\Omega) : \ \text{div } g = 0, \ g|_\Gamma = 0\right\} \quad \text{if } \frac{1}{q} < 2 - \frac{2}{p} < 2 \tag{15.3a}$$

$$\left(L^q_\sigma(\Omega), \mathcal{D}(A_q)\right)_{1-\frac{1}{p},p} = \left\{g \in B^{2-2/p}_{q,p}(\Omega) : \ \text{div } g = 0, \ g \cdot \nu|_\Gamma = 0\right\} \equiv \widetilde{B}^{2-2/p}_{q,p}(\Omega) \tag{15.3b}$$

$$\text{if } 0 < 2 - \frac{2}{p} < \frac{1}{q}; \ \text{or } 1 < p < \frac{2q}{2q-1}.$$

Notice that, in (15.3b), the condition $g \cdot \nu|_\Gamma = 0$ is an intrinsic condition of the space $L^q_\sigma(\Omega)$ in (13.4), not an extra boundary condition as $g|_\Gamma = 0$ in (15.3a).

Remark 15.1. In the analysis of well-posedness and stabilization of the nonlinear N-S translated feedback z-problem (14.18) = (24.1) with localized interior, tangential-like control u and localized tangential boundary control v - both expressed in feedback form - we shall need to impose the constrain $q > d = \dim \Omega$, in particular $q > 3$, see Eq (24.16), to obtain the embedding $W^{1,q} \hookrightarrow L^\infty(\Omega)$ in our case of interest $d = 3$. What is then the allowable range of the parameter p in such case $q > 3$?. The intended goal of the present paper is to obtain the sought-after stabilization result in a function space, such as a $B^{2-2/p}_{q,p}(\Omega)$-space, that does not recognize boundary conditions of the I.C. Thus, we need to avoid the case in (15.3a), as this implies a Dirichlet homogeneous B.C. Instead, we need to fit into the case (15.3b), where the conditions div $g \equiv 0$ and $g \cdot \nu|_\Gamma = 0$ are just features of the underlying space $L^q_\sigma(\Omega)$, see (13.4). We shall then impose the condition $2 - 2/p < 1/q < 1/3$ and then obtain that p must satisfy $p < 6/5$ for $d = 3$. Moreover, analyticity and maximal regularity of the Stokes problem will require $p > 1$. Thus, in conclusion, the allowed range of the parameters p, q under which we shall solve the well-posedness and stabilization problem of the nonlinear N-S feedback z-system (14.18) = (24.1) for $d = 3$, in a space $B^{2-2/p}_{q,p}(\Omega)$. This - as intended - does not recognize boundary conditions is: $q > 3,\ 1 < p < 6/5$. See Theorem A and B to be proved as Theorem 24.1 through Theorem 24.3 in Section 24.

(b) **The Stokes and Oseen operators generate strongly continuous analytic semigroups on $L^q_\sigma(\Omega)$, $1 < q < \infty$.**

Theorem 15.1. *Let $d \geq 2, 1 < q < \infty$ and let Ω be a bounded domain in $\mathbb{R}^d$ of class C^3. Then*

(i) the Stokes operator $-A_q = P_q \Delta$ in (14.20), repeated here as

$$- A_q \psi = P_q \Delta \psi, \quad \psi \in \mathcal{D}(A_q) = W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega) \cap L^q_\sigma(\Omega) \tag{15.4}$$

generates a s.c analytic semigroup $e^{-A_q t}$ on $L^q_\sigma(\Omega)$. See the review paper [H-S, Theorem 2.8.5 p 17].

(ii) The Oseen operator $\mathcal{A}_q$ in (14.22)

$$\mathcal{A}_q = -(\nu_o A_q + A_{o,q}), \quad \mathcal{D}(\mathcal{A}_q) = \mathcal{D}(A_q) \subset L^q_\sigma(\Omega) \tag{15.5}$$

generates a s.c analytic semigroup $e^{\mathcal{A}_q t}$ on $L^q_\sigma(\Omega)$. This follows as $A_{o,q}$ is relatively bounded with respect to $A_q^{1/2}$, to be formally defined in (15.9): thus a standard theorem on perturbation of an analytic semigroup generator applies [Pazy, Corollary 2.4, p 81].

(iii)

$$\begin{cases} 0 \in \rho(A_q) = \text{ the resolvent set of the Stokes operator } A_q & (15.6a) \\ A_q^{-1} : L^q_\sigma(\Omega) \longrightarrow L^q_\sigma(\Omega) \text{ is compact.} & (15.6b) \end{cases}$$

The s.c. analytic Stokes semigroup $e^{-A_q t}$ *is uniformly stable on* $L^q_\sigma(\Omega)$*: there exist constants* $M \geq 1, \delta > 0$ *(possibly depending on q) such that*

$$\left\| e^{-A_q t} \right\|_{\mathcal{L}(L^q_\sigma(\Omega))} \leq M e^{-\delta t}, \ t > 0. \tag{15.7}$$

(c) **Domains of fractional powers,** $\mathcal{D}(A_q^\alpha), 0 < \alpha < 1$ **of the Stokes operator** A_q **on** $L^q_\sigma(\Omega), 1 < q < \infty$,

Theorem 15.2. *For the domains of fractional powers* $\mathcal{D}(A_q^\alpha), 0 < \alpha < 1$, *of the Stokes operator* A_q *in (14.20) = (15.4), the following complex interpolation relation holds true [H-S, Theorem 2.8.5, p 18]*

$$[\mathcal{D}(A_q), L^q_\sigma(\Omega)]_{1-\alpha} = \mathcal{D}(A_q^\alpha), \ 0 < \alpha < 1, \ 1 < q < \infty; \tag{15.8}$$

in particular, see (14.21)

$$[\mathcal{D}(A_q), L^q_\sigma(\Omega)]_{\frac{1}{2}} = \mathcal{D}(A_q^{1/2}) \equiv W_0^{1,q}(\Omega) \cap L^q_\sigma(\Omega). \tag{15.9}$$

Thus, on the space $\mathcal{D}(A_q^{1/2})$*, the norms*

$$\|\nabla \cdot\|_{L^q(\Omega)} \text{ and } \| \ \|_{L^q(\Omega)} \tag{15.10}$$

are equivalent via the Poincaré inequality.

(d) **The Stokes operator** $-A_q$ **and the Oseen operator** $\mathcal{A}_q, 1 < q < \infty$ **generate s.c. analytic semigroups on the Besov space**

$$\left(L^q_\sigma(\Omega), \mathcal{D}(A_q) \right)_{1-\frac{1}{p},p} = \left\{ g \in B^{2-2/p}_{q,p}(\Omega) : \ \text{div } g = 0, \ g|_\Gamma = 0 \right\} \ \text{ if } \frac{1}{q} < 2 - \frac{2}{p} < 2; \tag{15.11a}$$

$$\left(L^q_\sigma(\Omega), \mathcal{D}(A_q) \right)_{1-\frac{1}{p},p} = \left\{ g \in B^{2-2/p}_{q,p}(\Omega) : \ \text{div } g = 0, \ g \cdot \nu|_\Gamma = 0 \right\} \equiv \widetilde{B}^{2-2/p}_{q,p}(\Omega) \tag{15.11b}$$

$$\text{if } 0 < 2 - \frac{2}{p} < \frac{1}{q}, \ \text{ or } 1 < p < \frac{2q}{2q-1}.$$

In fact, Theorem 3.1(i) states that the Stokes operator $-A_q$ generates a s.c analytic semigroup on the space $L^q_\sigma(\Omega)$, $1 < q < \infty$, hence on the space $\mathcal{D}(A_q)$ in (15.4), with norm $\|\cdot\|_{\mathcal{D}(A_q)} = \|A_q \cdot\|_{L^q_\sigma(\Omega)}$ as $0 \in \rho(A_q)$. Then, one obtains that the Stokes operator $-A_q$ generates a s.c. analytic semigroup on the real interpolation spaces in (15.11). Next, the Oseen operator $\mathcal{A} = -(\nu_o A_q + A_{o,q})$ likewise generates a s.c. analytic semigroup $e^{\mathcal{A}_q t}$ on $L^q_\sigma(\Omega)$ by Theorem 3.1(ii). Moreover $\mathcal{A}_q$ generates a s.c. analytic semigroup on $\mathcal{D}(\mathcal{A}_q) = \mathcal{D}(A_q)$ (equivalent norms). Hence $\mathcal{A}_q$ generates a s.c. analytic semigroup on the real interpolation spaces (15.11). Here below, however, we shall formally state the result only in the case $2 - {}^2/_p < {}^1/_q$. i.e. $1 < p < {}^{2q}/_{2q-1}$, in the space $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$, as this does not contain B.C. The objective of the present paper is precisely to obtain stabilization results on spaces that do not recognize B.C.

Theorem 15.3. *Let* $1 < q < \infty, 1 < p < {}^{2q}/_{2q-1}$

(i) The Stokes operator $-A_q$ *in (15.4) generates a s.c analytic semigroup* $e^{-A_q t}$ *on the space* $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$ *defined in (15.3b) = (15.11b) which moreover is uniformly stable, as in (15.7),*

$$\left\| e^{-A_q t} \right\|_{\mathcal{L}\left(\widetilde{B}^{2-2/p}_{q,p}(\Omega)\right)} \leq M e^{-\delta t}, \quad t > 0. \tag{15.12}$$

(ii) The Oseen operator $\mathcal{A}_q$ *in (15.5) generates a s.c. analytic semigroup* $e^{\mathcal{A}_q t}$ *on the space* $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$ *in (15.3b) = (15.11b).*

(e) **Space of maximal L^p regularity on $L^q_\sigma(\Omega)$ of the Stokes operator** $-A_q$, $1 < p < \infty$, $1 < q < \infty$ **up to $T = \infty$.** We shall use the notation of [Dore] and write $-A_q \in MReg(L^p(0,\infty; L^q(\Omega)))$. We return to the dynamic Stokes problem in $\{\varphi(t,x), \pi(t,x)\}$

$$\begin{cases} \varphi_t - \Delta\varphi + \nabla\pi = F & \text{in } (0,T] \times \Omega \equiv Q & (15.13a) \\ div\ \varphi \equiv 0 & \text{in } Q & (15.13b) \\ \varphi|_\Sigma \equiv h_0 & \text{in } (0,T] \times \Gamma \equiv \Sigma & (15.13c) \\ \varphi|_{t=0} = \varphi_0 & \text{in } \Omega, & (15.13d) \end{cases}$$

rewritten in abstract form, after applying the Helmholtz projection P_q to (15.13a) and recalling A_q in (15.4) as

$$\varphi' + A_q\varphi = F_\sigma \equiv P_q F, \quad \varphi_0 \in \left(L^q_\sigma(\Omega), \mathcal{D}(A_q)\right)_{1-\frac{1}{p},p}, \tag{15.14}$$

recall (15.11). Next, we introduce the space of maximal regularity for $\{\varphi, \varphi'\}$ as [H-S, p 2; Theorem 2.8.5.iii, p 17], [G-G-H.1, p 1404-5], with T up to ∞:

$$X^T_{p,q,\sigma} = L^p(0,T;\mathcal{D}(A_q)) \cap W^{1,p}(0,T;L^q_\sigma(\Omega)) \tag{15.15}$$

(recall (14.20) for $\mathcal{D}(A_q)$) and the corresponding space for the pressure as

$$Y^T_{p,q} = L^p(0,T;\widehat{W}^{1,q}(\Omega)), \quad \widehat{W}^{1,q}(\Omega) = W^{1,q}(\Omega)/\mathbb{R}. \tag{15.16}$$

The following embedding, also called trace theorem, holds true [Amann.2, Theorem 4.10.2, p 180, BUC for $T = \infty$].

$$X^T_{p,q,\sigma} \subset X^T_{p,q} \equiv L^p(0,T;W^{2,q}(\Omega)) \cap W^{1,p}(0,T;L^q(\Omega)) \hookrightarrow C\Big([0,T]; B^{2-2/p}_{q,p}(\Omega)\Big). \tag{15.17}$$

For a function g such that $div\ g \equiv 0,\ g|_\Gamma = 0$ we have $g \in X^T_{p,q} \iff g \in X^T_{p,q,\sigma}$, by (13.4).
The solution of Eq(15.14) is

$$\varphi(t) = e^{-A_q t}\varphi_0 + \int_0^t e^{-A_q(t-\tau)} F_\sigma(\tau) d\tau. \tag{15.18}$$

The following is the celebrated result on maximal regularity on $L^q_\sigma(\Omega)$ of the Stokes problem due originally to Solonnikov [Sol.2] reported in [H-S, Theorem 2.8.5(iii) for $\varphi_0 = 0$ and Theorem 2.10.1 p24], [Saa], [G-G-H.1, Proposition 4.1 , p 1405], [P-S]. See also [B-L-T.1, Theorem 3.1 p 31 for $p = q = 2$ case]

Theorem 15.4. *Let $1 < p, q < \infty, T \leq \infty$. With reference to problem (15.13) = (15.14), assume*

$$F_\sigma \in L^p(0,T;L^q_\sigma(\Omega)),\ \varphi_0 \in \Big(L^q_\sigma(\Omega), \mathcal{D}(A_q)\Big)_{1-\frac{1}{p},p}. \tag{15.19}$$

Then there exists a unique solution $\varphi \in X^T_{p,q,\sigma}$ continuously on the data: there exist constants C_0, C_1 independent of T, F_σ, φ_0 such that via (15.17)

$$\begin{aligned} C_0 \|\varphi\|_{C\left([0,T];B^{2-2/p}_{q,p}(\Omega)\right)} &\leq \|\varphi\|_{X^T_{p,q,\sigma}} + \|\pi\|_{Y^T_{p,q}} \\ &= \|\varphi'\|_{L^p(0,T;L^q_\sigma(\Omega))} + \|A_q\varphi\|_{L^p(0,T;L^q_\sigma(\Omega))} + \|\pi\|_{Y^T_{p,q}} \\ &\leq C_1 \Big\{ \|F_\sigma\|_{L^p(0,T;L^q_\sigma(\Omega))} + \|\varphi_0\|_{\left(L^q_\sigma(\Omega),\mathcal{D}(A_q)\right)_{1-\frac{1}{p},p}} + \|h_0\|_{L^p(0,\infty;W^{1-1/q,q}(\Gamma))} \Big\}. \end{aligned} \tag{15.20}$$

In particular,

(i) *With reference to the variation of parameters formula (15.18) of problem (15.14) arising from the Stokes problem (15.13), we have recalling (15.15): the map*

$$F_\sigma \longrightarrow \int_0^t e^{-A_q(t-\tau)} F_\sigma(\tau) d\tau \quad : \textit{continuous} \tag{15.21}$$

$$L^p(0,T;L^q_\sigma(\Omega)) \longrightarrow X^T_{p,q,\sigma} \equiv L^p(0,T;\mathcal{D}(A_q)) \cap W^{1,p}(0,T;L^q_\sigma(\Omega)). \tag{15.22}$$

(ii) *The s.c. analytic semigroup $e^{-A_q t}$ generated by the Stokes operator $-A_q$ (see (15.4)) on the space $\Big(L^q_\sigma(\Omega), \mathcal{D}(A_q)\Big)_{1-\frac{1}{p},p}$ satisfies*

$$e^{-A_q t}: \textit{ continuous } \quad \Big(L^q_\sigma(\Omega), \mathcal{D}(A_q)\Big)_{1-\frac{1}{p},p} \longrightarrow X^T_{p,q,\sigma} \equiv L^p(0,T;\mathcal{D}(A_q)) \cap W^{1,p}(0,T;L^q_\sigma(\Omega)). \tag{15.23a}$$

In particular via (15.11b), for future use, for $1 < q < \infty, 1 < p < \frac{2q}{2q-1}$, the s.c. analytic semigroup $e^{-A_q t}$ on the space $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$, satisfies

$$e^{-A_q t}: \textit{ continuous } \quad \widetilde{B}^{2-2/p}_{q,p}(\Omega) \longrightarrow X^T_{p,q,\sigma}. \tag{15.23b}$$

(iii) *Moreover, setting $\nabla \pi = (Id - P_q)(\Delta + F)$, it follows that $\{\varphi, \pi\} \in X^T_{p,q,\sigma} \times Y^T_{p,q}$, see (15.16), solves problem (15.13) and there is a constant $C > 0$ independent of T, F_σ, ϕ_0 s.t.*

$$\|\varphi\|_{X^T_{p,q,\sigma}} + \|\pi\|_{Y^T_{p,q}} \le C \Big\{ \|F_\sigma\|_{L^p(0,T;L^q_\sigma(\Omega))} + \|\varphi_0\|_{\big(L^q_\sigma(\Omega), \mathcal{D}(A_q)\big)_{1-\frac{1}{p},p}} \Big\} \tag{15.24a}$$

while, for future use, for $1 < q < \infty, 1 < p < \frac{2q}{2q-1}$, then (15.24a) specializes to

$$\|\varphi\|_{X^T_{p,q,\sigma}} + \|\pi\|_{Y^T_{p,q}} \le C \Big\{ \|F_\sigma\|_{L^p(0,T;L^q_\sigma(\Omega))} + \|\varphi_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \Big\}. \tag{15.24b}$$

(f) **Maximal L^p regularity on $L^q_\sigma(\Omega)$ of the Oseen operator $\mathcal{A}_q$, $1 < p < \infty$, $1 < q < \infty$: $\mathcal{A}_q \in MReg(L^p(0,T;L^q_\sigma(\Omega)))$, T finite arbitrary.** We next transfer the maximal regularity of the Stokes operator $(-A_q)$ on $L^q_\sigma(\Omega)$-asserted in Theorem 15.4 into the maximal regularity of the Oseen operator $\mathcal{A}_q = -\nu_o A_q - A_{o,q}$ exactly on the same space $X^T_{p,q,\sigma}$ defined in (15.15), except now only on $T < \infty$.

Thus, consider the dynamic Oseen problem in $\{\psi(t,x), \pi(t,x)\}$ with equilibrium solution y_e, see

Theorem 13.1 on (13.2) :

$$\begin{cases} \psi_t - \nu_o \Delta\psi + L_e(\psi) + \nabla\pi = F & \text{in } (0,T] \times \Omega \equiv Q & (15.25a) \\ div\ \psi \equiv 0 & \text{in } Q & (15.25b) \\ \psi|_{\Sigma} \equiv 0 & \text{in } (0,T] \times \Gamma \equiv \Sigma & (15.25c) \\ \psi|_{t=0} = \psi_0 & \text{in } \Omega, & (15.25d) \end{cases}$$

$$L_e(\psi) = (y_e.\nabla)\psi + (\psi.\nabla)y_e \tag{15.26}$$

rewritten in abstract form, after applying the Helmholtz projector P_q to (15.25a) and recalling $\mathcal{A}_q$ in (15.5)

$$\psi_t = \mathcal{A}_q\psi + P_qF = -\nu_o A_q\psi - A_{o,q}\psi + F_\sigma, \quad \psi_0 \in \big(L^q_\sigma(\Omega), \mathcal{D}(A_q)\big)_{1-\frac{1}{p},p} \tag{15.27}$$

whose solution is

$$\psi(t) = e^{\mathcal{A}_q t}\psi_0 + \int_0^t e^{\mathcal{A}_q(t-\tau)}F_\sigma(\tau)d\tau. \tag{15.28}$$

$$\psi(t) = e^{-\nu_o A_q t}\psi_0 + \int_0^t e^{-\nu_o A_q(t-\tau)}F_\sigma(\tau)d\tau - \int_0^t e^{-\nu_o A_q(t-\tau)}A_{o,q}\psi(\tau)d\tau. \tag{15.29}$$

Theorem 15.5. *Let $1 < p, q < \infty$, $0 < T < \infty$. Assume (as in (15.19))*

$$F_\sigma \in L^p\big(0,T; L^q_\sigma(\Omega)\big), \quad \psi_0 \in \big(L^q_\sigma(\Omega), \mathcal{D}(A_q)\big)_{1-\frac{1}{p},p} \tag{15.30}$$

where $\mathcal{D}(A_q) = \mathcal{D}(\mathcal{A}_q)$, see (15.5). Then there exists a unique solution $\psi \in X^T_{p,q,\sigma}$ of the dynamic Oseen problem (15.25), continuously on the data: that is, there exist constants C_{0T}, C_{1T} independent of F_σ, ψ_0 such that (recall (15.17)):

$$\begin{aligned} C_{0T} \|\psi\|_{C\left([0,T]; B^{2-2/p}_{q,p}(\Omega)\right)} &\leq \|\psi\|_{X^T_{p,q,\sigma}} + \|\pi\|_{Y^T_{p,q}} \\ &\equiv \|\psi'\|_{L^p(0,T;L^q(\Omega))} + \|A_q\psi\|_{L^p(0,T;L^q(\Omega))} + \|\pi\|_{Y^T_{p,q}} \qquad (15.31) \\ &\leq C_{1T}\Big\{ \|F_\sigma\|_{L^p(0,T;L^q_\sigma(\Omega))} + \|\psi_0\|_{\left(L^q_\sigma(\Omega),\mathcal{D}(A_q)\right)_{1-\frac{1}{p},p}} \Big\} \qquad (15.32) \end{aligned}$$

Equivalently, for $1 < p, q < \infty$

i. The map

$$F_\sigma \longrightarrow \int_0^t e^{\mathcal{A}_q(t-\tau)}F_\sigma(\tau)d\tau \ \ : \textit{continuous}$$
$$L^p(0,T; L^q_\sigma(\Omega)) \longrightarrow L^p\big(0,T; \mathcal{D}(\mathcal{A}_q) = \mathcal{D}(A_q)\big) \tag{15.33}$$

where then automatically, see (15.27)

$$L^p(0,T;L^q_\sigma(\Omega)) \longrightarrow W^{1,p}(0,T;L^q_\sigma(\Omega)) \tag{15.34}$$

and ultimately

$$L^p(0,T;L^q_\sigma(\Omega)) \longrightarrow X^T_{p,q,\sigma} \equiv L^p\big(0,T;\mathcal{D}(A_q)\big) \cap W^{1,p}(0,T;L^q_\sigma(\Omega)). \tag{15.35a}$$

Thus,

$$\mathcal{A}_q \in MReg(L^p(0,T;L^q_\sigma(\Omega))),\ 1 < T < \infty \tag{15.35b}$$

and the operator $\mathcal{A}_q$ has maximal L^p regularity on $L^q_\sigma(\Omega)$.

ii. *The s.c. analytic semigroup $e^{\mathcal{A}_q t}$ generated by the Oseen operator $\mathcal{A}_q$ (see (15.5)) on the space $\big(L^q_\sigma(\Omega),\mathcal{D}(A_q)\big)_{1-\frac{1}{p},p}$ satisfies for $1 < p, q < \infty$*

$$e^{\mathcal{A}_q t}: \text{ continuous } \quad \big(L^q_\sigma(\Omega),\mathcal{D}(A_q)\big)_{1-\frac{1}{p},p} \longrightarrow L^p\big(0,T;\mathcal{D}(\mathcal{A}_q) = \mathcal{D}(A_q)\big) \tag{15.36}$$

and hence automatically

$$e^{\mathcal{A}_q t}: \text{ continuous } \quad \big(L^q_\sigma(\Omega),\mathcal{D}(A_q)\big)_{1-\frac{1}{p},p} \longrightarrow X^T_{p,q,\sigma}. \tag{15.37}$$

In particular, for future use, for $1 < q < \infty, 1 < p < \frac{2q}{2q-1}$, we have that the s.c. analytic semigroup $e^{\mathcal{A}_q t}$ on the space $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$, satisfies

$$e^{\mathcal{A}_q t}: \text{ continuous } \quad \widetilde{B}^{2-2/p}_{q,p}(\Omega) \longrightarrow L^p\big(0,T;\mathcal{D}(\mathcal{A}_q) = \mathcal{D}(A_q)\big),\ T < \infty, \tag{15.38}$$

and hence automatically

$$e^{\mathcal{A}_q t}: \text{ continuous } \quad \widetilde{B}^{2-2/p}_{q,p}(\Omega) \longrightarrow X^T_{p,q,\sigma},\ T < \infty. \tag{15.39}$$

iii. *An estimate such as the one in (15.24a) applies to the pressure π, where now $\nabla\pi(Id-P_q)(\Delta - L_e + F)$.*

A proof of Theorem 15.5 is given in Appendix A.

(g) We return to the Dirichlet map D introduced in Section 14.1, and extract an important result [to be used e.g. in Section 21 to claim that the feedback generator $\mathbb{A}_F$ generates a s.c. analytic semigroup in $L^q_\sigma(\Omega)$]. All this is a perfect counterpart of results in Hilbert spaces ($q = 2$), which have been used in [L-T.3] etc.

We first quote a known result

Proposition 15.6. *With reference to the Stokes operator A_q introduced in (15.4) on $L^q_\sigma(\Omega)$, we have, for $1 < q < \infty$.*

(i)

$$W^{s,q}(\Omega) \equiv W_0^{s,q}(\Omega), \quad 0 \le s \le \frac{1}{q} \tag{15.40a}$$

(ii)

$$W_0^{2s,q}(\Omega) \cap L^q_\sigma(\Omega) \subset \mathcal{D}\big(A_q^\gamma\big), \quad 0 \le \gamma < s,\ 0 \le s \le 1,\ q \ge 2,$$
$$2s \ne \frac{1}{q},\ 2s \ne \frac{1}{q} + 1 \tag{15.40b}$$

(iii) In particular, for $\varepsilon > 0$ arbitrary, $q \ge 2$, via (i):

$$W^{1/q,q}(\Omega) \cap L^q_\sigma(\Omega) = W_0^{1/q,q}(\Omega) \cap L^q_\sigma(\Omega) \subset \mathcal{D}\Big(A_q^{1/2q - \varepsilon}\Big) \quad \square \tag{15.40c}$$

Indeed, for (i) we invoke [Wahl, (0.2.17) p XX1]. For (ii), we quote [Wahl, Theorem III.2.3 p 91] where, in this reference, the space $H_q(\Omega)$ is our space $L^q_\sigma(\Omega)$, and the space $\mathring{H}^{2s,q}(\Omega)$ can be replaced (see proof) by the space $W_0^{2s,q}(\Omega)$ in our notation. For (iii), we apply (i) and (ii) with $2s = {}^1/_q$, hence $\gamma < {}^1/_{2q}$. $\square$

Corollary 15.7. For the Dirichlet map $D : g \longrightarrow \psi$ defined in reference to problem (14.1) and the paragraph below it, we have complementing (14.13) = (14.28)

$$g \in U_q = \big\{g \in L^q(\Gamma);\ g \cdot \nu = 0 \text{ on } \Gamma\big\} \longrightarrow Dg \in W^{1/q,q}(\Omega) \cap L^q_\sigma(\Omega) \subset \mathcal{D}\Big(A_q^{1/2q-\varepsilon}\Big) \tag{15.41a}$$

$$\text{or } A_q^{1/2q-\varepsilon} D \in \mathcal{L}(U_q, L^q_\sigma(\Omega)) \tag{15.41b}$$

We shall invoke this property in Theorem 21.1.

Remark 15.2. As noted in Remark 13.2, The literature reports physical situations where the volumetric force f in (13.1a) or (13.2a), is actually replaced by $\nabla g(x)$; that is, f is a time dependent conservative vector field. In this case, a solution to the stationary problem (13.2) is: $y_e \equiv 0, \pi_e = g$.

Taking $y_e \equiv 0$ (hence $L_e(\phi) = 0$) and returning to Eq (13.1a) with $f(x)$ replaced now by $\nabla g(x)$ and applying to the resulting equation the projection operator P_q, one obtains in this case the projected equation

$$y_t - \nu_o P_q \Delta y + P_q\big[(y \cdot \nabla)y\big] = P_q(mu) \quad \text{in } Q. \tag{15.42}$$

This, along with the solenoidal and boundary conditions (13.1b), (13.1c), yields the corresponding abstract form

$$y_t + \nu_o A_q(y - Dv) + \mathcal{N}_q y = P_q(mu) \quad \text{in } L^q_\sigma(\Omega). \tag{15.43}$$

Then y-problem (15.43) is the same as the z-problem (14.24) or (14.25), except without the Oseen term $A_{o,q}$ see (14.22). The linearized version of problem (15.43) is then

$$\eta_t + \nu_o A_q(\eta - Dv) = P_q(mu) \quad \text{in } L^q_\sigma(\Omega), \tag{15.44}$$

which is the same as the w-problem (14.26), except without the Oseen term $A_{o,q}$. The s.c. analytic semigroup $e^{-\nu_o A_q t}$ driving the linear equation (15.44) is uniformly stable in $L^q_\sigma(\Omega)$, see (15.7), as well as in $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$, see (15.12) with decay rate $-\delta$. Then, in the case of the present Remark and as anticipated in the Orientation, the present paper may be used to <u>enhance at will the uniform stability</u> of the corresponding problem by use only of the tangential boundary feedback finite dimensional control v, as acting on the entire boundary Γ. It is of the <u>form</u> given by (17.8), with boundary vectors f_k now acting on the entire boundary Γ. Thus one can take the interior tangential-like control $u \equiv 0$, or vectors $q_k \equiv 0$ in (17.9). Given the original decay rate $-\delta$ of the Stokes semigroup in (15.7) or (15.12), and preassigned a desirable decay rate $-k^2$ (arbitrary), the procedure of the present paper can be adopted to construct such a tangential boundary finite dimensional feedback control v on all of Γ that yields the decay rate $-k^2$. Thus there is no need to perform the translation $y \longrightarrow z$ of Section 13.5, when f in (13.2a) is replaced by $\nabla g(x)$; i.e. $y_e = 0$ in this case. The corresponding required "unique continuation property" holds true for the Stokes problem ($y_e = 0$), see Problem #3 of Appendix F, [RT.4], [RT.5].

16 Introducing the Problem of Feedback Stabilization of the Linearized w-Problem (14.26) on the Complexified $L^q_\sigma(\Omega)$-space.

Preliminaries: In this subsection we take q fixed, $1 < q < \infty$ throughout. Accordingly, to streamline the notation in the preceding setting of Section 14, we shall drop the dependence on q of all relevant quantities and thus write $P, A, A_o, \mathcal{A}$ instead of $P_q, A_q, A_{o,q}, \mathcal{A}_q$. We return to the linearized system (14.26).

Moreover, as in [B-T.1], [B-L-T.1], we shall henceforth let $L^q_\sigma(\Omega)$ denote the complexified space $L^q_\sigma(\Omega)+ iL^q_\sigma(\Omega)$, whereby then we consider the extension of the linearized problem (14.26) to such complexified space. Thus, henceforth, w will mean $w + i\widetilde{w}$, u will mean $u + i\widetilde{u}$, v will mean $v + i\widetilde{v}$, w_0 will mean $w_0 + i\widetilde{w}_0$. Thus, henceforth, the abstract model (14.26) is rewritten with the same symbols as

$$w_t - \mathcal{A}w = -\mathcal{A}Dv + P((mu)\tau) \in [\mathcal{D}(\mathcal{A}^*)]', \quad w(0) \in L^q_\sigma(\Omega), \quad v \cdot \nu = 0 \text{ on } \Sigma \tag{16.1}$$

to mean however the complexified version of (14.26). As noted in Theorem 3.1(iii), the Oseen operator $\mathcal{A}$ has compact resolvent on $L^q_\sigma(\Omega)$. It follows that $\mathcal{A}$ has a discrete point spectrum $\sigma(\mathcal{A}) = \sigma_p(\mathcal{A})$ consisting of isolated eigenvalues $\{\lambda_j\}_{j=1}^\infty$, which are repeated according to their (finite) algebraic multiplicity ℓ_j. However, since $\mathcal{A}$ generates a C_0 analytic semigroup on $L^q_\sigma(\Omega)$, Theorem 3.1(ii), its eigenvalues $\{\lambda_j\}_{j=1}^\infty$ lie in a triangular sector of a well-known type.

The case of interest in stabilization occurs where $\mathcal{A}$ has a finite number, say N, of eigenvalues $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_N$ on the complex half plane $\{\lambda \in \mathbb{C} : Re\ \lambda \geq 0\}$ which we then order according to their real parts, so that

$$\ldots \leq Re\ \lambda_{N+1} < 0 \leq Re\ \lambda_N \leq \ldots \leq Re\ \lambda_1, \tag{16.2}$$

each $\lambda_i,\ i = 1, \dots, N$ being an unstable eigenvalue repeated according to its geometric multiplicity ℓ_i. Let M denote the number of distinct unstable eigenvalues λ_j of $\mathcal{A}$. Denote by P_N and P_N^* the projections given explicitly by [K-1, p 178], [B-T.1], [B-L-T.1]

$$P_N = -\frac{1}{2\pi i}\int_{\Gamma} (\lambda I - \mathcal{A})^{-1}\, d\lambda : L^q_\sigma(\Omega) \text{ onto } W^u_N \subset L^q_\sigma(\Omega) \tag{16.3a}$$

$$P^*_N = -\frac{1}{2\pi i}\int_{\bar{\Gamma}} (\lambda I - \mathcal{A}^*)^{-1}\, d\lambda : (L^q_\sigma(\Omega))^* = L^{q'}_\sigma(\Omega) \text{ onto } (W^u_N)^* \subset L^{q'}_\sigma(\Omega), \tag{16.3b}$$

${}^1/_q + {}^1/_{q'} = 1$, recall (13.8b), where Γ (respectively, its conjugate counterpart $\bar{\Gamma}$) is a smooth closed curve that separates the unstable spectrum from the stable spectrum of $\mathcal{A}$ (respectively, $\mathcal{A}^*$).

As in [B-L-T.1, Sect 3.4, p 37], following [RT.1],we decompose the space $L^q_\sigma(\Omega)$ into the sum of two complementary subspaces (not necessarily orthogonal):

$$L^q_\sigma(\Omega) = W^u_N \oplus W^s_N; \quad W^u_N \equiv P_N L^q_\sigma(\Omega); \quad W^s_N \equiv (I - P_N) L^q_\sigma(\Omega); \quad \dim W^u_N = N, \tag{16.4}$$

where each of the spaces W^u_N and W^s_N is invariant under $\mathcal{A}$, and let

$$\mathcal{A}^u_N = P_N\mathcal{A} = \mathcal{A}|_{W^u_N}; \quad \mathcal{A}^s_N = (I - P_N)\mathcal{A} = \mathcal{A}|_{W^s_N} \tag{16.5}$$

be the restrictions of $\mathcal{A}$ to W^u_N and W^s_N respectively. The original point spectrum (eigenvalues) $\{\lambda_j\}_{j=1}^\infty$ of $\mathcal{A}$ is then split into two sets

$$\sigma(\mathcal{A}^u_N) = \{\lambda_j\}_{j=1}^N; \quad \sigma(\mathcal{A}^s_N) = \{\lambda_j\}_{j=N+1}^\infty, \tag{16.6}$$

and W^u_N is the generalized eigenspace of ($\mathcal{A}$, hence of) $\mathcal{A}^u_N$, corresponding to its unstable eigenvalues. The system (16.1) on $L^q_\sigma(\Omega)$ with $v \cdot \nu = 0$ on Σ can accordingly be decomposed as

$$w = w_N + \zeta_N, \quad w_N = P_N w, \quad \zeta_N = (I - P_N)w. \tag{16.7}$$

After applying P_N and $(I - P_N)$ (which commute with $\mathcal{A}$) to (16.1), we obtain via (16.5)

$$\text{on } W^u_N : w'_N - \mathcal{A}^u_N w_N = -P_N(\mathcal{A}Dv) + P_N P((mu)\tau) \tag{16.8a}$$

$$= -\mathcal{A}^u_N P_N Dv + P_N P((mu)\tau); \; w_N(0) = P_N w_0 \tag{16.8b}$$

$$\text{on } W_N^s : \zeta_N' - \mathcal{A}_N^s \zeta_N = -(I - P_N)(\mathcal{A} Dv) + (I - P_N)P((mu)\tau) \tag{16.9a}$$

$$= -\mathcal{A}_N^s (I - P_N) Dv + (I - P_N)P((mu)\tau); \ \zeta_N(0) = (I - P_N) w_0 \tag{16.9b}$$

respectively. [In (16.8a), (16.9a), actually P_N is the extension from original $L^q_\sigma(\Omega)$ to $[\mathcal{D}(\mathcal{A}^*)]'$ [B-L-T.1, Appendix A.1]]. For each distinct $\lambda_i, i = 1, \dots, M$, let $P_{N,i}, P_{N,i}^*$ be the projection corresponding to λ_i and $\bar{\lambda}_i$, respectively, given by a similar integral of $(\lambda I - \mathcal{A})^{-1}$, or $(\lambda I - \mathcal{A}^*)^{-1}$, respectively, as in (16.3a-16.3b), this time over a curve that encircles only λ_i, or $\bar{\lambda}_i$, respectively, and no other eigenvalue. Let $(W_N^u)_i = P_{N,i} L^q_\sigma(\Omega)$, and $(A_N^u)_i = \mathcal{A}^u\big|_{(W_N^u)_i}$.

