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Solvability of Nonlinear Wave Equation with Nonlinear Integral Neumann Conditions

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Abstract. In this paper, we examine a nonlinear hyperbolic equation with a nonlinear integral condition. In particular, we prove the existence and the uniqueness of the linear problem by the Fadeo Galerkin method, and by applying an iterative process to some significant results obtained for the linear problem, the existence and the uniqueness of the weak solution for the nonlinear problem are additionally examined.

1. Introduction

The nonlinear hyperbolic equations describes important processes in the nonlinear evolution equation basis of mathematical models of diverse phenomena and processes in mechanics, physics, technology, biophysics, biology, ecology, and many other areas [1–4]. Such ubiquitous occurrence of nonlinear hyperbolic equations is to be explained, first of all, by the fact that they are derived from fundamental laws in the real world [5, 6].

Let us remark only that for broad classes of equations, the fundamental questions of solvability and uniqueness of solutions of various boundary value problems have been solved and that the differentiability properties of the solutions have been studied in detail [7–9]. General results of the solvability and uniqueness were inferred by different methods such as the energy method, upper lower method

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and the Fadeo Galerkin methods. The later one is regarded one of the most important methods that were mainly developed in the 1960s, but they are still powerful tools today to deal with nonlinear evolution equations, especially those who are modeled by non-classical boundary conditions that consist of integral conditions [10–13]. Non-local and integral partial differential equations are used to solve a vast range of current physics and technology challenges [14–19]. When it is hard to directly measure the minimum and maximum values on the border, the overall value or average is known. This method might be utilized for modeling where we can model more complicated domain with nonlinear integral condition.

Motivated by the above perspective, we trait in this work to discuss a nonlinear evolution equation with a nonlinear integral condition. In particular, we aim to focus on the solvability of the solution of nonlinear hyperbolic problems with the integral condition of the second type by the method of Fadeo-Galerkin. In the following sections, we present first the existence of the linear problem, and then by applying an iterative process based on the results obtained for the linear problem, we prove the existence and the uniqueness of the weak solution of the nonlinear problem.

2. The statement of the main problem

In this section, we let $Q = \left\{(x,t) \in \mathbb{R}^2,\ x \in \Omega =]0,l[\text{ and } 0 < t < T\right\}$, besides we consider the main following initial boundary value problem for a nonlinear hyperbolic equation:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - a\dfrac{\partial^2 u}{\partial x^2} = f(x,t,u,u_x) \\ u(x,0) = \varphi(x) \\ u_t(x,0) = \psi(x) \\ \dfrac{\partial u}{\partial x}(0,t) = \int_0^l k(x,t)g(u_t)(x,t)dx \\ \dfrac{\partial u}{\partial x}(l,t) = \int_0^l k(x,t)h(u_t)(x,t)dx. \end{cases} \tag{P_1}$$

Assuming that $f \in L^2(Q)$ and $\varphi, \psi \in L^2(\Omega)$. The nonlinear hyperbolic equation is given as follows:

$$\mathcal{L}u = \frac{\partial^2 u}{\partial t^2} - a\frac{\partial^2 u}{\partial x^2} = f(x,t,u,u_x), \tag{2.1}$$

which satisfies the following identities:

- The initial conditions

$$\ell u = \begin{cases} u(x,0) = \varphi(x) \\ u_t(x,0) = \psi(x) \end{cases}, \qquad x \in (0,l).$$

- The boundary conditions are integral conditions of the second type defined as:

$$\begin{aligned} \frac{\partial u}{\partial x}(0,t) &= \int_0^l k(x,t)g(u_t)(x,t)dx, \quad t \in (0,T), \\ \frac{\partial u}{\partial x}(l,t) &= \int_0^l k(x,t)h(u_t)(x,t)dx, \quad t \in (0,T), \end{aligned}$$

where

$$k(x,t) \geq 0 \ \ \forall (x,t) \in Q \text{and } g(u_t)(x,t) \leq h(u_t)(x,t) \ \forall (x,t) \in Q,$$

and for all $v \in L^2(Q)$, we have:

$$\|g(v)\|_{L^2(Q)} \leq C \|v\|_{L^\infty(0,T,L^2(\Omega))},$$

in which we define the space V by $V = H^1(\Omega)$.

Actually, the space V is provided with the norm $\|v\|_V = \|v\|_{H^1(\Omega)}$, and hence it is a Hilbert space. From this point of view, we are now able to formulate problem (P_2) in order to precisely study it. From this fact, we will need to the following hypothesis:

$$(H) : \begin{cases} f \in L^2\left(0,T;\ L^2(\Omega)\right) & (H.1) \\ \varphi \in H^1(\Omega) & (H.2) \end{cases}.$$

3. Position of problem (P_1)

In the rectangular area $Q = \Omega \times (0,T)$, and $T < \infty$, we consider the following linear problem (P_2):

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - a\dfrac{\partial^2 u}{\partial x^2} = f(x,t) & \forall (x,t) \in Q \\ u(x,0) = \varphi(x) & \forall x \in (0,l) \\ u_t(x,0) = \psi(x) & \forall x \in (0,l) \\ \dfrac{\partial u}{\partial x}(0,t) = \int_0^l k(x,t)g(u_t)(x,t)dx & \forall t \in (0,T) \\ \dfrac{\partial u}{\partial x}(l,t) = \int_0^l k(x,t)g(u_t)(x,t)dx & \forall t \in (0,T) \end{cases} \qquad (P_2)$$

in which the hyperbolic equation is given as follows:

$$\mathcal{L}u = \frac{\partial^2 u}{\partial t^2} - a\frac{\partial^2 u}{\partial x^2} = f(x,t), \qquad (3.1)$$

with the initial conditions:

$$\ell u = \begin{cases} u(x,0) = \varphi(x) \\ u_t(x,0) = \psi(x) \end{cases}, \qquad x \in (0,l),$$

and with the integral condition of the second type:

$$\begin{aligned} \frac{\partial u}{\partial x}(0,t) &= \int_0^l k(x,t)g(u_t)(x,t)dx, \quad t \in (0,T), \\ \frac{\partial u}{\partial x}(l,t) &= \int_0^l k(x,t)h(u_t)(x,t)dx, \quad t \in (0,T), \end{aligned}$$

where

$$k(x,t) \geq 0 \ \ \forall (x,t) \in Q \text{ and } 0 \leq g(u_t)(x,t) \leq h(u_t)(x,t) \ \forall (x,t) \in Q,$$

and

$$\|g(v)\|_{L^2(Q)} \leq C \|v\|_{L^\infty(0,T,L^2(Q))},$$

such that the space $V = H^1(\Omega)$ provided with the norm $\|v\|_V = \|v\|_{H^1(\Omega)}$ is a Hilbert space. We are now able to formulate the problem (P_2), precisely to study it, according to the following hypothesis:

$$(H):\begin{cases} f \in L^2\left(0,T;L^2(\Omega)\right) & (H.1) \\ \varphi \in H^1(\Omega) & (H.2) \end{cases}.$$

Definition 3.1. *The weak solution of problem (P_2) is a function that satisfies:*

