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Picture Fuzzy Knowledge Measure with Application to MADM

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Abstract
The complementary dual of entropy is termed "knowledge measure" in recent studies concerning fuzzy and intuitionistic fuzzy sets. A picture fuzzy set is an extended and generalized form of fuzzy and intuitionistic fuzzy sets. The broader perspective of the picture fuzzy set inculcated the possibility of the formulation of a picture fuzzy knowledge measure and its potential implications. In this paper, we set up an axiomatic framework for obtaining a complementary dual of the picture fuzzy entropy. Subsequently, we derive two new knowledge measures that strictly follow the axiomatic requirements. Some empirical investigations establish the advantages of our proposed knowledge measure over the existing measures. We also present a novel multiple attribute decision-making (MADM) algorithm, wherein the proposed knowledge measure computes attribute weights and exhibits encouraging performance. The comparative analysis shows the potential implications and advantages of the proposed measures.

Keywords- Fuzzy set, Picture fuzzy set, Knowledge measure, MADM.

1. Introduction

Precision is a virtue in science, and every scientific advancement strives to achieve more precision and efficiency. But in certain computations based on expert and knowledge-based systems, vagueness or ambiguity are inevitable. Zadeh (1965) formally represented this ambiguity through the conception of the fuzzy set (FS). In an FS, an element of the universal set is characterized by independent membership and a dependent non-membership degree. Atanassov (1986) proposed an intuitionistic fuzzy set (IFS) as an extension of FS that considers membership and non-membership independently with some hesitancy degree. Cường (2015) introduced the term picture fuzzy set (PFS), which includes more dimensions of vagueness. The picture fuzzy set is a direct generalization of IFS and FS that comprises membership, non-membership, and neutrality. Also, refusal degrees may be calculated as one minus the sum of membership, non-membership, and neutrality degrees. PFS is an important mechanism for soft computations in problems concerning pattern analysis, decision-making, clustering, and medical diagnosis.

The term "entropy measure" is used to calculate the amount of ambiguity, while the term "knowledge measure" quantifies the precision in an FS. De-Luca and Termini (1993) pioneered an axiomatic framework to define a fuzzy entropy measure. Yager (1979) determined an FS's entropy by measuring the distance between the fuzzy set and its complement. After that, much investigation on entropy has been done regarding classical and non-standard fuzzy sets (Xuecheng, 1992; Burillo and Bustince, 1996; Hung and Yang, 2006; Li et al., 2012; Farhadinia, 2013; Pal et al., 2013). Recent attempts to study the knowledge measures associated with FS (Singh et al., 2019), IFS (Szmidt and Kacprzyk, 2001; Szmidt and Kacprzyk, 2007; Szmidt et al., 2014), and PFS (Arya and Kumar, 2020; Joshi, 2020a). In the context of IFS, Szmidt and Kacprzyk (2001) proposed an axiomatic definition for intuitionistic fuzzy entropy. Singh et al. (2019) introduced a knowledge measure of a fuzzy set and obtained its non-parametric generalization. Singh et al. (2020) and Singh and Ganie (2022) further obtained one- and two-parametric generalizations of fuzzy knowledge measure. Guo (2016) introduced a novel knowledge measure and gave an axiomatic definition of an IFS knowledge measure. Singh and Ganie (2021) and Lalotra and Singh (2020) studied the knowledge measures concerning hesitant fuzzy sets. Khan et al. (2020) studied the knowledge measure of q-rung ortho pair fuzzy sets by utilizing the inverse tangent function with a graphical representation. An application to multiple attribute group decision-making is also discussed by utilizing their proposed knowledge measures. Ganie (2022) presented some knowledge measures based on the distance measure of Fermetean fuzzy sets and suggested their application to decision-making problems. Recently, Joshi (2023) studied a new entropy-based knowledge measure for fuzzy sets and applied it to a decision-making problem. Singh (2023) studied the knowledge measure of dual hesitant fuzzy sets and investigated their application in the site selection of solar power plants.

Knowledge generally refers to precise or valuable information in a specific situation. In a fuzzy or non-standard fuzzy system, a measure of knowledge is, in fact, a complementary dual of entropy, implying that the lower the entropy, the more the amount of knowledge. A dual of fuzzy entropy is applicable to solve real-life problems in which fuzzy entropy is used. Because fuzzy techniques are adaptive, obtaining an independent complementary dual of fuzzy entropy is difficult. A soft complementary dual of fuzzy entropy can be obtained and used in situations where fuzzy entropy fails to compute an appropriate result. To the best of our knowledge, no study in picture fuzzy framework investigated the complementary dual of entropy and its applications. Because of these facts, some natural queries arise.

Can we define a knowledge measure for a picture fuzzy set? How to set up a general methodology to define a picture fuzzy knowledge measure? What will be the potential significance and advantage of such a measure?

These queries and the research gap motivated us to consider the present study. The novel contribution in this article is as follows:

- We define a knowledge measure of a picture fuzzy set axiomatically as the complementary dual of an entropy measure.
- We proposed two new picture-fuzzy set knowledge measures as well as their axiomatic validation.
- We present a MADM algorithm that utilizes picture fuzzy knowledge measures to compute attribute weights.
- We present a comparative analysis to demonstrate the benefits of the proposed knowledge measures.

The remaining content of the paper is structured as follows: We provide the basic terminologies in Section 2. In Section 3, we present an axiomatic framework for measuring knowledge. Section 4 offers some empirical investigations of the proposed knowledge measure with a comparative analysis. Section 5 presents the PF-knowledge-based attribute weight computation in the MADM problem, compares the proposed knowledge measure with some existing measures based on attribute weight computation, and investigates a MADM problem. Finally, Section 6 concludes the whole study and highlights the scope for future work.

2. Preliminaries

This section presents necessary concepts that are essential for the progress of the present study.

***Definition* 1** (Zadeh, 1965): Let $U = \{u_1, u_2, u_3, \dots, u_n\}$ be a finite universal set. Then a fuzzy set N in U is represented as follows:

$$N = \{(u_i, \alpha_N(u_i)) \big| \, u_i \in U, i = 1, 2, 3, \dots, n\}.$$

where, $\alpha_N : U \to [0, 1]$ is called a membership function on U and captures the degree of the membership of the generic element of U in the fuzzy set N. In addition, $\gamma_N(u_i) = 1 - \alpha_N(u_i)$, is known as a non-membership degree of u_i to N $\forall\, u_i \in U$. The values $\alpha_N(u_i)$ and $\gamma_N(u_i)$, denote the extent of presence and non-presence of u_i, to N, respectively.

***Definition* 2** (Atanassov, 1986): Let $U = \{u_1, u_2, u_3, \dots, u_n\}$ be a finite universal set. Then an intuitionistic fuzzy set O is represented as follows:

$$O = \{(u_i, \alpha_O(u_i), \beta_O(u_i)) |\, u_i \in U, i = 1, 2, 3, \dots, n\}.$$

such that $0 \le \alpha_O(u_i) + \beta_O(u_i) \le 1$. Where $\alpha_O : U \to [0, 1]$ and $\beta_O : U \to [0, 1]$ are the membership and non-membership functions, respectively. In addition, $\pi_O(u_i) = 1 - \alpha_O(u_i) - \beta_O(u_i)$, is known as the hesitancy degree of u_i to O $\forall\, u_i \in U$. The values $\alpha_O(u_i)$ and $\beta_O(u_i)$, denote the extent of presence and non-presence of u_i, in O, respectively.

***Definition* 3** (Cường, 2015): The picture fuzzy set represents an element of a finite universal set $U = \{u_1, u_2, \dots, u_n\}$ with a specific membership degree (α), neutrality degree (β), and non-membership degree (γ) as long as the sum of all membership, non-membership, and neutrality degrees is less than or equal to one. Mathematically, we write it as:

$$P = \{(u_i, \alpha_P(u_i), \beta_P(u_i), \gamma_P(u_i)) | u_i \in U, i = 1,2, \dots, n\}.$$

such that, $0 \le \alpha_P(u_i) + \beta_P(u_i) + \gamma_P(u_i) \le 1$. Where $\alpha_P : U \to [0, 1]$, $\beta_P : U \to [0, 1]$ and $\gamma_P : U \to [0, 1]$ represents membership, neutrality, and non-membership function, respectively. In addition, $\delta_P(u_i) = 1 - (\alpha_P(u_i) + \beta_P(u_i) + \gamma_P(u_i))$, is known as a refusal of u_i to P $\forall\, u_i \in U$.

The function $\delta_P(u_i) = 1 - (\alpha_P(u_i) + \beta_P(u_i) + \gamma_P(u_i))$ is termed as a refusal of an element u_i.

Remark 1: When $\beta_P(u_i) = 0$, then PFS becomes IFS, and when $\beta_P(u_i) = 0$ and $\gamma_P(u_i) = 0$, then PFS becomes FS.

Operation on PF-sets (Cường, 2015): For any $P, Q \in PFS(U)$, the following are the definitions for operations like union, intersection, complement, and encompassing relation:

(i) Union of P and Q:

$$P \cup Q = \left\{\left(u_i, \alpha_{P\cup Q}(u_i), \beta_{P\cup Q}(u_i), \gamma_{P\cup Q}(u_i)\right) \middle| u_i \in U, i = 1,2, \dots, n\right\}.$$

where, $\alpha_{P\cup Q}(u) = \max\{\alpha_P(u_i), \alpha_Q(u_i)\}$, $\beta_{P\cup Q}(u_i) = \min\{\beta_P(u_i), \beta_Q(u_i)\}$.

and $\gamma_{P\cup Q}(u_i) = \min\{\gamma_P(u_i), \gamma_Q(u_i)\}$.

(ii) Intersection of P and Q:

$$P \cap Q = \left\{\left(u_i, \alpha_{P\cap Q}(u_i), \beta_{P\cap Q}(u_i), \gamma_{P\cap Q}(u_i)\right) \middle| u_i \in U, i = 1,2, \dots, n\right\}.$$

$$\text{where, } t_{P\cap Q}(u_i) = \min\{\alpha_P(u_i), \alpha_Q(u_i)\}, \beta_{P\cap Q}(u_i) = \min\{\beta_P(u_i), \beta_Q(u_i)\}.$$
$$\text{and } \gamma_{P\cap Q}(u_i) = \max\{\gamma_P(u_i), \gamma_Q(u_i)\}.$$

(iii) Complement of P:

$$P^c = \{(u_i, \gamma_P(u_i), \beta_P(u_i), \alpha_P(u_i)) | u_i \in U, i = 1,2,\ldots,n\}.$$

(iv) Subset:
P is said to be a subset of Q ($P \subseteq Q$) if and only if $\forall\, u_i \in U$ it holds that:
$\alpha_P(u_i) \le \alpha_Q(u_i)$, $\beta_P(u_i) \le \beta_Q(u_i)$ and $\gamma_P(u_i) \ge \gamma_Q(u_i)$.