We anticipate the fact (noted below (18.3) and in the first paragraph of Section 19) that, for $1 < p, q < \infty$:

$$W_N^u = \left\{ \begin{array}{c} \text{space of generalized} \\ \text{eigenfunctions of } \mathcal{A}_q (= \mathcal{A}_N^u) \\ \text{corresponding to its distinct} \\ \text{unstable eigenvalues} \end{array} \right\} \subset \left\{ \begin{array}{l} \left(L^q_\sigma(\Omega), \mathcal{D}(A_q) \right)_{1-\frac{1}{p}, p} \\ \left[\mathcal{D}(A_q), L^q_\sigma(\Omega) \right]_{1-\alpha} = \mathcal{D}(A_q^\alpha), \ 0 < \alpha < 1 \end{array} \right\} \subset L^q_\sigma(\Omega). \tag{16.10}$$

17 Main results

17.1 Orientation

Having introduced the necessary background in the L^q-setting, we can finally list the main uniform stabilization results by a pair $\{v, u\}$ of feedback tangential controllers, both finite dimensional: with v acting tangentially on an arbitrary small connected portion $\widetilde{\Gamma}$ of the boundary Γ, and u acting tangentially on a patch ω supported by $\widetilde{\Gamma}$. We do this first, globally, for the linear case. More precisely, we do this first for the uniform feedback stabilization of the finite dimensional w_N-system in (16.8) in the space $W_N^u \subset L^q_\sigma(\Omega)$, for which we obtain an arbitrarily large decay rate in Theorem 17.1; then for the linearized w-problem in (14.26) = (16.1), in both the $L^q_\sigma(\Omega)$-setting as well as the Besov space $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$ in Theorem 17.2. These linear global uniform stabilization results are next employed to obtain the sought-after goal: local uniform stabilization, near an unstable equilibrium solution y_e, by a finite dimensional pair $\{v, u\}$ of feedback tangential controllers, also for $d = 3$, first for the z-dynamics (14.25) in Theorem 17.4; finally, for the original N-S dynamics (13.1) in Theorem 17.5. The linear/linearized Theorems 17.1 and 17.2 will require $q \geq 2$, to guarantee that the boundary vectors $\dfrac{\partial \varphi^*_{ij}}{\partial \nu} \in W^{2-1/q,q}(\Gamma) \subset L^q(\Gamma)$, see Appendix E, in particular Eq (C.5). In addition, in the case of interest $d = 3$, the non-linear Theorem 17.3, 17.4, 17.5 will require $q > 3, 1 < p < {}^{2q}/_{2q-1}$ by (24.6).

All the main results of this paper, Theorems 17.1 through 17.5, are stated (at first) in the complex state space setting $L^q_\sigma(\Omega) + iL^q_\sigma(\Omega)$. Thus, the finitely many stabilizing feedback vectors $p_k \in (W_N^u)^* \subset L^{q'}_\sigma(\Omega)$, $u_k \in W_N^u \subset L^q_\sigma(\Omega)$ constructed in the subsequent proofs are related to the complex finite dimensional unstable subspace W_N^u. The question then arises as to transfer back these results into the original real setting. This issue was resolved in [B-T.1]. Here, the translation, taken from [B-T.1], from the results in the complex setting (Theorems 17.1 through 17.5) into corresponding results in the original real setting is given in Section 17.6.

17.2 Arbitrary decay rate of the finite dimensional w_N-dynamics (16.8) by suitable finite-dimensional boundary feedback tangential localized control v and interior localized tangential-like feedback control u. Constructive proof with $q \geq 2$.

The following is the key desired control theoretic result of the dynamic w_N in (16.8) over the finite dimensional space $W_N^u \subset L^q_\sigma(\Omega)$. We shall henceforth impose the condition $q \geq 2$, due to requirement (C.5) in Appendix E.

Theorem 17.1. *Let $\lambda_1, \dots, \lambda_M$ be the unstable distinct eigenvalues of the Oseen operator $\mathcal{A}$ $(= \mathcal{A}_q)$ as in (16.2), with geometric multiplicity ℓ_i, $i = 1, \dots, M$, and set $K = \sup\{\ell_i; i = 1, \dots, M\}$. Let $\widetilde{\Gamma}$ be an open connected subset of the boundary Γ of positive surface measure and ω be a localized collar supported by $\widetilde{\Gamma}$ (Fig. 2). Let $q \geq 2$. Given $\gamma_1 > 0$ arbitrarily large, we can construct two K-dimensional controllers: a boundary tangential control $v = v_N$ acting with support on $\widetilde{\Gamma}$, of the form given by*

$$v = v_N = \sum_{k=1}^{K} \nu_k(t) f_k, \;\; f_k \in \mathcal{F} \subset W^{2-\frac{1}{q},q}(\Gamma), \;\; q \geq 2, \;\; \text{so that } f_k \cdot \nu = 0, \;\; \text{hence } v_N \cdot \nu = 0 \text{ on } \Gamma \quad (17.1)$$

$\mathcal{F}$ defined in (13.23), $q \geq 2$, f_k supported on $\widetilde{\Gamma}$, and an interior tangential-like control $u = u_N$ acting on ω, of the form given by

$$u = u_N = \sum_{k=1}^{K} \mu_k(t) u_k, \quad u_k \in W_N^u \subset L^q_\sigma(\Omega), \quad \mu_k(t) = \text{scalar}, \tag{17.2}$$

thus with interior vectors $[u_1, \dots, u_K]$ in the smooth subspace W_N^u of $L^q_\sigma(\Omega)$, $2 \leq q < \infty$, supported on ω, such that, once inserted in the finite dimensional projected w_N-system in (16.8), yields the system

$$w_N' - \mathcal{A}_N^u w_N = -\mathcal{A}_N^u P_N D\left(\sum_{k=1}^{K} \nu_k(t) f_k\right) + P_N P\left(m\left(\sum_{k=1}^{K} \mu_k(t) u_k\right)\tau\right), \tag{17.3}$$

whose solution then satisfies the estimate

$$\|w_N(t)\|_{L^q_\sigma(\Omega)} + \|v_N(t)\|_{L^q(\widetilde{\Gamma})} + \|v_N'(t)\|_{L^q(\widetilde{\Gamma})} + \\ \|u_N(t)\|_{L^q_\sigma(\omega)} + \|u_N'(t)\|_{L^q_\sigma(\omega)} \leq C_{\gamma_1} e^{-\gamma_1 t} \|P_N w_0\|_{L^q_\sigma(\Omega)}, \quad t \geq 0. \tag{17.4}$$

In (17.4) we may replace the $L^q_\sigma(\Omega)$-norm, $2 \leq q < \infty$, alternatively either with the $\left(L^q_\sigma(\Omega), \mathcal{D}(A_q)\right)_{1-\frac{1}{p},p}$ norm, $2 \leq q < \infty$; or else with the $\left[\mathcal{D}(A_q), L^q_\sigma(\Omega)\right]_{1-\alpha} = \mathcal{D}(A_q^\alpha)$-norm, $0 \leq \alpha \leq 1$, $2 \leq q < \infty$. In *particular, we also have*

$$\|w_N(t)\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} + \|v_N(t)\|_{L^q(\widetilde{\Gamma})} + \left\|v'_N(t)\right\|_{L^q(\widetilde{\Gamma})} + \|u_N(t)\|_{\widetilde{B}^{2-2/p}_{q,p}(\omega)} + \left\|u'_N(t)\right\|_{\widetilde{B}^{2-2/p}_{q,p}(\omega)} \leq C_{\gamma_1} e^{-\gamma_1 t} \|P_N w_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)}, \quad t \geq 0, \quad (17.5)$$

in the $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$-norm, $2 \leq q < \infty$, $p < 2q/_{2q-1}$.[Estimate (17.4) (in the weaker form (8.1) = (10.3), i.e. without the derivative terms) will be invoked in the nonlinear stabilization proof of Section 22, see (22.11)].

Moreover, such controllers $v = v_N$ and $u = u_N$ may be chosen in feedback form: that is, with references to the explicit expressions (17.1) for v and (17.2) for u, of the form $\nu_k(t) = \left\langle w_N(t), p_k \right\rangle_{W^u_N}$ and $\mu_k(t) = \left\langle w_N(t), q_k \right\rangle_{W^u_N}$ for suitable vectors $p_k \in (W^u_N)^* \subset L^{q'}_\sigma(\Omega)$, $q_k \in (W^u_N)^* \subset L^{q'}_\sigma(\Omega)$ depending on γ_1, where $\langle\ ,\ \rangle$ denotes the duality pairing $W^u_N \times (W^u_N)^*$.

In conclusion, w_N in (17.5) is the solution of the equation (17.3) on W^u_N rewritten explicitly as

$$w'_N - \mathcal{A}^u_N w_N = -\mathcal{A}^u_N P_N D\left(\sum_{k=1}^{K} \left\langle w_N(t), p_k \right\rangle_{W^u_N} f_k\right) + P_N P\left(m\left(\sum_{i=1}^{K} \left\langle w_N(t), q_k \right\rangle_{W^u_N} u_k\right)\tau\right), \quad (17.6)$$

f_k supported on $\widetilde{\Gamma}$, u_k supported on ω, rewritten in turn as

$$w'_N = \overline{A}^u w_N,\ w_N(t) = e^{\overline{A}^u t} P_N w_0,\ w_N(0) = P_N w_0 \quad \text{on } W^u_N. \quad (17.7)$$

The technical proof will be given in Section 20

17.3 Global well-posedness and Uniform Exponential Stabilization of the Linearized w-problem (13.11) or (14.26)=(16.1) in various $L^q_\sigma(\Omega)$-based spaces, $q \geq 2$, by means of the same feedback controls $\{v, u\}$ obtained for the w_N-problem in Section 17.2

We let again, $2 \leq q < \infty$ throughout this section.

Theorem 17.2. *With reference to the unstable, possibly repeated, eigenvalues $\{\lambda_j\}_{j=1}^N$, as in (16.2), M of which are distinct, let $\varepsilon > 0$ and set $\gamma_0 = |Re\ \lambda_{N+1}| - \varepsilon$. Let $q \geq 2$. Consider the same*

K-dimensional feedback controllers constructed in Theorem 17.1 and yielding estimate (17.4), (17.5) for the finite-dimensional projected w_N-system (16.8) in feedback form (17.6); that is, the tangential boundary controller $v = v_N$ supported on $\widetilde{\Gamma}$, and the tangential-like interior controller $u = u_N$ supported on ω

$$v = v_N = \sum_{k=1}^{K} \nu_k(t) f_k = \sum_{k=1}^{K} \big\langle w_N(t), p_k \big\rangle_{W_N^u} f_k, \quad f_k \in \mathcal{F} \subset W^{2-1/q,q}(\Gamma),\ p_k \in (W_N^u)^* \subset L^{q'}_\sigma(\Omega),\ q \geq 2$$
$$f_k \cdot \nu|_\Gamma = 0;\ \textit{hence}\ v \cdot \nu|_\Gamma = 0,\ f_k\ \textit{supported on}\ \widetilde{\Gamma} \tag{17.8}$$

$$u = u_N = \sum_{k=1}^{K} \mu_k(t) u_k = \sum_{k=1}^{K} \big\langle w_N(t), q_k \big\rangle_{W_N^u} u_k, \quad q_k \in (W_N^u)^* \subset L^{q'}_\sigma(\Omega),\ u_k\ \textit{supported on}\ \omega. \tag{17.9}$$

(a) *(Well-posedness) Once inserted, this time, in the full linear w-problem (13.11) or (14.26) = (16.1), such v and u in (17.8), (17.9) yield the linearized feedback dynamics ($w_N = P_N w$) driven by the dynamical feedback stabilizing operator $\mathbb{A}_{_{F,q}}$ below*

$$\frac{dw}{dt} = \mathcal{A}_q w - \mathcal{A}_q D\left(\sum_{k=1}^{K} \big\langle P_N w, p_k \big\rangle_{W_N^u} f_k \right) + P_q\left(m\left(\sum_{k=1}^{K} \big\langle P_N w, q_k \big\rangle_{W_N^u} u_k \right)\tau \right) \equiv \mathbb{A}_{_{F,q}} w, \tag{17.10}$$

where $\mathbb{A}_{_{F,q}}$ is the generator of a s.c. analytic semigroup in the space $L^q_\sigma(\Omega)$. More specifically $\mathbb{A}_{_{F,q}}$ is rewritten as in the subsequent Section 21, Eqts (21.1), (21.11), (21.12) as

$$\mathbb{A}_{_{F,q}} = A_{_{F,q}} + G : L^q_\sigma(\Omega) \supset \mathcal{D}\big(\mathbb{A}_{_{F,q}}\big) \longrightarrow L^q_\sigma(\Omega),\ q \geq 2 \tag{17.11}$$

$$\begin{cases} A_{_{F,q}} = \mathcal{A}_q(I - DF)\ :\ L^q_\sigma(\Omega) \supset \mathcal{D}(A_{_{F,q}}) \longrightarrow L^q_\sigma(\Omega),\ q \geq 2 & \text{(17.12a)} \\ \mathcal{D}(A_{_{F,q}}) = \big\{ h \in L^q_\sigma(\Omega) : h - DFh \in \mathcal{D}(\mathcal{A}_q) = W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega) \cap L^q_\sigma(\Omega) \big\} & \text{(17.12b)} \end{cases}$$

$$F(\cdot) = \sum_{k=1}^{K} \big\langle P_N\ \cdot, p_k \big\rangle_{W_N^u} f_k \in W^{2-1/q,q}(\widetilde{\Gamma});$$
$$G(\cdot) = P_q\left(m\left(\sum_{k=1}^{K} \big\langle P_N\ \cdot, q_k \big\rangle_{W_N^u} u_k \right)\tau \right) \in L^q_\sigma(\Omega),\ q \geq 2 \tag{17.13a}$$
$$F \in \mathcal{L}(L^q_\sigma(\Omega), L^q(\widetilde{\Gamma})); \quad G \in \mathcal{L}(L^q_\sigma(\Omega)). \tag{17.13b}$$

(b) *(Uniform stabilization) Moreover, such dynamics w in (17.10) (equivalently, such generator $\mathbb{A}_{F,q}$ in (17.10)) is uniformly stable in the space $L^q_\sigma(\Omega)$ with decay rate $\gamma_0 > 0$: there exists $C_{\gamma_0} > 0$ such that*

$$\left\| e^{\mathbb{A}_F t} w_0 \right\|_{L^q_\sigma(\Omega)} = \|w(t; w_0)\|_{L^q_\sigma(\Omega)} \leq C_{\gamma_0} e^{-\gamma_0 t} \|w_0\|_{L^q_\sigma(\Omega)}, \quad t \geq 0, \ q \geq 2 \tag{17.14}$$

or for $0 < \theta < 1, \delta > 0$ arbitrarily small, $q \geq 2$

$$\left\| A_q^\theta \, e^{\mathbb{A}_F t} w_0 \right\|_{L^q_\sigma(\Omega)} = \left\| A_q^\theta \, w(t; w_0) \right\|_{L^q_\sigma(\Omega)} \leq \begin{cases} C_{\gamma_0,\theta} e^{-\gamma_0 t} \left\| A_q^\theta \ w_0 \right\|_{L^q_\sigma(\Omega)}, t \geq 0, & (17.15a) \\ \qquad\qquad w_0 \in \mathcal{D}(A_q^\theta). & \\ C_{\gamma_0,\theta,\delta} e^{-\gamma_0 t} \|w_0\|_{L^q_\sigma(\Omega)}, \quad t \geq \delta > 0. & (17.15b) \end{cases}$$

As in the case of Theorem 17.1, we may replace the $L^q_\sigma(\Omega)$-norm in (17.14), $2 \leq q < \infty$, with the $\left(L^q_\sigma(\Omega), \mathcal{D}(A_q)\right)_{1-\frac{1}{p},p}$-norm, $1 < p < \infty$; in particular, with the $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$-norm and obtain

$$\left\| e^{\mathbb{A}_F t} w_0 \right\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} = \|w(t; w_0)\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \leq C_{\gamma_0} e^{-\gamma_0 t} \|w_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)}, \ t \geq 0$$
$$2 \leq q < \infty, \ 1 < p < \frac{2q}{2q-1} \tag{17.16}$$

The proof of well-posedness will be given in Section 21, while the proof of uniform stabilization will be given in Section 22.

17.4 Local well-posedness and uniform (exponential) null-stabilization of the translated nonlinear z-problem (13.10) or (14.25) by means of a finite dimensional explicit, spectral based tangential feedback control pair $\{v, u\}$ localized on $\widetilde{\Gamma}$ and ω. Now $q > 3$ for $d = 3$.

Starting with the present section, the nonlinearity of problem (13.1) will impose for $d = 3$ the requirement $q > 3$, while $q > 2$ for $d = 2$, see (24.16) below. As our deliberate goal is to obtain the stabilization result in the space $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$ which does not recognize boundary conditions, then the limitation $p < 2q/_{2q-1}$, of this space applies. In conclusion, our well-posedness and stabilization results will hold under the restriction $q > 3, 1 < p < 6/5$ for $d = 3$; $q > 2, 1 < p < 4/3$ for $d = 2$. As throughout this paper, $\widetilde{\Gamma}$ is an open connected subset of the boundary Γ of positive surface measure and ω in a localized collar.

Theorem 17.3. *(Well-posedness) Let $d = 3; 1 < p < \frac{6}{5}$ and $q > 3$. Consider the nonlinear z-problem (13.10) or (14.25) in the following feedback form in the notation of Theorem 17.2:*

$$\frac{dz}{dt} - \mathcal{A}_q\left[z - D\left(\sum_{k=1}^{K} \langle P_N z, p_k \rangle_{W_N^u} f_k\right)\right] + \mathcal{N}_q z = P_q\left(m\left(\sum_{k=1}^{K} \langle P_N z, q_k \rangle_{W_N^u} u_k\right)\tau\right); \; z_0 = z(0) \quad (17.17)$$

i.e. subject to a feedback controls of the same structure as in the linear w-dynamics (17.10) of Theorem 17.2, Here p_k, q_k, f_k, u_k are the same vectors as those constructed in Theorem 17.1, and appearing in (17.6), (17.8)-(17.10); f_k supported on $\widetilde{\Gamma}$, u_k supported on ω. There exists a positive constant $\rho > 0$ such that, if the initial condition z_0 satisfies

$$\|z_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} < \rho, \quad (17.18)$$

then problem (17.17) defines a unique solution z in the space (see (15.15), (15.17))

$$z \in X^{\infty}_{p,q,\sigma} \equiv L^p(0,\infty; \mathcal{D}(A_q)) \cap W^{1,p}(0,\infty; L^q_\sigma(\Omega)) \quad (17.19)$$

$$\hookrightarrow C([0,\infty); \widetilde{B}^{2-2/p}_{q,p}(\Omega)) \quad (17.20)$$

where $\mathcal{D}(A_q) = W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega) \cap L^q_\sigma(\Omega)$, see (15.4).

Theorem 17.4. *(Uniform Stabilization) In the situation of Theorem 17.3, $d = 3, 1 < p < \frac{6}{5}, q > 3$, we have that such solution is uniformly stable on the space $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$: there exist constants $\widetilde{\gamma} > 0, M_{\widetilde{\gamma}} \geq 1$, such that said solution satisfies*

$$\|z(t; z_0)\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \leq M_{\widetilde{\gamma}} e^{-\widetilde{\gamma} t} \|z_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)}. \quad (17.21)$$

A proof of Theorem 17.3 will be given in Section 24: it will be critically based on the maximal regularity property of Section 23. A proof of Theorem 17.4 will be given in Section 25. Remark 25.1 will provide insight on the relationship between $\widetilde{\gamma}$ in the nonlinear case in (17.21) and γ_0 in the corresponding linear case in (17.14).

17.5 Local well-posedness and uniform (exponential) stabilization of the original nonlinear y-problem (13.1) in a neighborhood of an unstable equilibrium solution y_e, by means of a finite dimensional explicit, spectral based, tangential feedback control pair $\{v, u\}$ localized on $\widetilde{\Gamma}$ and ω. Now $q > 3$ for $d = 3$.

The results of this subsection are an immediate corollary of Theorems 17.3 and 17.4 via the change of variable in (13.9). They are listed as Theorem A and Theorem B in Section 13.7.

Theorem 17.5. *Let $d = 3; 1 < p < \frac{6}{5}, q > 3$, consider the original N-S problem (13.1). Let y_e be a given unstable equilibrium solution as guaranteed by Theorem 13.1 for the steady state problem (13.2): i.e. assume (16.2). For a constant $\rho > 0$, let the initial condition y_0 in (13.1d) be in $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$ and satisfy*

$$\mathcal{V}_\rho \equiv \left\{ y_0 \in \widetilde{B}^{2-2/p}_{q,p}(\Omega) : \|y_0 - y_e\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \le \rho \right\}, \quad \rho > 0. \tag{17.22}$$

If $\rho > 0$ is sufficiently small, then

(i) for each $y_0 \in \mathcal{V}_\rho$, there exist bounded finite-dimensional operators $F \in \mathcal{L}\big(L^q_\sigma(\Omega), L^q(\widetilde{\Gamma})\big)$ and $\widetilde{G} \in \mathcal{L}\big(L^q_\sigma(\Omega)\big)$ have the following form with p_k, q_k, f_k, u_k defined in, say, (17.8); f_k supported on $\widetilde{\Gamma}$ and u_k supported on ω:

$$F(y - y_e) = \sum_{k=1}^{K} \big\langle P_N(y - y_e), p_k \big\rangle_{W^u_N} f_k, \text{ supported on } \widetilde{\Gamma}, \tag{17.23}$$

$$\widetilde{G}(y - y_e) = \sum_{k=1}^{K} \big\langle P_N(y - y_e), q_k \big\rangle_{W^u_N} u_k, \text{ supported on } \omega, \tag{17.24}$$

that is, of the same structure as in the translated N-S z-problem (17.17), such that the closed loop problem corresponding to (13.1)

$$\begin{cases} y_t - \nu_o \Delta y + (y \cdot \nabla) y + \nabla \pi = (m(\widetilde{G}(y - y_e))\tau) + f(x) & \text{in } Q \quad (17.25a) \\ \text{div } y = 0 & \text{in } Q \quad (17.25b) \\ y = F(y - y_e) & \text{on } \Sigma \quad (17.25c) \\ y|_{t=0} = y_0 & \text{in } \Omega \quad (17.25d) \end{cases}$$

rewritten abstractly after application of the Helmholtz projection P_q as

$$\frac{dy}{dt} - \mathcal{A}_q\big[y - DF(y - y_e)\big] + \mathcal{N}_q y = P_q\Big[(m(\widetilde{G}(y - y_e))\tau)\Big] + f(x) \tag{17.26a}$$

explicitly

$$\frac{dy}{dt} - \mathcal{A}_q\left[y - D\left(\sum_{k=1}^{K} \big\langle P_N(y - y_e), p_k \big\rangle_{W^u_N} f_k\right)\right] + \mathcal{N}_q y = P_q\left[m\left(\sum_{k=1}^{K} \big\langle P_N(y - y_e), q_k \big\rangle_{W^u_N} u_k\right)\tau\right] + f(x) \tag{17.26b}$$

$$y(0) = y_0 \in \widetilde{B}^{2-2/p}_{q,p}(\Omega) \tag{17.26c}$$

has a unique (nonlinear semigroup) solution $y \in C\big([0, \infty); \widetilde{B}^{2-2/p}_{q,p}(\Omega)\big)$.

(ii) Moreover, such solution exponentially stabilizes the equilibrium solution y_e in the space $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$: there exist constants $\widetilde{\gamma} > 0$ and $M_{\widetilde{\gamma}} \geq 1$. such that said solution satisfies

$$\|y(t) - y_e\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \leq M_{\widetilde{\gamma}} e^{-\widetilde{\gamma}t} \|y_0 - y_e\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)}, \quad t \geq 0, \; y_0 \in \mathcal{V}_\rho. \tag{17.27}$$

Once the neighborhood $\mathcal{V}_\rho$ is obtained to ensure the well-posedness, then the values of $M_{\widetilde{\gamma}}$ and $\widetilde{\gamma}$ do not depend on $\mathcal{V}_\rho$ and $\widetilde{\gamma}$ can be made arbitrarily large through a suitable selection of the feedback operator F.

17.6 Results on the real space setting

Here we shall complement the results of Theorems 17.1 through 17.5 by giving their version in the real space setting. We shall quote from [B-T.1]. In the complexified setting $L^q_\sigma(\Omega) + iL^q_\sigma(\Omega)$ we have that the complex unstable subspace W^u_N is,

$$\begin{aligned} W^u_N &= W^1_N + iW^2_N & (17.28) \\ &= \text{space of generalized eigenfunctions } \{\phi_j\}_{j=1}^N \text{ of the operator } \mathcal{A}_q (= \mathcal{A}^u_q) \\ &\quad \text{corresponding to its } N \text{ unstable eigenvalues.} & (17.29) \end{aligned}$$

Set $\phi_j = \phi^1_j + i\phi^2_j$ with ϕ^1_j, ϕ^2_j real. Then:

$$W^1_N = \text{Re } W^u_N = \text{span}\{\phi^1_j\}_{j=1}^N; \quad W^2_N = \text{Im } W^u_N = \text{span}\{\phi^2_j\}_{j=1}^N. \tag{17.30}$$

The stabilizing vectors $p_k, u_k, k = 1, \dots, K$ are complex valued and belong to W^u_N.

The complex-valued uniformly stable linear w-system in (14.26) with K complex valued stabilizing vectors admits the following real-valued uniformly stable counterpart

$$\begin{aligned} \frac{dw}{dt} = \mathcal{A}_q w - \mathcal{A}_q D \Bigg(\sum_{k=1}^K \text{Re}\big(w_N(t), p_k\big)_{W^u_N} \text{Re} f_k - \sum_{k=1}^K \text{Im}\big(w_N(t), p_k\big)_{W^u_N} \text{Im} f_k \Bigg) \\ + P_q \Bigg(m \Bigg(\sum_{k=1}^K \text{Re}(w_N(t), q_k)_{W^u_N} \text{ Re } u_k - \sum_{k=1}^K \text{Im } (w_N(t), q_k)_{W^u_N} \text{ Im } u_k \Bigg) \cdot \tau \Bigg) \end{aligned} \tag{17.31}$$

with $2K \leq N$ real stabilizing vectors, see [B-T.1, Eq 3.52a, p 1472].If $K = \sup \{\ell_i, i = 1, \dots, M\}$ is achieved for a real eigenvalue λ_i (respectively, a complex eigenvalue λ_i), then the *effective* number of stabilizing controllers is $K \leq N$, as the generalized functions are then real, since y_e is real; respectively, $2K \leq N$, for, in this case, the complex conjugate eigenvalue $\bar{\lambda}_j$ contributes an equal number of components in terms of generalized eigenfunctions $\phi_{\bar{\lambda}_j} = \bar{\phi}_{\lambda_j}$. In all cases, the actual (*effective*) upper bound $2K$ is $2K \leq N$. For instance, if all unstable eigenvalues were real and simple then $K = 1$, and only one stabilizing controller is actually needed.

Similarly, the complex-valued locally (near y_e) uniformly stable nonlinear y-system (14.28) with K complex-valued stabilizing vectors admits the following real-valued locally uniformly stable counterpart

$$\begin{aligned}\frac{dy}{dt} - \nu A_q y + \mathcal{N}_q y = -\mathcal{A}_q D\Bigg(\sum_{k=1}^{K} \ \mathrm{Re}\big(w_N(t), p_k\big)_{W_N^u} \ \mathrm{Re} f_k - \sum_{k=1}^{K} \ \mathrm{Im}\big(w_N(t), p_k\big)_{W_N^u} \ \mathrm{Im} f_k \Bigg) \\ + P_q\Bigg(m\Bigg(\sum_{k=1}^{K} \mathrm{Re}\ (y - y_e, p_k)_\omega \ \mathrm{Re}\ u_k - \sum_{k=1}^{K} \mathrm{Im}\ (y - y_e, p_k)_\omega \ \mathrm{Im}\ u_k \Bigg) \cdot \tau \Bigg) \qquad (17.32)\end{aligned}$$

with $2K \leq N$ real stabilizing vectors, see [B-L-T.1, p 43].

18 First step in the proof of Theorem 17.1 for the w_N-system in (16.8): verification of the controllability algebraic rank conditions under the Finite-Dimensional Spectral Assumption (FDSA) [B-L-T.1, Section 3.6], based on the unique continuation property of Lemma 18.2

Orientation The first challenging step in the proof of Theorem 5.1 consists in showing that the N-dimensional w_N-problem (16.8) is controllable in W_N^u by using a finite dimensional pair $\{v, u\}$ of localized tangential controllers, in particular in feedback form. Verification of the corresponding algebraic rank conditions of Kalman and Hautus style runs into a peculiar unique continuation property for the Oseen eigenvalue problem related to the unstable eigenvalues. Without the injection of the internal, localized (on ω), finite-dimensional, tangential-like control u, the resulting unique continuation property which is required if only v were to be employed on the subportion $\widetilde{\Gamma}$ of Γ is false. This was already explained in the Orientation of Section 13, with the technical help of Appendix F. In short and repeating, due to reference [F-L], the unique continuation property of Lemma 18.2 is false, if one omits the interior condition $\varphi \cdot \tau \equiv 0$ on ω, in (18.36c). See problem #1 in Appendix F: over-determination $\varphi|_{\widetilde{\Gamma}} = 0$ and $\partial_\nu \varphi|_{\widetilde{\Gamma}} \equiv 0$ only in the portion $\widetilde{\Gamma}$ is not enough. On the other hand, for a general unstable equilibrium solution y_e, the unique continuation property with over-determination $\varphi|_\Gamma \equiv 0$, $\partial_\nu \varphi|_\Gamma \equiv 0$ on the whole boundary is unknown at present. See Appendix F. This leaves open if one could only employ the tangential boundary control v as applied to all of Γ (with $u \equiv 0$), in the case of a general unstable equilibrium solution y_e. The modified version as in Lemma 18.2 from [L-T.2] of the unique continuation problem of the Oseen eigenfunction problem that results from the extra condition $\varphi \cdot \tau \equiv 0$ on ω imposed by such internal, tangential-like control u has however a positive answer. The related algebraic rank conditions for controllability can then be established in full generality for the Oseen operator restriction $\mathcal{A}_N^u$ on W_N^u, in a constructive way, in terms only of the eigenvectors $\{\varphi_{ij}^*\}_{i=1,j=1}^{M\ \ \ell_i}$ of the adjoint operator $(\mathcal{A}_N^u)^*$ on $(W_N^u)^* \subset (L_\sigma^q(\Omega))' = L_\sigma^{q'}(\Omega)$ by (13.8c). This is done in Section 19. In the present Section 18, we provide a preliminary analysis of a special - generically true case - characterized by the Finite Dimensional Spectral Assumption below.

For $i = 1, \dots, M$, we now denote by $\{\varphi_{ij}\}_{j=1}^{\ell_i}$, $\{\varphi^*_{ij}\}_{j=1}^{\ell_i}$ the (normalized) linearly independent (on $L^q_\sigma(\Omega)$) eigenfunctions corresponding to the (possibly unstable) distinct eigenvalues $\lambda_1, \dots, \lambda_M$ of $\mathcal{A}$ $(= \mathcal{A}_q)$ and $\overline{\lambda}_1, \dots \overline{\lambda}_M$ of $\mathcal{A}^*$ $(= \mathcal{A}^*_q)$, respectively:

$$\begin{aligned} \mathcal{A}_q \varphi_{ij} &= \lambda_i \varphi_{ij} \in \mathcal{D}(\mathcal{A}_q) = W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega) \cap L^q_\sigma(\Omega) \subset L^q_\sigma(\Omega), \\ \mathcal{A}^* \varphi^*_{ij} &= \overline{\lambda}_i \varphi^*_{ij} \in \mathcal{D}(\mathcal{A}^*_q) = W^{2,q'}(\Omega) \cap W^{1,q'}_0(\Omega) \cap L^{q'}_\sigma(\Omega) \subset L^{q'}_\sigma(\Omega). \end{aligned} \tag{18.1a}$$

The eigenvectors φ_{ij} and φ^*_{ij} are in $\mathcal{D}(\mathcal{A}^n_q)$ and $\mathcal{D}((\mathcal{A}^*_q)^n)$, for any n, hence they are arbitrarily smooth in $L^q_\sigma(\Omega)$ and $L^{q'}_\sigma(\Omega)$, respectively. For our purposes, it will suffice to take $q \geq 2$, hence $q' \leq 2$, ${}^1/_q + {}^1/_{q'} = 1$, and view eigenvectors henceforth as follows, see Appendix E, Eq (C.5)

$$\varphi_{ij} \in W^{3,q}(\Omega) \cap L^q_\sigma(\Omega); \quad \varphi^*_{ij} \in W^{3,q}(\Omega) \cap L^q_\sigma(\Omega). \tag{18.1b}$$

Hence, for $i = 1, \dots, M$, we may view φ_{ij} and φ^*_{ij} as elements of the generalized eigenspace W^u_N in (16.4) and its dual $(W^u_N)^*$, corresponding to the unstable eigenvalues as in (16.2). For $h_1 \in W^u_N, h_2 \in (W^u_N)^*$, we set $\left\langle h_1, h_2 \right\rangle_{W^u_N} = \displaystyle\int_\Omega h_1 h_2 \, d\Omega$, as a duality pairing.

FDSA: We henceforth assume in this section that for each of the distinct eigenvalues $\lambda_1, \dots, \lambda_M$ of the Oseen operator $(\mathcal{A}_q =)$ $\mathcal{A}$, algebraic and geometric multiplicity coincide:

$$(W^u_N)_i \equiv P_{N,i} L^q_\sigma(\Omega) = \operatorname{span}\{\varphi_{ij}\}_{j=1}^{\ell_i}; \quad (W^u_N)^*_i = P^*_{N,i} L^{q'}_\sigma(\Omega) = \operatorname{span}\{\varphi^*_{ij}\}_{j=1}^{\ell_i}. \tag{18.2}$$

The space $(W^u_N)_i$ = range of $P_{N,i}$ is the algebraic/geometric eigenspace of the eigenvalue λ_i, and $\ell_i = \dim(W^u_N)_i$ is the algebraic/geometric multiplicity of λ_i, so that $\ell_1 + \ell_2 + \dots + \ell_M = N$. Here $P_{N,i}$, $P^*_{N,i}$ are the projections corresponding to the eigenvalue λ_i and $\overline{\lambda}_i$, respectively. For instance, $P_{N,i}$ is given by an integral such as that on the RHS of (16.3a), where now Γ is a closed smooth curve encircling the eigenvalue λ_i and no other. Similarly, for $P^*_{N,i}$. As a consequence of the FDSA, we obtain

$$W^u_N = P_N L^q_\sigma(\Omega) = \operatorname{span}\{\varphi_{ij}\}_{i=1,j=1}^{M \ \ \ell_i}; \quad (W^u_N)^* = P^*_N L^{q'}_\sigma(\Omega) = \operatorname{span}\{\varphi^*_{ij}\}_{i=1,j=1}^{M \ \ \ell_i} \tag{18.3}$$

[without the FDSA, W^u_N is the span of the generalized eigenfunctions of $\mathcal{A}$, corresponding to its (possibly unstable) distinct eigenvalues $\{\lambda_j\}_{j=1}^M$ as in (16.2); and similarly for $(W^u_N)^*$ (see the subsequent, more general Section 19). In other words, the FDSA says that the restriction $\mathcal{A}^u_N$ in (16.5) is diagonalizable or that $\mathcal{A}^u_N$ is a semisimple operator on W^u_N in the terminology of [K-1, p. 43]. Under the FDSA assumption, any vector $w \in W_N^u$ admits the following unique expansion [K-1, p.12], [B-T.1, p.1453],

$$W_N^u \ni w = \sum_{i,j}^{M,\ell_i} \left\langle w, \varphi_{ij}^* \right\rangle_{W_N^u} \varphi_{ij}, \quad \left\langle \varphi_{ij}, \varphi_{hk}^* \right\rangle_{W_N^u} = \begin{cases} 1, & \text{if } i = h,\ j = k, \\ 0, & \text{otherwise.} \end{cases} \tag{18.4}$$

that is, the system consisting of $\{\varphi_{ij}\} \in L^q_\sigma(\Omega)$ and $\{\varphi^*_{ij}\} \in L^{q'}_\sigma(\Omega)$, $i = 1, \dots, M$, $j = 1, \dots, \ell_i$, –each system being viewed also as made of elements of W_N^u for $q \geq 2$, see (18.1b)– can be chosen to form bi-orthogonal sequences. Next, we return to the w_N-dynamics in (16.8a), rewritten here for convenience

$$\text{on } W_N^u : w_N' - \mathcal{A}_N^u w_N = -P_N(\mathcal{A}Dv) + P_N P_q((mu)\tau),\ w_N(0) = P_N w_0. \tag{18.5}$$

We next express (18.5) component-wise, using the expansion (18.4).