- $u \in L^2\left(0,T;\ H^1(\Omega)\right) \cap L^\infty\left(0,T;\ H^1(\Omega)\right)$.
- *u admits a strong derivative* $\dfrac{\partial u}{\partial t} \in L^2\left(0,T;\ L^2(\Omega)\right)$.
- $u(0) = \varphi$, $u_t(0) = \psi$.
- *The following identity:*

$$(u_{tt},v) + a(u_x,v_x) = (f,v) + u_x(l,t)v(l) - u_x(0,t)v(0) \quad \forall v \in V,\ \forall t \in [0,T].$$

3.1. **Variational formulation.** By multiplying the equation:

$$\frac{\partial^2 u}{\partial t^2} - a\frac{\partial^2 u}{\partial x^2} = f(x,t) \tag{3.2}$$

by an element $v \in V$, and the by integrating the result over Ω, we obtain:

$$\int_\Omega \frac{\partial^2 u}{\partial t^2}\cdot v dx - a\int_\Omega \frac{\partial^2 u}{\partial x^2}\cdot v dx = \int_\Omega f\cdot v dx. \tag{3.3}$$

By using the boundary conditions and using Green's formula, (3.3) becomes

$$(u_{tt},v) + a(u_x,v_x) = (f,v_t) + u_x(l,t)v(l) - u_x(0,t)v(0), \quad \forall v \in V, \tag{3.4}$$

where $(\cdot,\cdot)$ denotes the scalar product $L^2(\Omega)$.

3.2. **Study of the existence of weak solution of problem (P_2).** The demonstration of the existence of the solution of problem (P_2) can be discussed based on the Faedo-Galerkin method which consists of carrying out the next three steps.

3.2.1. *Step 1: Construction of the approximate solutions.* As the space V is separable, then there exists a sequence $w_1, w_2, \cdots, w_m$ having the following properties:

$$\begin{cases} w_i \in V, & \forall i, \\ \forall m, w_1, w_2, ..., w_m & \text{are linearly independent,} \\ V_m = \langle\{w_1, w_2, ..., w_m\}\rangle & \text{is dense in } V. \end{cases} \tag{3.5}$$

In particular, we can say:

$$\forall\varphi \ \in \ V \implies \exists(\alpha_{km})_m \in IN^*,\ \varphi_m = \sum_{k=1}^m \alpha_{km}w_k \longrightarrow \varphi \text{ when } m \longrightarrow +\infty. \tag{3.6}$$

$$\forall\psi \ \in \ V \implies \exists(\beta_{km})_m \in IN^*,\ \psi_m = \sum_{k=1}^m \beta_{km}w_k \longrightarrow \varphi \text{ when } m \longrightarrow +\infty. \tag{3.7}$$

Now, the Faedo Galerkin's approximation aims to search about a function in which

$$t \mapsto u_m(x,t) = \sum_{i=1}^{m} g_{im}(t)\, w_i(x)$$

verifies

$$\begin{cases} u_m(t) \in V_m, \qquad \forall t \in [0,T] \\ \left((u_m(t))_{tt}, w_k\right) + A\left(u_m(t), w_k\right) = \left(f(t), w_k\right) \qquad \forall k = \overline{1,m} \end{cases}, \tag{P_3}$$

for any integer $m \geq 1$, where

$$\begin{aligned} \left((u_m(t))_{tt}, w_k\right) &= \left(\left(\sum_{i=1}^{m} g_{im}(t)\, w_i\right)_{tt}, w_k\right) \\ &= \left(\sum_{i=1}^{m} \frac{\partial^2 g_{im}}{\partial t}(t)\, w_i(x), w_k\right) = \sum_{i=1}^{m} (w_i, w_k) \frac{\partial^2 g_{im}}{\partial t^2}(t), \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} A\left(u_m(t), w_k\right) &= A\left(\sum_{i=1}^{m} g_{im}(t)\, w_i, w_k\right) \\ &= a\sum_{i=1}^{m} g_{im}(t)\left[\int_\Omega \frac{\partial w_i}{\partial x}\frac{\partial w_k}{\partial x}dx - \frac{\partial w_i}{\partial x}(l)\, w_k(l) + \frac{\partial w_i}{\partial x}(0)\, w_k(0)\right] \\ &= a\sum_{i=1}^{m} g_{im}(t)\int_\Omega \frac{\partial w_i(x)}{\partial x}\frac{\partial w_k(x)}{\partial x}dx - a\sum_{i=1}^{m} g_{im}(t)\frac{\partial w_i}{\partial x}(l)\, w_k(l) \\ &+ a\sum_{i=1}^{m} g_{im}(t)\frac{\partial w_i}{\partial x}(0)\, w_k(0) = \sum_{i=1}^{m} A(w_i, w_k)\, g_{im}(t). \end{aligned} \tag{3.9}$$

In addition, we have

$$u_m(0) = \sum_{i=1}^{m} g_{im}(0)\, w_i(x) = \varphi_m = \sum_{i=1}^{m} \alpha_{im} w_i(x).$$

and

$$u_m'(0) = \sum_{i=1}^{m} g_{im}'(0)\, w_i(x) = \beta_m = \sum_{i=1}^{m} \beta_{im} w_i(x).$$

We obtain consequently a system of first-order nonlinear differential equations:

$$\begin{cases} \sum\limits_{i=1}^{m} (w_i, w_k)\frac{\partial^2 g_{im}}{\partial t^2}(t) + a\sum\limits_{i=1}^{m}\left(\frac{\partial w_i}{\partial x}, \frac{\partial w_k}{\partial x}\right) g_{im}(t) \\ = (f(t), w_k) + a\sum\limits_{i=1}^{m} g_{im}(t)\frac{\partial w_i}{\partial x}(l)\, w_k(l) - a\sum\limits_{i=1}^{m} g_{im}(t)\frac{\partial w_i}{\partial x}(0)\, w_k(0) \\ g_{im}(0) = \alpha_{im} \qquad \forall i = \overline{1,m}. \\ g_{im}'(0) = \beta_{im} \qquad \forall i = \overline{1,m}. \end{cases}, \tag{P_4}$$

From this view, we consider the vector:

$$g_m = \left(g_{1m}(t), \cdots, g_{mm}(t)\right), f_m = \left((f, w_1), \cdots, (f, w_m)\right),$$

coupled with the matrices:

$$B_m = \left((w_i, w_j) \right)_{\substack{1 \leq i \leq m \\ 1 \leq j \leq m}}, A_m = \left(\left(\frac{\partial w_i}{\partial x}, \frac{\partial w_j}{\partial x} \right) \right)_{\substack{1 \leq i \leq m \\ 1 \leq j \leq m}},$$

and

$$C_m = \left(\frac{\partial w_i}{\partial x} (l) \cdot w_j(l) \right)_{\substack{1 \leq i \leq m \\ 1 \leq j \leq m}}, D_m = \left(\frac{\partial w_i}{\partial x} (0) \cdot w_j(0) \right)_{\substack{1 \leq i \leq m \\ 1 \leq j \leq m}}.$$

