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Dynamic Advertising-based Goodwill Incorporating Fuzzy Environment in Segment-Specific Market

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Abstract
A new product's introduction to the market is greatly influenced by effective and enough advertising. In this paper, we have considered a problem in which the firm partitions the market into various segments to reduce costs associated with advertising and targets to maximize the total profit. By integrating single channel and differentiated advertising to segmented market, we extend Nerlove-goodwill Arrow's dynamic model in which advertising variables are control variables. It is assumed that the whole available budget is imprecise and fuzzy in nature in order to create a realistic model. Using necessity and possibility constraints, the optimal control model with fuzzy parameters is transformed into crisp form, and the Pontryagin Maximum principle is then used to solve the problem. Numerical examples are provided to support the theoretical analysis.

Keywords- Fuzzy parameter, Goodwill, Budget constraint, Maximum principle, Optimal control problem.

1. Introduction

Marketing seeks to maximise profit by creating value and a stock of goodwill for the firm, which is impossible to achieve if one wishes to survive with existing products and services. In the competitive global economy of today, every business aims to surpass its competitors. By creating customer relationships that are rich in value, marketing seeks to profitably increase customer happiness as consumer preferences are constantly shifting. Since there is a large amount of risk involved for the firm when introducing a new product to the market, market penetration is crucial for the product's future dissemination. A firm must decide the group of consumers who would be interested in purchasing the product, to whom a new product should be targeted. The target market refers to this particular consumer group. Market segmentation is the first stage in achieving this goal. It is carried out by dividing the entire market into different consumer groups, each of which is defined by a unique set of attribute values and whose members have the same requirements and behaviours. Specific products and marketing strategies are needed for each consumer category. Geographic variables (for example as Nations, states, countries, cities, and region etc.), Demographic variables (like income, gender, age and education etc.), Psychographic variables that include personality, social class and lifestyle etc., and Behavioural variables (for example as Usage rate, purchase occasion and attitude towards product etc.) are some of the key variables used for segmentation (Kotler, 2003). Following market segmentation, the firm must choose which market segment the product should be pitched to and how to reach each group utilising the available marketing mix. The focus of many marketing initiatives is on advertising strategies that encourage the introduction of new products and have a favourable impact on the diffusion curve. In actuality, the target market may consist of a variety of customer groups that can be addressed through a variety of advertising channels.

To obtain optimal techniques to control a dynamic system led to the development of the mathematical field known as optimal control theory. The exceptionally broad range of conceivable applications for optimal control is one of its most intriguing characteristics. The dynamic behaviour of sales-advertising control analysis makes the application of optimal control theory viable. When it comes to studying the behaviour of sales-advertising systems where dynamic aspects are crucial, continuous optimal control models are a valuable tool.

As a dynamic process, goodwill formation provides an opportunity to explore the dynamic nature of advertising process by incorporating single channel and differentiated advertising into an optimal control model of goodwill formation. The evolution of the product's goodwill stock in the market is mostly dependent on the advertising efforts. The demands and attributes of the target audience must be considered when deciding how much money will be spent on advertising. In this paper, an advertising model for a product is considered. The market is divided into different groups and advertising is specific to the groups. The expenses for advertising are classified into two categories. One for the individual segments and one at the single channel level with fixed proportions for the segments. The concept of goodwill as given by Nerlove and Arrow (1962) is incorporated and the revenue it generates contributes to the total profit. It is assumed that the stock of goodwill depends on the level of advertising made. To achieve a more realistic model, the budget available for advertising is taken as a fuzzy parameter. This parameter is represented as triangular fuzzy number. The budget constraint is converted to a crisp one by the use of concept of necessity and possibility as given by Dubois and Prade (1988, 1997) and Liu and Iwamura (1998a).

The next Section 2 explain literature review. The basic concept and definitions that are related to fuzzy number are described in Section 3. Section 4 provides notations and the development of the proposed model. Section 5 describes local stability analysis in the state and adjoint phase plane. In section 6, numerical examples and sensitivity analysis are provided. Section 7 provides result and discussion of the proposed model. Managerial implication is discussed in Section 8 and finally conclusion is described in Section 9.

2. Literature Review

Nerlove and Arrow (1962) have put forward a pricing and advertising model with the introduction of concept of goodwill associated with the sales of product. Buratto and Viscolani (2002) have applied the goodwill concept given by Nerlove and Arrow in advertising models, aiming at minimizing the time and cost of launching a new product. A similar application of the concept of goodwill has been witnessed in the study of an advertising model by Grosset and Viscolani (2009). Nair and Narasimhan (2006) have developed a model which studies the effect of factors like advertising and product quality on the formation of goodwill. The problem of advertising and production for a seasonal product in a segmented market incorporating the concept of goodwill, has been taken up by Favaretto and Viscolani (2010). An optimal control model of product goodwill is given by Gorajski and Machowska (2017). A goodwill based dynamic model which focusses on decision making in the area of advertising and film making has been considered by Feng and Liu (2018). Effect of advertising on the goodwill of companies is given by Machowska (2019).

Chenavaz and Eynan (2021) have analyzed how goodwill and advertising influence the presence of Veblen effect. De Giovanni (2020) has considered a model formulated to develop advertising strategies for a firm in view of the fact that defects in products may cause the goodwill to deteriorate. Kumar et al. (2022) proposed a model to determine the optimal price and advertising strategy that achieves the maximum integrated profit for a product in segmented market involving price dependent market potential. Chaudhary et al. (2022) have addressed the optimal control problem to determine the optimal promotional policies of a diffusion model in segmented market under the assumption that the new product's additional demand improves by brand image of the firm in the form of goodwill. Little and Lodish (1969) have proposed a model for formulating advertising policies with different media options in a segmented market. A sales and advertising model has been considered by Seidman et al. (1987), where the population is divided into segments and advertising is done at different levels. Various advertising models have been proposed by Buratto et al. (2006a, 2006b) and Jha et al. (2009), which assume the market to be partitioned into homogeneous groups. Innovation-diffusion models designed by Huang et al. (2012) and Helmes et al. (2013) are for markets that are dynamic and segmented.

Optimizing promotional efforts in a segmented market has been dealt with the model given by Mehta et al. (2020), where the potential customers are assumed to be dynamic in nature. The above-mentioned authors have assumed that the parameters considered are deterministic. In real world problems this may not be the situation. There may be some parameters which are vaguely defined. Zadeh (1965) has given the concepts of fuzzy sets and introduced the concepts of fuzzy numbers and intervals used for modelling such parameters. Researchers in the past have given models where some parameters are fuzzy in nature, thereby making them more realistic. A fuzzy optimal control problem has been given by Filev and Angelov (1992) and Zhu (2009) in which the objective function is taken to be fuzzy in nature. The models given by Maity and Maiti (2005, 2007) and Mandal et al. (2010) deal with production inventory systems in which the holding and production costs are represented by fuzzy numbers. Later Zhu (2011) considered a multistage fuzzy optimal control problem. Ramezanzadeh and Heydari (2011) have considered an optimal control problem with constraints having fuzzy parameters. Roul et al. (2015, 2019) have considered fuzzy optimal control problems with fuzzy budget and fuzzy costs. Optimal Control problem which uses fuzzy intervals to represent fuzziness has been studied by Campos et al. (2020).

2.1 Research Gap and Motivation

To the best of our knowledge, the advertising models in literature have considered various situations but, budget has not been a variant and various other parameters are also considered as fixed while developing advertising strategies. Fuzziness has also been introduced in various studies relating to inventory and production and innovation diffusion. However, in this paper we have discussed an advertising-based goodwill model in which the of a new product is studied in an environment where advertising efforts are distributed over various market segments. Moreover, the total available budget for advertising is one of the variants and is additionally considered to be imprecise in nature, like in most real-world problems. In this paper, real-life possibility or necessity constraints in the context of marketing problem under advertising budget constraint novel aspect which has not seen in the recent research. The proposed work's motivations are derived from the current literature. In this section, we point out the identified gaps that are found from the review of the literature. Following are the gaps identified in the related work:

- From the methodological point of view, Nerlove-goodwill Arrow's dynamic model did not consider market segmentation. Therefore, there is a strong need to extend the model when the market is segmented.