The term $P_N(\mathcal{A}Dv)$. For $v \cdot \nu \equiv 0$ on Σ, v supported on $\widetilde{\Gamma}$, we compute by (18.4) in the duality pairing between $\mathcal{D}(\mathcal{A}^*)$ and $[\mathcal{D}(\mathcal{A}^*)]'$ [B-L-T.1, Eqns. (3.6.7), (3.6.8)], since $P_N^* \varphi^*_{ij} = \varphi^*_{ij} \in \mathcal{D}(\mathcal{A}^*)$:

$$W_N^u \ni P_N(\mathcal{A}Dv) = \sum_{i,j=1}^{M,\ell_i} \left\langle P_N(\mathcal{A}Dv), \varphi^*_{ij} \right\rangle_{W_N^u} \varphi_{ij} = \sum_{i,j} \left\langle \mathcal{A}Dv, \varphi^*_{ij} \right\rangle_{W_N^u} \varphi_{ij} \tag{18.6}$$

$$= \sum_{i,j} \left\langle v, D^* \mathcal{A}^* \varphi^*_{ij} \right\rangle_{L^q(\widetilde{\Gamma})} \varphi_{ij}$$

$$\text{(by (14.36))} \quad = \nu_0 \sum_{i,j=1}^{M,\ell_i} \left(v, \left. \frac{\partial \varphi^*_{ij}}{\partial \nu} \right|_{\Gamma} \right)_{L^q(\widetilde{\Gamma})} \varphi_{ij}, \tag{18.7}$$

by invoking (14.36) since $v \cdot \nu = 0$, on Γ by assumption, and v is supported only on $\widetilde{\Gamma}$.

Motivated by (18.7), we introduce the following subspace as in (13.23) (see Appendix E, Eq (C.5)).

$$\begin{cases} \mathcal{F} \equiv \operatorname{span} \left\{ \dfrac{\partial}{\partial \nu} \varphi^*_{ij},\ i = 1, \dots, M;\ j = 1, \dots, \ell_i \right\} \subset W^{2-1/q,q}(\Gamma), & \text{(18.8a)} \\ \text{for } \varphi^*_{ij} \in W^{3+m,q'}(\Omega) \cap L^q_\sigma(\Omega) \subset W^{3,q}(\Omega) \cap L^q_\sigma(\Omega) & \text{(18.8b)} \end{cases}$$

$q \geq 2,\ m \geq d\left(1/q' - 1/q\right) \geq 0,\ 1/q' + 1/q = 1,\ 1 < q' \leq 2 \leq q < \infty$, via a Sobolev embedding Theorem [Adams, Theorem 5.4 p97]. We also recall that (as a consequence of $\varphi^*_{ij}|_\Gamma = 0$ and div $\varphi^*_{ij} \equiv 0$ in Ω), we have, as recalled in Proposition 14.5 and from [B-L-T.1, Lemma 3.3.1, p. 35], [RT.4, Lemma 5.1, p. 495]

$$\left.\frac{\partial\varphi^*_{ij}}{\partial\nu}\right|_{\Gamma} \text{ is tangential on } \Gamma,\ \frac{\partial\varphi^*_{ij}}{\partial\nu}\cdot\nu = 0 \text{ on } \Gamma,\ \text{thus } \mathcal{F}\cdot\nu = 0 \text{ on } \Gamma. \tag{18.9}$$

Next, we pick boundary vectors $f_1, f_2, \dots, f_K \in \mathcal{F}$, $K \geq \ell_i$, $i = 1, \dots, M$, and select the tangential boundary control v of the form

$$v = \sum_{k=1}^{K} \nu_k(t) f_k \in W^{2-1/q,q}(\Gamma), \quad f_k \in \mathcal{F},\ \text{so that } f_k\cdot\nu = 0 \text{ on } \Gamma \text{ by (18.9)},\ q \geq 2 \tag{18.10}$$

and the condition $v\cdot\nu = 0$ on Γ for v in (18.10) is then satisfied. Substituting (18.10) into (18.7) yields

$$W^u_N \ni P_N(\mathcal{A}Dv) = \nu_0 \sum_{i,j=1}^{M,\ell_i} \left\{ \sum_{k=1}^{K} \left(f_k, \left.\frac{\partial\varphi^*_{ij}}{\partial\nu}\right|_{\Gamma} \right)_{L^q(\widetilde{\Gamma})} \nu_k(t) \right\} \varphi_{ij}. \tag{18.11}$$

Accordingly, by (18.11), we introduce the $\ell_i \times K$ matrix W_i, $i = 1, \dots, M$:

$$W_i = \begin{bmatrix} (f_1, \partial_\nu\varphi^*_{i1}|_\Gamma)_{\widetilde{\Gamma}}, & \cdots, & (f_K, \partial_\nu\varphi^*_{i1}|_\Gamma)_{\widetilde{\Gamma}} \\ (f_1, \partial_\nu\varphi^*_{i2}|_\Gamma)_{\widetilde{\Gamma}}, & \cdots, & (f_K, \partial_\nu\varphi^*_{i2}|_\Gamma)_{\widetilde{\Gamma}} \\ \vdots & & \vdots \\ (f_1, \partial_\nu\varphi^*_{i\ell_i}|_\Gamma)_{\widetilde{\Gamma}}, & \cdots, & (f_K, \partial_\nu\varphi^*_{i\ell_i}|_\Gamma)_{\widetilde{\Gamma}} \end{bmatrix} : \ell_i \times K;\ \partial_\nu = \frac{\partial}{\partial\nu},\ (\ ,\ )_{\widetilde{\Gamma}} = (\ ,\ )_{L^q(\widetilde{\Gamma}), L^{q'}(\widetilde{\Gamma})}. \tag{18.12}$$

Define by β_i and β the following ordered bases of length ℓ_i and N, respectively:

$$\begin{aligned} \beta_i &= [\varphi_{i1}, \dots \varphi_{i\ell_i}] \\ \beta &= \beta_1 \cup \beta_2 \dots \cup \beta_M = [\varphi_{11}, \dots, \varphi_{1\ell_1}, \varphi_{21} \dots, \varphi_{2\ell_2}, \dots, \varphi_{M1}, \dots, \varphi_{M\ell_M}]. \end{aligned} \tag{18.13}$$

Moreover, denote by $[P_N(\mathcal{A}Dv)]_{\beta_i}$ and $[P_N(\mathcal{A}Dv)]_{\beta}$ the coordinates (as column vectors) of $P_N(\mathcal{A}Dv)$ with respect to the basis β_i and the basis β, respectively. Then for $i = 1, \dots M$ by (18.11):

$$[P_N(\mathcal{A}Dv)]_{\beta_i} = \nu_0 W_i \hat{v}_K : \ell_i \times 1; \quad \hat{v}_K = \begin{bmatrix} \nu_1 \\ \nu_2 \\ \vdots \\ \nu_K \end{bmatrix} : K \times 1; \tag{18.14}$$

$$[P_N(\mathcal{A}Dv)]_\beta = \nu_0 \begin{bmatrix} W_1 \\ W_2 \\ \vdots \\ W_M \end{bmatrix} \hat{\nu}_K = \begin{bmatrix} \nu_o W_1 \hat{\nu}_K \\ \nu_o W_2 \hat{\nu}_K \\ \vdots \\ \nu_o W_M \hat{\nu}_K \end{bmatrix} = \nu_o W \hat{\nu}_K : N \times 1; \quad W = \begin{bmatrix} W_1 \\ \vdots \\ W_M \end{bmatrix}. \tag{18.15}$$

The term $P_N P_q((mu)\tau)$. Next, for $mu \in L^q_\sigma(\omega)$, we compute via (18.4), noticing that $P_q^* P_N^* \varphi_{ij}^* = P_{q'} P_N^* \varphi_{ij}^* = \varphi_{ij}^* \in \mathcal{D}(\mathcal{A}^*)$, $\varphi_{ij}^* \in W^{3,q}(\Omega)$, as well, $q \geq 2$, see (18.8) and Appendix E Eq (C.5):

$$W_N^u \ni P_N P_q((mu)\tau) = \sum_{i,j=1}^{M,\ell_i} \left\langle P_N P_q((mu)\tau), \varphi_{ij}^* \right\rangle_{W_N^u} \varphi_{ij} \tag{18.16}$$

$$= \sum_{i,j=1}^{M,\ell_i} \left\langle (mu)\tau, \varphi_{ij}^* \right\rangle_{W_N^u} \varphi_{ij} = \sum_{i,j=1}^{M,\ell_i} \left\langle u\tau, \varphi_{ij}^* \right\rangle_{L^q_\sigma(\omega)} \varphi_{ij} \tag{18.17}$$

$$= \sum_{i,j=1}^{M,\ell_i} \left\langle u, \varphi_{ij}^* \cdot \tau \right\rangle_{L^q_\sigma(\omega)} \varphi_{ij}, \tag{18.18}$$

since $m \equiv 0$ on $\Omega \backslash \omega$ and $(u\tau) \cdot \varphi_{ij}^* = u(\varphi_{ij}^* \cdot \tau)$ for u scalar . Next, we select the scalar interior control function u of the separated form

$$u = \sum_{k=1}^{K} \mu_k(t) u_k, \qquad u_k \in W_N^u \subset L^q_\sigma(\Omega), \quad \mu_k(t) = \text{ scalar}. \tag{18.19}$$

Substituting (18.19) into (18.18) yields

$$W_N^u \ni P_N P_q((mu)\tau) = \sum_{i,j=1}^{M,\ell_i} \left\{ \sum_{k=1}^{K} \left\langle u_k, \varphi_{ij}^* \cdot \tau \right\rangle_{L^q_\sigma(\omega)} \mu_k(t) \right\} \varphi_{ij}. \tag{18.20}$$

Accordingly, by (18.20), we introduce the $\ell_i \times K$ matrix U_i, $i = 1, \dots, M$; $K \geq \ell_i$, $i = 1, \dots, M$:

$$U_i = \begin{bmatrix} \left\langle u_1, \varphi_{i1}^* \cdot \tau \right\rangle_\omega, & \cdots, & \left\langle u_K, \varphi_{i1}^* \cdot \tau \right\rangle_\omega \\ \left\langle u_1, \varphi_{i2}^* \cdot \tau \right\rangle_\omega & \cdots, & \left\langle u_K, \varphi_{i2}^* \cdot \tau \right\rangle_\omega \\ \vdots & & \vdots \\ \left\langle u_1, \varphi_{i\ell_i}^* \cdot \tau \right\rangle_\omega & \cdots, & \left\langle u_K, \varphi_{i\ell_i}^* \cdot \tau \right\rangle_\omega \end{bmatrix} : \ \ell_i \times K; \quad \left\langle \,\cdot\,, \cdot\, \right\rangle_\omega = \left\langle \,\cdot\,, \cdot\, \right\rangle_{L^q_\sigma(\omega)}. \tag{18.21}$$

duality pairing between $L^q_\sigma(\omega), L^{q'}_\sigma(\omega)$. Then, in the notation of (18.13)–(18.15), we can write for

$i = 1, \dots M$:

$$[P_N P_q((mu)\tau)]_{\beta_i} = U_i \hat{\mu}_K : \ell_i \times 1; \quad \hat{\mu}_K = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_K \end{bmatrix}; \tag{18.22}$$

$$[P_N P_q((mu)\tau)]_{\beta} = \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_M \end{bmatrix} \hat{\mu}_K = \begin{bmatrix} U_1\hat{\mu}_K \\ U_2\hat{\mu}_K \\ \vdots \\ U_M\hat{\mu}_K \end{bmatrix} = U\hat{\mu}_K : N \times 1; \quad U = \begin{bmatrix} U_1 \\ \vdots \\ U_M \end{bmatrix}. \tag{18.23}$$

Substituting (18.11) and (18.20) on the RHS of (18.5), we obtain

$$\text{on } W_N^u: \; w_N' - \mathcal{A}_N^u w_N$$
$$= \sum_{i,j=1}^{M,\ell_i} \left\{ \sum_{k=1}^{K} \left[(-\nu_0) \left(f_k, \frac{\partial \varphi_{ij}^*}{\partial \nu}\bigg|_{\Gamma} \right)_{L^q(\widetilde{\Gamma})} \nu_k(t) + \left\langle u_k, \varphi_{ij}^* \cdot \tau \right\rangle_{L^q_\sigma(\omega)} \mu_k(t) \right] \right\} \varphi_{ij}. \tag{18.24}$$

Next, we represent the N-dimensional vector $w_N \in W_N^u$ as column vector $\hat{w}_N = [w_N]_\beta$ i.e.,

$$w_N = \sum_{i,j=1}^{M,\ell_i} w_N^{ij} \varphi_{ij}; \text{ and set } \hat{w}_N = \text{col}\left[w_N^{11}, \dots, w_N^{1,\ell_1}, \dots, w_N^{i,1}, \dots, w_N^{i,\ell_i}, \dots, w_N^{M,1}, \dots, w_N^{M,\ell_M} \right]. \tag{18.25}$$

$N = \ell_1 + \cdots + \ell_M$. Then, in $\mathbb{C}^N$, with respect to the basis $\{\varphi_{ij}\}_{i=1,j=1}^{M \;\; \ell_i}$ of normalized eigenfunctions of $\mathcal{A}_N^u$, we may rewrite system (18.5) = (18.24) = (16.8) as

$$(\hat{w}_N)' - \Lambda \hat{w}_N = -\nu_o W \hat{\nu}_K + U \hat{\mu}_K = [-\nu_0 W, U] \begin{bmatrix} \hat{\nu}_K \\ \hat{\mu}_K \end{bmatrix} = B \begin{bmatrix} \hat{\nu}_K \\ \hat{\mu}_K \end{bmatrix}, \tag{18.26}$$

where, similarly, $\hat{\nu}_K = \text{col}[\nu_1, \nu_2, \dots, \nu_K]$; $\hat{\mu}_K = \text{col}[\mu_1, \mu_2, \dots, \mu_K]$, see (18.14), (18.12), (18.15). In (18.26), we have introduced $N \times K$ matrices

$$W = \|(f_r, \partial_\nu \varphi_{ij}^*|_\Gamma)_{\widetilde{\Gamma}}\| = \begin{bmatrix} W_1 \\ W_2 \\ \vdots \\ W_M \end{bmatrix}; \quad U = \|(u_r, \varphi_{ij}^* \cdot \tau)_\omega\| = \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_M \end{bmatrix}, \tag{18.27a}$$

and

$$B_i = [-\nu_0 W_i, U_i] : \ell_i \times 2K; \qquad B = [-\nu_0 W, U] : N \times 2K, \tag{18.27b}$$

based on the $\ell_i \times K$ matrices W_i and U_i in (18.12) and (18.21), accounting for the control action v on the boundary and the tangential control $u\tau$ acting in the interior collar ω, and one $N \times N$ matrix

$$\Lambda = \begin{bmatrix} \lambda_1 I_1 & & & 0 \\ & \lambda_2 I_2 & & \\ & & \ddots & \\ 0 & & & \lambda_M I_M \end{bmatrix} : \ N \times N, \quad I_i : \ell_i \times \ell_i \text{ identity}, \tag{18.27c}$$

accounting for the free dynamics operator on W^u_N (see also [RT.1], [RT.2], [B-L-T.1, Sect. 3.6], [B-T.1, (3.37), (3.38)]. The following is the main result of the present section, from which the desirable and sought-after control-theoretic results will follow as corollaries in Remark 18.1 below.

Theorem 18.1. *Assume the FDSA. With reference to (18.12), (18.21), it is possible to select boundary vectors $f_1, \dots, f_K$ in $\mathcal{F} \subset W^{2-1/q,q}(\Gamma)$, $\mathcal{F}$ defined in (18.8) with support on $\widetilde{\Gamma}$, and interior vectors $u_1, \dots, u_K \in L^q(\omega), K = sup\{\ell_i, i = 1 \dots M\}$, such that for the matrix $B_i = [-\nu_0 W_i \,\vdots\, U_i]$ of size $\ell_i \times 2K$, we have*

$$rank\, [-\nu_0 W_i \,\vdots\, U_i] = full = \ell_i, \quad i = 1, \dots, M. \tag{18.28a}$$

In fact, explicitly and more precisely, for each $i = 1, \dots, M$, we have via (18.12), (18.21):

$$rank \left[\begin{array}{c|c} (f_1, \partial_\nu \varphi^*_{i1})_{\widetilde{\Gamma}} \cdots (f_{\ell_i}, \partial_\nu \varphi^*_{i1})_{\widetilde{\Gamma}} & \langle u_1, \varphi^*_{i1} \cdot \tau \rangle_\omega \cdots \langle u_{\ell_i}, \varphi^*_{i1} \cdot \tau \rangle_\omega \\ (f_1, \partial_\nu \varphi^*_{i2})_{\widetilde{\Gamma}} \cdots (f_{\ell_i}, \partial_\nu \varphi^*_{i2})_{\widetilde{\Gamma}} & \langle u_1, \varphi^*_{i2} \cdot \tau \rangle_\omega \cdots \langle u_{\ell_i}, \varphi^*_{i2} \cdot \tau \rangle_\omega \\ \vdots & \vdots \\ (f_1, \partial_\nu \varphi^*_{i\ell_i})_{\widetilde{\Gamma}} \cdots (f_{\ell_i}, \partial_\nu \varphi^*_{i\ell_i})_{\widetilde{\Gamma}} & \langle u_1, \varphi^*_{i\ell_i} \cdot \tau \rangle_\omega \cdots \langle u_{\ell_i}, \varphi^*_{i\ell_i} \cdot \tau \rangle_\omega \end{array} \right] = \ell_i, \tag{18.28b}$$

where the matrix in (6.28b) is $\ell_i \times 2\ell_i$ and the boundary terms are only evaluated on $\widetilde{\Gamma}$.

Proof. The proof will be critically based on a unique continuation result, Lemma 18.2 below.

Step 1: If the set of vectors $\{\partial_\nu \varphi^*_{i1}, \dots, \partial_\nu \varphi^*_{i\ell_i}\}$ were linearly independent in $L^q(\widetilde{\Gamma})$ recall (18.8a), $q \geq 2$, or else if the set of functions $\{\varphi^*_{i1} \cdot \tau, \dots, \varphi^*_{i\ell_i} \cdot \tau\}$ were linearly independent in $L^q(\omega)$, conclusion (6.28a) would follow for infinitely many choices of the vectors $f_1, \ldots, f_K \in \mathcal{F}$ and $u_1, \ldots u_K \in L^q(\omega)$. In general, in seeking that the ℓ_i rows (of length $2\ell_i$) of the matrix in (6.28b) be linearly independent, we see that the full rank statement (6.28b) will hold true if and only if we can exclude that each of the two sets of vectors

$$\{\partial_\nu \varphi^*_{i1}, \ldots, \partial_\nu \varphi^*_{i\ell_i}\} \text{ in } L^q(\widetilde{\Gamma}) \text{ and } \{\varphi^*_{i1} \cdot \tau, \ldots, \varphi^*_{i\ell_i} \cdot \tau\} \text{ in } L^q(\omega) \tag{18.29}$$

are linearly dependent, with the same linear dependence relation in the two cases; that is, if and only if we establish that we cannot have simultaneously,

$$\partial_\nu \varphi^*_{i\ell_i} = \sum_{j=1}^{\ell_i - 1} \alpha_j \partial_\nu \varphi^*_{ij} \text{ in } L^q(\widetilde{\Gamma}) \quad \text{and} \quad \varphi^*_{i\ell_i} \cdot \tau = \sum_{j=1}^{\ell_i - 1} \alpha_j \varphi^*_{ij} \cdot \tau \text{ in } L^q(\omega), \; q \geq 2 \tag{18.30}$$

with the same constants $\alpha_1, \ldots, \alpha_{\ell_i - 1}$ in both expansions. In fact, in one direction, validity of both expansions in (18.30) would imply a similar expansion, with the same common constants $\alpha_1, \ldots, \alpha_{\ell_i-1}$, of the last row of the $\ell_i \times 2\ell_i$ matrix in (6.28b) in terms of its preceding rows, thus violating condition (6.28a) for all choices of the w_r and u_r. Conversely, failure of 18.30 would imply that the last row of this matrix cannot be a linear combination of its preceding rows, yielding (6.28a).

It remains to show the following **Claim:** Statement (18.30) is false. By contradiction, suppose that both linear combinations in (18.30) hold true. Define the function (depending on i) in $L^q(\Omega)$

$$\varphi^* = \left[\sum_{j=1}^{\ell_i - 1} \alpha_j \varphi^*_{ij} - \varphi^*_{i\ell_i} \right] \in L^q(\Omega), \; i = 1, \ldots, M, \; q \geq 2, \tag{18.31a}$$

so that by (18.30),

$$\partial_\nu \varphi^* |_{\widetilde{\Gamma}} = 0, \text{ in } \widetilde{\Gamma} \text{ and } \varphi^* \cdot \tau \equiv 0 \text{ in } \omega. \tag{18.31b}$$

Then we have that φ^* satisfies the following eigenvalue problem for the operator $\mathcal{A}^*$ (or $(\mathcal{A}^u_N)^*$):

$$\begin{cases} \mathcal{A}^* \varphi^* = \overline{\lambda}_i \varphi^*, \text{ div } \varphi^* \equiv 0 \text{ in } \Omega; & (18.32a) \\ \varphi^*|_{\widetilde{\Gamma}} = 0; \; \partial_\nu \varphi^*|_{\widetilde{\Gamma}} = 0; \; \varphi^* \cdot \tau \equiv 0 \text{ on } \omega. & (18.32b) \end{cases}$$

Both statements in (6.32a) hold true for the function φ^* in (18.31a), since they are true for the eigenfunctions φ^*_{ij}, see (18.1) and $\mathcal{D}(\mathcal{A}^*) \subset V$. Similarly, for the Dirichlet B.C. $\varphi^*|_\Gamma = 0$ in (6.32a) as

$\varphi^*_{ij} \in \mathcal{D}(\mathcal{A}^*) \subset V$. Finally, the remaining two conditions $\partial_\nu \varphi^*|_\Gamma = 0$ and $\varphi^* \cdot \tau \equiv 0$ in ω are due to (18.31b). Explicitly, the PDE version of problem (6.32a–b) is

$$\begin{cases} -\nu_o \Delta \varphi^* - (L_e)^*(\varphi^*) + \nabla p^* = \overline{\lambda}_i \varphi^* & \text{in } \Omega; \quad (18.33a) \\ \operatorname{div} \varphi^* \equiv 0 & \text{in } \Omega; \quad (18.33b) \\ \varphi^*|_{\widetilde{\Gamma}} = 0; \ \left.\dfrac{\partial \varphi^*}{\partial \nu}\right|_{\widetilde{\Gamma}} = 0; \ \varphi^* \cdot \tau = 0 & \text{in } \omega; \quad (18.33c) \end{cases}$$

$$(L_e)^*(\varphi^*) = (y_e \cdot \nabla)\varphi^* + (\varphi^* \cdot \nabla)^* y_e, \tag{18.34}$$

where $(f \cdot \nabla)^* y_e$ is a d-vector whose ith component is $\sum_{j=1}^{d} (D_i y_{e_j}) f_j$ [B-L-T.1, p. 55], [Fur.1].

Step 2: The critical point is now that the over-determined problem (6.33a–c) implies (see subsequent Step 3).

$$\varphi^* \equiv 0 \text{ in } L^q(\Omega); \text{ or } \varphi^*_{i\ell_i} = \alpha_1 \varphi^*_{i1} + \alpha_2 \varphi^*_{i2} + \cdots + \alpha_{\ell_i - 1} \alpha^*_{i\ell_i - 1} \text{ in } L^q(\Omega), \tag{18.35}$$

i.e., the set $\{\varphi^*_{i1}, \ldots, \varphi^*_{i\ell_i}\}$ is linearly dependent on $(L^q(\Omega)$. But this is false, by the very selection of such eigenfunctions, see (18.1) and statement preceding it. Thus, the two conditions (18.30) cannot hold simultaneously. The **Claim** is established.

Hence, it is possible to select, in infinite many ways, boundary functions $f_1, \ldots, f_K \in \mathcal{F} \subset L^q(\Gamma)$ for $q \geq 2$, see (18.8a) and interior functions $u_1, \ldots, u_K$ in $L^q(\omega)$ such that the algebraic full rank condition (18.28b) hold true for each $i = 1, \ldots, M$.

Step 3: Here we shall establish the unique continuation property that is needed in Step 2 to conclude with statement (18.35). Actually we shall do this for the original problem in φ rather than for the problem (18.33) for φ^* in order to fall readily in results of [RT.4, Theorem 3.2, p. 489].

Lemma 18.2. *Assume that* $\{\varphi, p\} \in W^{2,q}(\Omega) \times W^{1,q}(\Omega)$ *satisfies*

$$-\nu_o \Delta \varphi + L_e(\varphi) + \nabla p = \lambda \varphi \qquad \text{in } \Omega; \tag{18.36a}$$

$$\operatorname{div} \varphi \equiv 0 \qquad \text{in } \Omega; \tag{18.36b}$$

$$\varphi|_{\widetilde{\Gamma}} = 0; \ \left.\frac{\partial \varphi}{\partial \nu}\right|_{\widetilde{\Gamma}} = 0; \quad \varphi \cdot \tau \equiv 0 \qquad \text{in } \omega, \tag{18.36c}$$

Then, in fact,

$$\varphi \equiv 0 \ \text{in } \Omega, \quad p \equiv const. \tag{18.37}$$

where $\widetilde{\Gamma}$ is an open subset of Γ of positive surface measure and ω is a local collar of $\widetilde{\Gamma}$ (Fig. 2).

Proof. Step 1: First, condition $\varphi \cdot \tau \equiv 0$ in a collar ω of the sub-portion $\widetilde{\Gamma}$ of the boundary Γ implies

$$\frac{\partial^2}{\partial \nu^2} \varphi(\xi) \cdot \tau(\xi) \equiv 0, \;\; \xi \in \omega, \;\; \text{hence } \frac{\partial^2}{\partial \nu^2} \varphi|_{\widetilde{\Gamma}} \cdot \tau = 0 \text{ on } \widetilde{\Gamma}; \tag{18.38}$$

this is justified in Appendix D, Lemma D.1, Eqns. (B.10), (B.11).

Step 2: Next, recalling [L-T.2, Prop. 3C.6, p. 305], [S-Z.1, Prop. 2.68, p. 94], we deduce from the first two boundary conditions in (18.36c) and from (18.38) that

$$\Delta \varphi \cdot \tau|_{\Gamma} = \left[\frac{\partial^2 \varphi}{\partial \nu^2} + \Delta_{\Gamma} \varphi + (\text{div } \nu) \frac{\partial \varphi}{\partial \nu} \right]_{\widetilde{\Gamma}} \cdot \tau = 0. \tag{18.39}$$

Step 3: We next return to Eqn. (6.36a) and restrict it on the portion of $\widetilde{\Gamma}$ of the boundary. We use $\Delta \varphi|_{\widetilde{\Gamma}} \cdot \tau = 0$ from (18.39), $L_e(\varphi)|_{\widetilde{\Gamma}} \cdot \tau = 0$ from the definition (1.2) of L_e combined with $y_e|_{\Gamma} = 0$ in (1.3c) and $\varphi|_{\widetilde{\Gamma}} = 0$ in (6.36c). We thus obtain for the tangential derivative on $\widetilde{\Gamma}$:

$$\frac{\partial p}{\partial \tau}|_{\widetilde{\Gamma}} = \nabla p \cdot \tau|_{\widetilde{\Gamma}} = 0 \Rightarrow p \text{ constant on } \widetilde{\Gamma}. \tag{18.40}$$

Since p is identified up to a constant, we may then take

$$p \equiv 0 \text{ on } \widetilde{\Gamma}. \tag{18.41}$$

Step 4: We now return to the BC in (18.36c) together with the B.C. in (6.36c) to obtain

$$\varphi|_{\widetilde{\Gamma}} \equiv 0, \quad \left[\frac{\partial \varphi}{\partial \nu} - p \right]_{\widetilde{\Gamma}} \equiv 0 \text{ on } \widetilde{\Gamma}. \tag{18.42}$$

We then invoke [RT.4, Thm 3.2, p.489] or [RT.5, Thm 1.3, p.647] to system (6.36a–c), combined with the B.C. in (18.42) to conclude that

$$\varphi \equiv 0 \text{ in } \Omega, \tag{18.43}$$

as desired. Lemma 18.2 is proved. The proof of (18.43) is along classical lines (based also on [B-L-T.1, Sect. 3.6]) in elliptic equations. □

Remark 18.1. The following alternative (less direct) route is available which we illustrate for ω being a collar of the entire boundary Γ, so that $\widetilde{\Gamma} = \Gamma$ (Fig. 1).

We already know from [RT.4, Lemma 5.5, p. 496] that the three conditions in (6.36b–c):

$$\text{div } \varphi \equiv 0 \text{ in } \Omega; \;\; \varphi|_\Gamma = 0; \;\; \left.\frac{\partial \varphi}{\partial \nu}\right|_\Gamma = 0 \tag{18.44}$$

imply

$$\left.\frac{\partial^2 \varphi}{\partial \nu^2}\right|_\Gamma \cdot \nu = 0 \text{ on } \Gamma. \tag{18.45}$$

The property (18.45), combined with (18.38) with $\widetilde{\Gamma} = \Gamma$, i.e., $\frac{\partial^2 \varphi}{\partial \nu^2} \cdot \tau \equiv 0$ on Γ yields then

$$\left.\frac{\partial^2 \varphi}{\partial \nu^2}\right|_\Gamma = 0 \text{ on } \Gamma. \tag{18.46}$$

Next, recalling [L-T.1, Prop. 3C.6, p. 305], [S-Z.1, Prop. 2.68, p. 94], we deduce from the first two boundary conditions in (18.36c) with $\widetilde{\Gamma} = \Gamma$ as in (18.44) and from (18.46) that

$$\Delta \varphi|_\Gamma = \left[\frac{\partial^2 \varphi}{\partial \nu^2} + \Delta_\Gamma \varphi + (\text{div } \nu)\frac{\partial \varphi}{\partial \nu}\right]_\Gamma = 0. \tag{18.47}$$

Then returning to Eqn. (6.36c), restricting it on Γ, and invoking (18.4), as well as $L_e(\varphi)|_\Gamma = 0$ from (1.2) with $y_e|_\Gamma = 0$ and $\varphi|_\Gamma = 0$, we then obtain

$$\nabla p|_\Gamma = 0, \quad \text{or } p|_\Gamma = \text{const}, \;\; \left.\frac{\partial p}{\partial \nu}\right|_\Gamma = 0 \text{ in } \Gamma, \;\; \text{in fact, } p|_\Gamma \equiv 0 \text{ on } \Gamma, \tag{18.48}$$

since p is identified up to a constant. So all Cauchy data for φ and p vanish on Γ. Results (18.47) and (18.48) are stronger than necessary in order to invoke [RT.5, Thm. 3.2, p. 489] and include that $\varphi = 0$ in Ω, as desired. □

19 First step in the proof of Theorem 17.1 for the w_N-system in (16.8): verification of the controllability algebraic rank conditions in the general case

In the present section we dispense with the FDSA (18.2). More precisely, we shall obtain Theorem 18.1 without assuming the FDSA (18.2). This is Theorem 19.2 below. Thus, in the present general case W_N^u is the space of generalized eigenfunctions of $\mathcal{A}_q$ $(= \mathcal{A}_N^u)$ corresponding to its unstable eigenvalues, see (16.9). Also in Section 18, we have used the notation $\mathcal{A}\varphi_{ij} = \lambda_i\varphi_{ij}$ and $\mathcal{A}^*\varphi_{ij}^* = \bar{\lambda}_i\varphi_{ij}^*$ for the eigenvectors of $\mathcal{A}$ and $\mathcal{A}^*$ corresponding to the distinct unstable eigenvalues $\lambda_1, \dots, \lambda_M$ or $\bar{\lambda}_1, \dots, \bar{\lambda}_M$ respectively. In the present section we shall introduce a more complicated notation to deal with generalized eigenvectors. The relationship between the notation for eigenvectors in the present section and in Section 18 is identified in (19.7b) below. We recall Appendix E, in particular (C.5) yielding (18.8), $q \geq 2$.

Step 1: To treat this computationally more complicated case we shall, essentially invoke the classical result on controllability of a finite-dimensional, time-invariant system $\{\mathbb{A}, \mathbb{B}\}, \mathbb{A} : N \times N, \mathbb{B} : N \times p$, where $\mathbb{A}$ is given in Jordan form J. Controllability is invariant under any equivalent transformation. Let again $\lambda_1, \lambda_2, \cdots, \lambda_M$ be the distinct eigenvalues of $\mathbb{A} = J$. Let $\mathbb{A}_i$ denote all the Jordan blocks associated with the eigenvalue λ_i; let ℓ_i be the number of Jordan blocks of $\mathbb{A}$ (i.e., the number of linearly independent eigenvectors associated with the eigenvalue λ_i or geometric multiplicity of λ_i). Let $\mathbb{A}_{ij}$ be the j^{th} Jordan block in $\mathbb{A}_i$ corresponding to a Jordan cycle of length N_j^i. That is:

$$\mathbb{A} = \text{diag}\{\mathbb{A}_1, \mathbb{A}_2, \cdots, \mathbb{A}_M\}; \mathbb{A}_i = \text{diag}\{\mathbb{A}_{i1}, \mathbb{A}_{i2}, \cdots, \mathbb{A}_{i\ell_i}\}. \tag{19.1}$$

Partition the matrix $\mathbb{B}$ accordingly:

$$\underset{(N\times N)}{\mathbb{A}} = \begin{bmatrix} \mathbb{A}_1 & & & 0 \\ & \mathbb{A}_2 & & \\ & & \ddots & \\ 0 & & & \mathbb{A}_M \end{bmatrix}, \quad \underset{(N\times p)}{\mathbb{B}} = \begin{bmatrix} \mathbb{B}_1 \\ \mathbb{B}_2 \\ \vdots \\ \mathbb{B}_M \end{bmatrix} \tag{19.2}$$

$$\underset{(N_i \times N_i)}{\mathbb{A}_i} = \begin{bmatrix} \mathbb{A}_{i1} & & & 0 \\ & \mathbb{A}_{i2} & & \\ & & \ddots & \\ 0 & & & \mathbb{A}_{i\ell_i} \end{bmatrix}, \qquad \underset{(N_i \times p)}{\mathbb{B}_i} = \begin{bmatrix} \mathbb{B}_{i1} \\ \mathbb{B}_{i2} \\ \vdots \\ \mathbb{B}_{i\ell_i} \end{bmatrix} \tag{19.3}$$

$$\underset{(N_j^i \times N_j^i)}{\mathbb{A}_{ij}} = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \lambda_i & 1 & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix}, \qquad \underset{(N_j^i \times p)}{\mathbb{B}_{ij}} = \begin{bmatrix} b_{1ij} \\ b_{2ij} \\ \vdots \\ b_{Lij} \end{bmatrix} \tag{19.4}$$

If E_{λ_i} and K_{λ_i} denote the eigenspace and generalized eigenspace associated with the eigenvalue λ_i, $i = 1, \dots M$, then dim $E_{\lambda_i} = \ell_i = \#$ of Jordan blocks in $\mathbb{A}_i$, dim $K_{\lambda_i} = N_i$, $N_j^i =$ length of j^{th}-cycle associated with λ_i, $j = 1, \dots \ell_i$. We have dim $W_N^u = N = \sum_{i=1}^{M} N_i = \sum_{i=1}^{M} \sum_{j=1}^{\ell_i} N_j^i$. In (19.4), the last row of $\mathbb{B}_{ij}$ is denoted by b_{Lij}. The following result is classical [K-H-N.1, p. 204], [L-M.1, Ex. #7, p. 102], [Chen.1, p. 211], [Chen.2, p. 165], [B-M, Theorem 3.3-4, p 148].