Now, we can immediately write problem (P_4) in the matrix form as:

$$\begin{cases} B_m \frac{\partial g_m}{\partial t} (t) + aA_m g_m + aD_m g_m = f_m + aC_m g_m \\ g_m (0) = (\alpha_{im})_{1 \leq i \leq m} \\ g'_m (0) = (\beta_{im})_{1 \leq i \leq m} \end{cases},$$

As the matrix entries B_m are linearly independent (because it is a diagonal matrix), then $\det B_m \neq 0$. So, it is invertible, and then g_m is the solution of the following states:

$$\begin{cases} \dfrac{\partial^2 g_m}{\partial t^2} (t) + \left(aB_m^{-1} A_m + bB_m^{-1} D_m - aB_m^{-1} C_m \right) g_m = B_m^{-1} f_m \\ g_m (0) = (\alpha_{im})_{1 \leq i \leq m}. \\ g'_m (0) = (\beta_{im})_{1 \leq i \leq m}. \end{cases} \qquad (P_5)$$

Now, it is easy to verify that this ordinary differential system has a solution where the matrix:

$$\left(aB_m^{-1} A_m + bB_m^{-1} B_m - aB_m^{-1} C_m \right)$$

is of constant coefficients and the vector $B_m^{-1} f_m$ are continuous functions and majorized by integrable functions on $(0, T)$. Consequently, we can conclude that there exists a t_m that depends only on $|\alpha_{im}|$ and $|\beta_{im}|$.

3.2.2. *Step 2: A priori estimate.* Herein, we intend to begin this step with state and prove the next result.

Lemma 3.1. *For all $m \in \mathbb{N}^*$, if*

$$\frac{1}{8a} > k^2,$$

the solution $u_m \in L^2 (0, T \ ; V_m)$ of problem (P_2) satisfies:

$$\begin{aligned} \|u_m\|_{L^2(0,T;\ H^1(\Omega))} &\leq c_1, \\ \left\| \frac{\partial u_m}{\partial t} \right\|_{L^2(0,T;\ L^2(\Omega))} &\leq c_2, \end{aligned}$$

where c_1 and c_2 are two positive constants independent of m.

Proof. By multiplying the equation of (P_3) by $g_{km}(t)$, and then by summing the result over k, we obtain:

$$\begin{aligned}
&\sum_{k=1}^{m}\left(\left(u_m(t)\right)_{tt}, w_k\right) g'_{km}(t) + a\sum_{k=1}^{m}\left(\frac{\partial u_m}{\partial x}(t), \frac{\partial w_k}{\partial x \partial t}\right) g'_{km}(t) \\
&\quad = \sum_{k=1}^{m}\left(f(t), w_k\right)\cdot g'_{km}(t) + a\sum_{i=1}^{m} g_{im}(t)\frac{\partial w_i}{\partial x}(l)\sum_{k=1}^{m} g'_{km}(t) w_k(l) \\
&\quad - a\sum_{i=1}^{m} g_{im}(t)\frac{\partial w_i}{\partial x}(0)\sum_{k=1}^{m} g'_{km}(t) w_k(0).
\end{aligned}$$

So, we obtain

$$\begin{aligned}
&\left(\left(u_m(t)\right)_{tt}, \left(u_m(t)\right)_t\right) + a\left(\frac{\partial u_m}{\partial x}(t), \frac{\partial u_m}{\partial x \partial t}(t)\right) \\
&\qquad = \left(f(t), \left(u_m(t)\right)_t\right) + a\sum_{i=1}^{m} g_{im}(t)\frac{\partial w_i}{\partial x}(l)\sum_{k=1}^{m} g'_{km}(t) w_k(l) \\
&\qquad - a\sum_{i=1}^{m} g_{im}(t)\frac{\partial w_i}{\partial x}(0)\sum_{k=1}^{m} g'_{km}(t) w_k(0).
\end{aligned}$$

Thus, we get

$$\begin{aligned}
&\left(\left(u_m(t)\right)_{tt}, \left(u_m(t)\right)_t\right) + a\left\|\frac{\partial u_m}{\partial x}\right\|^2_{L^2(\Omega)} = \left(f(t), u_m(t)\right) \\
&+ a\sum_{i=1}^{m} g_{im}(t)\frac{\partial w_i}{\partial x}(l)\sum_{k=1}^{m} g'_{km}(t) w_k(l) - a\sum_{i=1}^{m} g_{im}(t)\frac{\partial w_i}{\partial x}(0)\sum_{k=1}^{m} g'_{km}(t) w_k(0).
\end{aligned}$$

Integrating the above equality over 0 to t coupled wit using the Cauchy inequality with ε, i.e.

$$|ab| \le \frac{a^2}{2\varepsilon} + \frac{\varepsilon b^2}{2},$$

we get

$$\begin{aligned}
&\frac{1}{2}\left\|\frac{\partial u_m}{\partial t}\right\|^2_{L^2(\Omega)} + \frac{a}{2}\left\|\nabla u_m\right\|^2_{L^2(\Omega)} \\
\le& \frac{1}{2\varepsilon}\left\|f\right\|^2_{L^2(Q)} + \frac{\varepsilon}{2}\left\|\frac{\partial u_m}{\partial t}\right\|^2_{L^2(Q)} + a\sum_{i=1}^{m} g_{im}(t)\frac{\partial w_i}{\partial x}(l)\sum_{k=1}^{m} g'_{km}(t) w_k(l) \\
&- a\sum_{i=1}^{m} g_{im}(t)\frac{\partial w_i}{\partial x}(0)\sum_{k=1}^{m} g'_{km}(t) w_k(0) + \frac{1}{2}\left\|\varphi_m\right\|^2_{L^2(\Omega)} + \frac{a}{2}\left\|\nabla \psi_m\right\|^2_{L^2(\Omega)}, \\
\le& \frac{1}{2\varepsilon}\left\|f\right\|^2_{L^2(Q)} + \frac{\varepsilon}{2}\left\|\frac{\partial u_m}{\partial t}\right\|^2_{L^2(Q)} + a\frac{\partial u_m}{\partial x}(l,t)\frac{\partial u_m}{\partial t}(l,t) - a\frac{\partial u_m}{\partial x}(0,t)\frac{\partial u_m}{\partial t}(0,t), \\
\le& \frac{1}{2\varepsilon}\left\|f\right\|^2_{L^2(Q)} + \frac{\varepsilon}{2}\left\|\frac{\partial u_m}{\partial t}\right\|^2_{L^2(Q)} + \frac{1}{2}\left\|\varphi_m\right\|^2_{L^2(\Omega)} + \frac{a}{2}\left\|\nabla \psi_m\right\|^2_{L^2(\Omega)} \\
&+ a\int_0^{\tau}\left[\left(\int_{\Omega} k(x,t) g\left(\left(u_t\right)_m\right)(x,t)\,dx\right)\frac{\partial u_m}{\partial t}(l,t)\right.
\end{aligned}$$

$$
\begin{aligned}
&\left.-\left(\int_\Omega k(x,t)h\left((u_t)_m\right)(x,t)dx\right)\frac{\partial u_m}{\partial t}(0,t)\right]\\
\leq&\frac{1}{2\varepsilon}\|f\|^2_{L^2(Q)}+\frac{\varepsilon}{2}\left\|\frac{\partial u_m}{\partial t}\right\|^2_{L^2(Q)}+\frac{1}{2}\|\varphi_m\|^2_{L^2(\Omega)}+\frac{a}{2}\|\nabla\psi_m\|^2_{L^2(\Omega)}\\
&+a\int_0^\tau\left[\left(\int_\Omega k(x,t)g\left((u_t)_m\right)(x,t)dx\right)\frac{\partial u_m}{\partial t}(l,t)\right.\\
&\left.-\left(\int_\Omega k(x,t)h\left((u_t)_m\right)(x,t)dx\right)\frac{\partial u_m}{\partial t}(0,t)\right].
\end{aligned}
$$

