1-1-2019

Problems in graph theory and partially ordered sets

Stephen Gene Zane Smith

PROBLEMS IN GRAPH THEORY AND PARTIALLY ORDERED SETS

by

Stephen G. Z. Smith

A Dissertation

Submitted in Partial Fulfillment of the

Requirements for the Degree of

Doctor of Philosophy

Major: Mathematical Sciences

The University of Memphis

May 2019

ACKNOWLEDGEMENTS

First, and foremost, I would like to thank my supervisors, Professors Béla Bollobás and Paul Balister.

I must express my deepest gratitude for the opportunities afforded to me by Béla. Without Béla, there is no doubt that I would not have been able to travel to so many wonderful places, not to mention being able to complete some of this work at universities and institutes as a guest. I am also indebted to Béla for the influence in my taste in mathematics. Everything contained in this dissertation is a direct result of exposure to a topic in a mini-seminar or a particular lecture on a topic that I found amazing. Finally, I am most grateful for the people that I have met through him. Many of these people have become good friends.

I would also express my deepest gratitude to Paul Balister for all of the help over the years. I have been lucky to have such an amazing supervisor with seemingly infinite patience. Beyond his abilities, both in mathematics and supervising, he has been a truly outstanding lecturer in such a varied number of topics, and I aspire to be as good a lecturer as he is. He has most definitely influenced in my teaching style during my time as a teaching assistant. I am also grateful for the indispensable help and feedback during this process.

I am also very grateful to have Tricia Simmons. She has been absolutely amazing in all things, especially helping me navigate the strange waters of academia. Outside of that realm, she has been incredibly supportive through her empathy, encouragement, and ability to help me remain focused. She is truly a cornerstone of the combinatorics group in Memphis.

One of the unseen benefits of belonging to this group is the ability to create friendships with people that I would never have met otherwise. Their varied backgrounds and interests have caused my own interests to grow as a result. Specifically, I am grateful to Tomas Juškevičius for sparking an interest in antichains, Michał Przykucki for percolation-type processes, Richard Johnson for helping me prepare for exams and our time together as housemates, and Rebekah Herrman for sitting through preparations for the defense of this disseration. I would also like to express my gratitutde to my collaborators, István Tomon, Gábor Mészáros, António Girão, and Peter van Hintum. I would especially like to thank Peter, who has been able to make things click very quickly, for being a great house mate during some of the aforementioned trips, and for all his help with proof reading.

Outside of the research area, I would like to thank Fernanda Botelho for being amazing in her duties as director of graduate studies and graduate coordinator as well as her support and encouragement in all things. I would also like to thank John Haddock for his support and encouragement while fulfilling a plethora of duties in the department.

I would also like to thank Gabriella Bollobás for her warmth, kindness, and hospitality that she extends to all of Bélas students, as well as to our significant others.

I am very lucky to have the support of my parents, without whom I doubt I would have been able to accomplish this. Their support through this has been immeasurable. I would also like to express my love and gratitude to Amanda LeQuire for her support, love, and understanding during this process. I know that whatever comes next, she will be there.

Finally, and most certainly not least, I would like to thank Dr. Moacir Schnapp. Long before I thought about pursuing this PhD, I was diagnosed with painful peripheral neuropathy, and I strongly doubt that I would have been able to get through those first few years without his expert medical treatment and care. His empathy, professionalism, and support has helped by keeping me on my feet and walking as much as possible, making it possible to find a balance in pain management and mathematics.

PREFACE

This dissertation features research from two publications. The first publication is joint work with Gábor Mészáros, António Girão, [20], is titled, ”On a conjecture of Gentner and Rautenbach”, and appears in Discrete Mathematics. This research appears in Section I, Chapter 3. The second publication is joint work with István Tomon, [36] is titled ”The poset of connected graphs is Sperner”, appears in The Journal of Combinatorial Theory, Series A. This research is the entirety of Section III.

ABSTRACT

Smith, Stephen G. Z. Ph.D. The University of Memphis. May 2019. Graphs and Partially Ordered Sets. Major Professor: Béla Bollobás, Ph.D and Paul Balister, Ph.D.

This dissertation answers problems in three areas of combinatorics - processes on graphs, graph coloring, and antichains in a partially ordered set.

First we consider Zero Forcing on graphs, an iterative infection process introduced by AIM Minimum Rank - Special Graphs Workgroup in 2008. The Zero Forcing process is a graph infection process obeying the following rules: a white vertex is turned black if it is the only white neighbor of some black vertex. The Zero Forcing Number of a graph is the minimum cardinality over all sets of black vertices such that, after a finite number of iterations, every vertex is black. We establish some results about the zero forcing number of certain graphs and provide a counter example of a conjecture of Gentner and Rautenbach. This chapter is joint with Gábor Mészáros, António Girão, and Chapter 3 appears in Discrete Math, Vol. 341(4).

In the second part, we consider problems in the area of Dynamic Coloring of graphs. Originally introduced by Montgomery in 2001, the r-dynamic chromatic number of a graph G is the least k such that $V(G)$ is properly colored, and each vertex is adjacent to at least r different colors. In this coloring regime, we prove some bounds for graphs with lattice like structures, hypercubes, generalized intervals, and other graphs of interest. Next, we establish some of the first results in the area of r-dynamic coloring on random graphs. The work in this section is joint with Peter van Hintum.

In the third part, we consider a question about the structure of the partially ordered set of all connected graphs. Let $\mathcal{G}$ be the set of all connected graphs on vertex set $[n]$. Define the partial ordering $<$ on $\mathcal{G}$ as follows: for $G, H \in \mathcal{G}$ let $G < H$ if $E(G) \subset E(H)$. The poset $(\mathcal{G}, <)$ is graded, each level containing the connected graphs with the same number of edges. We prove that $(\mathcal{G}, <)$ has the Sperner property, namely that the largest antichain of $(\mathcal{G}, <)$ is equal to its largest sized level. This chapter is in collaboration with István Tomon and appears in The Journal Theory Series A, Vol. 150

TABLE OF CONTENTS

LIST OF FIGURES

Figure | Page

Part I

Zero Forcing

CHAPTER 1

INTRODUCTION

The Zero Forcing Number of a graph was first introduced by Burgarth and Giovannetti in 2007 [15] and independently by the AIM Minimum Rank - Special Graphs Workgroup in 2008 [1]. The original motivation for the latter came about, due to the problem of bounding the minimum rank of adjacency matrices of a graph, while the former used the model to describe the controllability of certain quantum systems. Despite its beginnings in linear algebra and small applications in physics, the model has received considerable attention from combinatorialists due to its obvious ties to graph theory ([11, 12, 18, 13]).
The Zero Forcing process is a discrete-time process in which we have a set of vertices of a graph G that is initially colored black, while the remaining vertices are colored white. At each time step, the color change rule is applied, according to Definition 1. Once a vertex has been changed to black, it remains black forever. The process terminates if all of the vertices are colored black in finite time. Otherwise, if there are some white vertices that are unable to be colored in finite time, then we may artificially halt the process.
For the purposes of this paper, the Zero Forcing Process is always on a finite, simple, undirected graph.
In this chapter, we prove that the forcing number of a graph, $F(G)$, can be affected by the removal of edges. Namely, if we remove k edges from a graph, then $F(G)$ can drop by at most k, or it can increase by at most k. We also determine the forcing number for complete graphs when a triangle is removed, as well as the forcing number for complete d-ary trees. Finally, we end this chapter with a counter example to a conjecture of Gentner and Rautenbach [19]. Genter and Rautenbach conjectured that for all graphs with maximum degree 3, the forcing number is at most $\frac{1}{3} \mid V(G) \mid +2$. We construct an infinite family of graphs with maximum degree 3 such that the forcing number is *at least* $\frac{4}{9} \mid V(G) \mid$.

Definitions and Preliminaries

Definition 1. The *Color Change Rule* states that if u is a white vertex in G, it will be recolored if it is the only white neighbor of some black vertex v. We say that the vertex v *forced* u to change color.

Definition 2. An Initial Set or Forcing Set, is a set of black vertices of G such that, after a finite number of applications of the Color Change Rule, every vertex in G is colored black.

Definition 3. The Zero Forcing Number for a graph G, written $F(G)$, is the minimum cardinality of all zero forcing sets of G. In other words, $F(G) = \min\{|Z| : Z \subset V(G), Z \text{ forces } G\}$. It is clear there might be distinct forcing sets with minimum cardinality, however we shall refer to any such set as a *Zero Forcing* Set.

Definition 4. A black vertex u is said to be an *active* vertex if it has only one white neighbor.

Definition 5. A black vertex u is said to be a *support* vertex of v, $u \in \text{supp}(v)$, if v is active and u is a black neighbor. Equivalently, recoloring u from black to white results in v being unable to force any vertex.

Definition 6. A *forcing sequence* is a sequence of vertices, $\mathcal{S} = (v_0, v_1, \ldots, v_k)$, such that v_{i-1} forces v_i, for $i = 0, \ldots, k-1$.

Definition 7. A *forcing chain* is a list of all forces performed; $\mathcal{C} = \{(z_0 \to v_0), \ldots (z_i \to v_i), \ldots, (z_k \to v_k)\}$.

Definition 8. A *derived set* is a chronological list of forces performed, $\mathcal{D} = (S_0, S_1, \ldots S_l)$, where S_0 is the set of verticies initially colored black, and S_i is the set of vertices colored black at time i.

CHAPTER 2

ALTERATIONS

Introduction

In this section, we explore the effect of some elementary graph operations on the zero forcing number $F(G)$. In [33], Row introduced a technique for computing the forcing number of a graph with a cut-vertex. In particular, it was shown that one may be able to more easily compute the forcing number of a graph G, with a cut-vertex v, by considering the forcing numbers of the components of $G-v$. In [32], Owens showed that, for a given edge e, the forcing number of $G-e$ satisfies $F(G)-1 \leq F(G-e) \leq F(G)+1$. In this same spirit, we explore the effect that some graph operations have on the forcing number. The following lemma shall be quite useful as it gives us a clear and obvious way to identify some of the edges that we may add or remove from a graph while avoiding a change in the zero forcing number.

Lemma 9. Let G be a graph with derived set $\mathcal{D} = (S_0, \ldots S_l)$, of length l, generated by minimum zero forcing set $S_0 = Z$. If u and v are two distinct vertices that appear in S_i, then we may add edge $e = uv$, or if it already exists, remove it, without affecting the zero forcing number.

Proof.
Consider two vertices in the S_i. Adding an edge between any two vertices in S_i does not affect whether vertices in S_{i-1} were active. Once the vertices in S_i are active, no edge in S_i affects which vertices of S_i are active. □

Corollary 10. If k vertices are forced simultaneously, we can add enough edges to create a *clique* on those vertices, and similarly, we may remove all of the edges between any pair of the vertices.

F(G) Under Edge Removal

Let G be a simple, undirected, loopless graph. Let $F(G)$ be its forcing number. We begin with a simple lemma for the lower bound of $F(G)$ and few questions about edge and vertex removal.

Lemma 11. Let $\delta(G)$ be the minimum degree of a graph G, then $F(G) \geq \delta(G)$.

Proof.
Let v be a vertex. If v is active, it must have $deg(v) - 1$ neighbors that are black, as well as itself. Therefore, at time zero, we have at least $\delta(G)$ black vertices. □

Question 12. Let G be a graph, $F(G)$ its forcing number, and $0 \leq k \leq F(G)$. How many edges may we remove from G such that its forcing number decreases or increases by k?

Question 13. How many edges may we add to a graph without affecting its forcing number?

Indeed, given a vertex v, its addition to, or deletion from a graph can cause the zero forcing number of drop by one, increase by one, or remain unchanged. Similarly, edge removal or addition can wildly alter the forcing number. The following two propositions are due to K. Owens [32].

Proposition 14. Let v be a vertex of the graph G. Then
$F(G-v) - 1 \leq F(G) \leq F(G-v) + 1$.

Proposition 15. Let e be an edge of the graph G. Then $F(G) - 1 \leq F(G-e) \leq F(G) + 1$.

We now know that we should be a bit more restrictive with the base graph we begin with and which operations we execute. To that end, let us modify Question 12 by asking, "If e is any edge in $G = K_n$, and $G' = K_n - e$, what is $F(G')$?" It is easy to see that $F(G') = n - 2$. Take the same initial set as in K_n , label the white vertex w_1, and remove any edge e. Let $u, v \in V(G)$. There are two cases: $e = uv$, or $e = uw_1$. For the former case, without loss of generality, recolor u from black to white and observe that v is now the only vertex without two white neighbors. Thus, v forces w_1, followed by the forcing of u. For the latter, recolor any other black vertex that is not u. This results in every vertex neighboring two white vertices, except for u, which only neighbors the recently recolored. This is clearly enough to force the graph.

Next, $F(G')$ can not be less than $n-2$ by Lemma 11. If not, then there exists a forcing set of size $n-3$. This clearly does not work as now every vertex has at least two white neighbors, contradicting the assumption that a set of $n-3$ vertices forces the graph.

Removal of Independent Sets

Theorem 16. Let G be a complete graph on n vertices. Let M be a partial matching. Let $G' = G - M$. Then $F(G') = n-2$.

Proof.

Let A be the set of initially colored vertices such that A forces G. Let $G' = G - M$. The removal of M causes the degree of every vertex that is an incident to an edge in M to decrease by 1. Let $u, v \in A$ such that the edge $uv \in M$, and let $w \notin \{u, v\}$ be the only white vertex of G. Without loss of generality, suppose u is white. Then v forces w, followed by w forcing u. Thus, $|A| = n-2$ and $F(G') = n-2$.

Finally, we can not improve this as the minimum degree is a lower bound for the forcing number of any graph.

□

Before proceeding further, it is worth pointing out that a graph with forcing number $n-1$ is both a necessary and sufficient condition for a graph to be complete. In other words, removing any edge e_1 from a K_n causes the forcing number to decrease by one. However, if we have two independent edges in K_n that we call e_1 and e_2, their simultaneous removal does not decrease the forcing number by 2. In other words, $F(K_n - e_1) = F(K_n - e_1 - e_2)$ here.

In Theorem 16, we showed that the forcing number drops by one after a matching, even a partial matching, is removed from a complete graph. However, Propositions 14 and 15 hint that it is not always the case for any set of edges. To that end, we explore effect that removal of more edges from a graph has on the forcing number.

Theorem 17. Let G be a graph with two edges, $e_1 = uv, e_2 = xy$. Then $F(G) - 2 \leq F(G - e_1 - e_2) \leq F(G) + 2$

Proof.

Apply Proposition 14 twice. □

Theorem 18. Let G be a graph with a set M of k edges. Then $F(G) - k \leq F(G - M) \leq F(G) + k$.

Proof.

Apply Proposition 15 k times. □

Theorem 19. Let $e_1 = ux_1, e_2 = ux_2, e_3 = ux_3 \ldots, e_n = ux_k$ be n dependent edges in a graph G. Let $G' = G - \{\cup_i^k e_i\}$. Then $F(G) - k \leq F(G') \leq F(G) + 2$.

Proof.

For the inequality on the left hand side, apply Theorem 18.

For the inequality on the right hand side, let B be a forcing set for G. Again, any force by a vertex that is not u or one of the x_i's is still a valid force in G', as any force by u or x_i to a vertex not in $\{u, x_1, \ldots, x_k\}$ is still a valid force. If u is the only vertex from the end points of dependent edges that forces an x_i, then we can add that x_i to B to guarantee that the process completes as all other x_i must already be black when u forces x_i. Therefore, $B \cup \{x_i\}$ is a forcing set for G'. Suppose, without loss of generality, that $x_1 \to u$ is a force and is followed by $u \to x_2$. Then it must have been the case that for all $i \in \{3, \ldots, k\}$, x_i was either in the initial forcing set, or that it was forced by another vertex. Therefore, $B \cup \{u, x_2\}$ is a forcing set for G'. Thus, $F(G') \leq F(G) + 2$. □

Removal of Triangles

Theorem 20. Let G be a graph with a triangle Δ, whose edges are $e_1 = xy, e_2 = xz$, and $e_3 = yz$. Let $G' = G - \Delta$. Then $F(G) - 2 \leq F(G') \leq F(G) + 1$.

Proof.

First, we show $F(G) - 2 \leq F(G')$. Suppose that $x \to x_1, y \to y_1$, and $z \to z_1$ in G'. Let B be a forcing set for G'. Then $B \cup \{y, z\}$ is a forcing set for G where x is the first black vertex in the forcing process on G'. For the other inequality, the forcing number only changes when one of the vertices of the triangle forces another. Again, without loss of generality, suppose $x \to y$. Then z must have already been colored black, so removing the three edges requires us to only color y. Thus, the forcing number of G' is at most $F(G) + 1$. □

Edge Additions to Cycles

Theorem 21. If C_n is a cycle on n vertices, then at most $n - 3$ edges can be added such that the forcing number is unaffected.

Proof.

Let G be a graph on n vertices with forcing number $F(G) = 2$. We will show by induction on the number of vertices that $| E(G) | \leq 2n - 3$. For the case $n = 3$, there can be at most 3 edges, thus satisfying $| E(G) | \leq 2n - 3$. This means that G must be a K_3 since we have assumed that $F(G) = 2$, otherwise G would be a path and have forcing number 1.
Henceforth, we assume that $n \geq 4$ and $| E(G) | \leq 2n - 3$.
Let $v \in G$ be a vertex forced at the last time step t. Then, at time $t - 1$, v is the only white neighbor to some black vertex. Since a black vertex can force at most one white vertex, and $F(G) = 2$, then at most 2 vertices are active at each time step.
There are two possibilities here. Since we can not have more than 2 vertices forced at any time step, either v was the only white vertex at time step $t - 1$, or there was at most one other vertex that was white at time step $t - 1$, say u.

We first consider the case when v was the sole white vertex at time step $t-1$ in the forcing process on G. Let $G' = G - v$. Since there can be at most two active vertices for each time step, v was adjacent to at most two vertices in G, hence
$\mid E(G') \mid \leq 2n-3-2 = 2(n-1)-3$.
If u and v were the remaining white vertices at time step $t-1$, then each was adjacent to only 1 active vertex at time step $t-1$ since we have assumed they were both forced at time t. Therefore, at most 2 edges can be removed and
$\mid E(G') \mid \leq 2n-3-2 = 2(n-1)-3$. □

Remark 22. The run-time of the forcing process as in Theorem 21 is lengthened from $\lfloor n/2 \rfloor$ steps to $n-2$ steps.

On graphs with $F(G) \geq n-2$

It was proved in [16] that, for any graph G, $F(G) = n-1$ if and only if it is a K_n and $F(G) = 1$ if and only if it is a path. We continue this investigation by giving a constructive characterization of graphs with $F(G) \geq n-2$. We will use the number of vertices contained in a path to refer to its length.
Let H be an induced subgraph of the graph G, and let $N \subset V(H)$ denote the set of vertices with neighbors in $V(G) \backslash V(H)$. The following lemma establishes a bound on the number of black vertices within the induced subgraph H.

Lemma 23. Let G,H,N be defined as above and let $S \subset V(G)$ a forcing set. Then

$$|S \cap V(H)| \geq F(H) - |N|.$$

Proof.
Assume the converse inequality holds for a particular choice of $G,H,N,$ and, S. Observe that $S' = (S \cap V(H)) \cup N$ will be a forcing set in H with $|S'| < F(H)$, a contradiction. □

Definition 24. A graph G is complement reducible *(cograph)* if every induced subgraph of G with at least two vertices is either disconnected or is the complement of a disconnected graph.

The characterization of cographs as induced P_4-free graphs is a folklore theorem that we use in our later proofs.

Theorem 25. A graph G is a cograph if and only if it does not contain an induced subgraph isomorphic to P_4.

Proof.

First, P_4 itself is not a cograph as it and its complement, $\overline{P_4}$, are both connected. If G contains P_4 as an induced subgraph, it is not a cograph since every cograph has an induced subgraph on at least 2 vertices that is either disconnected, or is the complement of a disconnected graph.

Assume that G has no induced P_4. We will proceed by induction on $\mid V(G) \mid$ and assume that every graph on fewer vertices is a cograph. Trivially, we note that any graph on 3 vertices is a cograph, so we may assume that $\mid V(G) \mid \geq 4$. If G is not connected, then we are done, as G contains an induced subgraph that is disconnected. Thus, we may assume that G is connected.

For $v \in V(G)$, let $G' = G - v$ and observe that by induction hypothesis, G' is a cograph. Here, we consider two cases; when G' is disconnected or connected. When G' is disconnected, let $A_1, \ldots, A_n$ be its connected components. Since G is connected, there is a vertex $a_i \in A_i$ such that a_i is a neighbor of v in G, for $1 \leq i \leq n$. If v is connected to every vertex in G, then $G = \overline{\overline{G'} \cup v}$, a cograph, and we are done. Suppose that v is not connected to every vertex. Then, without loss of generality, there is a vertex $b_1 \in A_1$ such that v is not a neighbor of b_1. Since A_1 is connected, we may assume that a_1 is a neighbor of b_1. But this means that for some vertex $a_2 \in A_2$, $b_1 a_1 v a_2$ is an induced path of length 4. This contradicts our assumption that there is no induced P_4 in G.

Finally, we consider the case when G' is connected. Since G' was assumed to be a cograph, $\overline{G'}$ must be disconnected. We now apply the same argument to $\overline{G'}$ to see that there is no induced P_4 in G. Thus, G is a cograph. □

Lemma 26. If G is a graph with an induced path of length at least k then $F(G) \leq n-k+1$.

Proof.
Let G be a graph on n vertices with an induced path P_k of length at least k. If we color every vertex of G black except the first $k-1$ vertices in P_k (that is, $n-k+1$ vertices in total), G will clearly be forced. □

Remark 27. Equality in the last lemma may occur. For instance, take a path P_k of length k and a complete graph K_m and connect all vertices of K_m to all vertices of P_k.

Corollary 28. A graph G with $F(G) = n-2$ is a cograph.

While the constructive characterization of cographs is well known, observe that not every cograph G on n vertices has forcing number $F(G) \geq n-2$.

On the forcing number of complete d-ary trees

For any tree T, let $L(T)$ denote the number of leaves in T. The upper bound $F(T) \leq L(T)-1$, originally shown in [1], may be sharp for certain classes of trees (e. g. paths and stars), but may also be very far from the zero-forcing number. For further results pertaining to the relationship between $F(T)$ and $L(T)$, see [1].

