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Existence of Solutions via C-Class Functions in A_b-Metric Spaces With Applications

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Abstract. Using C-class functions, we demonstrate a few popular common coupled fixed point theorems on A_b-metric spaces and discuss some implications of the main findings. Additionally, we provide examples to illustrate the findings and their applications to both homotopy theory and integral equations.

1. Introduction

Fixed point theory plays a significant role in many parts of the development of nonlinear analysis. It has been applied to various fields of engineering and research. This research was inspired by recent work on the extension of the Banach contraction principle on A_b metric spaces, which was carried out by M. Ughade et al. [1] and studied various fixed point results on these spaces. In the further, N. Mlaiki et al. [2] and K. Ravibabu et al. [3], [4] and P. Naresh et al. [5] succeeded in deriving unique coupled common fixed point theorems in A_b-metric spaces.

Sessa [6] began researching common fixed point theorems for weakly commuting pair of mappings in 1982. Later, in 1986, Jungck [7] expanded the idea of weakly commuting mappings to compatible mappings in metric spaces and proved compatible pair mappings commute on the sets of coincidence point of the involved mappings. When they commute at their coincidence sites, Jungck and Rhoades [8]

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introduced the idea of weak compatibility in 1998 and demonstrated that compatible mappings are weakly compatible but the reverse is not true.

However, Khan et al. [9] first proposed the idea of modifying distance function, which is a control function that modifies the distance between two locations in a metric space. Ansari [10] presented the idea of C-class functions in 2014 and proved the unique fixed point theorems for certain contractive mappings with regard to the C-class functions, which started a lot of work in this field (See. [11], [12], [13], [14], [15], [16], [17])

The idea of coupled fixed point was first developed by Guo and Lakshmikantham [18] in 1987 . Later, employing a weak contractivity type assumption, Bhaskar and Lakshmikantham [19] developed a novel fixed point theorem for a mixed monotone mapping in a metric space driven with partial ordering. See study results in ([20], [21], [22], [23], [24]) and related references for additional results on coupled fixed point outcomes.

In the framework of A_b-metric spaces, the goal of the current study is to establish original common coupled fixed point theorems using C-class functions. Additionally, we may provide relevant applications for homotopy, integral equations, and appropriate examples.

First we recall some basic results.

2. Preliminaries

Definition 2.1. *([1]) Let $\Im$ be a non-empty set and $b \geq 1$ be given real number. A mapping $A_b : \Im^n \to [0, \infty)$ is called an A_b-metric on $\Im$ if and only if for all $\Upsilon_i, a \in \Im$ $i = 1, 2, 3, ..n$; the following conditions hold :*

(A_b1) $A_b(\Upsilon_1, \Upsilon_2, ..., \Upsilon_{n-1}, \Upsilon_n) \geq 0$,

(A_b2) $A_b(\Upsilon_1, \Upsilon_2, ..., \Upsilon_{n-1}, \Upsilon_n) = 0 \Leftrightarrow \Upsilon_1 = \Upsilon_2 = \cdots\cdots = \Upsilon_{n-1} = \Upsilon_n$,

$$(A_b3)\ A_b(\Upsilon_1, \Upsilon_2, ..., \Upsilon_{n-1}, \Upsilon_n) \leq b \left(\begin{array}{c} A_b(\Upsilon_1, \Upsilon_1, ..., (\Upsilon_1)_{n-1}, a) \\ +A_b(\Upsilon_2, \Upsilon_2, ..., (\Upsilon_2)_{n-1}, a) \\ +\cdots\cdots + A_b(\Upsilon_{n-1}, \Upsilon_{n-1}, ..., (\Upsilon_{n-1})_{n-1}, a) \\ +A_b(\Upsilon_n, \Upsilon_n, ..., (\Upsilon_n)_{n-1}, a) \end{array} \right)$$

Then the pair $(\Im, A_b)$ is called an A_b-metric space.

Remark 2.1. *([1]) The class of A_b-metric spaces is actually larger than that of A-metric spaces, it should be emphasised. Each A-metric space is, in fact, a A_b-metric space with $b = 1$. The opposite isn't always true, though. A n-dimensional S_b-metric space is also a A_b-metric space. As a result, a A_b-metric with $n = 3$ is a particular instance of a S_b-metric.*

The example below demonstrates that an A_b-metric on $\Im$ need not be an A-metric on $\Im$.

Example 2.1. *([1]) Let $\Im = [0, +\infty)$, define $A_b : \Im^n \to [0, +\infty)$ as $A_b(\Upsilon_1, \Upsilon_2, ..., \Upsilon_{n-1}, \Upsilon_n) = \sum_{i=1}^{n} \sum_{i<j} |\Upsilon_i - \Upsilon_j|^2$ for all $\Upsilon_i \in \Im, i = 1, 2, \cdots$. Then $(\Im, A_b)$ is an A_b-metric space with $b = 2 > 1$.*

Definition 2.2. *([1]) Let $(\Im, A_b)$ be a A_b-metric space. Then, for $\Upsilon \in \Im$, $r > 0$ we defined the open ball $B_{A_b}(\Upsilon, r)$ and closed ball $B_{A_b}[\Upsilon, r]$ with center Υ and radius r as follows respectively:*

$$B_{A_b}(\Upsilon, r) = \{\mho \in \Im : A_b(\mho, \mho, \cdots, (\mho)_{n-1}, \Upsilon) < r\},$$

and

$$B_{A_b}[\Upsilon, r] = \{\mho \in \Im : A_b(\mho, \mho, \cdots, (\mho)_{n-1}, \Upsilon) \leq r\}.$$

Lemma 2.1. *([1]) In a A_b-metric space, we have*

(1) $A_b(\Upsilon, \Upsilon, \cdots, (\Upsilon)_{n-1}, \mho) \leq bA_b(\mho, \mho, \cdots, (\mho)_{n-1}, \Upsilon)$;
(2) $A_b(\Upsilon, \Upsilon, \cdots, (\Upsilon)_{n-1}, \eth) \leq b(n-1)A_b(\Upsilon, \Upsilon, \cdots, (\Upsilon)_{n-1}, \mho) + b^2 A_b(\mho, \mho, \cdots, (\mho)_{n-1}, \eth)$.

Definition 2.3. *([1]) If $(\Im, A_b)$ be a A_b-metric space. A sequence $\{\Upsilon_k\}$ in $\Im$ is said to be:*

(1) *A_b-Cauchy sequence if, for each $\epsilon > 0$, there exists $n_0 \in \mathcal{N}$ such that $A_b(\Upsilon_k, \Upsilon_k, \cdots\cdots (\Upsilon_k)_{n-1}, \Upsilon_m) < \epsilon$ for each $m, k \geq n_0$.*
(2) *A_b-convergent to a point $\Upsilon \in \Im$ if, for each $\epsilon > 0$, there exists a positive integer n_0 such that $A_b(\Upsilon_k, \Upsilon_k, \cdots\cdots (\Upsilon_k)_{n-1}, \Upsilon) < \epsilon$ for all $n \geq n_0$ and we denote by $\lim_{k\to\infty} \Upsilon_k = \Upsilon$.*
(3) *If every A_b-Cauchy sequence is A_b-convergent in $\Im$, then the A_b-metric space $(\Im, A_b)$ is said to be complete.*

Definition 2.4. *[10] A continuous mapping $\Gamma : [0, +\infty) \times [0. + \infty) \to R$ is called a C-class function if for all $s^*, t^* \in [0, \infty)$,*

(a) $\Gamma(s^*, t^*) \leq s$;
(b) $\Gamma(s^*, t^*) = s^*$ *implies that either* $s^* = 0$ *or* $t^* = 0$.

The family of all C-class functions is denoted by C.

Example 2.2. *[10] Each of the functions $\Gamma : [0, +\infty) \times [0. + \infty) \to R$ defined below are elements of C.*

(a) $\Gamma(s^*, t^*) = s^\star - t^\star$;
(b) $\Gamma(s^*, t^*) = ms^*$ *where* $m \in (0, 1)$.
(c) $\Gamma(s^*, t^*) = \frac{s^*}{(1+t^\star)^r}$ *where* $r \in (0, \infty)$.
(d) $\Gamma(s^*, t^*) = s^\star \eta(s^\star)$ *where* $\eta : [0, \infty) \to [0, \infty)$ *is continuous function.*
(e) $\Gamma(s^*, t^*) = s^\star - \varphi(s^\star)$ *for all* $s^*, t^* \in [0, +\infty)$ *where, the continuous function* $\varphi : [0, \infty) \to [0, \infty)$ *such that* $\varphi(s^\star) = 0 \Leftrightarrow s^\star = 0$.
(f) $\Gamma(s^*, t^*) = s\Omega(s^*, t^\star)$ *for all* $s^*, t^* \in [0, +\infty)$ *where, the continuous function* $\Omega : [0, \infty)^2 \to [0, \infty)$ *such that* $\Omega(s^\star, t^\star) < 1$.