This means

$$
\begin{aligned}
&\frac{1}{2}\left\|\frac{\partial u_m}{\partial t}\right\|^2_{L^2(\Omega)}+\frac{a}{2}\|\nabla u_m\|^2_{L^2(\Omega)}\\
\leq&\frac{1}{2\varepsilon}\|f\|^2_{L^2(Q)}+\frac{\varepsilon}{2}\left\|\frac{\partial u_m}{\partial t}\right\|^2_{L^2(Q)}+\frac{1}{2}\|\varphi_m\|^2_{L^2(\Omega)}+\frac{a}{2}\|\nabla\psi_m\|^2_{L^2(\Omega)}\\
&+a\int_0^\tau\left[\left(\int_\Omega k(x,t)g\left((u_t)_m\right)(x,t)dx\right)\frac{\partial u_m}{\partial t}(l,t)\right.\\
&\left.-\left(\int_\Omega k(x,t)g\left((u_t)_m\right)(x,t)dx\right)\frac{\partial u_m}{\partial t}(0,t)\right]\\
\leq&\frac{1}{2\varepsilon}\|f\|^2_{L^2(Q)}+\frac{\varepsilon}{2}\left\|\frac{\partial u_m}{\partial t}\right\|^2_{L^2(Q)}+\frac{1}{2}\|\varphi_m\|^2_{L^2(\Omega)}+\frac{a}{2}\|\nabla\psi_m\|^2_{L^2(\Omega)}\\
&+aCk\left\|\frac{\partial u_m}{\partial t}\right\|_{L^\infty(0,T;L^2(\Omega))}\left[\int_0^\tau\int_\Omega\frac{\partial^2 u_m}{\partial t\partial x}dxdt\right]\\
\leq&\frac{1}{2\varepsilon}\|f\|^2_{L^2(Q)}+\frac{\varepsilon}{2}\left\|\frac{\partial u_m}{\partial t}\right\|^2_{L^2(Q)}+\frac{1}{2}\|\varphi_m\|^2_{L^2(\Omega)}+\frac{a}{2}\|\nabla\psi_m\|^2_{L^2(\Omega)}\\
&+akC\left\|\frac{\partial u_m}{\partial t}\right\|_{L^\infty(0,T;L^2(\Omega))}\left[\int_\Omega\frac{\partial u_m}{\partial x}dx-\int_\Omega\frac{\partial\psi_m}{\partial x}dx\right],
\end{aligned}
$$

which yields

$$
\begin{aligned}
&\frac{1}{2}\left\|\frac{\partial u_m}{\partial t}\right\|^2_{L^2(\Omega)}+\frac{a}{2}\|\nabla u_m\|^2_{L^2(\Omega)}\\
\leq&\frac{1}{2\varepsilon}\|f\|^2_{L^2(Q)}+\frac{\varepsilon}{2}\left\|\frac{\partial u_m}{\partial t}\right\|^2_{L^2(Q)}+\Big[\frac{(aCk)^2}{2\delta}\left\|\frac{\partial u_m}{\partial t}\right\|^2_{L^\infty(0,T;L^2(\Omega))}\\
&+\delta\left[\|\nabla u_m\|^2_{L^\infty(0,T;L^2(\Omega))}+\|\nabla\varphi_m\|^2_{L^\infty(0,T;L^2(\Omega))}\right]\Big]\\
&+\frac{1}{2}\|\varphi_m\|^2_{L^2(\Omega)}+\frac{a}{2}\|\nabla\psi_m\|^2_{L^2(\Omega)}\\
\leq&\frac{1}{2\varepsilon}\|f\|^2_{L^2(Q)}+\frac{\varepsilon}{2}\left\|\frac{\partial u_m}{\partial t}\right\|^2_{L^2(Q)}\frac{(aCk)^2}{2\delta}\left\|\frac{\partial u_m}{\partial t}\right\|^2_{L^\infty(0,T;L^2(\Omega))}\\
&+\delta\|\nabla u_m\|^2_{L^\infty(0,T;L^2(\Omega))}+\frac{a}{2}\|\nabla\psi_m\|^2_{L^2(\Omega)}+\max(\frac{1}{2},\delta)\|\varphi_m\|^2_{H^1(\Omega)},
\end{aligned}
$$

or

$$\begin{aligned}
&\frac{1}{2}\left\|\frac{\partial u_m}{\partial t}\right\|^2_{L^2(\Omega)} + \frac{a}{2}\left\|\nabla u_m\right\|^2_{L^2(\Omega)} \\
&\quad \le \frac{1}{2\varepsilon}\left\|f\right\|^2_{L^2(Q)} + \left(\frac{\varepsilon C_T}{2} + \frac{(aCk)^2}{2\delta}\right)\left\|\frac{\partial u_m}{\partial t}\right\|^2_{L^\infty(0,T;L^2(\Omega))} \\
&\quad + \delta\left\|\nabla u_m\right\|^2_{L^\infty(0,T;L^2(\Omega))} + \frac{a}{2}\left\|\nabla\psi_m\right\|^2_{L^2(\Omega)} + \max(\frac{1}{2}, aCk\delta)\left\|\varphi_m\right\|^2_{H^1(\Omega)},
\end{aligned} \tag{3.10}$$

where $K = \max\int_Q k^2(x,t)\,dxdt$. Consequently, we obtain

$$\begin{aligned}
&\frac{1}{2}\left\|\frac{\partial u_m}{\partial t}\right\|^2_{L^2(\Omega)} + \frac{a}{2}\left\|\nabla u_m\right\|^2_{L^2(\Omega)} \\
&\qquad\qquad \le \frac{1}{2\varepsilon}\left\|f\right\|^2_{L^2(Q)} + \left(\frac{\varepsilon C_T}{2} + \frac{(aCk)^2}{2\delta}\right)\left\|\frac{\partial u_m}{\partial t}\right\|^2_{L^\infty(0,T;L^2(\Omega))} \\
&\qquad\qquad + \delta\left\|\nabla u_m\right\|^2_{L^\infty(0,T;L^2(\Omega))} + \frac{a}{2}\left\|\nabla\psi_m\right\|^2_{L^2(\Omega)} \\
&\qquad\qquad + \max(\frac{1}{2}, \delta)\left\|\varphi_m\right\|^2_{H^1(\Omega)},
\end{aligned}$$

which gives:

$$\begin{aligned}
&\left(\frac{1}{2} - \left(\frac{\varepsilon C_T}{2} + \frac{(aCk)^2}{2\delta}\right)\right)\left\|\frac{\partial u_m}{\partial t}\right\|^2_{L^\infty(0,T;L^2(\Omega))} + \left(\frac{a}{2} - \delta\right)\left\|\nabla u_m\right\|^2_{L^\infty(0,T;L^2(\Omega))} \\
&\qquad\qquad \le \frac{1}{2\varepsilon}\left\|f\right\|^2_{L^2(Q)} + \frac{a}{2}\left\|\nabla\psi_m\right\|^2_{L^2(\Omega)} + \max(\frac{1}{2}, \delta)\left\|\varphi_m\right\|^2_{H^1(\Omega)}
\end{aligned}$$