In this section, we determine the zero-forcing number of the complete d-ary trees. Let $T_{d,n}$ denote the complete d-ary tree of depth n on $1+d+d^2+\cdots+d^n$ vertices (d^n leaves) and let r_n denote the root of $T_{d,n}$. The study of $F(T_{d,n})$ in the binary case $d=2$, for small values, already reveals an intriguing parity-pattern. For a better understanding, we also calculate the forcing number $\widehat{F}(T_{2,n})$ of $T_{2,n}$ with r_n passive; in this setting a forcing set

$S \subseteq V(T_{2,n})$ has to force the tree without the root r_n ever becoming active. In other words, a passive vertex is one that may be forced, but may never force any vertex. Note, however, that r_n once turned to black, may then support further forcing in the tree.

n	1	2	3	4	5	6	7
$F(T_{2,n})$	1	3	5	11	21	43	85
$\widehat{F}(T_{2,n})$	2	3	6	11	22	43	86

Figure 1: Passive roots only increase the forcing number for odd indexes.

Observe, $F(T_{2,2n}) = \widehat{F}(T_{2,2n})$ and $F(T_{2,2n+1}) + 1 = \widehat{F}(T_{2,2n+1})$ for $n = 1, 2, 3$. We generalize this observation by an inductive proof. Consider the sequence (t_n^d) defined as follows:

$$t_n^d = \frac{d^{n+1} + (-1)^n}{d+1}.$$

Furthermore, observe that (t_n^d) satisfies the following recursions:

$$t_{2k+1}^d = d \cdot t_{2k}^d - 1,$$
$$t_{2k+2}^d = d \cdot t_{2k+1}^d + 1.$$

We prove the following statement:

Theorem 29.

i) $F(T_{d,2k}) = \widehat{F}(T_{d,2k}) = t_{2k}^d$. There exist forcing sets of size t_{2k}^d that contain r_{2k}.

ii) $F(T_{d,2k+1}) = t_{2k+1}^d$ and $F(\widehat{T_{d,2k+1}}) = t_{2k+1}^d + 1$. There exists no forcing set of size t_{2k+1}^d that contains r_{2k+1}.

Proof.

For a shorthand notation we write T_n and t_n instead of $T_{d,n}$ and t_n^d unless the notation is equivocal. Both statements are obviously true for T_0 and T_1. We prove our claim by induction on n in two steps:

Step 1: $T_{2k} \to T_{2k+1}$

Let $v_1, \ldots, v_d$ denote the neighbors of u in T_{2k+1}. These vertices may be viewed as roots of subraphs of T_{2k+1} isomorphic to T_{2k}. We denote these subgraphs by $T_{2k}^1, \ldots, T_{2k}^d$. Obviously, if $S \subseteq V(T_{2k+1})$ is a forcing set, then by Lemma 23, it follows that $|S \cap T_{2k}^i| \geq t_{2n} - 1$. Moreover, we may assume due to symmetry that u does not force any of the roots $u_1, \ldots, u_{d-1}$. If $|S \cap T_{2k}^i| = t_{2k} - 1$ for $i = 1, \ldots, d-1$, that would imply $F(T_{2k}) \leq t_{2k} - 1$, contradicting the inductive assumption. Thus, $|S| \geq (d-1) \cdot t_{2k} + (t_{2k} - 1) = t_{2k+1}$. In particular, if $|S| = t_{2k+1}$ then $d-1$ of the corresponding subtrees contain exactly t_{2k} vertices, initially colored black, while the one remaining subtree has exactly $t_{2k} - 1$ vertices initially colored black, hence $u \not\in S$.

Equality can be achieved as follows. Let $S_i \subseteq V(T_{2k}^i)$ be forcing sets of size t_{2n} containing v_i that force T_{2k}^i with their roots passive. Such sets are guaranteed to exist by induction. It is easy to see that

$$\left[\bigcup_{i=1}^{d} S_{2k}^i\right] \setminus \{u_d\}$$

is a forcing set of T_{2k+1}. Then u will get be forced by any neighbor but v_d, and afterwards it will force v_d. Now, the initial forcing set in T_{2k} together with v_d will force the rest of the tree.

Step 2: $T_{2k+1} \to T_{2k+2}$

Let $S \subseteq V(T_{2k+2})$ be a forcing set. We investigate two potential cases: $u \in S$ and $u \notin S$.

Case 1: Let $u \in S$. Observe that in this case $\widehat{F}(T_{2k+2}) \geq 1 + d \cdot F(T_{2k+1}) = t_{2k+2}$. Also, a minimum forcing set of T_{2k+1}^i does not contain v_i by induction, thus, even with u forcing v_i, the respective subgraph has to contain t_{2k+1} vertices initially colored black, implying $F(T_{2k+2}) \geq t_{2k+2}$.

Case 2: Let $u \notin S$ and assume and that it is forced by v_d. In this case

$$|S \cap V(T^d_{2k+1})| \geq t_{2k+1} + 1$$

since the root v_d cannot force another vertex in T^d_{2k+1}, implying v_d is passive. As before, we know that

$$|S \cap V(T^i_{2k+1})| \geq t_{2k+1}, i = 1 \ldots, d-1,$$

hence $|S| \geq (t_{2k+1} + 1) + (d-1) \cdot t_{2k+1} = t_{2k+2}$.

Finally, observe that coloring u black initially, together with arbitrary minimum forcing sets of every T^i_{2k+1}, will result in a forcing set of T_{2k+2} of size t_{2k+2} that forces every vertex, even with u passive.

□

CHAPTER 3

A COUNTEREXAMPLE TO A CONJECTURE OF GENTNER AND RAUTENBACH

Introduction

Amos, Caro, Davila, and Pepper [4] proved that for a connected graph G of order n and maximum degree $\Delta \geq 2$

$$F(G) \leq \frac{\Delta-2}{\Delta-1}n + \frac{2}{\Delta+1}.$$

It is not difficult to show that this bound is attained exactly when G is either $K_{\Delta+1}$, the complete bipartite graph $K_{\Delta,\Delta}$ or a cycle. Later, pushing this bound a little further, Gentner and Rautenbach [19] were able to remove the additive constant $\frac{2}{\Delta+1}$ (for $\Delta \geq 3$). Namely, they showed that $F(G) \leq \frac{\Delta-2}{\Delta-1}n$ holds for every connected graph G of order n and maximum degree $\Delta \geq 3$, unless when G is one of five exceptional graphs $K_{\Delta+1}, K_{\Delta,\Delta}, K_{\Delta-1,\Delta}$ or two other specific graphs (we do not exhibit them, for full details see [19]). Note that the zero forcing number of a connected graphs with maximum degree 2 is completely understood. Indeed, for such graphs the forcing number is either 1 in the case of a path or 2 in the case of a cycle. However, even when the maximum degree is 3, the limit

$$z_3 = \lim_{n\to\infty} \sup \left\{ \frac{F(G)}{|V(G)|} : G \text{ connected, } |V(G)| \geq n \text{ and } \Delta(G) \leq 3 \right\}$$

is not known. The currently best known upper bound for z_3 is $1/2$, proved by Amos, Caro, Davila, and Pepper, and follows from the result mentioned above. Furthermore, Gentner and Rautenbach ([19]), have proved that the upper bound of $n/2$ is far off when G has maximum degree 3 and girth at least 5, where n is the order of G. They showed that such graphs have zero forcing number at most $\frac{n}{2} - \frac{n}{24\log_2 n+6} + 2$. We remark that this result does not affect the best known upper bound for z_3 but suggests $1/2$ might not be the correct value. Motivated by this, the same authors conjectured that $F(G) \leq \frac{1}{3}n+2$ for every connected graph G with maximum degree 3 [19].

In this section, we disprove this conjecture by presenting an infinite family of connected graphs $\{G_n\}$, with maximum degree 3, such that the zero forcing number of G_n is at least $\frac{4}{9}|V(G_n)|$, thus proving $z_3 \geq \frac{4}{9}$.

Main Result

We create our counterexamples by substituting each leaf of a complete binary tree B_d on on $2^d - 1$ vertices by a complete graph on 4 vertices with one of its edges subdivided (see Figure 1). Indeed, let G_n ($n \geq 1$) be the graph obtained by replacing every leaf of B_{2n-1} by the aforementioned subdivided K_4. We also denote y^1_{n-1}, y^2_{n-1} to be the neighbors of r_n in G_n and H^1_{n-1}, H^2_{n-1} to be the corresponding connected components of $G_n - r_n$. Observe that both subgraphs are isomorphic to the binary tree B_{2n-2} with their leaves replaced by the subdivided K_4. Moreover, let $\widehat{G}_n$ be the graph obtained from G_n by attaching a new leaf y_n to the root r_n of the underlying binary tree in G_n. Throughout this note, we will view G_n as a subgraph of $\widehat{G}_n$ and containing 4 induced copies of G_{n-1}. Observe that the maximum degree of G_n and $\widehat{G}_n$ is 3, for all $n \geq 1$.

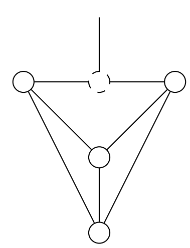

Figure 2: We substitute every leaf of B_d by a subdivided K_4 (the dashed vertex denotes a leaf in B_d).

We take a closer look at the structure of $\widehat{G}_n$ to obtain the required lower bound on $F(\widehat{G}_n)$. First, let the sequence t_n be defined inductively as follows: $t_1 = 2$ and $t_{n+1} = 4t_n + 2$ for every $n \geq 1$. Now we shall prove the following lemma.

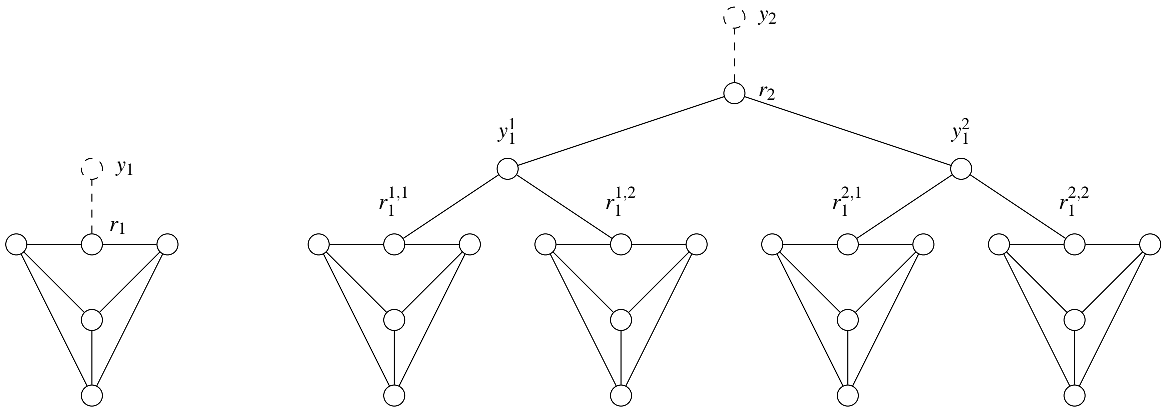

Figure 3: The graphs G_1 (left), and G_2 (right). The graphs $\widehat{G}_1$(left) and $\widehat{G}_2$(right) are the graphs containing G_1 and G_2 respectively, with the added dashed edge and vertex.

Lemma 30. Let F be a graph containing $\widehat{G}_n$ as an induced subgraph and such that there is no edge between $V(G_n)$ and $V(F)\setminus V(\widehat{G})$. Then, for every zero forcing set P of F, the following holds

i) $|V(G_n)\cap P| \geq t_n$.

ii) If $|V(G_n)\cap P| = t_n$ then $r_n \notin P$ and $V(G_n)\cap P$ does not force r_n within G_n.

Proof.

For $i \in \{1,2\}$, denote the induced left subtree of $H_n^i - y_n^i$ by $G_n^{i,1}$, and the right subtree by $G_n^{i,2}$. The root of $G_n^{i,j}$ will be denoted by $r_n^{i,j}$.

Both statements are straightforward for $n = 1$. For the inductive step, observe that if $|V(G_{n+1})\cap P| \leq t_{n+1} - 1 = 4\cdot t_n + 1$, then we may assume $|V(H_n^1)\cap P| \leq 2\cdot t_n$. Since $|V(G_n^{1,1})\cap P|$, $|V(G_n^{1,2})\cap P| \geq t_n$ by induction, we must have $|V(G_n^{1,1})\cap P| = |V(G_n^{1,2})\cap P| = t_n$. From (ii) we may deduce $r_n^{1,1}, r_n^{1,2} \notin P$. Moreover, during the process neither of these vertices can be forced by the vertices of $V(G_n^{1,1})$ or $V(G_n^{1,2})$, respectively. As a corollary, $r_n^{1,1}$ and $r^{1,2}$ must be forced by y_n^1, yet y_n^1 clearly can not force them both. This is a contradiction and it concludes the proof of part i). Note that we have proved $|V(H_n^i)\cap P| \geq 2\cdot t_n + 1$.

Assume now that $|V(G_{n+1})\cap P| = t_{n+1}$. Therefore, by the above, $|V(H_n^i)\cap P| = 2\cdot t_n + 1$

($i \in \{1,2\}$), which implies $r_{n+1} \notin P$. Finally, suppose that $r_{n+1} \notin P$ but it is forced during the process by a vertex in G_{n+1}. As $N_{G_{n+1}}(r_{n+1}) = \{y_n^1, y_n^2\}$, we may assume y_n^1 forced r_{n+1}. We proceed with a case by case analysis:

a) If $y_n^1 \in P$, then we must have $|V(G_n^{1,1}) \cap P| = |V(G_n^{2,1}) \cap P| = t_n$. By the induction hypothesis, neither $r_n^{1,1}$ nor $r_n^{2,1}$ belong to P, and neither of them is forced by a vertex in their respective subgraph $G_n^{1,i}$. Thus y_n^1 cannot force r_{n+1} as it has two unforced neighbors throughout the forcing process.

b) If $y_n^1 \notin P$, then it must be forced by $r_n^{1,1}$ or $r_n^{1,2}$. Let us assume $r_n^{1,1}$ forced y_n^1, then we must have $|V(G_n^{1,1}) \cap P| \geq t_n + 1$ and we may deduce

$$|V(G_n^{1,1}) \cap P| = t_n + 1 \text{ and}$$
$$|V(G_n^{1,2}) \cap P| = t_n$$

Hence, again by induction, $r_n^{1,2}$ does not belong to P and can not be forced within $G_n^{1,2}$. Although y_n^1 might indeed be forced by $r_n^{1,1}$, it still has two white neighbors $r_n^{1,2}$ and r_{n+1} thus it cannot force r_{n+1}, which is a contradiction. This completes our case check and the proof of the lemma.

□

Corollary 31. $F(\widehat{G}_n) \geq \frac{4}{9}|V(\widehat{G}_n)|$, for all $n \geq 1$.

Proof.

Observe that $t_n + 1 = \frac{8 \cdot 4^{n-1}+1}{3}$ and $|V(\widehat{G}_n)| = 6 \cdot 4^{n-1}$. Now, by Lemma 30, $F(\widehat{G}_n) \geq t_n + 1$ and therefore $F(\widehat{G}_n) \geq \left(\frac{4}{9} + \frac{1}{18 \cdot 4^{n-1}}\right)|V(\widehat{G}_n)|$. □

We end this section by determining the exact value of the zero forcing numbers of G_n and $\widehat{G}_n$.

Proposition 32. $F(G_n) = F(\widehat{G}_n) = t_n + 1$.

Proof.

Lemma 30 implies both $F(G_n)$ and $F(\widehat{G}_n)$ are greater or equal to $t_n + 1$. We shall prove equality holds, by induction on n. To do so, we will prove a stronger assertion, namely that G_n has a *zero forcing set* P_n of size $t_n + 1$ satisfying the following properties:

a) it contains r_n,

b) r_n does not need to force any of its neighbors.

The set P_1 can easily be found in G_1. For the inductive step, let P_{n+1} be the union of r_{n+1} with four ismorphic copies of the *zero forcing set* P_n inside each $G_n^{i,j}$ $(i, j \in \{1,2\})$, but with the two roots $r_n^{1,2}$ and $r_n^{2,2}$ removed. Clearly P_{n+1} has size $4 \cdot t_n + 3 = t_{n+1} + 1$ and satifies i). It is also easy to see that the vertices of both subgraphs $G_n^{1,1}$ and $G_n^{2,1}$ will be forced by the vertices in $P_{n+1} \cap G_n^{1,1}$ and $P_{n+1} \cap G_n^{1,1}$, respectively. (observe that this step requires the forcing to be completed without the active involvement of the root). Now, as r_{n+1} is black, y_n^1 and y_n^2 will force $r_n^{1,2}$ and $r_n^{2,2}$, respectively. Using induction again it follows both $G_n^{1,2}$ and $G_n^{2,2}$ will become black. Hence, P_{n+1} is a *zero forcing set* and r_{n+1} does not need to force any of its neighbours. From ii) we deduce P_{n+1} is also a *zero forcing set* of $\widehat{G}_n$. □

Additional Remarks

One of the most interesting remaining questions in the field is to find the value of z_3. Knowing our constructions, we believe the result of Amos et al. gives the correct value of z_3. We formulate this belief as a conjecture:

Conjecture 33. $z_3 = 1/2$.

The counterexamples we presented in this note used the idea of an appropriate "injection" of a subdivided K_4 in certain base graphs; we mention that, although the bound we obtained used binary trees as base graphs, we were able to beat the conjectured upper bound of $\frac{1}{3}n+2$ using different base graphs. For example, we state the following result (without proof):

Proposition 34. Let n be divisible by 6 and let C_n denote the cycle on n vertices. Furthermore, set $\widehat{C_n}$ to be the graph obtained by attaching a distinct leaf to every vertex in C_n, and finally, let G_n be constructed from $\widehat{C_n}$ by replacing every leaf with the subdivided K_4 graph. Then, $\frac{F(G_n)}{|V(G_n)|} \geq \frac{5}{12}$.

Before we prove Proposition 34, we will establish notation similar to that used in Lemma 30. Let $\{y_1,\ldots,y_n\}$ be the vertices of the subgraph C_n and let $\{r_1,\ldots,r_n\}$ be the vertices that lie on the subdivided edges of the K_4's such that r_iy_i is an edge in $E(G_n)$ for every $1 \leq i \leq n$. Since we will always be dealing with an attached subdivided K_4 to each y_i, we will denote by K^i, the subdivided K_4 attached to vertex y_i in C_n. Furthermore, we define $\widehat{K^i}$ to be the induced subgraph of G_n on vertex set $V(K^i) \cup y_i$. See graph $\widehat{G_1}$ on the left hand side of Figure 3.

We make the following observation on the number of vertices in G_n. Since n is divisible by 6 and since every vertex y_i is attached to a graph with 5 vertices (K^i), the number of vertices in G_n is $|V(G_n)| = 6^2k$, where $n = 6k$ and $k \geq 1$.

Lemma 35. Every forcing set of $\widehat{K^i}$ contains at least one neighbor of r_i in K^i, for every $1 \leq i \leq n$.

Proof.

Suppose not. Let u_i, v_i be the two neighbors of r_i in K^i. Let every vertex in $\widehat{K_i}$ be initially colored black except u_i, v_i. Every vertex now has two white neighbors, except y_i, which is of degree 1 in $\widehat{K_i}$ and is already adjacent to a black vertex. Since u_i and v_i are the only two remaining white vertices, and both are adjacent to the same three black vertices, $\widehat{K^i}$ never forces. Therefore, any forcing set must contain at least one of u_i or v_i in K^i. □

Suppose y_i is a passive vertex. By Lemma 35, it is easy to see that if y_i is passive, and u_i is in the initial set of forcing vertices for $\widehat{K^i}$, then any two of u_i's neighbors in K^i also being initially colored black will result in every vertex in $\widehat{K^i}$ being forced. In particular, at least 2 vertices in our forcing set must be vertices in $V(K^i) - r_i$. Therefore, in each K^i, we need at most 3 vertices initially colored black to force $\widehat{K^i}$, even if y_i is not passive. Moreover, if $S = \{u_i, r_i, v_i\}$ is a set of initially colored black vertices for K^i, no force occurs as both u_i and v_i have two white nieghbors in K^i.

Lemma 36. For any $\widehat{K^i} \in G_n$, if only 2 vertices in $\widehat{K^i}$ are initially colored black, and G_n is forced, then only one of $\{u_i, v_i\}$ is black and only one of their mutual neighbors in K^i is black. Moreover, y_i forces r_i.

Proof.

Let every vertex in G_n be initially colored except for $V(\widehat{K^i})$. At the first time step, since both y_{i-1} and y_{i+1} are black, y_i is forced. The next step in the forcing sequence sees y_i forcing r_i. Since r_i is now black, but has two white neighbors, u_i, v_i, we color one black. Without loss of generality, color u_i black and observe that r_i now forces v_i at the next step. However, as $\{u_i, r_i, v_i\}$ is not a forcing set for any K^i, we must color another vertex black. As there are only two remaining white vertices, and they are both adjacent to u_i and v_i, we can color either of them black, and observe that there is only one remaining white vertex, which gets forced by any of its neighbors. □

Proof of Proposition 34:

Without loss of generality, for each $\widehat{K^i}$, color u_i and one of its neighbors that is not r_i black. Observe that with this coloring, each $\widehat{K^i}$ has only 2 black vertices and G_n will never force since every black vertex has two white neighbors.

Choose an $i \in [n]$ and consider $\widehat{K^i}$ and $\widehat{K^{i+1}}$. Add a new black vertex to each of $\widehat{K^i}$ and $\widehat{K^{i+1}}$ so that y_i and y_{i+1} are forced by r_i and r_{i+1}, respectively. As both y_i and y_{i+1} are now black, they support one another during the next step of the forcing process, thus y_i forces

y_{i-1}, and y_{i+1} forces y_{i+2}.

For $\widehat{K^{i-2}}$ and $\widehat{K^{i+3}}$, we recolor a vertex in each so that y_{i-2} and y_{i+3} are forced by r_{i-2} and r_{i+3}, respectively. Then, y_{i-2} and y_i are supporting vertices for y_{i-1}, and y_{i-1} forces r_{i-1}. Similarly, y_{i+2} now has enough support to force r_{i+2}. By Lemma 36, $\widehat{K^{i-1}}$ and $\widehat{K^{i+2}}$ are forced.

We conclude by repeating this process of adding a new black vertex to every other K^i until we find that we have a $\widehat{K^j}$ and $\widehat{K^{j+1}}$ remaining, both with two black vertices. Since $\widehat{K^{j-1}}$ and $\widehat{K^{j+2}}$ both had a new black vertex added, both y_{j-1} and y_{j+2} were forced by r_{j-1} and r_{j+2}, respectively. Finally, y_{j-1} forces y_j, and y_{j+2} forces y_{j+1}, and thus, r_j is forced, as well as r_{j+1}. Therefore, by Lemma 36, both $\widehat{K^j}$ and $\widehat{K^{j+1}}$ are forced.