- Nerlove-goodwill Arrow's dynamic model consider only advertising as a control variable and did not provide any information/optimal policy when there is single channel and differentiated advertising policy adopted.
- Past research has mostly assumed that the stock of goodwill depends on the level of advertising made and did not provide the path to handle the situation when the decision makers are not completely sure about the amount of money that may be set aside for advertisement of his product.

In this paper we have filled the above-mentioned gaps in the literature by developing a model which is the extension of Nerlove-goodwill Arrow's dynamic model.

2.2 Research Contributions
The key research contributions of this study are:
- Tackles with the situation when the decision maker is not completely sure about the amount of money that may be set aside for advertisement of his product. Under these circumstances, use of fuzzy parameters has been made to come up with a feasible solution.
- Focuses on the optimal advertising policies of the diffusion model in a segment-specific market.
- Focuses on single channel and differentiated advertising efforts to improve the goodwill of the firm.

3. Basic Concept and Definitions
Triangular Fuzzy Number: A triangular fuzzy number $\tilde{A}$ in R is a fuzzy number which is denoted by a triplet (a_1, a_2, a_3) and whose membership function takes the following form:

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & x < a_1 \\ \frac{x-a_1}{a_2-a_1}, & a_1 \le x \le a_2 \\ \frac{a_3-x}{a_3-a_2}, & a_2 < x \le a_3 \\ 0, & x > a_3 \end{cases}$$

The graphical depiction of the membership function is given by a triangle as shown in Figure 1.

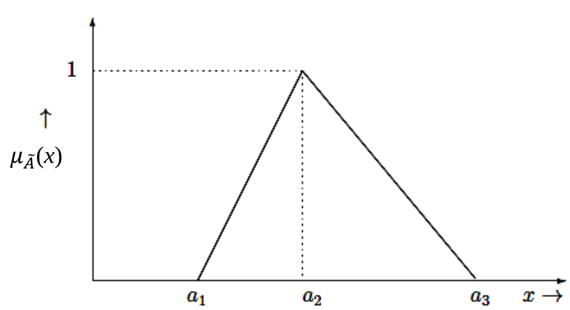

Figure 1. Membership function for the triangular fuzzy number $\tilde{A}$.

Possibility and Necessity on Fuzzy Numbers
Researchers (Dubois and Prade, 1988, 1997; Liu and Iwamura, 1998a, 1998b; Zadeh, 1999) have given the concept of Possibility and Necessity of fuzzy events. For two fuzzy numbers $\tilde{A}$ and $\tilde{B}$ in R with membership functions $\mu_{\tilde{A}}$ and $\mu_{\tilde{B}}$, the Possibility of a certain event is given as follows:

$$\text{Pos}(\tilde{A} \le \tilde{B}) = sup\{min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)): x, y \in R, x \le y\}.$$

The above possibility for a crisp real number $\tilde{A} = a$ becomes

$$\text{Pos}\,(a \leq \tilde{B}) = sup\{\, \mu_{\tilde{B}}(y):\ y \in \mathrm{R},\, a \leq y\}.$$

Necessity for a certain event in the fuzzy environment is defined as follows

$$\text{Nes}\,(\tilde{A} * \tilde{B}) = inf\,\{max(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)): x, y \in \mathrm{R},\, x * y\}.$$

where, * represents any one of the relational operators <, ≤, =, >, ≥.

Necessary and possible events share a dual relationship. This gives us the following result

$$\text{Nes}\,(\tilde{A} * \tilde{B}) = 1 - \text{Pos}\,(\overline{\tilde{A} * \tilde{B}}).$$

Programming Problem in Fuzzy Environment

We now consider a programming problem which has some fuzzy parameters.

Maximize Z

subject to $g \leq \tilde{Y}$

where, Z denotes the objective function and g is the constraint involves a fuzzy parameter $\tilde{Y}$.

The constraints are required to be converted to their crisp versions. For this the techniques given by Liu and Iwamura (1998a, 1998b) are used. The above problem can be rewritten in the crisp form using the Possibility and necessity constraints as

Maximize Z

subject to $Nes\{a \leq \tilde{Y}\} \geq \eta_1$ and/or $\text{Pos}\{a \leq \tilde{Y}\} \geq \eta_2$.

where, η_1 and η_2 denote predetermined confidence levels for the original fuzzy constraint.

4. Optimal Control Model and Framework

In this section we explain all the used notations and the development of advertising-based goodwill problem.

4.1 Notations

The notations used in the paper are as follows:

M	:	the number of market segments and a discrete variable.
$u_i(t)$	:	the differentiated advertising effort rate for ith segment at time t.
$u(t)$	:	the single channel market advertising effort rate at time t.
α_i	:	the segment spectrum of advertising for ith segment.
ϵ_i, ϵ	:	Magnitude of advertising effort rates for differentiated and single channel.
$\phi_i\big(u_i(t)\big) = \frac{\epsilon_i}{2} u_i^2(t)$	:	the differentiated market advertising effort cost for ith segment.
$\varphi\big(u(t)\big) = \frac{\epsilon}{2} u^2(t)$	:	the single channel market advertising effort cost.
$G_i(t)$	:	the stock of goodwill of the product at time $t \in [0, T]$ for the i^{th} segment.
δ_i	:	the goodwill depreciation rate for the i^{th} segment.
π_i	:	the revenue of goodwill for the i^{th} segment.
$\widetilde{W}$	:	fuzzy variable representing the total available budget.

4.2 Proposed Advertising-based Goodwill Problem in Fuzzy Environment
4.2.1 Goodwill Dynamics
Here it is assumed that the monopolistic firm considered has a target market in segment specific population and carry out an advertising process for a new product introduction in different segment. As given by Buratto et al. (2006a), Fruchter et al. (2006) and Chaudhary et al. (2022), Kumar et al. (2023), it is suggested that goodwill needs an advertising effort to increase and subjected to decay. Here, it is assumed that the stock of goodwill improves with the increase in single channel and differentiated advertising efforts and reduces by a depreciation rate δ_i. The evolution of goodwill dynamics is given by

$$\frac{dG_i(t)}{dt} = \left(u_i(t) + \alpha_i\, u(t)\right) - \delta_i G_i(t) \tag{1}$$

Equation (1) represents that the rate of stock of goodwill is increases by current advertising efforts of differentiated and single channel advertising process and depreciates with constant rate δ_i over time.

4.2.2 Advertising Cost
The objective involves maximizing the stock of goodwill for new product introduction while minimizing the total advertising cost. The firm could start advertising the new product introduction by using single channel and differentiated advertising process. This type of advertising process is costly, and that advertising cost is quadratic. Unlike the cost functions of advertising effort in other advertising models in the marketing literature (Teng and Thompson, 1983; Jorgensen et al., 2006; Helmes et al., 2013; Chaudhary et al., 2022), we assume that advertising costs are the functions of advertising effort rates and described by the quadratic cost functions:

$$c_i\left(u_i(t)\right) = \frac{\epsilon_i}{2} u_i^2(t),\ c\left(w(t)\right) = \frac{\epsilon}{2} u^2(t)\ \ where\ \epsilon_i, \epsilon > 0 \tag{2}$$

where, $\epsilon_i, \epsilon > 0$ are constants and denote the magnitude of the advertising effort rate for differentiated and single channel process.

4.2.3 Fuzzy Budget Constraint
The total available budget may not be crisply defined. This may be due to some unforeseen expenses, fluctuation in interest rates or unstable market conditions. Due to this the decision maker may use some fuzzy sets to denote the budget parameter. In this model, a triangular fuzzy number is used to denote the budget as can also be seen in Maity and Maiti (2007). Since we assume M market segments and quadratic advertising costs, the budget constraint can be written as

$$\int_0^T \left(\sum_{i=1}^{M} \frac{\epsilon_i}{2} u_i^2(t) + \frac{\epsilon}{2} u^2(t)\right) dt\ \le \widetilde{W} \tag{3}$$

where, $\widetilde{W} = (W_1, W_2, W)$ is a triangular fuzzy number that denotes the imprecise budget.