By putting $\varepsilon = \frac{1}{4C_T}$ and $\delta = 4(aCk)^2$, we get

$$\begin{aligned}
&\left\|\frac{\partial u_m}{\partial t}\right\|^2_{L^\infty(0,T;L^2(\Omega))} + \left\|\nabla u_m\right\|^2_{L^\infty(0,T;L^2(\Omega))} \\
&\qquad\qquad \le C_1\left[\left\|f\right\|^2_{L^2(Q)} + \left\|\nabla\psi_m\right\|^2_{L^2(\Omega)} + \left\|\varphi_m\right\|^2_{H^1(\Omega)}\right],
\end{aligned} \tag{3.11}$$

or

$$C_1 = \frac{\max\left(2C_T, \frac{a}{2}, \max(\frac{1}{2}, (aCk)^2)\right)}{\min\left\{\frac{1}{4}, \left(\frac{a}{2} - 4(aCk)^2\right)\right\}}.$$

From (3.11), we can also get:

$$\left\|\frac{\partial u_m}{\partial t}\right\|_{L^2(\Omega)} \le \sqrt{C_1}\left[\left\|f\right\|^2_{L^2(Q)} + \left\|\nabla\psi_m\right\|^2_{L^2(\Omega)} + \left\|\varphi_m\right\|^2_{H^1(\Omega)}\right]^{\frac{1}{2}}. \tag{3.12}$$

By integrating (3.12) over $[0,T]$, we obtain:

$$\begin{aligned}
\left\|\int_0^\tau \frac{\partial u_m}{\partial t}\right\|_{L^2(\Omega)} &\le \int_0^\tau \left\|\frac{\partial u_m}{\partial t}\right\|_{L^2(\Omega)} \\
&\le T\sqrt{C_1}\left[\left\|f\right\|^2_{L^2(Q)} + \left\|\nabla\psi_m\right\|^2_{L^2(\Omega)} + \left\|\varphi_m\right\|^2_{H^1(\Omega)}\right]^{\frac{1}{2}},
\end{aligned}$$

or

$$\left\| \int_0^{\tau} \frac{\partial u_m}{\partial t} \right\|_{L^2(\Omega)} \leq T\sqrt{C_1}\left[\|f\|^2_{L^2(Q)} + \|\nabla \psi_m\|^2_{L^2(\Omega)} + \|\varphi_m\|^2_{H^1(\Omega)} \right]^{\frac{1}{2}},$$

which implies:

$$\|u_m - \varphi_m\|_{L^2(\Omega)} \leq T\sqrt{C_1}\left[\|f\|^2_{L^2(Q)} + \|\nabla \psi_m\|^2_{L^2(\Omega)} + \|\varphi_m\|^2_{H^1(\Omega)} \right]^{\frac{1}{2}},$$

i.e.,

$$\begin{aligned} &\|u_m\|^2_{L^2(\Omega)} + \|\varphi_m\|^2_{L^2(\Omega)} \\ &\leq TC_1\left[\|f\|^2_{L^2(Q)} + \|\nabla \psi_m\|^2_{L^2(\Omega)} + \|\varphi_m\|^2_{H^1(\Omega)} \right] + 2\|u_m\|_{L^2(\Omega)}\|\varphi_m\|_{L^2(\Omega)}. \end{aligned}$$

By applying Cauchy inequality with γ, we get:

$$\begin{aligned} &\|u_m\|^2_{L^2(\Omega)} + \|\varphi_m\|^2_{L^2(\Omega)} \\ &\leq TC_1\left[\|f\|^2_{L^2(Q)} + \|\nabla \psi_m\|^2_{L^2(\Omega)} + \|\varphi_m\|^2_{H^1(\Omega)} \right] + \frac{1}{\gamma}\|u_m\|^2_{L^2(\Omega)} + \gamma\|\varphi_m\|^2_{L^2(\Omega)}. \end{aligned}$$

By putting $\gamma = 2$, we can have:

$$\|u_m\|^2_{L^2(\Omega)} \leq 4TC_1\left[\|f\|^2_{L^2(Q)} + \|\nabla \psi_m\|^2_{L^2(\Omega)} + \|\varphi_m\|^2_{H^1(\Omega)} \right]. \tag{3.13}$$

Now, it follows from (3.11) and (3.13) that the solution of the initial value problem for system (P_4) can be extended to $[0, T]$. This confirms what we have demonstrated in the first step. Consequently, when $m \to +\infty$ in (3.13), we obtain:

$$\begin{cases} u_m \text{ uniformly bounded in } L^2(0,T;\ H^1(\Omega)) \\ (u_m)_t \text{ uniformly bounded in } L^2\left(0,T;\ L^2(\Omega)\right) \end{cases}. \tag{3.14}$$

□

3.2.3. *Step 3: Convergence and the result of existence.*

Theorem 3.1. *There is a function $u \in L^2(0,T;\ H^1(\Omega)) \cap L^\infty(0,T;\ L^2(\Omega))$ with $\frac{\partial u}{\partial t} \in L^2\left(0,T;L^2(\Omega)\right)$ and a subsequence denoted by $\left(u_{m_k}\right)_k \subseteq (u_m)_m$ such that*

$$\begin{cases} u_{m_k} \rightharpoonup u & \text{in } L^2(0,T;H^1(\Omega)) \\ \dfrac{\partial u_{m_k}}{\partial t} \rightharpoonup \dfrac{\partial u}{\partial t} & \text{in } L^2\left(0,T;L^2(\Omega)\right) \end{cases},$$

as $m \longrightarrow +\infty$.