Finally, as half of the subgraphs $\widehat{K^i}$ have 3 black vertices, and the other half have 2 black vertices, for every 12 vertices, there are 5 initially colored black. Therefore, $\frac{F(G_n)}{|V(G_n)|} \geq \frac{5}{12}$. □

It would be interesting to know if the presented injection technique with the appropriate choice of a base graph can imply even better lower bounds on z_3.

Part II

The Dynamic Chromatic Number

CHAPTER 1

INTRODUCTION

The chromatic number of a graph G on n vertices is the least number of colors need to color the vertices such that no two adjacent vertices have the same color. The r-dynamic chromatic number of a graph, introduced in 2001 by Montgomery [31], is a variation of the usual chromatic number for a graph with the additional requirement that each vertex sees at least r different colors in its neighborhood. Let G be a graph and $v \in V(G)$. Denote the neighborhood of v by $\Gamma(v)$. Then an r-dynamic k-coloring of G is a proper k-coloring f of G such that $| f(\Gamma(v)) | \geq \min\{r, \deg(v)\}$.

In Chapter 1, we determine the r-dynamic chromatic number of the Hexagonal, Triangular, and Integer lattices. For these lattices, we provide maps that give easy-to-check bounds on the number of colors needed. Additionally, we establish results for the n-dimensional hypercube, Q_n, and its generalization to products of intervals. In particular, we show that the 2-dynamic chromatic number is 4, and when r and n are both $2^n - 1$ or $2^n - 2$, the r-dynamic chromatic number is 2^n. Finally, we extend a result of Akbari, Ghanbari, and Jahanbekam [2], who determined the 2-dynamic chromatic number of cartesian products of paths and cycles to $r = 3, 4$, and use this to prove that the 4-dynamic chromatic number of Möbius graphs of sufficiently large length is at least 6.

In Chapter 2, we study the r-dynamic chromatic number of Erdős-Rényi random graph model, $\mathcal{G}(n, p)$. Moreover, we study the behavior of $\chi_r(G)$ for three separate regimes of r: when $r \ll \chi(G)$, when r is roughly the same as $\chi(G)$, and when $r \gg \chi(G)$. Let $p \in (0,1)$ and $G \in \mathcal{G}(n, p)$. When $r \ll \chi(G)$, we show that with high probability, $\chi_r(G) = \chi(G)$. When $r \gg \chi(G)$, we show that $\chi_r(G) = r\left(1 + (1-p)^{\frac{n}{r}(1+o(1))}\right)$. Finally, when r is roughly the same as $\chi(G)$, we show that $\chi_r(G) = \chi(G) + o(1)$ and, with high probability, $\chi_r(G) = \chi(G)$.

CHAPTER 2

THE R-DYNAMIC CHROMATIC NUMBER OF CERTAIN GRAPHS

For the entirety of this section, G is a finite, simple, undirected graph on n vertices.

Definition 37. Given a graph G, the square of G, denoted by G^2, is the graph on $V(G)$ vertices in which two vertices are adjacent if and only if they have distance at most 2.

From the definition of the r-dynamic coloring, we state the following immediate observations.

Observation 38. $\chi_{r+1}(G) \geq \chi_r(G)$

Observation 39. If $r \geq \Delta(G)$, then $\chi_r(G) = \chi_{\Delta(G)}(G) = \chi(G^2)$

Observation 40. $\chi_r(G) \geq \min\{\Delta(G), r\} + 1$

From these observations, we note that $r = 1$ is the usual chromatic number of a graph. In other words, $\chi_1(G) = \chi(G)$.

We begin this section with a result about regular graphs. Regular graphs became a topic of interest for 2-dynamic colorings due a conjecture of Montgomery that stated, for all regular graphs G, $\chi_2(G) \leq \chi(G) + 2$ [31]. This conjecture remained open until a recent paper of Bowler, Erde, Lehner, Merker, Pitz, Stavropoulos, [10], found a construction for an infinite family of graphs with a sharp upper bound satisfying $\chi_2(G) \leq 2\chi(G)$. Despite this counter example, we find that the regularity property of the graph to still be of interest for values of $r > 2$. This leads us to the following lemma.

Lemma 41. If G is an r-regular graph of order n, and $\chi_r(G) = r + 1$, then $(r+1) \mid n$.

Proof.

Assume G is r-regular and let f be the r-dynamic $(r+1)$-coloring of G. For any vertex v, let $\Gamma(v)$ denote its neighborhood. For each color c_i, $0 \leq i \leq r$, define

$A_i = \{v \in V(G) : f(v) = c_i\}$.

Let v be chosen from $V(G)$ uniformly at random and choose u from $(v \cup \Gamma(v))$ uniformly at random. As every color appears exactly once among $v \cup \Gamma(v)$ for any $v \in V(G)$, the probability that u is color c_i is $\mathbb{P}(f(u) = c_i) = \frac{1}{r+1}$.

On the other hand, as G is regular, choosing a random vertex in the way we choose u is equivalent to choosing a vertex uniformly at random. Therefore, we find $\mathbb{P}(f(u) = c_i) = \frac{|A_i|}{n}$. Combining these gives $\mathbb{P}(f(u) = c_i) = \frac{|A_i|}{n} = \frac{1}{r+1}$. Therefore, $| A_i | = \frac{n}{r+1}$, and since $|A_i|$ is integral, $(r+1) \mid n$. $\square$

Lemma 42. $\chi_2(C_5) = 5$.

Proof.

By Observation 39, if $r \geq \Delta(G)$, then $\chi_r(G) = \chi_{\Delta(G)}(G) = \chi(G^2)$. Then, we have that $C_5^2 = K_5$, and the claim follows. $\square$

Lemma 43. If C_5 is an induced subgraph of a cubic graph G, then $\chi_3(G) \geq 5$.

Let $V(C_5) = \{v_0, v_1, v_2, v_3, v_4\}$ be the vertices of the induced 5-cycle, C_5. Define u_i to be the neighbor of v_i not on C_5. $N = \{u_0, \ldots, u_4\}$. Note that not all u_i need to be distinct.

Proof.

Consider the square of the graph G. As there is an induced C_5 in G, there is an induced K_5 in G^2. Following Observation 39, $\chi_3(G) = \chi_{\Delta(G)}(G) = \chi(G^2) \geq 5$ $\square$

Hypercubes and Tori

Hypercubes

In this section, we study the r-dynamic chromatic number for n-dimensional hypercubes Q_n, with vertices of Q_n identified with $\mathbb{Z}_2^n$.. We note that trivially, Q_n is n regular, so that $\chi_r(Q_n) = \chi_n(Q_n)$ for all $r \geq n$. Assume henceforth that $r \leq n$.

Definition 44. Let the *Hamming code* $H \subset Q_{2^n-1}$ be the set $\{x \in Q_{2^n-1} : \exists y \in Q_{2^n-1}\}$ with

$$x_i = \begin{cases} y_i & \text{if } i \text{ is not a power of 2} \\ \displaystyle\sum_{\substack{i \neq j \\ \lfloor j/i \rfloor \equiv 1 \mod 2}} y_j \mod 2 & \text{if } i \text{ is a power of 2} \end{cases}$$

Lemma 45. Let $n \geq 2$. Then $\chi_2(Q_n) > 3$.

Proof.

For a contradiction, let f be a 2-dynamic 3-coloring of Q_n. Define e_i to be the n-dimensional vector with 0s in every coordinate except 1 in the i^{th} coordinate. We consider all coordinates modulo 2, such that $x+e_i$ is just the neighbor of x in the i^{th} coordinate, either $x+e_i$ or $x-e_i$.

We'll show by induction on k that for all $1 \leq k \leq n$, there exists a vertex x with at least k neighbors that belong to the same color class. Note that for $k=1$ this is trivial. Define $I = \{i : f(x+e_i) = c\}$ for some $x \in Q_n$. Assume by the induction hypothesis, $|I| \geq k$. Consider a neighbor of x belonging to a different color class, say $y = x+e_j$, with $j \notin I$. For each $i \in I$, the neighbor of y, $y+e_i$, has neighbor $x+e_i$.

Thus, $f(y+e_i) \neq c$, and $f(y+e_i) \neq f(y)$, hence $f(y+e_i) = f(x)$ so y has at least $k+1$ neighbors of the color $f(x)$. This completes induction. Finally, setting $k=n$ contradicts our assumption that f is a 2-dynamic 3-coloring.

□

In the following proposition, we generalize the result in Lemma 45 to the product of intervals by showing that for any $n > 1$ and for every $a_i \in \mathbb{N}$ such that $a_i > 1$ for every $i \in [n]$, the 2-dynamic chromatic number of $\prod_{i=1}^{n}[a_i]$ is 4.

Proposition 46. Let $a_i > 1$. If $n \geq 2$, then $\chi_2\left(\prod_{i=1}^{n}[a_i]\right) = 4$

Proof.

We prove the lower bound using a similar argument to Lemma 45. Assume, without loss of generality, $a_i > 1$.

For a contradiction, let f be a 2-dynamic 3-coloring of $\prod_{i=1}^{n}[a_i]$.

Let k be the greatest number such that there exists a vertex x with at least k neighbors that belong to the same color class. Define $I = \{\pm e_i : f(x \pm e_i) = c\}$ for some $x \in \prod_{i=1}^{n}[a_i]$. Assume $\mid I \mid = k$.

We distinguish two cases; either for every $i \in [n]$, at least one of e_i and $-e_i$ is in I, or not. In the latter case, choose an i for which $\pm e_i$ is not in I and consider $y = x + e_i$ (or $y = x - e_i$ if $x_i = a_i$). Note that $f(y) \neq f(x)$ and $f(y) \neq c$. For every $a \in I$, we have that the neighbor $y + a = (x + a) + e_i$ is a neighbor of y and of $x + a$, so must have the same color as x, that gives $k + 1$ neighbors of y with color $f(x)$, a contradiction.

Alternatively, assume that for every $i \in [n]$ at least one of e_i and $-e_i$ is in I. Let $J = \{\pm e_i : i \in [n]\} \setminus I$. Note that if $J = \emptyset$ all neighbors of x have the same color contradicting 2-dynamic nature of f.

Let $b \in \mathbb{Z}_{\geq 0}^{J}$. We will show by induction on $\sum_{j \in J} b_j$ that if $y = x + \sum_{j \in J} b_j j \in \prod_{i=1}^{n}[a_i]$, then we have that $\{y + a : a \in I\}$ have the same color. For $b = \vec{0}$, this is trivially true by our assumption on x. Assume it is true for b.

Consider any $v \in J$ such that $y + v$ is in $\prod_{i=1}^{n}[a_i]$. First, we see that $f(y + v) \neq f(y)$ as they are neighbors and $f(y + v) \neq f(y + a)$ for any $a \in I$ as $v \notin I$ and we have assumed that there is no vertex with more than k neighbors of the same color. Furthermore, all $f(y + a)$ have the same color by induction.

Note $-v \in I$ and let $I' = I \setminus \{-v\}$. For all $a \in I'$, we have that $(y + v) + a$ is a neighbor to both $y + v$ and $y + a$, and as these have different colors, we find that $f(y + v + a) = f(y)$. Hence, all elements in $\{(y + v) + a : a \in I\}$ have the same color, namely $f(y)$. This concludes the induction.

We now consider the maximal b such that $\left(x + \sum_{j \in J} b_j j\right) + v$ is outside of $\prod_{i=1}^{n}[a_i]$ for any $v \in J$. That implies that all neighbors of $x + \sum_{j \in J} b_j j$ are of the form $\left(x + \sum_{j \in J} b_j j\right) + a$ for some $a \in I$, but we just proved that these all have the same color. Hence, $x + \sum_{j \in J} b_j j$

does not see two colors and f was not 2-dynamic. This is a contradiction.

The upper bound comes from a construction. Setting $f(x) = 2x_1 + \sum_{i>1} x_i \pmod 4$ yields a proper 2-dynamic 4-coloring of $\prod_{i=1}^n [a_i]$. □

Corollary 47. Let $a_i > 1$. If $n \geq 3$, then $\chi_3\left(\prod_{i=1}^n [a_i]\right) = 4$

Proof.

The lower bound is given by Observation 39, since the maximum degree is at least 3. For the upper bound, define

$$f : [2]^3 \to [4], x \mapsto \begin{cases} 0 \text{ if } x = (0,0,0),(1,1,1) \\ 1 \text{ if } x = (1,0,0),(0,1,1) \\ 2 \text{ if } x = (0,1,0),(1,0,1) \\ 3 \text{ if } x = (0,0,1),(1,1,0) \end{cases}$$

We extend this to a coloring of $\prod_{i=1}^n [a_i]$. Let $x \in \prod_{i=1}^n [a_i]$ and let $y \in [2]^3$ such that $y_i \equiv x_i \mod 2$. Then define the coloring by $x \mapsto f(y) + \sum_{i>3} x_i \mod 4$.

□

Proposition 48. Let $a_i > 1$ and $n \geq 1$. Then $\chi_{2n}\left(\prod_{i=1}^n [a_i]\right) = 2n+1$

Proof.

Lower bound is Observation 39. For the upper bound consider the coloring $x \mapsto \sum ix_i \mod 2n+1$ This coloring has the property that for any vertex x, $f(x+e_i) = f(x)+i \mod 2n+1$, which is always distinct from $f(x)+j$, for some $j \neq i$, or even $f(x)-i$. Therefore, given a vertex x and this mapping, the colors in the neighborhood of x are always distinct from $f(x)$. □

Proposition 49. If $r \neq 2^n - 1$ for any $n \in \mathbb{N}$, then $\chi_r(Q_r) \geq r+2$

Proof.

Observe that Q_n is regular and apply Lemma 41. □

Proposition 50. Let $n \geq 1$. Then $\chi_{2^n-1}(Q_{2^n-1}) = 2^n$

Proof.

Lower bound is the trivial lower bound given by Observation 40. Consider a Hamming code $H \subset Q_{2^n-1}$. We color the vertices of H with the same color. After this first coloring, we take another distinct Hamming code and color its vertices with a new color. Since the minimum distance between Hamming codes is 3, we are in no danger of coloring a vertex from a previously used code. Because the codes do not intersect, we can do this 2^n times, partitioning Q_{2^n-1}. This code has the property that H is independent and every vertex in $Q_{2^n-1} \setminus H$ has exactly one neighbour in H [22]. If every vertex sees every color class at most once, then it must see at most $2^n - 1$ colors. Hence, the following coloring will be $2^n - 1$ dynamic:

$$f(x) = \begin{cases} 0 \text{ if } x \in H \\ i \text{ if } x + e_i \in H \end{cases}$$

□

Corollary 51. Let $n > 2$. Then $\chi_{2^n-2}(Q_{2^n-2}) = 2^n$

Proof.

Observe that Q_{2^n-2} is a subgraph of Q_{2^n-1}. Apply Proposition 50 for the upper bound and Proposition 49 for the lower bound. □

Tori

For $a = (a_1, \ldots a_n) \in \mathbb{N}^n$, let $\mathbb{T}(a)$ denote the n-dimensional torus, i.e. the graph on vertex set $V = \prod_i [a_i]$ where x and y are adjacent if and only if there exists $j \in [n]$ such that for every $i \neq j$, $x_i = y_i$ and $|x_j - y_j| \equiv 1 \mod a_i$.

Note that we always work modulo a_i in direction e_i. For instance, one would write $(0,0) - (0,1) = (0,4) \in \mathbb{T}(5,5)$.

Proposition 52. If $n \geq 3$ and $a_i > 2$, then $\chi_2(\mathbb{T}(a)) = \begin{cases} 3 & \text{if } \exists i : 3|a_i \\ 4 & \text{otherwise} \end{cases}$

Proof. First consider the latter case. For the lower bound, let, for a contradiction, f be a 2-dynamic 3-coloring of $\mathbb{T}(a)$.

Let k be the greatest number such that there exists a vertex x with at least k neighbors that belong to the same color class. Define $I = \{\pm e_i : f(x + \pm e_i) = c\}$ for some $x \in \prod_{i=1}^{n}[a_i]$. Assume $|I| = k$.

Distinguish two cases; either for every $i \in [n]$ at least one of e_i and $-e_i$ is in I or not.

In the latter case, choose an i for which it isn't and consider $y = x + e_i$ (or $y = x - e_i$ if $x_i = a_i$). Note that $f(y) \neq f(x)$ and $f(y) \neq c$. For every $a \in I$, we have that the neighbor $y + a = (x + a) + e_i$ is a neighbor of y and of $x + a$, so must have the same color as x, that gives $k + 1$ neighbors of y with color $f(x)$, a contradiction.

Alternatively assume for every $i \in [n]$ at least one of e_i and $-e_i$ is in I. Note that if $\{\pm e_i : i \in [n]\} \setminus I = \emptyset$ all neighbors of x have the same color contradicting 2-dynamic nature of f. Let v be any element of $\{\pm e_i : i \in [n]\} \setminus I$. We'll show by induction that for all $b \geq 0$, we have that if $y = x + bv$, then $\{y + a : a \in I\}$ have the same color. For $b = 0$ this is trivially true. Assume it is true for b.

Let $y = x + bv$. First, we see that $f(y + v) \neq f(y)$ as they're neighbors and $f(y + v) \neq f(y + a)$ for any $a \in I$ as $v \notin I$ and we have assumed that there is no vertex with more than k neighbors of the same color.

Note $-v \in I$ and let $I' = I \setminus \{-v\}$. For all $a \in I'$, we have that $(y + v) + a$ is a neighbor to both $y + v$ and $y + a$, and as these have different colors, we find that $f(y + v + a) = f(y)$. Hence, all elements in $\{(y + v) + a : a \in I\}$ have the same color, viz $f(y)$. This concludes the induction.

Moreover, we know that $x + bv$, $x + (b + 1)v$ and $x + (b + 2)v$ all have distinct colors since we have a 2-dynamic coloring. Hence, $x + bv$ and $x + (b + 3)v$ must have the same color. Thus, by the induction, if $b \not\equiv b' \mod 3$, then $f(x + bv) \neq f(x + b'v)$. However, if v is in the ith direction (i.e. $v \in \{\pm e_i\}$), this implies $f(x) \neq f(x+a_i v)$ as $3 \nmid a_i$, which is evidently false as $x = x + a_i v$.

Hence, at least 4 colors are needed if $3 \nmid a_i$.

Claim. For $l \neq 5$, there exists a 2-dynamic 4 coloring of the l-cycle.

Proof If $a_i = 5$ for all i, consider the following coloring. Let

$$f : [5] \to [4]; \quad x \mapsto \begin{cases} 2 & \text{if } x = 5 \\ x & \text{otherwise} \end{cases}$$

Then the coloring $x \mapsto 2f(x_1) + \sum_i f(x_i) \mod 4$ is a 2-dynamic 4-coloring of $\mathbb{T}(a)$.

Otherwise, consider some coordinate i such that $a_i \neq 5$ and let $f : [a_i] \to [4]$ be a 2-dynamic 4-coloring of the a_i-cycle. Let $g_j : [a_j] \to [4]$ for $j \neq i$ be proper colorings of the a_j-cycles. Then the coloring $x \mapsto f(x_i) + \sum_{j \neq i} g_j(x_j) \mod 4$ is a 2-dynamic 4-coloring of $\mathbb{T}(a)$. This proves the claim.

Finally, if $3|a_i$, let again $g_j : [a_j] \to [3]$ for $j \neq i$ be proper colorings of the a_j-cycles. Then $x \mapsto x_i + \sum_{j \neq i} g_j(x_j) \mod 3$ is a 2-dynamic 6-coloring of $\mathbb{T}(a)$ that only uses 3 colors..

□

Extension of Cartesian Products to a Möbius graph

In this section, we build upon work completed by Kang, Müller, and West [25]. The $m - by - n$ Möbius Strip, denoted $M_{m,n}$, is the graph with vertex set $[m] \times [n]$ where vertices (i, j) and (i', j') are adjacent if and only if $i = i'$ and $|j - j'| = 1$, $j = j'$ and $|i - i'| = 1$, or $i = m + 1 - i'$, $j = n$ and $j' = 1$.

The purpose of this section is to extend some results on Cartesian products of some graphs and use those to show the main result of this section. Namely, that we can find a family of graphs that is always at least 4-dynamically 6-colorable with a very regular structure. We now state the main result of this section.

Theorem 53. For m,n sufficiently large, $\chi_4(M_{m,n}) = 6$.

In keeping with the statement of Theorem 53, we will always assume that $m,n \geq 5$ unless otherwise specified. We will also let f denote our valid 4-dynamic coloring of $M_{m,n}$. We now state an observation that is immediate from these assumptions.

Observation 54. Any two entries that have the same color must have distance at least 2 from one another.

Observation 54 may also be read to mean that any 3×3 subgrid of $M_{m,n}$ must have a uniquely colored 'center' with respect to the remainder of the 3×3. Applying this same observation, we arrive at the following set of excluded entries for repeating colors displayed in Figure 4.

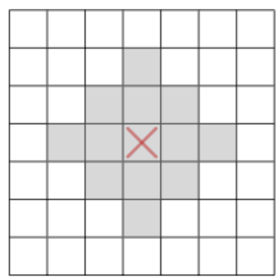

Figure 4: Excluded Colors

Before proving Theorem 53, we present some related results and small lemmas that will be of use. For the remainder of this section, we represent colorings of our graphs by a matrix X, with $x_{i,j} = f(i,j)$, where f is the specified dynamic coloring.

The Cartesian product of two graphs $G = (V(G),E(G))$ and $H = (V(H),E(H))$, written as $G \square H$, is the graph with vertex set $V(G) \times V(H)$ where vertex (a,x) is adjacent to vertex (b,y) whenever $ab \in E(G)$ and $x = y$, or $a = b$ and $xy \in E(H)$. Akbari, Ghanbari, and Jahanbekam calculated the 2-dynamic Cartesian product of cycles with paths in [2]. We calculate the 3 and 4-dynamic chromatic numbers for these graphs.

In the following lemma, we note that if $n = 1$, we are left with simply calculating the dynamic chromatic number of a cycle. In the case $m = n = 2$, we can not color this graph with a 3 or 4 dynamic coloring as in any case, every vertex is a border vertex, but sees the same neighbor twice. Therefore, we will only consider the case when at least m or n is at least 3.