***Definition* 4** (Hessian matrix): Let $F(\mu_1, \mu_2, \mu_3, \mu_4)$ be a function of four variables, then the Hessian matrix of a function F is given by,

$$\widetilde{HEN}(F) = \begin{bmatrix} \frac{\partial^2 F}{\partial \mu_1^2} & \frac{\partial^2 F}{\partial \mu_2 \partial \mu_1} & \frac{\partial^2 F}{\partial \mu_3 \partial \mu_1} & \frac{\partial^2 F}{\partial \mu_4 \partial \mu_1} \\ \frac{\partial^2 F}{\partial \mu_1 \partial \mu_2} & \frac{\partial^2 F}{\partial \mu_2^2} & \frac{\partial^2 F}{\partial \mu_3 \partial \mu_2} & \frac{\partial^2 F}{\partial \mu_4 \partial \mu_2} \\ \frac{\partial^2 F}{\partial \mu_1 \partial \mu_3} & \frac{\partial^2 F}{\partial \mu_2 \partial \mu_3} & \frac{\partial^2 F}{\partial \mu_3^2} & \frac{\partial^2 F}{\partial \mu_4 \partial \mu_3} \\ \frac{\partial^2 F}{\partial \mu_1 \partial \mu_4} & \frac{\partial^2 F}{\partial \mu_2 \partial \mu_4} & \frac{\partial^2 F}{\partial \mu_3 \partial \mu_4} & \frac{\partial^2 F}{\partial \mu_4^2} \end{bmatrix}.$$

The function F is convex and concave at a point in its domain according as $\widetilde{HEN}(F)$ is positive semi-definite and negative semi-definite, respectively.

***Definition* 5** (Idempotent matrix): A square matrix M of order n is said to be idempotent if $M^2 = M$.

***Note* 1:** Matrix $I - M = \frac{1}{n}\begin{bmatrix} n-1 & \cdots & -1 \\ \vdots & \ddots & \vdots \\ -1 & \cdots & n-1 \end{bmatrix}_{n\times n}$ is also an idempotent matrix of order n, where I is an identity matrix of order n.

***Note* 2:** Eigenvalues of an idempotent matrix always equal 0 and 1.

The entropy of a fuzzy set measures the amount of information present in it. Referring to Shannon's probabilistic entropy, (De Luca and Termini, 1972) pioneered a fuzzy entropy to estimate the amount of vagueness in a fuzzy set.

***Definition* 6** (De Luca and Termini, 1993): A real-valued function $\tilde{E}: FS(U): \to [0, \infty]$ is called an entropy for FS if for all $P \in FS(U)$ it satisfies the following set of axioms:
F-Ẽ1: $\tilde{E}(P) = 0$ if and only if P is a crisp set;
F-Ẽ2: $\tilde{E}(P)$ is maximum if and only if $\alpha_P(u_i) = \frac{1}{2}$;
F-Ẽ3: $\tilde{E}(P) \le \tilde{E}(Q)$ if P is crisper than Q, that is, $\alpha_{P(u_i)} \le \alpha_{Q(u_i)}$ if $\alpha_{Q(u_i)} \le \frac{1}{2}$ and $\alpha_{P(u_i)} \ge \alpha_{Q(u_i)}$ if $\alpha_{Q(u_i)} \ge \frac{1}{2}$;
F-Ẽ4: $\tilde{E}(P) = \tilde{E}(P^c)$, where P^c represents the complement of P.

***Definition* 7** (Szmidt and Kacprzyk, 2001): A real-valued function $\breve{E}: IFS(U): \rightarrow [0, \infty]$ is called an entropy for *IFS* if for all $P \in IFS(U)$ it satisfies the following set of axioms:
IF-$\breve{E}$1: $\breve{E}(P) = 0$ if and only if *P* is a crisp set;
IF-$\breve{E}$2: $\breve{E}(P)$ assumes a unique maximum if $\alpha_P(u_i) = \gamma_P(u_i)$ for all i ;
IF-$\breve{E}$3:$\breve{E}(P) \leq \breve{E}(Q)$ if *P* is less fuzzy than Q, that is, $\alpha_{P(u_i)} \leq \alpha_Q(u_i),\ \gamma_P(u_i) \geq \gamma_Q(u_i),$ for $\alpha_Q(u_i) \leq \gamma_Q(u_i)$ or $\alpha_{P(u_i)} \geq \alpha_Q(u_i),\ \gamma_P(u_i) \geq \gamma_Q(u_i),$ for $\alpha_Q(u_i) \leq \gamma_Q(u_i)$;
IF-$\breve{E}$4: $\breve{E}(P) = \breve{E}(P^c)$, where P^c represents the complement of *P*.

Since, in an intuitionistic fuzzy representation, we have three components (α, γ, π) with the condition $\alpha + \gamma + \pi = 1$, assuming (α, γ, π) as a probability distribution, Hung and Yang (2006) extended the concept of fuzzy entropy by (Luca and Termini, 1972) to intuitionistic fuzzy entropy given in the following way.

***Definition* 8** (Hung and Yang, 2006): A real-valued function $\tilde{E}: IFS(U): \rightarrow [0, \infty]$ is called an entropy for *IFS* if for all $P \in IFS(U)$ it satisfies the following set of axioms:
IF-Ẽ1: $\tilde{E}(P) = 0$ if and only if *P* is a crisp set;
IF-Ẽ2: $\tilde{E}(P)$ assumes a unique maximum if $\alpha_P(u_i) = \gamma_P(u_i) = \pi_P(u_i) = \frac{1}{3}$;
IF-Ẽ3:$\tilde{E}(P) \leq \tilde{E}(Q)$ if *P* is crisper than Q, that is, $\alpha_P(u_i) \geq \alpha_Q(u_i),\ \gamma_P(u_i) \geq \gamma_Q(u_i),$ for $\min\left(\alpha_Q(u_i), \gamma_Q(u_i)\right) \geq \frac{1}{3}$ and $\alpha_P(u_i) \leq \alpha_Q(u_i),\ \gamma_P(u_i) \leq \gamma_Q(u_i),$ for $\max\left(\alpha_Q(u_i), \gamma_Q(u_i)\right) \leq \frac{1}{3}$;
IF-Ẽ4: $\tilde{E}(P) = \tilde{E}(P^c)$, where P^c represents the complement of *P*.

Since, PFSs are the generalization of IFSs, which have four components $(\alpha_P, \beta_P, \gamma_P, \delta_P)$ in their full form with the condition $\delta_P + \alpha_P + \beta_P + \gamma_P = 1$. Consequently, taking into account the four parametric descriptions of PFSs, Joshi (2020a) expanded the definition of Hung and Yang (2006) to the following definition of picture fuzzy entropy.

***Definition* 9** (Joshi, 2020a): A real-valued function $\tilde{E}: PFS(U): \rightarrow [0, \infty]$ is called an entropy for *PFS* if for all $P \in PFS(U)$ it satisfies the following set of axioms:
PF-Ẽ1 (Sharpness): $\tilde{E}(P) = 0$ if and only if *P* is a crisp set;
PF-Ẽ2 (Maximality): $\tilde{E}(P)$ assumes a unique maximum if $\alpha_P(u_i) = \beta_P(u_i) = \gamma_P(u_i) = \delta_P(u_i) = \frac{1}{4}$;
PF-Ẽ3 (Resolution): $\tilde{E}(P) \leq \tilde{E}(Q)$ if *P* is crisper than Q , *i.e.*, $\alpha_{P(u_i)} \geq \alpha_Q(u_i),\ \beta_P(u_i) \geq \beta_Q(u_i)$, $\gamma_P(u_i) \geq \gamma_Q(u_i),$ for $\min\left(\alpha_Q(u_i), \beta_Q(u_i), \gamma_Q(u_i)\right) \geq \frac{1}{4}$ and $\alpha_P(u_i) \leq \alpha_Q(u_i),\ \beta_P(u_i) \leq \beta_Q(u_i)$, $\gamma_P(u_i) \leq \gamma_Q(u_i),$ for $\max\left(\alpha_Q(u_i), \beta_Q(u_i), \gamma_Q(u_i)\right) \leq \frac{1}{4}$;
PF-Ẽ4 (Symmetry):$\tilde{E}(P) = \tilde{E}(P^c)$ where P^c represents the complement of *P*.

***Definition* 10** (Bustince et al., 2006): A real-valued strictly increasing function $\varphi: [0,1] \rightarrow [0,1]$ with $\varphi(0) = 0,$ and $\varphi(1) = 1$ is known as automorphism of the unit interval.

In conventional fuzzy theory, the crispness of the set means the membership degree of elements of the universal set is either 0 or 1. While dealing with the picture fuzzy set, the presence of an element of the universal set can be thought of positively or negatively. Mathematically, in the picture fuzzy framework, the representations of these two situations are (1,0,0) and (0,0,1), respectively. Clearly, (1,0,0) and (0,0,1) complement each other in a picture-fuzzy environment. Furthermore, in conventional fuzzy theory, a set *P* is the fuzziest if each element of the universal set has a membership value equal to 0.5. In the picture fuzzy representation, this situation is equivalent to the fact that all the components of the picture fuzzy representation are equal to 0.25, and the sum of all these components equals one.

In the next section, we present an axiomatic framework for the knowledge measure of a picture fuzzy set.

3. Proposed Framework for Knowledge Measure of a PFS

We discuss the theoretical background of non-probabilistic entropy before introducing the axiomatic definition of picture fuzzy knowledge measure.

3.1 Background

A measure of entropy in a standard or non-standard fuzzy set *N* estimates the amount of vagueness present in *N*. The natural requirements for valid fuzzy/non-standard fuzzy entropy are:
NE1(Sharpness): Entropy is zero for a crisp set.
NE2 (Maximality): Entropy is the maximum in the equilibrium state or most fuzzy state.
NE3 (Resolution): Entropy is smaller for the crisper set.
NE4 (Symmetry): Entropy of a fuzzy set equals its complement.