Theorem 19.1. *The pair $\{J, \mathbb{B}\}$, $J : N \times N$, Jordan form, $\mathbb{B} : N \times p$ is controllable if and only if, for each $i = 1, 2, \cdots, M$ (that is for each distinct eigenvalue) the rows of the $\ell_i \times p$ matrix constructed with all "last" rows $b_{Li1}, \dots b_{Li\ell_i}$*

$$\mathbb{B}_i^L = \begin{bmatrix} \mathbb{B}_{Li1} \\ \mathbb{B}_{Li2} \\ \vdots \\ \mathbb{B}_{Li\ell_i} \end{bmatrix} : \ell_i \times p \tag{19.5}$$

are linearly independent (in the field of complex number). [A direct proof uses Hautus criterion for controllability [Chen.1], [Chen.2].]

We next apply the above Theorem 19.1 to the w_N-problem (16.8) and (18.5). To this end, we select a Jordan basis β for the operator $\mathcal{A}_N^u$ on W_N^u and β_i for the operator $(\mathcal{A}_N^u)_i$ on $W_{N,i}^u$ given by

Jordan Basis

$$\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_M \tag{19.6a}$$

$$\beta_i = \left\{ e_1^1(\lambda_i), e_2^1(\lambda_i), \cdots, e^1_{N^i_1}(\lambda_i) \vdots e_1^2(\lambda_i), \cdots, e^2_{N^i_2}(\lambda_i) \vdots \cdots \vdots e_1^{\ell_i}(\lambda_i), \cdots, e^{\ell_i}_{N^i_{\ell_i}}(\lambda_i) \right\}. \tag{19.6b}$$

Here the first vector of each cycle: $e_1^1(\lambda), e_1^2(\lambda_i), \cdots, e_1^{\ell_i}(\lambda_i)$ are eigenvectors of $(\mathcal{A}_N^u)_i$ corresponding to the eigenvalue λ_i, while the remaining vectors in β_i are corresponding generalized eigenvectors. Thus, in the notation (18.1) of Section 18, we have:

$$\varphi_{i1} = e_1^1(\lambda_i);\ \varphi_{i2} = e_1^2(\lambda_i), \ldots \varphi_{i\ell_i} = e_1^{\ell_i}(\lambda_i). \tag{19.6c}$$

Next, we can choose a bi-orthogonal basis β_i^* of $((\mathcal{A}_N^u)^*)_i$ corresponding to its eigenvalue $\bar{\lambda}_i$ given by

Bi-orthogonal Basis

$$\beta_i^* = \left\{ \Phi_1^1(\overline{\lambda}_i), \Phi_2^1(\overline{\lambda}_i), \cdots, \Phi^1_{N^i_1}(\overline{\lambda}_i) \vdots \Phi_1^2(\overline{\lambda}_i), \Phi_2^2(\overline{\lambda}_i), \cdots, \Phi^2_{N^i_2}(\overline{\lambda}_i) \vdots \right.$$
$$\left. \cdots \vdots \Phi_1^{\ell_i}(\overline{\lambda}_i), \Phi_2^{\ell_i}(\overline{\lambda}_i), \cdots, \Phi^{\ell_i}_{N^i_{\ell_i}}(\overline{\lambda}_i) \right\}. \tag{19.7a}$$

Thus, in the notation (18.1) of Section 18, we have

$$\varphi_{i1}^* = \Phi_1^1(\bar{\lambda}_i),\ \varphi_{i2}^* = \Phi_1^2(\bar{\lambda}_i), \ldots \varphi_{i\ell_i}^* = \Phi_1^{\ell_i}(\bar{\lambda}_i). \tag{19.7b}$$

In the bi-orthogonality relationship between the vectors in (19.6) and those in (19.7), the first eigenvector $e_1^1(\lambda_i)$ of the first cycle in β_i is associated with the last generalized eigenvector $\Phi^1_{N^i_1}(\bar{\lambda}_i)$ of the first cycle in β_i^*; etc., the last generalized eigenvector $e^1_{N^i_1}(\lambda_i)$ of the first cycle in β_i is associated with the first eigenvector $\Phi_1^1(\bar{\lambda}_i)$ of the first cycle in β_i^*; etc.

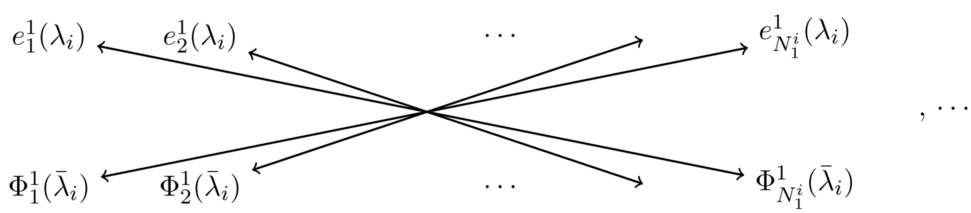


Fig. 7.1: Relation between the generalized eigenvectors of $\mathcal{A}_N^u$ and $(\mathcal{A}_N^u)^*$

Thus, if $f \in W^u_{N,i}$, the following expression holds true:

$$f = (f, \Phi^1_{N^i_1}(\overline{\lambda}_i))e^1_1(\lambda_i) + \cdots + (f, \Phi^1_1(\bar{\lambda}_i))e^1_{N^i_1} + \cdots + (f, \Phi^{\ell_i}_{N^i_{\ell_i}}(\bar{\lambda}_i))e^{\ell_i}_1(\lambda_i) + \cdots + (f, \Phi^{\ell_i}_1(\overline{\lambda}_i))e^{\ell_i}_{N^i_{\ell_i}}(\lambda_i). \quad (19.8)$$

This is the counterpart of $\sum_{j=1}^{\ell_i}(w, \varphi^*_{ij})_{W^u_N}\varphi_{ij} \in W^u_{N,i}$ in (18.4) under the FDSA. Next, we apply (19.8) with $f = P_N(\mathcal{A}Dv)$, with v compactly supported on $\widetilde{\Gamma}$. More specifically, we shall write the vector representation of $P_N(\mathcal{A}Dv)$ with respect to the basis β_i in (19.5), and moreover, in line with Theorem 19.1, we shall explicitly note only the coordinates corresponding to the vectors $e^1_{N^i_1}(\lambda_i), e^2_{N^i_2}(\lambda_i), \cdots, e^{\ell_i}_{N^{\ell_i}_i}(\lambda_i)$, each being the last vector of each cycle in (19.6):

$$[P_N(\mathcal{A}Dv)]_{\beta_i} = \nu_0 \begin{bmatrix} \times\times\times \\ \left(v, \partial_\nu \Phi^1_1(\bar{\lambda}_i)\right)_{\widetilde{\Gamma}} \\ \cdots\cdots\cdots\cdots\cdots \\ \times\times\times \\ \left(v, \partial_\nu \Phi^2_1(\bar{\lambda}_i)\right)_{\widetilde{\Gamma}} \\ \cdots\cdots\cdots\cdots\cdots \\ \times\times\times \\ \left(v, \partial_\nu \Phi^{\ell_i}_1(\bar{\lambda}_i)\right)_{\widetilde{\Gamma}} \end{bmatrix} = \nu_0 \begin{bmatrix} \times\times\times \\ \left(v, \partial_\nu \varphi^*_{i1}\right)_{\widetilde{\Gamma}} \\ \cdots\cdots\cdots\cdots \\ \times\times\times \\ \left(v, \partial_\nu \varphi^*_{i2}\right)_{\widetilde{\Gamma}} \\ \cdots\cdots\cdots\cdots \\ \times\times\times \\ \left(v, \partial_\nu \varphi^*_{i\ell_i}\right)_{\widetilde{\Gamma}} \end{bmatrix} \begin{matrix} \longleftarrow \text{last row of the } 1^{st} \text{ cycle} \\ \\ \longleftarrow \text{last row of the } 2^{nd} \text{ cycle} \\ \\ \longleftarrow \text{last row of the } \ell_i^{th} \text{ cycle} \end{matrix} \quad (19.9)$$

with $(\ ,\ )_{\widetilde{\Gamma}}$ the $L^q(\widetilde{\Gamma}), L^{q'}(\widetilde{\Gamma})$ duality paring. The symbol $\times\times\times$ refers to terms which we do not care about. (19.9) is the relevant counterpart of the expansion $\nu_0 \sum_{j=1}^{\ell_i}(v, \partial_i \varphi^*_{ij})_{\widetilde{\Gamma}}$ in (18.7) under the FDSA. Notice that (19.9) involves only the eigenvectors $\varphi^*_{i1} = \Phi^1_1(\bar{\lambda}_i), \varphi^*_{i2} = \Phi^2_1(\bar{\lambda}_i), \dots, \varphi^*_{i\ell_i} = \Phi^{\ell_i}_1(\bar{\lambda}_i)$ of $(\mathcal{A}^u_N)^*$ corresponding to the eigenvalue $\overline{\lambda}_i$.

The controls v and u

Next, we choose a tangential boundary control v in (18.10) of the form

$$v = \sum_{k=1}^{K} \nu_k(t) f_k, \quad f_k \in \mathcal{F} \subset W^{2-1/q,q}(\Gamma), \text{ so that } f_k \cdot \nu = 0 \text{ on } \Gamma, \quad (19.10)$$

as in (18.10). We then get the relevant counterpart of (18.12), which we write by omitting the explicit dependence on $\overline{\lambda}_i$:

$$
W_i = \begin{bmatrix}
\times & \times & & \times \\
(f_1, \partial_\nu \Phi_1^1)_{\widetilde{\Gamma}} & (f_2, \partial_\nu \Phi_1^1)_{\widetilde{\Gamma}} & \cdots & (f_K, \partial_\nu \Phi_1^1)_{\widetilde{\Gamma}} \\
\cdots & \cdots & \cdots & \cdots \\
(f_1, \partial_\nu \Phi_1^2)_{\widetilde{\Gamma}} & (f_2, \partial_\nu \Phi_1^2)_{\widetilde{\Gamma}} & \cdots & (f_K, \partial_\nu \Phi_1^2)_{\widetilde{\Gamma}} \\
\cdots & \cdots & \cdots & \cdots \\
 & \vdots & & \\
(f_1, \partial_\nu \Phi_1^{\ell_i})_{\widetilde{\Gamma}} & (f_2, \partial_\nu \Phi_1^{\ell_i})_{\widetilde{\Gamma}} & \cdots & (f_K, \partial_\nu \Phi_1^{\ell_i})_{\widetilde{\Gamma}}
\end{bmatrix}
\begin{matrix}
 \\
\leftarrow \text{last row of first cycle: } b_{Li1}(f) \\
 \\
\leftarrow \text{last row of second cycle: } b_{Li2}(f) \\
 \\
 \\
\leftarrow \text{last row of last cycle: } b_{Li\ell_i}(f)
\end{matrix}
\tag{19.11}
$$

where we recall from (19.7b) $\varphi_{i1}^* = \Phi_1^1(\bar{\lambda}_i), \varphi_{i2}^* = \Phi_1^2(\bar{\lambda}_i), \dots, \varphi_{i\ell_i}^* = \Phi_1^{\ell_i}(\bar{\lambda}_i)$. The relevant rows exhibited in (19.11) correspond to the last rows of each Jordan sub-block $\mathbb{A}_{i1}, \mathbb{A}_{i2}, \cdots, \mathbb{A}_{i\ell_i}$ in (19.3), that is to the row $b_{Li1}, b_{Li2}, \cdots, b_{Li\ell_i}$ related to the vector f. Similarly we select a scalar interior $(d = 2)$ or a two component control vector $u = [u^1, u^2]$, $(d = 3)$ of the separated form

$$
u = \sum_{k=1}^{K} \mu_k(t) v_k, \qquad u_k \in (L^q(\omega))^{d-1}, \quad u_k = \begin{cases} \text{scalar}, & d = 2 \\ \begin{vmatrix} u_k^1 \\ u_k^2 \end{vmatrix}, & d = 3. \end{cases} \tag{19.12}
$$

likewise obtain that the relevant counterpart of (18.21) is

$$
U_i = \begin{bmatrix}
\times & \times & & \times \\
\langle u_1, \Phi_1^1 \cdot \tau \rangle_\omega & \langle u_2, \Phi_1^1 \cdot \tau \rangle_\omega & \cdots & \langle u_K, \Phi_1^1 \cdot \tau \rangle_\omega \\
\cdots & \cdots & \cdots & \cdots \\
\times & \times & & \times \\
\langle u_1, \Phi_1^2 \cdot \tau \rangle_\omega & \langle u_2, \Phi_1^2 \cdot \tau \rangle_\omega & \cdots & \langle u_K, \Phi_1^2 \cdot \tau \rangle_\omega \\
\cdots & \cdots & \cdots & \cdots \\
\times & \times & & \times \\
\langle u_1, \Phi_1^{\ell_i} \cdot \tau \rangle_\omega & \langle u_2, \Phi_1^{\ell_i} \cdot \tau \rangle_\omega & \cdots & \langle u_K, \Phi_1^{\ell_i} \cdot \tau \rangle_\omega
\end{bmatrix}
\begin{matrix}
 \\
\leftarrow \text{ row } b_{Li1}(u) \\
 \\
 \\
\leftarrow \text{ row } b_{Li2}(u) \\
 \\
 \\
\leftarrow \text{ row } b_{Li\ell_i}(u)
\end{matrix}
\tag{19.13}
$$

Again, the relevant rows exhibited in (19.12) correspond to the last rows of each Jordan sub-block $\mathbb{A}_{i1}, \mathbb{A}_{i2}, \cdots, \mathbb{A}_{i_{\ell_i}}$ in (19.3). As a consequence, setting as in (18.27b), $B_i = [-\nu_o W_i, U_i]$, this time with W_i and U_i defined by (19.11) and (19.13), we obtain for $i = 1, 2, \ldots, M$:

$$B_i = \left[\begin{array}{cc|cc} \times & \times & \times & \times \\ (f_1, \partial_\nu \Phi_1^1)_{\widetilde{\Gamma}} \cdots & (f_K, \partial_\nu \Phi_1^1)_{\widetilde{\Gamma}} & \left\langle u_1, \Phi_1^1 \cdot \tau \right\rangle_\omega \cdots & \left\langle u_K, \Phi_1^1 \cdot \tau \right\rangle_\omega \\ \times & \times & \times & \times \\ (f_1, \partial_\nu \Phi_1^2)_{\widetilde{\Gamma}} \cdots & (f_K, \partial_\nu \Phi_1^2)_{\widetilde{\Gamma}} & \left\langle u_1, \Phi_1^2 \cdot \tau \right\rangle_\omega \cdots & \left\langle u_K, \Phi_1^2 \cdot \tau \right\rangle_\omega \\ \times & \times & \times & \times \\ (f_1, \partial_\nu \Phi_1^{\ell_i})_{\widetilde{\Gamma}} \cdots & (f_K, \partial_\nu \Phi_1^{\ell_i})_{\widetilde{\Gamma}} & \left\langle u_1, \Phi_1^{\ell_i} \cdot \tau \right\rangle_\omega \cdots & \left\langle u_K, \Phi_1^{\ell_i} \cdot \tau \right\rangle_\omega \end{array}\right]. \tag{19.14}$$

In (19.13), we have displayed only the relevant rows: $b_{Li1}, b_{Li2}, \ldots, b_{Li\ell_i}$. According to Theorem 19.1 the test for controllability as applied to system (18.5), i.e., to the pair $\{\mathcal{A}_N^u, B\}$, $B = \text{col}[B_1, B_2, \ldots B_M]$, is

$$\text{rank} \begin{bmatrix} \text{row } b_{Li1} \text{ of } B_i \\ \text{row } b_{Li2} \text{ of } B_i \\ \cdots \\ \text{row } b_{Li\ell_i} \text{ of } B_i \end{bmatrix}$$

$$= \quad \text{rank} \left[\begin{array}{c|c} (f_1, \partial_\nu \Phi_1^1)_{\widetilde{\Gamma}} \cdots (f_K, \partial_\nu \Phi_1^1)_{\widetilde{\Gamma}} & \left\langle u_1, \Phi_1^1 \cdot \tau \right\rangle_\omega \cdots \left\langle u_K, \Phi_1^1 \cdot \tau \right\rangle_\omega \\ (f_1, \partial_\nu \Phi_1^2)_{\widetilde{\Gamma}} \cdots (f_K, \partial_\nu \Phi_1^2)_{\widetilde{\Gamma}} & \left\langle u_1, \Phi_1^2 \cdot \tau \right\rangle_\omega \cdots \left\langle u_K, \Phi_1^2 \cdot \tau \right\rangle_\omega \\ \vdots & \vdots \\ (f_1, \partial_\nu \Phi_1^{\ell_i})_{\widetilde{\Gamma}} \cdots (f_K, \partial_\nu \Phi_1^{\ell_i})_{\widetilde{\Gamma}} & \left\langle u_1, \Phi_1^{\ell_i} \cdot \tau \right\rangle_\omega \cdots \left\langle u_K, \Phi_1^{\ell_i} \cdot \tau \right\rangle_\omega \end{array}\right] \tag{19.15a}$$

$$= \quad \text{rank} \left[\begin{array}{c|c} (f_1, \partial_\nu \varphi_{i1}^*)_{\widetilde{\Gamma}} \cdots (f_{\ell_i}, \partial_\nu \varphi_{i1}^*)_{\widetilde{\Gamma}} & \left\langle u_1, \varphi_{i1}^* \cdot \tau \right\rangle_\omega \cdots \left\langle u_{\ell_i}, \varphi_{i1}^* \cdot \tau \right\rangle_\omega \\ (f_1, \partial_\nu \varphi_{i2}^*)_{\widetilde{\Gamma}} \cdots (f_{\ell_i}, \partial_\nu \varphi_{i2}^*)_{\widetilde{\Gamma}} & \left\langle u_1, \varphi_{i2}^* \cdot \tau \right\rangle_\omega \cdots \left\langle u_{\ell_i}, \varphi_{i2}^* \cdot \tau \right\rangle_\omega \\ \vdots & \vdots \\ (f_1, \partial_\nu \varphi_{i\ell_i}^*)_{\widetilde{\Gamma}} \cdots (f_{\ell_i}, \partial_\nu \varphi_{i\ell_i}^*)_{\widetilde{\Gamma}} & \left\langle u_1, \varphi_{i\ell_i}^* \cdot \tau \right\rangle_\omega \cdots \left\langle u_{\ell_i}, \varphi_{i\ell_i}^* \cdot \tau \right\rangle_\omega \end{array}\right] = \ell_i, \tag{19.15b}$$

$i = 1, 2, \cdots, M$. But this is exactly the test obtained in (6.28b) via the identification in (19.7b). Thus the remainder of the proof in Section 18 past (6.28b) applies and shows Theorem 18.1 without the FDSA. We have

Theorem 19.2. *With reference to the W_i (19.11) and U_i in (19.13), it is possible to select boundary vectors $f_1, \dots, f_K$ in $\mathcal{F} \subset W^{2-1/q,q}(\Gamma)$, see (18.8b), $\mathcal{F}$ defined in (18.8), f_i supported on $\widetilde{\Gamma}$ and interior vectors $u_1, \dots, u_K \in L^q(\omega)$, $K = sup\{\ell_i, i = 1, \dots, M\}$, such that the algebraic conditions (19.15b) hold true, $i = 1, \dots, M$.*

Thus Theorem 19.2 states that we may require that the full boundary Γ (Fig. 1) be replaced by an arbitrarily small portion $\widetilde{\Gamma}$ of positive measure (Fig. 2) for the tangential boundary control v, with an associated internal tangential-like control u on ω. We close this section by writing down the counterpart of expansion (18.24) for the w_N-dynamics in terms this time of the basis β (see (19.8)) of generalized eigenvectors in the present general case:

The w_N-dynamics in (16.8) in Jordan basis β. Then the w_N-dynamics in (16.8) (projection of the dynamics (14.26) onto the unstable subspace W_N^u) may be given in terms of the basis β in (19.6) of generalized eigenvectors of $\mathcal{A}$ (or $\mathcal{A}_N^u$) as follows [L-T.2, Eq (85)], recalling (19.6), (19.7) and $(\ , \ )_{\widetilde{\Gamma}}$

the $L^q(\widetilde{\Gamma}), L^{q'}(\widetilde{\Gamma})$ duality paring, $q \geq 2$:

$$
\begin{aligned}
&\text{on } W_N^u : w_N' - \mathcal{A}_N^u z_N \\
&= (-\nu_o)\sum_{i=1}^{N}\Bigg\{\sum_{k=1}^{K}\Big[(f_k, \partial_\nu \Phi^1_{N_1^i}(\bar{\lambda}_i))_{\tilde{\Gamma}} v_k(t) + \big\langle u_k, \Phi^1_{N_1^i}(\bar{\lambda}_i)\cdot\tau\big\rangle_\omega \mu_k(t)\Big] e_1^1(\lambda_i) \\
&\quad + \cdots + \cdots \\
&\quad + \sum_{k=1}^{K}\Big[(f_k, \partial_\nu \Phi^1_1(\bar{\lambda}_i))_{\tilde{\Gamma}} v_k(t) + \big\langle u_k, \Phi^1_1(\bar{\lambda}_i)\cdot\tau\big\rangle_\omega \mu_k(t)\Big] e^1_{N_1^i}(\lambda_i) \\
&\quad + \cdots + \cdots \\
&\quad + \sum_{k=1}^{K}\Big[(f_k, \partial_\nu \Phi^{\ell_i}_{N^i_{\ell_1}}(\bar{\lambda}_i))_{\tilde{\Gamma}} v_k(t) + \big\langle u_k, \Phi^{\ell_i}_{N^i_{\ell_i}}(\bar{\lambda}_i)\cdot\tau\big\rangle_\omega \mu_k(t)\Big] e_1^{\ell_i}(\lambda_i) \\
&\quad + \cdots + \cdots \\
&\quad + \sum_{k=1}^{K}\Big[(f_k, \partial_\nu \Phi^{\ell_i}_1(\bar{\lambda}_i))_{\tilde{\Gamma}} v_k(t) + \big\langle u_k, \Phi^{\ell_i}_1(\bar{\lambda}_i)\cdot\tau\big\rangle_\omega \mu_k(t)\Big] e^{\ell_i}_{N^i_{\ell_i}}(\lambda_i)\Bigg\}.
\end{aligned}
\tag{19.16}
$$

where, for $d = 3$, recalling (18.19)

$$
(u_k, \Phi^i_j \cdot \tau)_{L^q(\omega)} = \left(\begin{vmatrix} u_k^1 \\ u_k^2 \end{vmatrix}, \begin{vmatrix} \Phi^i_j \cdot \tau_1 \\ \Phi^i_j \cdot \tau_2 \end{vmatrix} \right)_{L^q(\omega)}. \tag{19.17}
$$

The above expansion (19.17) is across the 1st, 2nd, $\cdots$, ℓ_i-th cycle of β_i and β_i^*.

20 Proof of Theorem 17.1: Arbitrary decay rate of the w_N-dynamics (19.16) (or (18.24) under FDSA) by suitable finite-dimensional boundary tangential localized control v on $\widetilde{\Gamma}$ and interior localized tangential-like control u in feedback form as in (17.6) on ω

We are now in a position to obtain the desired control-theoretic result of Theorem 17.1, which we now restate for convenience. This will be a corollary of Theorem 19.2. Let $1 < q < \infty$.

Theorem 20.1. *Let $\lambda_1, \dots, \lambda_M$ be the unstable distinct eigenvalues of $\mathcal{A}$ as in (16.2). Let $\widetilde{\Gamma}$ be an open subset of the boundary Γ of positive surface measure and ω be a localized collar of $\widetilde{\Gamma}$ (Fig. 2). By virtue of Theorem 19.2, pick vectors $[f_1, \dots, f_K]$ in $\mathcal{F} \subset W^{2-1/q,q}(\Gamma)$, see (18.8b) and interior vectors $[u_1, \dots, u_K]$ in $L^q(\omega)$ such that the rank conditions (19.15) hold true, with $K = \sup \ell_i$, $i = 1, \dots, M$, ℓ_i = geometric multiplicity of λ_i.*

Then: Given $\gamma_1 > 0$ arbitrarily large, there exist two K-dimensional controllers: a boundary tangential control $v = v_N$ acting on $\widetilde{\Gamma}$ of the form given by (18.10) = (19.10), so that $v_N \cdot \nu|_\Gamma = 0$, and an interior tangential-like control $u = u_N$ acting on ω, of the form given by (18.19) or (19.12), such that, once inserted in (16.8) or (19.16) yield the estimate

$$\|w_N(t)\|_{L^q_\sigma(\Omega)} + \|v_N(t)\|_{L^q(\widetilde{\Gamma})} + \|u_N(t)\|_{L^q(\omega)} \leq C_{\gamma_1} e^{-\gamma_1 t} \|P_N w_0\|_{L^q_\sigma(\Omega)}, \quad t \geq 0. \tag{20.1}$$

Here, w_N is the solution of (19.16), i.e., (18.5) or (16.8) corresponding to such controls $v = v_N$ and $u = u_N$. Moreover, such controls $v = v_N$ and $u = u_N$ can be given in feedback form $\nu_k(t) = \langle w_N(t), p_k \rangle_{W_N^u}$ and $\mu_k(t) = \langle w_N(t), q_k \rangle_{W_N^u}$ for suitable (constructed) vectors p_k and q_k , so that, in conclusion, w_N in (20.1) is the solution of the equation on W_N^u (see (17.6)):

$$w_N' - \mathcal{A}_N^u w_N = -\mathcal{A}_N^u P_N D \left(\sum_{k=1}^{K} \langle w_N(t), p_k \rangle_{W_N^u} f_k \right) + P_N P_q \left(m \left(\sum_{i=1}^{K} \langle w_N(t), q_k \rangle_{W_N^u} u_k \right) \cdot \tau \right), \tag{20.2}$$

rewritten as

$$w_N' = \overline{A}^u w_N, \; w_N(t) = e^{\overline{A}^u t} P_N w_0, \; w_N(0) = P_N w_0. \tag{20.3}$$

Proof. Step 1: Following [RT.1], [RT.2], [B-T.1], [B-L-T.1] the proof consists in testing the controllability of the linear, finite-dimensional system (2.5), in short, the pair

$$\{J, B\}, \quad B = [-\nu_0 W, U]\} : \quad N \times 2K, K = \sup\{\ell_i, i = 1 \dots M\}, \tag{20.4}$$

J being the Jordan form of $\mathcal{A}_N^u$ with respect to the Jordan basis $\beta = \beta_1 \cup \beta_2 \cup \dots \cup \beta_M$, β_i being given by (19.6). But the rank conditions (19.15) precisely assert such controllability property of the pair $\{\mathcal{A}_N^u = J, B\}$, in light of Theorem 19.1.

Step 2: Having established the controllability condition for the pair $\{J = \mathcal{A}_N^u, B\}$, then by the well-known Popov's criterion in finite-dimensional theory, there exists a feedback matrix $Q : 2K \times N$, such that the spectrum of the matrix $(J + BQ) = (J + [-\nu_0 W, U]Q)$ may be arbitrarily preassigned; in particular, to lie in the left half-plane $\{\lambda : \text{ Re } \lambda < -\gamma_1 < - \text{ Re } \lambda_{N+1}\}$, as desired. The resulting closed-loop system

$$(\hat{w}_N)' - J\hat{w}_N = [-\nu_o W, U] \begin{bmatrix} v_N \\ u_N \end{bmatrix}, \tag{20.5}$$

is obtained with $[v_N, u_N] = Q\hat{w}_N$, Q being the $2K \times N$ matrix with row vectors $[\hat{p}_1, \dots, \hat{p}_K, \hat{q}_1, \dots, \hat{q}_K]$, $v_N^k = (\hat{w}_N, \hat{p}_k)$, $\mu_N^k = (\hat{w}_N, \hat{q}_k)$ in the $\mathbb{C}^N$-inner product. Thus, returning from $\mathbb{C}^N \times \mathbb{C}^N$ back to $W_N^u \times (W_N^u)^*$, there exist suitable vectors $p_1, \dots, p_K$ and $q_1, \dots, q_K$ in W_N^u, such that $v_N^k = \langle z_N, p_k \rangle$, $\mu_N^k = \langle z_N, q_k \rangle$, whereby the closed-loop system (20.2) corresponds precisely to (19.16) via $P_N(\mathcal{A}Dv)$ and $P_N P((mu)\tau)$ written in terms of the Jordan basis of generalized eigenvectors β in (19.6). □

Remark 20.1. In the easier case of Section 2 under the FDSA, checking controllability of system (18.26) is easier. To this end, we can pursue, as usual, two strategies.

A first strategy invokes the well-known Kalman controllability criterion by constructing the $N \times (2K)N$ Kalman controllability matrix

$$\mathcal{K} = [B, \Lambda B, \Lambda^2 B, \dots, \Lambda^{N-1} B] = \begin{bmatrix} B_1 & J_1 B_1, & \cdots, & J_1^{N-1} B_1 \\ B_2 & J_2 B_2, & \cdots, & J_2^{N-1} B_2 \\ \dots & \dots & \dots & \dots \\ B_M & J_M B_M, & \cdots, & J_M^{N-1} B_M \end{bmatrix}, \tag{20.6}$$

$$B = \mathrm{col}[B_1, B_2, \dots B_M], \quad B_i = [-\nu_0 W_i, U_i] : \ell_i \times 2\ell_i \tag{20.7}$$

of size $N \times (2K)N$, $N = \dim W_N^u$, $J_i = \lambda_i I_i$: $\ell_i \times \ell_i$, $B_i = [\nu_0 W_i, U_i] : \ell_i \times 2\ell_i$, and requiring that it be full rank.

$$\mathrm{rank}\, \mathcal{K} = \mathrm{full} = N. \tag{20.8}$$

In view of generalized Vandermond determinants, this is the case if and only if rank B_i = rank $[-\nu_0 W_i, U_i] = \ell_i$ (full), $i = 1, \dots, M$, as assumed.

A second strategy invokes the Hautus controllability criterion:

$$\mathrm{rank}\ [\Lambda - \lambda_i I, B] = \mathrm{rank}\ [\Lambda - \lambda_i I, [-\nu_0 W, U]] = N \ \text{(full)}, \tag{20.9}$$

for all unstable eigenvalues λ_i, $i = 1, \dots, M$, yielding again the condition that rank $[-\nu_0 W_i, U_i] = \ell_i$, $i = 1, \dots, M$.

In conclusion, with the present Section 20, Theorem 17.1 is proved.

21 Proof of Theorem 17.2: The feedback operator $\mathbb{A}_{_{F,q}}$ in (17.10) generates a s.c analytic semigroup in $L^q_\sigma(\Omega)$, $2 < q < \infty$ or in $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$, $1 < p < 2q/_{2q-1}$, $q > d$, $d = 2, 3$.

We return to the feedback operator $\mathbb{A}_{_{F,q}} = A_{F,q} + G$ in (17.11), driving the feedback w-dynamics (17.10). We also refer to (17.15), (17.16) which require $q \geq 2$ in order to have the finite dimensional feedback operators $F : L^q_\sigma(\Omega) \longrightarrow L^q(\Gamma)$ and G on $L^q_\sigma(\Omega)$ bounded. Thus in turn, is due to Appendix E, in particular Eq (C.5) which yields (18.8): $\phi^*_{ij} \in W^{3,q}(\Omega)$, $q \geq 2$.

$$\left.\frac{\partial \varphi^*_{ij}}{\partial \nu}\right|_\Gamma \in W^{2-1/q,q}(\Gamma) \subset L^q(\Gamma), \quad \text{and } \mathcal{F} \subset W^{2-1/q,q}(\Gamma)$$

Thus, it suffices to consider the operator $A_{_{F,q}}$, below in (21.1), which differs from $\mathbb{A}_{_{F,q}}$ by the bounded operator G.

Theorem 21.1. *Let F be the bounded operator $L^q_\sigma(\Omega) \longrightarrow L^q(\widetilde{\Gamma})$ in (17.13), $q \geq 2$. Then, with reference to the Oseen operator $\mathcal{A}_q$ introduced in (15.5) on $L^q_\sigma(\Omega)$ and the Dirichlet map D introduced in Section 14, with reference to problem (14.1), we have*
The operator, see (17.12)

$$A_{_{F,q}} = \mathcal{A}_q(I - DF) \;:\; L^q_\sigma(\Omega) \supset \mathcal{D}(A_{_{F,q}}) \longrightarrow L^q_\sigma(\Omega) \tag{21.1a}$$

$$\mathcal{D}(A_{_{F,q}}) = \left\{h \in L^q_\sigma(\Omega) : h - DFh \in \mathcal{D}(\mathcal{A}_q) = W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega) \cap L^q_\sigma(\Omega)\right\} \tag{21.1b}$$

generates a s.c. analytic semigroup on $L^q_\sigma(\Omega)$ and $R(\lambda, A_{_{F,q}})$ is compact in $L^q_\sigma(\Omega)$, $q \geq 2$.

Proof. (i). [B-L-T.1]. We shall critically use property (15.41) for the Dirichlet map D in the L^q-setting, $1 < q < \infty$, just as it was done in these references in the Hilbert setting; namely that, with $\varepsilon > 0$, recalling (15.41), we have:

$$D:\ \text{continuous } U_q = \left\{g \in L^q(\Omega) = W^{0,q}(\Omega),\ g \cdot \nu = 0 \text{ on } \Gamma\right\} \longrightarrow W^{1/q,q}(\Omega) \cap L^q_\sigma(\Omega) \subset \mathcal{D}\left(A_q^{1/2q - \varepsilon}\right), \tag{21.2}$$

$1 < q < \infty$, where A_q is the Stokes operator. We transfer relation (21.2) to the Oseen operator $\mathcal{A}_q$ in (15.5). To do this, we just translate it. Let $k > 0$ be suitably large, then, via (21.2)

$$\hat{\mathcal{A}}_q^{1/2q-\varepsilon} D = \left(kI - \mathcal{A}_q\right)^{1/2q - \varepsilon} D:\ \text{continuous } \left\{g \in L^q(\Gamma),\ g \cdot \nu = 0 \text{ on } \Gamma\right\} \longrightarrow L^q_\sigma(\Omega) \tag{21.3}$$

(ii) We shall next give a containment relation for $\mathcal{D}(A_{F,q})$. Since $DF \in \mathcal{L}(L^q_\sigma(\Omega)), q \geq 2$, then, recalling (21.1a), we have

$$\mathcal{D}(A_{F,q}) = \mathcal{D}\Big(\big(kI - \mathcal{A}_q\big)^{1-1/2q+\varepsilon}\Big[\big(kI - \mathcal{A}_q\big)^{1/2q-\varepsilon} - \big(kI - \mathcal{A}_q\big)^{1/2q-\varepsilon} DF\Big]\Big) \tag{21.4}$$

$$= \Big\{x \in \mathcal{D}\big(kI - \mathcal{A}_q\big)^{1/2q-\varepsilon} : \big(kI - \mathcal{A}_q\big)^{1/2q-\varepsilon} x - \big(kI - \mathcal{A}_q\big)^{1/2q-\varepsilon} DFx \in \mathcal{D}\big(kI - \mathcal{A}_q\big)^{1-1/2q+\varepsilon}\Big\} \tag{21.5}$$

$$\subset \mathcal{D}\big(kI - \mathcal{A}_q\big)^{1/2q-\varepsilon} = \mathcal{D}\big(\hat{\mathcal{A}}_q\big)^{1/2q-\varepsilon} = \mathcal{D}\big(A_q\big)^{1/2q-\varepsilon} \subset W^{1/q-4\varepsilon,q}(\Omega) \cap L^q_\sigma(\Omega) \tag{21.6}$$

where in the last step we have invoked the relationship [Wahl, p 93]

$$\mathcal{D}(A_q^\gamma) \subset W^{2s,q}(\Omega) \cap L^q_\sigma(\Omega), \quad 1 \geq \gamma > s, \; q \geq 2, \tag{21.7}$$

with $\gamma = 1/2q - \varepsilon$ and $s = 1/2q - 2\varepsilon$. Relation (21.6) says that $\mathcal{D}\big(A_{F,q}\big)$ is contained in $\mathcal{D}\big(A_q^\gamma\big)$ up to the level $\gamma = 1/2 - \varepsilon$ that does not recognize boundary conditions.