4.2.4 Optimal Fuzzy Control Problem
Assuming that planning period is finite $[0, T]$ and the firm seek an advertising effort policy which leads to maximal value of total profit. The revenue of goodwill contributes to the net profit. Based on above assumptions, the optimal fuzzy control problem is formulated to obtain the optimal advertising effort rates $u_i(t)$ and $u(t)$ for all segments as follows:

$$\left.\begin{aligned} &Max\, J = \int_0^T \left[\sum_{i=1}^M \left(\pi_i\big(G_i(t)\big) - \frac{\epsilon_i}{2} u_i^2(t)\right) - \frac{\epsilon}{2} u^2(t)\right] dt \\ &\text{subject to} \qquad \frac{dG_i(t)}{dt} = \big(u_i(t) + \alpha_i\, u(t)\big) - \delta_i G_i(t) \\ &\qquad \int_0^T \left(\sum_{i=1}^M \frac{\epsilon_i}{2} u_i^2(t) + \frac{\epsilon}{2} u^2(t)\right) dt \le \widetilde{W} \\ &u_i^l \le u_i(t) \le u_i^u, \forall\, i = 1,2, \dots M;\ u^l \le u(t) \le u^u \end{aligned}\right\} \tag{4}$$

In the above proposed optimal fuzzy control model, the control variables are advertising effort rates $u_i(t)$ and $u(t)$; the state variables are the stock of goodwill $G_i(t)$. As a result of the characteristics of definite integrals, we obtain from Equation (3),

$$\sum_{i=1}^M \frac{\epsilon_i}{2} u_i^2(t) + \frac{\epsilon}{2} u^2(t) \le \frac{\widetilde{W}}{T} \tag{5}$$

The fuzzy constraint (3) represents the fuzzy relations. There are several ways to represent fuzzy relations. These relationships are interpreted in the context of possibility theory (Dubois and Prade,1983), where fuzzy numbers will interpret the degree of uncertainty. The constraint (5) is reduced to necessity and possibility constraints (Liu and Iwamura, 1998a, 1998b).

$$Nes\left\{\sum_{i=1}^M \frac{\epsilon_i}{2} u_i^2(t) + \frac{\epsilon}{2} u^2(t) < \frac{\widetilde{W}}{T}\right\} \ge \eta_1.$$

$$Pos\left\{\sum_{i=1}^M \frac{\epsilon_i}{2} u_i^2(t) + \frac{\epsilon}{2} u^2(t) \le \frac{\widetilde{W}}{T}\right\} \ge \eta_2.$$

4.2.5 Equivalent Crisp Optimal Control Problem

The above model assumes the budget for advertising to be represented by a triangular fuzzy number.

$$Let\ \widetilde{W} = (W_1, W_2, W_3),\ \ then\ \frac{\widetilde{W}}{T} = \left(\frac{W_1}{T}, \frac{W_2}{T}, \frac{W_3}{T}\right) = (W_1', W_2', W_3').$$

The constraint is transformed into a crisp one by using the concept of necessity and possibility and thereby the problem represented by (4)-(5) reduces to its equivalent crip version as follows

$$Max\, J = \int_0^T \left[\sum_{i=1}^M \left(\pi_i\big(G_i(t)\big) - \frac{\epsilon_i}{2} u_i^2(t)\right) - \frac{\epsilon}{2} u^2(t)\right] dt \tag{6}$$

subject to constraint (1) and $u_i^l \le u_i(t) \le u_i^u, \forall\, i = 1,2, \dots M;\ u^l \le u(t) \le u^u$ for all scenarios and

for scenario 1 (Necessity Constraint):

$$Nes\left\{\sum_{i=1}^M \frac{\epsilon_i}{2} u_i^2(t) + \frac{\epsilon}{2} u^2(t) < \frac{\widetilde{W}}{T}\right\} \ge \eta_1 \tag{7}$$

which can be written in the crisp form as

$$\frac{\sum_{i=1}^M \frac{\epsilon_i}{2} u_i^2(t) + \frac{\epsilon}{2} u^2(t) - W_1'}{W_2' - W_1'} \le 1 - \eta_1.$$

Scenario 2 (Possibility Constraint)

$$Pos\left\{\sum_{i=1}^M \frac{\epsilon_i}{2} u_i^2(t) + \frac{\epsilon}{2} u^2(t) \le \frac{\widetilde{W}}{T}\right\} \ge \eta_2 \tag{8}$$

which can be written in the crisp form as

$$\frac{W_3' - \left(\sum_{i=1}^{M} \frac{\epsilon_i}{2} u_i^2(t) + \frac{\epsilon}{2} u^2(t)\right)}{W_3' - W_2'} \geq \eta_2.$$

4.3 Mathematical Approach to Solve the Crisp Optimal Control Problem

To solve the above problem in the two scenarios, we use the technique of optimal control theory. The mathematical technique of optimization used to find optimal policy to control a dynamical system is known as optimal control theory and important area of dynamic optimization. Many operational research fields, including economics, finance, inventory production and natural resource consumption, have applied optimal control theory (Nerlove and Arrow, 1962; Davis and Elzinga, 1971; Pierskalla and Voelker, 1976; Derzko and Sethi, 1981; Chaudhary et al., 2011; Singh et al., 2013; Kumar and Sahni, 2016). The Hamiltonian function for the above optimal control problem is

$$H = \sum_{i=1}^{M} \left(\pi_i\big(G_i(t)\big) - \frac{\epsilon_i}{2} u_i^2(t)\right) - \frac{\epsilon}{2} u^2(t) + \sum_{i=1}^{M} \lambda_i(t) \frac{dG_i}{dt} \tag{9}$$

where, λ_i are adjoint variables associated with their state equations. The corresponding Lagrangian function for Scenario 1

$$L = H + \mu\left[(1-\eta_1)W_2' + \eta_1 W_1' - \left(\sum_{i=1}^{M} \frac{\epsilon_i}{2} u_i^2(t) + \frac{\epsilon}{2} u^2(t)\right)\right] \tag{10}$$

and for Scenario 2

$$L = H + \mu\left[(1-\eta_2)W_3' + \eta_2 W_2' - \left(\sum \frac{\epsilon_i}{2} u_i^2(t) + \frac{\epsilon}{2} u^2(t)\right)\right] \tag{11}$$

Now, the adjoint functions are defined by the following differential equation

$$\frac{d}{dt}\lambda_i(t) = -\left[\frac{\partial \pi_i}{\partial G_i} + \delta_i \lambda_i(t)\right],\ \lambda_i(T) = 0 \tag{12}$$

Solution of the first order differential equation (12) is given by

$$\lambda_i(t) = \int_t^T \left(\frac{\partial \pi_i}{\partial G_i}\right) e^{-\delta_i(t-\tau)} d\tau \tag{13}$$

Unlike the rate of profit margin gross of goodwill in Sethi's (1977) and Buratto et al. (2006a, 2006b), in this study we assume that $\pi_i(G_i(t)) = \theta_i(G_i(t))^{\beta}\ ; 0 < \beta \leq 1$. Where, $\theta_i > 0$ are given parameters for each segment. Now, Equation (13) becomes

$$\lambda_i(t) = \theta_i \beta_i \int_t^T \big(G_i(\tau)\big)^{(\beta_i - 1)} e^{-\delta_i(t-\tau)} d\tau \tag{14}$$

The shadow price of stock of goodwill $\lambda_i(t)$ is the net result of the effects of advertising adjustment to stock of goodwill. The Pontryagin's maximum principle states that Lagrangian function should be maximised with respect to the control variables $u_i(t)\ \forall\, i = 1,2, \dots M$ and $u(t)$. The necessary optimality conditions are $\frac{\partial L}{\partial u_i(t)} = 0$ and $\frac{\partial L}{\partial u(t)} = 0$. Thus,