In the following, we present mathematical interpretations of the fuzziest state and relative crispness in various versions of fuzzy set.

A. Interpretations of Most Fuzzy State for Classical Fuzzy Set

We know that a standard or classical fuzzy set *P* is most fuzzy if membership degree = non-membership degree $=\frac{1}{2}$, *i.e.*,

$$\alpha_P(u_i) = \gamma_P(u_i) = 1 - \alpha_P(u_i) = \frac{1}{2}.$$

In other words, *P* becomes crisper as soon as $\alpha_P(u_i)$ approaches 0 or 1. In fact, *P* becomes crisper as $\alpha_P(u_i)$ approaches 0 whenever $\alpha_P(u_i) \le \frac{1}{2}$, and $\alpha_P(u_i)$ approaches whenever $\alpha_P(u_i) \ge \frac{1}{2}$.

We can also interpret this as follows:
P becomes crisper as $\alpha_P(u_i)$ approaches 0, and $\gamma_P(u_i)$ approaches 1 whenever $\alpha_P(u_i) \le \frac{1}{2} \le \gamma_P(u_i)$.
or, *P* becomes crisper as $\alpha_P(u_i)$ approaches 1, and $\gamma_P(u_i)$ approaches 0 whenever $\alpha_P(u_i) \ge \frac{1}{2} \ge \gamma_P(u_i)$.

B. Comparison of the Crispness of two Fuzzy Sets (Interpretation I)

If *P* and *Q* be two fuzzy sets in a universe *U*, then *P* is said to be crisper than *Q* in two situations.
Case I: When $\alpha_P(u_i) \le \frac{1}{2}$.
P is said to be crisper than *Q* if $\alpha_P(u_i) \le \alpha_Q(u_i)$.

Case II: When $\alpha_P(u_i) \ge \frac{1}{2}$.
P is said to be crisper than *Q* if $\alpha_P(u_i) \ge \alpha_Q(u_i)$.

C. Comparison of the Crispness of Two Fuzzy Sets (Interpretation II)

If *P* and *Q* be two fuzzy sets in a universe *U*, then *P* is said to be crisper than *Q* in two situations.
Case I: When $\alpha_P(u_i) \le \frac{1}{2}$.
P is said to be crisper than *Q* if $\alpha_P(u_i) \le \alpha_Q(u_i)$ for $\max\{\alpha_P(u_i), \alpha_Q(u_i)\} \le \frac{1}{2}$.

Case II: When $\alpha_P(u_i) \geq \frac{1}{2}$.
P is said to be crisper than Q if $\alpha_P(u_i) \geq \alpha_Q(u_i)$ for $\min\{\alpha_P(u_i), \alpha_Q(u_i)\} \geq \frac{1}{2}$.

Remark 2: The above two interpretations reveal that more the membership value of a set is away from the equilibrium point $\frac{1}{2}$ as shown in Figure 1(A), the crisper the fuzzy set.

D. Interpretation of Most Fuzzy State for an Intuitionistic Fuzzy Set
We know that an intuitionistic fuzzy set P is most fuzzy if membership degree = non-membership degree, $i.e.$,

$$\alpha_P(u_i) = \gamma_Q(u_i).$$

In other words, P becomes crisper as soon as $(\alpha_P(u_i), \gamma_P(u_i))$ moves away from the line $\alpha_P(u_i) = \gamma_P(u_i)$ within the right-angled triangle shown in Figure 1(B).

E. Comparison of the Crispness of Two Intuitionistic Fuzzy Sets (Interpretation I)
If P and Q be two intuitionistic fuzzy sets in a universe U, then P is said to be crisper than Q.
if $\alpha_P(u_i) \leq \alpha_Q(u_i)$ and $\gamma_P(u_i) \geq \gamma_Q(u_i)$ for $\alpha_P(u_i) \leq \gamma_Q(u_i)$.
or $\alpha_P(u_i) \geq \alpha_Q(u_i)$ and $\gamma_P(u_i) \leq \gamma_Q(u_i)$ for $\alpha_P(u_i) \geq \gamma_Q(u_i)$.

Remark 3: Interpretations I reveal that the more the membership-non-membership value pair of a set is away from the equilibrium line as shown in Figure 1(B), the crisper the fuzzy set.

F. Comparison of the Crispness of Two Intuitionistic Fuzzy Sets (Interpretation II)
If P and Q be two fuzzy sets in a universe U, then P is said to be crisper than Q.
if $\alpha_P(u_i) \leq \alpha_Q(u_i)$ and $\gamma_P(u_i) \leq \gamma_Q(u_i)$ for $\max\{\alpha_Q(u_i), \gamma_Q(u_i)\} \leq \frac{1}{3}$.
or $\alpha_P(u_i) \geq \alpha_Q(u_i)$ and $\gamma_P(u_i) \geq \gamma_Q(u_i)$ for $\min\{\alpha_Q(u_i), \gamma_Q(u_i)\} \geq \frac{1}{3}$.

Remark 4: The interpretations-II reveals that more the intuitionistic fuzzy set P is away from the equilibrium point $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, the crisper the fuzzy set. The point $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ falls within the cube shown in Figure 1(C). This interpretation is better than interpretation-I in the sense that the resolution property of standard fuzzy sets can directly be deduced from this interpretation.

It is interesting to note that the resolution property in interpretation-I has an infinite number of points of maximum fuzziness and ignores the role of hesitancy. However, interpretation II encompasses the role of hesitancy and considers a unique point of maximum hesitancy.

Because of these observations, the resolution property for picture fuzzy sets can be considered as follows.

G. Resolution Property in Picture Fuzzy Sets (RPFSs)

A picture fuzzy set P is said to be most fuzzy if $\alpha_P(u_i) = \beta_P(u_i) = \gamma_P(u_i) = \delta_P(u_i) = \frac{1}{4}$.
If P and Q be two fuzzy sets in a universe U, then P is said to be crisper than Q.

If $\alpha_P(u_i) \leq \alpha_Q(u_i)$, $\beta_P(u_i) \leq \beta_Q(u_i)$ and $\gamma_P(u_i) \leq \gamma_Q(u_i)$ for $\max\{\alpha_Q(u_i), \beta_Q(u_i), \gamma_Q(u_i)\} \leq \frac{1}{4}$.
or

$\alpha_P(u_i) \geq \alpha_Q(u_i)$, $\beta_P(u_i) \geq \beta_Q(u_i)$ and $\gamma_P(u_i) \geq \gamma_Q(u_i)$ for $\min\{\alpha_Q(u_i), \beta_Q(u_i), \gamma_Q(u_i)\} \geq \frac{1}{4}$.

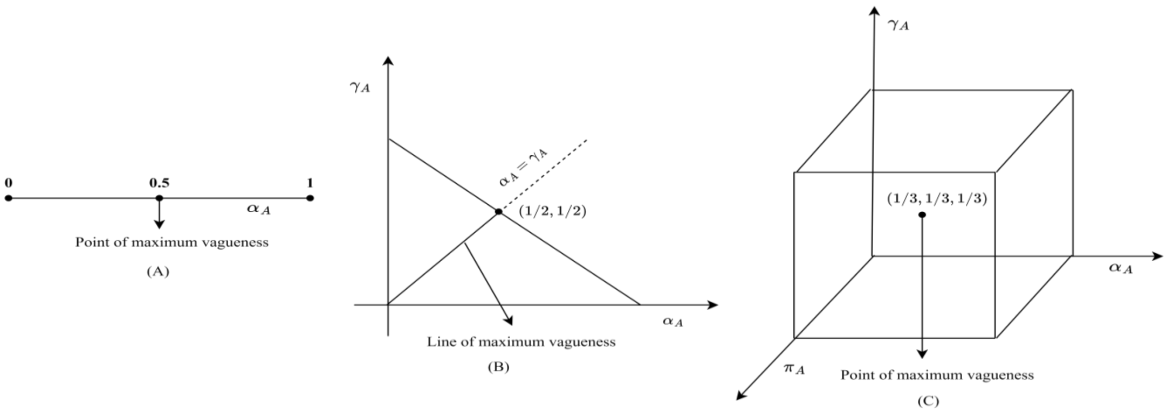

Figure 1. Interpretation of the fuzziest state.

Property* 1*: If RPFSs hold then

$$\left|\alpha_P(u_i) - \frac{1}{4}\right| + \left|\beta_P(u_i) - \frac{1}{4}\right| + \left|\gamma_P(u_i) - \frac{1}{4}\right| + \left|\delta_P(u_i) - \frac{1}{4}\right| \geq \left|\alpha_Q(u_i) - \frac{1}{4}\right| + \left|\beta_Q(u_i) - \frac{1}{4}\right| + \left|\gamma_Q(u_i) - \frac{1}{4}\right| + \left|\delta_Q(u_i) - \frac{1}{4}\right| \quad (1)$$

and

$$\left(\alpha_P(u_i) - \frac{1}{4}\right)^2 + \left(\beta_P(u_i) - \frac{1}{4}\right)^2 + \left(\gamma_P(u_i) - \frac{1}{4}\right)^2 + \left(\delta_P(u_i) - \frac{1}{4}\right)^2 \geq \left(\alpha_Q(u_i) - \frac{1}{4}\right)^2 + \left(\beta_Q(u_i) - \frac{1}{4}\right)^2 + \left(\gamma_Q(u_i) - \frac{1}{4}\right)^2 + \left(\delta_Q(u_i) - \frac{1}{4}\right)^2 \quad (2)$$

Proof: If $\alpha_P(u_i) \leq \alpha_Q(u_i)$, $\beta_P(u_i) \leq \beta_Q(u_i)$ and $\gamma_P(u_i) \leq \gamma_Q(u_i)$ with $\max\{\alpha_Q(u_i), \beta_Q(u_i), \gamma_Q(u_i)\} \leq \frac{1}{4}$ then, $\alpha_P(u_i) \leq \alpha_Q(u_i) \leq \frac{1}{4}$; $\beta_P(u_i) \leq \beta_Q(u_i) \leq \frac{1}{4}$; $\gamma_P(u_i) \leq \gamma_Q(u_i) \leq \frac{1}{4}$ and $\delta_P(u_i) \geq \delta_Q(u_i) \geq \frac{1}{4}$ which implies that Eq. (1) and Eq. (2) hold.