(iii) **Analyticity: first proof.** We let throughout $q \geq 2$. Finally we establish by classical perturbation theory based on the resolvent $R\big(\lambda, A_{F,q}\big)$ of $A_{F,q}$ and property (21.3) that, for $F \in \mathcal{L}\big(L^q_\sigma(\Omega), L^q(\Gamma)\big)$, the operator $A_{F,q}$ generates a s.c. analytic semigroup $e^{A_{F,q}t}$ on $L^q_\sigma(\Omega)$, $t > 0$; and moreover that $R\big(\lambda, A_{F,q}\big)$ is compact on $L^q_\sigma(\Omega)$. Both statements rely on the classical perturbation formula [Pazy] written for $A_{F,q}$ in (21.1).

$$R\big(\lambda, A_{F,q}\big) = [I + R(\lambda, \mathcal{A}_q)\mathcal{A}_q DF]^{-1} R(\lambda, \mathcal{A}_q) \tag{21.8}$$

where by property (21.3) $\hat{\mathcal{A}}_q^{1/2q-\varepsilon} DF \in \mathcal{L}(L^q_\sigma(\Omega))$. Moreover, since $\mathcal{A}_q$ generates a s.c. analytic semigroup in $L^q_\sigma(\Omega)$ (Theorem 3.1.ii), a well-known formula [Pazy] gives for $\varepsilon > 0, \; \theta = 1 - 1/2q - \varepsilon$

$$\Big\| R(\lambda, \hat{\mathcal{A}}_q)\hat{\mathcal{A}}_q^\theta \Big\|_{\mathcal{L}(L^q_\sigma(\Omega))} \leq \frac{C}{|\lambda|^{1-\theta}} = \frac{C}{|\lambda|^{1/2q+\varepsilon}} \to 0 \text{ as } |\lambda| \to \infty. \tag{21.9}$$

Then, (21.9) in (21.8) yields

$$\big\| R\big(\lambda, A_{F,q}\big) \big\|_{\mathcal{L}(L^q_\sigma(\Omega))} \leq C_{\rho_o} \left\| R(\lambda, \mathcal{A}_q) \right\|_{\mathcal{L}(L^q_\sigma(\Omega))}, \quad \forall \; \lambda, \; |\lambda| \geq \text{ some } \rho_o > 0 \tag{21.10}$$

and hence, via (21.10) the properties of $R(\lambda, \mathcal{A}_q)$ of Theorem 15.1 [generation of s.c. analytic semigroup on $L^q_\sigma(\Omega)$ and, respectively, compactness] transfer into corresponding properties for $R(\lambda, A_{F,q})$.

Theorem 21.1 is proved. $\square$

(iv) **Analyticity: second proof.** One may provide a second proof that $A_{F,q}$ generates a s.c. analytic semigroup on $L^q_\sigma(\Omega), q \geq 2$. This is still a perturbation argument, however perturbation of an original analytic generator, not of the resolvent. In fact, the present perturbation argument applies to the adjoint operator $A^*_{F,q}$ on $L^{q'}_\sigma(\Omega)$, not to $A_{F,q}$ on $L^q_\sigma(\Omega), 1 < q' \leq 2, 2 \leq q$. Eqts (23.13), (23.14) in the argument below of Proposition 23.2 dealing with $\mathbb{A}_{F,q} = A_{F,q} + G$ show that $A^*_{F,q}$ can be written as $A^*_{F,q} = -A^*_q + \Pi$, where the perturbation Π is $\big(A^*_q\big)^\theta$-bounded, with $\theta = 1 - {}^1\!/_{2q} + \varepsilon < 1$, see (23.17). Thus, since $-A^*_q$ generates s.c. analytic semigroup on $L^{q'}_\sigma(\Omega)$ (by adjointness on Theorem 3.1(i) on $-A_q$), then a standard semigroup result [Pazy] implies that the perturbed operator $A^*_{F,q}$ is an analytic semigroup generator on $L^{q'}_\sigma(\Omega), 1 < q' \leq 2$. But this is equivalent ($L^q_\sigma(\Omega)$ being reflexive, $1 < q < \infty$) to the original operator $A_{F,q}$ begin an analytic semigroup generator on $L^q_\sigma(\Omega), q \geq 2$, as desired. The quoted Proposition 23.2 shows more. In fact: that is, that $A_{F,q}$ (actually $\mathbb{A}_{F,q} = A_{F,q} + G$) has L^p-maximal regularity on $L^q_\sigma(\Omega)$, in symbols, $A_{F,q} \in MReg\big(L^p(0,\infty; L^q_\sigma(\Omega))\big), q \geq 2$ (23.18). And maximal regularity implies analyticity [Dore], The argument of Proposition 23.2 is of the perturbation type described above, however tuned to the notion of maximal regularity, which is stronger than analyticity. $\square$

The desired result follows next for the operator $\mathbb{A}_{F,q}$ in (17.11), describing the evolution of the w-linearized dynamics (17.10) under feedback control, see (17.13)

$$F(\cdot) = \sum_{k=1}^{K} \big\langle P_N \cdot, p_k \big\rangle_{W^u_N} f_k \in W^{2-{}^1\!/_q,q}(\widetilde{\Gamma}); \quad G(\cdot) = P_q\bigg(m\bigg(\sum_{k=1}^{K} \big\langle P_N \cdot, q_k \big\rangle_{W^u_N} u_k\bigg)\tau\bigg) \in L^q_\sigma(\Omega), \ q \geq 2 \tag{21.11}$$

both bounded: $F \in \mathcal{L}(L^q_\sigma(\Omega), L^q(\widetilde{\Gamma}))$, $G \in \mathcal{L}(L^q_\sigma(\Omega))$. In going from (21.12a) to (21.12c) below, we recall that $f_k \in W^{2-{}^1\!/_q,q}(\Gamma)$, so that $DFh \in W^{2,q}(\Omega) \cap L^q_\sigma(\Omega)$, for $h \in L^q_\sigma(\Omega)$, $q \geq 2$, by Corollary C.2(v) in Appendix E.

Corollary 21.2. Let $q \geq 2$. With reference to the feedback operator $\mathbb{A}_{F,q}$ in (17.11) describing the feedback w-system in (17.10), repeated here as

$$\mathbb{A}_{F,q} = \mathcal{A}_q(I - DF) + G = A_{F,q} + G : L^q_\sigma(\Omega) \supset \mathcal{D}\big(\mathbb{A}_{F,q}\big) \longrightarrow L^q_\sigma(\Omega) \tag{21.12a}$$

$$\mathcal{D}(\mathbb{A}_{F,q}) = \mathcal{D}(A_{F,q}) \overset{(21.1b)}{=} \Big\{ h \in L^q_\sigma(\Omega): \; h - DFh \in \mathcal{D}(\mathcal{A}_q)$$
$$\equiv \mathcal{D}(A_q) \equiv W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega) \cap L^q_\sigma(\Omega) \Big\} \tag{21.12b}$$
$$= \Big\{ \varphi \in W^{2,q}(\Omega) \cap L^q_\sigma(\Omega): \; \varphi\big|_\Gamma = F\varphi \Big\} \tag{21.12c}$$

with F and G as in (21.11), we have:

(i) $\mathbb{A}_{F,q}$ generates a s.c. analytic semigroup $e^{\mathbb{A}_{F,q}t}$ on $L^q_\sigma(\Omega), \; t > 0, \; q \geq 2$;

(ii) $\mathbb{A}_{F,q}$ has a compact resolvent on $L^q_\sigma(\Omega), q \geq 2$. □

We can next extend Corollary 21.2 to the Besov space $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$ in (13.13) $1 < p < 2q/_{2q-1}, \; q \geq 2$, of interest.

To this end, we need the following result.

Proposition 21.3. *Let* $1 < p < \dfrac{2q}{2q-1}, q \geq 2$. *Then (recall (15.3b))*

$$\big(L^q_\sigma(\Omega), \mathcal{D}(\mathbb{A}_{F,q})\big)_{1-\frac{1}{p},p} = \widetilde{B}^{2-2/p}_{q,p}(\Omega) \tag{21.13}$$
$$= \Big\{ g \in B^{2-2/p}_{q,p}(\Omega): \; div \; g \equiv 0, \; g \cdot \nu|_\Gamma = 0 \Big\}. \tag{21.14}$$

Proof. Step 1: From the characterization of $\mathcal{D}(\mathbb{A}_{F,q})$ in (21.12b) we obtain for $0 < \theta < 1, \; p > 1, \; q \geq 2$

$$\big(L^q_\sigma(\Omega), \mathcal{D}(\mathbb{A}_{F,q})\big)_{\theta,p} \subset \big(L^q_\sigma(\Omega), W^{2,q}(\Omega) \cap L^q_\sigma(\Omega)\big)_{\theta,p} = B^{2\theta}_{q,p}(\Omega) \cap L^q_\sigma(\Omega) \tag{21.15}$$

recalling the definition/characterization (15.1) of $B^s_{q,p}(\Omega)$ with $m = 2, \; s/_2 = \theta$. Next we take $1 < p < 2q/_{2q-1}, \; q \geq 2, \; \theta = 1 - 1/_p$, so that - for these parameters - (21.15) specializes to

$$\big(L^q_\sigma(\Omega), \mathcal{D}(\mathbb{A}_{F,q})\big)_{1-\frac{1}{p},p} \subset \widetilde{B}^{2-2/p}_{q,p}(\Omega) = \text{ defined in (21.14)} = \text{(15.3b)} \tag{21.16}$$

Step 2: But $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$ does not recognize boundary conditions [the conditions div $g = 0, \; g \cdot \nu|_\Gamma = 0$ are included in the definition of the underlying space $L^q_\sigma(\Omega)$, see (13.4)].

Hence, the space in the LHS of (21.16) does not recognize boundary conditions. Thus, for the indexes $\big\{\theta = 1 - 1/_p, p\big\}$, with $1 < p < 2q/_{2q-1}, q \geq 2$, we have recalling (15.3b)

$$\big(L^q_\sigma(\Omega), \mathcal{D}(\mathbb{A}_{F,q})\big)_{1-\frac{1}{p},p} = \big(L^q_\sigma(\Omega), \mathcal{D}(A_q)\big)_{1-\frac{1}{p},p} = \widetilde{B}^{2-2/p}_{q,p}(\Omega) \tag{21.17}$$

as $\mathcal{D}(\mathbb{A}_{F,q})$ and $\mathcal{D}(A_q)$ both consist of $W^{2,q}(\Omega) \cap L^q_\sigma(\Omega)$ functions, subject only to different boundary conditions. Thus (21.17) proves the desired conclusion (21.13), (21.14). □

Theorem 21.4. *The operator $\mathbb{A}_{F,q}$ in (21.12), where the bounded operators F and G are defined by (21.11), generates a s.c. analytic semigroup $e^{\mathbb{A}_{F,q}t}$ on the Besov space $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$, $1 < p < 2q/_{2q-1}$, $q \geq 2$ defined in (21.14).*

Proof. The operator $\mathbb{A}_{F,q}$ generates a s.c. analytic semigroup $e^{\mathbb{A}_{F,q}t}$ on $L^q_\sigma(\Omega)$ by Corollary 21.2 for $q \geq 2$. Then, it generates a s.c. analytic semigroup on $\mathcal{D}(\mathbb{A}_{F,q})$. Hence the conclusion follows by (21.13). □

22 Proof of Theorem 17.2: The feedback operator $\mathbb{A}_{_{F,q}}$ in (17.11) is uniformly stable on $L^q_\sigma(\Omega)$, $2 \le q < \infty$; or on $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$, $1 < p < 2q/_{2q-1}$, $q \ge 2$: Feedback stabilization of the linearized w-system (16.1) by suitable finite-dimensional localized boundary tangential control v and interior localized tangential-like control u

The feedback analytic generator $\mathbb{A}_{_{F,q}}$ in (17.11) is, moreover, uniformly stable. We restate for convenience the feedback stabilization part of Theorem 17.2.

Theorem 22.1. *Consider the setting of Theorem 20.1, so that, in particular, the feedback finite-dimensional control pair acting in Eqn (14.26) and resulting in Eqn (17.10), (17.13) is given by*

$$v = v_N \; = \; \sum_{k=1}^{K} \langle w_N(t), p_k \rangle_{W^u_N} f_k, \quad f_k \in \mathcal{F} \subset W^{2-\frac{1}{q},q}(\Gamma), \; q \ge 2, \; f_k \cdot \nu|_\Gamma = 0; \tag{22.1a}$$

$$u = u_N \; = \; \sum_{k=1}^{K} \langle w_N(t), q_k \rangle_{W^u_N} u_k, \tag{22.1b}$$

and satisfies estimate (17.4) = (20.1). The vectors $p_k, q_k \in (W^u_N)^* \subset L^{q'}_\sigma(\Omega)$ and $f_k \in W^{1-1/q,q}(\Gamma)$, see (17.1), are constructed in the proof of Theorem 17.1. This results on the feedback w-problem (17.10). Then, with $\gamma_0 < \gamma_1$, where γ_1 was picked up in Theorem 20.1, the corresponding feedback solution w of (14.26) satisfies the (uniform stabilization) estimate

$$\left\| A^\theta_q e^{\mathbb{A}_{F,q} t} w_0 \right\| = \left\| A^\theta_q w(t) \right\|_{L^q_\sigma(\Omega)} \le C_{\gamma_0,\delta,\theta} e^{-\gamma_0 t} \left\| y_0 \right\|_{L^q_\sigma(\Omega)}, \quad t \ge \delta > 0, \; 0 \le \theta < \frac{1}{4}, \; q \ge 2, \tag{22.2}$$

$\delta > 0$ arbitrary, where we can take $\delta = 0$ for $\theta = 0$.

Proof. Step 1: According to Theorem 20.1, the finite-dimensional system w^u_N in (16.8) can be uniformly feedback stabilized with an arbitrarily preassigned decay rate $\gamma_1 > 0$ (in particular, $-\gamma_1 <$ Re $\lambda_{N+1} \le 0$, see (16.2)) by a pair of finite-dimensional feedback controllers $\{v = v_N, u = u_N\}$ as to obtain the feedback system (20.2) (rewritten coordinate-wise as in (19.16)), as quantified by inequality (20.1), (17.4):

$$\left\| w_N(t) \right\|_{L^q_\sigma(\Omega)} + \left\| v_N(t) \right\|_{L^q(\widetilde{\Gamma})} + \left\| u_N(t) \right\|_{L^q(\omega)} \le C_{\gamma_1} e^{-\gamma_1 t} \left\| P_N w_0 \right\|_{L^q_\sigma(\Omega)}, t \ge 0, \; q \ge 2 \tag{22.3}$$

((17.4) includes also v'_N and u'_N.) Here, as in (20.2), (and (10.1a-b))

$$v_N(t) = \sum_{k=1}^{K} (w_N(t), p_k))_{W_N^u} f_k, \quad u_N(t) = \sum_{k=1}^{K} (w_N(t), q_k)_{W_N^u} u_k, \tag{22.4}$$

are the tangential boundary feedback control, $v_N \cdot \nu|_\Gamma \equiv 0$, and the interior tangential-like control $u_N(t) \cdot \tau$; the first acting on the arbitrary sub-portion $\tilde{\Gamma}$ of Γ of positive measure, the second acting on the corresponding collar ω based on $\tilde{\Gamma}$ (Fig. 2).

Step 2: Next, we examine the impact of such constructive feedback control pair $\{v_N, u_N \cdot \tau\}$ on the ζ_N-dynamics (16.9), whose explicit solution is given by the variation of parameter formula

$$\zeta_N(t) = e^{\mathcal{A}_N^s t} \zeta_N(0) + (I_{\text{int}})(t) + (I_{\text{bry}})(t); \tag{22.5}$$

$$\left\| e^{\mathcal{A}_N^s t} \right\|_{\mathcal{L}(L^q_\sigma(\Omega))} \leq C_{\gamma_0} e^{-\gamma_0 t}, \ 0 \leq t, \ 0 < \gamma_0 < |\text{Re } \lambda_{N+1}|; \tag{22.6}$$

$$\big(I_{\text{int}}\big)(t) = -\int_0^t e^{\mathcal{A}_N^s (t-r)} (I - P_N) P(m(u_N(r) \cdot \tau(r)) dr; \tag{22.7}$$

$$\big(I_{\text{bry}}\big)(t) = -\int_0^t e^{\mathcal{A}_N^s (t-r)} \mathcal{A}_N^s (I - P_N) D v_N(r) dr; \tag{22.8}$$

Here, I_{int} is the integral term driven by the interior control u_N, while I_{bry} is the integral term driven by the tangential boundary control v_N.

We now recall from Section 3(b) and 3(d) that the Oseen operator $\mathcal{A}_q$ generates a s.c. analytic semigroup not only on $L^q_\sigma(\Omega)$ but also on $\big(L^q_\sigma(\Omega), \mathcal{D}(A_q)\big)_{1-\frac{1}{p},p}$, in particular on $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$. So we can estimate (22.5) in the norm of either of these spaces. Furthermore, the (point) spectrum of the generator $\mathcal{A}_N^s$ on W_N^s satisfies $\sup\{Re\ \sigma(\mathcal{A}_N^s)\} < -|\lambda_{N+1}| < -\gamma_0$ by assumption. We shall carry our the subsequent computations explicitly in the space $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$ for the case of greatest interest in the nonlinear analysis of sections 24, 25.

Step 3. The Interior-Driven Integral Term $\big(I_{\text{int}}\big)(t)$: We consider first the term $e^{\mathcal{A}_N^s t} \zeta_N(0) + (I_{\text{int}})(t)$ in (22.5) and provide estimates in the $L^q_\sigma(\Omega)$-norm first, $q \geq 2$. Selecting as we may, in view of Theorem 20.1, $\gamma_1 > \gamma_0$, we obtain the estimate

$$\left\| e^{\mathcal{A}_N^s t} \zeta_N(0) + (I_{\text{int}})(t) \right\|_{L^q_\sigma(\Omega)} \leq C_{\gamma_0} e^{-\gamma_0 t} \left\| w_0 \right\|_{L^q_\sigma(\Omega)}, \quad \forall\ t \geq 0, \ q \geq 2; \tag{22.9}$$

$$\left\| A_q^\theta \zeta_N(t) \right\|_{L^q_\sigma(\Omega)} \leq \begin{cases} C_{\gamma_0,\delta,\theta} e^{-\gamma_0(t-\delta)} \left\| w_0 \right\|_{L^q_\sigma(\Omega)}, \quad \forall\, t \geq \delta > 0;\ q \geq 2; & \text{(22.10a)} \\ C_{\gamma_0,\theta} e^{-\gamma_0 t} \left\| A_q^\theta w_0 \right\|_{L^q_\sigma(\Omega)}, \quad \forall\, t \geq 0,\ w_0 \in \mathcal{D}\Big(A_q^\theta\Big),\ q \geq 2. & \text{(22.10b)} \end{cases}$$

$0 < \gamma_0 < |\text{Re } \lambda_{N+1}|$, $0 < \theta < 1$. In fact invoking the estimate (22.3) of (20.1) for u_N, we obtain in the in the $L^q_\sigma(\Omega)$-norm since the operators $(I - P_N), P$ are bounded:

$$\left\| e^{\mathcal{A}_N^s t} \zeta_N(0) + I_{\text{int}}(t) \right\|_{L^q_\sigma(\Omega)} \leq \left\| e^{\mathcal{A}_N^s t} \zeta_N(0) \right\|_{L^q_\sigma(\Omega)} + C \int_0^t \left\| e^{\mathcal{A}_N^s (t-\tau)} \right\| \left\| u_N(\tau) \right\|_{L^q_\sigma(\Omega)} d\tau \tag{22.11}$$

$$\leq C e^{-\gamma_0 t} \left\| \zeta_N(0) \right\|_{L^q_\sigma(\Omega)} + C \int_0^t e^{-\gamma_0 (t-r)} e^{-\gamma r} dr \left\| P_N w_0 \right\|_{L^q_\sigma(\Omega)} \tag{22.12}$$

Since we may choose $\gamma_1 > \gamma_0$ by Theorem 17.1(or by Theorem 20.1), we may obtain

$$\left\| e^{\mathcal{A}_N^s t} \zeta_N(0) + I_{\text{int}}(t) \right\|_{L^q_\sigma(\Omega)} \leq C \left[e^{-\gamma_0 t} + e^{-\gamma_0 t} \frac{1 - e^{-(\gamma - \gamma_0)t}}{\gamma - \gamma_0} \right] \left\| w_0 \right\|_{L^q_\sigma(\Omega)} \tag{22.13}$$

$$\leq C e^{-\gamma_0 t} \left\| w_0 \right\|_{L^q_\sigma(\Omega)}, \quad \forall t > 0,\ q \geq 2. \tag{22.14}$$

Step 4. The Interior-Driven Integral Term $I_{\text{bry}}(t)$: We shall follow the computations given in [B-L-T.1, Prop. B.2.1 Eqn. (B.2.5) and its proof, p. 105] for the boundary-driven integral $I_{\text{bry}}(t)$ in (22.8), with $v = v_N$ given by (17.1) and obeying estimate (17.4) for v_N. We shall obtain

$$\left\| A_q^\theta (I_{\text{bry}})(t) \right\|_{L^q_\sigma(\Omega)} \leq C_{\gamma_0,\delta,\theta} e^{-\gamma_0 t} \left\| w_0 \right\|_{L^q_\sigma(\Omega)},\ t \geq \delta > 0,\ 0 \leq \theta < \frac{1}{4},\ g \geq 2, \tag{22.15}$$

where we can take $\delta = 0$ for $\theta = 0$. In fact, from (22.8) we obtain

$$\big(-\mathcal{A}_N^s \big)^\theta I_{\text{bry}}(t) = \int_0^t \big(-\mathcal{A}_N^s \big)^{1 - 1/2q + \varepsilon + \theta} e^{\mathcal{A}_N^s (t-\tau)} \big(-\mathcal{A}_N^s \big)^{1/2q - \varepsilon} (I - P_N) D v_N(\tau) d\tau \tag{22.16}$$

where we choose $\theta + 1 - {}^1/_{2q} + \varepsilon = 1 - \varepsilon',\ \varepsilon > 0,\ \varepsilon' > 0.$

But, by Theorem 15.5, $\mathcal{A}_N^s$ generates a s.c. analytic semigroup on W_N^s, which therefore satisfies the spectrum determined growth condition. Thus, choose $\varepsilon_1 > 0$ s.t. $\gamma_0 + \varepsilon_1 < |\text{Re } \lambda_{N+1}|$, see (16.2) and then $\left\| e^{\mathcal{A}_N^s} t \right\| \leq C e^{-(\gamma_0 + \varepsilon_1)t},\ t \geq 0$ in the $L^q_\sigma(\Omega)$-norm. We then compute form (22.8):

$$\left\|\big(-\mathcal{A}_N^s\big)^\theta I_{\rm bry}\right\|_{L^q_\sigma(\Omega)} = C\int_0^t \frac{e^{-(\gamma_0+\varepsilon_1)(t-\tau)}}{(t-\tau)^{1-\varepsilon'}}\,\|v_N(t)\|_{L^q_\sigma(\Omega)}\;d\tau \tag{22.17}$$

$$\text{(by (22.3))} \;\le \int_0^t \frac{e^{-(\gamma_0+\varepsilon_1)(t-\tau)}}{(t-\tau)^{1-\varepsilon'}} e^{-(\gamma_1+\varepsilon_1)\tau}\;d\tau\,\|P_N w_0\|_{L^q_\sigma(\Omega)} \tag{22.18}$$

$$\le Ce^{-(\gamma_0+\varepsilon_1)t}\int_0^t \frac{e^{-(\gamma_1-\gamma_0)\tau}}{(t-\tau)^{1-\varepsilon'}}\;d\tau\,\|P_N w_0\|_{L^q_\sigma(\Omega)} \tag{22.19}$$

$$\le -Ce^{-(\gamma_0+\varepsilon_1)t}\frac{(t-\tau)^{\varepsilon'}}{\varepsilon'}\bigg|_{\tau=0}^{\tau=1}\|P_N w_0\|_{L^q_\sigma(\Omega)} \tag{22.20}$$

$$\le \frac{C}{\varepsilon'}t^{\varepsilon'}e^{-(\gamma_0+\varepsilon_1)t}\,\|P_N w_0\|_{L^q_\sigma(\Omega)} \tag{22.21}$$

$$\le C_{\varepsilon'}e^{-\gamma_0 t}\,\|P_N w_0\|_{L^q_\sigma(\Omega)}\,,\quad t\ge 0,\;\; q\ge 2. \tag{22.22}$$

In (22.16) we have used (22.3) with γ_1 replaced by $\gamma_1+\varepsilon_1$, as this is an arbitrarily large preassigned number. But the domain of the $\big(-\mathcal{A}_N^s\big)^\theta$-powers coincide in norm with the domain of the A_q^θ-powers [as $\mathcal{D}(A_q)=\mathcal{D}(\mathcal{A}_q)$]. Thus (22.10) yields

$$\left\|\mathcal{A}_q^\theta\; I_{\rm bry}\right\|_{L^q_\sigma(\Omega)} = \widetilde{C}e^{-\gamma_0 t}\,\|P_N w_0\|_{L^q_\sigma(\Omega)}\,,\quad t\ge 0 \tag{22.23}$$

Step 5: Combining 22.9 with (22.23) in (22.5) yields Theorem 22.1 □

We conclude this section with the counterpart of the Theorem 22.1 on the space $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$, a result to be needed in Section 23, Eq (23.44).

Theorem 22.2. *Under the same setting of Theorem 22.1, in particular for* $q\ge 2,\; 1<p<{}^{2q}/_{2q-1}$, *we have*

$$\left\|e^{\mathbb{A}_{F,q}t}w_0\right\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \le Ce^{-\gamma_0 t}\,\|w_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)}\,,\; t\ge 0 \tag{22.24}$$

Proof. We recall (15.39) for $e^{\mathcal{A}_q t}$, which restricted on the stable subspace W_N^s gives

$$\begin{aligned} &e^{\mathcal{A}_{N,q}^s t}:\;\text{continuous } \widetilde{B}^{2-2/p}_{q,p}(\Omega)\longrightarrow X^\infty_{p,q,\sigma}\\ &\left\|e^{\mathcal{A}_{N,q}^s t}\right\|_{\mathcal{L}\left(\widetilde{B}^{2-2/p}_{q,p}(\Omega)\right)} \le Ce^{-\gamma_0 t},\; t\ge 0 \end{aligned} \tag{22.25}$$

counterpart of (22.6). We now repeat the proof of Theorem 22.1, except on the space $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$ rather than $L^q_\sigma(\Omega)$., by using (22.25) instead of (22.6). We obtain:

(i)

$$\left\| e^{\mathcal{A}_N^s t} \zeta_N(0) + I_{\text{int}}(t) \right\|_{\widetilde{B}_{q,p}^{2-2/p}(\Omega)} \leq C e^{-\gamma_0 t} \left\| w_0 \right\|_{\widetilde{B}_{q,p}^{2-2/p}(\Omega)}, \quad \forall t > 0, \; q \geq 2. \tag{22.26}$$

counterpart of (22.14);

(ii)

$$\left\| I_{\text{bry}} \right\|_{\widetilde{B}_{q,p}^{2-2/p}(\Omega)} = \widetilde{C} e^{-\gamma_0 t} \left\| w_0 \right\|_{\widetilde{B}_{q,p}^{2-2/p}(\Omega)}, \quad t \geq 0 \tag{22.27}$$

counterpart of (22.23)

Combining (22.26) and (22.27) in (22.5) yields (22.24). □

23 Maximal L^p-regularity on $L^q_\sigma(\Omega)$, $q \geq 2$ and up to $T = \infty$ of the s.c. analytic semigroup $e^{\mathbb{A}_{F,q}t}$ yielding uniform decay of the linearized w-problem (17.10) of Theorem 17.2

Preliminaries

1. We recall the tangential boundary feedback operator $F \in \mathcal{L}\big(L^q_\sigma(\Omega), L^q(\widetilde{\Gamma})\big)$, for $q \geq 2$ and the interior tangential-like feedback operator $G \in \mathcal{L}\big(L^q_\sigma(\Omega)\big)$ from (21.11)

$$F(\cdot) = \sum_{k=1}^{K} \big\langle P_N \ \cdot, p_k \big\rangle_{W^u_N} f_k \in W^{2-1/q,q}(\widetilde{\Gamma}),\ q \geq 2;$$

$$G(\cdot) = P_q\bigg(m\bigg(\sum_{k=1}^{K} \big\langle P_N \ \cdot, q_k \big\rangle_{W^u_N} u_k\bigg)\tau\bigg) \in L^q_\sigma(\Omega),\ q \geq 2 \quad (23.1)$$

so that we rewrite the feedback w-equation (17.10) as

$$\frac{dw}{dt} = \mathcal{A}_q(I - DF)w + Gw = \mathbb{A}_{F,q} w \quad (23.2a)$$

$$\begin{cases} \mathbb{A}_{F,q} : L^q_\sigma(\Omega) \supset \mathcal{D}\big(\mathbb{A}_{F,q}\big) \longrightarrow L^q_\sigma(\Omega), \quad q \geq 2 \\ \mathcal{D}\big(\mathbb{A}_{F,q}\big) = \big\{h \in L^q_\sigma(\Omega) : (h - DFh) \in \mathcal{D}(\mathcal{A}_q) = \mathcal{D}(A_q)\big\}, \end{cases} \quad (23.2b)$$

since G is a bounded operator $G \in \mathcal{L}\big(L^q_\sigma(\Omega)\big)$. Recall from (13.23), or (17.1), or (18.8), or Appendix E that the boundary vectors f_k (= linear combinations of normal traces of eigenfunctions of $\mathcal{A}^* = \mathcal{A}^*_q$ in (18.1)) have the regularity $f_k \in W^{2-1/q,q}(\Gamma)$, so that $DFh \in W^{2,q}(\Omega) \cap L^q_\sigma(\Omega)$ for $h \in L^q_\sigma(\Omega)$, $q \geq 2$, see below (21.11), in light of Corollary C.2(v) in Appendix E. Thus, we can more specifically describe $\mathcal{D}(\mathbb{A}_{F,q})$ as follows.

$$\begin{cases} \mathbb{A}_{F,q} : L^q_\sigma(\Omega) \supset \mathcal{D}\big(\mathbb{A}_{F,q}\big) \longrightarrow L^q_\sigma(\Omega), \\ \mathcal{D}\big(\mathbb{A}_{F,q}\big) = \big\{\varphi \in W^{2,q}(\Omega) \cap L^q_\sigma(\Omega) : \varphi|_\Gamma = F\varphi\big\},\ q \geq 2 \end{cases} \quad (23.2c)$$

see (21.12c). Such characterization of $\mathcal{D}\big(\mathbb{A}_{F,q}\big)$ will be critical in using maximal L^p regularity of $\mathbb{A}_{F,q}$ in the analysis of the non-linear problem in Sections 24 and 25.

We also recall that $e^{\mathbb{A}_{F,q}t}$ is a s.c. analytic semigroup in $L^q_\sigma(\Omega)$, which moreover is uniformly stable here (Theorem 17.2, Eq (17.14) for $\mathbb{A}_F = \mathbb{A}_{F,q}$) as well as Theorem 22.1, Eq (22.2):

$$\left\| e^{\mathbb{A}_{F,q} t} \right\|_{\mathcal{L}\left(L^q_\sigma(\Omega)\right)} \leq C_{\gamma_0} e^{-\gamma_0 t}, \quad t \geq 0, \ q \geq 2. \tag{23.3}$$

2. We consider the system

$$\frac{d\eta}{dt} = \mathbb{A}_{F,q} \eta + f; \quad \eta(0) = \eta_0 \text{ in } L^q_\sigma(\Omega), \ q \geq 2 \tag{23.4}$$

$$\eta(t) = e^{\mathbb{A}_{F,q} t} \eta_0 + \int_0^t e^{\mathbb{A}_{F,q}(t-\tau)} f(\tau) \, d\tau. \tag{23.5}$$

Goal: The goal of the present section is to establish maximal L^p regularity on $L^q_\sigma(\Omega)$ and for $T = \infty$ of the feedback analytic generator $\mathbb{A}_{F,q}$ in (23.2), as described in the following result.

Theorem 23.1. *Let $q \geq 2$. With reference to the dynamics (23.4), (23.5) with $\eta_o = 0$, we have: the map*

$$f \longrightarrow \eta(t) = \int_0^t e^{\mathbb{A}_{F,q}(t-\tau)} f(\tau) \, d\tau : \textit{continuous} \tag{23.6a}$$

$$L^p(0, \infty; L^q_\sigma(\Omega)) \longrightarrow L^p\big(0, \infty; \mathcal{D}(\mathbb{A}_{F,q})\big), \ 1 < p < \infty, \tag{23.6b}$$

$$L^p(0, \infty; L^q_\sigma(\Omega)) \longrightarrow X^\infty_{p,q,\sigma} \equiv L^p\big(0, \infty; \mathcal{D}(\mathbb{A}_{F,q})\big) \cap W^{1,p}(0, \infty; L^q_\sigma(\Omega)), \tag{23.6c}$$

by (23.2c), so that, there exists a constant $C = C_{p,q} > 0$ such that

$$\|\eta_t\|_{L^p\left(0,\infty;L^q_\sigma(\Omega)\right)} + \left\|\mathbb{A}_{F,q}\eta\right\|_{L^p\left(0,\infty;L^q_\sigma(\Omega)\right)} \leq C \|f\|_{L^p\left(0,\infty;L^q_\sigma(\Omega)\right)}. \tag{23.7a}$$

In short:

$$\mathbb{A}_{F,q} \in MReg(L^p\big(0, \infty; L^q_\sigma(\Omega)\big)) \tag{23.7b}$$

If we introduce the space of maximal regularity for $\{\eta, \eta_t\}$, with $\eta_0 = 0$, as

$$X^\infty_{p,q,\sigma} \equiv L^p\big(0, \infty; \mathcal{D}(\mathbb{A}_{F,q})\big) \cap W^{1,p}(0, \infty; L^q_\sigma(\Omega)) \tag{23.8a}$$

$$\subset X^\infty_{p,q} \equiv L^p(0, \infty; W^{2,q}(\Omega)) \cap W^{1,p}(0, \infty; L^q(\Omega)), \tag{23.8b}$$

we rewrite (23.7) as

$$f \in L^p\big(0, \infty; L^q_\sigma(\Omega)\big) \longrightarrow \eta \in X^\infty_{p,q,\sigma} \hookrightarrow C\big([0, \infty]; B^{2-2/p}_{q,p}(\Omega)\big) \tag{23.8c}$$

where to justify the continuous embedding in (23.8c), we recall [Amann.2, Theorem 4.10.2; p180] and the characterization (23.2c) for $\mathcal{D}(\mathbb{A}_{F,q})$

Proof. Step 1: Because of the intrinsic presence of the operator DF (boundary feedback F followed by the Dirichlet map D) as a right factor in

$$\mathbb{A}_{F,q} = \mathcal{A}_q(I - DF) + G = (-\nu_o A_q - A_{o,q})(I - DF) + G \tag{23.9a}$$

$$: L^q_\sigma(\Omega) \supset \mathcal{D}(\mathbb{A}_{F,q}) \text{ in (23.2b)} \longrightarrow L^q_\sigma(\Omega),\ 2 \le q < \infty \tag{23.9b}$$

recall (15.5), we find it is necessary to consider instead the more amenable adjoint/dual operator (with $\nu_o = 1$ w.l.o.g)

$$\mathbb{A}^*_{F,q} = (I - DF)^* \mathcal{A}^*_q + G^* = -(I - DF)^* A^*_q - (I - DF)^* A^*_{o,q} + G^* \tag{23.10a}$$

$$\mathcal{D}\big(\mathbb{A}^*_{F,q}\big) = \mathcal{D}\big(\mathcal{A}^*_q\big) = \mathcal{D}\big(A^*_q\big) = \big\{h \in W^{2,q'}(\Omega) \cap W^{1,q'}_0(\Omega) \cap L^{q'}_\sigma(\Omega)\big\} \tag{23.10b}$$

$$\mathbb{A}^*_{F,q} : L^{q'}_\sigma(\Omega) \supset \mathcal{D}\big(\mathbb{A}^*_{F,q}\big) \text{ in (23.10b)} \longrightarrow L^{q'}_\sigma(\Omega),\ 1 < q' \le 2 \tag{23.10c}$$

Here ${}^1/_q + {}^1/_{q'} = 1$, where $q \ge 2$ for $\mathbb{A}_{F,q}$. In order to have F bounded $L^q_\sigma(\Omega) \longrightarrow L^q(\Gamma)$, we need to impose $1 < q' \le 2$, in which case $(I - DF)^* \in \mathcal{L}(L^{q'}_\sigma(\Omega)),\ 1 < q' \le 2$, see Appendix E, Eqt (C.11). We rewrite $\mathbb{A}^*_{F,q}$ in (23.10a) as

$$\mathbb{A}^*_{F,q} = -A^*_q + \big[F^* D^* A_q^{*\,{}^1/_{2q}-\varepsilon}\big] A_q^{*\,1-{}^1/_{2q}+\varepsilon} - \big[(I - DF)^* \big(A_q^{-{}^1/_2} A_{o,q}\big)^*\big] A_q^{*\,{}^1/_2} + G^* \tag{23.11}$$

$$: L^{q'}_\sigma(\Omega) \supset \mathcal{D}(\mathbb{A}^*_{F,q}) \longrightarrow L^{q'}_\sigma(\Omega), \quad \frac{1}{q} + \frac{1}{q'} = 1,\ q \ge 2,\ 1 < q' \le 2 \tag{23.12}$$

whereby the adjoint of the right factor becomes now a left factor. In obtaining in (23.10a) the form of $\mathbb{A}^*_{F,q}$ from that of $\mathbb{A}_{F,q}$ in (23.9a), we have used that $(I - DF) \in \mathcal{L}(L^q_\sigma(\Omega))$ $q \ge 2$ [Fat.1, p 14]. Moreover, to go from (23.9) to (23.11), we use $A_{o,q} = A_q^{{}^1/_2}\big(A_q^{-{}^1/_2} A_{o,q}\big)$, hence $A^*_{o,q} = \big(A_q^{-{}^1/_2} A_{o,q}\big)^* A_q^{*\,{}^1/_2}$, where the ( )–term is bounded by (14.21).