Definition 2.5. *[9] A function $\theta_\star : [0, \infty) \to [0, \infty)$ is called an altering distance function if the following properties are satisfied:*

(a) *$\theta_\star$ is nondecreasing and continuous;*

(*b*) $\theta_\star(t) = 0$ *if and only if* $t = 0$.

Here Θ *represents the family of all altering distance functions.*

We must take the following into consideration in order to get our outcomes.

3. Main Results

Definition 3.1. *Let* $(\Im, A_b)$ *be a* A_b*-metric spaces and suppose* $\Omega : \Im^2 \to \Im$ *be a mapping. If* $\Omega(\wp, \varpi) = \wp$, $\Omega(\varpi, \wp) = \varpi$ *for* $\wp, \varpi \in \Im$ *then* $(\wp, \varpi)$ *is called a coupled fixed point of* Ω.

Definition 3.2. *Let* $(\Im, A_b)$ *be a* A_b*-metric spaces and suppose* $\Omega : \Im^2 \to \Im$ *and* $\Lambda : \Im \to \Im$ *be two mappings. An element* $(\wp, \varpi)$ *is said to be a coupled coincident point of* Ω *and* Λ *if*
$F(\wp, \varpi) = \Lambda\wp, \ \ \Omega(\varpi, \wp) = \Lambda\varpi$

Definition 3.3. *Let* $(\Im, A_b)$ *be a* A_b*-metric spaces and suppose* $\Omega : \Im^2 \to \Im$, $\Lambda : \Im \to \Im$ *be two mappings. An element* $(\wp, \varpi)$ *is said to be a coupled common point of* Ω *and* Λ *if*
$\Omega(\wp, \varpi) = \Lambda\wp = \wp, \ \ \Omega(\varpi, \wp) = \Lambda\varpi = \varpi,$

Definition 3.4. *Let* $(\Im, A_b)$ *be a* A_b*-metric space.*

(*a*) *A pair* (Ω, Λ) *is called weakly compatible if* $\Lambda(\Omega(\wp, \varpi)) = \Omega(\Lambda\wp, \Lambda\varpi)$ *whenever for all* $\wp, \varpi \in \Im$ *such that* $F(\wp, \varpi) = \Lambda\wp, \ \ \Omega(\varpi, \wp) = \Lambda\varpi$

(*b*) *A pair* (Ω, Λ) *is called compatible if*

$$\lim_{p\to\infty} A_b(\Lambda\Omega(\imath_p, \jmath_p), \Lambda\Omega(\imath_p, \jmath_p)\cdots, \Omega(\Lambda\imath_p, \Lambda\jmath_p)) = \lim_{p\to\infty} A_b(\Lambda\Omega(\jmath_p, \imath_p), \Lambda\Omega(\jmath_p, \imath_p)\cdots, \Omega(\Lambda\jmath_p, \Lambda\imath_p)) = 0$$

wherever $\{\imath_p\}, \{\jmath_p\}$ *are sequences in* $\Im$, *such that*

$$\lim_{p\to\infty} \Omega(\imath_p, \jmath_p) = \Lambda\imath_p = \imath \text{ and } \lim_{p\to\infty} \Omega(\jmath_p, \imath_p) = \Lambda\jmath_p = \jmath.$$

Lemma 3.1. *If the pair* (Ω, Λ) *of mappings on the* A_b*-metric space* $(\Im, A_b)$ *is compatible, then it is weakly compatible. The converse does not hold.*

Proof. Let $\Omega(i, j) = \Lambda i$ and $\Omega(j, i) = \Lambda j$ for some $i, j \in \Im$. we have to prove that $\Lambda\Omega(i, j) = \Omega(\Lambda i, \Lambda j)$ and $\Lambda\Omega(j, i) = \Omega(\Lambda j, \Lambda i)$. Put $\imath_p = i$ and $\jmath_p = j$ for every $p \in N$. we have $\Omega(\imath_p, \jmath_p) = \Lambda\imath_p \to \Lambda i$ and $\Omega(\jmath_p, \imath_p) = \Lambda\jmath_p \to \Lambda j$. Since (Ω, Λ) is compatible

$$\lim_{p\to\infty} A_b(\Lambda\Omega(\imath_p, \jmath_p), \Lambda\Omega(\imath_p, \jmath_p)\cdots, \Omega(\Lambda\imath_p, \Lambda\jmath_p)) = \lim_{p\to\infty} A_b(\Lambda\Omega(\jmath_p, \imath_p), \Lambda\Omega(\jmath_p, \imath_p)\cdots, \Omega(\Lambda\jmath_p, \Lambda\imath_p)) = 0$$

Therefore, $\Lambda\Omega(i, j) = \Omega(\Lambda i, \Lambda j)$ and $\Lambda\Omega(j, i) = \Omega(\Lambda j, \Lambda i)$ and hence the pair (Ω, Λ) is weakly compatible. However, the opposite need not be the case.

For example, Let $\Im = [0, +\infty)$, define $A_b : \Im^n \to [0, +\infty)$ as
$A_b(\Upsilon_1, \Upsilon_2, \ldots\ldots, \Upsilon_{n-1}, \Upsilon_n) = \sum_{i=1}^{n} \sum_{i<j} |\Upsilon_i - \Upsilon_j|^2$ for all $\Upsilon_i \in \Im, i = 1, 2, \cdots$. Then $(\Im, A_b)$ is an A_b-metric space with $b = 2 > 1$.

Define two mappings $\Omega : \Im^2 \to \Im$ by $\Omega(i,j) = \begin{cases} \frac{6i-3j+6n-3}{6n} & \text{for} \quad i,j \in [0, \frac{1}{2}) \\ \frac{n}{4} & \text{for} \quad i,j \in [\frac{1}{2}, \infty) \end{cases}$ and $\Lambda : \Im \to \Im$ by $\Lambda(i) = \begin{cases} \frac{9i+6n-6}{6n} & \text{for} \quad i \in [0, \frac{1}{2}) \\ \frac{n}{4} & \text{for} \quad i \in [\frac{1}{2}, \infty) \end{cases}$

Now we define the two sequences $\{\imath_p\}, \{\jmath_p\}$ as $\imath_p = \frac{1}{p}$ and $\jmath_p = 1 + \frac{1}{p}$, then $\Omega(\imath_p, \jmath_p) = \frac{\frac{3}{p}+6n-6}{6n} \to \frac{n-1}{n}$ as $p \to \infty$ and $\Lambda(\imath_p) = \frac{\frac{9}{p}+6n-6}{6n} \to \frac{n-1}{n}$ as $p \to \infty$, also $\Omega(\jmath_p, \imath_p) = \frac{\frac{3}{p}+6n+3}{6n} \to \frac{2n+1}{2n}$ as $p \to \infty$ and $\Lambda(\jmath_p) = \frac{\frac{9}{p}+6n+3}{6n} \to \frac{2n+1}{2n}$ as $p \to \infty$. But

$$\begin{aligned} & \lim_{p\to\infty} A_b(\Lambda\Omega(\imath_p, \jmath_p), \Lambda\Omega(\imath_p, \jmath_p) \cdots, \Omega(\Lambda\imath_p, \Lambda\jmath_p)) \\ = & \lim_{p\to\infty} A_b(\frac{\frac{27}{p} + 36n^2 + 18n - 54}{36n^2}, \frac{\frac{27}{p} + 36n^2 + 18n - 54}{36n^2} \cdots, \frac{\frac{27}{p} + 36n^2 - 45}{36n^2}) \\ = & (n-1)|\frac{2n-1}{4n^2}|^2 \neq 0, \text{ if } n = 2. \end{aligned}$$

and

$$\begin{aligned} & \lim_{p\to\infty} A_b(\Lambda\Omega(\jmath_p, \imath_p), \Lambda\Omega(\jmath_p, \imath_p) \cdots, \Omega(\Lambda\jmath_p, \Lambda\imath_p)) \\ = & \lim_{p\to\infty} A_b(\frac{\frac{27}{p} + 36n^2 + 18n + 27}{36n^2}, \frac{\frac{27}{p} + 36n^2 + 18n + 27}{36n^2} \cdots, \frac{\frac{27}{p} + 36n^2 + 36}{36n^2}) \\ = & (n-1)|\frac{2n-1}{4n^2}|^2 \neq 0, \text{ if } n = 2. \end{aligned}$$

Thus the pair (Ω, Λ) is not compatible. Also,Then for any $i, j \in [\frac{1}{2}, \infty)$, $(\frac{n}{4}, \frac{n}{4})$is a coupled coincidence point of Ω and Λ it is namely that $i = j = \frac{n}{4}$, $\Omega(i,j) = \frac{7}{8} = \Lambda i$ and $\Omega(j,i) = \frac{7}{8} = \Lambda j$ for $n = 2$ and $\lambda\Omega(i,j) = \Omega(\Lambda i, \Lambda j)$, $\lambda\Omega(j,i) = \Omega(\Lambda j, \Lambda i)$, showing that Ω, Λ are weakly compatible maps on $\Im$.