Similarly, if
$\alpha_P(u_i) \geq \alpha_Q(u_i)$, $\beta_P(u_i) \geq \beta_Q(u_i)$ and $\gamma_P(u_i) \geq \gamma_Q(u_i)$ with $\min\{\alpha_Q(u_i), \beta_Q(u_i), \gamma_P(u_i)\} \geq \frac{1}{4}$ then, Eq. (1) and Eq. (2) hold.

The distance between two IFSs was proposed by Szmidt and Kacprzyk (2000) as the distance between their parameters (α, β, π). Two common distance measurements used to determine the separation between two IFSs are the Euclidean distance and the Hamming distance. As PFS are the generalizations of IFS with four parameters $(\alpha, \beta, \gamma, \delta)$ therefore extending the idea of IFSs to PFSs it may be concluded from Property-I that the PFS Q is closer to the fuzziest value $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$ than PFS P.

Now, we introduce an axiomatic definition of knowledge measure as a complementary dual of entropy of a picture fuzzy set.

3.2 Knowledge Measure of a PFS

Because of the duality of entropy and knowledge measure, we introduce the following axiomatic definition of a picture fuzzy knowledge measure.

Definition* 11*: Let $\widetilde{K}_1 : PFS(U) \rightarrow [0,1]$ be a real function and is known as a knowledge measure for PFS if for all $P \in PFS(U)$ it satisfies the following axioms:

PF-$\widetilde{K}_1^1$ (Sharpness): $\widetilde{K}_1(P) = 1 \Leftrightarrow P$ is a crisp set;

PF-$\widetilde{K}_1^2$ (Minimality): $\widetilde{K}_1(P)$assumes a unique minimum if $\alpha_P(u_i) = \beta_P(u_i) = \gamma_P(u_i) = \delta_P(u_i) = \frac{1}{4}$;

PF-$\widetilde{K}_1^3$ (Resolution): $\widetilde{K}_1(P) \geq \widetilde{K}_1(Q)$ if P is crisper than Q, that is, $\alpha_P(u_i) \geq \alpha_Q(u_i), \beta_P(u_i) \geq \beta_Q(u_i), \gamma_P(u_i) \geq \gamma_Q(u_i)$ for $\min\left(\alpha_Q(u_i), \beta_Q(u_i), \gamma_Q(u_i)\right) \geq \frac{1}{4}$ and $\alpha_P(u_i) \leq \alpha_Q(u_i),\ \beta_P(u_i) \leq \beta_Q(u_i)\ , \gamma_P(u_i) \leq \gamma_Q(u_i)$ for $\max\left(\alpha_Q(u_i), \beta_Q(u_i), \gamma_Q(u_i)\right) \leq \frac{1}{4}$;

PF-$\widetilde{K}_1^4$ (Symmetry): $\widetilde{K}_1(P) = \widetilde{K}_1(P^c)$ where P^crepresents the complement of P.

Following Theorem 1 introduces a novel knowledge measure.

Theorem* 1*: Let P be a picture fuzzy set, then $\widetilde{K}_1(P)$ defined in Eq. (3) is a valid knowledge measure of a picture fuzzy set.

$$\widetilde{K}_1(P) = \frac{1}{n}\sum_{i=1}^{n}\Big[\left(\alpha_P(u_i)\right)^3 + \left(\beta_P(u_i)\right)^3 + \left(\gamma_P(u_i)\right)^3 + \left(\delta_P(u_i)\right)^3 - \alpha_P(u_i)\beta_P(u_i)\gamma_P(u_i) - \beta_P(u_i)\gamma_P(u_i)\delta_P(u_i) - \alpha_P(u_i)\delta_P(u_i)\beta_P(u_i) - \alpha_P(u_i)\gamma_P(u_i)\delta_P(u_i)\Big]. \tag{3}$$

Proof: We verify the axioms PF-$\widetilde{K}_1^1$ to PF-$\widetilde{K}_1^4$ given in Definition 11.

PF-$\widetilde{K}_1^1$ (Sharpness): First, suppose $\widetilde{K}_1(P) = 1$, we have,

$$\left(\alpha_P(u_i)\right)^3 + \left(\beta_P(u_i)\right)^3 + \left(\gamma_P(u_i)\right)^3 + \left(\delta_P(u_i)\right)^3 - \alpha_P(u_i)\beta_P(u_i)\gamma_P(u_i) - \beta_P(u_i)\gamma_P(u_i)\delta_P(u_i) - \alpha_P(u_i)\delta_P(u_i)\beta_P(u_i) - \alpha_P(u_i)\gamma_P(u_i)\delta_P(u_i) = 1 \tag{4}$$

Eq. (4) will hold only if

*Case*1: Either, $\alpha_P(u_i) = 1,\ \beta_P(u_i) = 0, \gamma_P(u_i) = 0,\ \delta_P(u_i) = 0$;
*Case*2: or $\alpha_P(u_i) = 0,\ \beta_P(u_i) = 1, \gamma_P(u_i) = 0,\ \delta_P(u_i) = 0$;
*Case*3: or $\alpha_P(u_i) = 0,\ \beta_P(u_i) = 0, \gamma_P(u_i) = 1,\ \delta_P(u_i) = 0$;
*Case*4: or $\alpha_P(u_i) = 0,\ \beta_P(u_i) = 0, \gamma_P(u_i) = 0,\ \delta_P(u_i) = 1$.
In all of these cases $\widetilde{K}_1(P) = 1$ indicates that P is a crisp set.

Contrary, we suppose P is a crisp set; then, we have,
*Case*1: Either, $\alpha_P(u_i) = 1,\ \beta_P(u_i) = 0, \gamma_P(u_i) = 0,\ \delta_P(u_i) = 0$;
*Case*2: or $\alpha_P(u_i) = 0,\ \beta_P(u_i) = 1, \gamma_P(u_i) = 0,\ \delta_P(u_i) = 0$;
*Case*3: or $\alpha_P(u_i) = 0,\ \beta_P(u_i) = 0, \gamma_P(u_i) = 1,\ \delta_P(u_i) = 0$;
*Case*4: or $\alpha_P(u_i) = 0,\ \beta_P(u_i) = 0, \gamma_P(u_i) = 0,\ \delta_P(u_i) = 1$.
In all the cases, we have $\widetilde{K}_1(P) = 1$.
Thus, $\widetilde{K}_1(P) = 1$ if and only if P is a crisp set.

PF-$\widetilde{K}_1^2$(Minimality): $\widetilde{K}_1(P)$ assumes a unique minimum if $\alpha_P(u_i) = \beta_P(u_i) = \gamma_P(u_i) = \delta_P(u_i) = \frac{1}{4}$.

Since, $\left(\alpha_P(u_i)\right)^3 + \left(\beta_P(u_i)\right)^3 + \left(\gamma_P(u_i)\right)^3 + \left(\delta_P(u_i)\right)^3 - \alpha_P(u_i)\beta_P(u_i)\gamma_P(u_i) - \beta_P(u_i)\gamma_P(u_i)\delta_P(u_i) - \alpha_P(u_i)\delta_P(u_i)\beta_P(u_i) - \alpha_P(u_i)\gamma_P(u_i)\delta_P(u_i) = 1$.

To obtain the minimum value of the proposed function, we apply Lagrange's method of the undetermined multiplier. Thus, the Lagrangian of the proposed measure defined in Eq. (3) is written as,

$$\psi(\alpha_P,\beta_P,\gamma_P,\ \delta_P) = \phi(\alpha_P,\beta_P,\gamma_P,\delta_P) + \lambda\,\xi(\alpha_P,\beta_P,\gamma_P,\delta_P). \quad (5)$$

where, $\xi(\alpha_P,\ \beta_P,\ \gamma_P,\ \delta_P) = \alpha_P(u_i) + \beta_P(u_i) + \gamma_P(u_i) + \delta_P(u_i) - 1$ and λ is an undetermined multiplier. To calculate the smallest value of ψ, we differentiate Equation (5) partially concerning $\alpha_P,\ \beta_P,\ \gamma_P,\ \delta_P$ and by putting the partial derivative obtained is equal to zero, we find $\alpha_P(u_i) = \beta_P(u_i) = \gamma_P(u_i) = \delta_P(u_i) = \frac{1}{4}$ as a stationary point.

Now, we establish the stationary point as a point of convexity of the hypersurface $\psi(\alpha_P,\ \beta_P,\ \gamma_P,\ \delta_P)$ to prove the minimality of $\psi(\alpha_P,\ \beta_P,\ \gamma_P,\ \delta_P)$.

We use Definition 4 to find the convexity of the hypersurface $\psi(\alpha_P,\ \beta_P,\ \gamma_P,\ \delta_P)$.

The value of $\widetilde{HEN}(\psi(\alpha_P,\ \beta_P,\ \gamma_P,\ \delta_P))$ computed at the stationary point $\alpha_P(u_i) = \beta_P(u_i) = \gamma_P(u_i) = \delta_P(u_i) = \frac{1}{4}$ is given by

$$\widetilde{HEN}(\psi) = 2 \times \mathrm{M},$$

where $M = \frac{1}{4}\begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$ is an idempotent matrix with one Eigenvalue '0' and three.

Eigenvalues '1', which implies that *M* is a positive semi-definite matrix. Thus, $\psi(\alpha_P,\ \beta_P,\ \gamma_P,\delta_P)$ is a convex function with a stationary point $\alpha_P = \beta_P = \gamma_P = \delta_P = \frac{1}{4}$.
Hence, proves the condition of minimality for $\widetilde{K}_1(P)$.

PF-$\widetilde{K}_1^3$(Resolution): Let *P* is crisper than *Q*,
i.e., $\alpha_P(u_i) \le \alpha_Q(u_i)$, $\beta_P(u_i) \le \beta_Q(u_i)$ and $\gamma_P(u_i) \le \gamma_Q(u_i)$ for $\max\{\alpha_Q(u_i), \beta_Q(u_i), \gamma_Q(u_i)\} \le \frac{1}{4}$. In this case, $\delta_P(u_i) \ge \delta_Q(u_i) \ge \frac{1}{4}$.

or $\alpha_P(u_i) \ge \alpha_Q(u_i)$, $\beta_P(u_i) \ge \beta_Q(u_i)$ and $\gamma_P(u_i) \ge \gamma_Q(u_i)$ for $\min\{\alpha_Q(u_i), \beta_Q(u_i), \gamma_Q(u_i)\} \ge \frac{1}{4}$. In this case, $\delta_P(u_i) \le \delta_Q(u_i) \le \frac{1}{4}$.