Now for $u_i(t)$

$$\frac{\partial L}{\partial u_i(t)} = \lambda_i(t) - \epsilon_i u_i(t)(1+\mu).$$

and

$$\frac{\partial^2 L}{\partial u_i^2(t)} = -\epsilon_i(1+\mu) < 0.$$

Hence

$$\frac{\partial L}{\partial u_i(t)} = 0 \quad \Rightarrow \quad u_i(t) = \frac{\lambda_i(t)}{\epsilon_i\,(1+\mu)} \tag{15}$$

Here, following three cases can arise

Case 1: If $\frac{\partial L}{\partial u_i(t)} > 0$, then Lagrangian function is an increasing function of advertising effort rate $u_i(t)$ and optimal solution of $u_i^*(t) = Minimum\left\{\frac{\lambda_i(t)}{\epsilon_i\,(1+\mu)}, u_i^u\right\}$

Case 2: If $\frac{\partial L}{\partial u_i(t)} = 0$, then optimal advertising effort rate is $u_i^*(t) = \frac{\lambda_i(t)}{\epsilon_i\,(1+\mu)}$

Case 3: If $\frac{\partial L}{\partial u_i(t)} < 0$, then Lagrangian function is a decreasing function of advertising effort rate $u_i(t)$ and optimal solution of $u_i^*(t) = Maximum\left\{\frac{\lambda_i(t)}{\epsilon_i\,(1+\mu)}, u_i^l\right\}$

For $u(t)$

$$\frac{\partial L}{\partial u(t)} = \sum_{i=1}^{M} \alpha_i \lambda_i(t) - \epsilon\, u(t)(1+\mu).$$

$$\frac{\partial^2 L}{\partial u^2(t)} = -\epsilon\,(1+\mu) < 0.$$

Hence

$$\frac{\partial L}{\partial u(t)} = 0 \Rightarrow \quad u(t) = \frac{\sum \alpha_i \lambda_i(t)}{\epsilon\,(1+\mu)} \tag{16}$$

Similarly, three cases can arise for $u(t)$

Case 1: If $\frac{\partial L}{\partial u(t)} > 0$, then Lagrangian function is an increasing function of advertising effort rate $u(t)$ and optimal solution of $u^*(t) = Minimum\left\{\frac{\sum \alpha_i \lambda_i(t)}{\epsilon\,(1+\mu)}, u^u\right\}$

Case 2: If $\frac{\partial L}{\partial u(t)} = 0$, then optimal advertising effort rate is $u^*(t) = \frac{\sum \alpha_i \lambda_i(t)}{\epsilon\,(1+\mu)}$

Case 3: If $\frac{\partial L}{\partial u(t)} < 0$, then Lagrangian function is a decreasing function of advertising effort rate $u(t)$ and optimal solution of $u^*(t) = Maximum\left\{\frac{\sum \alpha_i \lambda_i(t)}{\epsilon\,(1+\mu)}, u^l\right\}$

The Lagrange multiplier μ must satisfy the complementary slackness conditions $\mu \geq 0$,
As well as for Scenario 1

$$\mu\left[(1-\eta_1)W_2' + \eta_1 W_1' - \left(\sum_{i=1}^{M} \frac{\epsilon_i}{2} u_i^2(t) + \frac{\epsilon}{2} u^2(t)\right)\right] = 0.$$

And for Scenario 2

$$\mu\left[(1-\eta_2)W_3' + \eta_2 W_2' - \left(\sum \frac{\epsilon_i}{2} u_i^2(t) + \frac{\epsilon}{2} u^2(t)\right)\right] = 0.$$

Case 1: If $\mu = 0$

$$u_i{}^*(t) = \frac{\lambda_i(t)}{\epsilon_i} \tag{17}$$

and

$$u^*(t) = \frac{\sum \alpha_i \lambda_i(t)}{\epsilon} \tag{18}$$

Case 2: If $\mu > 0$, it implies for

Scenario 1:

$$(1-\eta_1)W_2' + \eta_1 W_1' - \left(\sum_{i=1}^{M} \frac{\epsilon_i}{2} u_i^2(t) + \frac{\epsilon}{2} u^2(t)\right) = 0.$$

$$\Rightarrow \sum_{i=1}^{M} \frac{\epsilon_i}{2} u_i^2(t) + \frac{\epsilon}{2} u^2(t) = (1-\eta_1)W_2' + \eta_1 W_1'.$$

Substituting the values of $u_i(t)$ and $u(t)$ from (15) and (16) in above equation, Thus,

$$(1+\mu) = \left(\frac{\sum_{i=1}^{M} \frac{\lambda_i^2(t)}{2\epsilon_i} + \frac{1}{2}\frac{[\sum \alpha_i \lambda_i(t)]^2}{\epsilon}}{(1-\eta_1)W_2' + \eta_1 W_1'}\right)^{\frac{1}{2}} \tag{19}$$

Using (19), the optimal advertising effort rates for differentiated and single-channel advertising can be written as

$$u_i^*(t) = \frac{\lambda_i(t)}{\epsilon_i \left(\frac{\sum_{i=1}^{M} \frac{\lambda_i^2(t)}{2\epsilon_i} + \frac{1}{2}\frac{[\sum \alpha_i \lambda_i(t)]^2}{\epsilon}}{(1-\eta_1)W_2' + \eta_1 W_1'}\right)^{\frac{1}{2}}} \tag{20}$$

and

$$u^*(t) = \frac{\sum \alpha_i \lambda_i(t)}{\epsilon \left(\frac{\sum_{i=1}^{M} \frac{\lambda_i^2(t)}{2\epsilon_i} + \frac{1}{2}\frac{[\sum \alpha_i \lambda_i(t)]^2}{\epsilon}}{(1-\eta_1)W_2' + \eta_1 W_1'}\right)^{\frac{1}{2}}} \tag{21}$$

Scenario 2:

$$(1-\eta_2)W_3' + \eta_2 W_2' - \left(\sum \frac{\epsilon_i}{2} u_i^2(t) + \frac{\epsilon}{2} u^2(t)\right) = 0.$$

$$\Rightarrow \sum_{i=1}^{M} \frac{\epsilon_i}{2} u_i^2(t) + \frac{\epsilon}{2} u^2(t) = (1-\eta_2)W_3' + \eta_2 W_2'.$$

Similarly, using values of $u_i(t)$ and $u(t)$ from (15) and (16) in above equation, we obtain

$$(1+\mu) = \left(\frac{\sum_{i=1}^{M} \frac{\lambda_i^2(t)}{2\epsilon_i} + \frac{1}{2}\frac{[\sum \alpha_i \lambda_i(t)]^2}{\epsilon}}{(1-\eta_2)W_3' + \eta_2 W_2'}\right)^{\frac{1}{2}} \tag{22}$$

Using (22), the optimal advertising effort rates for differentiated and single-channel advertising can be written as

$$u_i^*(t) = \frac{\lambda_i(t)}{\epsilon_i \left(\frac{\sum_{i=1}^{M} \frac{\lambda_i^2(t)}{2\epsilon_i} + \frac{1}{2}\frac{[\sum \alpha_i \lambda_i(t)]^2}{\epsilon}}{(1-\eta_2)w_3' + \eta_2 w_2'}\right)^{\frac{1}{2}}} \tag{23}$$

and

$$u^*(t) = \frac{\sum \alpha_i \lambda_i(t)}{\epsilon\left(\frac{\sum_{i=1}^{M} \frac{\lambda_i^2(t)}{2\epsilon_i} + \frac{1}{2}\frac{\left[\sum \alpha_i \lambda_i(t)\right]^2}{\epsilon}}{(1-\eta_2)w_3' + \eta_2 w_2'}\right)^{\frac{1}{2}}} \tag{24}$$

The equilibrium point's local stability analysis will be covered in the next section to demonstrate how the future effects on profit can be stabilized locally when small modifications are made to stock of goodwill of the product using advertising effort rates.

5. Local Stability Analysis in the State and Adjoint Phase Plane

The local stability of the model for state and adjoint equations (1) and (12) for the interior equilibrium point would be covered in this part.