□

Theorem 3.1. *Let $(\Im, A_b)$ be a complete A_b-metric space. Suppose $T : \Im^2 \to \Im$ and $f : \Im \to \Im$ be a two mappings satisfying the following:*

$$\eta_\star\left(2b^2 A_b(T(\imath,\jmath), T(\imath,\jmath), \cdots, T(\eth,\mho))\right) \leq \Gamma\left(\eta_\star\left(A_b(f\imath, f\imath, \cdots, f\eth)\right), \theta_\star\left(A_b(f\jmath, f\jmath, \cdots, f\mho)\right)\right) \quad (3.1)$$

for all $\imath, \jmath, \eth, \mho \in \Im$, where $\eta_\star, \theta_\star \in \Theta$ and $\Gamma \in C$

a) $T(\Im^2) \subseteq f(\Im)$;

b) pair (T, f) is compatible;

c) f is continuous.

Then there is a unique common coupled fixed point of T and f in $\Im$.

Proof. Let $\imath_0, \jmath_0 \in \Im$ be arbitrary, and from (a), we construct the sequences $\{\imath_p\}, \{\jmath_p\}$, in $\Im$ as

$$T(\imath_p, \jmath_p) = f\imath_{p+1} = \aleph_p, \quad T(\jmath_p, \imath_p) = f\jmath_{p+1} = \Upsilon_p, \text{ where } p = 0, 1, 2, \ldots.$$

Now from (3.1), we have

$$\begin{aligned}\eta_\star\left(2b^2A_b(\aleph_1,\aleph_1,\cdots,\aleph_2)\right) &= \eta_\star\left(2b^2A_b(T(\imath_1,\jmath_1),T(\imath_1,\jmath_1),\cdots,T(\imath_2,\jmath_2))\right)\\ &\le \Gamma\left(\eta_\star\left(A_b(f\imath_1,f\imath_1,\cdots,f\imath_2)\right),\theta_\star\left(A_b(f\jmath_1,f\jmath_1,\cdots,f\jmath_2)\right)\right)\\ &\le \eta_\star\left(A_b(f\imath_1,f\imath_1,\cdots,f\imath_2)\right)\\ &\le \eta_\star\left(A_b(\aleph_0,\aleph_0,\cdots,\aleph_1)\right)\end{aligned}$$

By the definition of $\eta_\star$, we have that

$$A_b(\aleph_1,\aleph_1,\cdots,\aleph_2)\le\frac{1}{2b^2}A_b(\aleph_0,\aleph_0,\cdots,\aleph_1).$$

Also

$$\begin{aligned}\eta_\star\left(2b^2A_b(\aleph_2,\aleph_2,\cdots,\aleph_3)\right) &= \eta_\star\left(2b^2A_b(T(\imath_2,\jmath_2),T(\jmath_2,\jmath_2),\cdots,T(\imath_3,\jmath_3))\right)\\ &\le \Gamma\left(\eta_\star\left(A_b(f\imath_2,f\imath_2,\cdots,f\imath_3)\right),\theta_\star\left(A_b(f\jmath_2,f\jmath_2,\cdots,f\jmath_3)\right)\right)\\ &\le \eta_\star\left(A_b(f\imath_2,f\imath_2,\cdots,f\imath_3)\right)\\ &\le \eta_\star\left(A_b(\aleph_1,\aleph_1,\cdots,\aleph_2)\right)\end{aligned}$$

By the definition of $\eta_\star$, we have that

$$\begin{aligned}A_b(\aleph_2,\aleph_2,\cdots,\aleph_3) &\le \frac{1}{2b^2}A_b(\aleph_1,\aleph_1,\cdots,\aleph_2)\\ &\le \frac{1}{(4b^2)^2}A_b(\aleph_0,\aleph_0,\cdots,\aleph_1)\end{aligned}$$

Continuing this process, we can conclude that

$$A_b(\aleph_p,\aleph_p,\cdots,\aleph_{p+1}) \le \frac{1}{(2b^2)^p}A_b(\aleph_0,\aleph_0,\cdots,\aleph_1)\to 0 \text{ as } p\to\infty.$$

that is

$$\lim_{p\to\infty}A_b(\aleph_p,\aleph_p,\cdots,\aleph_{p+1})=0.$$

Similarly, we can prove that

$$\lim_{p\to\infty}A_b(\Upsilon_p,\Upsilon_p,\cdots,\Upsilon_{p+1})=0.$$

Now for $q>p$, by use of (A_b3), we have

$$A_b\left(\aleph_p,\aleph_p,\cdots,\aleph_{n-1},\aleph_q\right)\le b\begin{pmatrix} A_b\left(\aleph_p,\aleph_p,........,(\aleph_p)_{n-1},\aleph_{p+1}\right)\\ +A_b\left(\aleph_p,\aleph_p,........,(\aleph_p)_{n-1},\aleph_{p+1}\right)\\ +\cdots\cdots+A_b\left(\aleph_p,\aleph_p,........,(\aleph_p)_{n-1},\aleph_{p+1}\right)\\ +A_b\left(\aleph_q,\aleph_q,........,(\aleph_q)_{n-1},\aleph_{p+1}\right)\end{pmatrix}$$

$$
\begin{aligned}
&\leq b(n-1)A_b(\aleph_p,\aleph_p,........,(\aleph_p)_{n-1},\aleph_{p+1}) \\
&\quad +bA_b(\aleph_q,\aleph_q,........,(\aleph_q)_{n-1},\aleph_{p+1}) \\
&\leq b(n-1)A_b(\aleph_p,\aleph_p,........,(\aleph_p)_{n-1},\aleph_{p+1}) \\
&\quad +b^2A_b(\aleph_{p+1},\aleph_{p+1},........,(\aleph_{p+1})_{n-1},\aleph_q) \\
&\leq b(n-1)A_b(\aleph_p,\aleph_p,........,(\aleph_p)_{n-1},\aleph_{p+1}) \\
&\quad +b^3(n-1)A_b(\aleph_{p+1},\aleph_{p+1},........,(\aleph_{p+1})_{n-1},\aleph_{p+2}) \\
&\quad +b^4A_b(\aleph_{p+2},\aleph_{p+2},........,(\aleph_{p+2})_{n-1},\aleph_q)
\end{aligned}
$$

$$
\begin{aligned}
&\leq b(n-1)A_b(\aleph_p,\aleph_p,........,(\aleph_p)_{n-1},\aleph_{p+1}) \\
&\quad +b^3(n-1)A_b(\aleph_{p+1},\aleph_{p+1},........,(\aleph_{p+1})_{n-1},\aleph_{p+2}) \\
&\quad +b^5(n-1)A_b(\aleph_{p+2},\aleph_{p+2},........,(\aleph_{p+2})_{n-1},\aleph_{p+3}) \\
&\quad +b^7(n-1)A_b(\aleph_{p+3},\aleph_{p+3},........,(\aleph_{p+3})_{n-1},\aleph_{p+4}) \\
&\quad +\ldots+b^{2q-2p-2}(n-1)A_b(\aleph_{q-2},\aleph_{q-2},........,(\aleph_{q-2})_{n-1},\aleph_{q-1}) \\
&\quad +b^{2q-2p-3}A_b(\aleph_{q-1},\aleph_{q-1},........,(\aleph_{q-1})_{n-1},\aleph_q) \\
&\leq (n-1)\left(b\frac{1}{(2b^2)^p}+b^3\frac{1}{(2b^2)^{p+1}}+b^5\frac{1}{(2b^2)^{p+2}}+\ldots+b^{2q-2p-2}\frac{1}{(2b^2)^{q-2}}\right)A_b(\aleph_0,\aleph_0,\cdots,\aleph_1) \\
&\quad +b^{2q-2p-3}\frac{1}{(2b^2)^{q-1}}A_b(\aleph_0,\aleph_0,\cdots,\aleph_1) \\
&\leq (n-1)b\frac{1}{(2b^2)^p}\left(1+b^2\frac{1}{2b^2}+b^4(\frac{1}{2b^2})^2+\ldots+b^{2q-2p-4}(\frac{1}{2b^2})^{q-p-2}\right)A_b(\aleph_0,\aleph_0,\cdots,\aleph_1) \\
&\quad +b^{2q-2p-3}(\frac{1}{2b^2})^{q-p-1}A_b(\aleph_0,\aleph_0,\cdots,\aleph_1) \\
&\leq (n-1)b\frac{1}{(2b^2)^p}\left(1+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^{q-p-2}}+\cdots\right)A_b(\aleph_0,\aleph_0,\cdots,\aleph_1) \\
&\leq 2(n-1)b\frac{1}{(2b^2)^p}A_b(\aleph_0,\aleph_0,\cdots,\aleph_1)\to 0 \text{ as } p,q\to\infty.
\end{aligned}
$$