For the first case, we observe that $\widetilde{K}_1(P)$ is decreasing function in the first three variables $\alpha_p(u_i), \beta_p(u_i), \gamma_p(u_i)$ and increasing function in the fourth variable $\delta_P(u_i)$. Consequently, $\widetilde{K}_1(P) \ge \widetilde{K}_1(Q)$.

For the second case, we observe that $\widetilde{K}_1(P)$ is increasing function in the first three variables $\alpha_p(u_i), \beta_p(u_i), \gamma_p(u_i)$ and decreasing function in the fourth variable $\delta_P(u_i)$. Consequently, $\widetilde{K}_1(P) \ge \widetilde{K}_1(Q)$.

Thus, $\widetilde{K}_1(P) \ge \widetilde{K}_1(Q)$, whenever *P* is crisper than *Q*.
This satisfies PF-$\widetilde{K}_1^3$.

PF-$\widetilde{K}_1^4$(Symmetry): This follows directly from the axiom PF-$\widetilde{K}_1^4$ of Definition 13, that is, $\widetilde{K}_1(P) = \widetilde{K}_1(P^c)$.
Hence proves the theorem.

We now prove some properties of the proposed knowledge measure.

Theorem 2: For any two $P_1, P_2 \in PFS(U)$ such that either $P_1 \subseteq P_2$ or $P_2 \subseteq P_1$, then,

$$\widetilde{K}_1(P_1 \cup P_2) + \widetilde{K}_1(P_1 \cap P_2) = \widetilde{K}_1(P_1) + \widetilde{K}_1(P_2).$$

Proof: Let $P_1, P_2 \in PFS(U)$ such that,
$P_1 = \{u_i \in U, i = 1,2, \dots, n : P_1 \subseteq P_2\}$,
$P_2 = \{u_i \in Ui = 1,2, \dots, n : P_2 \subseteq P_1\}$,
Then for all $u_i \in P_1$, $\alpha_{P1}(u_i) \le \alpha_{P2}(u_i)$, $\beta_{P1}(u_i) \le \beta_{P2}(u_i)$, $\gamma_{P1}(u_i) \ge \gamma_{P2}(u_i)$ and for all $u_i \in P_2$, $\alpha_{P2}(u_i) \le \alpha_{P1}(u_i)$, $\beta_{P2}(u_i) \le \beta_{P1}(u_i)$, $\gamma_{P2}(u_i) \ge \gamma_{P1}(u_i)$.
Using Eq. (3), we have,

$$\widetilde{K}_1(P_1 \cup P_2) = \frac{1}{n}\sum_{i=1}^{n}\Big[\left(\alpha_{P_1\cup P_2}(u_i)\right)^3 + \left(\beta_{P_1\cup P_2}(u_i)\right)^3 + \left(\gamma_{P_1\cup P_2}(u_i)\right)^3 + \left(\delta_{P_1\cup P_2}(u_i)\right)^3 - \alpha_{P_1\cup P_2}(u_i)\beta_{P_1\cup P_2}(u_i)\gamma_{P_1\cup P_2}(u_i) - \beta_{P_1\cup P_2}(u_i)\gamma_{P_1\cup P_2}(u_i)\delta_{P_1\cup P_2}(u_i) - \beta_{P_1\cup P_2}(u_i)\alpha_{P_1\cup P_2}(u_i)\delta_{P_1\cup P_2}(u_i) - \alpha_{P_1\cup P_2}(u_i)\gamma_{P_1\cup P_2}(u_i)\delta_{P_1\cup P_2}(u_i)\Big].$$

$$\widetilde{K}_1(P_1 \cup P_2) = \frac{1}{n}\sum_{i=1}^{n}\Big[\left(\alpha_{P_2}(u_i)\right)^3 + \left(\beta_{P_2}(u_i)\right)^3 + \left(\gamma_{P_2}(u_i)\right)^3 + \left(\delta_{P_2}(u_i)\right)^3 - \alpha_{P_2}(u_i)\beta_{P_2}(u_i)\gamma_{P_2}(u_i) - \beta_{P_2}(u_i)\gamma_{P_2}(u_i)\delta_{P_2}(u_i) - \alpha_{P_2}(u_i)\delta_{P_2}(u_i)\beta_{P_2}(u_i) - \alpha_{P_2}(u_i)\gamma_{P_2}(u_i)\delta_{P_2}(u_i)\Big] \quad (6)$$

Similarly,

$$\widetilde{K}_1(P_1 \cap P_2) = \frac{1}{n}\sum_{i=1}^{n}\Big[\left(\alpha_{P_1}(u_i)\right)^3 + \left(\beta_{P_1}(u_i)\right)^3 + \left(\gamma_{P_1}(u_i)\right)^3 + \left(\delta_{P_1}(u_i)\right)^3 - \alpha_{P_1}(u_i)\beta_{P_1}(u_i)\gamma_{P_1}(u_i) - \beta_{P_1}(u_i)\gamma_{P_1}(u_i)\delta_{P_1}(u_i) - \alpha_{P_1}(u_i)\delta_{P_1}(u_i)\beta_{P_1}(u_i) - \alpha_{P_1}(u_i)\gamma_{P_1}(u_i)\delta_{P_1}(u_i)\Big] \quad (7)$$

From Eq. (6) and (7), we get,

$$\widetilde{K}_1(P_1 \cup P_2) + \widetilde{K}_1(P_1 \cap P_2) = \widetilde{K}_1(P_1) + \widetilde{K}_1(P_2).$$

Hence proves the theorem.

Remark 5: The result in Theorem 2 established that the proposed knowledge measure $\widetilde{K}_1$ satisfies the valuation property. Thus, we can generate similarity measures between two PFSs from the proposed knowledge measure $\widetilde{K}_1$.

In the next Theorem, we propose a knowledge measure of a picture fuzzy set wherein refusal is not considered.

Theorem 3: Let P be a picture fuzzy set, then $\widetilde{K}_2(P)$ defined in Eq. (8) is a knowledge measure of a picture fuzzy set.

$$\widetilde{K}_2(P) = \frac{1}{n}\sum_{i=1}^{n}\Big[\left(\alpha_P(u_i)\right)^2 + \left(\beta_P(u_i)\right)^2 + \left(\gamma_P(u_i)\right)^2 - \alpha_P(u_i)\beta_P(u_i) - \beta_P(u_i)\gamma_P(u_i) - \gamma_P(u_i)\alpha_P(u_i)\Big] \quad (8)$$

Proof: The validity of the proposed knowledge measure $\boldsymbol{\widetilde{K}_2(P)}$ can be tested as that of $\boldsymbol{\widetilde{K}_1(P)}$.

The following theorems obtain other PF-knowledge measures using the existing ones.

Theorem 4: If $\widetilde{K}_1(P)$ is a knowledge measure of a picture fuzzy set, then $\breve{K}(P) = \frac{2\widetilde{K}_1(P)}{1+\widetilde{K}_1(P)}$ is also a knowledge measure of a picture fuzzy set.

Proof: It is easy to prove (PF-$\widetilde{K}_1^1$), (PF-$\widetilde{K}_1^2$), and (PF-$\widetilde{K}_1^4$).
To verify (PF-$\widetilde{K}_1^3$), we consider the generating function for $\breve{K}(P)$ is $\acute{f}(p) = \frac{2p}{1+p}$ for all $p \in [0,1]$, having $\frac{d\acute{f}}{dp} = \frac{2}{(1+p)^2} > 0$ for all $p \in [0,1]$. It means that $\acute{f}(p)$ is an increasing function. Since, $\widetilde{K}_1(P)$ is a knowledge measure of PFS, then by using PF-$\widetilde{K}_1^3$, we have $\widetilde{K}_1(P) \geq \widetilde{K}_1(Q)$.

Consequently, $\breve{K}(P) \geq \breve{K}(Q)$.
Which proves the axiom (PF-$\widetilde{K}_1^3$).

Hence, $\breve{K}(P)$ is a valid knowledge measure of picture fuzzy set *P*.

Theorem 5: If $\widetilde{K}_2(P)$ is a knowledge measure of a picture fuzzy set, then $\breve{K}(P) = \frac{2\widetilde{K}_2(P)}{1+\widetilde{K}_2(P)}$ is again a knowledge measure of a picture fuzzy set.
Proof: Similar as carried out in Theorem 4.

In the next section, we contrast the performance of the proposed knowledge measures with some existing measures.

4. Comparative Analysis

In this section, we evaluate the performance of our proposed PF-knowledge measures $\widetilde{K}_1$ and $\widetilde{K}_2$ defined in Eq. (3) and Eq. (8) respectively. To demonstrate the efficacy and performance of our proposed PF-knowledge measures, we compare them with existing fuzzy entropy/knowledge measures. We present prominent existing PF entropy measures in Table 1.

Table 1. Existing entropy/knowledge measures.