Lemma 55. Let $m \geq 8$. $\chi_3(P_2 \square C_m) = \begin{cases} 4 & \text{if } 4 \mid m \\ 5 & \text{if } 4 \nmid m \end{cases}$

Proof.

Observe that every vertex in this graph has degree 3. By Observation 40, we must have at least 4 colors. For $4 \mid m$, the coloring $(i, j) \mapsto 2i + j \mod 4$ is 3-dynamic.

Assume henceforth that $4 \nmid m$.

For the lower bound, note that if it is possible to 3-dynamically 4-color the graph, then four consecutive vertices on the top row must all have distinct colors. For a contradiction, consider four consecutive vertices not satisfying this condition. As the coloring is 3-dynamic the first and the last must have the same color, say $x_{0,i} = x_{0,i+3} = a$, $x_{0,i+1} = b$ and $x_{0,i+2}$. However, that implies that in order for $v_{0,i+1}$ and $v_{0,i+2}$ to see three colors, both $x_{1,i+1} = d$ and $x_{1,i+2} = d$ which is a contradiction of the dynamic property. Hence, four consecutive vertices on the top row must all have distinct colors, and thus the length of the top row must be a multiple of 4.

For the upper bound consider the coloring of $P_2 \square C_4$ in Figure 5. Repeating this pattern will give a 3-dynamic coloring of $P_2 \square C_m$, very similar to the one for $4 \mid m$. Unfortunately, because $4 \mid m$, this pattern cannot be repeated until the entire graph is colored. We can use the fifth color to mitigate the problem we get at the point where we can no longer repeat the pattern in Figure 5. Figures 6 through 8 show how to recolor parts of the graph to get 3-dynamic 5 colorings, with the alterations boldfaced. $\square$

$$\begin{matrix} a & b & c & d \\ c & d & a & b \end{matrix}$$

Figure 5: A 3-dynamic coloring of $P_2 \square C_4$

$$\begin{matrix} a & b & c & d & \cdot & a & b & c & d \\ c & d & a & b & \cdot & c & d & a & b \end{matrix} \Rightarrow \begin{matrix} a & b & c & d & \mathbf{e} & a & b & c & d \\ c & d & \mathbf{e} & \mathbf{a} & \mathbf{b} & c & d & a & b \end{matrix}$$

Figure 6: A recoloring of $P_2 \square C_m$ when $m \equiv 1 \mod 4$

$$\begin{matrix} a & b & c & d & \cdot & \cdot & a & b & c & d \\ c & d & a & b & \cdot & \cdot & c & d & a & b \end{matrix} \Rightarrow \begin{matrix} \mathbf{e} & b & \mathbf{a} & d & \mathbf{c} & \mathbf{b} & a & \mathbf{e} & c & d \\ c & d & \mathbf{e} & b & \mathbf{a} & \mathbf{e} & c & d & a & b \end{matrix}$$

Figure 7: A recoloring of $P_2 \square C_m$ when $m \equiv 2 \mod 4$

Corollary 56. All 3-dynamic colorings in Lemma 55 may be extended to arbitrary n by repeating rows and shifting columns to the left or right by one position.

Remark 57. For 3-dynamic colorings of $C_m \square P_n$, any of the interior vertices, i.e. those not on the border, will see one color twice after applying Lemma 55.

Proposition 58. For $m \geq 7$, and $n \geq 3$ and $G = C_m \square P_n$

$$\chi_4(G) = \begin{cases} 5 & \text{if } 5 \,|\, m \\ 6 & \text{otherwise} \end{cases}$$

To prove this propositions, we need two little lemmas.

Lemma 59. If $C_m \square P_n$ is 4-dynamically 5-colored, then the coloring is 5-periodic on the middle rows.

Proof.

Let X be a 3×3 subgrid of $C_m \square P_n$, with m and n sufficiently large, and let f be a 4-dynamic 5-coloring. Without loss of generality, we color $x_{1,1} = a$, $x_{1,2} = b$, $x_{2,1} = c$, $x_{1,0} = d$, $x_{0,1} = e$. It suffices to show that $x_{1,5} = d$.

We are limited to two colors for each of the corners:

$x_{0,0} \in \{b,c\}, x_{0,2} \in \{c,d\}, x_{2,2} \in \{d,e\}$, and $x_{0,2} \in \{e,b\}$. Observe that selecting a color for one of the corners determines the colors for the other corners.

Assume without loss of generality $x_{2,0} = x_{0,1} = e$. Now note if $v_{1,2}$ is to see four colors, then entry $x_{1,3}$ must be e as well. From here, we are forced to set $x_{0,3} = x_{2,2} = d$, and

$$\begin{array}{cccccccccc} a & b & c & d & \cdot & \cdot & \cdot & a & b & c & d \\ c & d & a & b & \cdot & \cdot & \cdot & c & d & a & b \end{array} \Rightarrow \begin{array}{ccccccccccc} a & b & c & d & \mathbf{e} & \mathbf{b} & \mathbf{c} & a & b & \mathbf{e} & d \\ c & d & a & b & \mathbf{c} & \mathbf{a} & \mathbf{e} & \mathbf{d} & \mathbf{c} & a & b \end{array}$$

Figure 8: A recoloring of $P_2 \square C_m$ when $m \equiv 3 \mod 4$

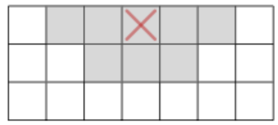

Figure 9: Excluded Colors for border entries in a subgrid with $r = 3$

$x_{2,3} = x_{1,1} = a$. We fill the next column similarly: if $v_{0,3}$ is to see four colors we need $x_{0,4} = x_{1,1} = a$, and then by exclusion $x_{1,4} = x_{2,1} = c$, and $x_{2,4} = x_{0,0} = b$. Thence, in the same way $x_{1,5} = d$ $\square$

$$\begin{array}{cccccc} \cdot & e & \cdot & \cdot & \cdot & \cdot \\ d & a & b & \cdot & \cdot & \cdot \\ \cdot & c & \cdot & \cdot & \cdot & \cdot \end{array} \Rightarrow \begin{array}{cccccc} b & e & c & d & a & b \\ d & a & b & e & c & d \\ e & c & d & a & b & e \end{array} \quad \text{or} \quad \begin{array}{cccccc} c & e & d & a & b & c \\ d & a & b & c & e & d \\ b & c & e & d & a & b \end{array}$$

Figure 10: Two 4-dynamic 5-colorings for a 3×5 subgrid of $C_m \square P_n$

Lemma 60. For every $l \geq 7$, there is a 2-dynamic $[6]$-coloring of the l-cycle such that adjacent vertices have colors differing by at least 2.

Proof.

Find $x, y \in \mathbb{N}$, such that $l = 3x + 2y$ and $3x > 2y$. Now consider some sequence $(a_i)_{i=1}^{l}$ of 2's and 3's such that there are $3x$ 2's and $3y$ 3's, $a_1 = a_l = 2$ and there are no two consecutive 3's. Now consider coloring

$$f : [l] \to [6], k \mapsto \sum_{i=1}^{k} a_i \mod 6$$

Note that the definition of the a_i's makes sure that consecutive vertices have colors differing by at least 2 and moreover that are distinct. The only cases to check are vertices 1 and l. Fortunately as $\sum_{i=1}^{l} a_i \equiv 3x \cdot 2 + 2y \cdot 3 \equiv 0 \mod 6$, we find that

$f(l) = 0 \neq f(2) = a_1 + a_2 = 2 + a_2$, so 1 sees two distinct colors differing at least 2 from its own color a_1. Finally $f(l-1) \equiv -a_l \not\equiv a_1 \equiv f(1) \mod 6$, so also l sees two distinct colors differing at least 2 from its own color. □

Proof of Proposition 58

Suppose $5 \mid m$. The trivial lower bound, as shown in Observation 39, holds. Our upper bound is given by $(i, j) \mapsto i + 2j \mod 5$.

Now, we suppose that $5 \nmid m$. To see that we need 6 colors, note that Lemma 59 implies that 5 colors cannot be enough. On the other hand, to see 6 colors suffice we construct the following coloring. We find a coloring f of the m-cycle using Lemma 60. Then we define the coloring $(i, j) \mapsto i + f(j) \mod 6$. By the definition of f, we know that for any vertex the neighbors in the second dimension have distinct colors differing by at least 2, while the neighbors in the first dimension are easily seen to differ by exactly 1.

□

Proposition 61. $\chi_r(M_{2,2}) = 4$ for every r.

Proof.

$M_{2,2}$ is a complete graph on 4 vertices. Therefore, each vertex is a neighbor to every other vertex and each must have a distinct color. □

Theorem 62. $\chi_2(M_{2,n}) = 4$ for each $n \in \mathbb{N}$

Proof.

First note that $M_{2,n}$ is isomorphic to the graph on vertex set $[2n]$ with edge set $\{ij : |i - j| \equiv 1, n \mod 2n\}$. We will work with this definition for simplicity in this theorem. Assume for a contradiction that f is a 2-dynamic 3-coloring of $M_{2,n}$.

Claim: f is 3-periodic on $[2n]$.

Note that every 2×2 square of vertices must have a diagonal of identically colored vertices. Say without loss of generality $f(i) = f(i+n+1) = 1$ and $f(i+1) = 2$, then

$f(i+2) = 3$ as $i+1$ must see two colors. Now $i+n+2$ has neighbors $i+n+1$ and $i+2$, so $f(i+n+2) = 2$. For $i+2$ to see two colors we need $f(i+3) = 1$, which proves the claim.

If f is 3-periodic, then $n = 3k$ for some k. However, that implies that $f(1+n) = f(1+3k) = f(1)$, which is obviously absurd as 1 and $1+n$ are adjacent.

For the upper bound, we consider simple alterations of the coloring $i \mapsto i \mod 3$. If $3 \mid n$, consider coloring

$$i \mapsto \begin{cases} i \mod 3 & \text{if } i \leq n \\ 4 & \text{if } i = n+1 \\ i+1 \mod 3 & \text{if } i > n+1 \end{cases}$$

If $n \equiv 1 \mod 3$, then $i \mapsto (i \mod 3)$, is in fact a proper coloring although not completely 2-dynamic. Figure 11 shows how the end of the strip makes the coloring not 2-dynamic, and how a simple recoloring can fix it. The italicized 1 indicates vertex 1. Similarly, Figure 12 shows how the following case, $n \equiv 2 \mod 3$, works:

$$i \mapsto \begin{cases} i \mod 3 & \text{if } i < 2n \\ 4 & \text{if } i = 2n \end{cases}$$

□

2	3	1	2	*1*	2	3	$\Rightarrow$	2	3	**4**	**3**	1	2	3
1	2	3	1	2	3	1		1	**4**	**2**	1	2	3	1

Figure 11: A recoloring of $M_{2,n}$ for $n \equiv 1 \mod 3$

1	2	3	1	*1*	2	3	$\Rightarrow$	1	2	3	**4**	1	2	3
2	3	1	2	3	1	2		2	3	1	2	3	1	2

Figure 12: A recoloring of $M_{2,n}$ for $n \equiv 2 \mod 3$

The method for the 3-dynamic chromatic number is very similar to the proof of Theorem 55.

Lemma 63. Let $n \geq 8$. $\chi_3(M_{2,n}) = \begin{cases} 4 & \text{if } n \equiv 2 \mod 4 \\ 5 & \text{if } n \not\equiv 2 \mod 4 \end{cases}$

Proof.

As in Theorem 62, let $M_{2,n} = ([2n], \{ij : |i-j| \equiv 1, n \mod 2n\})$. Observe that every vertex in this graph has degree 3. By Observation 40, we must have at least 4 colors. For $n \equiv 2 \mod 4$, the maps $i \mapsto i \mod 4$ is a 3-dynamic colouring as the numbers $i-1, i+1$ and $i+n$ are all distinct modulo 4.

Assume henceforth that $n \not\equiv 2 \mod 4$.

For the lower bound, note that if f is a 3-dynamic 4-coloring of the graph, then four consecutive vertices must all have distinct colors. For a contradiction, consider four consecutive vertices not satisfying this condition. As the coloring is 3-dynamic the first and the last must have the same color, say $f(i) = f(i+3) = a$, $f(i+1) = b$ and $f(i+2) = c$. However, that implies that for $i+1$ and $i+2$ to see three colors, both $f(i+1+n) = d$ and $f(i+2+n) = d$ which is absurd. Hence, four consecutive vertices must all have distinct colors, so $4|2n$. As $n \not\equiv 2 \mod 4$ this implies $f(i) = f(i+n)$ which is absurd. Hence, we need at least 5 colors.

To see that 5 colors suffice, consider slight alterations of the coloring

$$i \mapsto \begin{cases} i \mod 4 & \text{if } i \leq n \\ i-n+2 \mod 4 & \text{if } i > n \end{cases}$$

around vertex 1 (indicated by italics), as specified in Figures 14 through 16.

□

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & \mathit{1} & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 & 3 & 4 & 1 & 2 \end{array}$$

Figure 13: The coloring of $M_{2,n}$ for $n \equiv 2 \mod 4$

$$\begin{array}{cccccccccc} 2 & 3 & 4 & 1 & 2 & \mathit{1} & 2 & 3 & 4 & 1 \\ 4 & 1 & 2 & 3 & 4 & 3 & 4 & 1 & 2 & 3 \end{array} \Rightarrow \begin{array}{cccccccccc} 2 & 3 & 4 & \mathbf{5} & 2 & \mathit{1} & \mathbf{4} & 3 & \mathbf{5} & 1 \\ 4 & \mathbf{5} & 2 & \mathbf{1} & 4 & 3 & \mathbf{5} & 1 & 2 & 3 \end{array}$$

Figure 14: Recoloring of $M_{2,n}$ for $n \equiv 0 \mod 4$

Lemma 64.

$$\chi_r(M_{3,3}) = \begin{cases} 3 & \text{if } r = 2 \\ 6 & \text{if } r = 3 \\ 9 & \text{if } r = 4 \end{cases}$$

Proof.

Case $r = 2$ The lower bound is trivial. For the upper bound, consider the coloring given by $f(i, j) = i + j \mod 3$ for the first two rows. For the last row, color each entry the same as the color in the corresponding column from the first row.

Case $r = 3$

Consider the 3×4 matrix, where the 4^{th} column is the inverted 1^{st} column. Without loss of generality, color entry $(1,0)$ with color a. Notice that since entries $(1,0)$ and $(1,3)$ are identified with one another, we may not use the same color to either horizontal neighbor of $(1,0)$, as that would force two adjacent entries to be colored the same. Therefore, we color entries $(1,0), (1,1), (1,2)$ with colors a, b, and c respectively.

Observe that we are not able to reuse any of the colors from row 1, as that would force at least one entry in either row 0 or row 2 to be see only two colors, contradicting a 3-dynamic coloring. Therefore, we must use a new color to color an entry from either of the two rows. Again, without loss of generality, we color both entries $(0,0)$ and $(2,0)$ with color d. If both of these entries are not the same color, then we note that neither of the colors they have may be used again as it would force an entry in either row 0 or row 2 to

$$\begin{array}{cccccccccc} 4 & 1 & 2 & 3 & \mathit{1} & 2 & 3 & 4 & 1 & 2 \\ 2 & 3 & 4 & 1 & 3 & 4 & 1 & 2 & 3 & 4 \end{array} \Rightarrow \begin{array}{cccccccccc} 4 & 1 & 2 & 3 & \mathbf{2} & \mathbf{1} & 3 & 4 & \mathbf{5} & 2 \\ 2 & 3 & 4 & 1 & \mathbf{5} & 4 & \mathbf{2} & \mathbf{1} & 3 & 4 \end{array}$$

Figure 15: Recoloring of $M_{2,n}$ for $n \equiv 1 \mod 4$

$$\begin{array}{cccccccccc} 2 & 3 & 4 & 1 & \mathit{1} & 2 & 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 & 3 & 4 & 1 & 2 & 3 & 4 \end{array} \Rightarrow \begin{array}{cccccccccc} 2 & 3 & \mathbf{5} & \mathbf{4} & \mathit{1} & 2 & 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 & \mathbf{5} & 4 & 1 & 2 & 3 & 4 \end{array}$$

Figure 16: Recoloring of $M_{2,n}$ for $n \equiv 3 \mod 4$

see only two colors. Finally, it is clear that the remaining entries may not be the same color if they are in the same row, but they may repeat in the same column as every entry in row 1 has degree 4 and already sees two colors. Therefore, the remaining four may be colored with 2 new colors, e and f.

$$\begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \\ a & b & c & a \\ \cdot & \cdot & \cdot & \cdot \end{array} \Rightarrow \begin{array}{cccc} d & \cdot & \cdot & d \\ a & b & c & a \\ d & \cdot & \cdot & d \end{array} \Rightarrow \begin{array}{cccc} d & e & f & d \\ a & b & c & a \\ d & e & f & d \end{array}$$

Figure 17: The 3-dynamic coloring of a 3×3 subgrid and its inverted first column.

Case $r = 4$ By considering the entries that are excluded, as shown in Figure 4, it is clear to see that the entries must all be distinct.

□

Proof of Theorem 53

Proof.

For the lower bound, assume for a contradiction that f is a 4-dynamic 5-coloring of $M_{m,n}$. Lemma 59 implies that f has to be 5-periodic in the middle rows. In particular that implies that $n = 5k$ for some $k \in \mathbb{N}$. However, as in Theorem 62, this is problematic; for m odd, this implies that $f(\frac{m+3}{2}, i) = f(\frac{m-1}{2}, i)$, so $(\frac{m+1}{2}, i)$ doesn't see four colors. For m even, this implies that $f(\frac{m}{2}, i) = f(\frac{m+2}{2}, i)$ which is also absurd as $(\frac{m}{2}, i)$ and $(\frac{m+2}{2}, i)$ are adjacent. Therefore, f is not a 4-dynamic 5-coloring, a contradiction.

For the upper bound, note that the coloring $f(i, j) = i + 2j \mod 5$ is almost a 4-dynamic 5-coloring in the sense that all entries (i, j) with $1 < j < n$ see four distinct colors. We will take advantage of this by recoloring select vertices (i, j) with columns j around n.

Since $f : V(M_{m,n}) \to [0,1,2,3,4]$ □

CHAPTER 3

THE DYNAMIC CHROMATIC NUMBER OF RANDOM GRAPHS

Thus far in the study of the dynamic chromatic numbers of graphs, very little has been accomplished in the regime of random graphs. However, the probabilistic method has been employed successfully in a number of problem. Dehghan and Ahadi [14] used probabilistic tools to find upper bounds of $\chi_2(G)$ for regular graphs in terms of $\chi(G)$, as well as by Alishahi [3], who proved a similar result, but for d-regular graphs, further sharpened by Taherkhani [35].

The Erdős-Rényi random graph model, $\mathcal{G}(n,p)$, is a probability space of graphs on n vertices with edges included with probability $p \in (0,1)$. The first result for $\mathcal{G}(n,p)$ is due to Alishahi, who showed in [3] that almost all graphs in $\mathcal{G}(n,p)$ have the same chromatic number as dynamic chromatic number. This result, however, is limited to the case when $r = 2$. In this section, we extend this result. Namely, we use the classical result of Bollobás [7].

Question 65. Let $G \in \mathcal{G}(n,p)$. What is $\chi_r(G)$?

The behavior of χ_r might be separated into three different regimes of r, roughly, the regimes $r \ll \chi(G)$, $r \sim \chi(G)$ and $r \gg \chi(G)$. In the first, we find $\chi_r(G) = \chi(G)$ almost surely. In the second the behavior is still a bit unclear. In the last, we find
$\chi_r(G) = r\left(1 + (1-p)^{\frac{n}{r}(1+o(1))}\right)$.
In this section, we'll write $\log = \log_{\frac{1}{1-p}}$ and $\ln = \log_e$.
Note that if in a coloring every color class is to be seen by many vertices, a roughly equitable coloring might seem like an attractive way to color. A study by Krivelevich and Patkós [28] showed that finding an equitable color is almost as easy as finding a coloring. These equitable colorings will provide us with important machinery to find an upper bound to $\chi_r(G)$

The behavior of small r

Theorem 66. For every $p \in (0,1)$, there is a constant C, such that if $n \to \infty$, $r \leq \frac{Cn}{\log(n)}$ and $G \in \mathcal{G}(n,p)$, then $\mathbb{E}[\chi_r(G)] = \mathbb{E}[\chi(G)] + o(1)$. Moreover, $\chi_r(G) = \chi(G)$ with high probability.

Proof. We let the base of our log's be $\frac{1}{1-p}$. Let $\alpha, \varepsilon > 0$ be any constants and $k(n) = (2+\varepsilon)\log(n)$. Finally, let C be the solution to

$$(2+\varepsilon)C\left[\log\left(\frac{e}{(2+\varepsilon)C}\right)+1\right] = \alpha$$

Start by noting that $\mathbb{E}[\chi(G)] = \Theta(\frac{n}{\log(n)})$ [7]. Consider a $\chi(G)$ coloring of G, say f. Every color class is an independent set, so

$$\begin{aligned}
\mathbb{P}(\text{there exists a color class of size} \geq k) &\leq \mathbb{E}[\#\text{independent sets of size } k] \\
&= \binom{n}{k}(1-p)^{\binom{k}{2}} \\
&\leq n^k(1-p)^{\binom{k}{2}} \\
&= (1-p)^{-k\log(n)}(1-p)^{\frac{k(k-1)}{2}} \\
&= (1-p)^{k\left(\frac{k-1}{2}-\log(n)\right)}
\end{aligned}$$

Substituting $k = (2+\varepsilon)\log(n)$ back into the inequality,

$$\begin{aligned}
&= (1-p)^{(2+\varepsilon)\log(n)\left(\frac{(2+\varepsilon)\log(n)-1}{2}-\log(n)\right)} \\
&\leq \frac{1}{n^{\frac{(2+\varepsilon)\log(n)}{2}(\log(n)-1)}} \\
&\leq \frac{1}{n^2}
\end{aligned}$$

If a color class has size at least k, give each vertex in the graph a different color. This changes $\mathbb{E}[\chi_r(G)]$, by at most $\frac{1}{n^2} \cdot n = o(1)$. Henceforth, assume the biggest independent set has size at most k.