Hence $\{\aleph_p\}$ is a Cauchy sequence in $\Im$. We can also demonstrate that $\{\Upsilon_p\}$, is Cauchy sequence in $\Im$. Therefore,

$$\lim_{p,q\to\infty} A_b(\aleph_p,\aleph_p,\cdots,\aleph_q)=0, \text{ and } \lim_{p,q\to\infty} A_b(\Upsilon_p,\Upsilon_p,\cdots,\Upsilon_q)=0.$$

Since $(\Im, A_b)$ is complete, there exist $\aleph,\Upsilon\in\Im$ such that

$$\lim_{p\to\infty}\aleph_p=\lim_{p\to\infty}T(\imath_p,\jmath_p)=\lim_{p\to\infty}f\imath_{p+1}=\aleph \qquad \lim_{p\to\infty}\Upsilon_p=\lim_{p\to\infty}T(\jmath_p,\imath_p)=\lim_{p\to\infty}f\jmath_{p+1}=\Upsilon.$$

Since $f:\Im\to\Im$ is continuous

$$\lim_{p\to\infty}f^2\imath_{p+1}=f\aleph \text{ and } \lim_{p\to\infty}fT(\imath_p,\jmath_p)=f\aleph$$

$$\lim_{p\to\infty} f^2 \jmath_{p+1} = f\Upsilon \text{ and } \lim_{p\to\infty} fT(\jmath_p, \imath_p) = f\Upsilon$$

Since $\{T, f\}$ is compatible, we have $F(f\imath_p, f\jmath_p) \to f\aleph$ and $F(f\jmath_p, f\imath_p) \to f\Upsilon$

$$\lim_{p\to\infty} A_b(T(f\imath_p, f\jmath_p), T(f\imath_p, f\jmath_p)\cdots, fT(\imath_p, \jmath_p)) = 0. \tag{3.2}$$

$$\lim_{p\to\infty} A_b(T(f\jmath_p, f\imath_p), T(f\jmath_p, f\imath_p)\cdots, fT(\jmath_p, \imath_p)) = 0. \tag{3.3}$$

Now, we prove that $f\aleph = T(\aleph, \Upsilon)$ and $f\Upsilon = T(\Upsilon, \aleph)$.

For all $p \geq 0$, we have

$$\begin{aligned} & A_b\left(f\aleph, f\aleph, \cdots, (f\aleph)_{n-1}, T(f\imath_p, f\jmath_p)\right) \\ \leq\ & b\begin{pmatrix} (n-1)A_b\left(f\aleph, f\aleph, \ldots\ldots.., (f\aleph)_{n-1}, fT(\imath_p, \jmath_p)\right) \\ +A_b\left(fT(\imath_p, \jmath_p), fT(\imath_p, \jmath_p), \ldots\ldots.., (fT(\imath_p, \jmath_p))_{n-1}, T(f\imath_p, f\jmath_p)\right) \end{pmatrix} \\ \leq\ & (n-1)bA_b\left(f\aleph, f\aleph, \ldots\ldots.., (f\aleph)_{n-1}, fT(\imath_p, \jmath_p)\right) \\ & \quad +b^2A_b\left(T(f\imath_p, f\jmath_p), T(f\imath_p, f\jmath_p), \ldots\ldots.., (T(f\imath_p, f\jmath_p))_{n-1}, fT(\imath_p, \jmath_p)\right) \end{aligned}$$

On taking limits as $p \to \infty$ and from (3.2) we get

$$A_b\left(f\aleph, f\aleph, \cdots, (f\aleph)_{n-1}, T(\aleph, \Upsilon)\right) = 0.$$

Similarly it is easy to see that $A_b\left(f\Upsilon, f\Upsilon, \cdots, (f\Upsilon)_{n-1}, T(\Upsilon, \aleph)\right) = 0$.

Thus, $T(\aleph, \Upsilon) = f\aleph$ and $T(\Upsilon, \aleph) = f\Upsilon$. Hence $(\aleph, \Upsilon)$ is a coupled coincidence point of T and f. Now we prove that $f\aleph = \aleph$ and $f\Upsilon = \Upsilon$. Now consider

$$\begin{aligned} \eta_\star\left(2b^2A_b\left(f\aleph, f\aleph, \cdots, (f\aleph)_{n-1}, \aleph_p\right)\right) &= \eta_\star\left(2b^2A_b\left(T(\aleph, \Upsilon), T(\aleph, \Upsilon), \cdots, (T(\aleph, \Upsilon))_{n-1}, T(\imath_p, \jmath_p)\right)\right) \\ &\leq \Gamma\left(\eta_\star\left(A_b(f\aleph, f\aleph, \cdots, f\imath_p)\right), \theta_\star\left(A_b(f\Upsilon, f\Upsilon, \cdots, f\jmath)\right)\right) \\ &\leq \eta_\star\left(A_b(f\aleph, f\aleph, \cdots, f\imath_p)\right) \end{aligned}$$

By the definition of $\eta_\star$, we have

$$A_b\left(f\aleph, f\aleph, \cdots, (f\aleph)_{n-1}, \aleph_p\right) \leq \frac{1}{2b^2}A_b(f\aleph, f\aleph, \cdots, f\imath_p)$$

Letting $p \to \infty$, we get $A_b\left(f\aleph, f\aleph, \cdots, (f\aleph)_{n-1}, \aleph\right) \leq \frac{1}{2b^2}A_b(f\aleph, f\aleph, \cdots, \aleph)$ which implies that $f\aleph = \aleph$. Similarly, we can prove $f\Upsilon = \Upsilon$. Therefore, $T(\aleph, \Upsilon) = f\aleph = \aleph$ and $T(\Upsilon, \aleph) = f\Upsilon = \Upsilon$. Thus,$(\aleph, \Upsilon)$ is a common coupled point of T and f. In order to demonstrate uniqueness, we first assume that $(\aleph^\star, \Upsilon^\star)$ is a another coupled common fixed point of T and f.

Now

$$\begin{aligned} \eta_\star\left(2b^2A_b\left(\aleph, \aleph, \cdots, (\aleph)_{n-1}, \aleph^\star\right)\right) &= \eta_\star\left(2b^2A_b\left(T(\aleph, \Upsilon), T(\aleph, \Upsilon), \cdots, (T(\aleph, \Upsilon))_{n-1}, T(\aleph^\star, \Upsilon^\star)\right)\right) \\ &\leq \Gamma\left(\eta_\star\left(A_b(f\aleph, f\aleph, \cdots, f\aleph^\star)\right), \theta_\star\left(A_b(f\Upsilon, f\Upsilon, \cdots, f\Upsilon^\star)\right)\right) \\ &\leq \eta_\star\left(A_b(f\aleph, f\aleph, \cdots, f\aleph^\star)\right) \\ &\leq \eta_\star\left(A_b(\aleph, \aleph, \cdots, \aleph^\star)\right) \end{aligned}$$

By the definition of $\eta_\star$, we have $A_b(\aleph, \aleph, \cdots, \aleph^\star) \leq \frac{1}{2b^2} A_b(\aleph, \aleph, \cdots, \aleph^\star)$
Therefore, $A_b(\aleph, \aleph, \cdots, \aleph^\star) = 0$ implies $\aleph = \aleph^\star$. Similarly, we can shows that $\Upsilon = \Upsilon^\star$. Thus,$(\aleph, \Upsilon)$ is a unique common coupled point of T and f. Finally, we will show $\aleph = \Upsilon$.

$$\begin{aligned}
\eta_\star\left(2b^2 A_b\left(\aleph, \aleph, \cdots, (\aleph)_{n-1}, \Upsilon\right)\right) &= \eta_\star\left(2b^2 A_b\left(T(\aleph, \Upsilon), T(\aleph, \Upsilon), \cdots, (T(\aleph, \Upsilon))_{n-1}, T(\Upsilon, \aleph)\right)\right) \\
&\leq \Gamma\left(\eta_\star\left(A_b(f\aleph, f\aleph, \cdots, f\Upsilon)\right), \theta_\star\left(A_b(f\Upsilon, f\Upsilon, \cdots, f\aleph)\right)\right) \\
&\leq \eta_\star\left(A_b(f\aleph, f\aleph, \cdots, f\Upsilon)\right) \\
&\leq \eta_\star\left(A_b(\aleph, \aleph, \cdots, \Upsilon)\right)
\end{aligned}$$

By the definition of $\eta_\star$, we have $A_b(\aleph, \aleph, \cdots, \Upsilon) \leq \frac{1}{2b^2} A_b(\aleph, \aleph, \cdots, \Upsilon)$.
Therefore, $A_b(\aleph, \aleph, \cdots, \Upsilon) = 0$ implies $\aleph = \Upsilon$. Thus,$(\aleph, \aleph)$ is a common fixed point of T and f.