Step 2: By duality on Corollary 21.2 on a reflexive Banach space, the operator $\mathbb{A}^*_{F,q}$ in (23.10) generates a s.c. analytic semigroup $e^{\mathbb{A}^*_{F,q} t}$ on $L^{q'}_\sigma(\Omega)$, which moreover is uniformly stable in $\mathcal{L}\big(L^{q'}_\sigma(\Omega)\big)$, $1 < q' \le 2$, with the same decay rate $\gamma_0 > 0$ in (23.3) = (17.14) as $e^{\mathbb{A}_{F,q} t}$ in $\mathcal{L}\big(L^q_\sigma(\Omega)\big)$, $q \ge 2$ in Theorem 22.1.

Step 3:

Proposition 23.2. *For the generator $\mathbb{A}^*_{F,q}$ in (23.10) of a s.c. analytic, uniformly bounded semigroup $e^{\mathbb{A}^*_{F,q} t}$ on $L^{q'}_\sigma(\Omega)$, we have: $\mathbb{A}^*_{F,q} \in MReg\big(L^p(0,\infty; L^{q'}_\sigma(\Omega))\big)$, $1 < q' \le 2$.*

Proof. The proof is based on a perturbation argument. For $q \geq 2$, rewrite (23.11) as

$$\mathbb{A}^*_{F,q} = -A^*_q + \Pi \tag{23.13}$$

$$\Pi = \left[F^* D^* A_q^{*^{1/_{2q}-\varepsilon}}\right] A_q^{*^{1-1/_{2q}+\varepsilon}} - \left[(I - DF)^* \big(A_q^{-1/2} A_{o,q}\big)^*\right] A_q^{*^{1/2}} + G^*. \tag{23.14}$$

In (23.14), both terms in square brackets [ ] are bounded in $L^{q'}_\sigma(\Omega)$, and so is G^*, $1 < q' \leq 2$. To this end we use critically and recall (15.41b):

$$A_q^{1/_{2q}-\varepsilon} D \in \mathcal{L}(U_q, L^q_\sigma(\Omega)), \text{ so } D^* A_q^{*^{1/_{2q}-\varepsilon}} \in \mathcal{L}(L^{q'}_\sigma(\Omega), L^{q'}(\Gamma)),\ 1 \leq q' \leq 2:$$

while $A_q^{-1/2} A_{o,q} \in \mathcal{L}(L^q_\sigma(\Omega))$ by (14.21).

The following estimates then hold, $q \geq 2, 1 < q' \leq 2$:

$$i.\ \left\| \left[F^* D^* A_q^{*^{1/_{2q}-\varepsilon}}\right] A_q^{*^{1-1/_{2q}+\varepsilon}} x \right\|_{L^{q'}_\sigma(\Omega)} \leq C_q \left\| A_q^{*^{1-1/_{2q}+\varepsilon}} x \right\|_{L^{q'}_\sigma(\Omega)}, \quad \forall\, x \in \mathcal{D}\big(A_q^{*^{1-1/_{2q}+\varepsilon}}\big) \tag{23.15}$$

$$\begin{aligned} ii.\ \left\| \left[(I - DF)^* \big(A_q^{-1/2} A_{o,q}\big)^*\right] A_q^{*^{1/2}} x \right\|_{L^{q'}_\sigma(\Omega)} &\leq C_q \left\| A_q^{*^{1/2}} x \right\|_{L^{q'}_\sigma(\Omega)}, \\ &= C_q \left\| \left(A_q^{*^{-1/2+1/_{2q}-\varepsilon}}\right) A_q^{*^{1-1/_{2q}+\varepsilon}} x \right\|_{L^{q'}_\sigma(\Omega)} \\ &\leq \widetilde{C}_q \left\| A_q^{*^{1-1/_{2q}+\varepsilon}} x \right\|_{L^{q'}_\sigma(\Omega)} \end{aligned} \tag{23.16}$$

Hence, the perturbation Π in (23.14) satisfies $q \geq 2,\ 1 < q' \leq 2$:

$$\|\Pi x\|_{L^{q'}_\sigma(\Omega)} \leq C_q \left\| A_q^{*^{1-1/_{2q}+\varepsilon}} x \right\|_{L^{q'}_\sigma(\Omega)}, \quad x \in \mathcal{D}\big(A_q^{*^{1-1/_{2q}+\varepsilon}}\big) \tag{23.17}$$

$1/_q + 1/_{q'} = 1,\ \varepsilon > 0$. We draw now some consequences from (23.13), (23.17):

(a) The perturbation operator Π is $A_q^{*\theta}$-bounded on $L^{q'}_\sigma(\Omega)$ with $\theta = 1 - 1/_{2q} + \varepsilon < 1,\ 1 < q' \leq 2 \leq q$.

(b) On the other hand $A^*_q \in MReg\big(L^p(0,\infty; L^{q'}_\sigma(\Omega))\big)$, from Section 3(c) In fact, while A_q is the Stokes operator on $L^q_\sigma(\Omega)$, $1 < q < \infty$, A^*_q is the Stokes operator on $L^{q'}_\sigma(\Omega)$ by (14.31), $1/_q + 1/_{q'} = 1$. Then, properties (a), (b) imply -by the abstract perturbation theorem in the Appendix C, see also [Dore, Theorem 6.2. p 311] and [K-W.2, SNP Remark 1i, p 426 for $\beta = 1$], for related results, that $\mathbb{A}^*_{F,q} \in MReg\big(L^p(0,\infty; L^{q'}_\sigma(\Omega))\big)$, $1 < q' \leq 2$ and Proposition 23.2 is proved.

□

Step 4: We now prove Theorem 23.1 that $\mathbb{A}_{F,q}$ satisfies the maximal L^p regularity on $L^q_\sigma(\Omega)$:

$$\mathbb{A}_{F,q} \in MReg\Big(L^p\big(0,\infty; L^q_\sigma(\Omega)\big)\Big), \quad 2 \le q < \infty. \tag{23.18}$$

Step 4.i: We invoke the fundamental result of L. Weis [K-W.2, Theorem 1.11 p 76], [Weis, Theorem p 198]. Since $\mathbb{A}_{F,q}$ generates a bounded analytic semigroup $e^{\mathbb{A}_{F,q}t}$ on $L^q_\sigma(\Omega)$, $2 \le q < \infty$, on a UMD-space [K-W.2, p 75], then the sought-after property that $\mathbb{A}_{F,q} \in MReg\big(L^p\big(0,\infty; L^q_\sigma(\Omega)\big)\big)$ is equivalent to the property that the family $\tau \in \mathcal{L}\big(L^q_\sigma(\Omega)\big)$

$$\tau = \Big\{tR\big(it, \mathbb{A}_{F,q}\big), t \in \mathbb{R}\backslash\{0\}\Big\} \quad \text{be } R\text{-bounded} \tag{23.19}$$

Step 4.ii: By the complete duality for R-boundedness on $L^q(\Omega)$, $2 \le q < \infty$, we have [K-W.2, Corollary 2.11 p90] that the family τ in (23.19) is R-bounded if and only if the corresponding dual family τ' in $\mathcal{L}\big(L^{q'}_\sigma(\Omega)\big)$, $(L^q_\sigma(\Omega))^* = L^{q'}_\sigma(\Omega))$ by (13.8b)

$$\tau' = \Big\{tR\big(it, \mathbb{A}^*_{F,q}\big), t \in \mathbb{R}\backslash\{0\}\Big\} \quad \text{is } R\text{-bounded} \tag{23.20}$$

Step 4.iii: But the R-boundedness property in (23.20) is equivalent, by the same result [K-W.2] to the property that $\mathbb{A}^*_{F,q} \in MReg\big(L^p(0,\infty; L^{q'}_\sigma(\Omega))\big), 1 < q' \le 2$, ${}^1/_q + {}^1/_{q'} = 1$, and this is true by Proposition 23.2. In conclusion: $\mathbb{A}_{F,q} \in MReg\big(L^p\big(0,\infty; L^q_\sigma(\Omega)\big)\big)$, and Theorem 23.1 is proved. □

We next examine the regularity of the term $e^{\mathbb{A}_{F,q}t}\eta_0$ due to the initial condition η_0 in (23.5). For the same reasons noted in the Theorem 23.1, Eqts (23.10) through (23.12), we shall equivalently examine the regularity of the adjoint semigroup $e^{\mathbb{A}^*_{F,q}t}$. To this end, we need the counterpart of Proposition 21.3 this time for the adjoint/dual operator $\mathbb{A}^*_{F,q}$ on $L^{q'}_\sigma(\Omega)$, $1 < q' \le 2$, ${}^1/_q + {}^1/_{q'} = 1$.

Proposition 23.3. *Let* $1 < p < \dfrac{2q'}{2q'-1}$, $1 < q' \le 2$, $q \ge 2$, $\dfrac{1}{q} + \dfrac{1}{q'} = 1$. *Then*

$$\big(L^{q'}_\sigma(\Omega), \mathcal{D}(\mathbb{A}^*_{F,q})\big)_{1-\frac{1}{p},p} = \widetilde{B}^{2-2/p}_{q',p}(\Omega) \tag{23.21}$$

$$= \Big\{g \in B^{2-2/p}_{q',p}(\Omega) : div\ g \equiv 0,\ g \cdot \nu|_\Gamma = 0\Big\} \tag{23.22}$$

Proof. From (23.10b) we have: $\mathcal{D}\big(\mathbb{A}^*_{F,q}\big) \subset W^{2,q'}(\Omega) \cap L^{q'}_\sigma(\Omega)$, $1 < q' \le 2$. Thus, as in (21.15)

$$\big(L^{q'}_\sigma(\Omega), \mathcal{D}(\mathbb{A}^*_{F,q})\big)_{\theta,p} \subset \big(L^{q'}_\sigma(\Omega), W^{2,q'}(\Omega) \cap L^{q'}_\sigma(\Omega)\big)_{\theta,p} = B^{2\theta}_{q',p}(\Omega) \cap L^{q'}_\sigma(\Omega) \tag{23.23}$$

recalling the definition (15.2). Next we take $1 < p < {}^{2q'}/_{2q'-1}$, $\theta = 1 - {}^1/_p$, so that, for these parameters, (23.23) specializes to

$$\big(L^{q'}_\sigma(\Omega), \mathcal{D}(\mathbb{A}^*_{F,q})\big)_{1-\frac{1}{p},p} \subset \widetilde{B}^{2-{}^2/_p}_{q',p}(\Omega) = \text{ defined in (15.3b).} \tag{23.24}$$

But $\widetilde{B}^{2-{}^2/_p}_{q',p}(\Omega)$ does not recognize the boundary conditions so neither does the space on the LHS of (23.24). Thus for these parameters $\theta = 1 - {}^1/_p$, with $1 < p < {}^{2q'}/_{2q'-1}$, we have

$$\big(L^{q'}_\sigma(\Omega), \mathcal{D}(\mathbb{A}^*_{F,q})\big)_{1-\frac{1}{p},p} = \big(L^{q'}_\sigma(\Omega), \mathcal{D}(A^*_q)\big)_{1-\frac{1}{p},p} \tag{23.25}$$

$$= \widetilde{B}^{2-{}^2/_p}_{q',p}(\Omega) \tag{23.26}$$

recalling (14.31) (A^*_q is the Stokes operator on $L^{q'}_\sigma(\Omega)$) and (15.3b) as $\mathcal{D}\big(\mathbb{A}^*_{F,q}\big)$ and $\mathcal{D}\big(A^*_q\big)$ both consist of $W^{2,q'}(\Omega) \cap L^{q'}_\sigma(\Omega)$ functions, subject only to different boundary conditions. Thus (23.26) proves the desired conclusion. □

We conclude this section with results for the semigroup $e^{\mathbb{A}_{F,q}t}$ on $L^q_\sigma(\Omega)$ that yield the solution in the space $X^T_{p,q}$ of maximal regularity. This is the companion result of Theorem 23.1. It is done by duality on the adjoint semigroup $e^{\mathbb{A}^*_{F,q}t}$ as in the proof of Theorem 23.1.

Theorem 23.4. *(i) Let $1 < p < {}^{2q'}/_{2q'-1}$, $1 < q' \le 2$, $q \ge 2$, ${}^1/_q + {}^1/_{q'} = 1$. Consider the adjoint s.c. analytic semigroup $e^{\mathbb{A}^*_{F,q}t}$ on $L^{q'}_\sigma(\Omega)$, which is uniformly stable here, by duality on Eq (22.2), $\theta = 0$, or (17.14) of Theorem 22.1. Then (see (23.26))*

$$e^{\mathbb{A}^*_{F,q}t}: \text{ continuous } \widetilde{B}^{2-{}^2/_p}_{q',p}(\Omega) = \big(L^{q'}_\sigma(\Omega), \mathcal{D}(\mathbb{A}^*_{F,q})\big)_{1-\frac{1}{p},p} = \big(L^{q'}_\sigma(\Omega), \mathcal{D}(A^*_q)\big)_{1-\frac{1}{p},p} \tag{23.27}$$

$$\longrightarrow X^\infty_{p,q'} \equiv L^p\big(0,\infty; W^{2,q'}(\Omega)\big) \cap W^{1,p}\big(0,\infty; L^{q'}_\sigma(\Omega)\big). \tag{23.28}$$

(ii) Consider now the original s.c. analytic feedback semigroup $e^{\mathbb{A}_{F,q}t}$ on $L^q_\sigma(\Omega)$, which is uniformly stable here by (17.14) or (22.2), $\theta = 0$. Let $1 < p < {}^{2q}/_{2q-1}$, $q \ge 2$. Then, see (21.17)

$$e^{\mathbb{A}_{F,q}t}: \text{ continuous } \widetilde{B}^{2-{}^2/_p}_{q,p}(\Omega) = \big(L^q_\sigma(\Omega), \mathcal{D}(\mathbb{A}_{F,q})\big)_{1-\frac{1}{p},p} = \big(L^q_\sigma(\Omega), \mathcal{D}(A_q)\big)_{1-\frac{1}{p},p} \tag{23.29}$$

$$\longrightarrow X^\infty_{p,q} = L^p\big(0,\infty; W^{2,q}(\Omega)\big) \cap W^{1,p}\big(0,\infty; L^q_\sigma(\Omega)\big). \tag{23.30}$$

Proof. We shall prove (i) and then (ii) will follow by duality.

Step 1: Thus, consider $\mathbb{A}^*_{F,q}$ in $L^{q'}_\sigma(\Omega)$, $1 < q' \leq 2$. Write

$$\chi(t) = e^{\mathbb{A}^*_{F,q} t}\chi_o, \quad \chi_t = \mathbb{A}^*_{F,q}\chi, \ \chi(0) = \chi_o. \tag{23.31}$$

Recalling $\mathbb{A}^*_{F,q} = -A^*_q + B_1 A^{*\,1-1/2q+\varepsilon}_q + B_2 A^{*\,1/2}_q + G^*$ from (23.11), where B_1, B_2 are in $\mathcal{L}\big(L^{q'}_\sigma(\Omega)\big)$, we rewrite the equation in (23.31) as

$$\chi_t = -A^*_q\chi + B_1 A^{*\,1-1/2q+\varepsilon}_q\chi + B_2 A^{*\,1/2}_q\chi + G^*\chi \tag{23.32}$$

whose solution is

$$\chi(t) = e^{-A^*_q t}\chi_o + \int_0^t e^{-A^*_q(t-\tau)} B_1 A^{*\,1-1/2q+\varepsilon}_q \chi(\tau)\, d\tau + \int_0^t e^{-A^*_q(t-\tau)} B_2 A^{*\,1/2}_q \chi(\tau)\, d\tau + \int_0^t e^{-A^*_q(t-\tau)} G^*\chi(\tau)\, d\tau. \tag{23.33}$$

Hence apply A^*_q throughout,

$$A^*_q\chi(t) = A^*_q e^{-A^*_q t}\chi_o + A^*_q\int_0^t e^{-A^*_q(t-\tau)} B_1 A^{*\,1-1/2q+\varepsilon}_q \chi(\tau)\, d\tau + A^*_q\int_0^t e^{-A^*_q(t-\tau)} B_2 A^{*\,1/2}_q \chi(\tau)\, d\tau + A^*_q\int_0^t e^{-A^*_q(t-\tau)} G^*\chi(\tau)\, d\tau. \tag{23.34}$$

Step 2: We now recall from (14.31) that A^*_q is nothing but the Stokes operator on the space $L^{q'}_\sigma(\Omega)$. Thus, A^*_q enjoys the maximal regularity properties stated for A_q in Section 3(c) except on $(L^q_\sigma(\Omega))' = L^{q'}_\sigma(\Omega)$, see (13.8b). We shall use these for each of the four terms of the RHS of (23.34).

First term: By use of estimate (15.23b), or (15.20), we obtain changing q into q'

$$\left\| A^*_q e^{-A^*_q \cdot}\chi_o \right\|_{L^p\left(0,\infty;L^{q'}_\sigma(\Omega)\right)} \leq C\left\|\chi_o\right\|_{\widetilde{B}^{2-2/p}_{q',p}(\Omega)}. \tag{23.35}$$

Second Term: Again by the maximal regularity property of A^*_q in (15.22), except in $L^{q'}_\sigma(\Omega)$, we estimate since $B_1 \in \mathcal{L}\big(L^{q'}_\sigma(\Omega)\big)$

$$\left\| A^*_q \int_0^{\cdot} e^{-A^*_q(\,\cdot\,-\tau)} B_1 A^{*\,1-1/2q+\varepsilon}_q \chi(\tau)\, d\tau \right\|_{L^p\left(0,\infty;L^{q'}_\sigma(\Omega)\right)} \leq C\left\| B_1 A^{*\,1-1/2q+\varepsilon}_q \chi \right\|_{L^p\left(0,\infty;L^{q'}_\sigma(\Omega)\right)} \tag{23.36}$$

$$\leq \widetilde{C}\left\| A^{*\,1-1/2q+\varepsilon}_q \chi \right\|_{L^p\left(0,\infty;L^{q'}_\sigma(\Omega)\right)} \tag{23.37}$$

$$\leq \varepsilon_1 \left\| A^*_q\chi\right\|_{L^p\left(0,\infty;L^{q'}_\sigma(\Omega)\right)} + C_{\varepsilon_1}\left\|\chi\right\|_{L^p\left(0,\infty;L^{q'}_\sigma(\Omega)\right)} \tag{23.38}$$

after using an interpolation inequality [Triebel, Thm 5.3, Eq (3)], see (A.11) with $\theta = {}^1/_2$ in Appendix C, to go from (23.37) to (23.38).

Third term: Similarly, since $B_2 \in \mathcal{L}\big(L^{q'}_\sigma(\Omega)\big)$, via (15.20):

$$\begin{aligned}\left\| A_q^* \int_0^{\cdot} e^{-A_q^*(\,\cdot\, - \tau)} B_2 {A_q^*}^{1/2} \chi(\tau)\, d\tau \right\|_{L^p\left(0,\infty; L^{q'}_\sigma(\Omega)\right)} &\leq C \left\| B_2 {A_q^*}^{1/2} \right\|_{L^p\left(0,\infty; L^{q'}_\sigma(\Omega)\right)} \\ &\leq \widetilde{C} \left\| {A_q^*}^{1/2} \right\|_{L^p\left(0,\infty; L^{q'}_\sigma(\Omega)\right)} \qquad (23.39) \\ &\leq \varepsilon_2 \left\| A_q^* \chi \right\|_{L^p\left(0,\infty; L^{q'}_\sigma(\Omega)\right)} \\ &\qquad + C_{\varepsilon_2} \left\| \chi \right\|_{L^p\left(0,\infty; L^{q'}_\sigma(\Omega)\right)}. \qquad (23.40)\end{aligned}$$

Fourth term: Finally, since $G^* \in \mathcal{L}\big(L^{q'}_\sigma(\Omega)\big)$, via (15.20)

$$\left\| A_q^* \int_0^{\cdot} e^{-A_q^*(\,\cdot\, - \tau)} G^* \chi(\tau)\, d\tau \right\|_{L^p\left(0,\infty; L^{q'}_\sigma(\Omega)\right)} \leq C \left\| \chi \right\|_{L^p\left(0,\infty; L^{q'}_\sigma(\Omega)\right)}. \qquad (23.41)$$

Invoking (23.35), (23.38), (23.40), (23.41) in (23.34), we obtain

$$\left\| A_q^* \chi \right\|_{L^p\left(0,\infty; L^{q'}_\sigma(\Omega)\right)} \leq C \left\| \chi_o \right\|_{\widetilde{B}^{2-2/p}_{q',p}(\Omega)} + (\varepsilon_1 + \varepsilon_2) \left\| A_q^* \chi \right\|_{L^p\left(0,\infty; L^{q'}_\sigma(\Omega)\right)} + \widetilde{C} \left\| \chi \right\|_{L^p\left(0,\infty; L^{q'}_\sigma(\Omega)\right)} \qquad (23.42)$$

from which we obtain $1 < q' \leq 2,\ q \geq 2,\ 1 < p < {}^{2q'}/_{2q'-1}$, with $\varepsilon_1 + \varepsilon_2$ small,

$$\left\| A_q^* \chi \right\|_{L^p\left(0,\infty; L^{q'}_\sigma(\Omega)\right)} \leq C \left\| \chi_o \right\|_{\widetilde{B}^{2-2/p}_{q',p}(\Omega)} + \widetilde{C} \left\| \chi \right\|_{L^p\left(0,\infty; L^{q'}_\sigma(\Omega)\right)} \qquad (23.43)$$

Step 3: By returning to (23.31) with $\chi_o \in \widetilde{B}^{2-2/p}_{q',p}(\Omega)$, since $e^{\mathbb{A}^*_{F,q} t}$ is a s.c. semigroup, uniformly stable in such space $\widetilde{B}^{2-2/p}_{q',p}(\Omega) \subset L^{q'}_\sigma(\Omega)$, see (22.24) with q replaced by q', we obtain a-fortiori

$$\begin{cases} \chi_o \in \widetilde{B}^{2-2/p}_{q',p}(\Omega) \xrightarrow{\ e^{\mathbb{A}^*_{F,q} t}\ } \chi \in L^p\big(0,\infty; L^{q'}_\sigma(\Omega)\big) & (23.44a) \\ \left\| \chi \right\|_{L^p\left(0,\infty; L^{q'}_\sigma(\Omega)\right)} \leq C \left\| \chi_o \right\|_{\widetilde{B}^{2-2/p}_{q',p}(\Omega)}. & (23.44b) \end{cases}$$

Substituting (23.44b) in (23.43) yields the desired estimate.

$$\left\| A_q^* \chi \right\|_{L^p\left(0,\infty; L^{q'}_\sigma(\Omega)\right)} \leq C \left\| \chi_o \right\|_{\widetilde{B}^{2-2/p}_{q',p}(\Omega)} \qquad (23.45)$$

$$\begin{aligned}\chi_o \in \widetilde{B}^{2-2/p}_{q',p}(\Omega) \xrightarrow{\ e^{\mathbb{A}^*_{F,q} t}\ } \chi &\in L^p\big(0,\infty; \mathcal{D}(A_q^*)\big) \\ &= L^p\big(0,\infty; W^{2,q'}(\Omega) \cap W^{1,q'}_0(\Omega) \cap L^{q'}_\sigma(\Omega)\big) \qquad (23.46)\end{aligned}$$

continuously see (23.10b). Consequently, (23.46) gives

$$e^{\mathbb{A}^*_{F,q}t}: \text{ continuous } \widetilde{B}^{2-2/p}_{q',p}(\Omega) \to X^{\infty}_{q',p} = L^p\big(0,\infty; W^{2,q'}(\Omega)\big) \cap W^{1,p}\big(0,\infty; L^{q'}(\Omega)\big) \tag{23.47}$$

Thus (23.47) shows part (i) for $e^{\mathbb{A}^*_{F,q}t}$, based on $L^{q'}_{\sigma}(\Omega)$. As noted, part (ii) then follows by duality. Theorem 23.4 is proved. □

24 Proof of Theorem 17.3: Well-posedness on $X^{\infty}_{p,q}$ of the non-linear z-dynamics (17.17) in feedback form

In this section we return to the translated non-linear z-dynamics (13.10a) and apply to it the interior feedback (localized tangential-like) control $u = G_q z = P_q\left(m\left(\sum_{k=1}^{K}(P_N z, q_k)_{W^u_N} u_k\right)\tau\right)$ as well as the tangential localized boundary control $v = Fz = \sum_{k=1}^{K}(P_N z, p_k)_{W^u_N} f_k$ i.e. of the same structure as the feedback identified on the linearized w-dynamics (20.2), Here the vectors $p_k, q_k, u_k \in W^u_N$ and the boundary vectors $f_k \in \mathcal{F} \subset W^{2-1/q',q'}(\Gamma)$ are precisely those identified in Theorem 20.1. These feedback operators produced the s.c. analytic, uniformly stable feedback semigroup $e^{\mathbb{A}_{F,q}t}$ on $L^q_\sigma(\Omega)$ Eq (22.2), as well as on the space $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$, $1 < p < \infty$, $1 < q < \infty$, as described in Theorem 22.2, (22.24). In addition such semigroup $e^{\mathbb{A}_{F,q}t}$ possesses maximal L^p-regularity on $L^q_\sigma(\Omega)$ and up to $T = \infty$, as documented on Section 23. These properties will be critically used now in the analysis of the fully nonlinear z-dynamics (13.10a).

$$\frac{dz}{dt} - \mathcal{A}_q(I - DF)z + \mathcal{N}_q z - G_q z = 0; \quad z(0) = z_0 \tag{24.1a}$$

in detail, see (17.17)

$$\frac{dz}{dt} - \mathcal{A}_q\left[z - D\left(\sum_{k=1}^{K}(P_N z, p_k)_{W^u_N} f_k\right)\right] + \mathcal{N}_q z = P_q\left(m\left(\sum_{k=1}^{K}(P_N z, q_k)_{W^u_N} u_k\right)\tau\right); \quad z(0) = z_0 \tag{24.1b}$$

Recalling from (17.10) of Theorem 17.2 the feedback generator $\mathbb{A}_{F,q}$, we can rewrite (24.1b) as

$$z_t = \mathbb{A}_{F,q} z - \mathcal{N}_q z; \quad z(0) = z_0 \tag{24.2}$$

whose variation of parameters formula is

$$z(t) = e^{\mathbb{A}_{F,q}t} z_0 - \int_0^t e^{\mathbb{A}_{F,q}(t-\tau)} \mathcal{N}_q z(\tau) d\tau. \tag{24.3}$$

We already know from (22.24) of Theorem 22.2 that for $z_0 \in \widetilde{B}^{2-2/p}_{q,p}(\Omega), 1 < q < \infty,\ 1 < p < 2q/_{2q-1}$ we have

$$\left\| e^{\mathbb{A}_{F,q}t} z_0 \right\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \leq M_{\gamma_0} e^{-\gamma_0 t} \left\| z_0 \right\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)}, \quad t \geq 0, \tag{24.4}$$

with M_{γ_0} possibly depending on p, q. Maximal regularity properties corresponding to the solution operator formula in (24.3) were established in section 23. Accordingly, for $z_0 \in \widetilde{B}^{2-2/p}_{q,p}(\Omega)$ and $f \in$

$X^\infty_{p,q,\sigma} \equiv L^p(0,\infty;\mathcal{D}(\mathbb{A}_{F,q})) \cap W^{1,p}(0,\infty;L^q_\sigma(\Omega))$, see (23.8a), $\mathcal{D}(\mathbb{A}_{F,q})$ given by (23.2b), we define the operator $\mathbb{F}_q$ by

$$\mathbb{F}_q(z_0,f)(t) = e^{\mathbb{A}_{F,q}t}z_0 - \int_0^t e^{\mathbb{A}_{F,q}(t-\tau)}\mathcal{N}_q f(\tau)d\tau \tag{24.5}$$

The main result of this section is Theorem 17.3. restated as

Theorem 24.1. *Let $d = 3$, $q > 3$ and $1 < p < {}^6\!/_5$ (in order to satisfy the requirement $p < {}^{2q}\!/_{2q-1}$). There exists a positive constant $r_1 > 0$ (identified in the proof below in (24.24)), such that if*

$$\|z_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} < r_1, \tag{24.6}$$

then the operator $\mathbb{F}_q$ in (24.5) has a unique fixed point on $X^\infty_{p,q,\sigma} \equiv L^p(0,\infty;\mathcal{D}(\mathbb{A}_{F,q}))\cap W^{1,p}(0,\infty;L^q_\sigma(\Omega))$

$$\mathbb{F}_q(z_0,z) = z, \text{ or } z(t) = e^{\mathbb{A}_{F,q}t}z_0 - \int_0^t e^{\mathbb{A}_{F,q}(t-\tau)}\mathcal{N}_q z(\tau)d\tau \tag{24.7}$$

which therefore is the unique (nonlinear semigroup) solution of problem (24.2) (= (24.1)) in $X^\infty_{p,q,\sigma}$.

The proof of Theorem 24.1 is accomplished in two steps.

Step 1:

Theorem 24.2. *Let $d = 3$, $q > 3$ and $1 < p < {}^6\!/_5$ (in order to satisfy the requirement $p < {}^{2q}\!/_{2q-1}$). There exists a positive constant $r_1 > 0$ (identified below in (24.24)) and a subsequent constant $r > 0$ (identified below in (24.22)) depending on $r_1 > 0$ and the constant $C > 0$ in (24.20), such that with $\|z_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} < r_1$ as in (24.6), the operator $\mathbb{F}_q(z_0,f)$ maps the ball $B(0,r)$ in $X^\infty_{p,q,\sigma}$ into itself.* □

Theorem 24.1 will follow then from Theorem 24.2 after establishing that

Step 2:

Theorem 24.3. *Let $d = 3$, $q > 3$ and $1 < p < {}^6\!/_5$ (in order to satisfy the requirement $p < {}^{2q}\!/_{2q-1}$). There exists a positive constant $r_1 > 0$, such that if $\|z_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} < r_1$ as in (24.6), then there exists a constant $0 < \rho_0 < 1$, such that the operator $\mathbb{F}_q(z_0,f)$ defines a contraction in the ball $B(0,\rho_0)$ of $X^\infty_{p,q,\sigma}$* □

The Banach contraction principle then establishes Theorem 24.1, once we prove Theorems 24.2 and 24.3.

Proof of Theorem 24.2. *Step 1*: We start from definition (24.5) of $\mathbb{F}_q$ and invoke the maximal regularity properties (23.29), (23.30) for $e^{\mathbb{A}_{F,q}t}$ and (23.8c) for $\displaystyle\int_0^t e^{\mathbb{A}_{F,q}(t-\tau)}\mathcal{N}_q f(\tau)d\tau$. We obtain

$$\|\mathbb{F}_q(z_0,f)(t)\|_{X^\infty_{p,q,\sigma}} \leq \left\| e^{\mathbb{A}_{F,q}t}z_0 \right\|_{X^\infty_{p,q,\sigma}} + \left\| \int_0^t e^{\mathbb{A}_{F,q}(t-\tau)}\mathcal{N}_q f(\tau)d\tau \right\|_{X^\infty_{p,q,\sigma}} \tag{24.8}$$

$$\leq C\Big[\|z_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} + \|\mathcal{N}_q f\|_{L^p(0,\infty;L^q_\sigma(\Omega))} \Big]. \tag{24.9}$$

Step 2: By the definition $\mathcal{N}_q f = P_q[(f.\nabla)f]$ in (14.23), we estimate ignoring $\|P_q\|$ and using, $\sup_{\cdot} \big[\, |g(\cdot)|\, \big]^r = [\sup_{\cdot} (|g(\cdot)|)]^r$

$$\|\mathcal{N}_q f\|^p_{L^p(0,\infty;L^q_\sigma(\Omega))} \leq \int_0^\infty \|P_q[(f.\nabla)f]\|^p_{L^q_\sigma(\Omega)}\, dt$$

$$\leq \int_0^\infty \left\{ \int_\Omega |f(t,x)|^q\, |\nabla f(t,x)|^q\, d\Omega \right\}^{p/q} dt \tag{24.10}$$

$$\leq \int_0^\infty \left\{ \left[\sup_\Omega |\nabla f(t,\cdot)|^q \right]^{1/q} \left[\int_\Omega |f(t,x)|^q\, d\Omega \right]^{1/q} \right\}^p dt \tag{24.11}$$

$$\leq \int_0^\infty \|\nabla f(t,\cdot)\|^p_{L^\infty(\Omega)}\, \|f(t,\cdot)\|^p_{L^q_\sigma(\Omega)}\, dt \tag{24.12}$$

$$\leq \sup_{0\leq t\leq \infty} \|f(t,\cdot)\|^p_{L^q_\sigma(\Omega)} \int_0^\infty \|\nabla f(t,\cdot)\|^p_{L^\infty_\sigma(\Omega)}\, dt \tag{24.13}$$

$$= \|f\|^p_{L^\infty(0,\infty;L^q_\sigma(\Omega))}\, \|\nabla f\|^p_{L^p(0,\infty;L^q_\sigma(\Omega))} \tag{24.14}$$

Step 3: The following embeddings hold true:

(i) [G-G-H.1, Proposition 4.3, p 1406 with $\mu = 0, s = \infty, r = q$] so that the required formula reduces to $1 \geq {}^1\!/_p$, as desired

$$f \in X^\infty_{p,q,\sigma} \hookrightarrow f \in L^\infty(0,\infty;L^q_\sigma(\Omega)) \tag{24.15a}$$

$$\text{so that,}\ \ \|f\|_{L^\infty(0,\infty;L^q_\sigma(\Omega))} \leq C\, \|f\|_{X^\infty_{p,q,\sigma}} \tag{24.15b}$$

(ii) [Kes, Theorem 2.4.4, p 74 requiring C^1-boundary]

$$W^{1,q}(\Omega) \subset L^\infty(\Omega) \text{ for q} > \dim\, \Omega = d,\ \ d = 2,3, \tag{24.16}$$

so that, with $p > 1, q > 3$, in case $d = 3$:

$$\|\nabla f\|^p_{L^p(0,\infty;L^q_\sigma(\Omega))} \leq C\,\|\nabla f\|^p_{L^p(0,\infty;W^{1,q}(\Omega))} \leq C\,\|f\|^p_{L^p(0,\infty;W^{2,q}(\Omega))} \tag{24.17}$$

$$\leq C\,\|f\|^p_{X^\infty_{p,q,\sigma}} \tag{24.18}$$

In going from (24.17) to (24.18) we have recalled the definition of $f \in X^\infty_{p,q,\sigma}$ in (23.8a). Then, the sought-after final estimate of the non-linear term $\mathcal{N}_q f, f \in X^\infty_{p,q,\sigma}$, is obtained from substituting (24.15b) and (24.18) into the RHS of (24.14). We obtain

$$\|\mathcal{N}_q f\|_{L^p(0,\infty;L^q_\sigma(\Omega))} \leq C\,\|f\|^2_{X^\infty_{p,q,\sigma}}, \quad f \in X^\infty_{p,q,\sigma}. \tag{24.19}$$

Returning to (24.8), we finally, obtain by (24.19)

$$\|\mathbb{F}_q(z_0, f)\|_{X^\infty_{p,q,\sigma}} \leq C\Big\{\,\|z_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} + \|f\|^2_{X^\infty_{p,q,\sigma}}\Big\}. \tag{24.20}$$

Step 4: We now impose the restrictions on the data on the RHS of (24.20): z_0 is in a ball of radius $r_1 > 0$ in $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$ and f is in a ball of radius $r > 0$ in $X^\infty_{p,q,\sigma}$. We further demand that the final result $\mathbb{F}_q(z_0, f)$ shall lie in a ball of radius r in $X^\infty_{p,q,\sigma}$. Thus we obtain from (24.20)

$$\|\mathbb{F}_q(z_0, f)\|_{X^\infty_{p,q,\sigma}} \leq C\Big\{\,\|z_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} + \|f\|^2_{X^\infty_{p,q,\sigma}}\Big\} \leq C(r_1 + r^2) \leq r \tag{24.21}$$

This implies

$$Cr^2 - r + Cr_1 \leq 0 \quad \text{or} \quad \frac{1 - \sqrt{1 - 4C^2 r_1}}{2C} \leq r \leq \frac{1 + \sqrt{1 - 4C^2 r_1}}{2C} \tag{24.22}$$

whereby

$$\Big\{\text{ range of values of r }\Big\} \longrightarrow \text{ interval } \Big[0, \frac{1}{C}\Big], \text{ as } r_1 \searrow 0 \tag{24.23}$$

a constraint which is guaranteed by taking

$$r_1 \leq \frac{1}{4C^2},\ C \text{ being the constant in (24.20).} \tag{24.24}$$

We have thus established that by taking r_1 as in (24.24) and subsequently r as in (24.22), then the map

$$\mathbb{F}_q(z_0, f) \text{ takes: } \left\{\begin{array}{c}\text{ball in } \widetilde{B}^{2-2/p}_{q,p}(\Omega) \\ \text{of radius } r_1\end{array}\right\} \times \left\{\begin{array}{c}\text{ball in } X^\infty_{p,q,\sigma} \\ \text{of radius } r\end{array}\right\} \text{ into } \left\{\begin{array}{c}\text{ball in } X^\infty_{p,q,\sigma} \\ \text{of radius } r\end{array}\right\},$$

$$3 < q,\ 1 < p < \frac{2q}{2q-1} \tag{24.25}$$

This establishes Theorem 24.2. $\square$

Proof of Theorem 24.3 Step 1: For f_1, f_2 both in the ball of $X^\infty_{p,q,\sigma}$ of radius r obtained in (24.22) of the proof of Theorem 24.2, we estimate from (24.5):

$$\|\mathbb{F}_q(z_0, f_1) - \mathbb{F}_q(z_0, f_2)\|_{X^\infty_{p,q,\sigma}} = \left\| \int_0^t e^{\mathbb{A}_{F,q}(t-\tau)} \big[\mathcal{N}_q f_1(\tau) - \mathcal{N}_q f_2(\tau)\big] d\tau \right\|_{X^\infty_{p,q,\sigma}} \tag{24.26}$$

$$\leq \widetilde{m}\, \|\mathcal{N}_q f_1 - \mathcal{N}_q f_2\|_{L^p(0,\infty; L^q_\sigma(\Omega))} \tag{24.27}$$

after invoking the maximal regularity property (11.6).