Proof. From Lemma 1.2, we might deduce that there are subsequences denoted respectively by $\left(u_{m_k}\right)$, $\left(\frac{\partial u_{m_k}}{\partial t}\right)$ of (u_m) and $(u_m)_t$ such that

$$u_{m_k} \rightharpoonup u \quad \text{in } L^2(0,T;\ H^1(\Omega)), \tag{3.15}$$

and

$$\frac{\partial u_{m_k}}{\partial t} \rightharpoonup w \quad \text{in } L^2\left(0,T;\ L^2(\Omega)\right). \tag{3.16}$$

We know that according to Relikh-Kondrachoff's theorem the injection of $H^1(Q)$ into $L^2(Q)$ is compact. In addition, like the results of Rellich's theorem, any weakly convergent sequence in $H^1(Q)$ has a subsequence which converges strongly in $L^2(Q)$. So, we can assert:

$$u_{m_k} \longrightarrow u \qquad \text{in } L^2(Q) \ . \tag{3.17}$$

On the other hand, from Lemma 1.3, there is a subsequence of $\left(u_{m_k}\right)_k$, which is still denoted by u_{m_k}, converges almost everywhere to u such that

$$u_{m_k} \longrightarrow u \qquad \text{almost everywhere } Q \ . \tag{3.18}$$

It is still essential to demonstrate that $w = \frac{\partial u}{\partial t}$. This actually suffices to prove:

$$u(t) = \varphi + \int_0^t w(\tau)d\tau. \tag{3.19}$$

To this aim, we note that as

$$u_{m_k} \rightharpoonup u \qquad \text{in} L^2(0,T;L^2(\Omega)) \ ,$$

then the proof of (3.19) is equivalent to prove that

$$u_{m_k} \rightharpoonup \varphi + \chi \qquad \text{in } L^2(0,T\ ;L^2(\Omega)) \ ,$$

which means

$$\lim \left(u_{m_k} - \varphi - \chi, v\right)_{L^2(0,T;L^2(\Omega))} = 0, \ \forall v \in L^2(0,T;L^2(\Omega)),$$

as

$$\chi(t) = \int_0^t w(\tau)d\tau.$$

In fact, by using the equality

$$u_{m_k} - \varphi_{m_k} = \int_0^t \frac{\partial u_{m_k}}{\partial \tau} d\tau, \ \text{for all } t \in [0,T],$$

with the help of using $u_{m_k} \in L^2\left(0,T;V_{m_k}\right)$ and $\left(u_{m_k}\right)_t \in L^2\left(0,T;V_{m_k}\right)$, we can get:

$$\begin{aligned}
&\left(u_{m_k} - \varphi - \int_0^t w(\tau)d\tau, v\right)_{L^2(0,T;L^2(\Omega))} \\
&\quad = \left(u_{m_k} - \varphi_{m_k} - \int_0^t w(\tau)d\tau, v\right)_{L^2(0,T;L^2(\Omega))} + \left(\varphi_{m_k} - \varphi, v\right)_{L^2(0,T;L^2(\Omega))} \\
&\quad = \left(\int_0^t \left(\frac{\partial u_{m_k}}{\partial \tau} - w(\tau)\right) d\tau, v\right)_{L^2(0,T;L^2(\Omega))} + \left(\varphi_{m_k} - \varphi, v\right)_{L^2(0,T;L^2(\Omega))},
\end{aligned}$$

for all $t \in [0,T]$. By virtue of part (ii) of Lemma 1.6, it comes

$$\left(u_{m_k} - \varphi - \int_0^t w(\tau)d\tau, v\right)_{L^2(0,T;L^2(\Omega))}$$
$$= \int_0^t \left(\frac{\partial u_{m_k}}{\partial \tau} - w(\tau), v\right)_{L^2(0,T;L^2(\Omega))} d\tau + \left(\varphi_{m_k} - \varphi, v\right)_{L^2(0,T;L^2(\Omega))},$$

for all $t \in [0,T]$. On the one hand, we have

$$\lim_{k \longrightarrow \infty} \int_0^t \left(\frac{\partial u_{m_k}}{\partial \tau} - w(\tau), v\right)_{L^2(0,T;L^2(\Omega))} d\tau = 0, \tag{3.20}$$

for $t \in [0,T]$. Besides, we have:

$$\lim_{k \longrightarrow \infty} (\varphi_m - \varphi, v)_{L^2(0,T;L^2(\Omega))} = 0, \tag{3.21}$$

which implies:

$$\lim_{k \longrightarrow \infty} \left(u_{m_k} - \varphi - \chi, v\right)_{L^2(0,T\ ;\ L^2(\Omega))} = 0, \quad \forall v \in L^2\left(0,T; L^2\left(\Omega\right)\right).$$

□

Theorem 3.2. *The function u of the Theorem (3.1) is the weak solution to the problem (P_2) in the sense of the definition 3.1.*

Proof. From Theorem (3.1), we have shown that the limit function u satisfies the first two conditions of the definition 3.1. Now we will demonstrate (iii). According to the Theorem 3.1, we have:

$$u_{m_k}(0) \rightharpoonup u(0) \qquad \text{in } L^2(\Omega)\ .$$

On the other hand, we have

$$u_{m_k}(0) \longrightarrow \varphi \qquad \text{in } L^2(\Omega)\ ,$$

which implies:

$$u_{m_k}(0) \rightharpoonup \varphi \qquad \text{in } L^2(\Omega)\ .$$

From the uniqueness of the limit, we get $u(0) = \varphi$. By using the same previous steps, we demonstrate $u_t(0) = \psi$. It remains to demonstrate (iv). To this aim, we have:

$$(u_{tt}, v) + a(u, v) = (f, v) \qquad \forall v \in V, \text{ and } \forall t \in [0,T].$$

Integrating (P_3) over $(0,T)$, we obtain:

$$\int_0^t \left((u_m(t))_{tt}, w_k\right) d\tau + \int_0^t a\left(u_m(t), w_k\right) d\tau = \int_0^t \left(f(t), w_k\right) d\tau, \tag{3.22}$$

for all $k = \overline{1, m}$ and for all $t \in [0, T]$. Using (3.13) and that V_m dense in V and passing to the limit in (3.22), we get:

$$\int_0^T (u_{tt}, w_k)\, d\tau + \int_0^T a(u, w_k)\, d\tau = \int_0^T (f, w_k)\, d\tau, \quad \forall t \in [0, T].$$

This immediately implies that (iv) is verified. □

Corollary 3.1. *The uniqueness of the solution of problem (P_2) comes straight through the estimate (3.11).*

4. Weak solution of the nonlinear problem

Initially, we present the considered solution's concept. For this purpose, we let $v = v(x, t)$ be any function of V such that

$$V = \left\{ v \in C^1(Q),\ v_x(l, t) = v_x(0, t) = 0,\ t \in [0, T] \right\}.$$

By multiplying

$$\frac{\partial^2 u}{\partial t^2} - a\frac{\partial^2 u}{\partial x^2} = f(x, t, u, u_x)$$

by v and integrating the result over Q_τ, we obtain

$$\begin{aligned}\int_{Q_\tau} \frac{\partial^2 y}{\partial t^2}(x, t) \cdot \frac{\partial v}{\partial t}(x, t)dxdt - a\int_{Q_\tau} \Delta y(x, t) \cdot \frac{\partial v}{\partial t}(x, t)dxdt \\ = \int_{Q_\tau} G(x, t, y, y_x) \cdot \frac{\partial v}{\partial t}(x, t)dxdt.\end{aligned}$$

Now, by using integration by parts and the conditions on y and v, we get

$$\begin{aligned}\int_{Q_\tau} \frac{\partial^2 y}{\partial t^2}(x, t) \cdot \frac{\partial v}{\partial t}(x, t)dxdt + a\int_{Q_\tau} \frac{\partial y}{\partial x}(x, t) \cdot \frac{\partial^2 v}{\partial x \partial t}(x, t)dxdt \\ = \int_{Q_\tau} G(x, t, y, y_x) \cdot \frac{\partial v}{\partial t}(x, t)dxdt.\end{aligned} \tag{4.1}$$