□

Theorem 3.2. *Let $(\Im, A_b)$ be a complete A_b-metric space. Suppose $T : \Im^2 \to \Im$ and $f : \Im \to \Im$ be a two mappings satisfying the following:*

$$\eta_\star\left(2b^2 A_b(T(\imath, \jmath), T(\imath, \jmath), \cdots, T(\eth, \mho))\right) \leq \Gamma\left(\eta_\star\left(A_b(f\imath, f\imath, \cdots, f\eth)\right), \theta_\star\left(A_b(f\jmath, f\jmath, \cdots, f\mho)\right)\right) \quad (3.4)$$

for all $\imath, \jmath, \eth, \mho \in \Im$, where $\eta_\star, \theta_\star \in \Theta$ and $\Gamma \in C$
a) $T(\Im^2) \subseteq f(\Im)$;
b) pair (T, f) is weakly compatible;
c) $f(\Im)$ is closed in $\Im$.
Then there is a unique common coupled fixed point of T and f in $\Im$.

Proof. Let $\imath_0, \jmath_0 \in \Im$ and from Theorem 3.1, we construct the sequences $\{\aleph_p\}, \{\Upsilon_p\}$ in $\Im$ are Cauchy sequences. Since $(\Im, A_b)$ is complete, $\{\aleph_p\}, \{\Upsilon_p\}$ are convergent as follows

$$\lim_{p\to\infty} \aleph_p = \lim_{p\to\infty} T(\imath_p, \jmath_p) = \lim_{p\to\infty} f\imath_{p+1} = \aleph \qquad \lim_{p\to\infty} \Upsilon_p = \lim_{p\to\infty} T(\jmath_p, \imath_p) = \lim_{p\to\infty} f\jmath_{p+1} = \Upsilon.$$

Since $f(\Im)$ is closed in $(\Im, A_b)$, so $\{\aleph_p\}, \{\Upsilon_p\} \subseteq f(\Im)$ are converges in the complete A_b- metric spaces $(\Im, A_b)$, therefore, there exist $\aleph, \Upsilon \in f(\Im)$ with

$$\lim_{p\to\infty} \aleph_{p+1} = \aleph \qquad \lim_{p\to\infty} \Upsilon_{p+1} = \Upsilon$$

Since $f : \Im \to \Im$ and $\aleph, \Upsilon \in f(\Im)$, there exist $\mho, \wp \in \Im$ such that $f\mho = \aleph$ and $f\wp = \Upsilon$. We claim that $T(\mho, \wp) = \aleph$ and $T(\wp, \mho) = \Upsilon$. By using (3.4), we have

$$\begin{aligned}
&\eta_\star\left(2b^2 A_b\left(T(\imath_p, \jmath_p), T(\imath_p, \jmath_p), \cdots, (T(\imath_p, \jmath_p))_{n-1}, T(\mho, \wp)\right)\right) \\
\leq\ &\Gamma\left(\eta_\star\left(A_b(f\imath_p, f\imath_p, \cdots, f\mho)\right), \theta_\star\left(A_b(f\jmath_p, f\jmath_p, \cdots, f\wp)\right)\right) \\
\leq\ &\eta_\star\left(A_b(f\imath_p, f\imath_p, \cdots, f\mho)\right)
\end{aligned}$$

By the definition of $\eta_\star$,
$A_b\left(T(\imath_p, \jmath_p), T(\imath_p, \jmath_p), \cdots, (T(\imath_p, \jmath_p))_{n-1}, T(\mho, \wp)\right) \leq \frac{1}{2b^2} A_b(f\imath_p, f\imath_p, \cdots, f\mho)$

Letting $p \to \infty$, it yields that

$$\lim_{p\to\infty} A_b\left(T(\imath_p, \jmath_p), T(\imath_p, \jmath_p), \cdots, (T(\imath_p, \jmath_p))_{n-1}, T(\mho, \wp)\right) \leq \lim_{p\to\infty} \frac{1}{2b^2} A_b(f\imath_p, f\imath_p, \cdots, f\mho) = 0.$$

It follows that $A_b(\aleph, \aleph, \cdots, (\aleph)_{n-1}, T(\mho, \wp)) = 0$ implies that $T(\mho, \wp) = \aleph$. Similarly, we can prove $T(\wp, \mho) = \Upsilon$.

Hence, $T(\mho, \wp) = \aleph = f\mho$ and $T(\wp, \mho) = \Upsilon = f\wp$.

Since $\{T, f\}$ is weakly compatible pair, we have $T(\aleph, \Upsilon) = f\aleph$ and $T(\Upsilon, \aleph) = f\Upsilon$. Now we shall prove that $f\aleph = \aleph$ and $f\Upsilon = \Upsilon$. By using (3.4), we have

$$\begin{aligned}
\eta_\star\left(2b^2 A_b(f\aleph, f\aleph, \cdots, (f\aleph)_{n-1}, \aleph_p)\right) &= \eta_\star\left(2b^2 A_b(T(\aleph, \Upsilon), T(\aleph, \Upsilon), \cdots, (T(\aleph, \Upsilon))_{n-1}, T(\imath_p, \jmath_p))\right) \\
&\leq \Gamma\left(\eta_\star\left(A_b(f\aleph, f\aleph, \cdots, f\imath_p)\right), \theta_\star\left(A_b(f\Upsilon, f\Upsilon, \cdots, f\jmath_p)\right)\right) \\
&\leq \eta_\star\left(A_b(f\aleph, f\aleph, \cdots, f\imath_p)\right)
\end{aligned}$$

By the definition of $\eta_\star$, $A_b(f\aleph, f\aleph, \cdots, (f\aleph)_{n-1}, \aleph_p) \leq \frac{1}{2b^2} A_b(f\aleph, f\aleph, \cdots, f\imath_p)$

Letting $p \to \infty$, it yields that $A_b(f\aleph, f\aleph, \cdots, (f\aleph)_{n-1}, \aleph) \leq \frac{1}{2b^2} A_b(f\aleph, f\aleph, \cdots, \aleph)$, which possible holds only, $A_b(f\aleph, f\aleph, \cdots, (f\aleph)_{n-1}, \aleph) = 0$ implies that $f\aleph = \aleph$. Similarly, we shall show $f\Upsilon = \Upsilon$. It follows that $T(\aleph, \Upsilon) = f\aleph = \aleph$

and $T(\Upsilon, \aleph) = f\Upsilon = \Upsilon$. Therefore, the common coupled fixed point of T and f is $(\aleph, \Upsilon)$.

It is simple to demonstrate the connected fixed point's uniqueness and the common fixed point's uniqueness of T and f, just like in the proof of Theorem 3.1. □

Corollary 3.1. *Let $(\Im, A_b)$ be a complete A_b-metric space. Suppose a mapping $T : \Im^2 \to \Im$ be satisfies:*

$$\eta_\star\left(2b^2 A_b(T(\imath, \jmath), T(\imath, \jmath), \cdots, T(\eth, \mho))\right) \leq \Gamma\left(\eta_\star\left(A_b(\imath, \imath, \cdots, \eth)\right), \theta_\star\left(A_b(\jmath, \jmath, \cdots, \mho)\right)\right)$$

for all $\imath, \jmath, \eth, \mho \in \Im$, where $\eta_\star, \theta_\star \in \Theta$ and $\Gamma \in C$. Then T has a unique coupled fixed point in $\Im$.

Example 3.1. *Let $\Im = [0, +\infty)$, define $A_b : \Im^n \to [0, +\infty)$ as $A_b(\wp_1, \wp_2, \ldots\ldots\ldots, \wp_{n-1}, \wp_n) = \sum_{i=1}^n \sum_{i<j} |\wp_i - \wp_j|^2$ for all $\wp_i \in \Im, i = 1, 2, \cdots$. Then $(\Im, A_b)$ is an complete A_b-metric space with $b = 2$.*

Let $T : \Im^2 \to \Im$ and $f : \Im \to \Im$ be given by $T(\wp, \varpi) = \sin\left(\frac{4\wp - 2\varpi + 32n - 2}{32n}\right)$ and $f(\wp) = \frac{2\wp + 4n - 2}{4n}$ and $\Gamma(s^\star, t^\star) = \frac{s^\star}{1 + 2s^\star}\ \forall\ s^\star, t^\star \in [0, \infty)$. Let $\eta_\star(t^\star) = \frac{t^\star}{4}$ and $\theta_\star(t^\star) = t^{\star 2}$ Then obviously, $T(\Im^2) \subseteq f(\Im)$ and the pair (T, f) are ω-compatible and clearly for all $\imath, \jmath, \eth, \mho \in \Im$, where $\eta_\star, \theta_\star \in \Theta$ and $\Gamma \in C$, we have

$$\begin{aligned}
&\Gamma\left(\eta_\star\left(A_b(f\imath, f\imath, \cdots, f\eth)\right), \theta_\star\left(A_b(f\jmath, f\jmath, \cdots, f\mho)\right)\right) - \eta_\star\left(2b^2 A_b(T(\imath, \jmath), T(\imath, \jmath), \cdots, T(\eth, \mho))\right) \\
= &\frac{\eta_\star\left(A_b(f\imath, f\imath, \cdots, f\eth)\right)}{1 + 2\eta_\star\left(A_b(f\imath, f\imath, \cdots, f\eth)\right)} - 2b^2 \frac{A_b(T(\imath, \jmath), T(\imath, \jmath), \cdots, T(\eth, \mho))}{4} \\
= &\frac{\frac{1}{4} A_b(f\imath, f\imath, \cdots, f\eth)}{1 + \frac{1}{2} A_b(f\imath, f\imath, \cdots, f\eth)} - \frac{b^2}{2}(n-1)|\sin(\frac{4\imath - 2\jmath + 32n - 2}{32n}) - \sin(\frac{4\eth - 2\mho + 32n - 2}{32n})|^2
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\frac{1}{4}|\frac{2\imath-2\eth}{4n}|^2}{1+\frac{1}{2}|\frac{2\imath-2\eth}{4n}|^2} - 2b^2(n-1)|\cos(\frac{4\imath-2\jmath+64n-4+4\eth-2\mho}{64n})\sin(\frac{4\imath-2\jmath-4\eth+2\mho}{64n})|^2 \\
&\geq \frac{|\frac{2\imath-2\eth}{4n}|^2}{2\left(2+|\frac{2\imath-2\eth}{4n}|^2\right)} \geq 0.
\end{aligned}
$$