Theorem: The interior equilibrium point $E^*(G^*, \lambda^*)$ is locally stable if $\delta_i^2 > \frac{1}{4}\left(\frac{1}{\epsilon_i} + \frac{\alpha_i^2}{\epsilon}\right).\theta_i \beta(1 - \beta)\left(G_i(t)\right)^{(\beta-2)}$.

Proof: Consider the following system of state and adjoint differential equations (1) and (12) using equations (15) and (16). Thus

$$\frac{dG_i(t)}{dt} = \left(\frac{\lambda_i(t)}{\epsilon_i} + \alpha_i \frac{\sum \alpha_i \lambda_i(t)}{\epsilon}\right) - \delta_i G_i(t) \tag{25}$$

and

$$\frac{d\lambda_i(t)}{dt} = -\delta_i \lambda_i(t) - \theta_i \beta\left(G_i(\tau)\right)^{(\beta_i - 1)} \tag{26}$$

Since

$$\frac{\partial \dot{G}_i}{\partial G} = -\delta_i < 0, \; \frac{\partial \dot{G}_i}{\partial \lambda_i} = \left(\frac{1}{\epsilon_i} + \frac{\alpha_i^2}{\epsilon}\right) > 0 \tag{27}$$

and

$$\frac{\partial \dot{\lambda}_i}{\partial G} = \theta_i \beta(1-\beta)\left(G_i(t)\right)^{(\beta-2)} \geq 0, 0 < \beta \leq 1; \; \frac{\partial \dot{\lambda}_i}{\partial \lambda} = -\delta_i < 0 \tag{28}$$

Using (27) and (28), the Jacobian corresponding to system with $E^*(G^*, \lambda^*)$ is as follows:

$$J = \begin{bmatrix} -\delta_i & \left(\frac{1}{\epsilon_i} + \frac{\alpha_i^2}{\epsilon}\right) \\ \theta_i \beta(1-\beta)\left(G_i(t)\right)^{(\beta-2)} & -\delta_i \end{bmatrix} \tag{29}$$

The eigen values γ_{1i}, γ_{2i} corresponding to the Jacobian are

$$\gamma_{1i} = -\delta_i + \frac{1}{2}\sqrt{\left(\frac{1}{\epsilon_i} + \frac{\alpha_i^2}{\epsilon}\right)\theta_i \beta(1-\beta)\left(G_i(t)\right)^{(\beta-2)}} \tag{30}$$

$$\gamma_{2i} = -\delta_i - \frac{1}{2}\sqrt{\left(\frac{1}{\epsilon_i} + \frac{\alpha_i^2}{\epsilon}\right)\theta_i \beta(1-\beta)\left(G_i(t)\right)^{(\beta-2)}} \tag{31}$$

The eigen values $\gamma_{1i} < 0 \;\&\; \gamma_{2i} < 0$ if $\delta_i^2 > \frac{1}{4}\left(\frac{1}{\epsilon_i} + \frac{\alpha_i^2}{\epsilon}\right)\theta_i \beta(1-\beta)\left(G_i(t)\right)^{(\beta-2)}$ would be negative according to the Routh Hurwitz criteria. This establishes the interior equilibrium point's local stability.

Case 1: When put $\alpha_i = 1$ and $\beta = \frac{1}{2}$, from Sethi (1977), then eigen values $\gamma_{2i} < 0$ and $\gamma_{1i} < 0$ if $\delta_i^2 > \frac{1}{16}\left(\frac{1}{\epsilon_i} + \frac{1}{\epsilon}\right)\theta_i\left(G_i(t)\right)^{-\frac{3}{2}}$ are negative. Hence the system is locally stable.

Case 2: When put $\alpha_i = 1$ and $\beta = 1$, then eigen values $\gamma_{2i} < 0$ and $\gamma_{1i} < 0$. Hence the system is locally stable.

6. Numerical Illustration

To illustrate the procedure, we have considered numerical examples and sensitivity analysis has been done. We assume that the time horizon is divided into 12 equal time periods. We suppose that the number of market segments is two (i.e., $M = 2$). The triangular fuzzy number representing the budget, (W_1', W_2', W_3') is taken as (400, 700, 800). The values of remaining parameters which are used throughout in this numerical illustration are given in Table 1.

Table 1. Values of parameters used in numerical examples.

Segments	α_i	δ_i	ϵ_i	ϵ	η_i	θ_i	Initial Goodwill
S1	0.4	0.015	450	800	0.044	70	10
S2	0.6	0.01	460	800	0.67	71	11

6.1 Scenario 1

Case (a): When $\beta = \frac{1}{2}$

Using the value of $\theta_1 = 70$, the values of goodwill obtained for each segment of market are given in Table 2 and Table 3.

Table 2. The optimal goodwill in segmented market.

Time Periods	T1	T2	T3	T4	T5	T6	T7	T8	T9	T10	T11	T12
$G_1(t)$	10.318	10.681	11.099	11.577	12.125	12.749	13.451	14.234	15.094	16.025	17.013	17.967
$G_2(t)$	12.233	13.457	14.674	15.882	17.083	18.276	19.460	20.635	21.799	22.951	24.090	25.181

Table 3. Optimal advertising efforts for all market segments.

Time Periods	T1	T2	T3	T4	T5	T6	T7	T8	T9	T10	T11	T12
$u_1(t)$	0.260	0.313	0.374	0.443	0.521	0.606	0.695	0.785	0.873	0.954	1	0
$u_2(t)$	1	1	1	1	1	1	1	1	1	1	1	0
$u(t)$	0.585	0.592	0.599	0.606	0.613	0.619	0.624	0.627	0.627	0.625	0.622	0

When $\beta = \frac{1}{2}$ then the optimal value of total profit is 882.69504 units for Scenario 1.

Case (b): When $\beta = 1$

Using the value of $\theta_1 = 70$, the values of goodwill obtained for each segment of market are given in Table 4 and Table 5.

Table 4. The optimal goodwill in segmented market.

Time Periods	T1	T2	T3	T4	T5	T6	T7	T8	T9	T10	T11	T12
$G_1(t)$	10.313	10.671	11.082	11.554	12.096	12.713	13.41	14.189	15.047	15.977	16.965	17.919
$G_2(t)$	12.232	13.456	14.672	15.881	17.081	18.274	19.458	20.633	21.797	22.949	24.088	25.179

Table 5. Optimal advertising efforts for all market segments.

Time Periods	T1	T2	T3	T4	T5	T6	T7	T8	T9	T10	T11	T12
$u_1(t)$	0.255	0.307	0.367	0.437	0.515	0.600	0.690	0.781	0.870	0.953	1	0
$u_2(t)$	1	1	1	1	1	1	1	1	1	1	1	0
$u(t)$	0.584	0.591	0.598	0.606	0.613	0.619	0.624	0.626	0.627	0.625	0.622	0

When $\beta = 1$ then the optimal value of total profit is 1994.07494 units for Scenario 1.

6.2 Scenario 2

Case (a): When $\beta = \frac{1}{2}$

Using the value of $\theta_2 = 71$ in scenario 2 and the values of optimal goodwill obtained for each segment of market are given in Table 6 and Table 7.

Table 6. The optimal goodwill in segmented market.

Time Periods	T1	T2	T3	T4	T5	T6	T7	T8	T9	T10	T11	T12
$G_1(t)$	10.333	10.714	11.15	11.649	12.22	12.869	13.599	14.412	15.305	16.271	17.276	18.234
$G_2(t)$	12.244	13.48	14.708	15.928	17.141	18.346	19.542	20.728	21.903	23.067	24.217	25.318

Table 7. Optimal advertising efforts for all market segments.

Time Periods	T1	T2	T3	T4	T5	T6	T7	T8	T9	T10	T11	T12
$u_1(t)$	0.269	0.323	0.386	0.458	0.539	0.626	0.718	0.811	0.901	0.985	1	0
$u_2(t)$	1	1	1	1	1	1	1	1	1	1	1	0
$u(t)$	0.604	0.611	0.619	0.626	0.634	0.640	0.645	0.647	0.648	0.646	0.642	0

When $\beta = \frac{1}{2}$ then the optimal value of total profit is 724.653385 units for Scenario 2.