Entropy/knowledge measures	Reference
$K_{SK2} = \frac{1}{n}\sum_{i=1}^{n}\left(\max\{\lvert\alpha_P(u_i)\rvert^p, \lvert\beta_P(u_i)\rvert^p, (1- \pi_P(u_i))^p\}\right)$	Das et al. (2018)
$K_{SKB} = 1 - \frac{1}{2n}\sum_{i=1}^{n}\left(\frac{\min\{\alpha_P(u_i),\beta_P(u_i)\} + \pi_P(u_i)}{\max\{\alpha_P(u_i),\beta_P(u_i)\} + \pi_P(u_i)} + \pi_P(u_i)\right)$	Szmidt et al. (2014)
$K_{BB} = 1 - \frac{1}{n}\sum_{i=1}^{n} \pi_P(u_i)$	Burillo and Bustince (1996)
$K_{HC}^{\alpha} = \begin{cases} 1 - \sum_{i=1}^{n}\frac{1-\alpha_P(u_i)^v - \beta_P(u_i)^v - \pi_P(u_i)}{(\alpha-1)n} \\ 1 + \frac{1}{n}\sum_{i=1}^{n}\left(\alpha_P(u_i)\log(\alpha_P(u_i)) + \beta_P(u_i)\log(\beta_P(u_i)) + \pi_P(u_i)\log(\pi_P(u_i))\right) \end{cases}$	Hung and Yang (2006)

Table 1 continued…

$K_R^{\sigma} = 1 - \frac{1}{n}\sum_{i=1}^{n}\frac{\log(\alpha_P(u_i)^{\sigma} + \alpha_P(u_i)^{\sigma} + \alpha_P(u_i)^{\sigma})}{1-\sigma}$	Hung and Yang (2006)
$\tilde{E}_n(P) = 1 - \frac{1}{2n}\sum_{i=1}^{n}(\lvert\alpha_P(u_i) - \beta_P(u_i)\rvert + \lvert\alpha_P(u_i) - \gamma_P(u_i)\rvert + \lvert\beta_P(u_i) - \gamma_P(u_i)\rvert)$	Wang et al. (2018)
$e(P) = -\frac{1}{n}\sum_{i=1}^{n}[\alpha_P(u_i)\log_2(\alpha_P(u_i)) + \beta_P(u_i)\log_2(\beta_P(u_i)) + \gamma_P(u_i)\log_2(\gamma_P(u_i)) + \delta_P(u_i)\log_2(\delta_P(u_i))]$	Arya and Kumar (2020)
$\tilde{\psi}(P) = \frac{1}{n(1-v)}\sum_{i=1}^{n}\{(\alpha_P(u_i)^{v} + \beta_P(u_i)^{v} + \gamma_P(u_i)^{v} + \delta_P(u_i)^{v}) - 1\}, \quad v > 0(\neq 1)$	Joshi (2020b)
$\psi(P) = \frac{R}{n(R-1)}\sum_{i=1}^{n}\{1 - (\alpha_P(u_i)^{R} + \beta_P(u_i)^{R} + \gamma_P(u_i)^{R} + \delta_P(u_i)^{R})\}, \quad R > 0(\neq 1)$	Joshi (2020a)

Now, we perform a numerical experiment to show the advantage of the proposed knowledge measure by comparing it with the existing knowledge measures given in Table 1. For this, we consider the following numerical Example 2.

Example 2: Take into account the following picture fuzzy sets in a universe $U = \{u\}$ given in Table 2.

Table 2. Different picture fuzzy sets.

	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8
Picture fuzzy sets	$\{(u,1,0,0)\}$	$\{(u,0.9,0,0.1)\}$	$\{(u,0.8,0,0.1)\}$	$\{(u,0.8,0,0.2)\}$	$\{(u,0.7,0,0.3)\}$	$\{(u,0.6,0,0.4)\}$	$\{(u,0.5,0,0.4)\}$	$\{(u,0.5,0,0.5)\}$

It is clear that $0 = \beta_{F1}(u_i) = \beta_{F2}(u_i) = \beta_{F3}(u_i) < \beta_{F4}(u_i) < \beta_{F5}(u_i) < \beta_{F6}(u_i) = \beta_{F7}(u_i) < \beta_{F8}(u_i) = \gamma_{F1}(u_i) < \gamma_{F2}(u_i) = \gamma_{F3}(u_i) < \gamma_{F4}(u_i) < \gamma_{F5}(u_i) < \gamma_{F6}(u_i) = \gamma_{F7}(u_i) < \gamma_{F8}(u_i) = \alpha_{F8}(u_i) = \alpha_{F7}(u_i) < \alpha_{F6}(u_i) < \alpha_{F5}(u_i) < \alpha_{F4}(u_i) = \alpha_{F3}(u_i) < \alpha_{F2}(u_i) < \alpha_{F1}(u_i) = 1$.

Therefore, the right order of knowledge and entropy should be

$$\tilde{K}(F_1) > \tilde{K}(F_2) > \tilde{K}(F_3) > \tilde{K}(F_4) > \tilde{K}(F_5) > \tilde{K}(F_6) > \tilde{K}(F_7) > \tilde{K}(F_8) \quad (9)$$

and

$$\tilde{E}(F_1) < \tilde{E}(F_2) < \tilde{E}(F_3) < \tilde{E}(F_4) < \tilde{E}(F_5) < \tilde{E}(F_6) < \tilde{E}(F_7) < \tilde{E}(F_8) \quad (10)$$

Various entropy and knowledge measures are utilized to compare the performance of the proposed knowledge measures. The accuracy of each information measure is calculated according to Eq. (11) (Zhang et al., 2019).

$$\text{Accuracy}(\%) = \frac{\text{Count}\left(\text{Right order of}\frac{\tilde{\text{K}}(F_i)}{\tilde{\text{E}}(F_i)}\right)}{\text{No.}(F_i)}\text{x}100 \quad (11)$$

The accuracy variable refers to the percentage of values that were properly ranked across all knowledge/entropy measures. The highest accuracy depicts the better measure. The results for the picture fuzzy sets in Table 2 are generated due to various measures shown in Table 3.

From Table 3, results illustrate that all the knowledge/entropy measures without the order property in Eq. (9) and Eq. (10) do not meet the property (PF-$\tilde{\text{K}}$3). Also, the accuracy of the proposed knowledge measure $\widetilde{K}_1$ is the same as that of $K_{HC}^1, K_{r^{0.5}}$ and $\tilde{\psi}(P)$. However, the order of the results generated due to the proposed knowledge measure $\widetilde{K}_2$ is completely correct and indicates the highest level of accuracy than other measures. Thus, from Table 3, the proposed knowledge measure $\widetilde{K}_2$ is better than all other measures.

Table 3. Comparative results.

Measures	1	2	3	4	5	6	7	8	Right or Wrong	Number of wrongs	Accuracy (%)
K_{SK2}	1	1	0.81	1	1	1	0.81	1	Wrong	8	0
K_{SKB}	1	0.944	0.839	0.875	0.786	0.667	0.533	0.5	Wrong	1	87.5
K_{BB}	1	1	1	0.9	1	1	0.9	1	Wrong	8	0
K_{HC}^1	1	0.859	0.722	0.783	0.735	0.708	0.590	0.699	Wrong	2	75
$K_{r^{0.5}}$	1	0.796	0.632	0.745	0.717	0.703	0.562	0.699	Wrong	2	75
$e(P)$	0	0.8485	1.1662	1.1314	1.2961	1.3865	1.5232	1.4142	Wrong	3	62.5
$\tilde{\psi}(P)$	0	0.1800	0.3400	0.3200	0.4200	0.4800	0.5800	0.5000	Wrong	2	75
$\psi(P)$	*NaN*	*NaN*	*NaN*	*NaN*	*NaN*	*NaN*	*NaN*	*NaN*	Wrong	8	0
Proposed $\widetilde{K}_1$	1	0.7300	0.5060	0.5200	0.3700	0.2800	0.1700	0.2500	Wrong	2	75
Proposed $\widetilde{K}_2$	1	0.7300	0.5700	0.5200	0.3700	0.2800	0.2100	0.2500	Right	0	100

In Table 3, *NaN* represents that the concerned measure provides a result as not a number.

A MADM problem is one of the most widely studied problems receiving a lot of attention. In the next section, we present some novel methods and apply our proposed knowledge measures to investigate their potential applications.

5. Applications

In this section, we investigate the applications of the proposed knowledge measure and present a novel method for MADM.

5.1 PF Knowledge-based Attribute Weight Computation in MADM Problem with Conflicting Attributes

In a MADM problem, we provide a set of alternatives as well as a set of attributes. Sometimes the attributes are conflicting, *i.e.*, an attitude toward one attribute may lead to a loss in the significance of the other attribute. In such a scenario, selecting an effective attribute weight computation model is unavoidable to deliver the appropriate decision-making result. After analyzing the factors, we must choose the most desirable alternative. Weights are assigned by decision experts or approximated using a model to evaluate the importance of attributes. The use of entropy measures is frequent to estimate attribute weights. However, attribute weights based on entropy are not always suitable; hence, some alternative model is required. We consider some examples of attribute weight computation in a PF environment to investigate the underlying fact.

***Example* 3:** Consider the set of three alternatives $P_i (i = 1,2,3)$ and three attributes Q_j $(j = 1,2,3)$ in PF-decision matrix.

$$A_1 = \begin{matrix} P_1 \\ P_2 \\ P_3 \end{matrix}\begin{pmatrix} Q_1 & Q_2 & Q_3 \\ \langle 0.1, 0.3, 0.1\rangle & \langle 0.5, 0.2, 0.3\rangle & \langle 0.4, 0.4, 0.1\rangle \\ \langle 0.4, 0.2, 0.1\rangle & \langle 0.1, 0.4, 0.2\rangle & \langle 0.6, 0.1, 0.3\rangle \\ \langle 0.2, 0.1, 0.5\rangle & \langle 0.3, 0.4, 0.1\rangle & \langle 0.1, 0.5, 0.1\rangle \end{pmatrix}.$$

Example 4: Consider another set of three alternatives $P_i(i = 1,2,3)$and three attributes Q_j $(j = 1,2,3)$ in a PF-decision matrix.

$$A_2 = \begin{matrix} \\ P_1 \\ P_2 \\ P_3 \end{matrix} \begin{pmatrix} Q_1 & Q_2 & Q_3 \\ \langle 0.2, 0.3, 0.1 \rangle & \langle 0.5, 0.2, 0.3 \rangle & \langle 0.4, 0.4, 0.1 \rangle \\ \langle 0.4, 0.2, 0.1 \rangle & \langle 0.1, 0.4, 0.2 \rangle & \langle 0.4, 0.1, 0.2 \rangle \\ \langle 0.2, 0.1, 0.5 \rangle & \langle 0.3, 0.4, 0.1 \rangle & \langle 0.1, 0.4, 0.1 \rangle \end{pmatrix}.$$

Assume a MADM circumstance having n alternatives P_i $(i = 1,2,\dots,n)$ and m attributes Q_j $(j = 1,2,\dots,m)$. The following strategy (Wang et al. 2018) is used to compute the weight of each attribute.

$$w_j = \frac{1-\tilde{E}(Q_j)}{\sum_{j=1}^{m}\left(1-\tilde{E}(Q_j)\right)}, \quad j = 1,2,\dots,m \tag{12}$$

Here, $\tilde{E}$ represents the PF-entropy measure.

Now, by considering Examples 3 and 4 and utilizing the PF-entropy-based attribute weight method in Eq. (12), the weight of all the attributes is listed in Tables 3 and 4, respectively.

Table 3. Computed attribute weights using different measures for Example 3.