It then results from (4.1) that

$$A(y, v) = \int_{Q_\tau} G(x, t, y, y_x) \cdot \frac{\partial v}{\partial t}(x, t)dxdt, \tag{4.2}$$

or

$$A(y, v) = \int_{Q_\tau} \frac{\partial^2 y}{\partial t^2}(x, t) \cdot \frac{\partial v}{\partial t}(x, t)dxdt + a\int_{Q_\tau} \frac{\partial y}{\partial x}(x, t) \cdot \frac{\partial^2 v}{\partial x \partial t}(x, t)dxdt.$$

Thus, it is the time to build a recurring sequence starting with $y^{(0)} = 0$. The sequence $\left(y^{(n)}\right)_{n\in\mathbb{N}}$ is defined as follows: Given the element $y^{(n-1)}$, then for $n = 1, 2, 3, \cdots$, we can solve the following problem:

$$\begin{cases} \frac{\partial^2 y^{(n)}}{\partial t^2} - a\Delta y^{(n)} = G\left(x, t, y^{(n-1)}, y_x^{(n-1)}\right) \\ y^{(n)}(x, 0) = 0 \\ y_t^{(n)}(x, 0) = 0 \\ y_x^{(n)}(0, t) = 0 \\ y_x^{(n)}(l, t) = 0 \end{cases} . \tag{P_4}$$

According to the last linear problem, we fix the n each time. Problem (P_4) admits then a unique solution $y^{(n)}(x, t)$, which can be given by the Fadeo-Galarkin method. In this regard, we assume

$$z^{(n)}(x, t) = y^{(n+1)}(x, t) - y^{(n)}(x, t).$$

As a result, we have the following new problem:

$$\begin{cases} \frac{\partial^2 z^{(n)}}{\partial t^2} - a\Delta z^{(n)} = p^{(n-1)}(x, t) \\ z^{(n)}(x.0) = 0 \\ z_t^{(n)}(x.0) = 0 \\ z_x^{(n)}(0, t) = 0 \\ z_x^{(n)}(l, t)\, dx = 0 \end{cases} , \tag{P_5}$$

or

$$p^{(n-1)}(x, t) = G\left(x, t, y^{(n)}, y_x^{(n)}\right) - G\left(x, t, y^{(n-1)}, y_x^{(n-1)}\right).$$

Multiplying

$$\frac{\partial^2 z^{(n)}}{\partial t^2} - a\Delta z^{(n)} = p^{(n-1)}(x, t)$$

by $z^{(n)}$ and then integrating the result over Q_τ yield:

$$\begin{aligned} \int_{Q_\tau} \frac{\partial^2 z^{(n)}}{\partial t^2}(x, t) \cdot \frac{\partial z^{(n)}}{\partial t}(x, t) dxdt - a \int_{Q_\tau} \Delta z^{(n)}(x, t) \cdot \frac{\partial z^{(n)}}{\partial t}(x, t)\, dxdt \\ = \int_{Q_\tau} p^{(n-1)}(x, t) \cdot \frac{\partial z^{(n)}}{\partial t}(x, t)\, dxdt. \end{aligned}$$

If we apply an integration by parts for each term of the above equality, keeping in view the initial and boundary conditions, we get:

$$\frac{1}{2}\int_{\Omega} (\frac{\partial z^{(n)}}{\partial t}(x, \tau))^2 dx + \frac{a}{2}\int_{\Omega} \left(\frac{\partial z^{(n)}}{\partial x}(x, t)\right)^2 dxdt = \int_{Q_\tau} p^{(n-1)}(x, t) \cdot \frac{\partial z^{(n)}}{\partial t}(x, t)\, dxdt.$$

When the Cauchy Schwarz inequality is applied to the second portion of the above equation, the following result is obtained:

$$\frac{1}{2}\int\limits_{\Omega}(\frac{\partial z^{(n)}}{\partial t}(x,\tau))^2dx+\frac{a}{2}\int\limits_{\Omega}\left(\frac{\partial z^{(n)}}{\partial x}(x,t)\right)^2dxdt$$
$$\leqslant \frac{1}{2\varepsilon}\int_{Q^\tau}\mid p^{(n-1)}(x,t)\mid^2 dxdt+\frac{\varepsilon}{2}\int_{Q^\tau}\left(\frac{\partial z^{(n)}}{\partial t}(x,t)\right)^2dxdt,$$

or

$$\frac{1}{2}\int\limits_{\Omega}(\frac{\partial z^{(n)}}{\partial t}(x,\tau))^2dx+\frac{a}{2}\int\limits_{\Omega}\left(\frac{\partial z^{(n)}}{\partial x}(x,t)\right)^2dx\leqslant\frac{1}{2\varepsilon}\int_{Q^\tau}\mid G\left(x,t,y^{(n)},y_x^{(n)}\right)$$
$$-G\left(x,t,y^{(n-1)},y_x^{(n-1)}\right)\mid^2 dxdt+\frac{\varepsilon}{2}\int_{Q^\tau}\left(\frac{\partial z^{(n)}}{\partial t}(x,t)\right)^2dxdt.$$

We deduce consequently that:

$$\frac{1}{2}\int\limits_{\Omega}(\frac{\partial z^{(n)}}{\partial t}(x,\tau))^2dx+\frac{a}{2}\int\limits_{\Omega}\left(\frac{\partial z^{(n)}}{\partial x}(x,t)\right)^2dx$$
$$\leqslant\frac{k^2}{2\varepsilon}\int_{Q^\tau}(\mid y^{(n)}-y^{(n-1)}\mid+\mid y_x^{(n)}-y_x^{(n-1)}\mid)^2dxdt+\frac{\varepsilon}{2}\int_{Q^\tau}\left(\frac{\partial z^{(n)}}{\partial t}(x,t)\right)^2dxdt$$
$$\leqslant\frac{k^2}{2\varepsilon}\int_{Q^\tau}(\mid z^{(n-1)}\mid+\mid z_x^{(n-1)}\mid)^2dxdt+\frac{\varepsilon}{2}\int_{Q^\tau}\left(\frac{\partial z^{(n)}}{\partial t}(x,t)\right)^2dxdt$$
$$\leqslant\frac{k^2}{\varepsilon}\int_{Q^\tau}(\mid z^{(n-1)}\mid^2+\mid z_x^{(n-1)}\mid^2)dxdt+\frac{\varepsilon}{2}\int_{Q^\tau}\left(\frac{\partial z^{(n)}}{\partial t}(x,t)\right)^2dxdt$$
$$\leqslant\frac{k^2}{\varepsilon}\left\|z^{(n-1)}\right\|_{L^2(0,T,H^1(0,l))}+\frac{\varepsilon}{2}\int_{Q^\tau}\left(\frac{\partial z^{(n)}}{\partial t}(x,t)\right)^2dxdt.$$