If a vertex v does not satisfy the condition of seeing at least r colors then it can connect to at most $r-1$ color classes, i.e. to at most $O(r\log(n))$ vertices. The probability such a vertex exists is tiny: (note that the logs have base $\frac{1}{1-p}$)

$$\begin{aligned}
\mathbb{P}(\exists \text{a vertex with degree} \leq rk) &\leq \mathbb{E}[\#\text{vertices with degree} \leq rk] \\
&\leq n\binom{n}{rk}(1-p)^{n-rk} \\
&\leq n\left(\frac{en}{rk}\right)^{rk}(1-p)^{n-rk} \\
&= n(1-p)^{-rk\log(\frac{en}{rk})}(1-p)^{n-rk} \\
&\leq n(1-p)^{n-rk\log(\frac{en}{rk})-rk}
\end{aligned}$$

Before we proceed to the next step, note that:

$$\begin{aligned}
rk\log\left(\frac{en}{rk}\right)+rk &= (2+\varepsilon)n\frac{r\log(n)}{n}\left[\log\left(\frac{en}{(2+\varepsilon)r\log(n)}\right)+1\right] \\
&\leq n(2+\varepsilon)C\left[\log\left(\frac{e}{(2+\varepsilon)}\right)+1-\log(C)\right] \\
&\leq \alpha n
\end{aligned}$$

This shows that

$$n(1-p)^{n-rk\log(\frac{en}{rk})-rk} \leq n(1-p)^{(1-\alpha)n}$$

Hence, coloring all vertices a different color in this case only affects $\mathbb{E}[\chi_r(G)]$, by at most $n^2(1-p)^{(1-\alpha)n} = o(1)$.

Note that after these two conditionings, both of which occur almost never, the regular coloring of G is immediately also r-dynamic. □

Theorem 67. Let $n \to \infty$, $p \in [0,1]$ constant, $r = \Omega(\frac{n}{\log^3(n)})$, $\alpha \in (0,1)$ and $G \in \mathcal{G}(n,p)$, then $\mathbb{E}[\chi_r(G)] \leq Cr + \mathbb{E}[\chi(G)]$, where C is the solution of $Ce(1-p)^C = \alpha$.

Proof. Pick a set X of Cr vertices at random from the graph. Distinguish two cases; either all vertices in the graph connect to at least r of these, or not. First consider the latter. In this case give all vertices a distinct color, we claim that as this case has small probability, this won't seriously affect our bound on $\mathbb{E}[\chi_r(G)]$.

Note that for a vertex to connect to at most r vertices in X we can condition on what vertices it does connect to and use a union bound there on.

$$
\begin{aligned}
&\mathbb{P}(\exists \text{a vertex connecting to } < r \text{ vertices in } X) \\
&\quad \leq \mathbb{E}[\#\text{vertices connecting to } < r \text{ vertices in } X] \\
&\quad = n \cdot \mathbb{P}(\text{a given vertex connects to } < r \text{ vertices in } X) \\
&\quad \leq n \sum_{Y \subset X, |Y| = r} \mathbb{P}(\text{a given vertex connects only to some subset of the vertices in } Y) \\
&\quad \leq n \binom{Cr}{r} (1-p)^{Cr-r} \\
&\quad \leq n \left(\frac{Cre}{r}\right)^r (1-p)^{Cr-r} \\
&\quad = n \left(Ce(1-p)^{C-1}\right)^r \\
&\quad = \alpha^{r - \log_{\frac{1}{\alpha}}(n)} \\
&\quad \leq \alpha^{\frac{n}{\log^4(n)}}
\end{aligned}
$$

Hence the effect of this conditioning on our bound is at most

$$n\alpha^{\frac{n}{\log^4(n)}} = o(1)$$

For the other case, where every vertex connects to at least r elements of X, give every element in X a distinct color and extend this to a coloring of the whole of G. This requires at most $Cr + \chi(G)$ colors. Hence, we find $\mathbb{E}[\chi_r(G)] \leq Cr + \chi(G)$ and $\chi_r(G) \leq Cr + \chi(G)$

almost surely $\square$

This method also extends to $r = \Theta(n/\log(n))$, to give;

Corollary 68. Let $n \to \infty$, $p \in [0,1]$ constant, $r = \Theta(\frac{n}{\log(n)})$ and $G \in \mathcal{G}(n,p)$, then $\mathbb{E}[\chi_r(G)] \leq O(r)$

And to the regime $o(\frac{n}{\log(n)}) = r = \Omega(\frac{n}{\log^2(n)})$ to give

Corollary 69. Let $n \to \infty$, $p \in [0,1]$ constant, $o(\frac{n}{\log(n)}) = r = \Omega(\frac{n}{\log^2(n)})$ and $G \in \mathcal{G}(n,p)$, then $\mathbb{E}[\chi_r(G)] \leq \chi(G)(1+o(1))$

Upper Bound on large r from Equitable Colorings

For the purposes of analytic convenience, it would be easier to control the number of colors. Therefore, for the regime of larger r, we use an equitable coloring since this allows us maximize the expected number of color classes seen by a particular vertex. An equitable coloring a graph, written as $\chi_=(G)$ is a proper coloring in which the sizes of any two color classes differ by at most one. Following the work of Krivelevich and Patkós [28], we define $\chi_=^*(G)$ to be the minimal $k \in \mathbb{N}$ such that for all $k' \geq k$ there exists a coloring of G with k' colors with the biggest difference in sizes of color classes is at most one.

For sake of clarity, we state four results of Krivelevich and Patkós, since we use their ideas heavily.

Theorem 70. [28] For constant $p < 0.99$ and $G \in \mathcal{G}(n,p)$, with high probability $\chi(G) \leq \chi_=(G) \leq \chi(G)(1+o(1))$.

Theorem 71. [28] For constant $p < 0.99$ and $G \in \mathcal{G}(n,p)$, with high probability $\chi(G) \leq \chi_=^*(G) \leq \chi(G)(2+o(1))$.

The form that we need for the purposes of this section puts no upper bound on p, the following is an intermediate result from [28]

Lemma 72. [28] For any $p \in (0,1)$ constant, $l = o(n)$ and $G \in \mathcal{G}(n,p)$, we have

$$\mathbb{P}(\nexists \text{an equitable coloring of } G \text{ using } l \text{ colors}) \leq O(ne^{-l(1-p)^{\frac{n}{l}}})$$

and its corollary

Corollary 73. For any $\gamma > 0$, there is a $C > 0$, such that if $l \geq \frac{n}{\log(n)}(1+\gamma)$, then

$$\mathbb{P}(\nexists \text{an equitable coloring of } G \text{ using } l \text{ colors}) \leq O\left(\exp\left(-C\frac{n^{\frac{\gamma}{1+\gamma}}}{\log^2(n)}\right)\right)$$

Theorem 74. Let $p \in (0,1)$ constant and $G \in \mathcal{G}(n,p)$. Let $\gamma, \varepsilon > 0$, such that $\gamma(1+\varepsilon) > 1$ and eventually $r = r(n) \geq \frac{n}{\log(n)}(1+\gamma)$ and $r(n) = o(n)$, Then for sufficiently large n

$$\chi_r(G) \leq r\left(1 + (1-p)^{\frac{n}{(1+\varepsilon)r}}\right)$$

Proof.

First note that trivially $\chi_r(G) \geq r+1$. Let

$$\begin{aligned}
a &= \frac{\gamma}{1+\gamma} - \frac{3\log\log(n)}{\log(n)} \\
\delta &\in \left(\max\left\{\frac{1}{\gamma} - 1, 0\right\}, \varepsilon\right) \\
k &= 1 + \left[(1+(1-p)^{\frac{n}{(1+\delta)r}})r\right]^a \\
l &= \left[(1+(1-p)^{\frac{n}{(1+\delta)r}})r\right]^{1-a} \\
f &= \frac{kl}{r} - 1 = \left(1+(1-p)^{\frac{n}{(1+\delta)r}}\right)\left(1+\left[r\left(1+(1-p)^{\frac{n}{(1+\delta)r}}\right)\right]^{-a}\right) - 1
\end{aligned}$$

Here, we note that $\gamma(1+\delta) > 1$.

Consider the following process to handle random graphs. First, partition the graph deterministically in k parts of size $\frac{n}{k}$, $\{X_i : 1 \leq i \leq k\}$. Second, color each of the parts X_i in an equitable fashion as is possible by Lemma 72, with l colors, different colors for each part. Note that to be able to repeatedly invoke Lemma 72 with high probability, we need the probability of failure to be $o(1)$.

$$
\begin{aligned}
&\mathbb{P}(\nexists \text{an equitable coloring of one of the parts}) \\
&\quad \leq k \cdot \mathbb{P}(\nexists \text{an equitable coloring of a given part}) \\
&\quad \leq k \cdot O\left(\frac{n}{k}\, e^{-l(1-p)^{\frac{n}{kl}}}\right) \\
&\quad \leq O\left(\exp\left(\ln(n) - l(1-p)^{\frac{n}{kl}}\right)\right) \\
&\quad \leq O\left(\exp\left(\ln(n) - \frac{n^{1-a}}{\log(n)}(1-p)^{\frac{\log(n)}{(1+\gamma)}}\right)\right) \\
&\quad \leq O\left(\exp\left(\ln(n) - \frac{n^{\frac{3\log\log(n)}{\log(n)}}}{\log(n)}\right)\right) \\
&\quad \leq O\left(\exp\left(\ln(n) - \log^2(n)\right)\right) \\
&\quad = o(1)
\end{aligned}
$$

Third, consider for every vertex $v \in X_i$ in the graph the number of color classes outside X_i it connects to. Noting that the edges between v and these color classes have had no influence on the coloring as constructed in the second step.

Note that all color classes have at least $\lfloor \frac{n}{kl} \rfloor$ elements and note that none of the computations are affected by the ignoring the rounding.

Let $v \in X_i$ and $Y \subset X_j$ $(i \neq j)$ some other color class.

$$\mathbb{P}(v \sim Y) = 1 - (1-p)^{|Y|} \geq 1 - (1-p)^{\frac{n}{kl}}$$

Let this latter probability be $p_0 = p_0(n)$. Note that $p_0 \to 1$ as $n \to \infty$ since $r = o(n)$.

For a vertex v let Z_v denote the number of color classes v is connected to. Let $Z \sim \text{Bin}((k-1)l, p_0)$, by the obvious coupling checking only the color classes in other parts of the partition, we find that $Z_v \geq Z$. Let $k-1=k'$ for convenience. That prepares us for the big calculation.

$$
\begin{aligned}
\mathbb{P}(\exists v : Z_v < r) &\leq \mathbb{E}[\#v \text{ with } Z_v < r] \\
&= n\mathbb{P}(Z_v < r) \\
&\leq n\mathbb{P}(Z < r) \\
&\leq n\binom{k'l}{k'l-r}(1-p_0)^{k'l-r} \\
&\leq n\left(\frac{ek'l}{k'l-r}(1-p_0)\right)^{k'l-r} \\
&\leq n\left(\frac{ek'l}{k'l-r}(1-p)^{\frac{n}{kl}}\right)^{k'l-r} \qquad \text{evaluating } k'l-r \text{ in the denominator} \\
&= n\left(\frac{ek'l}{r}(1-p)^{\frac{n}{kl}-\frac{n}{(1+\delta)r}}\right)^{k'l-r} \\
&\leq n\left(\frac{ek'l}{r}(1-p)^{\frac{\delta n}{2r(1+\delta)}}\right)^{k'l-r} \\
&= n\left(\frac{ek'l}{r}(1-p)^{\frac{\delta n}{2r(1+\delta)}}\right)^{(1-p)^{\frac{n}{(1+\delta)r}}r}
\end{aligned}
$$

Note that taking logarithms we find;

$$
\begin{aligned}
\log(n) + &(1-p)^{\frac{n}{(1+\delta)r}} r\log\left(\frac{ek'l}{r}(1-p)^{\frac{\delta n}{2r(1+\delta)}}\right) \\
&= \log(n) + (1-p)^{\frac{n}{(1+\delta)r}} r\log\left(\frac{ek'l}{r}\right) + (1-p)^{\frac{n}{(1+\delta)r}} r\log((1-p)^{\frac{\delta n}{2r(1+\delta)}}) \\
&= \log(n) + o(1) + (1-p)^{\frac{n}{(1+\delta)r}} \frac{\delta n}{2(1+\delta)} \log(1-p) \\
&= \log(n) + o(1) - (1-p)^{\frac{n}{(1+\delta)r}} \frac{\delta n}{2(1+\delta)} \\
&\leq \log(n) + o(1) - \frac{\delta}{2(1+\delta)} n^{1-\frac{1}{(1+\delta)(1+\gamma)}} \qquad \text{as } r \geq \frac{n}{\log(n)}(1+\gamma) \\
&\leq -\omega(\log(n))
\end{aligned}
$$

Hence

$$\mathbb{P}(\exists v : Z_v < r) \leq n^{-\omega(1)}$$

Recoloring all vertices if this event occurs will affect $\mathbb{E}[\chi_r(G)]$ by at most $n^{-\omega(1)}n = o(1)$. Hence, we find

$$\begin{aligned}\chi_r(G) &\leq kl \\ &= r\left(1+(1-p)^{\frac{n}{(1+\delta)r}}\right) + \left[r\left(1+(1-p)^{\frac{n}{(1+\delta)r}}\right)\right]^{1-a}\end{aligned}$$

Finally, note that as $r = o(n)$ the last term is $r \cdot o(n^{-a+\xi})$ for any $\xi > 0$. Choose some $\xi \in (0, \frac{\gamma - \frac{1}{\delta+1}}{1+\gamma})$, which is possible as $\gamma(1+\delta) > 1$.

$$\begin{aligned}(1-p)^{\frac{n}{(1+\delta)r}} &\geq n^{\frac{-1}{(1+\gamma)(1+\delta)}} \\ &= \omega\left(n^{\frac{-\gamma}{1+\gamma} + \frac{3\log\log(n)}{\log(n)} + \xi}\right) \\ &= \omega\left(\left[r\left(1+(1-p)^{\frac{n}{(1+\delta)r}}\right)\right]^{1-a}\right)\end{aligned}$$

Thus,

$$\begin{aligned}\chi_r(G) &\leq r\left(1+(1-p)^{\frac{n}{(1+\delta)r}}(1+o(1))\right) \\ &\leq r\left(1+(1-p)^{\frac{n}{(1+\varepsilon)r}}\right)\end{aligned}$$

□

Theorem 75. Let $\gamma > 1$, $\varepsilon \in \left(0, \frac{1}{2}\left(1-\frac{1}{\gamma}\right)\right)$, eventually $r = r(n) \geq (1+\gamma)\frac{n}{\log(n)}$ and $r(n) = o(n)$. Define $\varphi(n,r) = 1 + 3\log(n)n^{\frac{-r\log(n)}{n+r\log(n)}}$, and $\psi(n,r,\varepsilon) = \frac{n+\varepsilon r\log(n)}{n+r\log(n)}$. Then for sufficiently large n

$$\chi_r(G) \leq r\left(1+(1-p)^{\frac{n}{r}(\varphi(n,r))^{-1}}\right) + \left[r\left(1+(1-p)^{\frac{n}{r}(\varphi(n,r))^{-1}}\right)\right]^{\psi(n,r,\varepsilon)}$$
$$= r\left(1+(1-p)^{\frac{n}{r}(1-o(1))}\right)$$

Proof.

First note that trivially $\chi_r(G) \geq r+1$. Let

$$a = \frac{\gamma(1-\varepsilon)}{(1+\gamma)}$$
$$f' = \left(1+(1-p)^{\frac{n}{1.001r}}\right)\left(1+\left[r\left(1+(1-p)^{\frac{n}{r}}\right)\right]^{-a}\right) - 1 = o(1)$$
$$\delta = f' + \frac{3\log(n)}{n^{\frac{\gamma}{1+\gamma}}} = o(1)$$
$$k = 1 + \left[(1+(1-p)^{\frac{n}{(1+\delta)r}})r\right]^{a}$$
$$l = \left[(1+(1-p)^{\frac{n}{(1+\delta)r}})r\right]^{1-a}$$
$$f = \frac{kl}{r} - 1 = \left(1+(1-p)^{\frac{n}{(1+\delta)r}}\right)\left(1+\left[r\left(1+(1-p)^{\frac{n}{(1+\delta)r}}\right)\right]^{-a}\right) - 1$$

We follow the same process as outlined in the proof of Theorem 74. Namely, that we partition the graph deterministically in k parts of size $\frac{n}{k}$, $\{X_i : 1 \leq i \leq k\}$ in the first step. For the second step, we color each of the parts X_i in an equitable fashion as is possible by invoking Lemma 72 to ensure that the probability of failure is $o(1)$.

$$
\begin{aligned}
&\mathbb{P}(\nexists \text{an equitable coloring of one of the parts}) \\
&\quad \leq k \cdot \mathbb{P}(\nexists \text{an equitable coloring of a given part}) \\
&\quad \leq k \cdot O\left(\frac{n}{k}\, e^{-l(1-p)^{\frac{n}{kl}}}\right) \\
&\quad \leq O\left(\exp\left(\ln(n) - l(1-p)^{\frac{n}{kl}}\right)\right) \\
&\quad \leq O\left(\exp\left(\ln(n) - \frac{n^{1-a}}{\log(n)}(1-p)^{\frac{\log(n)}{(1+\gamma)}}\right)\right) \\
&\quad \leq O\left(\exp\left(\ln(n) - \frac{n^{\frac{\varepsilon\gamma}{1+\gamma}}}{\log(n)}\right)\right) \\
&\quad = o(1)
\end{aligned}
$$

Again, we follow the third step outlined in the proof of Theorem 74, which is identical with the exception of the big calculation, which follows.

$$
\begin{aligned}
\mathbb{P}(\exists v : Z_v < r) &\leq \mathbb{E}[\#v \text{ with } Z_v < r] \\
&= n\mathbb{P}(Z_v < r) \\
&\leq n\mathbb{P}(Z < r) \\
&\leq n\binom{k'l}{k'l-r}(1-p_0)^{k'l-r} \\
&\leq n\left(\frac{ek'l}{k'l-r}(1-p_0)\right)^{k'l-r} \\
&\leq n\left(\frac{ek'l}{k'l-r}(1-p)^{\frac{n}{kl}}\right)^{k'l-r} && \text{evaluating } k'l-r \text{ in the denominator} \\
&= n\left(\frac{ek'l}{r}(1-p)^{\frac{n}{kl}-\frac{n}{(1+\delta)r}}\right)^{k'l-r} \\
&\leq n\left(\frac{ek'l}{r}(1-p)^{\frac{(\delta-f)n}{r(1+f)(1+\delta)}}\right)^{k'l-r} \\
&= n\left(\frac{ek'l}{r}(1-p)^{\frac{(\delta-f)n}{r(1+f)(1+\delta)}}\right)^{(1-p)^{\frac{n}{(1+\delta)r}}r}
\end{aligned}
$$

Note that taking logarithms we find;

$$
\begin{aligned}
\log(n)+&(1-p)^{\frac{n}{(1+\delta)r}} r\log\left(\frac{ek'l}{r}(1-p)^{\frac{(\delta-f)n}{r(1+f)(1+\delta)}}\right) \\
&= \log(n)+(1-p)^{\frac{n}{(1+\delta)r}} r\log\left(\frac{ek'l}{r}\right)+(1-p)^{\frac{n}{(1+\delta)r}} r\log((1-p)^{\frac{(\delta-f)n}{r(1+f)(1+\delta)}}) \\
&= \log(n)+o(1)+(1-p)^{\frac{n}{(1+\delta)r}}\frac{(\delta-f)n}{r(1+f)(1+\delta)}\log(1-p) \\
&= \log(n)+o(1)-(1-p)^{\frac{n}{(1+\delta)r}}\frac{(\delta-f)n}{(1+f)(1+\delta)} \\
&\leq \log(n)+o(1)-\frac{(\delta-f)}{(1+f)(1+\delta)}n^{1-\frac{1}{(1+\delta)(1+\gamma)}} && \text{as } r \geq \frac{n}{\log(n)}(1+\gamma) \\
&\leq \log(n)+o(1)-\frac{3}{(1+f)(1+\delta)}\log(n)-\frac{(f'-f)n^{1-\frac{1}{(1+\delta)(1+\gamma)}}}{(1+f)(1+\delta)} \\
&\leq -1.1\log(n)
\end{aligned}
$$

Hence

$$\mathbb{P}(\exists v : Z_v < r) \leq n^{-\omega(1)}$$

Recoloring all vertices if this event occurs will affect $\mathbb{E}[\chi_r(G)]$ by at most $n^{-1.1}n = o(1)$. Hence, we find

$$\begin{aligned}\chi_r(G) &\leq kl \\ &= r\left(1+(1-p)^{\frac{n}{(1+\delta)r}}\right) + \left[r\left(1+(1-p)^{\frac{n}{(1+\delta)r}}\right)\right]^{1-a}\end{aligned}$$

Finally, we compare the two error terms $r(1-p)^{\frac{n}{(1+\delta)r}}$ and $\left[r\left(1+(1-p)^{\frac{n}{(1+\delta)r}}\right)\right]^{1-a}$. Note that the latter is $r \cdot o(n^{-a+\xi})$ for any constant $\xi > 0$. Choose some $0 < \xi < \frac{\gamma}{\gamma+1}\left(\frac{1}{2} - \frac{1}{2\gamma}\right)$.

$$\begin{aligned}-a+\xi &= \frac{-\gamma(1-\varepsilon)}{(1+\gamma)} + \xi \\ &< \frac{-\gamma\left(\frac{1}{2}+\frac{1}{2\gamma}\right)}{(1+\gamma)} + \xi \\ &< \frac{-\gamma\left(\frac{1}{2}+\frac{1}{2\gamma}\right) + \gamma\left(\frac{1}{2}-\frac{1}{2\gamma}\right)}{(1+\gamma)} \\ &= \frac{-1}{1+\gamma} \\ &< \frac{-1}{(1+\gamma)(1+\delta)} \\ &\leq \log_n\left((1-p)^{\frac{n}{(1+\delta)r}}\right)\end{aligned}$$

Hence

$$\begin{aligned}\chi_r(G) &= r\left(1+(1-p)^{\frac{n}{(1+\delta)r}}\right) + \left[r\left(1+(1-p)^{\frac{n}{(1+\delta)r}}\right)\right]^{1-a} \\ &= r\left(1+(1-p)^{\frac{n}{r}(1-o(1))}\right)\end{aligned}$$

□

Lower bound on large r

Note that as $\chi(G) \sim \frac{n}{2\log(n)}$ which is only a factor two from the bound given in the theorem. Thus if there is an r such that $\chi_r(G) = r+1$ with non-vanishing probability, it's in the range $\frac{n}{2\log(n)} \leq r \leq \frac{n}{\log(n)}$.