Entropy/Knowledge measure	Weights corresponding to different attributes	Accuracy (%)
$\tilde{E}_n(P)$	$\boldsymbol{Q_1 = 0.30}$, $\boldsymbol{Q_2 = 0.30}$, $Q_3 = 0.40$	33.3
$e(P)$	$Q_1 = NaN$, $Q_2 = NaN$, $Q_3 = NaN$	0
$\tilde{\psi}(P)$	$\boldsymbol{Q_1 = 0.34}$, $\boldsymbol{Q_2 = 0.34}$, $Q_3 = 0.30$	33.3
$\psi(P)$	$\boldsymbol{Q_1 = 0.31}$ $\boldsymbol{Q_2 = 0.31}$, $Q_3 = 0.36$	33.3
$\tilde{K}_1$	$Q_1 = 0.28$, $Q_2 = 0.24$, $Q_3 = 0.47$	100
$\tilde{K}_2$	$Q_1 = 0.26, Q_2 = 0.23, Q_3 = 0.29$	100

In Table 3, bold values indicate the same weight to two distinct attributes and *NaN* indicates that the concerned measure provides a result as not a number.

Table 4. Computed attribute weights using different measures for Example 4.

Entropy/Knowledge measure	Weights corresponding to different attributes	Accuracy (%)
$\tilde{E}_n(P)$	$\boldsymbol{Q_1 = 0.33}$, $\boldsymbol{Q_2 = 0.33}$, $\boldsymbol{Q_3 = 0.33}$	0
$e(P)$	$Q_1 = NaN, Q_2 = NaN, Q_3 = NaN$	0
$\tilde{\psi}(P)$(Joshi 2020b)	$\boldsymbol{Q_1 = 0.33}, \boldsymbol{Q_2 = 0.33}, \boldsymbol{Q_3 = 0.33}$	0
$\psi(P)$	$Q_1 = 0.32$, $\boldsymbol{Q_2 = 0.33}$, $\boldsymbol{Q_3 = 0.33}$	33.3
$\tilde{K}_1$	$Q_1 = 0.29, Q_2 = 0.34, Q_3 = 0.36$	100
$\tilde{K}_2$	$Q_1 = 0.33$, $Q_2 = 0.30$, $Q_3 = 0.36$	100

In Table 4, bold values indicate the same weight to two distinct attributes and *NaN* indicates that the concerned measure provides a result as not a number.

From Tables 3 and 4, the attribute weights using different PF-entropy measures give identical weights to two distinct attributes and the entropy measure $e(P)$ is unable to compute the attribute weight. As a result, innovative approaches are always required to calculate attribute weights more effectively in MADM.

Now, we present a new method for determining attribute weights based on our suggested PF-knowledge measure. The new technique is defined as follows.

Knowledge-based attribute weights (Proposed)

$$w_j = \frac{\tilde{K}(Q_j)}{\sum_{j=1}^{m}\left(\tilde{K}(Q_j)\right)}, \; j = 1,2,\dots,m \tag{13}$$

Here, $\widetilde{K}$ represents the PF-knowledge measure.

$$\text{Accuracy}(\%) = \frac{\text{Count(No.of distinct weight to attributes)}}{\text{Total no. of attributes}} \text{x}100 \quad (14)$$

Now, we employ our proposed method and proposed knowledge measure to calculate the attribute weights for Examples 3 and 4 and the corresponding results are shown in Tables 3 and 4. From Tables 3 and 4, the proposed knowledge measure $\widetilde{K}_1$ and $\widetilde{K}_2$ assigns different weights to different attributes, making it useful when entropy-based weights produce erroneous findings. In comparison to the existing PF-entropy measure, the accuracy level of each measure is calculated using Eq. (14) and highlighted in a separate column of Table 3 and Table 4. It can be noticed from both Tables (3 and 4) that the accuracy level of the proposed knowledge measure is higher than the existing measures. Therefore, the problem of weight computation that cannot be resolved by the existing measure is effectively resolved by the proposed methodology. This demonstrates the advantage of the proposed methodology.

Now, we employ the proposed methodology of weight computation in a MADM algorithm, which is described as follows.

5.2 MADM- Algorithm

Here, we present a novel method for MADM defined as follows.

Scenario

Given a collection of *n* alternatives P_i $(i = 1,2,\dots,n)$ and *m* attributes $Q_j (j = 1,2,\dots,m)$ as well as the weight vector of an attribute $w = (w_1, w_2, \dots, w_m)$, $0 \le w_j \le 1, j = 1, 2, \dots, m$ and $\sum_{j=1}^{m} w_j = 1$.

Aim

Choosing the best alternative because of a given knowledge base.

The methodology for choosing the best option is as follows:

Algorithm

Step 1: Build the PF-decision matrix $Z = [z_{ij}]_{n\times m}$ in which $z_{ij} = (\alpha_{ij}, \beta_{ij}, \gamma_{ij})$ is a picture fuzzy value where α_{ij} denotes membership degree of alternative P_i under attribute Q_j , γ_{ij} denotes the non-membership degree of alternative P_i under attribute Q_j and β_{ij} denotes the neutrality degree of alternative P_i under attribute Q_j.

Step 2: Determine the normalized form of the decision matrix $S = [s_{ij}]_{n\times m}$ where

$$s_{ij} = \begin{cases} z_{ij} = (\alpha_{ij}, \beta_{ij}, \gamma_{ij}), \text{for benefit factors/attributes,} \\ (z_{ij})^c = (\gamma_{ij}, t_{ij}, \alpha_{ij}), \text{for cost factor/attribute.} \end{cases}$$

Step 3: Use one of the following techniques to determine the attribute weight.

(a) Entropy-based attribute weights computed using Eq. (12) (Wang et al., 2018).

(b) Knowledge-based attribute weights computed using Eq. (13) (Proposed).

Step 4: Evaluate each alternative's overall preference value $P_i, i = 1, 2, \dots, n$ using the (Wei 2016) picture fuzzy weighted averaging (PFWA) technique defined in Eq. (15).

$$PFWA(t_1, t_2, \dots, t_m) = \left(1 - \prod_{j=1}^{m}\left(1 - \alpha_{t_j}\right)^{w_j}, \prod_{j=1}^{m}\left(\beta_{t_j}\right)^{w_j}, \ \prod_{j=1}^{m}\left(\gamma_{t_j}\right)^{w_j}\right) \quad (15)$$

where $t_j = \left(\alpha_{l_j}, \beta_{l_j}, \gamma_{l_j}\right)$ represent a picture fuzzy value (PFV).

***Step* 5:** Using the score function developed by (Wang et al., 2017), determine each possible score with the help of Eq. (16).

$$SC(P_i) = \alpha_{P_i} - \beta_{P_i}, \ i = 1, 2, \dots, n \tag{16}$$

***Step* 6:** Rank/Arrange the alternatives by their score values in decreasing order.
Figure 2 depicts the flowchart for the above-mentioned MADM technique.

Now, in a picture fuzzy environment, we use the abovementioned approach to solve a numerical example concerning a MADM problem.

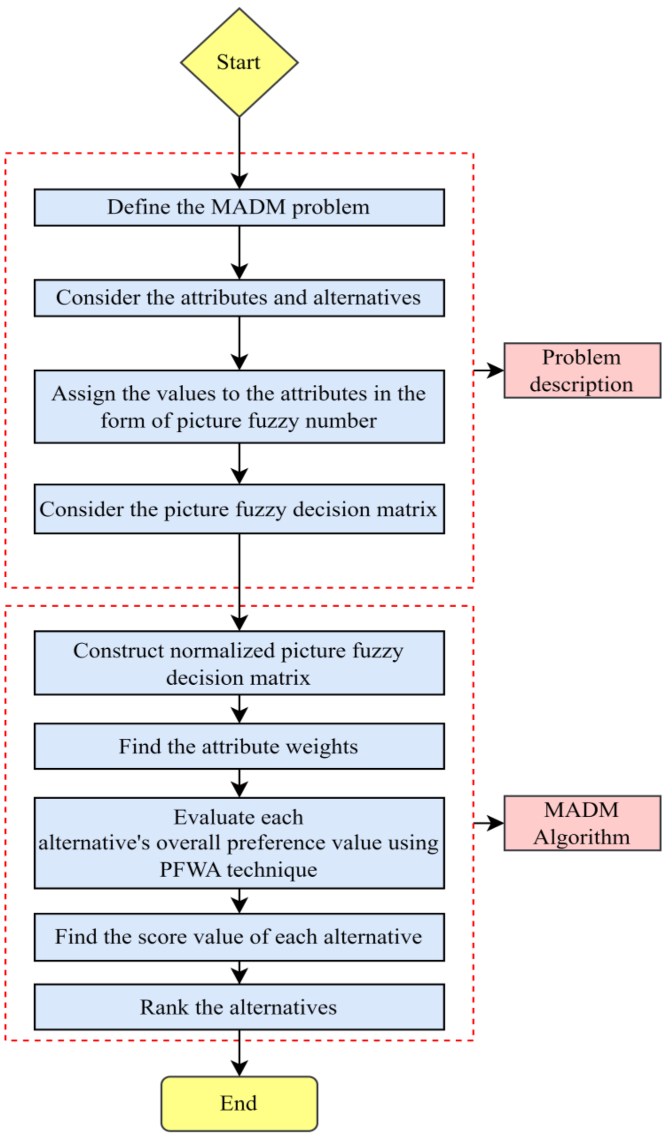

Figure 2. Flow chart of the MADM method.

5.3 Numerical Example

***Example* 5:** Consider the difficulty of choosing a solar panel from a set of five leading solar manufacturers in the world: (P_1) Company A, (P_2) Company B (P_3) Company C, (P_4) Company D and (P_5) Company E. A decision maker considers four attributes listed below while evaluating these five solar panels: (Q_1) Durability, (Q_2) Price, (Q_3) Silicon grade and (Q_4) Material used. A brief description of the problem is shown in Figure 3.