Then we have

$\eta_\star\left(2b^2A_b(T(\imath,\jmath),T(\imath,\jmath),\cdots,T(\eth,\mho))\right) \leq \Gamma\left(\eta_\star\left(A_b(f\imath,f\imath,\cdots,f\eth)\right),\theta_\star\left(A_b(f\jmath,f\jmath,\cdots,f\mho)\right)\right)$. *Thus, all assumptions of Theorem 3.1 are satisfied and* $(1,1)$ *is unique common coupled fixed point of* T *and* f.

4. APPLICATION TO INTEGRAL EQUATIONS

In this part, we examine the existence of a singular initial value solution with reference to Corollary 3.1

Theorem 4.1. *Consider the initial value problem*

$$\wp^1(t) = T(t,(\wp,\varpi)(t)),\ t\in I=[0,1],\ (\wp,\varpi)(0)=(\wp_0,\varpi_0) \tag{4.1}$$

where $T: I\times[\frac{\wp_0}{4},\infty)\to[\frac{\wp_0}{4},\infty)$ *and* $\wp_0\in\mathcal{R}$ *Then there exists unique solution in* $C\left(I,[\frac{\wp_0}{4},\infty)\right)$ *for the initial value problem (4.1).*

Proof. The integral equation for the initial value problem (4.1) is

$$\wp(t) = \wp_0 + 2b^2\sqrt{n-1}\int_0^t T(s,(\wp,\varpi)(s))ds.$$

Let $\Im = C\left(I,[\frac{\wp_0}{4},\infty)\right)$ and $A_b(\wp_1,\wp_2,........,\wp_{n-1},\wp_n) = \sum_{i=1}^n\sum_{i<j}|\wp_i-\wp_j|^2$ for all $\wp_i\in\Im$, $i=1,2,\cdots$, define $\eta_\star,\theta_\star:[0,\infty)\to[0,\infty)$ as $\eta_\star(t)=t$, $\theta_\star(t)=\frac{3t}{5}$ and $\Gamma:[0,\infty)\to[0,\infty)$ as $\Gamma(s^\star,t^\star)=ms^\star$ where $m\in(0,1)$ $R:\Im^2\to\Im$ by

$$R(\wp,\varpi)(t) = \frac{\wp_0}{2b^2\sqrt{(n-1)}} + \int_0^t T(s,(\wp,\varpi)(s))ds.$$

Now for all $\wp,\varpi\in\Im$, we have

$$
\begin{aligned}
&\eta_\star\left(2b^2A_b\left(R(\wp,\varpi)(t),R(\wp,\varpi)(t),\cdots,R(\eth,\mho)(t)\right)\right) \\
&= 2b^2(n-1)|R(\wp,\varpi)(t)-R(\eth,\mho)(t)|^2 \\
&= 2b^2(n-1)|\frac{\wp_0}{2b^2\sqrt{(n-1)}} + \int_0^t T(s,(\wp,\varpi)(s))ds - \frac{\eth_0}{2b^2\sqrt{(n-1)}} + \int_0^t T(s,(\eth,\mho)(s))ds|^2 \\
&= \frac{1}{2b^2}|\wp(t)-\eth(t)|^2 \leq \frac{1}{2}(n-1)|\wp(t)-\eth(t)|^2 \leq \frac{1}{2}A_b(\wp,\wp,\cdots,\eth) \\
&\leq \Gamma\left(\eta_\star\left(A_b(\imath,\imath,\cdots,\eth)\right),\theta_\star\left(A_b(\jmath,\jmath,\cdots,\mho)\right)\right)
\end{aligned}
$$

It follows from Corollary 3.1, we conclude that R has a unique solution in $\Im$. □

5. Application to Homotopy

In this part, we examine the possibility that homotopy theory has a single solution.

Theorem 5.1. *Let $(\Im, A_b)$ be complete A_b-metric space, V and $\overline{V}$ be an open and closed subset of $\Im$ such that $V \subseteq \overline{V}$. Suppose $\mathcal{H}_b : \overline{V}^2 \times [0,1] \to \Im$ be an operator with following conditions are satisfying,*

τ_0) $\wp \neq \mathcal{H}_b(\wp, \varpi, s)$, $\varpi \neq \mathcal{H}_b(\varpi, \wp, s)$, for each $\wp, \varpi \in \Upsilon V$ and $s \in [0,1]$ (Here ΥV is boundary of V in $\Im$);

τ_1) for all $\wp, \varpi, \imath, \jmath \in \overline{V}$, $s \in [0,1]$ and $\eta_\star, \theta_\star \in \Theta$, $\Gamma \in C$ such that

$$\eta_\star\left(2b^2 A_b(\mathcal{H}_b(\wp,\varpi,s), \mathcal{H}_b(\wp,\varpi,s), \cdots, \mathcal{H}_b(\imath,\jmath,s))\right) \leq \Gamma\left(\eta_\star\left(A_b(\wp,\wp,\cdots,\imath)\right), \theta_\star\left(A_b(\varpi,\varpi,\cdots,\jmath)\right)\right).$$

τ_2) $\exists\, M \geq 0 \ni A_b(\mathcal{H}_b(\wp,\varpi,s), \mathcal{H}_b(\wp,\varpi,s), \cdots, \mathcal{H}_b(\wp,\varpi,t)) \leq M|s-t|$
for every $\wp, \varpi \in \overline{V}$ and $s, t \in [0,1]$.
Then $\mathcal{H}_b(.,0)$ has a coupled fixed point $\iff$ $\mathcal{H}_b(.,1)$ has a coupled fixed point.

Proof. Let the set

$$\Im = \left\{\ s \in [0,1] : \mathcal{H}_b(\wp,\varpi,s) = \wp, \mathcal{H}_b(\varpi,\wp,s) = \varpi \text{ for some } \wp, \varpi \in V\ \right\}.$$

We have that $(0,0)$ in $\Im^2$ if $\mathcal{H}_b(.,0)$ has a coupled fixed point in V^2. Therefore, the set $\Im$ is not empty. We now demonstrate that $\Im$ is both closed and open in $[0,1]$ and that $\Im = [0,1]$ is the result of this connectivity.

As a result, V^2 has a coupled fixed point for $\mathcal{H}_b(.,1)$. We begin by demonstrating that $\Im$ closed in $[0,1]$. Observing this, Let $\{s_p\}_{p=1}^{\infty} \subseteq \Im$ with $s_p \to \kappa \in [0,1]$ as $p \to \infty$. We must show that $s \in \Im$. Since $s_p \in \Im$ for $p = 0,1,2,3,\cdots$, there exists sequences $\{\wp_p\}, \{\varpi_p\}$ with $\wp_p = \mathcal{H}_b(\wp_p, \varpi_p, s_p)$, $\varpi_p = \mathcal{H}_b(\varpi_p, \wp_p, s_p)$.