Step 2: Next recalling $\mathcal{N}_q f = P_q[(f\cdot\nabla)f]$ from (14.23), we estimate the RHS of (24.27). In doing so, we add and subtract $(f_2\cdot\nabla)f_1$, set $A = (f_1\cdot\nabla)f_1 - (f_2\cdot\nabla)f_1,\ B = (f_2\cdot\nabla)f_1 - (f_2\cdot\nabla)f_2$, and use $|A+B|^q \leq 2^q[|A|^q + |B|^q]$. [T-L.1, p 12] We obtain, again ignoring $\|P_q\|$:

$$\|\mathcal{N}_q f_1 - \mathcal{N}_q f_2\|_{L^p(0,\infty;L^q_\sigma(\Omega))} \leq \int_0^\infty \left\{ \left[\int_\Omega |(f_1\cdot\nabla)f_1 - (f_2\cdot\nabla)f_2|^q \, d\Omega \right]^{1/q} \right\}^p dt \tag{24.28}$$

$$= \int_0^\infty \left[\int_\Omega |A+B|^q \, d\Omega \right]^{p/q} dt \tag{24.29}$$

$$\leq 2^q \int_0^\infty \left\{ \int_\Omega \big[\, |A|^q + |B|^q \,\big] d\Omega \right\}^{p/q} dt \tag{24.30}$$

$$= 2^q \int_0^\infty \left\{ \left[\int_\Omega |A|^q \, d\Omega + \int_\Omega |B|^q \, d\Omega \right]^{1/q} \right\}^p dt \tag{24.31}$$

$$= 2^q \int_0^\infty \left\{ \left[\|A\|^q_{L^q_\sigma(\Omega)} + \|B\|^q_{L^q_\sigma(\Omega)} \right]^{1/q} \right\}^p dt \tag{24.32}$$

$$\leq 2^q \cdot 2^{1/q} \int_0^\infty \left\{ \|A\|_{L^q_\sigma(\Omega)} + \|B\|_{L^q_\sigma(\Omega)} \right\}^p dt \tag{24.33}$$

$$\leq 2^{p+q+1/q} \int_0^\infty \left[\|A\|^p_{L^q_\sigma(\Omega)} + \|B\|^p_{L^q_\sigma(\Omega)} \right] dt \tag{24.34}$$

$$= 2^{p+q+1/q} \int_0^\infty \Big[\|((f_1 - f_2)\cdot\nabla)f_1\|^p_{L^q_\sigma(\Omega)} + \|(f_2\cdot\nabla)(f_1-f_2)\|^p_{L^q_\sigma(\Omega)} \Big] dt \tag{24.35}$$

$$= 2^{p+q+1/q} \int_0^\infty \Big\{ \|f_1 - f_2\|^p_{L^q_\sigma(\Omega)} \|\nabla f_1\|^p_{L^q_\sigma(\Omega)} + \|f_2\|^p_{L^q(\Omega)} \|\nabla(f_1 - f_2)\|^p_{L^q(\Omega)} \Big\} dt \tag{24.36}$$

Step 3: We now notice that regarding each of the integral term in the RHS of (24.36) we are structurally and topologically as in the RHS of (24.12), except that in (24.36) the gradient terms $\nabla f_1, \nabla(f_1 - f_2)$ are penalized in the $L^q_\sigma(\Omega)$-norm which is dominated by the $L^\infty(\Omega)$-norm, as it occurs for the gradient term ∇f in (24.12). Thus we can apply to each integral term on the RHS of (24.36) the same argument as in going from (24.12) to the estimates (24.15b) and (24.18) with $q > \dim \Omega = 3$. We obtain

$$
\begin{aligned}
\|\mathcal{N}_q f_1 - \mathcal{N}_q f_2\|^p_{L^p(0,\infty;L^q_\sigma(\Omega))} &\leq \text{RHS of (24.36)} \\
\text{(see (24.14))} \qquad &\leq C_{p,q}\Big\{ \|f_1 - f_2\|^p_{L^\infty(0,\infty;L^q(\Omega))} \|\nabla f_1\|^p_{L^p(0,\infty;L^\infty(\Omega))} \\
&\qquad + \|f_2\|^p_{L^\infty(0,\infty;L^q(\Omega))} \|\nabla (f_1 - f_2)\|^p_{L^p(0,\infty;L^\infty(\Omega))} \Big\} \qquad (24.37) \\
\text{(see (24.15b) and (24.18))} \qquad &\leq C_{p,q}\Big\{ \|f_1 - f_2\|^p_{X^\infty_{p,q,\sigma}} \|f_1\|^p_{X^\infty_{p,q,\sigma}} \qquad (24.38) \\
&\qquad + \|f_2\|^p_{X^\infty_{p,q,\sigma}} \|f_1 - f_2\|^p_{X^\infty_{p,q,\sigma}} \Big\} \qquad (24.39) \\
&= C_{p,q}\Big\{ \|f_1 - f_2\|^p_{X^\infty_{p,q,\sigma}} \left(\|f_1\|^p_{X^\infty_{p,q,\sigma}} + \|f_2\|^p_{X^\infty_{p,q,\sigma}} \right) \Big\} \qquad (24.40)
\end{aligned}
$$

with $C_{p,q} = 2^{p+q+1/q}$, finally (24.40) yields

$$
\begin{aligned}
\|\mathcal{N}_q f_1 - \mathcal{N}_q f_2\|_{L^p(0,\infty;L^q_\sigma(\Omega))} &\leq C_{p,q} \|f_1 - f_2\|_{X^\infty_{p,q,\sigma}} \left(\|f_1\|^p_{X^\infty_{p,q,\sigma}} + \|f_2\|^p_{X^\infty_{p,q,\sigma}} \right)^{1/p} \qquad (24.41) \\
&\leq 2^{1/p} C_{p,q} \|f_1 - f_2\|_{X^\infty_{p,q,\sigma}} \left(\|f_1\|_{X^\infty_{p,q,\sigma}} + \|f_2\|_{X^\infty_{p,q,\sigma}} \right) \qquad (24.42)
\end{aligned}
$$

Step 4: Using estimate (24.42) on the RHS of estimate (24.27) yields

$$
\|\mathbb{F}_q(z_0, f_1) - \mathbb{F}_q(z_0, f_2)\|_{X^\infty_{p,q,\sigma}} \leq K_p \|f_1 - f_2\|_{X^\infty_{p,q,\sigma}} \left(\|f_1\|_{X^\infty_{p,q,\sigma}} + \|f_2\|_{X^\infty_{p,q,\sigma}} \right) \qquad (24.43)
$$

$K_{p,q} = \widetilde{m} C_{p,q} = \widetilde{m} 2^{p+1/p+q+1/q}$ ($\widetilde{m}$ as in (24.27)). Next, pick f_1, f_2 in the ball of $X^\infty_{p,q,\sigma}$ of radius R:

$$
\|f_1\|_{X^\infty_{p,q,\sigma}}, \|f_2\|_{X^\infty_{p,q,\sigma}} \leq R \qquad (24.44)
$$

Then

$$
\|\mathbb{F}_q(z_0, f_1) - \mathbb{F}_q(z_0, f_2)\|_{X^\infty_{p,q,\sigma}} \leq \rho_0 \|f_1 - f_2\|_{X^\infty_{p,q,\sigma}} \qquad (24.45)
$$

and $\mathbb{F}_q(z_0, f)$ is a contraction on the space $X^\infty_{p,q,\sigma}$ as soon as

$$
\rho_0 \equiv 2K_{p,q} R < 1 \text{ or } R < 1/2K_{p,q}, \; K_{p,q} = \widetilde{m} 2^{p+1/p+q+1/q}. \qquad (24.46)
$$

In this case, the map $\mathbb{F}_q(z_0, f)$ defined in (24.5) has a fixed point z in $X^\infty_{p,q,\sigma}$

$$
\mathbb{F}_q(z_0, z) = z, \;\text{ or } z = e^{\mathbb{A}_{F,q} t} z_0 - \int_0^t e^{\mathbb{A}_{F,q}(t-\tau)} \mathcal{N}_q z(\tau) d\tau \qquad (24.47)
$$

and such fixed point $z \in X^{\infty}_{p,q,\sigma}$ is the unique solution of the translated non-linear equation (24.1), or (24.2) with finite dimensional control u in feedback form, as described by the RHS of (24.1). Theorem 24.1 is proved. $\square$

25 Proof of Theorem 17.4. Local exponential decay of the non-linear translated z-dynamics (17.17) = (24.1) with finite dimensional localized feedback control $\{v, u\}$, case $d = 3$.

In this section we return to the feedback problem (24.1) rewritten equivalently as in (24.3)

$$z(t) = e^{\mathbb{A}_{F,q}t} z_0 - \int_0^t e^{\mathbb{A}_{F,q}(t-\tau)} \mathcal{N}_q z(\tau) d\tau. \tag{25.1}$$

For z_0 in a small ball of $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$, Theorem 24.1 provides a unique solution in a ball of $X^{\infty}_{p,q,\sigma}$. We recall from (22.24) = (24.4)

$$\left\| e^{\mathbb{A}_{F,q}t} z_0 \right\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \leq M_{\gamma_0} e^{-\gamma_0 t} \left\| z_0 \right\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)}, \ t \geq 0. \tag{25.2}$$

Our goal now is to show that for z_0 in a small ball of $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$, problem (25.1) satisfies the exponential decay

$$\left\| z(t) \right\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \leq C e^{-at} \left\| z_0 \right\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)}, \ t \geq 0, \text{ for some constants, } a > 0, C_a \geq 1.$$

Step 1: Starting from (25.1) and using (25.2), we estimate

$$\left\| z(t) \right\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \leq M e^{-\gamma_0 t} \left\| z_0 \right\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} + \sup_{0 \leq t \leq \infty} \left\| \int_0^t e^{\mathbb{A}_{F,q}(t-\tau)} \mathcal{N}_q z(\tau) d\tau \right\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \tag{25.3}$$

$$\leq M e^{-\gamma_0 t} \left\| z_0 \right\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} + C \left\| \int_0^t e^{\mathbb{A}_{F,q}(t-\tau)} \mathcal{N}_q z(\tau) d\tau \right\|_{X^{\infty}_{p,q,\sigma}} \tag{25.4}$$

$$\leq M e^{-\gamma_0 t} \left\| z_0 \right\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} + C \left\| \mathcal{N}_q z \right\|_{L^p(0,\infty; L^q_\sigma(\Omega))} \tag{25.5}$$

$$\left\| z(t) \right\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \leq M e^{-\gamma_0 t} \left\| z_0 \right\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} + C_1 \left\| z \right\|^2_{X^{\infty}_{p,q,\sigma}}. \tag{25.6}$$

In going from (25.3) to (25.4) we have recalled the embedding $X^{\infty}_{p,q,\sigma} \hookrightarrow L^{\infty}\big(0, \infty; \widetilde{B}^{2-2/p}_{q,p}(\Omega)\big)$ from (23.8c) or (15.17) with $T = \infty$. Next, in going from (25.4) to (25.5) we have used the maximal regularity property (11.6). Finally, to go from (25.5) to (25.6) we have invoked estimate (24.19).

Step 2: We shall next establish that

$$\left\| z \right\|_{X^{\infty}_{p,q,\sigma}} \leq M \left\| z_0 \right\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} + K \left\| z \right\|^2_{X^{\infty}_{p,q,\sigma}}, \text{ hence } \left\| z \right\|_{X^{\infty}_{p,q,\sigma}} \left(1 - K \left\| z \right\|_{X^{\infty}_{p,q,\sigma}} \right) \leq M \left\| z_0 \right\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \tag{25.7}$$

In fact, to this end, we take the $X^\infty_{p,q,\sigma}$ estimate of equation (25.1). We obtain

$$\|z\|_{X^\infty_{p,q,\sigma}} \leq \left\| e^{\mathbb{A}_{F,q}t} z_0 \right\|_{X^\infty_{p,q,\sigma}} + \left\| \int_0^t e^{\mathbb{A}_{F,q}(t-\tau)} \mathcal{N}_q z(\tau) d\tau \right\|_{X^\infty_{p,q,\sigma}} \tag{25.8}$$

from which then (25.7) follows by invoking the maximal regularity property (23.34), (23.35) on $e^{\mathbb{A}_{F,q}t}$ as well as the maximal regularity estimate (11.6) followed by use of of (24.19), as in going from (25.4) to (25.6)

$$\left\| \int_0^t e^{\mathbb{A}_{F,q}(t-\tau)} \mathcal{N}_q z(\tau) d\tau \right\|_{X^\infty_{p,q,\sigma}} \leq \widetilde{m} \left\| \mathcal{N}_q z \right\|_{L^p(0,\infty;L^q_\sigma(\Omega))} \tag{25.9}$$

$$\leq \widetilde{m} C \left\| z \right\|^2_{X^\infty_{p,q,\sigma}}. \tag{25.10}$$

Thus (25.7) is proved with $K = \widetilde{m}C$ where C is the same constant occurring in (24.19), hence in (24.21), (24.22).

Step 3: The well-posedness Theorem 24.1 says that

$$\left\{ \begin{array}{c} \text{If } \left\| z_0 \right\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \leq r_1 \\ \text{for } r_1 \text{ sufficiently small} \end{array} \right\} \Longrightarrow \left\{ \begin{array}{c} \text{The solution } z \text{ satisfies} \\ \left\| z \right\|_{X^\infty_{p,q,\sigma}} \leq r \end{array} \right\} \tag{25.11}$$

where r satisfies the constraint (24.22) in terms of r_1 and some constant C that occurs for $K = \widetilde{m}C$ in (25.10). We seek to guarantee that we can obtain

$$\begin{cases} \left\| z \right\|_{X^\infty_{p,q,\sigma}} \leq r < \frac{1}{2K} = \frac{1}{2\widetilde{m}C} \left(< \frac{1}{2C} \right) \\ \\ \text{hence } \frac{1}{2} < 1 - K \left\| z \right\|_{X^\infty_{p,q,\sigma}}, \end{cases} \tag{25.12}$$

where w.l.o.g. we can take the maximal regularity constant $\widetilde{m}$ in (24.27) to satisfy $\widetilde{m} \geq 1$. Again, the constant C arises from application of estimate (24.19). This is indeed possible by choosing $r_1 > 0$ sufficiently small. In fact, as $r_1 \searrow 0$, (24.23) shows that the interval $r_{min} \leq r \leq r_{max}$ of corresponding values of r tends to the interval $\left[0, \dfrac{1}{C} \right]$. Thus (25.12) can be achieved as $r_{min} \searrow 0$: $0 < r_{min} < r < \dfrac{1}{2\widetilde{m}C}$. Next, (25.12) implies that (25.7) holds true and yields then

$$\left\| z \right\|_{X^\infty_{p,q,\sigma}} \leq 2M \left\| z_0 \right\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \leq 2M r_1. \tag{25.13}$$

Substituting (25.13) in estimate (25.6) then yields

$$\|z(t)\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \leq Me^{-\gamma_0 t}\|z_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} + 4C_1M^2\|z_0\|^2_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \tag{25.14}$$

$$= M\left[e^{-\gamma_0 t} + 4MC_1\|z_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)}\right]\|z_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \tag{25.15}$$

$$\|z(t)\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \leq M\left[e^{-\gamma_0 t} + 4MC_1 r_1\right]\|z_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \tag{25.16}$$

recalling the constant $r_1 > 0$ in (25.11).

Step 4: Now take T sufficiently large and $r_1 > 0$ sufficiently small such that

$$\beta \equiv Me^{-\gamma_0 T} + 4M^2C_1r_1 < 1 \tag{25.17}$$

Then (25.15) implies by (25.17)

$$\|z(T)\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \leq \beta\|z_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \quad \text{and hence} \tag{25.18a}$$

$$\|z(nT)\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \leq \beta\|z((n-1)T)\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \leq \beta^n\|z_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)}. \tag{25.18b}$$

Since $\beta < 1$, the semigroup property of the evolution implies that there are constants $\widetilde{M} \geq 1, \widetilde{\gamma} > 0$ such that

$$\|z(t)\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \leq \widetilde{M}e^{-\widetilde{\gamma}t}\|z_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)}, \quad t \geq 0 \tag{25.19}$$

This proves Theorem 17.4. □

Remark 25.1. The above computations - (25.17) through (25.19) - can be used to support qualitatively the intuitive expectation that "the larger the decay rate γ_0 in (22.24) of the linearized feedback w-dynamics (17.10), the larger the decay rate $\widetilde{\gamma}$ in (25.19) of the nonlinear feedback z-dynamics (13.24) = (24.1b); hence the larger the rate $\widetilde{\gamma}$ in (13.20) of the original y-dynamics in (13.16)".

The following considerations are somewhat qualitative. Let $S(t)$ denote the non-linear semigroup in the space $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$, with infinitesimal generator $\left[\mathbb{A}_{F,q} - \mathcal{N}_q\right]$ describing the feedback z-dynamics (13.24)=(24.1b), as guaranteed by the well posedness Theorem B.(i) = Theorem 24.1. Thus, $z(t; z_0) = S(t)z_0$ on $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$. By (25.17), we can rewrite (25.18a) as:

$$\|S(T)\|_{\mathcal{L}\left(\widetilde{B}^{2-2/p}_{q,p}(\Omega)\right)} \leq \beta < 1. \tag{25.20}$$

It follows from [Bal, p 178] via the semigroup property that

$$-\,\widetilde{\gamma}\ \text{ is just below }\ \frac{\ln \beta}{T} < 0. \tag{25.21}$$

Pick $r_1 > 0$ in (25.17) so small that $4M^2C_1r_1$ is negligible, so that β is just above $Me^{-\gamma_0 T}$, so $\ln \beta$ is just above $\big[\ln M - \gamma_0 T\big]$, hence

$$\frac{\ln \beta}{T} \text{ is just above } (-\gamma_0) + \frac{\ln M}{T}. \tag{25.22}$$

Hence, by (25.21), (25.22),

$$\widetilde{\gamma} \sim \gamma_0 - \frac{\ln M}{T} \tag{25.23}$$

and the larger γ_0, the larger is $\widetilde{\gamma}$, as desired.

26 Well-posedness of the pressure χ for the z-problem (13.24), (17.17) = (24.1b) in feedback form, and of the pressure π for the y-problem (13.16) in feedback form

The z-problem in feedback form: We return to the translated z problem (13.24) = (17.17), with $Ł_e(z)$ given by (15.26)

$$z_t - \nu\Delta z + L_e(z) + (z\cdot\nabla)z + \nabla\chi = m(\widetilde{G}z)\tau \qquad \text{in } Q \tag{26.1a}$$

$$\text{div } z = 0 \qquad \text{in } Q \tag{26.1b}$$

$$z = Fz \qquad \text{on } \Sigma \tag{26.1c}$$

$$z(0,x) = y_0(x) - y_e(x) \qquad \text{on } \Omega \tag{26.1d}$$

with Fz and $m(\widetilde{G}z)\tau$ given in the feedback form as in (17.23) = (17.13) and (17.24) respectively

$$m(\widetilde{G}z)\tau = m\left(\sum_{k=1}^{K}(P_N z, q_k)_{W_N^u} u_k\right)\tau, \quad Fz = \sum_{k=1}^{K}\langle P_N z, p_k\rangle_\Gamma f_k, \tag{26.1e}$$

for which Theorem B = Theorem 17.3 provides a local well-posedness result (17.19), (17.20) for the z variable. We now complement such well-posedness for z with a corresponding local well-posedness result for the pressure χ.

Here we recall maximal regularity result in (15.20) for problem (15.13) which accounts for inhomogeneous no-slip Dirichlet boundary conditions [P-S]. We present it for convenience

$$\begin{cases} \psi_t - \nu_o\Delta\psi + \nabla\pi = F & \text{in } (0,T]\times\Omega \equiv Q \qquad (26.2a)\\ div\ \psi \equiv 0 & \text{in } Q \qquad (26.2b)\\ \psi|_\Sigma \equiv h_0 & \text{in } (0,T]\times\Gamma \equiv \Sigma \qquad (26.2c)\\ \psi|_{t=0} = \psi_0 & \text{in } \Omega, \qquad (26.2d)\end{cases}$$

Then there exists a unique solution $\varphi \in X^T_{p,q,\sigma}, \pi \in Y^T_{p,q}$ to the dynamic Stokes problem (26.2) or (15.20), continuously on the data: there exist constants C_0, C_1 independent of $T, F_\sigma = P_qF, \varphi_0$ such that via (15.17)

$$\begin{aligned} C_0 \|\varphi\|_{C\left([0,T];B^{2-2/p}_{q,p}(\Omega)\right)} &\leq \|\varphi\|_{X^T_{p,q,\sigma}} + \|\pi\|_{Y^T_{p,q}} \\ &\equiv \|\varphi'\|_{L^p(0,T;L^q_\sigma(\Omega))} + \|A_q\varphi\|_{L^p(0,T;L^q_\sigma(\Omega))} + \|\pi\|_{Y^T_{p,q}} \\ &\leq C_1\Big\{ \|F_\sigma\|_{L^p(0,T;L^q_\sigma(\Omega))} + \|\varphi_0\|_{\left(L^q_\sigma(\Omega),\mathcal{D}(A_q)\right)_{1-\frac{1}{p},p}} + \|h_0\|_{L^p(0,\infty;W^{1-1/q,q}(\Gamma))} \Big\}. \end{aligned} \tag{26.3}$$

Theorem 26.1. *Consider the setting of Theorem A for problem (13.16). Then the following well-posedness result for the pressure χ holds true, where we recall the spaces $Y^\infty_{p,q}$ for $T=\infty$ and $\widehat{W}^{1,q}(\Omega)$ in (15.15), (15.16) as well as the steady state pressure π_e from Theorem 13.1:*

$$\|\chi\|_{Y^\infty_{p,q}} \leq \widetilde{C}\, \|y_0 - y_e\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \Big\{ \|y_0 - y_e\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} + 1 \Big\}. \tag{26.4}$$

Proof. We first apply the full maximal-regularity up to $T=\infty$ (26.3) to the Stokes component of problem (26.1) with $F_q = P_q\big(mG(z) - L_e(z) - (z\cdot\nabla)z\big)$ and $h_0 = Fz$ to obtain

$$\begin{aligned} \|z\|_{X^\infty_{p,q,\sigma}} + \|\chi\|_{Y^\infty_{p,q}} &\leq C\Big\{ \|P_q[m(Gz) - (z\cdot\nabla)z - L_e(z)]\|_{L^p\left(0,\infty;L^q_\sigma(\Omega)\right)} + \|z_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \\ &\qquad + \|Fz\|_{L^p(0,\infty;W^{1-1/q,q}(\Gamma))} \Big\} \\ &\leq C\Big\{ \|P_q[m(Gz)]\|_{L^p\left(0,\infty;L^q_\sigma(\Omega)\right)} + \|P_q(z\cdot\nabla)z\|_{L^p\left(0,\infty;L^q_\sigma(\Omega)\right)} \\ &\quad + \|P_qL_e(z)\|_{L^p\left(0,\infty;L^q_\sigma(\Omega)\right)} + \|z_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} + \|Fz\|_{L^p(0,\infty;W^{1-1/q,q}(\Gamma))} \Big\}. \end{aligned} \tag{26.5}$$

But $P_q[mG(z)] = mG(z)$ as the vectors u_k in the definition of $\widetilde{G}$ in (21.11) are $u_k \in W^u_N \subset L^q_\sigma(\Omega)$. Moreover $G \in \mathcal{L}(L^q_\sigma(\Omega))$, we obtain

$$\left\|P_q[m(\widetilde{G}z)]\right\|_{L^p\left(0,\infty;L^q_\sigma(\Omega)\right)} \leq C_1 \|z\|_{X^\infty_{p,q,\sigma}}, \tag{26.6a}$$

$$\|Fz\|_{L^p(0,\infty;W^{1-1/q,q}(\Gamma))} \leq C_1' \|z\|_{X^\infty_{p,q,\sigma}} \tag{26.6b}$$

recalling the space $X^\infty_{p,q,\sigma}$ from (15.15). Next, recalling (24.19) for $\mathcal{N}_q z = P_q\big[(z\cdot\nabla)z\big]$, see (14.23), we obtain

$$\|P_q(z\cdot\nabla)z\|_{L^p\left(0,\infty;L^q_\sigma(\Omega)\right)} \leq C_2 \|z\|^2_{X^\infty_{p,q,\sigma}}. \tag{26.7}$$

The equilibrium solution $\{y_e, \pi_e\}$ is given by Theorem 13.1 as satisfying

$$\|y_e\|_{W^{2,q}(\Omega)} + \|\pi_e\|_{\widehat{W}^{q,1}} \leq c\, \|f\|_{L^q(\Omega)}, \quad 1<q<\infty. \tag{26.8}$$

We next estimate the term $P_q L_e(z) = P_q[(y_e \cdot \nabla)z + (z \cdot \nabla)y_e]$ in (26.5)

$$
\begin{aligned}
\|P_q L_e(z)\|_{L^p\left(0,\infty;L^q_\sigma(\Omega)\right)} &= \|P_q(y_e.\nabla)z + P_q(z.\nabla)y_e\|_{L^p\left(0,\infty;L^q_\sigma(\Omega)\right)} && (26.9)\\
&\le \|P_q(y_e.\nabla)z\|_{L^p\left(0,\infty;L^q_\sigma(\Omega)\right)} + \|P_q(z.\nabla)y_e\|_{L^p\left(0,\infty;L^q_\sigma(\Omega)\right)} && (26.10)\\
&\le \|y_e\|_{L^q(\Omega)} \|\nabla z\|_{L^p\left(0,\infty;L^q_\sigma(\Omega)\right)} + \|z\|_{L^p\left(0,\infty;L^q_\sigma(\Omega)\right)} \|\nabla y_e\|_{L^q(\Omega)} && (26.11)\\
&\le 2C_2 \|f\|_{L^q(\Omega)} \|z\|_{L^p\left(0,\infty;L^q_\sigma(\Omega)\right)} && (26.12)\\
&\le C_3 \|z\|_{X^\infty_{p,q,\sigma}} && (26.13)
\end{aligned}
$$

with the constant C_3 depending on the $L^q(\Omega)$-norm of the datum f. Setting now $C_4 = C \cdot \{C_1, C_2, C_3\}$ and substituting (26.6a), (26.7), (26.13) in (26.5), we obtain

$$
\|z\|_{X^\infty_{p,q,\sigma}} + \|\chi\|_{Y^\infty_{p,q}} \le C_4 \Big\{ \|z\|^2_{X^\infty_{p,q,\sigma}} + 2\|z\|_{X^\infty_{p,q,\sigma}} + \|z_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \Big\} \tag{26.14}
$$

Next we drop the term $\|z\|_{X^\infty_{p,q,\sigma}}$ on the left hand side of (26.14) and invoking (25.13) to estimate $\|z\|_{X^\infty_{p,q,\sigma}}$. Thus we obtain

$$
\begin{aligned}
\|\chi\|_{Y^\infty_{p,q}} &\le C_5 \Big\{ \|z_0\|^2_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} + 2\|z_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} + \|z_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \Big\} && (26.15)\\
&\le \widetilde{C} \|z_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \Big\{ \|z_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} + 1 \Big\}, \quad \widetilde{C} = 3C_5 && (26.16)
\end{aligned}
$$

and (26.16) proves (26.4), as desired, recalling (13.9). □

<u>The y-problem in feedback form</u> We return to the original y-problem however in feedback form as in (13.16), (13.21), (13.22) for which Theorem A in Secntion 13.7 proves a local well-posedness result. We now complement such well-posedness result for y with the corresponding local well-posedness result for the pressure π.

Theorem 26.2. *Consider the setting of Theorem A for the y-problem in (13.16), (13.21), (13.22). Then, the following well-posedness result for the pressure π holds true.*

$$
\begin{aligned}
\|\pi - \pi_e\|_{Y^T_{p,q}} \le \|\pi - \pi_e\|_{Y^\infty_{p,q}} &\le \widetilde{C} \|y_0 - y_e\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \Big\{ \|y_0 - y_e\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} + 1 \Big\} && (26.17)\\
&\le \widehat{C} \Big\{ \|y_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} + \|y_e\|_{W^{2,q}(\Omega)} \Big\} \Big\{ \|y_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} + \|y_e\|_{W^{2,q}(\Omega)} + 1 \Big\} && (26.18)\\
&\le \widehat{C} \Big\{ \|y_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} + \|f\|_{L^q(\Omega)} \Big\} \Big\{ \|y_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} + \|f\|_{L^q(\Omega)} + 1 \Big\} && (26.19)
\end{aligned}
$$

$$\|\pi\|_{Y^T_{p,q}} \leq \widehat{C}\,\|y_0 - y_e\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \left\{ \|y_0 - y_e\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} + 1 \right\} + cT^{1/p}\,\|\pi_e\|_{\widehat{W}^{1,q}(\Omega)}\,, \ \ 0 < T < \infty \tag{26.20}$$

$$\leq \widehat{C}\Big\{ \|y_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} + \|f\|_{L^q(\Omega)} \Big\}\Big\{ \|y_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} + \|f\|_{L^q(\Omega)} + 1 \Big\}$$

$$+ cT^{1/p}\,\|f\|_{L^q(\Omega)}\,, \ \ 0 < T < \infty \tag{26.21}$$

Proof. We return to the estimate (26.4) for χ and recall $\chi = \pi - \pi_e$ from (13.9) to obtain (26.17). We next estimate $y - y_e$ by

$$\|y_0 - y_e\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \leq C\big\{ \|y_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} + \|y_e\|_{W^{2,q}(\Omega)} \big\}. \tag{26.22}$$

which substituted in (26.17) yields (26.18). In turn, (26.18) leads to (26.19) by means of (26.8). □

Appendix C Proof of Theorem 15.5: maximal regularity of the Oseen operator $\mathcal{A}_q$ on $L^q_\sigma(\Omega)$, $1 < p, q < \infty$

Part I: (15.33). By (15.28) with $\psi_0 = 0$

$$\psi(t) = \int_0^t e^{\mathcal{A}_q(t-\tau)} F_\sigma(\tau) d\tau. \tag{A.1}$$

where by Theorem 3.1(ii)

$$\left\| e^{\mathcal{A}_q(t-\tau)} \right\|_{\mathcal{L}(L^q_\sigma(\Omega))} \leq M e^{b(t-\tau)}, \quad 0 \leq \tau \leq t \tag{A.2}$$

for $M \geq 1$, b possibly depending on q.

Step 1: We have the following estimate

$$\int_0^T \|\psi(t)\|^p_{L^q_\sigma(\Omega)} dt \leq C_T \int_0^T \|F_\sigma(t)\|^p_{L^q_\sigma(\Omega)} dt \tag{A.3}$$

where the constant C_T may depend also on p, q, b. This follows at once from Young's inequality for convolutions: [Sa]

$$\|\psi(t)\|_{L^q_\sigma(\Omega)} \leq M \int_0^t e^{b(t-\tau)} \|F_\sigma(\tau)\|_{L^q_\sigma(\Omega)} d\tau \in L^p(0,T)$$

and the convolution of the $L^p(0,T)$-function $\|F_\sigma\|_{L^q_\sigma(\Omega)}$ and the $L^1(0,T)$-function e^{bt} is in $L^p(0,T)$. More elementary, one can use Hölder inequality with $1/p + 1/\tilde{p} = 1$ and obtain an explicit constant.

Step 2: Claim: Here we shall next complement (A.3) with the estimate

$$\int_0^T \|A_q\psi(t)\|^p_{L^q_\sigma(\Omega)} dt \leq C \int_0^T \|\psi(t)\|^p_{L^q_\sigma(\Omega)} dt + C \int_0^T \|F_\sigma(t)\|^p_{L^q_\sigma(\Omega)} dt \tag{A.4}$$

to be shown below. Using (A.3) in (A.4) then yields

$$\int_0^T \|A_q\psi(t)\|^p_{L^q_\sigma(\Omega)} dt \leq C_T \int_0^T \|F_\sigma(t)\|^p_{L^q_\sigma(\Omega)} dt. \tag{A.5}$$

With respect to (15.27) with $\psi_0 = 0$, then (A.5) says

$$F_\sigma \in L^p(0,T; L^q_\sigma(\Omega)) \longrightarrow \psi \in L^p(0,T; \mathcal{D}(A_q) = \mathcal{D}(\mathcal{A}_q)) \tag{A.6}$$

while (15.28) then yields via (A.6)

$$F_\sigma \in L^p(0,T; L^q_\sigma(\Omega)) \longrightarrow \psi_t \in L^p(0,T; L^q_\sigma(\Omega)) \tag{A.7}$$

continuously. This is part (i) of Theorem 13.6.