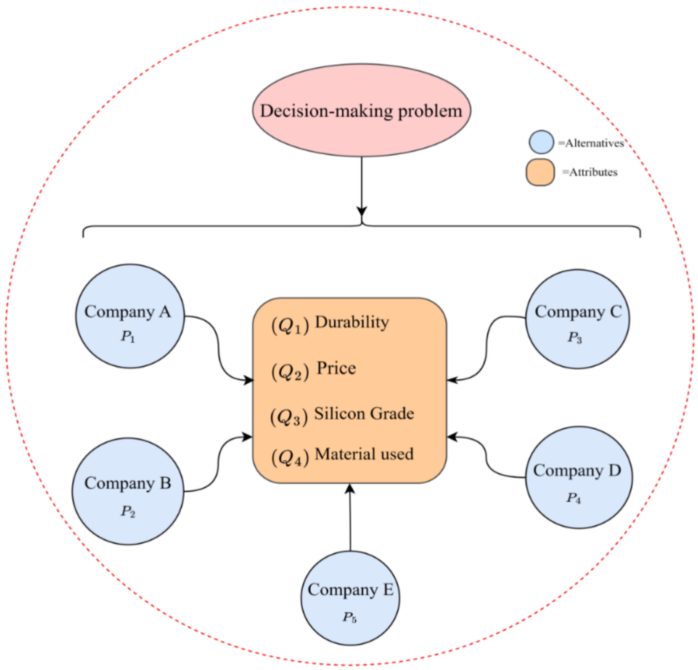

Figure 3. Brief description of the MADM problem in Example 5.

***Step* 1:** A decision expert provides the evaluation values of the five solar panels under the four attributes in the form of $PFVs$. Table 5 shows the results as a PF-decision matrix.

Table 5. PF-decision matrix indicates the evaluation values of solar panels for Example 5.

	Q_1	Q_2	Q_3	Q_4
P_1	$\langle 0.62, 0.23, 0.15\rangle$	$\langle 0.20, 0.56, 0.12\rangle$	$\langle 0.51, 0.15, 0.34\rangle$	$\langle 0.47, 0.10, 0.33\rangle$
P_2	$\langle 0.50, 0.22, 0.28\rangle$	$\langle 0.18, 0.43, 0.25\rangle$	$\langle 0.64, 0.19, 0.17\rangle$	$\langle 0.54, 0.17, 0.29\rangle$
P_3	$\langle 0.58, 0.30, 0.12\rangle$	$\langle 0.08, 0.60, 0.32\rangle$	$\langle 0.80, 0.05, 0.15\rangle$	$\langle 0.62, 0.28, 0.11\rangle$
P_4	$\langle 0.61, 0.26, 0.13\rangle$	$\langle 0.10, 0.58, 0.22\rangle$	$\langle 0.67, 0.07, 0.26\rangle$	$\langle 0.55, 0.18, 0.27\rangle$
P_5	$\langle 0.65, 0.25, 0.10\rangle$	$\langle 0.37, 0.50, 0.13\rangle$	$\langle 0.47, 0.18, 0.35\rangle$	$\langle 0.76, 0.24, 0.00\rangle$

***Step* 2:** Q_2 is the only cost attribute. We create the normalized form of the PF-decision matrix as in Table 6.

Table 6. Normalized PF-decision matrix.

Alternatives\Attributes	Q_1	Q_2	Q_3	Q_4
P_1	$\langle 0.62, 0.23, 0.15\rangle$	$\langle 0.56, 0.20, 0.12\rangle$	$\langle 0.51, 0.15, 0.34\rangle$	$\langle 0.47, 0.10, 0.33\rangle$
P_2	$\langle 0.50, 0.22, 0.28\rangle$	$\langle 0.43, 0.18, 0.25\rangle$	$\langle 0.64, 0.19, 0.17\rangle$	$\langle 0.54, 0.17, 0.29\rangle$
P_3	$\langle 0.58, 0.30, 0.12\rangle$	$\langle 0.60, 0.08, 0.32\rangle$	$\langle 0.80, 0.05, 0.15\rangle$	$\langle 0.62, 0.28, 0.11\rangle$
P_4	$\langle 0.61, 0.26, 0.13\rangle$	$\langle 0.58, 0.10, 0.22\rangle$	$\langle 0.67, 0.07, 0.26\rangle$	$\langle 0.55, 0.18, 0.27\rangle$
P_5	$\langle 0.65, 0.25, 0.10\rangle$	$\langle 0.50, 0.37, 0.13\rangle$	$\langle 0.47, 0.18, 0.35\rangle$	$\langle 0.76, 0.24, 0.00\rangle$

***Step* 3:** We compute the knowledge measure-based attribute weights in Table 7. We use the proposed PF-knowledge measure given in Eq. (3) and Eq. (8) for computation.

Table 7. Attribute weights using our proposed knowledge measures.

Knowledge measure↓ Weights→	w_1	w_2	w_3	w_4
$\tilde{E}_n(P)$	0.24	0.22	**0.26**	**0.26**
$e(P)$	*NaN*	*NaN*	*NaN*	*NaN*
$\tilde{\psi}(P)$	0.25	0.28	0.21	0.24
$\psi(P)$	**0.25**	0.21	0.27	**0.25**
$\widetilde{K}_1$	0.24	0.18	0.31	0.26
$\widetilde{K}_2$	0.22	0.19	0.31	0.26

In Table 7, bold values indicate the same weight to two distinct attributes and *NaN* indicates that the concerned measure provides a result as not a number.

***Step* 4:** Table 8 shows the overall preference value for each alternative.

Table 8. Alternative's overall preference value.

Measures	Alternatives	Overall preference value	Measures	Alternatives	Overall preference value
$\tilde{E}_n(P)$	P_1	⟨0.5339,0.1655,0.2253⟩	$\psi(P)$	P_1	⟨0.5350,0.1664,0.2258⟩
	P_2	⟨0.5312,0.1953,0.2483⟩		P_2	⟨0.5329,0.1959,0.2473⟩
	P_3	⟨0.6603,0.1416,0.1609⟩		P_3	⟨0.6623,0.1411,0.1599⟩
	P_4	⟨0.5986,0.1399,0.2202⟩		P_4	⟨0.6001,0.1399,0.2189⟩
	P_5	⟨0.6096,0.2624,0.0000⟩		P_5	⟨0.6079,0.2604,0.0000⟩
$e(P)$	P_1	-	$\widetilde{K}_1$	P_1	⟨0.5352,0.1605,0.2323⟩
	P_2	-		P_2	⟨0.5444,0.1925,0.2402⟩
	P_3	-		P_3	⟨0.6749,0.1349,0.1532⟩
	P_4	-		P_4	⟨0.6068,0.1343,0.2187⟩
	P_5	-		P_5	⟨0.6162,0.2507,0.0000⟩
$\tilde{\psi}(P)$	P_1	⟨0.5388,0.1705,0.2100⟩	$\widetilde{K}_2$	P_1	⟨0.5300, 0.1627, 0.2326⟩
	P_2	⟨0.5237,0.1922,0.2482⟩		P_2	⟨0.5406, 0.1951, 0.2430⟩
	P_3	⟨0.6478,0.1433,0.1691⟩		P_3	⟨0.6722, 0.1347, 0.1580⟩
	P_4	⟨0.5946,0.1421,0.2163⟩		P_4	⟨0.6028, 0.1348, 0.2243⟩
	P_5	⟨0.6064,0.2727,0.0000⟩		P_5	⟨0.6057, 0.2551, 0.0000⟩

***Step* 5:** Table 9 presents the alternative's score value.

Table 9. Score values of the alternatives.

Measures	Score values of the alternatives				
	P_1	P_2	P_3	P_4	P_5
$\tilde{E}_n(P)$	0.3684	0.3359	0.5187	0.4587	0.3472
$e(P)$	-	-	-	-	-
$\tilde{\psi}(P)$	0.3683	0.3315	0.5045	0.4525	0.3337
$\psi(P)$	0.3686	0.3370	0.5212	0.4602	0.3475
$\widetilde{K}_1$	0.3747	0.3519	0.5400	0.4725	0.3655
$\widetilde{K}_2$	0.3673	0.3455	0.5375	0.4880	0.3506

***Step* 6:** Table 10 presents the rank of all the alternatives in decreasing order of their score values.

From Table 10, it can be seen that the entropy measure $e(P)$ is unable to give the ranking results. The other entropy measures and proposed knowledge measures suggest P_3 as the best alternative. We also noticed that the ranking pattern is the same for all the measures. Thus, the proposed technique for weight computation is compatible with the existing entropy-based attribute weight computation in a MADM problem. Therefore, we can use knowledge-based attribute weights instead of entropy-based attribute weights, whenever the entropy-based method gives unacceptable findings.

Table 10. Results of ranking.

Measures	Ranking
$\tilde{E}_n(P)$	$P_3 > P_4 > P_1 > P_5 > P_2$
$e(P)$	-
$\tilde{\psi}(P)$	$P_3 > P_4 > P_1 > P_5 > P_2$
$\psi(P)$	$P_3 > P_4 > P_1 > P_5 > P_2$
$\widetilde{K}_1$	$P_3 > P_4 > P_1 > P_5 > P_2$
$\widetilde{K}_2$	$P_3 > P_4 > P_1 > P_5 > P_2$

6. Conclusion

The axiomatic framework of an entropy or knowledge measure in standard and non-standard fuzzy environments includes the resolution property. The interpretation of resolution in the higher dimension (more aspects of vagueness) is very cumbersome. In this article, we have generalized the resolution (PF-$\widetilde{K}_1^1$) to picture fuzzy framework, and successfully proposed an axiomatic framework to define picture fuzzy knowledge measures. Furthermore, we defined two picture fuzzy knowledge measures and validated them using this framework. The proposed knowledge measure was found to have better accuracy in resolving the picture fuzziness of two sets (Section 3). Since our proposed knowledge measures have a better capability of resolution of picture fuzziness, hence they perform more accurately in the attribute weight computation process in MADM problems (Section 5.2). Moreover, in the MADM problem, the consistency of the ranking results with existing picture fuzzy measures confirms the credibility of the proposed measures.

A picture fuzzy set consists of four dimensions of vagueness: membership, non-membership, neutrality, and refusal. We have theoretically investigated the resolution property in four dimensions and defined the picture fuzzy knowledge measure. This study further seems to be significant for explorations of more dimensions of vagueness and subsequent formulations of information-theoretic measures.

The drawback of this research is that the numerical examples have been worked out using artificially generated data. An analysis of the proposed measures with real data can make this study more impactful. Additionally, because of the present work, the possibility of obtaining a class of similarity and dissimilarity measures between picture fuzzy sets using the proposed knowledge measure may be investigated. Also, some more theoretical interconnections among various picture fuzzy information measures may be studied in the future.

Conflict of interest
The authors declare that there is no conflict of interest.

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