Case (b): When $\beta = 1$

Using the value of $\theta_2 = 71$ in scenario 2 and the values of optimal goodwill obtained for each segment of market are given in Table 8 and Table 9.

Table 8. The optimal goodwill in segmented market.

Time Periods	T1	T2	T3	T4	T5	T6	T7	T8	T9	T10	T11	T12
$G_1(t)$	10.328	10.703	11.132	11.625	12.189	12.831	13.556	14.365	15.256	16.221	17.226	18.185
$G_2(t)$	12.244	13.479	14.707	15.927	17.139	18.343	19.539	20.725	21.901	23.064	24.214	25.316

Table 9. Optimal advertising efforts for all market segments.

Time Periods	T1	T2	T3	T4	T5	T6	T7	T8	T9	T10	T11	T12
$u_1(t)$	0.264	0.317	0.38	0.451	0.532	0.62	0.713	0.807	0.899	0.984	1	0
$u_2(t)$	1	1	1	1	1	1	1	1	1	1	1	0
$u(t)$	0.603	0.61	0.618	0.626	0.633	0.639	0.644	0.647	0.648	0.646	0.642	0

When $\beta = 1$ then the optimal value of total profit is 1864.47771 units for Scenario 2.

In Table 10, we represent the profit in both scenarios when we take $\beta = \frac{1}{2}$ and $\beta = 1$ respectively. When $\beta = \frac{1}{2}$ then the difference between scenario 1 and scenario 2 is 158.042 and when $\beta = 1$ then the difference between scenario 1 and scenario 2 is 129.597. This indicates that for both the values of β, scenario 1 generates higher profits, thereby suggesting that it is more beneficial for the decision maker to consider the constraint in the necessity form.

Table 10. Profit values.

	Scenario 1	Scenario 2
$\beta = 1/2$	882.695	724.653
$\beta = 1$	1994.075	1864.478

6.3 Sensitivity Analysis

Here, we have discussed the sensitivity analysis in Table 11 for proposed optimal control model to evaluate the used important parameters whose variation can change the value of objective function. If the values of parameter η_1 are increased/ decreased by 10%, 5%, -5%, -10%, then the values of the objective function change by 0.52%, 0.26%, -0.26% and -0.52% respectively for scenario 1. Similarly, if the values of parameter η_2 are increased/ decreased by 10%, 5%, -5%, -10%, then the values of the objective function change by 0.18%, 0.09%, -0.09% and -0.18% respectively for scenario 2.

Table 11. Variation in the value of objective function.

Scenario 1		Scenario 2	
Change of η_1 (%)	Change of J (%)	Change of η_2 (%)	Change of J (%)
10%	0.52%	10%	0.18%
5%	0.26%	5%	0.09%
0%	0	0%	0
-5%	-0.26%	-5%	-0.09%
-10%	-0.52%	-10%	-0.18%

7. Discussion

From the sensitivity analysis we see that the values of J behave linearly with respect to both η_1 and η_2. The total expenditure on advertising in the possibility and necessity constraint scenarios are given as follows:

Table 12. Total expenditure on advertising.

	Scenario 1	Scenario 2
$\beta = 1/2$	6760.5887	6893.3103
$\beta = 1$	5273.7739	5443.21

This indicates that for both the scenarios in Table 12, the capital spend on advertising lies in the interval (4800, 8400), which is the lower range of the available advertising budget. Hence the decision maker has the advantage to spend on the lower side. We also note that for both the scenarios, β=1 requires less expenditure.

7.1 Scenario 1

The optimal values of total profit for the parameters $\beta = \frac{1}{2}$ and $\beta = 1$ are 882.695 and 1994.075 units respectively in this scenario. The numerical solution for advertising efforts and brand goodwill are illustrated in Figures 2-4, respectively for the parameters $\beta = \frac{1}{2}$ and $\beta = 1$. Figure 2 represents the graph of adjoint variables for each segment for the parameters $\beta = \frac{1}{2}$ and $\beta = 1$. Figure 3 shows the optimal values of advertising effort rates $\left(u(t), u_i(t)\right)$ for both mass and differentiated advertising and optimal values of brand goodwill are shown in Figure 4 for the parameters $\beta = \frac{1}{2}$ and $\beta = 1$.

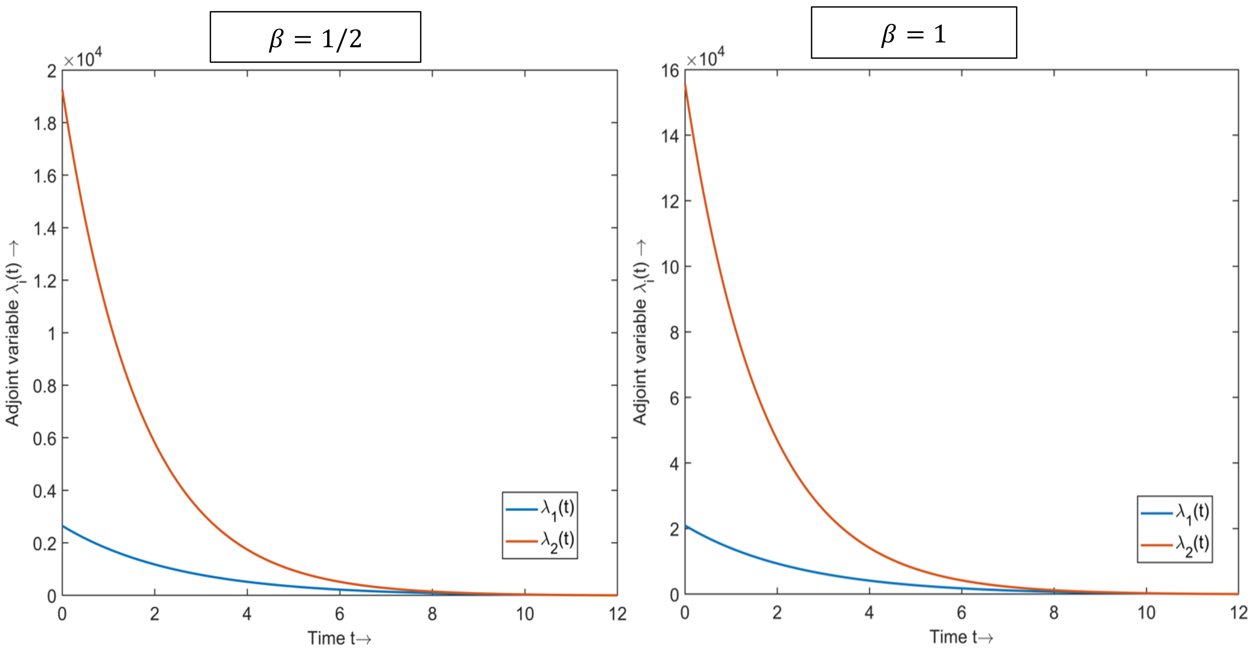

Figure 2. Adjoint trajectories.