By multiplying by 2 and applying Grenwell's Lemma, we get

$$\int\limits_{\Omega}(\frac{\partial z^{(n)}}{\partial t}(x,\tau))^2dx+a\int\limits_{\Omega}\left(\frac{\partial z^{(n)}}{\partial x}(x,t)\right)^2dx$$
$$\leqslant\frac{k^2}{\varepsilon}\left\|z^{(n-1)}\right\|_{L^2(0,T,H^1(0,l))}+\varepsilon\int_{Q^\tau}\left(\frac{\partial z^{(n)}}{\partial t}(x,t)\right)^2dxdt$$
$$\leq\frac{k^2}{\varepsilon}\exp(\varepsilon T)\left\|z^{(n-1)}\right\|_{L^2(0,T,H^1(0,l))}.$$

Integrating over t yields:

$$\int\limits_{Q_T} (\frac{\partial z^{(n)}}{\partial t}(x,\tau))^2 dxdt + a\int\limits_{Q_T} \left(\frac{\partial z^{(n)}}{\partial x}(x,t)\right)^2 dxdt$$
$$\leqslant \frac{Tk^2}{\varepsilon}\left\| z^{(n-1)} \right\|_{L^2(0,T,H^1(0,l))} \exp(\varepsilon T),$$

or

$$\int\limits_{Q_T} (\frac{\partial z^{(n)}}{\partial t}(x,\tau))^2 dxdt + \int\limits_{Q_T} \left(\frac{\partial z^{(n)}}{\partial x}(x,t)\right)^2 dxdt$$
$$\leqslant \frac{Tk^2 \exp(\varepsilon T)}{\varepsilon \min(1,a)}\left\| z^{(n-1)} \right\|_{L^2(0,T,H^1(0,1))}.$$

By putting

$$c = \frac{Tk^2 \exp(\varepsilon T)}{\varepsilon \min(1,a)},$$

we get:

$$\| \frac{\partial z^{(n)}}{\partial t} \|^2_{L^2(0,T,H^1(0,1))} + \| \frac{\partial z^{(n)}}{\partial x} \|^2_{L^2(0,T,H^1(\Omega))} \leqslant c \left\| z^{(n-1)} \right\|^2_{L^2(0,T,H^1(\Omega))}.$$

Thus, by applying Pointcarre, we have

$$\| z^{(n)} \|^2_{L^2(0,T,H^1(\Omega))} \leqslant Tc \left\| z^{(n-1)} \right\|^2_{L^2(0,T,H^1(\Omega))},$$

and

$$\sum_{i=1}^{n-1} z^{(i)} = y^{(n)}.$$

According to the convergence criterion of the series $\sum\limits_{n=1}^{\infty} z^{(n)}$ that converges if $|c| < 1$, we obtain:

$$\left| \frac{(Tk)^2 \exp(\varepsilon T)}{\varepsilon \min(1,a)} \right| < 1 \ \Rightarrow\ Tk\sqrt{\frac{\exp(\varepsilon T)}{\varepsilon \min(1,a)}} < 1.$$

Consequently, we get:

$$k < \sqrt{\frac{\varepsilon \min(1,a) \exp(-\varepsilon T)}{T}}.$$

Then $(y^{(n)})_n$ converges to an element of $L^2(0,T,H^1(\Omega))$, say y. Now, we will show that

$$\lim_{n\longrightarrow\infty} y^{(n)}(x,t) = y(x,t)$$

is a solution to the problem (P_5) by showing that y satisfies:

$$A(y,v) = \int\limits_{Q_\tau} G(x,t,y,y_x)\cdot v(x,t)dxdt.$$

We therefore consider the weak formulation of the problem (P_1) as follows:

$$A\left(y^{(n)},v\right) = \int\limits_{Q_\tau} \frac{\partial^2 y^{(n)}}{\partial t^2}(x,t)\cdot\frac{\partial v}{\partial t}(x,t)dxdt + \frac{a}{2}\int\limits_{Q_\tau} \frac{\partial y^{(n)}}{\partial x}(x,t)\cdot\frac{\partial^2 v}{\partial t \partial x}(x,t)dxdt.$$

From the linearity of A, we can have:

$$\begin{aligned}
&A\left(y^{(n)}, v\right) = A\left(y^{(n)} - y, v\right) + A(y, v) \\
&= \int\limits_{Q_\tau} \frac{\partial^2 (y^{(n)} - y)}{\partial t^2}(x,t) \cdot \frac{\partial v}{\partial t}(x,t)dxdt + \frac{a}{2}\int\limits_{Q_\tau} \frac{\partial (y^{(n)} - y)}{\partial x}(x,t) \cdot \frac{\partial^2 v}{\partial t \partial x}(x,t)dxdt \\
&- \int\limits_{Q_\tau} \frac{\partial^2 y}{\partial t^2}(x,t) \cdot \frac{\partial v}{\partial t}(x,t)dxdt + \frac{a}{2}\int\limits_{Q_\tau} \frac{\partial y}{\partial x}(x,t) \cdot \frac{\partial^2 v}{\partial t \partial x}(x,t)dxdt,
\end{aligned}$$

which implies

$$\begin{aligned}
A\left(y^{(n)} - y, v\right) = &\int\limits_{Q_\tau} \frac{\partial^2 (y^{(n)} - y)}{\partial t^2}(x,t) \cdot \frac{\partial v}{\partial t}(x,t)dxdt \\
&+ \frac{a}{2}\int\limits_{Q_\tau} \frac{\partial (y^{(n)} - y)}{\partial x}(x,t) \cdot \frac{\partial^2 v}{\partial t \partial x}(x,t)dxdt.
\end{aligned}$$

Now, by applying the Cauchy Schwartz inequality, the following results can be obtained:

$$\begin{aligned}
A\left(y^{(n)} - y, v\right) \leqslant &\| v_t \|_{L^2(Q_\tau)} \left\|(y^{(n)} - y)_{tt}\right\|_{L^2(0,T,L^2(\Omega))} \\
&+ \frac{a}{2} \| v_{xt} \|_{L^2(Q_\tau)} \left\|(y^{(n)} - y)_x\right\|_{L^2(0,T,L^2(\Omega))}.
\end{aligned}$$

Then, we can find

$$\begin{aligned}
A\left(y^{(n)} - y, v\right) \leqslant C &\left(\left\|(y^{(n)} - y)_{tt}\right\|_{L^2(0,T,L^2(\Omega))} + \left\|(y^{(n)} - y)_x\right\|_{L^2(0,T,L^2(\Omega))}\right) \\
&\times \left(\| v_t \|_{L^2(Q_\tau)} + \| v_{xt} \|_{L^2(Q_\tau)}\right),
\end{aligned}$$

or

$$C = \max\left(1, \frac{a}{2}\right).$$

Now, as $y^{(n)} \longrightarrow y$ in $L^2\left(0, T, H^1(0, l)\right) \cong H^1(Q)$, we get:

$$\begin{aligned}
y^{(n)} &\longrightarrow y \quad \text{in } L^2(Q), \\
y_t^{(n)} &\longrightarrow y_t \quad \text{in } L^2(Q), \\
y_x^{(n)} &\longrightarrow y_x \quad \text{in } L^2(Q).
\end{aligned}$$

Consequently, we note as $n \longrightarrow +\infty$, we find:

$$\lim_{n \longrightarrow +\infty} A\left(y^{(n)} - y, v\right) = 0.$$

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