Consider

$$\begin{aligned}
& A_b(\wp_p, \wp_p, \cdots, (\wp_p)_{n-1}, \wp_{p+1}) \\
= \ & A_b\left(\mathcal{H}_b(\wp_p,\varpi_p,s_p), \mathcal{H}_b(\wp_p,\varpi_p,s_p), \cdots, (\mathcal{H}_b(\wp_p,\varpi_p,s_p))_{n-1}, \mathcal{H}_b(\wp_{p+1},\varpi_{p+1},s_{p+1})\right) \\
\leq \ & \begin{array}{c} b(n-1)A_b\left(\ \mathcal{H}_b(\wp_p,\varpi_p,s_p), \mathcal{H}_b(\wp_p,\varpi_p,s_p), \cdots, \mathcal{H}_b(\wp_{p+1},\varpi_{p+1},s_p)\ \right) \\ +b^2 A_b\left(\ \mathcal{H}_b(\wp_{p+1},\varpi_{p+1},s_p), \mathcal{H}_b(\wp_{p+1},\varpi_{p+1},s_p), \cdots, \mathcal{H}_b(\wp_{p+1},\varpi_{p+1},s_{p+1})\ \right) \end{array} \\
\leq \ & \begin{array}{c} b(n-1)A_b\left(\ \mathcal{H}_b(\wp_p,\varpi_p,s_p), \mathcal{H}_b(\wp_p,\varpi_p,s_p), \cdots, \mathcal{H}_b(\wp_{p+1},\varpi_{p+1},s_p)\ \right) \\ +b^2 M|s_p - s_{p+1}| \end{array}
\end{aligned}$$

Letting $p \to \infty$, we get

$$\begin{aligned}
& \lim_{p\to\infty} A_b(\wp_p, \wp_p, \cdots, (\wp_p)_{n-1}, \wp_{p+1}) \\
\leq \ & \lim_{p\to\infty} b(n-1)A_b\left(\mathcal{H}_b(\wp_p,\varpi_p,s_p), \mathcal{H}_b(\wp_p,\varpi_p,s_p), \cdots, \mathcal{H}_b(\wp_{p+1},\varpi_{p+1},s_p)\right) + 0
\end{aligned}$$

Since $\eta_\star, \theta_\star$ are continuous and non-decreasing, we obtain

$$\begin{aligned} & \lim_{p\to\infty} \eta_\star\left(\frac{2b}{n-1} A_b(\wp_p, \wp_p, \cdots, (\wp_p)_{n-1}, \wp_{p+1})\right) \\ \leq & \lim_{p\to\infty} \eta_\star\left(2b^2 A_b\left(\mathcal{H}_b(\wp_p, \varpi_p, s_p), \mathcal{H}_b(\wp_p, \varpi_p, s_p), \cdots, \mathcal{H}_b(\wp_{p+1}, \varpi_{p+1}, s_p)\right)\right) \\ \leq & \lim_{p\to\infty} \Gamma\left(\eta_\star\left(A_b(\wp_p, \wp_p, \cdots, \wp_{p+1})\right), \theta_\star\left(A_b(\varpi_p, \varpi_p, \cdots, \varpi_{p+1})\right)\right) \\ \leq & \lim_{p\to\infty} \eta_\star\left(A_b(\wp_p, \wp_p, \cdots, \wp_{p+1})\right). \end{aligned}$$

By the definition of $\eta_\star$, it follows that

$$\lim_{p\to\infty}(\frac{2b}{n-1} - 1) A_b(\wp_p, \wp_p, \cdots, (\wp_p)_{n-1}, \wp_{p+1}) \leq 0$$

So that

$$\lim_{p\to\infty} A_b(\wp_p, \wp_p, \cdots, (\wp_p)_{n-1}, \wp_{p+1}) = 0$$

Now for $q > p$, by use of (A_b3), we have

$$\begin{aligned} A_b\left(\wp_p, \wp_p, \cdots, \wp_{n-1}, \wp_q\right) \leq\ & b(n-1) A_b\left(\wp_p, \wp_p, ..., (\wp_p)_{n-1}, \wp_{p+1}\right) \\ & + b^2 A_b\left(\wp_{p+1}, \wp_{p+1}, ..., (\wp_{p+1})_{n-1}, \wp_q\right) \\ \leq\ & b(n-1) A_b\left(\wp_p, \wp_p, ..., (\wp_p)_{n-1}, \wp_{p+1}\right) \\ & + b^3(n-1) A_b\left(\wp_{p+1}, \wp_{p+1}, ..., (\wp_{p+1})_{n-1}, \wp_{p+2}\right) \\ & + b^4 A_b\left(\wp_{p+2}, \wp_{p+2}, ..., (\wp_{p+2})_{n-1}, \wp_q\right) \\ \leq\ & b(n-1) A_b\left(\wp_p, \wp_p, ..., (\wp_p)_{n-1}, \wp_{p+1}\right) \\ & + b^3(n-1) A_b\left(\wp_{p+1}, \wp_{p+1}, ..., (\wp_{p+1})_{n-1}, \wp_{p+2}\right) \end{aligned}$$

$$\begin{aligned} & + b^5(n-1) A_b\left(\wp_{p+2}, \wp_{p+2}, ..., (\wp_{p+2})_{n-1}, \wp_{p+3}\right) \\ & + b^7(n-1) A_b\left(\wp_{p+3}, \wp_{p+3}, ..., (\wp_{p+3})_{n-1}, \wp_{p+4}\right) \\ & + \ldots + b^{2q-2p-2}(n-1) A_b\left(\wp_{q-2}, \wp_{q-2}, ..., (\wp_{q-2})_{n-1}, \wp_{q-1}\right) \\ & + b^{2q-2p-3} A_b\left(\wp_{q-1}, \wp_{q-1}, ..., (\wp_{q-1})_{n-1}, \wp_q\right) \to 0 \text{ as } p, q \to \infty. \end{aligned}$$

Hence $\{\wp_p\}$ is a Cauchy sequence in A_b metric spaces $(\Im, A_b)$.Similarly, we may demonstrate that the Cauchy sequence in $(\Im, A_b)$ is $\{\varpi_p\}$ and by the fact that $(\Im, A_b)$ is complete, there exist $u, v \in \Im$ with

$$\lim_{p\to\infty} \wp_{p+1} = u \lim_{p\to\infty} \wp_p \quad \lim_{p\to\infty} \varpi_{p+1} = v = \lim_{p\to\infty} \varpi_p$$

We have

$$\begin{aligned} \eta_\star\left(2b^2 A_b\left(u, u, \cdots, \mathcal{H}_b(u, v, s)\right)\right) =\ & \lim_{p\to\infty} \eta_\star\left(2b^2 A_b\left(\mathcal{H}_b(\wp_p, \varpi_p, s), \mathcal{H}_b(\wp_p, \varpi_p, s), \cdots, \mathcal{H}_b(u, v, s)\right)\right) \\ \leq\ & \lim_{n\to\infty} \Gamma\left(\eta_\star\left(A_b(\wp_p, \wp_p, \cdots, u)\right), \theta_\star\left(A_b(\varpi_p, \varpi_p, \cdots, v)\right)\right) \\ \leq\ & \lim_{n\to\infty} \eta_\star\left(A_b(\wp_p, \wp_p, \cdots, u)\right) = 0 \end{aligned}$$

It follows that $\mathcal{H}_b(u,v,s)=u$. Similarly, we can prove $\mathcal{H}_b(v,u,s)=v$. Thus $s\in\Im$. Hence $\Im$ is closed in $[0,1]$. Let $s_0\in\Im$, then there exist $\wp_0,\varpi_0\in V$ with $\wp_0=\mathcal{H}_b(\wp_0,\varpi_0,s_0)$, $\varpi_0=\mathcal{H}_b(\varpi_0,\wp_0,s_0)$. Since V is open, then there exist $r>0$ such that $B_{A_b}(\wp_0,r)\subseteq V$. Choose $s\in(s_0-\epsilon,s_0+\epsilon)$ such that $|s-s_0|\le\frac{1}{M^p}<\frac{\epsilon}{2}$, then for $\wp\in\overline{B_{A_b}(\wp_0,r)}=\{\wp\in\Im/A_b(\wp,\wp,\cdots,\wp_0)\le r+A_b(\wp_0,\wp_0,\cdots,\wp_0)\}$. Now we have

$$\begin{aligned}
& A_b\left(\mathcal{H}_b(\wp,\varpi,s),\mathcal{H}_b(\wp,\varpi,s),\cdots,\wp_0\right)\\
= \;& A_b\left(\mathcal{H}_b(\wp,\varpi,s),\mathcal{H}_b(\wp,\varpi,s),\cdots,\mathcal{H}_b(\wp_0,\varpi_0,s_0)\right)\\
\le \;& (n-1)bA_b\left(\mathcal{H}_b(\wp,\varpi,s),\mathcal{H}_b(\wp,\varpi,s),\cdots,\mathcal{H}_b(\wp,\varpi,s_0)\right)\\
& +b^2A_b\left(\mathcal{H}_b(\wp,\varpi,s_0),\mathcal{H}_b(\wp,\varpi,s_0),\cdots,\mathcal{H}_b(\wp_0,\varpi_0,s_0)\right)\\
\le \;& b(n-1)M|s-s_0|+b^2A_b\left(\mathcal{H}_b(\wp,\varpi,s_0),\mathcal{H}_b(\wp,\varpi,s_0),\cdots,\mathcal{H}_b(\wp_0,\varpi_0,s_0)\right)\\
\le \;& b(n-1)\frac{1}{M^{p-1}}+b^2A_b\left(\mathcal{H}_b(\wp,\varpi,s_0),\mathcal{H}_b(\wp,\varpi,s_0),\cdots,\mathcal{H}_b(\wp_0,\varpi_0,s_0)\right)
\end{aligned}$$