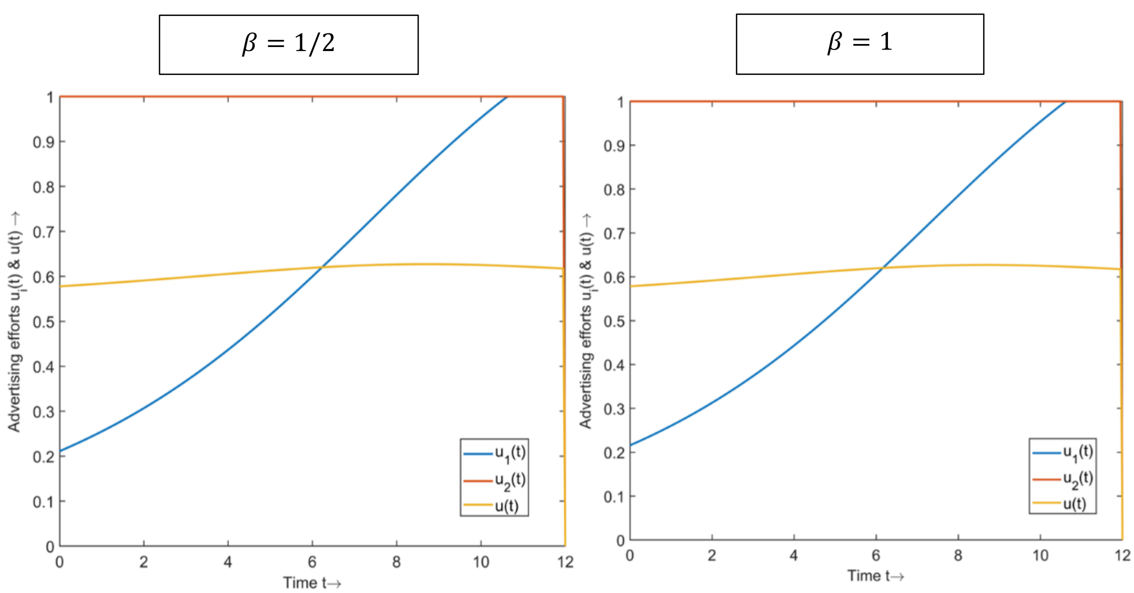

Figure 3. Optimal advertising effort rates $(u_i^*(t), u^*(t))$.

From Figure 3, we can see that optimal advertising effort rates are increasing over time and are decreasing at end of the planning period. This indicates that the advertising efforts that are used to enhance the stock of goodwill of the product during the whole planning period, are not required much towards the end of the planning period.

Figure 4, shows the optimal values of stock of goodwill over time in each segment. Figure 4 illustrates how the optimal values of the goodwill stock rise over time in each segment. This is due to the fact that an increased advertising effort at the start will boost goodwill more quickly and word-of-mouth near the end of the planning period.

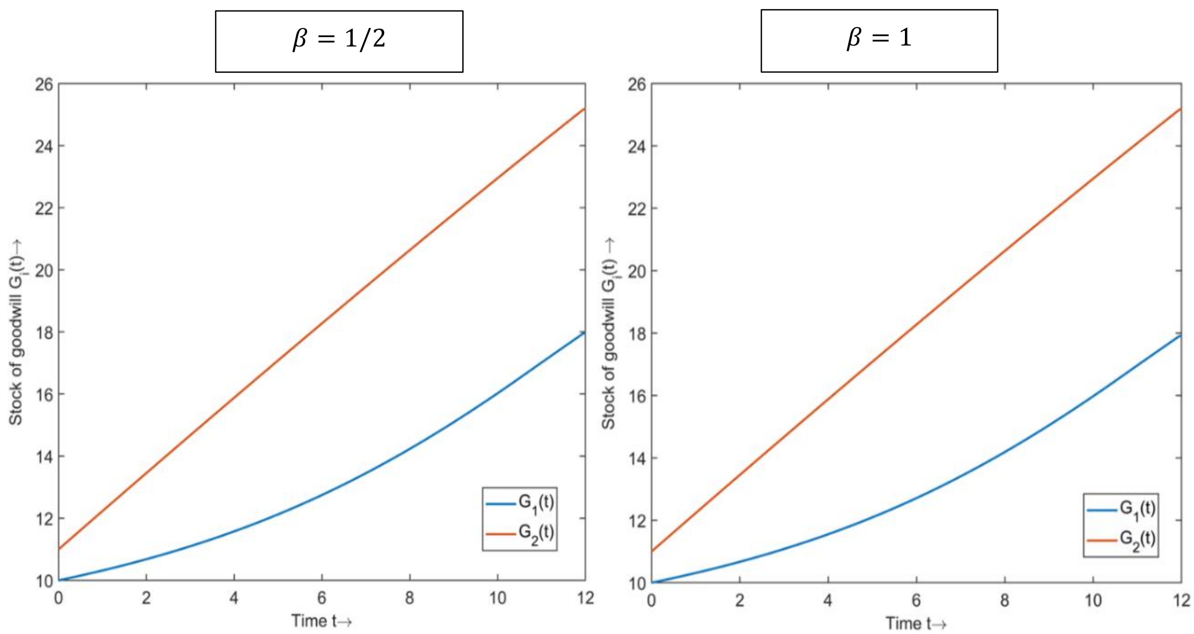

Figure 4. State trajectories.

7.2 Scenario 2

The optimal values of total profit for the parameters $\beta = \frac{1}{2}$ and $\beta = 1$ are 724.653 and 1864.478 units respectively in this scenario. The numerical solution for advertising efforts and brand goodwill are illustrated in Figures 5-7, respectively for the parameters $\beta = \frac{1}{2}$ and $\beta = 1$. Figure 6 shows the optimal values of advertising effort rates $(u(t), u_i(t))$ for both mass and differentiated advertising and optimal values of brand goodwill are shown in Figure 7 for the parameters $\beta = \frac{1}{2}$ and $\beta = 1$.

Figure 5 represents the graph of adjoint variables for each segment for the parameters $\beta = \frac{1}{2}$ and $\beta = 1$. Adjoint variable represents the shadow price (at time t) of a marginal increase in cumulative stock of goodwill.

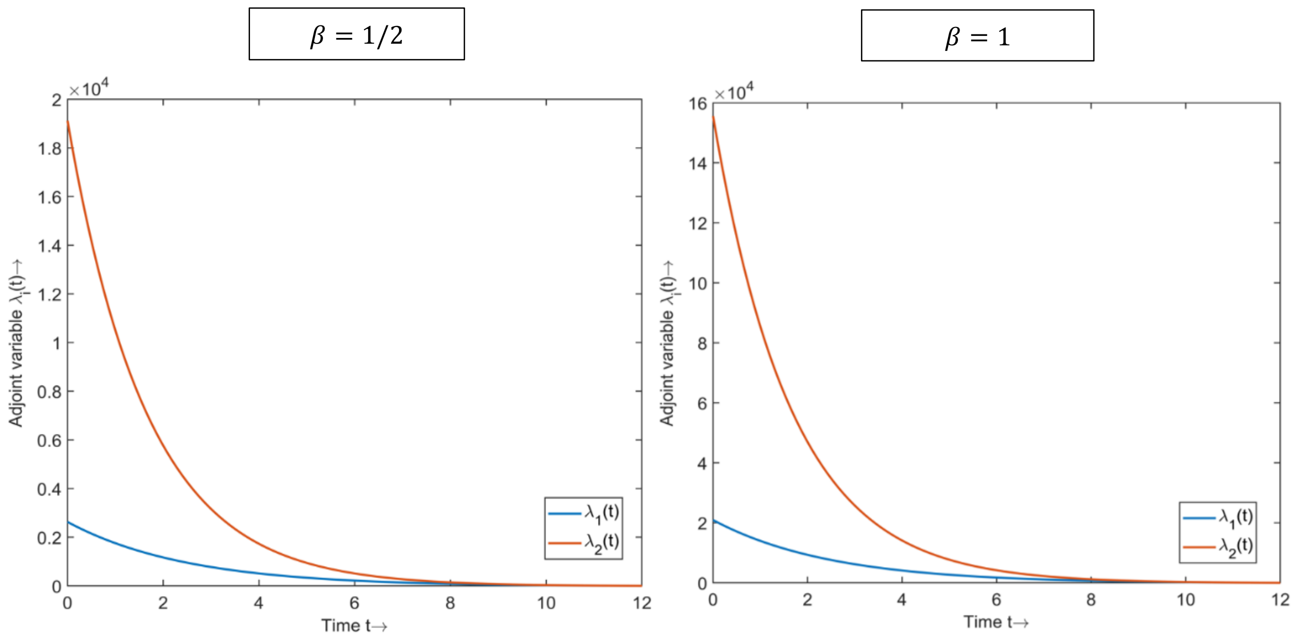

Figure 5. Adjoint trajectories.