Proof of (A.4): . In this step, with $\psi_0 = 0$, we shall employ the alternative formula, via (15.27)

$$\psi(t) = \int_0^t e^{-A_q(t-\tau)}(-A_{o,q})\psi(\tau)d\tau + \int_0^t e^{-A_q(t-\tau)} F_\sigma(\tau)d\tau. \tag{A.8}$$

where by maximal regularity of the Stokes operator $-A_q$ on the space $L^q_\sigma(\Omega)$, as asserted in Theorem 3.1(ii), Eq (15.22), we have in particular

$$F_\sigma \in L^p(0,T; L^q_\sigma(\Omega)) \longrightarrow \int_0^t e^{-A_q(t-\tau)} F_\sigma(\tau)d\tau \in L^p(0,T;\mathcal{D}(A_q)) \quad \text{continuously.} \tag{A.9}$$

Regarding the first integral term in (A.8) we shall employ the (complex) interpolation formula (15.9), and recall from (13.9) that $\mathcal{D}(A_{o,q}) = \mathcal{D}(A_q^{1/2})$:

$$\mathcal{D}(A_{o,q}) = \mathcal{D}(A_q^{1/2}) = [\mathcal{D}(A_q), L^q_\sigma(\Omega)]_{1/2} \tag{A.10}$$

so that the interpolation inequality [Triebel, Theorem p 53, Eq(3)] with $\theta = 1/2$ yields from (A.10)

$$\begin{aligned} \|a\|_{\mathcal{D}(A_{o,q})} = \|a\|_{\mathcal{D}\left(A_q^{1/2}\right)} &\leq C \|a\|^{1/2}_{\mathcal{D}(A_q)} \|a\|^{1/2}_{L^q_\sigma(\Omega)} \\ &\leq \varepsilon \|a\|_{\mathcal{D}(A_q)} + C_\varepsilon \|a\|_{L^q_\sigma(\Omega)} \end{aligned} \tag{A.11}$$

$\Big[$Since $\mathcal{D}(A_q^{1/2}) = W^{1,q}_0(\Omega) \cap L^q_\sigma(\Omega)$ by (14.21), then for $a \in \mathcal{D}(A_q) = W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega) \cap L^q_\sigma(\Omega)$, see (14.20), we may as well invoke the interpolation inequality for W-spaces. [Adams, Theorem 4.13, p 74]:

$$\|a\|_{W^{1,q}_0(\Omega)} \leq \varepsilon \|a\|_{W^{2,q}(\Omega)} + C_\varepsilon \|a\|_{L^q_\sigma(\Omega)} \Big]$$

We return to (A.8) and obtain

$$A_q\psi(t) = A_q \int_0^t e^{-A_q(t-\tau)}(-A_{o,q})\psi(\tau)d\tau + A_q \int_0^t e^{-A_q(t-\tau)} F_\sigma(\tau)d\tau. \tag{A.12}$$

Hence via the maximal regularity of the uniformly stable Stokes semigroup $e^{-A_q t}$, Eqts (15.22), (15.7)

$$\|A_q\psi\|_{L^p(0,T;L^q_\sigma(\Omega))} \leq C\Big\{ \|A_{o,q}\psi\|_{L^p(0,T;L^q_\sigma(\Omega))} + \|F_\sigma\|_{L^p(0,T;L^q_\sigma(\Omega))} \Big\} \tag{A.13}$$

$$\text{by (A.11)} \qquad \leq \varepsilon' \|A_q\psi\|_{L^p(0,T;L^q_\sigma(\Omega))} + C_{\varepsilon'} \|\psi\|_{L^p(0,T;L^q_\sigma(\Omega))} + C\|F_\sigma\|_{L^p(0,T;L^q_\sigma(\Omega))} \tag{A.14}$$

$\varepsilon' = \varepsilon C > 0$ arbitrarily small. Hence (A.14) yields

$$\|A_q\psi\|_{L^p(0,T;L^q_\sigma(\Omega))} \le \frac{C_{\varepsilon'}}{1-\varepsilon'}\|\psi\|_{L^p(0,T;L^q_\sigma(\Omega))} + \frac{C}{1-\varepsilon'}\|F_\sigma\|_{L^p(0,T;L^q_\sigma(\Omega))} \tag{A.15}$$

and estimate (A.4) of Step 2 is established. Part I of Theorem 15.5 is proved.

Part II: (15.36). For simplicity of notation, we shall write the proof on $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$ i.e. for $1 < q, p < {}^{2q}/_{2q-1}$. The proof on $\left(L^q_\sigma(\Omega), \mathcal{D}(A_q)\right)_{1-\frac{1}{p},p}$ in the other case ${}^{2q}/_{2q-1} < p$ is exactly the same.

Step 1: Let $\eta_0 \in \widetilde{B}^{2-2/p}_{q,p}(\Omega)$ and consider the s.c. analytic Oseen semigroup $e^{\mathcal{A}_q t}$ on the space $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$, as asserted by Theorem 3.3.ii:

$$\eta(t) = e^{\mathcal{A}_q t}\eta_0, \quad \text{or } \eta_t = \mathcal{A}_q\eta = -A_q\eta - A_{o,q}\eta \tag{A.16}$$

Then we can rewrite η as

$$\eta(t) = e^{-A_q t}\eta_0 + \int_0^t e^{-A_q(t-\tau)}(-A_{o,q})\eta(\tau)\,d\tau \tag{A.17}$$

$$A_q\eta(t) = A_q e^{-A_q t}\eta_0 + A_q\int_0^t e^{-A_q(t-\tau)}(-A_{o,q})\eta(\tau)\,d\tau \tag{A.18}$$

We estimate, recalling the maximal regularity (15.22), (15.23) as well as the uniform decay (13.25) of the Stokes operator.

$$\|A_q\eta\|_{L^p(0,T;L^q(\Omega))} \le C\|\eta_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} + C\|A_{o,q}\eta\|_{L^p(0,T;L^q_\sigma(\Omega))} \tag{A.19}$$

$$\le C\|\eta_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} + \varepsilon\widetilde{C}\|A_q\eta\|_{L^p(0,T;L^q_\sigma(\Omega))} + C_\varepsilon\|\eta\|_{L^p(0,T;L^q_\sigma(\Omega))} \tag{A.20}$$

after invoking, in the last step, the interpolation inequality (A.11). Thus (A.20) yields via (15.5)

$$\|A_q\eta\|_{L^p(0,T;L^q_\sigma(\Omega))} = \|\mathcal{A}_q\eta\|_{L^p(0,T;L^q_\sigma(\Omega))}$$
$$\le \frac{C}{1-\varepsilon\widetilde{C}}\|\eta_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} + \frac{C_\varepsilon}{1-\varepsilon\widetilde{C}}\|\eta\|_{L^p(0,T;L^q_\sigma(\Omega))} \tag{A.21}$$

Step 2: With $\eta_0 \in \widetilde{B}^{2-2/p}_{q,p}(\Omega)$, since $e^{\mathcal{A}_q t}$ generates a s.c (analytic) semigroup on $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$, Theorem 3.3.ii, we have

$$\eta(t) = e^{\mathcal{A}_q t}\eta_0 \in C\Big(0,T;\widetilde{B}^{2-2/p}_{q,p}(\Omega)\Big) \subset L^p\Big(0,T;\widetilde{B}^{2-2/p}_{q,p}(\Omega)\Big) \subset L^p\big(0,T;L^q_\sigma(\Omega)\big) \tag{A.22}$$

continuously, where in the last step, we have recalled that $\widetilde{B}^{2-2/p}_{q,p}(\Omega)$ is the interpolation between $L^q(\Omega)$ and $W^{2,q}(\Omega)$, see (15.3b). (A.22) says explicitly

$$\|\eta\|_{L^p\left(0,T;L^q_\sigma(\Omega))\right)} \leq C\,\|\eta_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \tag{A.23}$$

Step 3: Substituting (A.23) in (A.21) yields

$$\|A_q\eta\|_{L^p\left(0,T;L^q_\sigma(\Omega)\right)} \leq C\,\|\eta_0\|_{\widetilde{B}^{2-2/p}_{q,p}(\Omega)} \tag{A.24}$$

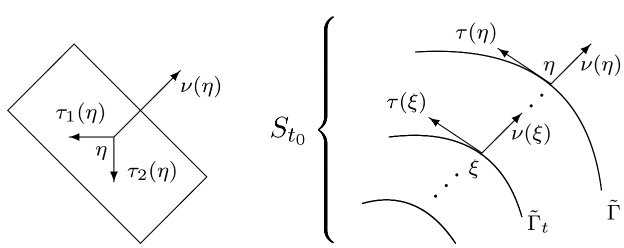
and (15.36) is established, from which (15.37) follows at once. Thus Theorem 15.5 is proved. □

Appendix D Justification of (18.38)

Step 0. Selection of tangential vector field.

$\tau(\xi)$ for $d = 3$ in ω. For $d = 3$, we may make a selection of the tangential vector field $\tau(\xi) = [\tau_1(\xi), \tau_1(xi)]$, an orthonormal system, $\xi \in \omega$, at the outset, following standard procedure [Car, pp 52-54]. We start with an open set $\mathcal{O}$ in $\mathbb{R}^2$ with canonical basis $e_1 = \{1, 0\}$, $e_2 = \{0, 1\}$, and corresponding coordinates $\{\alpha, \beta\}$. In $\mathcal{O}$, we fix two families oriented straight segments: family $\mathcal{F}_{e_1}$ with β = constant (segments parallel to e_1) and family $\mathcal{F}_{e_2}$ with α = constant (segments parallel to e_1). Let now $\mathcal{M}$ be a smooth, one-to-one conformal(parametrization) mapping of $\mathcal{O}$ onto $\widetilde{\Gamma} \subset \Gamma$, mapping the families $\mathcal{F}_{e_1}$ and $\mathcal{F}_{e_2}$ in $\mathcal{O}$ into two families $\mathcal{C}_{e_1}$ and $\mathcal{C}_{e_2}$ of oriented coordinate curves, respectively: $\mathcal{C}_{e_1}$ is obtained from $\alpha \to M(\alpha, \beta_0) = \{y_1(\alpha, \beta_0), y_2(\alpha, \beta_0), y_3(\alpha, \beta_0)\}$, while $\mathcal{C}_{e_2}$ is obtained from $\beta \to M(\alpha_0, \beta) = \{y_1(\alpha_0, \beta), y_2(\alpha_0, \beta), y_3(\alpha_0, \beta)\}$. Recall that any C^3-surface is conformal to the plane [Gug, p. 247], [Sto, p. 257], see also [Car, p.227]. For each point $x \in \widetilde{\Gamma}$, there exist only one coordinate curve of $\mathcal{C}_{e_1}$ and one coordinate curve of $\mathcal{C}_{e_2}$ meeting at x at an orthogonal angle. This allows us to define at each x of Γ an orthonormal system $\tau(x) = \{\tau_1(x), \tau_2(x)\}$. (Technically, we only need that $\tau_1(x)$ and $\tau_2(x)$ be a basis rather than an orthonormal basis.) Finally, we transport each such coordinate system $\tau(x)$, $x \in \widetilde{\Gamma}$, in a parallel fashion to points $\xi \in \omega$ by taking the normal line at x passing through ξ, so that $v(x) = v(\xi)$ for the two normal vectors.

Step 1. Preliminaries. [L-T.1, Appendix 3C, p. 297] The following considerations are actually local in character, and we may as well focus on a portion $\tilde{\Gamma}$ of Γ. Let $\eta \in \tilde{\Gamma}$, of class C^2. Let $\nu(\eta)$ denote the unit outward normal vector at η. On the tangent plane M_η of Γ at η, we let $[\tau_1(\eta), \tau_2(\eta)]$ denote an orthonormal system of tangent vectors



We then define the vector or point in Ω:

$$\xi = r(t; \eta) = \eta + t\nu(\eta), \quad -t_0 < t < 0, \ \eta \in \tilde{\Gamma}, \tag{B.1}$$

$|t_0|$ sufficiently small, that for t fixed and η running over $\tilde{\Gamma}$, describes the parallel translation surface $\tilde{\Gamma}_t$ of $\tilde{\Gamma}$ in Ω; moreover, as t runs over $(-t_0, 0)$, the family of surfaces $\tilde{\Gamma}_t$ sweeps a collar, or strip, S_{t_0} of $\tilde{\Gamma}$:

$$\tilde{\Gamma}_t = \{r(t;\eta) : \eta \in \tilde{\Gamma}\}, \qquad S_{t_0} = \bigcup_{-t_0<t<0} \tilde{\Gamma}_t. \tag{B.2}$$

The map $\eta \in \tilde{\Gamma} \mapsto \xi \in S_{t_0}$ is one-to-one (the Jacobian is $\neq 0$). For each $\eta \in \tilde{\Gamma}$ and corresponding $\xi = \eta + t\nu(\eta) \in S_{t_0}$, we let $\nu(\xi)$ be the unit outward normal to the surface $\tilde{\Gamma}_t$ passing through ξ, and let $[\tau_1(\xi), \tau_2(\xi)]$ be the corresponding orthonormal system of tangent vectors. Thus, we have

$$\nu(\xi) = \nu(\eta) \text{ and } [\tau_1(\xi), \tau_2(\xi)] = [\tau_1(\eta), \tau_2(\eta)] \quad \text{for all } \xi = \eta + t\nu(\eta),\ -t_0 < t < 0, \tag{B.3}$$

that is, the normal unit vector $\nu(\eta)$ at the boundary point $\eta \in \tilde{\Gamma}$ generates a constant vector field $\nu(\xi)$ for all points ξ of the normal line to η in the collar; and similarly for the orthonormal system $[\tau_1(\eta), \tau_2(\eta)]$ of tangent vectors. In this way, smooth vector fields $\nu(\xi)$ and $[\tau_1(\xi), \tau_2(\xi)]$ are defined at all points ξ of the collar, by parallel translation of $\nu(\eta)$ and the pair $[\tau_1(\eta), \tau_2(\eta)]$, $\eta \in \tilde{\Gamma}$, along the normal line to η. Thus, we may define the normal derivative and tangential derivatives of a sufficiently smooth vector $w = [w_1, \dots, w_d]$, $d = 2, 3$, to $\tilde{\Gamma}_t$ for each point $\xi = \eta + t\nu(\eta)$ of the collar S_{t_0} beside the case $\eta \in \tilde{\Gamma}$:

$$\frac{\partial w}{\partial \nu}(\xi) = \nabla w(\xi) \cdot \nu(\xi);\ \frac{\partial w}{\partial \tau_i}(\xi) = \nabla w(\xi) \cdot \tau_i(\xi),\ i = 1, 2, ; \tag{B.4a}$$

$$\nabla_\tau w(\xi) = \frac{\partial w}{\partial \tau_1}(\xi) \cdot \tau_1(\xi) + \frac{\partial w}{\partial \tau_2}(\xi) \cdot \tau_2(\xi); \quad \nabla w(\xi) = \frac{\partial w}{\partial \nu}(\xi) \cdot \nu(\xi) + \nabla_\tau w(\xi). \tag{B.4b}$$

Thus, given vector $w = [w_1, \dots, w_d]$, $d = 2, 3$, sufficiently smooth, the following two relations are well known to hold true pointwise at each point $\xi = Q$ of the collar of Γ:

(a) [L-T.1, Prop. 3C.6, p. 305], [S-Z.1, Proposition 2.68, p. 94]:

$$\Delta w|_Q = \left.\frac{\partial^2 w}{\partial \nu^2}\right|_Q + \left.\Delta_{\Gamma(\xi)} w\right|_Q + \left[\left(\frac{\partial w}{\partial \nu}\right)(\operatorname{div} \nu)\right]_Q, \quad Q \in \text{a collar of } \Gamma; \tag{B.5}$$

where Δ_Γ is the tangential Laplacian ($\Delta_\Gamma w|_Q = \left.\frac{\partial^2 w}{\partial \tau^2}\right|_Q$ when $d = 2$, where $\tau = [-\nu_2, \nu_1]$ is the corresponding unit tangential vector).

(b) [A-T, Prop. A.1, Appendix A] For $d = 2, 3$,

$$[\operatorname{div} w]_Q = \left[\frac{\partial w_1}{\partial x_1} + \cdots + \frac{\partial w_d}{\partial x_d}\right]_Q$$

$$= \left[\frac{\partial w}{\partial \nu} \cdot \nu\right]_Q + \left[\frac{\partial w}{\partial \tau_1} \cdot \tau_1\right]_Q + \left[\frac{\partial w}{\partial \tau_2} \cdot \tau_2\right]_Q, Q \in \text{a collar of } \Gamma. \quad \text{(B.6)}$$

(c) [L-T.1, Eqn. 3C.68, p. 309] The following property holds true:

$$\text{on } \Gamma : \frac{\partial}{\partial \tau_i} \frac{\partial f}{\partial \nu} = \frac{\partial}{\partial \nu} \frac{\partial f}{\partial \tau_i} + (\operatorname{div} \nu) \frac{\partial f}{\partial \tau_i}, \quad i = 1, 2. \quad \text{(B.7)}$$

Thus, the commutator of $\frac{\partial}{\partial \tau_i}$ and $\frac{\partial}{\partial \nu}$ is a first-order tangential operator.

The Needed Result. In the next lemma, we start with a function $\varphi(\xi)$ sufficiently smooth in the collar ω of Ω [not necessarily the function φ in problem (6.36a-b-c) subject to the assumption [the third condition in (18.36c)] that

$$\varphi(\xi) \cdot \tau(\xi) \equiv 0, \quad \xi \in \omega \subset \Omega; \quad \text{(B.8a)}$$

that is,

$$\varphi(\xi) = (\varphi(\xi) \cdot \nu(\xi))\nu(\xi) + (\varphi(\xi) \cdot \tau(\xi))\tau(\xi) \quad \text{(B.8b)}$$

$$= (\varphi(\xi) \cdot \nu(\eta))\nu(\eta). \quad \text{(B.8c)}$$

Lemma D.1. *(i) Let φ be a sufficiently smooth function defined on the collar ω such that assumption (A.8) holds true. Then, we have that:*

(i)

$$\frac{\partial \varphi(\xi)}{\partial \nu} = \textit{parallel to } \nu(\xi) \equiv \nu(\eta); \textit{ i.e., } \frac{\partial \varphi(\xi)}{\partial \nu} \cdot \tau(\xi) \equiv 0, \quad \xi \in \omega; \quad \text{(B.9)}$$

(ii)

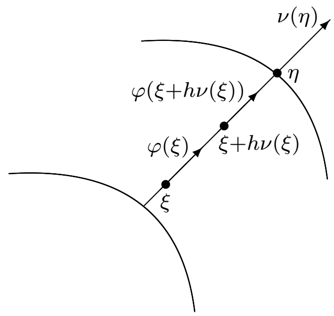
$$\frac{\partial \varphi^2(\xi)}{\partial \nu^2} = \textit{parallel to } \nu(\xi) \equiv \nu(\eta); \textit{ i.e., } \frac{\partial^2 \varphi(\xi)}{\partial \nu^2} \cdot \tau(\xi) \equiv 0, \quad \xi \in \omega, \quad \text{(B.10)}$$

hence

$$\left.\frac{\partial^2 \varphi}{\partial \nu^2}\right|_{\tilde{\Gamma}} \cdot \tau = 0 \textit{ on } \tilde{\Gamma}, \quad \text{(B.11)}$$

and $\frac{\partial^2 \varphi}{\partial \nu^2}$ is parallel to ν on $\tilde{\Gamma}$.

Proof.



Assumption (B.8) implies that the vectors $\varphi(\xi)$ and $\varphi(\xi+h\nu(\xi)) = \varphi(\xi+h\nu(\eta))$ are along such normal axis $\xi + h\nu(\xi) = \xi + h\nu(\eta)$, h scalar > 0, that is, orthogonal to $\tau(\xi) = \tau(\xi + h\nu(\eta)) = \tau(\eta)$. Thus, via (A.8c) and $\nu(\xi) = \nu(\eta)$:

$$\begin{aligned} \lim_{h\searrow 0} \frac{\varphi(\xi + h\nu(\xi)) - \varphi(\xi)}{h} &= \lim_{h\searrow 0} \left[\frac{(\varphi(\xi + h\nu(\eta)) - \varphi(\xi))}{h} \cdot \nu(\eta) \right] \nu(\eta) \\ &= \left(\frac{\partial\varphi(\xi)}{\partial\nu} \cdot \nu(\eta) \right) \nu(\eta), \end{aligned}$$

and thus (B.9), i.e., part (i), is established.

Part (ii) now follows by replacing $\varphi(\xi)$ with $\frac{\partial\varphi(\xi)}{\partial\nu}$, which by part (i) satisfies the counterpart of assumption (B.8). □

Appendix E The eigenvectors $\varphi^*_{ij} \in W^{2,q'}(\Omega) \cap W^{1,q'}_0(\Omega) \cap L^{q'}_\sigma(\Omega)$ of $\mathcal{A}^*(= \mathcal{A}^*_q)$ in $L^{q'}(\Omega)$ may be viewed also as $\varphi^*_{ij} \in W^{3,q}(\Omega)$, so that $\left.\dfrac{\partial \varphi^*_{ij}}{\partial \nu}\right|_\Gamma \in W^{2-1/q,q}(\Gamma),\ q \geq 2$

The eigenvectors φ^*_{ij} of $\mathcal{A}^*(= \mathcal{A}^*_q)$, defined in (18.1), are in $\mathcal{D}((\mathcal{A}^*_q)^n)$ for any n, so the are arbitrarily smooth in $L^{q'}_\sigma(\Omega)$, say $\varphi^*_{ij} \in W^{s,q'}(\Omega)$, with s as large as we please. We seek to view φ^*_{ij} in an $L^q(\Omega)$-based space. To this end, we recall a Sobolev embedding theorem.

Theorem E.1. *[Triebel, p328], For a more restricted version [Adams] Let Ω be an arbitrary bounded domain, dim $\Omega = d$. Let $0 \leq t \leq s < \infty$ and $\infty > q \geq \widetilde{q} > 1$. Then, the following embedding holds true:*

$$W^{s,\widetilde{q}}(\Omega) \subset W^{t,q}(\Omega),\ s - \frac{d}{\widetilde{q}} \geq t - \frac{d}{q} \quad \square \tag{C.1}$$

Corollary E.2. With $2 \leq q < \infty,\ ^1/_q + {}^1/_{q'} = 1$, so that $1 < q' \leq 2 \leq q,\ 0 \leq r$, we have

(i)

$$\varphi^*_{ij} \in W^{r+m,q'}(\Omega) \subset W^{r,q}(\Omega), \quad m \geq d\left(\frac{1}{q} + \frac{1}{q'}\right) = \begin{cases} 0, & q' = q = 2 \\ d, & q' = 1, q = \infty \end{cases} \tag{C.2}$$

(ii)

$$\left.\frac{\partial \varphi^*_{ij}}{\partial \nu}\right|_\Gamma \in W^{r-1-1/q,q}(\Gamma), \quad r > 1 + \frac{1}{q} \tag{C.3}$$

(iii) With reference to the sub-space $\mathcal{F}$ based on Γ, as defined in (13.23) = (18.8), we have

$$\mathcal{F} \equiv \text{span}\left\{\frac{\partial}{\partial \nu}\varphi^*_{ij},\ i = 1, \dots, M;\ j = 1, \dots, \ell_i\right\} \subset W^{r-1-1/q,q}(\Gamma),\ r > 1 + \frac{1}{q} \tag{C.4}$$

In particular, for our purposes, if will suffice to take $r = 3$ in (C.2), so that (C.2)-(C.4) become

$$\varphi^*_{ij} \in W^{3,q}(\Omega), \quad \left.\frac{\partial \varphi^*_{ij}}{\partial \nu}\right|_\Gamma \in W^{2-1/q,q}(\Gamma), \quad \mathcal{F} \subset W^{2-1/q,q}(\Gamma). \tag{C.5}$$

(iv) Thus, with reference to the boundary vector $v = v_N$ introduced in (17.1) = (18.10), we have

$$v = \sum_{k=1}^K \nu_k(t) f_k \in W^{2-1/q,q}(\Gamma),\ f_k \in \mathcal{F},\ f_k \cdot \nu|_\Gamma = 0,\ v \cdot \nu|_\Gamma = 0 \tag{C.6}$$

(v) Recalling the Dirichlet map D introduced to describe the solution of problem (14.1), we have

$$Dv \in W^{2,q}(\Omega), \quad 2 \le q < \infty \tag{C.7}$$

Proof. (i) Apply Theorem E.1 with $s = r+m \ge t = r$, $\widetilde{q} = q'$, ${}^1/_q + {}^1/_{q'} = 1$, $q \ge 2$, so that $q' = \widetilde{q} \le q$, to verify that the required condition (C.1)

$$s - \frac{d}{\widetilde{q}} = r + m - \frac{d}{q'} \ge t - \frac{d}{q} = r - \frac{d}{q}, \text{ or } m \ge d\left(\frac{1}{q'} - \frac{1}{q}\right) \ge 0 \tag{C.8}$$

can always be satisfied by taking $m \ge 0$ suitable as in (C.8). This is possible, since φ^*_{ij} is arbitrarily smooth.

(ii) Then (C.3) follows by the usual trace theory [Adams].

Then, (iii)-(v) readily follow.

□

Next, we return to the operator $F : L^q(\Omega) \subset L^q_\sigma(\Omega) \longrightarrow L^q(\Gamma)$ in (21.11) = (17.13). Its adjoint F^* is

$$F^* g = \sum_{k=1}^{K} (f_k, g)_\Gamma p_k \in (W^u_N)^* \subset L^{q'}_\sigma(\Omega), \quad g \in L^q(\Gamma) \tag{C.9}$$

where we have seen in (15.41) that $D : L^q(\Gamma) \supset U_q \longrightarrow W^{1/q,q}(\Omega) \cap L^q_\sigma(\Omega) \subset \mathcal{D}\Big(A_q^{1/2q-\varepsilon}\Big)$

$$F^* D^* h = \sum_{k=1}^{K} (f_k, D^* h)_\Gamma p_k = \sum_{k=1}^{K} \big\langle D f_k, h \big\rangle_{W^u_N} p_k \in (W^u_N)^* \tag{C.10}$$

where we have conservatively: $f_k \in L^{q'}(\Gamma)$, $f_k \cdot \nu = 0$ on Γ, thus by (14.13), $Df_k \in W^{1/q',q'}(\Omega) = W_0^{1/q',q'}(\Omega)$ by (15.40a) since ${}^1/_{q'} \le q'$ for $1 < q' \le 2$. Thus, in (C.10) we can take $h \in W^{-1/q,q}(\Omega)$. In particular

$$F^* D^* \in \mathcal{L}\big(L^{q'}(\Omega)\big), \quad 1 < q' \le 2. \tag{C.11}$$

Appendix F Relevant unique continuation properties for overdetermined Oseen eigenvalue problems

In this Appendix F, we assemble a comprehensive account of unique continuation problems for Oseen eigenproblems, as they pertain to the problem of controllability of finite dimensional projected system (4.8a-b) of the linearized w-problem (13.11a) (with interior, tangential-like localized control $u \equiv 0$). Positive solution, or lack thereof, of this finite dimensional problem is a key step, or obstruction, for the uniform stabilization of the Navier Stokes equations. This issue has been known since the study of boundary feedback stabilization of a parabolic equation with Dirichlet boundary trace in the feedback loop, as acting on the Neumann boundary conditions [L-T.4]. We return to the bounded domain $\Omega,\ d = 2, 3$, with boundary $\Gamma = \partial\Omega$. As before, $\widetilde{\Gamma}$ is a subportion of Γ.

Problem #1 (over-determination only on a portion $\widetilde{\Gamma}$ of Γ) Let $\{\varphi, p\} \in W^{2,q}(\Omega) \times W^{1,q}(\Omega)$ solve the over-determined problem

$$\begin{cases} (-\nu_o\Delta)\varphi + L_e(\varphi) + \nabla\pi = \lambda\varphi & \text{in } \Omega \quad \text{(D.1a)} \\ \operatorname{div} \varphi = 0 & \text{in } \Omega \quad \text{(D.1b)} \\ \varphi|_{\widetilde{\Gamma}} \equiv 0, \quad \left.\dfrac{\partial\varphi}{\partial\nu}\right|_{\widetilde{\Gamma}} \equiv 0 & \text{on } \widetilde{\Gamma} \quad \text{(D.1c)} \end{cases}$$

with over-determination only on the portion $\widetilde{\Gamma}$ of Γ. Does (D.1a-b-c) imply

$$\varphi \equiv 0 \quad \text{and} \quad p \equiv \text{const} \quad \text{in } \Omega\ ? \tag{D.2}$$

The answer is <u>negative</u> even in the Stokes case: $L_e(\varphi) \equiv 0$. This follows from [F-L], where the following counterexample is given in the 2-dimensional half-space $\Omega = \{(x, y) : x \in \mathbb{R}^+, y \in \mathbb{R}\}$ with boundary $\Gamma = \{x = 0\}$. On Ω take

$$u_1(x, y) \equiv 0, \quad u_2(x, y) = ax^2, \quad p = 2ay, \quad a \neq 0, \tag{D.3}$$

so that with $u = \{u_1, u_2\}$, it follows that

$$\Delta u = \nabla p \text{ in } \Omega, \quad u|_\Gamma = \nabla u|_\Gamma = 0, \tag{D.4}$$

to obtain a nontrivial solution of the Stokes overdetermined eigenproblem with $\lambda = 0$. Such half-space example can then be transformed into a counterexample over the bounded domain Ω where the over-determination is active on any subset $\widetilde{\Gamma}$ of the boundary $\Gamma = \partial\Omega$.

Implications of failure of unique continuation under Problem #1: A negative consequence of the lack of unique continuation (D.1) $\implies$ (D.2) with over-determination only in a portion $\widetilde{\Gamma}$ of the boundary Γ is as follows: that global uniform stabilization of the linearized w-problem (1.11a-d) by means of a purely tangential (finite or infinite dimensional) feedback boundary control v (as given by (17.8) in the finite dimensional case) acting only on a small subportion $\widetilde{\Gamma}$ of the boundary Γ (and thus with localized interior tangential-like control $u \equiv 0$) is not possible. This is so since the algebraic rank condition (6.28b) (with $u \equiv 0$) fails, as boundary traces

$$\left\{ \left.\frac{\partial \varphi^*_{i1}}{\partial \nu}\right|_{\widetilde{\Gamma}}, \left.\frac{\partial \varphi^*_{i2}}{\partial \nu}\right|_{\widetilde{\Gamma}}, \dots, \left.\frac{\partial \varphi^*_{i\ell_i}}{\partial \nu}\right|_{\widetilde{\Gamma}} \right\} \tag{D.5}$$

fail to be linearly independent on $\widetilde{\Gamma}$ since, equivalently, the implication (D.1) $\implies$ (D.2) fails. See Orientation.

Problem #2 (same as the statement of Lemma 18.2): necessity to complement the localized control v on $\widetilde{\Gamma}$ with a localized interior tangential-like control u supported on ω in terms of $\widetilde{\Gamma}$. Let now $\{\varphi, p\} \in W^{2,q}(\Omega) \times W^{1,q}(\Omega)$ solve the problem

$$\begin{cases} (-\nu_o \Delta)\varphi + L_e(\varphi) + \nabla \pi = \lambda \varphi & \text{in } \Omega \quad \text{(D.6a)} \\ \text{div } \varphi = 0 & \text{in } \Omega \quad \text{(D.6b)} \\ \varphi|_{\widetilde{\Gamma}} \equiv 0, \quad \left.\frac{\partial \varphi}{\partial \nu}\right|_{\widetilde{\Gamma}} \equiv 0, \quad \varphi \cdot \tau \equiv 0 & \text{in } \omega \quad \text{(D.6c)} \end{cases}$$

Then, [L-T.3, Theorem 6.2],

$$\varphi \equiv 0 \quad \text{and} \quad p \equiv \text{const} \quad \text{in } \Omega. \tag{D.7}$$

It is as a consequence of such unique continuation property that the Kalman algebraic rank conditions (6.28b) are satisfied. This is the basic result upon which the uniform stabilization of the present paper relies. Thus we can conclude that the results of the present paper (as in [L-T.3]) are optimal in terms of the required extra condition of the localized interior, tangential-like control needed to supplement the insufficient role of the localized tangential boundary control v on $\widetilde{\Gamma}$. Optimality is in terms of the smallness of the required control action for v and u.

Problem #3 (over-determination on the entire boundary $\Gamma = \partial\Omega$). Let now $\{\varphi, p\} \in W^{2,q}(\Omega) \times W^{1,q}(\Omega)$ solve the over-determined problem

$$\begin{cases} (-\nu_o\Delta)\varphi + L_e(\varphi) + \nabla\pi = \lambda\varphi & \text{in } \Omega \qquad \text{(D.8a)} \\ \text{div } \varphi = 0 & \text{in } \Omega \qquad \text{(D.8b)} \\ \varphi|_\Gamma \equiv 0, \quad \left.\dfrac{\partial\varphi}{\partial\nu}\right|_\Gamma \equiv 0 & \text{on } \Gamma \qquad \text{(D.8c)} \end{cases}$$

with over-determination on all of Γ. Then, does (D.8a-b-c) imply

$$\varphi \equiv 0 \quad \text{and} \quad p \equiv \text{const} \quad \text{in } \Omega \ ? \tag{D.9}$$

It seems that a general definitive answer is not known at present. Only partial results are known.

The desired unique continuation (D.8) $\Longrightarrow$ (D.9) holds true, if the equilibrium solution $y_e \equiv 0$ (Stokes eigenproblem) or, more generally, if y_e is sufficiently small in the $W^{1,q}(\Omega)$-norm. Several different proofs are given in [RT.4] and [RT.5].

The case $y_e \equiv 0$ is actually physically quite important as it occurs for instance when the forcing function in (13.1a) or (13.2a) is a conservative vector field (say an electrostatic or gravitational field) $f = \nabla g$. In this case, a solution (1.2a-b-c) is: $y_e \equiv 0,\ \pi_e = g$.

When $y_e \equiv 0$ (or y_e small) the tangential boundary feedback control v alone, in the form such as (17.13), as acting on the entire boundary Γ produces enhancement of stability at will for the linearized w-problem.

Of course, with $y_e \equiv 0$, the corresponding Oseen problems reduces to the Stokes problem. The Stokes semigroup is already uniformly stable, see (15.7), with margin of stability $\delta > 0$. When $y_e \equiv 0$ a most valuable variation of the problem under investigation of the present paper is to enhance the original margin of stability $\delta > 0$ of the original linearized uncontrolled w-problem (13.11) (with $u \equiv 0,\ v \equiv 0$) to obtain an arbitrary decay rate, say k^2, by means of only a tangential boundary finite dimensional feedback control, of the same form as the operator F in (17.13) but applied to all of Γ. To this, it suffices to apply the procedure of the present paper to a finite dimensional projected space spanned by the eigenvectors of the Stokes operator corresponding to finitely many eigenvalues $\lambda_i,\ i = 1, \dots, I$,

$$- k^2 \le -Re\ \lambda_I \le \cdots \le Re\ \lambda_1 \le -\delta \tag{D.10}$$

Problem #4 over-determination on a portion of the boundary $\widetilde{\Gamma}$ involving also the pressure p. Let $\{\varphi, p\} \in W^{2,q}(\Omega) \cap W^{1,q}(\Omega)$ solve the over-determined problem

$$\begin{cases} (-\nu_o\Delta)\varphi + L_e(\varphi) + \nabla\pi = \lambda\varphi & \text{in } \Omega & \text{(D.11a)} \\ \text{div } \varphi = 0 & \text{in } \Omega & \text{(D.11b)} \\ \varphi|_\Gamma \equiv 0, \quad \left[\dfrac{\partial\varphi}{\partial\nu} - p\nu\right]_\Gamma \equiv 0 & & \text{(D.11c)} \end{cases}$$

Does this imply

$$\varphi \equiv 0 \quad \text{and} \quad p \equiv \text{const} \quad \text{in } \Omega\ ? \tag{D.12}$$

This answer is in the affirmative. The argument, given in the [RT.4] is along more classical elliptic arguments [Ko]. Here however the new condition in (D.11c) contains the pressure, which must be viewed as unknown in general. Application of this result to the present paper will result in substituting $\partial_\nu \varphi^*_{ij}|_{\widetilde{\Gamma}}$ with $\partial_\nu \varphi^*_{ij} - p_i\nu|_{\widetilde{\Gamma}}$ in the matrix W_i in (18.12) or (19.11), which then - with this modification - becomes full rank, as desired. Thus, the stabilizing control will be expressed in terms of the pressure on the boundary, which is typically unknown.

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