Theorem 76. For all $\varepsilon > 0$, $\alpha > 1$, $r > \frac{\alpha n}{(1-\varepsilon)\log(n)}$, we have
$\chi_r(G) > r\left(1 + \left[\frac{\varepsilon^2}{3}\frac{(1-p)^{\frac{n}{(1-\varepsilon)r}}}{\frac{-n}{(1-\varepsilon)r}\ln(1-p)}\right]\right)$ almost surely for sufficiently large n.

Proof.
Let $l = r\left[\frac{\varepsilon^2}{3}\frac{(1-p)^{\frac{n}{(1-\varepsilon)r}}}{\frac{-n}{(1-\varepsilon)r}\ln(1-p)}\right]$ Let $w = \frac{1}{1-\varepsilon}$. Note that $\frac{r}{w} \geq l$ and $\frac{(r+l)}{w} \leq r$ for sufficiently large n.

Consider an r-dynamic $(r+l)$-coloring of G. The number of color classes with at least $\frac{w(n)n}{r+l}$ vertices is at most $\frac{r+l}{w}$, so at least $(1-\frac{1}{w})$ color classes have less vertices. On average these classes must be seen by $\varphi(r,w) = \frac{r(1-\frac{1}{w})}{r(1-\frac{1}{w})+l}$ of the vertices. In particular, there must be a color class of size at most $\frac{wn}{r+l}$ vertices seen by at least $\varphi(r,w)$ vertices.

$$\begin{aligned}
&\mathbb{P}(\exists \text{ color class of size} \leq \frac{wn}{r+l} \text{ seen by at least } \varphi(r,w)n \text{ vertices})\\
&\quad\leq \mathbb{P}(\exists \text{a set of size } \frac{wn}{r+l} \text{ seen by at least } \varphi(r,w)n \text{ vertices})\\
&\quad\leq \binom{n}{\frac{wn}{r+l}}\binom{n}{\varphi(r,w)n}(1-(1-p)^{\frac{wn}{r+l}})^{\varphi(r,w)n}\\
&\quad\leq \left(\frac{e(r+l)}{w}\right)^{\frac{wn}{r+l}}\binom{n}{\frac{l}{r(1-\frac{1}{w})+l}n}\left[\left(1-\frac{1}{\frac{1}{(1-p)}^{\frac{wn}{r+l}}}\right)^{\frac{1}{(1-p)}^{\frac{wn}{r+l}}}\right]^{(1-p)^{\frac{wn}{r+l}}\varphi(r,w)n}\\
&\quad\leq \left(\frac{e(r+l)}{w}\right)^{\frac{wn}{r+l}}\left(\frac{e(r(1-\frac{1}{w})+l)}{l}\right)^{\frac{l}{r(1-\frac{1}{w})+l}n} e^{-(1-p)^{\frac{wn}{r+l}}\varphi(r,w)n}
\end{aligned}$$

Taking logarithms on both sides, we have

$$\begin{aligned}
&\frac{wn}{r+l}\ln\left(\frac{e(r+l)}{w}\right)+\frac{ln}{r(1-\frac{1}{w})+l}\ln\left(\frac{e(r(1-\frac{1}{w})+l)}{l}\right)-(1-p)^{\frac{wn}{r+l}}\varphi(r,w)n\\
&\leq\frac{wn}{r}\ln(er)+\frac{ln}{r(1-\frac{1}{w})}\ln\left(\frac{er}{l}\right)-(1-p)^{\frac{wn}{r}}\left(1-\frac{1}{w}\right)n\\
&=\frac{n}{r}\left[w\ln(er)+\frac{l}{1-\frac{1}{w}}\ln\left(\frac{er}{l}\right)-r(1-p)^{\frac{wn}{r}}(1-\frac{1}{w})\right]\\
&\leq\frac{n}{r}\left[\frac{2l}{1-\frac{1}{w}}\ln\left(\frac{er}{l}\right)-r(1-p)^{\frac{wn}{r}}(1-\frac{1}{w})\right]\\
&=\frac{n}{r}\left[\frac{2l}{\varepsilon}\ln\left(\frac{er}{l}\right)-\varepsilon r(1-p)^{\frac{n}{(1-\varepsilon)r}}\right]\\
&=\frac{n}{r}\left[\frac{2l}{\varepsilon}\ln\left(\frac{er}{l}\right)-\varepsilon r(1-p)^{\frac{n}{(1-\varepsilon)r}}\right]\\
&=\frac{n}{r}\left[\frac{2\varepsilon r}{3}\left[\frac{\ln\left(\frac{er}{l}\right)}{\frac{-n}{(1-\varepsilon)r}\ln(1-p)}\right](1-p)^{\frac{n}{(1-\varepsilon)r}}-\varepsilon r(1-p)^{\frac{n}{(1-\varepsilon)r}}\right]\\
&=\varepsilon n(1-p)^{\frac{n}{(1-\varepsilon)r}}\left[\frac{2}{3}\left[\frac{\ln\left(\frac{re}{l}\right)}{\frac{-n}{(1-\varepsilon)r}\ln(1-p)}\right]-1\right]
\end{aligned}$$

$$
\begin{aligned}
&= \varepsilon n(1-p)^{\frac{n}{(1-\varepsilon)r}} \left[\frac{2}{3} \left[\frac{\ln\left(\frac{e \frac{n}{(1-\varepsilon)r} \ln(1-p)}{\frac{\varepsilon^2}{3}} \right) - \ln\left((1-p)^{\frac{n}{(1-\varepsilon)r}} \right)}{\frac{-n}{(1-\varepsilon)r} \ln(1-p)} \right] - 1 \right] \\
&= \varepsilon n(1-p)^{\frac{n}{(1-\varepsilon)r}} \left[\frac{2}{3} \left[\frac{r}{n} \ln\left(\frac{n}{r}\right) + O\left(\frac{r}{n}\right) + 1 \right] - 1 \right] \\
&\leq -\frac{\varepsilon}{4} n(1-p)^{\frac{n}{(1-\varepsilon)r}} \\
&\leq -\frac{\varepsilon}{4} n^{1-\frac{1}{\alpha}} \\
&\to -\infty
\end{aligned}
$$

Hence, with high probability such a color class doesn't exist and such a coloring doesn't exist. □

Corollary 77. For all $C > 1$, exists $\delta > 0$, such that if $r \geq C\frac{n}{\log(n)}$, then $\chi_r(G) = r + \omega(n^\delta)$

Proof.

Choose any $\alpha > 1$ and $\varepsilon > 0$ such that $\frac{\alpha}{1-\varepsilon} < C$, i.e. such that $r \geq \frac{\alpha n}{(1-\varepsilon)\log(n)}$. Then applying the theorem gives

$$
\begin{aligned}
\chi_r(G) &\geq r\left(1 + \left[\frac{\varepsilon^2}{3} \frac{(1-p)^{\frac{n}{(1-\varepsilon)r}}}{\frac{-n}{(1-\varepsilon)r}\ln(1-p)} \right]\right) \\
&\leq r + \frac{\alpha n}{(1-\varepsilon)\log(n)} \left[\frac{\varepsilon^2}{3} \frac{(1-p)^{\frac{n}{(1-\varepsilon)\frac{\alpha n}{(1-\varepsilon)\log(n)}}}}{\frac{-n}{(1-\varepsilon)\frac{\alpha n}{(1-\varepsilon)\log(n)}}\ln(1-p)} \right] \\
&= r + \frac{\varepsilon^2 \alpha^2}{3(1-\varepsilon)\ln(1-p)} \frac{n^{1-\frac{1}{\alpha}}}{\log(n)^2} \\
&= r + \omega(n^{\frac{1}{2}(1-\frac{1}{\alpha})})
\end{aligned}
$$

□

Part III

The Poset on Connected Graphs is Sperner

Introduction

Let $(P,\leq)$ be a partially ordered set (poset). We only consider partially ordered sets with finitely many elements. A *chain* in P is a set $C \subset P$ of pairwise comparable elements. An *antichain* $A \subset P$ is a set of pairwise incomparable elements. The poset $(P,\leq)$ is *graded* if there exists a partition of P into subsets $A_0,\dots,A_m$ such that A_0 is the set of minimal elements of P, and whenever $x \in A_i$ and $y \in A_j$ with $x < y$ and there is no $z \in P$ with $x < z < y$, then we have $j = i+1$. If such a partition exists, it is unique and the sets $A_0,\dots,A_m$ are the *levels* of P.

A graded poset $(P,\leq)$ is *Sperner* if the largest antichain in P is the largest sized level.

Let m be a positive integer, $[m] = \{1,\dots,m\}$. The Boolean lattice $2^{[m]}$ is the power set of $[m]$ ordered by inclusion, and $[m]^{(k)} = \{A \subset [m] : |A| = k\}$. By the well known theorem of Sperner [34], the poset $(2^{[m]},\subset)$ is Sperner, the largest antichains being equal to $[m]^{(\lfloor m/2 \rfloor)}$ and $[m]^{(\lceil m/2 \rceil)}$. The question whether certain posets are Sperner is widely studied. For a short list of such results, see [5]. In this paper, we investigate the Sperner property of the following poset.

Let n be a positive integer and let $\mathcal{C}$ denote the set of all connected graphs on vertex set $[n]$. (In other words, $\mathcal{C}$ is the family of labeled connected graphs on n vertices.) The family $\mathcal{C}$ is endowed with the following natural partial ordering: for $G,H \in \mathcal{C}$, let $G \leq H$ if G is a subgraph of H, or more formally, if $E(G) \subset E(H)$. When there is no risk of confusion, we shall simply write $\mathcal{C}$ when referring to the poset $(\mathcal{C},\leq)$. Observe that $\mathcal{C}$ is graded, the levels of $\mathcal{C}$ being the families $\mathcal{C}^{(k)} = \{G \in \mathcal{C} : |E(G)| = k\}$ for $k = n-1,\dots,m$. The following question originates from Katona [26].

Question 78. Is $(\mathcal{C},\leq)$ Sperner?

We prove that the answer is yes. More precisely, setting $m = \binom{n}{2}$ and $M = \lceil m/2 \rceil$, the main result of this paper is the following theorem.

Theorem 79. If n is sufficiently large, the unique largest antichain in $\mathcal{C}$ is $\mathcal{C}^{(M)}$.

Let us make a remark about how this result compares to Sperner's theorem [34]. Let $\mathcal{G}$ be the set of all graphs on vertex set $[n]$ and extend the ordering $\leq$ to $\mathcal{G}$ in the obvious way. Also, for $k = 0, \ldots, m$, let $\mathcal{G}^{(k)}$ be the set of graphs in $\mathcal{G}$ with k edges. Observe that $(\mathcal{G}, <)$ is isomorphic to $(2^{[m]}, \subset)$, hence $(\mathcal{G}, <)$ is Sperner. Note that $\mathcal{C}$ is a very dense subset of $\mathcal{G}$. As we shall see in Section III, the size of $\mathcal{C}$ is at least $2^m(1 - 2^{-n-o(n)})$. This corresponds to the well known statement that a graph chosen uniformly at random among all graphs with n vertices (that is an element of $G(n, 1/2)$ in the Erdős-Rényi random graph model) is disconnected with a probability that is exponentially small.

A problem similar to Question 78 has been considered in a paper of Jacobson, Kézdy and Seif [23]. Let G be a connected graph and let $(C(G), <)$ be the poset, whose elements are the connected, vertex-induced subgraphs of G, and $H < H'$ if H is an induced subgraph of H'. In [23], it was proved that this poset need not be Sperner, even if G is a tree.

This paper is organized as follows. In Section III, we discuss our notation and prove a few technical results. In Section III, we shall prove various bounds on the number of connected graphs with certain properties. These bounds provide us with some of the ingredients needed for the proof of Theorem 79 in Section III. In Section III, we propose some open problems.

Preliminaries

Let us say a few words about our notation, which is mostly conventional. If G is a graph, $V(G)$ is the vertex set of G, $E(G)$ is the set of its edges, and $e(G) = |E(G)|$. If $U \subset V(G)$, $G[U]$ denotes the subgraph of G induced on the vertex set U. If $F \subset E(G)$, then $G - F$ is the graph on vertex set $V(G)$ and edge set $E(G) \setminus F$. If $e \in E(G)$, we simply write $G - e$ instead of $G - \{e\}$.

For the sake of readability, we use the notation $\exp_2(x) = 2^x$, when necessary.

Furthermore, log denotes base 2 logarithm.

Our paper contains a lot of technical computations that are made more convenient by the following extension of the binomial coefficient. We define the binomial coefficient $\binom{x}{k}$ for any $k \in \mathbb{N}$, $x \in \mathbb{R}$ such that

$$\binom{x}{k} = \begin{cases} \frac{x(x-1)\dots(x-k+1)}{k!} & \text{if } k \leq x, \\ 0 & \text{otherwise.} \end{cases}$$

We collect some of the simple properties of $\binom{x}{k}$ in the following lemma.

Lemma 80. Let $k \in \mathbb{N}$, $x \in \mathbb{R}$.

(i) If $x \geq k$, we have $\binom{x}{k-1} / \binom{x}{k} = \frac{k}{x-k+1}$.

(ii) Let δ be a non-negative integer and suppose that $k \leq x \leq 2k - \delta$. Then $\binom{x+\delta}{k} \geq 2^{\delta}\binom{x}{k}$.

(iii) $\binom{x}{k} \leq \binom{x+1}{k}$ and $\binom{x}{k} \leq \binom{x+1}{k+1}$.

Proof.

(i) and (iii) easily follows from the definition.

Now let us prove (ii). If we prove the case $\delta = 1$, that is $\binom{x+1}{k} \geq 2\binom{x}{k}$ for $k \leq x \leq 2k-1$, the result follows by induction on δ. But in this case, we have

$\binom{x+1}{k} / \binom{x}{k} = (x+1)/(x-k+1) \geq 2$.

□

We remark that by continuity, for any fixed positive integer k and a real number $r \geq 1$, there is a unique $x \in \mathbb{R}$ such that $r = \binom{x}{k}$.

Throughout this paper, we shall also use the following simple inequalities.

Lemma 81. Let $a_1, \dots, a_s$ be positive integers and let $a_1 + \dots + a_s = n$. We have

$$\sum_{i=1}^{s} \binom{a_i}{2} \leq \binom{n-s+1}{2}, \tag{i}$$

and

$$\sum_{1\leq i<j\leq s} a_i a_j \geq (n-s+1)(s-1)+\binom{s-1}{2}. \qquad \text{(ii)}$$

Also, if $a_i \leq k$ for $i \in [s]$, where $n/2 < k \leq n-s+1$, then

$$\sum_{i=1}^{s}\binom{a_i}{2} \leq \binom{n-k-s+2}{2}+\binom{k}{2}, \qquad \text{(iii)}$$

and

$$\sum_{1\leq i<j\leq s} a_i a_j \geq k(n-k). \qquad \text{(iv)}$$

Proof.

The function $f(x) = x^2$ is convex, so $\sum_{i=1}^{s} a_i^2$ attains its maximum under the conditions $\sum_{i=1}^{s} a_i = n$ and $a_i \in \mathbb{Z}^+$ when $a_1 = \ldots = a_{s-1} = 1$ and $a_s = n-s+1$. Note that the left hand side of (i) is $\sum_{i=1}^{s} a_i^2/2 - n/2$, and the left hand side of (ii) is $(n^2 - \sum_{i=1}^{s} a_i^2)/2$, while the right hand sides of these inequalities are the respective values when
$a_1 = \ldots = a_{s-1} = 1$ and $a_s = n-s+1$.

For the inequalities (iii) and (iv), notice that with the additional condition that $a_i \leq k$, $\sum_{i=1}^{s} a_i^2$ attains its maximum when $a_1 = \ldots = a_{s-2} = 1$, $a_{s-1} = n-k-s+2$, $a_s = k$. The right hand side of (iii) is exactly $\sum_{i=1}^{s}\binom{a_i}{2}$ with these values inserted. On the other hand, we have

$$\sum_{1\leq i<j\leq s} a_i a_j \geq a_s(a_1+\ldots+a_{s-1}) = a_s(n-a_s) = k(n-k),$$

which proves (iv).

$\square$

Connectivity of graphs

In this section, we investigate the following problems. How many edges can a graph G have, whose removal destroys the connectivity, or 2-edge-connectivity of G? Also, what is the number of 2-edge-connected graphs G on vertex set $[n]$ in which there are exactly r edges, whose removal destroys the 2-edge-connectivity of G?

Let us start this section with the following well known result about the number of disconnected graphs. For completeness, we shall provide a short proof. A stronger form of this result can be found in [17], p. 138 as well.

Lemma 82. The number of disconnected graphs on vertex set $[n]$ is less than $exp_2\left(\binom{n-1}{2}+o(n)\right)$.

Proof.

A graph G is disconnected if there is a partition of $[n]$ into two nonempty sets A and B such that there are no edges between A and B. The number of disconnected graphs, where $|A|=1$ and $|B|=n-1$ is at most $n\cdot\exp_2\left(\binom{n-1}{2}\right)$, as we have n choices for the partition $\{A,B\}$, and $\exp_2\left(\binom{n-1}{2}\right)$ number of different choices for the edges in B.

The number of disconnected graphs where $|A|,|B|\geq 2$ is at most $\exp_2\left(n+\binom{n-2}{2}+1\right)=\exp_2\left(\binom{n-1}{2}+3\right)$, as there are at most $\exp_2(n)$ number of choices for the partition (A,B), and the number of ways to choose the edges inside A and B is at most $\exp_2\left(\binom{|A|}{2}+\binom{|B|}{2}\right)\leq\exp_2\left(\binom{n-2}{2}+1\right)$. Hence, the total number of disconnected graphs is at most $\exp_2\left(\binom{n-1}{2}+o(n)\right)$.

□

We define the *block tree* of a connected graph G as follows. An edge $e\in E(G)$ is a *bridge*, if $G-e$ is disconnected. Let B be the set of bridges in G and let $A_1,\ldots,A_t$ be the vertex sets of the components of $G-B$. Then the block tree of G is $Bt(G)=(B,\{A_1,\ldots,A_t\})$.

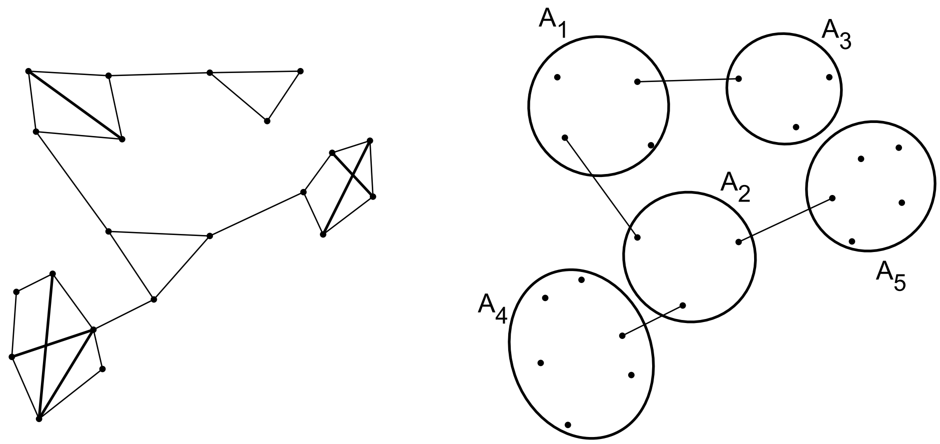

Figure 18: A graph and an illustration of its blocktree

The following lemma lists the main properties of the block tree, which may be easily verified by the reader.

Lemma 83. Let G be a connected graph with block tree $(B, \{A_1, ..., A_t\})$. Then $|B| = t - 1$ and $G[A_i]$ is 2-edge-connected for $i \in [t]$.

If G is a 2-edge-connected graph, let $R(G)$ be the set of edges $f \in E(G)$ such that $G - f$ is not 2-edge-connected. Lemma 84 gives an upper bound on the size of $R(G)$.

Lemma 84. Let G be a 2-edge-connected graph and let $H = G - R(G)$. Denote the number of components of $G - R(G)$ by q. Then $|R(G)| \leq 2q - 2$.

To make our proof more convenient, we shall work with *multi-graphs*. A multi-graph is a graph where we allow multiple edges between a pair of vertices, but no loops. We extend the definition of a *cycle* as follows: a cycle is either 2 vertices connected by 2 edges or a simple graph that is a cycle. A *chord* in a cycle C is an edge not in $E(C)$ connecting two vertices of C. For example, if the vertices x and y are connected by 3 edges, any two edges form a cycle and the third edge is a chord of this cycle.

Proof.

Let the components of H be $H_1, ..., H_q$. Every edge in $R(G)$ connects two different components in H. Define the multi-graph K on vertex set $[q]$ as follows: if H_i and H_j are connected by l edges in G, then i and j are connected by l edges in K.

Note that the graph K cannot contain a cycle with a chord. Otherwise, suppose that there is a cycle with vertices $i_1, \ldots, i_s$ and a chord $i_a i_b$. Let $e \in R(G)$ be an edge connecting H_{i_a} and H_{i_b} in G. Then H_{i_a} and H_{i_b} are still connected by at least 2 disjoint paths in $G - e$, hence e cannot be an element of $R(G)$.

Our lemma follows from the following result about multi-graphs without cycles with a chord.

Claim 85. If L is a multi-graph on q vertices without a cycle with a chord, then $e(L) \leq 2q - 2$.

Proof.

We proceed by induction on q. If $q = 1$, $E(L)$ is empty, so we are done. Suppose that $q > 1$. If L has a vertex v of degree at most 2, then let $L' = L - v$. Then L' has $q - 1$ vertices, at least $e(L) - 2$ edges, and does not contain a chorded cycle. Hence, by induction, $e(L) - 2 \leq 2q - 4$, which gives $e(L) \leq 2q - 2$. Now suppose that every vertex of L has degree at least 3. Let $v_1, \ldots, v_s$ be the consecutive vertices of a longest path in L. Every neighbor of v_1 is contained in the set $\{v_2, \ldots, v_s\}$, otherwise we can find a longer path in L. Hence, there exist i, j satisfying $2 \leq i \leq j \leq s$ such that the multi-set $E(L)$ contains three different edges, $v_1 v_2, v_1 v_i$ and $v_1 v_j$. But then $v_1, \ldots, v_j$ forms a cycle, and $v_1 v_i$ is a chord of this cycle. □

As K does not contain a cycle with a chord and has $|R(G)|$ edges, we get $|R(G)| \leq 2q - 2$. This completes the proof of Lemma 84.

□

We remark that if $R(G)$ is non-empty, it has at least 2 elements. This is true because if $e \in R(G)$, then $G - e$ contains a bridge f. But then $f \in R(G)$ as well.