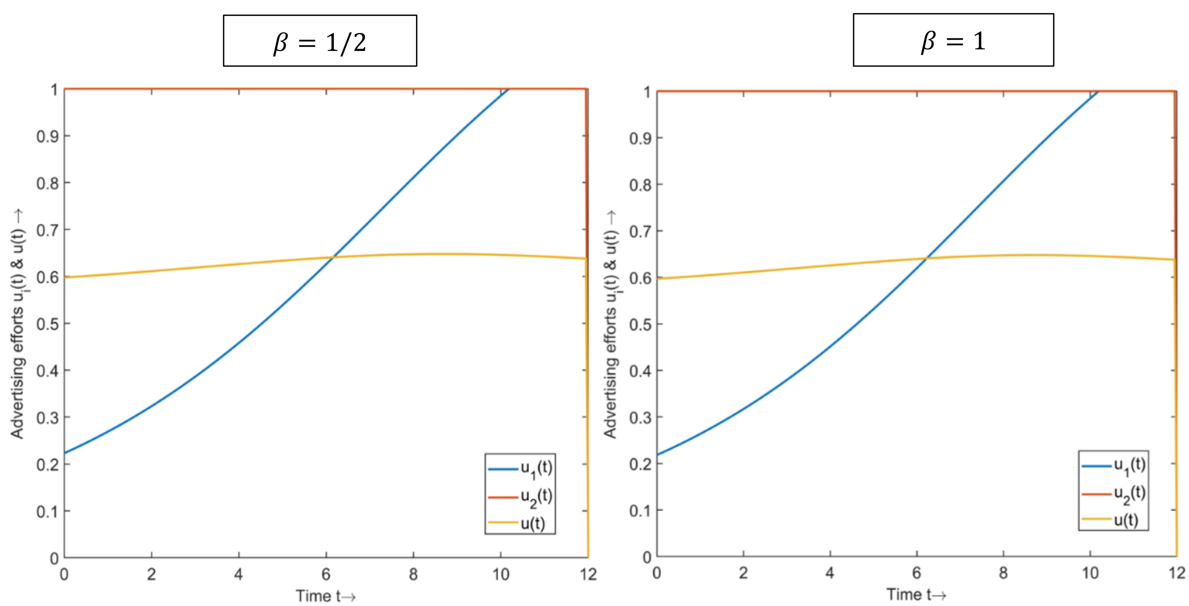

Figure 6. Optimal advertising effort rates $(u_i^*(t), u^*(t))$.

Figure 6 depict similar optimal advertising effort strategies as shown in scenario 2. Optimal advertising effort rates are increasing with time and falling at the end of the planning period. This shows that towards the end of the planning period, less promotion is needed than what is used to increase the product's goodwill over the entire planning time.

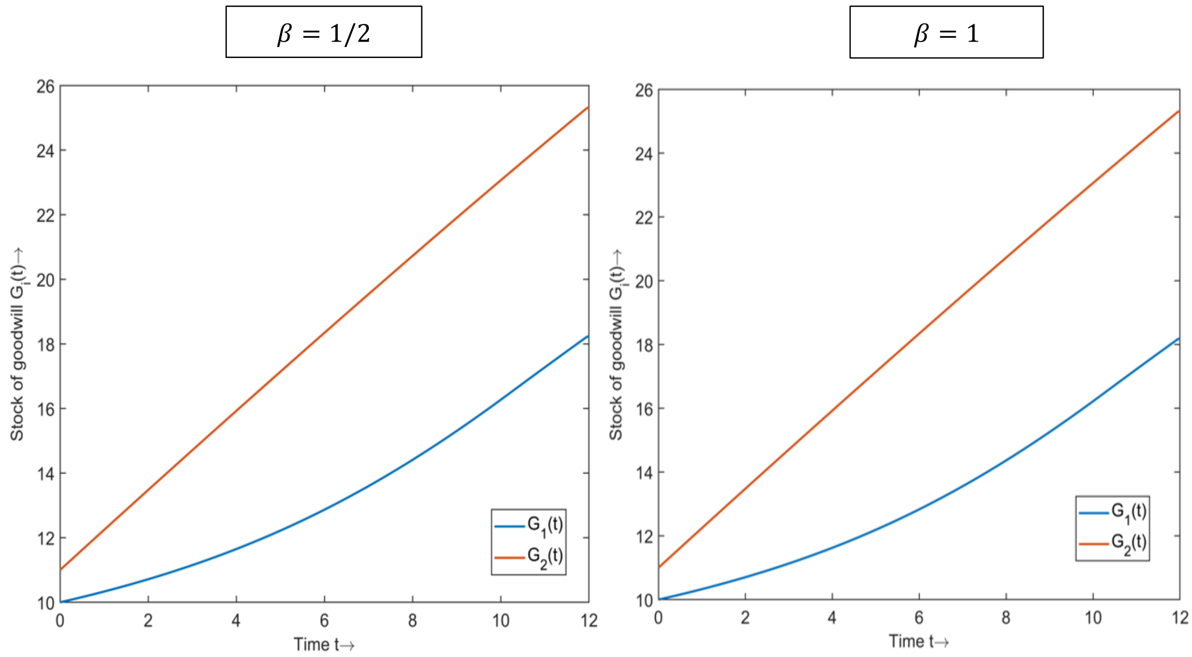

Figure 7. State trajectories.

Figure 7 depicts the curve of the optimal stock of the goodwill with optimal advertising strategy. It can be seen from Figure 7 that the optimal values of the stock of goodwill increase over time in each segment. This is because a greater advertising effort implied at the beginning will increase the stock of goodwill more quickly and word of mouth will gain momentum by the end of the planning period.

8. Managerial Implications

The competition between the firms has increased because of the challenging expansion of new items and the increase in customer demand. It is necessary in any industry to advertise products all over the world. To plan advertising, allot a budget, identify regions/markets, and execute the advertising plan while exhausting the financial support sanctioned in the budget. However, real-world situations are rarely straightforward and rarely exhibit a linear relationship as illustrated in the preceding simple example. In real-time situations, the budget may not be fixed and may be presented as a range with some components being variable. Similarly, markets may not be homogeneous, and advertising may need to be segment-specific. Thus, by simulating real-life scenarios, we develop a method for increasing efficiency and optimising budget utilisation for both homogeneous and segmented markets, while keeping the budget in mind.

Furthermore, the firm can set aside some funds to advertise its product to the entire market. To deal with fluctuating funds, the available budget is represented as a triangular fuzzy number. These constraints incorporate real-world situations encountered by any industry. Mathematical models have proven to be the best tools for developing advertising policies for long-term or new product sales, as well as understanding the growth and strategies of a market system. The main objective of this research is to develop a more comprehensive optimal control theoretic model in a fuzzy environment that can be used to decide on advertising policies.

9. Conclusion and Future Scope

With the rapid advancement of technology, we see more and more new items being created or outdated products being improved. A company must work harder under these conditions to make its products more well known. Here, we have chosen restrictions based on actual situations and apply optimal control theory approach to ascertain how a business might decide how to allocate advertising expenditures for the aim of advertising its product. For effective advertising, the market has been segmented into groups of clients with comparable inclinations. In most cases, the total budget available for advertising may not be completely known. This vagueness is modelled by employing the concept of fuzzy numbers. In this study, we developed a dynamic fuzzy optimal control model that considers how a firm's goodwill affects its profit. The evolution of goodwill is calculated on the assumption that advertising efforts increase goodwill and that it depreciates at a constant rate over time. A single channel and differentiated advertising procedures for each segment are allowed across the entire market. The maximum principle's analytical application yielded the analytical answer for the optimal dynamic control policy, which helps managers decide on advertising and better understand the firm's lucrative chances. The advertising policy shows that optimal advertising effort rates rise over time and decline at the end of the planning period. This indicates that the advertising efforts used to increase the product's stock of goodwill throughout the planning period are not as necessary near the end of the planning period. Next, local stability of the proposed model is also discussed. The stability analysis in context to goodwill-based advertising in marketing problems is a novel aspect but only few papers focus on stability analysis in recent research. Numerical examples to justify the theoretical analysis are provided. For instance, factors, such as quality and can be incorporated for future extension of proposed optimal control model.

Conflict of Interest

The authors confirm that there is no conflict of interest to declare for this publication.

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