Letting $p\to\infty$, we obtain

$$A_b\left(\mathcal{H}_b(\wp,\varpi,s),\mathcal{H}_b(\wp,\varpi,s),\cdots,\wp_0\right)\le b^2A_b\left(\mathcal{H}_b(\wp,\varpi,s_0),\mathcal{H}_b(\wp,\varpi,s_0),\cdots,\mathcal{H}_b(\wp_0,\varpi_0,s_0)\right)$$

Since $\eta_\star,\theta_\star$ are continuous and non-decreasing, we obtain

$$\begin{aligned}
& \eta_\star\left(A_b\left(\mathcal{H}_b(\wp,\varpi,s),\mathcal{H}_b(\wp,\varpi,s),\cdots,\wp_0\right)\right)\\
\le \;& \eta_\star\left(2b^2A_b\left(\mathcal{H}_b(\wp,\varpi,s_0),\mathcal{H}_b(\wp,\varpi,s_0),\cdots,\mathcal{H}_b(\wp_0,\varpi_0,s_0)\right)\right)\\
\le \;& \Gamma\left(\eta_\star(A_b(\wp,\wp,\cdots,\wp_0)),\theta_\star(A_b(\varpi,\varpi,\cdots,\varpi_0))\right)\\
\le \;& \eta_\star(A_b(\wp,\wp,\cdots,\wp_0))
\end{aligned}$$

Since $\eta_\star$ is non-decreasing, we have

$$\begin{aligned}
A_b\left(\mathcal{H}_b(\wp,\varpi,s),\mathcal{H}_b(\wp,\varpi,s),\cdots,\wp_0\right) &\le A_b(\wp,\wp,\cdots,\wp_0)\\
&\le r+A_b(\wp_0,\wp_0,\cdots,\wp_0).
\end{aligned}$$

Similarly, we can prove,

$$A_b\left(\mathcal{H}_b(\varpi,\wp,s),\mathcal{H}_b(\varpi,\wp,s),\cdots,\varpi_0\right)\le r+A_b(\varpi_0,\varpi_0,\cdots,\varpi_0).$$

Thus for each fixed $s\in(s_0-\epsilon,s_0+\epsilon)$, $\mathcal{H}_b(.,s):\overline{B_{A_b}(\wp_0,r)}\to\overline{B_{A_b}(\wp_0,r)}$, $\mathcal{H}_b(.,s):\overline{B_{A_b}(\varpi_0,r)}\to\overline{B_{A_b}(\varpi_0,r)}$. All of the Theorem 5.1's requirements are then met. Accordingly, we deduce that $\mathcal{H}_b(.,s)$ has a coupled fixed point in $\overline{V}^2$. But it has to be in V^2. Since (τ_0) is true. The result is that $s\in\Im$ for any $s\in(s_0-\epsilon,s_0+\epsilon)$. Because of this, $(s_0-\epsilon,s_0+\epsilon)\subseteq\Im$. In $[0,1]$, $\Im$ is obviously open. We follow the same approach for the opposite inference. □

Conclusion: This paper wraps up a few applications to integral equations and homotopy theory using coupled fixed point theorems for C-class functions in the framework of A_b-metric spaces.

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References

[1] M. Ughade, D. Turkoglu, S.R. Singh, et al. Some Fixed Point Theorems in A_b-Metric Space, Br. J. Math. Comput. Sci. 19 (2016), 1-24.

[2] N. Mlaiki, Y. Rohen, Some Coupled Fixed Point Theorems in Partially Ordered A_b-Metric Space, J. Nonlinear Sci. Appl. 10 (2017), 1731?1743. https://doi.org/10.22436/jnsa.010.04.35.

[3] R. Konchada, S.R. Chindirala, R.N. Chappa, A Novel Coupled Fixed Point Results Pertinent to A_b-Metric Spaces With Application to Integral Equations, Math. Anal. Contemp. Appl. 4 (2022), 63-83. https://doi.org/10.30495/maca.2022.1949822.1046.

[4] K. Ravibabu, C.S. Rao, C.R. Naidu, Applications to Integral Equations with Coupled Fixed Point Theorems in A_b-Metric Space, Thai J. Math. Spec. Iss. (2018), 148-167.

[5] P. Naresh, G.U. Reddy, B.S. Rao, Existence Suzuki Type Fixed Point Results in A_b-Metric Spaces With Application, Int. J. Anal. Appl. 20 (2022), 67. https://doi.org/10.28924/2291-8639-20-2022-67.

[6] S. Sessa, On a Weak Commutativity Condition of Mappings in Fixed Point Considerations, Publ. l'Inst. Math. 32 (1982), 149-153. http://eudml.org/doc/254762.

[7] G. Jungck, Compatible Mappings and Common Fixed Points, Int. J. Math. Math. Sci. 9 (1986), 771-779. https://doi.org/10.1155/s0161171286000935.

[8] G. Jungck, B.E. Rhoades, Fixed Point for Set Valued Functions Without Continuity, Indian J. Pure Appl. Math. 29 (1998), 227-238.

[9] M.S. Khan, M. Swaleh, S. Sessa, Fixed Point Theorems by Altering Distances Between the Points, Bull. Austral. Math. Soc. 30 (1984), 1-9. https://doi.org/10.1017/s0004972700001659.

[10] A.H. Ansari, Note on $\varphi-\psi$-Contractive Type Mappings and Related Fixed Point, in: The 2nd Regional Conference on Mathematics and Applications, Payame Noor University, 2014.

[11] A.H. Ansari, A. Kaewcharoen, C-Class Functions and Fixed Point Theorems for Generalized $\aleph-\eta-\psi-\varphi-F$-Contraction Type Mappings in $\aleph-\eta$-Complete Metric Spaces, J. Nonlinear Sci. Appl. 9 (2016), 4177-4190. https://doi.org/10.22436/jnsa.009.06.60.

[12] H. Huang, G. Deng, S. Radenovic, Fixed Point Theorems for C-Class Functions in b-Metric Spaces and Applications, J. Nonlinear Sci. Appl. 10 (2017), 5853-5868. https://doi.org/10.22436/jnsa.010.11.23.

[13] A.H. Ansari, W. Shatanawi, A. Kurdi, G. Maniu, Best Proximity Points in Complete Metric Spaces With (P)-Property via C-Class Functions, J. Math. Anal. 7 (2016), 54-67.

[14] V. Ozturk, A.H. Ansari, Common Fixed Point Theorems for Mappings Satisfying (E.A)-Property via C-Class Functions in b-Metric Spaces, Appl. Gen. Topol. 18 (2017), 45-52. https://doi.org/10.4995/agt.2017.4573.

[15] T. Hamaizia, Common Fixed Point Theorems Involving C-Class Functions in Partial Metric Spaces, Sohag J. Math. 8 (2021), 23-28. https://doi.org/10.18576/sjm/080103.

[16] G.S. Saluja, Common Fixed Point Theorems on S-Metric Spaces Via C-Class Functions, Int. J. Math. Comb. 3 (2022), 21-37.

[17] W. Shatanawi, M. Postolache, A.H. Ansari, W. Kassab, Common Fixed Points of Dominating and Weak Annihilators in Ordered Metric Spaces via C-Class Functions, J. Math. Anal. 8 (2017), 54-68.

[18] D. Guo, V. Lakshmikantham, Coupled Fixed Points of Nonlinear Operators With Applications, Nonlinear Anal.: Theory Methods Appl. 11 (1987), 623-632. https://doi.org/10.1016/0362-546x(87)90077-0.

[19] T.G. Bhaskar, V. Lakshmikantham, Fixed Point Theorems in Partially Ordered Metric Spaces and Applications, Nonlinear Anal.: Theory Methods Appl. 65 (2006), 1379-1393. https://doi.org/10.1016/j.na.2005.10.017.

[20] M. Abbas, M.A. Khan, S. Radenovic, Common Coupled Fixed Point Theorems in Cone Metric Spaces for ω-Compatible Mappings, Appl. Math. Comput. 217 (2010), 195-202. https://doi.org/10.1016/j.amc.2010.05.042.

[21] A. Aghajani, M. Abbas, E.P. Kallehbasti, Coupled Fixed Point Theorems in Partially Ordered Metric Spaces and Application, Math. Commun. 17 (2012), 497-509. https://hrcak.srce.hr/93280.

[22] E. Karapinar, Coupled Fixed Point on Cone Metric Spaces, Gazi Univ. J. Sci. 1 (2011), 51-58.

[23] M. Abbas, B. Ali, Y.I. Suleiman, Generalized Coupled Common Fixed Point Results in Partially Ordered A-Metric Spaces, Fixed point theory Appl. 2015 (2015), 64.

[24] W. Long, B.E. Rhoades, M. Rajovic, Coupled Coincidence Points for Two Mappings in Metric Spaces and Cone Metric Spaces, Fixed Point Theory Appl. 2012 (2012), 66. https://doi.org/10.1186/1687-1812-2012-66.