Let I_r be the set of 2-edge-connected graphs G such that $|R(G)| = r$, and let $I_r^{(k)} = I_r \cap \mathcal{C}^{(k)}$. In the next lemma, we give an upper bound on the size of $I_r^{(k)}$. Recall that $M = \lceil \binom{n}{2}/2 \rceil$.

Lemma 86. Let ε be a positive real number. There exists $n_1(\varepsilon)$ such that if $n > n_1(\varepsilon)$, the following holds. For any positive integers r and k satisfying $2 \leq r \leq n$ and $M \leq k \leq M + n$, we have

$$|I_r^{(k)}| \leq \binom{\binom{n-r/2}{2} + \varepsilon rn}{k}.$$

Proof.

Let $q = \lceil r/2 \rceil + 1$. If G is a 2-edge-connected graph with $|R(G)| = r$, then $G - R(G)$ has at least q components by Lemma 84. We now count the number of graphs G where $G - R(G)$ has exactly s components. Note that $s \leq r$, otherwise the edges of $R(G)$ could not connect all the components of $G - R(G)$.

The number of graphs G, for which $|R(G)| = s$, and where the components in $G - R(G)$ have sizes $a_1, \ldots, a_s$ with $e_1, \ldots, e_s$ edges inside them, respectively, is at most

$$\binom{n^2}{r} \binom{n}{a_1, \ldots, a_s} \prod_{i=1}^{s} \binom{\binom{a_i}{2}}{e_i}. \tag{1}$$

Here, $\binom{n^2}{r}$ is an upper bound on the number of ways to pick the edges of $R(G)$, $\binom{n}{a_1,\ldots,a_s}$ is the number of ways to partition $[n]$ into parts of size $a_1, \ldots, a_s$, and $\binom{\binom{a_i}{2}}{e_i}$ is the number of ways to choose the e_i edges in a component of size a_i. We shall prove that (1) is at most $\binom{\binom{n-s+1}{2}}{k} \exp_2(3\varepsilon rn/6)$. Let us bound the terms in (1).

First, $\binom{n^2}{r} \leq \exp_2(2r \log n) < \exp_2(\varepsilon rn/6)$, if n is sufficiently large given ε.

Also, $\binom{n}{a_1,\ldots,a_s} \leq s^n = \exp_2(n \log s)$. Unfortunately, if r is small, we cannot bound this term by $\exp_2(c\varepsilon rn)$, where c is some fixed constant. We shall overcome this obstacle later in the proof.

Finally,

$$\prod_{i=1}^{s} \binom{\binom{a_i}{2}}{e_i} \leq \binom{\sum_{i=1}^{s} \binom{a_i}{2}}{k-r} \leq \binom{\binom{n-s+1}{2}}{k-r},$$

where the last inequality holds by (i) in Lemma 81. Here,

$$\binom{\binom{n-s+1}{2}}{k-r} < \binom{r+\binom{n-s+1}{2}}{k},$$

see (iii) in Lemma 80. Hence, we have

$$\prod_{i=1}^{s}\binom{\binom{a_i}{2}}{e_i} \leq \binom{\binom{n-s+1}{2}+\varepsilon rn/6}{k},$$

provided $n > 6/\varepsilon$.

First, suppose that r is such that $\log r < \varepsilon r/6$. In this case, we have $\binom{n}{a_1,\ldots,a_s} \leq \exp_2(\varepsilon rn/6)$. Hence, (1) is at most $\binom{\binom{n-s+1}{2}+\varepsilon rn/6}{k} \cdot \exp_2(2\varepsilon rn/6)$.

Now consider the case when $\log r > \varepsilon r/6$. Then $r < R(\varepsilon)$, where $R(\varepsilon)$ is a constant only depending on ε. In this case, we shall bound the product

$$\binom{n}{a_1,\ldots,a_s}\prod_{i=1}^{s}\binom{\binom{a_i}{2}}{e_i}. \tag{2}$$

Without loss of generality, suppose that $a_1 \geq \ldots \geq a_s$ and observe that $\binom{n}{a_1,\ldots,a_s} < n^{a_2+\ldots+a_s}$. Thus, if $a_1 \geq n-4r$, then $\binom{n}{a_1,\ldots,a_s} < n^{4r} < \exp_2(\varepsilon rn/6)$, if n is sufficiently large given ε. Now suppose that $a_1 < n-4r$. Applying (iii) in Lemma 81, we get

$$\sum_{i=1}^{s}\binom{a_i}{2} \leq \binom{4r-s-2}{2}+\binom{n-4r}{2}$$
$$\leq 8r^2+\binom{n-4r}{2}.$$

Suppose $n > 20R(\varepsilon)$, then the inequality

$$8r^2+\binom{n-4r}{2} \leq \binom{n-s+1}{2}-2rn$$

holds as well. Hence,

$$\prod_{i=1}^{s}\binom{\binom{a_i}{2}}{e_i} < \binom{\sum_{i=1}^{s}\binom{a_i}{2}}{k-r} \leq \binom{\binom{n-s+1}{2}-2rn}{k-r} < \binom{\binom{n-s+1}{2}-2rn+r}{k},$$

where the last inequality holds by (iii) in Lemma 80. Also, using (ii) in Lemma 80,

$$\binom{\binom{n-s+1}{2}-2rn+r}{k} \leq \binom{\binom{n-s+1}{2}}{k}\exp_2(-rn).$$

Thus, we can bound (2) from above by $\binom{\binom{n-s+1}{2}}{k}$, and so (1) is at most

$$\binom{\binom{n-s+1}{2}+\varepsilon rn/6}{k}\exp_2(2\varepsilon rn/6)$$

in this case as well.

Now let us bound the number of all 2-edge-connected graphs with k edges, for which $|R(G)| = r$ and $G - R(G)$ has s components. The number of such graphs is at most

$$\sum_{a_1+\ldots+a_s=n}\sum_{e_1+\ldots+e_s=k-r}\binom{n^2}{r}\binom{n}{a_1,\ldots,a_s}\prod_{i=1}^{s}\binom{\binom{a_i}{2}}{e_i}. \tag{3}$$

The first sum has exactly $\binom{n}{s-1}$ terms since $a_i \geq 1$ for every $i \in [s]$, while the second sum has $\binom{k-r+s}{s-1}$ terms. Therefore, (3) is at most

$$\binom{n}{s-1}\binom{k-r+s}{s-1}\binom{\binom{n-s+1}{2}+\varepsilon rn/6}{k}\exp_2(2\varepsilon rn/6).$$

Here, $\binom{n}{s-1} \leq \exp_2(r\log n)$ and $\binom{k-r+s}{s-1} < \exp_2(2r\log n)$. Thus, (3) is at most

$$\binom{\binom{n-s+1}{2}+\varepsilon rn/6}{k}\exp_2(3\varepsilon rn/6),$$

provided n is sufficiently large given ε.

Finally, the number of 2-edge-connected graphs with $|R(G)| = r$ and k edges is at most

$$\sum_{i=q}^{r}\binom{\binom{n-i+1}{2}+\varepsilon rn/6}{k}\exp_2(3\varepsilon rn/6) < \binom{\binom{n-q+1}{2}+\varepsilon rn/6}{k}\exp_2(4\varepsilon rn/6).$$

Applying (ii) in Lemma 80, we get

$$|I_r^{(k)}| \leq \binom{\binom{n-q+1}{2}+\varepsilon rn}{k}.$$

□

In the proof of Theorem 79, we shall also use the following technical lemma. Again, recall that $M = \lceil \binom{n}{2}/2 \rceil$.

Lemma 87. Let $n > 150$. Let G be a connected graph on vertex set $[n]$ such that $e(G) \geq M$ and $Bt(G) = (B, \{A_1,...,A_t\})$. Suppose that $|A_i| \leq n-2$ for $i \in [t]$. Then

$$\sum_{1\leq i<j\leq t} |A_i||A_j| - 2(t-1) - \sum_{i=1}^{t} |R(G[A_i])| \geq n. \tag{4}$$

Proof.

By Lemma 84, we have $|R(G[A_i])| < 2|A_i|$. Hence, $\sum_{i=1}^{t} |R(G[A_i])| < 2n$.

First, suppose that $\max\{|A_1|,..,|A_t|\} \leq n-6$. By (iv) in Lemma 81, we have

$$\sum_{1\leq i<j\leq t} |A_i||A_j| \geq 6(n-6) \geq 5n.$$

Hence, using the trivial bound $t-1 < n$, we have that (4) holds.

Now suppose that $|A_1| \geq n-5$. In this case, we have $t \leq 6$. Let $H = G[A_1]$. Every edge of G not contained in H is either in B or it is an edge of $G[[n] \setminus A_1]$. Hence, the number of edges not contained in H is at most 20, so $e(H) \geq M - 20$.

Let $H_1,...,H_q$ be the vertex sets of the components of $H - R(H)$. Then, by Lemma 84, the number of edges of H is at most

$$2q-2+\sum_{i=1}^{q}\binom{|V(H_i)|}{2}<2n+\binom{n-q+1}{2},$$

where the inequality holds by (i) in Lemma 81. Comparing the lower and upper bounds on $e(H)$ we get the inequality

$$M-20<2n+\binom{n-q+1}{2}.$$

If $q>n/3$, the right hand side of the inequality is at most $2n^2/9+3n$, while the left hand side is larger than $n^2/4-n$. This is a contradiction, noting that $2n^2/9+3n<n^2/4-n$ for $n>150$. Hence, we have $q<n/3$, implying $|R(H)|<2n/3$. This gives

$$\sum_{i=1}^{t}|R(G[A_i])|\le|R(H)|+2(|A_2|+\ldots+|A_t|)<2n/3+10.$$

Since $|A_1|\le n-2$, we have $\sum_{1\le i<j\le t}|A_i||A_j|\ge 2(n-2)$ by (iv) in Lemma 81, so (4) holds.

□

Matchings between levels

In this section, we prove Theorem 79.

Let $n-1\le k,l\le m$. We say that there is a complete matching from $\mathcal{C}^{(k)}$ to $\mathcal{C}^{(l)}$, if there is an injection $f:\mathcal{C}^{(k)}\to\mathcal{C}^{(l)}$ such that G and $f(G)$ are comparable for all $G\in\mathcal{C}^{(k)}$. The next lemma states that to prove Theorem 79, it is enough to find a complete matching from the smaller sized level to the larger sized level for any two consecutive levels. Due to its simplicity, we shall only sketch the proof of this lemma.

Lemma 88. Suppose that there is a complete matching from $\mathcal{C}^{(k)}$ to $\mathcal{C}^{(k+1)}$ for $k=n-1,\ldots,M-1$, and there is a complete matching from $\mathcal{C}^{(l+1)}$ to $\mathcal{C}^{(l)}$ for

$l = M, ..., m-1$. Then the largest antichain in $\mathcal{C}$ is $\mathcal{C}^{(M)}$.

Proof.

Using the complete matchings, one can build a chain partition of $\mathcal{C}$ into $|\mathcal{C}^{(M)}|$ chains. But the size of the maximal antichain in $\mathcal{C}$ is at most the number of chains in any chain partition of $\mathcal{C}$.

□

First, we show that if we are below the middle level $\mathcal{C}^{(M)}$, or at least n above the middle level, then it is easy to prove the existence of a complete matching between consecutive levels.

Let $X \subset \mathcal{C}^{(k)}$ for some $n-1 \leq k \leq m$. The *lower shadow* of X is

$$\Delta(X) = \{G \in \mathcal{C}^{(k-1)} : \exists H \in X, G < H\},$$

and the *upper shadow* of X is

$$\nabla(X) = \{G \in \mathcal{C}^{(k+1)} : \exists H \in X, H < G\}.$$

In our proofs, we shall apply the well known theorem of Hall [21].

Theorem 89. (Hall's theorem) Let $G = (A, B; E)$ be a bipartite graph. There is a complete matching in G from A to B if and only if $|X| \leq |\Gamma(X)|$ for all $X \subset A$, where $\Gamma(X)$ denotes the set of vertices adjacent to some element of X.

First, let us deal with the levels below $\mathcal{C}^{(M)}$.

Lemma 90. There is a complete matching from $\mathcal{C}^{(k)}$ to $\mathcal{C}^{(k+1)}$ for $k = n-1, ..., M-1$.

Proof.

Let $X \subset \mathcal{C}^{(k)}$. By Hall's theorem, it is enough to show that $|X| \leq |\nabla(X)|$. Let B be the bipartite graph with vertex partition $(X, \nabla(X))$, and the edges of B being the comparable pairs. If $G \in X$, the degree of G is $m-k$. Also, if $H \in \nabla(X)$, the degree of H is at most $k+1$.

Let e be the number of edges of B. Then, counting e from X, and then from $\nabla(X)$, we have

$$|X|(m-k) = e,$$

and

$$e \leq |\nabla(X)|(k+1).$$

Hence,

$$|X| \leq |\nabla(X)|(k+1)/(m-k) \leq |\nabla(X)|.$$

□

Using similar ideas, we now show that if we are above the middle level by at least n, then there is a matching from $\mathcal{C}^{(k+1)}$ to $\mathcal{C}^{(k)}$.

Lemma 91. There is a complete matching from $\mathcal{C}^{(k+1)}$ to $\mathcal{C}^{(k)}$ for $k = M+n,\ldots,m$.

Proof.

Let $X \subset \mathcal{C}^{(k+1)}$. By Hall's theorem, it is enough to show that $|X| \leq |\Delta(X)|$. Let B be the bipartite graph with vertex partition $(X, \Delta(X))$, and the edges of B being the comparable pairs. If $G \in \Delta(X)$, then the degree of G in B is at most $m-k$.

Now let $G \in X$. If $e \in E(G)$ such that $G-e$ is not an element of $\mathcal{C}$, then e is a bridge of G. However, by Lemma 83, the number of bridges of G is at most $n-1$. Hence, the degree of G is at least $k+2-n$. Counting the number of edges of B two ways, we get

$$|X|(k+2-n) \leq |E(B)|,$$

and

$$|\Delta(X)|(m-k) \geq |E(B)|.$$

Hence,

$$\frac{|\Delta(X)|}{|X|} \geq \frac{k+2-n}{m-k} \geq 1.$$

□

Proving that there is a matching from $\mathcal{C}^{(k)}$ to $\mathcal{C}^{(k-1)}$ for the values of k that are slightly larger than M is more difficult. The remainder of this section is devoted to this problem. Before showing the details, we briefly outline the strategy for showing that there exists a complete matching from $\mathcal{C}^{(k)}$ to $\mathcal{C}^{(k-1)}$, where $M+1 \leq k < M+n$.

Our goal is to show that for every $X \subset \mathcal{C}^{(k)}$, we have $|\Delta(X)| \geq |X|$. To accomplish this, we write X as $Y \cup Z$, where Y is the set of 2-edge-connected graphs in X and Z is the set of the non-2-edge-connected graphs in X. We first show that if the two sets, Y and Z, do *not* have roughly the same size, then the larger of the two has a lower shadow that is already larger than $|X|$.

Now suppose that $|Y| \approx |Z|$. We show the existence of three functions $c_1, c_2, c_3 : \mathbb{N} \to \mathbb{R}^+$ satisfying the following properties:

1. $|\Delta(Y)| \geq |Y|(1+c_1(|Y|))$,
2. $|\Delta(Z)| \geq |Z|(1+c_2(|Z|))$,
3. if U is the set of 2-edge-connected graphs in $\Delta(Y)$, then $|U| \geq |Y|(1-c_3(|Y|))$,
4. $c_1(|Y|)c_2(|Z|) \geq c_3(|Y|)$, if $|Y| \approx |Z|$.

Roughly, 1. and 2. state that the lower shadow of Y and Z is slightly larger than Y and Z, respectively. Now, we would like to guarantee that $\Delta(X) = \Delta(Y) \cup \Delta(Z)$ is also larger than $Y \cup Z$. If this is not the case, then we must have that $\Delta(Y) \setminus \Delta(Z)$ is too small. But note that as $\Delta(Z)$ contains only non-2-edge-connected graphs, U is contained in $\Delta(Y) \setminus \Delta(Z)$. Hence, $|U|$ is a lower bound on the size of the set $\Delta(Y) \setminus \Delta(Z)$. Thus, 3. tells us that $\Delta(Y) \setminus \Delta(Z)$ cannot be much smaller than Y, and property 4. guarantees (as we shall see later) that we truly have $|Y \cup Z| \leq |\Delta(Y) \cup \Delta(Z)|$.

We remind the reader that $\mathcal{G}$ is the family of all graphs on vertex set $[n]$. For $X \subset \mathcal{C}^{(k)}$, let $\partial(X) = \{H \in \mathcal{G}^{(k-1)} : \exists G \in X, H < G\}$. As $(\mathcal{G}, <)$ is isomorphic to $(2^{[m]}, \subset)$, the Kruskal-Katona theorem [27, 29] tells us which subfamily of $\mathcal{G}^{(k)}$ of given size minimizes the lower shadow. Instead of using this, however, we use a weaker form of the Kruskal-Katona theorem, proved by Lovász [30]. This affords us a computationally more convenient way to obtain a lower bound on the size of $\partial(X)$.

Lemma 92. (Lovász [30]) Let $X \subset \mathcal{C}^{(k)}$ be nonempty and let x be a real number such that $|X| = \binom{x}{k}$. Then

$$|\partial(X)| \geq \binom{x}{k-1}.$$

In particular,

$$\frac{|\partial(X)|}{|X|} \geq \frac{k}{x-k+1}.$$

We remind the reader that we use the extended definition of binomial coefficients introduced in Section III, so both in the previous lemma and in what comes, x need not to be an integer in $\binom{x}{k}$.

Let $\mathcal{B}$ be the set of 2-edge-connected graphs in $\mathcal{C}$ and let $\mathcal{B}^{(k)} = \mathcal{C}^{(k)} \cap \mathcal{B}$. If $X \subset \mathcal{B}^{(k)}$, then $\Delta(X) = \partial(X)$. Hence, we can use Lemma 92 to get a lower bound for the size of $\Delta(X)$.

In the next lemma we show that if the size of $X \in \mathcal{C}^{(k)}$ is sufficiently large, then we have $|\Delta(X)| \geq |X|$.

Lemma 93. Let $\varepsilon > 0$. There exists $n_2(\varepsilon)$ such that if $n > n_2(\varepsilon)$ the following holds. Let $M+1 \leq k < M+n$ and let $|X| = \binom{x}{k}$, where $x > \binom{n-1}{2} + \varepsilon n$. We have $|\Delta(X)| > |X|$.

Proof.

By Lemma 92,

$$|\partial(X)| \geq \binom{x}{k-1}.$$

Let D be the set of disconnected graphs with $k-1$ edges. By Lemma 82,

$$|D| \leq \exp_2\left(\binom{n-1}{2} + o(n)\right).$$

Also,

$$|\Delta(X)| = |\partial(X) \setminus D| \geq |\partial(X)| - |D| \geq \binom{x}{k-1} - \exp_2\left(\binom{n-1}{2} + o(n)\right).$$

Thus, we get

$$|\Delta(X)| - |X| \geq \binom{x}{k-1} - \binom{x}{k} - \exp_2\left(\binom{n-1}{2} + o(n)\right) =$$

$$= \binom{x}{k-1}\frac{2k-x-1}{k} - \exp_2\left(\binom{n-1}{2} + o(n)\right) > \binom{\binom{n-1}{2}+\varepsilon n}{k-1}\frac{1}{n^2} - \exp_2\left(\binom{n-1}{2} + o(n)\right).$$

By (ii) in Lemma 80, we have $\binom{\binom{n-1}{2}+\varepsilon n}{k-1} \geq \binom{\binom{n-1}{2}}{k-1} \cdot \exp_2(\varepsilon n)$. Also, $\binom{\binom{n-1}{2}}{k-1} = \exp_2(\binom{n-1}{2} + o(n))$ holds by Stirling's formula. Hence, we have

$$|\Delta(X)| - |X| \geq \exp_2\left(\binom{n-1}{2} + \varepsilon n + o(n)\right) - \exp_2(\binom{n-1}{2} + o(n)).$$

Thus, if n is sufficiently large given ε, $|\Delta(X)| > |X|$. □

Now we show that if X is a set of 2-edge-connected graphs in $\mathcal{C}^{(k)}$, then the number of 2-edge-connected graphs in the shadow of X cannot be much less than $|X|$.

Lemma 94. Let $0 < \varepsilon < 1/4$. There exists $n_3(\varepsilon)$ such that if $n > n_3(\varepsilon)$, the following holds. Let $M < k < M+n$ and let $X \subset \mathcal{B}^{(k)}$. Let $|X| = \binom{x}{k}$ and let r be a positive integer satisfying $r < n$. If $x > \binom{n-(r+1)/2}{2} + \varepsilon rn$, then

$$\frac{|\Delta(X) \cap \mathcal{B}^{(k-1)}|}{|X|} > 1 - \frac{4r}{n^2}.$$

Proof.

Define $U = \Delta(X) \cap \mathcal{B}^{(k-1)}$ and let B be the bipartite graph with vertex partition (X, U), the edges being the comparable pairs. Every element of U has degree at most $m-k+1$ in B. Also, the degree of a graph G in X is exactly $k - |R(G)|$ in B. Let a be the number of graphs in X with degree at most $k-r-1$ and let a' be the number of graphs in $\mathcal{B}^{(k)}$ with $|R(G)| \geq r+1$. Then $a < a'$ and by Lemma 86, we have $a' < \binom{\binom{n-(r+1)/2}{2} + \varepsilon rn/2}{k}$, provided $n > n_1(\varepsilon/2)$. Moreover, we have the following bounds on the number of edges of B:

$$(k-r)(|X| - a') \leq e(B) \leq (m-k+1)|U|.$$

Hence,

$$\frac{|U|}{|X| - a'} \geq \frac{k-r}{m-k+1}.$$

Here, $k \geq m/2 + 1$, so

$$\frac{|U|}{|X| - a'} \geq \frac{m/2 - r + 1}{m/2} \geq 1 - \frac{4r-4}{n(n-1)}.$$

If $|X| > 8n^3 a'$, we get $\frac{|U|}{|X|} \geq 1 - \frac{4r}{n^2}$, using that $r \leq n-1$. But note that if n is sufficiently large given ε, then $8n^3 < \exp_2(\varepsilon n/3)$, which means that

$$8n^3 a' < \binom{\binom{n-(r+1)/2}{2} + \varepsilon rn/2}{k} \cdot \exp_2(\varepsilon n/3) < \binom{\binom{n-(r+1)/2}{2} + \varepsilon rn}{k} < \binom{x}{k},$$

where the second inequality is a consequence of (ii) from Lemma 80. $\square$

We remark that we do not have to consider the case when $r \geq n$. If $|X| = \binom{x}{k} \geq 1$, then $x \geq k$, and we can always find $r < n$ satisfying $x > \binom{n-(r+1)/2}{2} + \varepsilon rn$. This remark holds true for the upcoming lemmas as well.

In the next lemma, we show that if $X \subset \mathcal{C}^{(k)}$ is a set of non-2-edge-connected graphs, then the size of the shadow of X is slightly larger than $|X|$.

Lemma 95. Let ε be a positive real number such that $\varepsilon < 1/2$. There exists $n_4(\varepsilon)$ such that if $n > n_4(\varepsilon)$, the following holds. Let k be a positive integer with $M < k < M + n$ and let $X \subset \mathcal{C}^{(k)} \setminus \mathcal{B}^{(k)}$. Let $|X| = \binom{x}{k}$ and let r be a positive integer such that $r < n$ and $x > \binom{n-(r+1)/2}{2} + \varepsilon rn$. Then

$$\frac{|\Delta(X)|}{|X|} > 1 + \frac{4 - 4r/n}{n}.$$

Proof.

Define the bipartite graph B between X and $U = \Delta(X)$ as follows. Let $G \in X$ and $H \in \Delta(X)$ be connected by an edge if $H < G$ and $Bt(G) = Bt(H)$. If $T = (C, \{A_1, ..., A_t\})$ is the block tree of some graph, let $X(T)$ be the set of graphs in X with block tree T, and define $U(T)$ similarly. Let $B(T)$ be the bipartite subgraph of B induced on $X(T) \cup U(T)$, and let us estimate $|U(T)|/|X(T)|$. If $H \in U(T)$ and $e \in [n]^{(2)} \setminus E(H)$ is an edge connecting A_i and A_j with $i \neq j$, then the block tree of $H' = H \cup \{e\}$ differs from T. Hence, the degree of H in this bipartite graph is at most

$$u_T = m - k + 1 - \sum_{1 \leq i < j \leq t} |A_i||A_j| + t - 1.$$

Note that the term $t - 1$ corresponds to the number of edges in C. Now let $G \in X(T)$ and $e \in E(G)$. We have $Bt(G - e) = T$ if and only if $e \in G[A_i] \setminus R(G[A_i])$ for some $i \in [t]$. Hence, the degree of G in $B(T)$ is

$$x_T(G) = k - \sum_{i=1}^{t} |R(G[A_i])| - (t - 1).$$

Suppose that $t \geq 3$ or $\min\{|A_1|,|A_2|\} > 1$. Then by Lemma 87, we have

$$x_T(G) - u_T = \sum_{1\leq i<j\leq t} |A_i||A_j| - 2(t-1) - \sum_{i=1}^{t} |R(G[A_i])| \geq n.$$

Setting $x_T = u_T + n$, we have $x_T(G) \geq x_T$.

Bounding the edges of $B(T)$ in two different ways, we get

$$|X(T)|x_T \leq e(B(T)) \leq |U(T)|u_T.$$

We now consider the remaining case, when $t = 2$ and $\min\{|A_1|,|A_2|\} = 1$. Note that we need not consider the case $t = 1$ as T is not the block tree of a 2-edge-connected graph.

Without loss of generality, let $|A_1| = 1$. We have $u_T \leq M - (n-2)$, while $x_T(G) \geq M - |R(G[A_2])|$ for every $G \in X(T)$. Let a be the number of graphs G in $X(T)$ with $|R(G[A_2])| \geq r+3$. By Lemma 86, we have

$$a < \binom{\binom{(n-1)-(r+3)/2}{2} + \varepsilon rn/2}{k},$$

if $n > n_1(\varepsilon/2)$.

Counting the number of edges of $B(T)$ two ways, we get the following bounds:

$$(|X(T)| - a)(M - (r+2)) \leq e(B(T)) \leq (M - (n-2))|U(T)|.$$

Hence,

$$\frac{|U(T)|}{|X(T)| - a} \geq \frac{M - (r+2)}{M - (n-2)} = 1 + \frac{(n-r)}{M - (n-2)} > 1 + \frac{(4n-4r)}{n(n-1)}.$$

If $|X(T)| > 2n^3 a$, this implies $\frac{|U(T)|}{|X(T)|} \geq 1 + \frac{4n-4r}{n^2-1}$.

Let $\mathcal{T}_0$ be the set of pairs $T = (C, \{A_1, A_2\})$ satisfying the following conditions: T is the block tree of some graph in $\mathcal{C}$, $|A_1| = 1$, and $|X(T)| \leq 2n^3 a$. Let $X_0 = \bigcup_{T\in\mathcal{T}_0} X(T)$. Note that $|\mathcal{T}_0| < n^2$ as we have at most n choices for A_1 and at most $n-1$ choices for the one edge in C. Hence, we have $|X_0| \leq 2n^5 a$. This gives the following bound on the size of $\Delta(X)$.

$$|\Delta(X)| \geq \left(1 + \frac{4n-4r}{n^2-1}\right)(|X| - |X_0|) \geq$$

$$\geq \left(1 + \frac{4n-4r}{n^2-1}\right)|X| - 4n^5 \binom{\binom{n-(r+1)/2}{2} + \varepsilon rn/2}{k}.$$

Therefore, if $|X| \geq 4n^9 \binom{\binom{n-(r+1)/2}{2} + \varepsilon rn/2}{k}$, then

$$\frac{|\Delta(X)|}{|X|} \geq 1 + \frac{4n-4r}{n^2}.$$

But if n is sufficiently large given ε, we have

$$4n^9 \binom{\binom{n-(r+1)/2}{2} + \varepsilon rn/2}{k} < \binom{\binom{n-(r+1)/2}{2} + \varepsilon rn}{k} \leq |X|.$$

□

In the next lemma, we show that if the number of 2-edge-connected graphs in X is not in the same range as the number of non-2-edge-connected graphs in X, then $|X| < |\Delta(X)|$.

Lemma 96. There exists n_5 such that if $n > n_5$, the following holds. Let $M + 1 \leq k < M + n$, $X \subset \mathcal{C}^{(k)}$ and $Y = X \cap \mathcal{B}$, $Z = X - Y$. Suppose that $|Z| > n|Y|$ or $|Y| > n|Z|$. Then $|\Delta(X)| > |X|$.

Proof.

If $|X| \geq \binom{m-n/2}{k}$, we are done by Lemma 93. So we can suppose that $|X| < \binom{m-n/2}{k}$. Firstly, consider the case when $|Y| > n|Z|$. Let $Y = \binom{y}{k}$, then $y < m - n/2$. As $\partial(Y) = \Delta(Y)$, we can apply Lemma 92 to get

$$\frac{|\Delta(Y)|}{|Y|} > \frac{k}{m - n/2 - k} \geq 1 + \frac{2}{n}.$$

Hence, $|\Delta(X)| \geq |\Delta(Y)| > |Y| + 2|Y|/n > |Y| + |Z|$.

Now consider the case when $|Z| > n|Y|$. Let $|Z| = \binom{z}{k}$, then $z \geq k = n^2/4 + O(n)$.

Set $\varepsilon = 1/40$ and $r = \lceil 2n/3 \rceil$. We choose r and ε such that $r < n$ and $z > \binom{n-(r+1)/2}{2} + \varepsilon rn$ holds. Hence, by Lemma 95, we have

$$\frac{|\Delta(Z)|}{|Z|} \geq 1 + \frac{4 - 4r/n}{n} \geq 1 + \frac{4}{3n},$$

for n sufficiently large. Estimating the size of the shadow of X with $|\Delta(Z)|$, we get

$$|\Delta(X)| \geq |\Delta(Z)| \geq |Z| + \frac{4|Z|}{3n} \geq |Z| + |Y| = |X|.$$

□

We also need the following technical lemma, which tells us what conditions need to be satisfied for the sizes of the shadows of Y, Z to have $|X| < |\Delta(X)|$.

Lemma 97. Let a, b, c_1, c_2, c_3 be positive real numbers and $A = a(1+c_1)$, $B = b(1+c_2)$ and $C = a(1-c_3)$. If $c_3 \leq c_1 c_2$, then

$$a + b \leq C + \max\{B, A - C\}.$$

Proof.

We need to show that $ac_3 + b < \max\{B, A-C\}$. Observe that we can suppose that $B = A - C$. Otherwise, if $B < A - C$, we can substitute b with $b' > b$,and B with $B' = b'(1+c_3)$, satisfying $B' = A - C$. Then the left hand side of the inequality increases, while the right hand side does not change. We can proceed similarly if $A - C < B$.

If $B = A - C$, then $b = \frac{c_1+c_3}{1+c_2}a$. Hence, our inequality becomes

$$ac_3 + \frac{c_1 + c_3}{1 + c_2}a \leq (c_1 + c_3)a.$$

Simplifying this inequality, we get that it is equivalent with $c_3 \leq c_1 c_2$. □

Now we are ready to show the existence of a complete matching between the levels close to the middle level.

Theorem 98. There exists n_6 such that if $n > n_6$, the following holds. If $M+1 \leq k < M+n$, then there exists a complete matching from $\mathcal{C}^{(k)}$ to $\mathcal{C}^{(k-1)}$.

Proof.

By Hall's theorem, it is enough to prove that for any $X \subset \mathcal{C}^{(k)}$, we have $|X| \leq |\Delta(X)|$. Fix $\varepsilon = 1/18$. Let $|X| = \binom{x}{k}$. By Lemma 93, if $x > \binom{n-1}{2} + \varepsilon n$, then we are done if $n > n_2(\varepsilon)$. Now suppose that $x \leq \binom{n-1}{2} + \varepsilon n$. Let $Y = X \cap \mathcal{B}$ and $Z = X - Y$. Let $|Y| = \binom{y}{k}$, $|Z| = \binom{z}{k}$, and suppose that $n > n_5$. By Lemma 96, if $|Y| > n|Z|$ or $|Z| > n|Y|$, we are done. Hence, we can suppose that $x - \varepsilon n < y, z \leq x$, if n is sufficiently large.

Let $U = \Delta(Y) \cap \mathcal{B}$ and let r be a positive integer satisfying

$$\binom{n-(r+1)/2}{2} + \varepsilon(r+1)n \leq x < \binom{n-r/2}{2} + \varepsilon rn$$

One can easily check that as $k \leq x < \binom{n-1}{2} + \varepsilon$ and $\varepsilon < 1/4$, such an r always exists, it is unique, and $r < n$. Furthermore, $y, z > \binom{n-(r+1)/2}{2} + \varepsilon rn$.

By Lemma 94, if $n > n_3(\varepsilon)$, we have

$$\frac{|U|}{|Y|} > 1 - \frac{4r}{n^2}.$$

Also, by Lemma 92

$$\frac{|\Delta(Y)|}{|Y|} \geq \frac{k}{y-k+1} > \frac{k}{\binom{n-r/2}{2} + 2\varepsilon rn - k},$$

where the term $2\varepsilon rn$ comes from bounding $1 + \varepsilon rn$ above by $2\varepsilon rn$. Using that $k > m/2$, we have

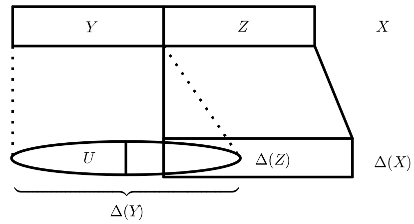

Figure 19: The comparability graph between X and its shadow

$$\frac{k}{\binom{n-r/2}{2}+2\varepsilon rn-k} > \frac{m/2}{\binom{n-r/2}{2}+2\varepsilon rn-m/2} =$$

$$= \frac{1}{1-r(2n-1)/n(n-1)+r^2/2n(n-1)+8\varepsilon r/(n-1)} >$$

$$> \frac{1}{1-2r/n+r^2/2n^2+9\varepsilon r/n},$$

where the last inequality holds if n is sufficiently large.

Finally, by Lemma 95, if $n > n_4(\varepsilon)$, we have

$$\frac{|\Delta(Z)|}{|Z|} > 1+\frac{4-4r/n}{n}.$$

Now we are ready to estimate $|\Delta(X)|$. We have

$$\Delta(X) = U \cup ((\Delta(Y)\setminus U)\cup \Delta(Z)),$$

where $\cup$ denotes disjoint union. Hence,

$$|\Delta(X)| \geq |U| + \max\{|\Delta(Y)|-|U|, |Z|\}.$$

Also, $|X| = |Y|+|Z|$. Let $c_1 = \frac{2r/n-r^2/2n^2-9\varepsilon r/n}{1-2r/n+r^2/2n^2+9\varepsilon}$, $c_2 = \frac{4-4r/n}{n}$ and $c_3 = 4r/n^2$. We have $|\Delta(Y)| > (1+c_1)|Y|$, $|\Delta(Z)| > (1+c_2)|Z|$ and $|U| > (1-c_3)|Y|$. Hence, by Lemma 97, our task is reduced to proving that $c_3 \leq c_1c_2$. Namely,

$$\frac{4r}{n^2} \leq \frac{2r/n - r^2/2n^2 - 9\varepsilon r/n}{1 - 2r/n + r^2/2n^2 + 9\varepsilon r/n} \frac{4 - 4r/n}{n}.$$

Simplifying this inequality, we get

$$1 - 2r/n + r^2/2n^2 + 9\varepsilon r/n \leq (2 - r/2n - 9\varepsilon)(1 - r/n).$$

For our convenience, let $\alpha = r/n$. Then the previous inequality can be written as

$$1 - 2\alpha + \alpha^2/2 + 9\varepsilon\alpha \leq (2 - \alpha/2 - 9\varepsilon)(1 - \alpha),$$

which reduces to

$$\alpha + 18\varepsilon \leq 2.$$

As $\alpha < 1$ and $\varepsilon = 1/18$, this inequality holds. Hence, if n is sufficiently large, we have $|X| \leq |\Delta(X)|$. □

We are now ready to prove our main theorem.

Proof of Theorem 79. Let $n > n_6$, where n_6 is the constant given in Lemma 98. By Lemma 88, it is enough to prove that for $k = 1, \ldots, M-1$ there is a complete matching from $\mathcal{C}^{(k)}$ to $\mathcal{C}^{(k+1)}$, and for $k = M+1, \ldots, m$, there is a complete matching from $\mathcal{C}^{(k)}$ to $\mathcal{C}^{(k-1)}$. But we proved exactly this statement in Lemma 90, Lemma 91 and Theorem 98.

□

As a final remark, we observe that the proof also shows that $\mathcal{C}^{(M)}$ is the unique largest antichain, as the strict inequality $|\Delta(X)| > |X|$ holds.

Open problems

In this section, we propose several open problems.

The first problem we propose is inspired by the question investigated in [23], which we mentioned in the Introduction. Let G be a connected graph and let $C'(G)$ be the family of subgraphs of G that are connected on the vertex set $V(G)$. Define the partial ordering $<$ on $C'(G)$ as usual: $H < H'$ if $E(H) \subset E(H')$.

Question 99. Let G be a connected graph. Is $(C'(G), <)$ Sperner?

We believe that there should be graphs G for which $(C'(G), <)$ is not Sperner. Unfortunately, even for small graphs, it is difficult to check this property.

We also propose another variation of Question 78. Let GP be a monotone graph property (a family of graphs closed under isomorphism, and adding edges) and let GP_n denote the family of graphs in GP with vertex set $[n]$. Also, for $k = 0, ..., \binom{n}{2}$ let $GP_n^{(k)}$ be the set of graphs in GP_n with k edges. Define the partial ordering $<$ on GP_n as usual. The poset $(GP_n, <)$ might not be graded, however it still makes sense to ask the following question. For which graph properties GP is it true that the largest antichain in $(GP_n, <)$ is $GP_n^{(k)}$ for some k? To ask a more specific question, we propose the following problem.

Question 100. Let H be the family of Hamiltonian graphs. Is $(H_n, <)$ Sperner?

Finally, we suggest the following variation of Question 78. Suppose we do not distinguish graphs that are isomorphic. More precisely, define the equivalence relation $\sim$ on $\mathcal{C}$ such that $G \sim H$ if G and H are isomorphic, and let $\mathcal{C}_0$ be the set of equivalence classes of $\mathcal{C}$. Define $<$ on $\mathcal{C}_0$ such that for $\widetilde{G}, \widetilde{H} \in \mathcal{C}_0$ we have $\widetilde{G} < \widetilde{H}$ if there exists $G \in \widetilde{G}$ and $H \in \widetilde{H}$ satisfying $G < H$ in $(\mathcal{C}, <)$.

Question 101. Is $(\mathcal{C}_0, <)$ Sperner?

Acknowledgements

We would like to thank the anonymous referees for their useful comments and suggestions, and Andrew Thomason for drawing our attention to the simple proof presented in Claim 85.

REFERENCES

[1] AIM Minimum Rank - Special Graphs Work Group (F. Barioli, W. Barrett, S. Butler, S. M. Cioabă, D. Cvetković, S. M. Fallat, C. Godsil, W. Haemers, L. Hogben, R. Mikkelson, S. Narayan, O. Pryporova, I. Sciriha, W. So, D. Stevanović, H. van der Holst, K. Vander Meulen, and A. Wangsness). *Zero forcing sets and the minimum rank of graphs*, Linear Algebra and its Applications, 428/7: 1628-1648, 2008.

[2] S. Akbari, M. Ghanbari, and S. Jahanbekam, *On the Dynamic Coloring of Cartesian Product Graphs*, Ars Combinatoria. 114. 161-168.

[3] M. Alishahi, *On the dynamic coloring of graphs*, Discrete Appl. Math., 159 (2011), 152156.

[4] D. Amos, Y. Caro, R. Davila, and R. Pepper, *Upper bounds on the k-forcing number of a graph*, Discrete Appl. Math., 181 (2015) 1-10.

[5] I. Anderson, *Combinatorics of Finite Sets*, Oxford University Press (1987).

[6] J. Balogh, R. Mycroft, and A. Treglown, *A random version of Sperner's theorem*, Journal of Combinatorial Theory, Series A, 128 (2014), 104-110.

[7] B. Bollobás, *The chromatic number of random graphs*, Combinatorica, 8 (1) (1988) 49-55.

[8] B. Bollobás, *Random Graphs*, Cambridge University Press (2001).

[9] B. Bollobás, *Modern Graph Theory*, Springer (1998).

[10] N. Bowler, J. Erde, F. Lehner, M. Merker, M. Pitz, and K. Stavropoulo, *A counterexample to Montgomerys conjecture on dynamic colourings of regular graphs*, Discrete Appl. Math., 229 (1) 2017, 151-153

[11] Y. Caro and R. Pepper, *Dynamic approach to k-forcing*, Theory and Applications of Graphs, 2 (2), Article 2, 2015.

[12] R. Davila, T. Kalinowski, and S. Stephen, *A lower bound on the zero forcing number*, Discrete Appl. Math., 250 (11) 2018, Pages 363-367

[13] R. Davila, F. Kenter, *Bounds for the Zero-Forcing Number of Graphs with Large Girth*, Theory and Applications of Graphs. Volume 2: Iss. 2, Article 1, 2015

[14] Dehghan, A. and Ahadi, A. *Upper bounds for the 2-hued chromatic number of graphs in terms of the independence number*, Discrete Appl. Math., 160 (2012) 2142-2146

[15] D. Burgarth and V. Giovannetti. *Full control by locally induced relaxation.* Physical Review Letters PRL 99, 100501 (2007).

[16] L. Eroh, C. X. Kang, and E. Yi, *A Comparison between the Metric Dimension and Zero Forcing Number of Trees and Unicyclic Graphs* Acta. Math. Sin.-English Ser. (2017) 33: 731.

[17] P. Flajolet, R. Sedgewick, *Analytic Combinatorics* Cambridge University Press (2009).

[18] M. Gentner, L. D. Penso, D. Rautenbach, and U. S. Souzab, *Extremal values and bounds for the zero forcing number.* Discrete Applies Math. 214 (2016), 196 -200.

[19] M. Gentner and D. Rautenbach. *Some Bounds on the Zero Forcing Number of a Graph.* Discrete Applied Mathematics. 236 (2018), 203-213

[20] A. Girão, G. Mészáros, and S.G.Z. Smith. *On a conjecture of Gentner and Rautenbach*. Discrete Math. 341 (4) (2018) 1094-1097

[21] P. Hall, *On Representatives of Subsets,* J. London Math. Soc., 10 (1) (1935): 26-30.

[22] R.W. Hamming, *Error detecting and error correcting codes* The Bell system technical journal, 29 (2) (1950): 147-160

[23] M. S. Jacobson, A. E. Kézdy, S. Seif, *The poset on connected induced subgraphs of a graph need not be Sperner,* Order, 12 (3) (1995): 315-318.

[24] S. Jahanbekam, J. Kim, Suil O, D.B. West, *On r-dynamic coloring of graphs*, Discrete Appl. Math., 206 (2016) 6572

[25] R. Kang, T. Müller, D.B. West, *On r-dynamic coloring of grids*, Discrete Appl. Math., 186 (2015) 286-290

[26] Gy. O. H. Katona, *Personal communication.*

[27] Gy. O. H. Katona, *A theorem of finite sets*, Theory of Graphs, Akadémia Kiadó, Budapest (1968): 187-207.

[28] M. Krivelevich, and B. Patkós, *Equitable coloring of random graphs*, Random Structures & Algorithms (2009) 35 (1), pp. 83-99

[29] J. B. Kruskal, *The number of simplicies in a complex*, Mathematical Optimization Techniques, Univ. of California Press (1963) 251-278.

[30] L. Lovász, *Combinatorial Problems and Exercises,* North-Holland, Amsterdam (1993).

[31] B. Montgomery, *Dynamic Coloring of Graphs* (Ph.D Dissertation), West Virginia University, 2001.

[32] K. Owens, *Properties of the zero forcing number*. (Ph.D. Dissertation), Brigham Young University, 2009.

[33] D. D. Row, *A technique for computing the zero forcing number of a graph with a cut-vertex* Linear Algebra and its Applications 436 (2012) 4423-4432.

[34] E. Sperner, *"Ein Satz über Untermengen einer endlichen Menge"*, Mathematische Zeitschrift (in German), 27 (1) (1928): 544-548.

[35] A. Taherkhani, *On R -dynamic Chromatic Number of Graphs* Discrete Appl. Math., 201 (2016) 222-227

[36] I. Tomon, S. G. Z. Smith, *The poset on connected graphs is Sperner*. Journal of Combinatorial Theory, Series A 150 (2017) 162-181