DEEP LEARNING FOR LINEAR INVERSE PROBLEMS USING THE PLUG-AND-PLAY PRIORS FRAMEWORK

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ABSTRACT

Linear inverse problems appear in many applications, which was commonly addressed by designing a specific algorithm for each problem. Innumerable attempts have been carried out to solve different variants of the linear inverse problem in different applications. Nowadays, the rapid development of deep learning (DL) provides a fresh perspective for solving the linear inverse problem, which has various well-designed network architectures results in state-of-the-art performance in many applications. In this overview paper, we present the combination of the DL and the Plug-and-Play priors (PPP) framework, which allows solving various inverse problems by leveraging the impressive capabilities of existing DL based denoising algorithms. Open challenges and potential future directions along this line of research are discussed in this paper.

***Index Terms*—** Deep learning, linear inverse problems, plug-and-play priors

1. INTRODUCTION

The linear inverse problem is fundamental to the development of various scientific areas such as astronomy, remote sensing, medical imaging, and telecommunications to name a few. In recent years, innumerable attempts have been carried out to solve different variants of the linear inverse problem in different applications.

Mathematically, the linear inverse problem can be described as the estimation of a finite number of hidden parameters $\mathbf{x} \in \mathbb{R}^N$ from the observed data $\mathbf{y} \in \mathbb{R}^M$ with some forward linear mapping $\mathbf{A}$:

$$\mathbf{y} = \mathbf{A}\mathbf{x}. \tag{1}$$

We have $M < N$ in many applications, e.g., compressive sensing and super-resolution, which makes the problem undetermined. In this case, the observed data does not contain enough information and additional information is required. Furthermore, the observed data is often corrupted by some noise $\mathbf{n} \in \mathbb{R}^M$:

$$\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{n}, \tag{2}$$

which makes finding the original solution more difficult.

As a long-standing problem, a number of algorithms have been proposed in literature to solve linear inverse problems. There are two broad classes of approaches to tackle inverse problems, i.e., model based approaches and learning based approaches. Traditional model based approaches have a few drawbacks: i) imperfect modeling of the real-world problem leads to the model error; ii) the approximation (e.g., using convex relaxation) of the original objective function leads to the structure error; and iii) the convergence to sub-optimal solutions leads to the convergence error, as illustrated in Fig. 1. Nowadays, the rapid development of deep learning (DL) provides a fresh perspective for solving the linear inverse problem. For example, by unfolding an iterative algorithm into a neural network, we can learn the parameters of iterative algorithms from training data, which differs from traditional algorithms that employ predetermined parameters. Using DL to solve linear inverse problems has several advantages. Instead of dealing with the imperfect mathematical models and approximated optimization problems, the DL based method learns the mapping from the input to the output directly and has the potential to overcome or relieve challenges brought by the model error, the structure error and the convergence error in traditional model based approaches. Furthermore, in

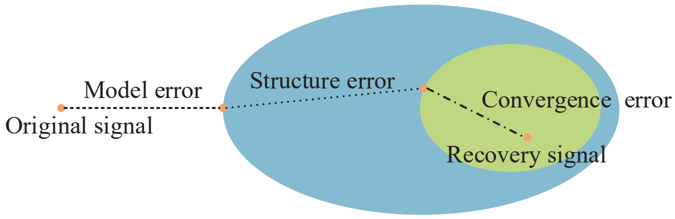

Fig. 1. The decomposition of the error in the solution of linear inverse problems [1].

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comparison to traditional iterative algorithms, DL can significantly increase the speed of convergence.

DL has reached outstanding performance for solving various ill-posed linear inverse problems. However, in comparison to classical optimization based methods, drawbacks of end-to-end learning approaches require expensive retraining whenever the specific problem, the noise level, noise type, or desired measure of fidelity changes. In addition, in some applications, it is difficult to acquire sufficient amount of training data in the same form as the test data. Some recent works tackle these problems by using neural networks trained for denoising as generic plug-and-play regularizers in optimization algorithms.

This overview paper focuses on the plug-and-play priors (PPP) framework, which has attracted significant attentions owing to its flexibility and effectiveness in handling various inverse problems. Different from the traditional optimization procedure that directly deals with a hand-crafted regularization term incorporating prior information, the PPP framework unrolls the cost function by variable splitting techniques and replaces the subproblem related to the regularization term by some off-the-shelf powerful operators that are not in the explicit form of the original optimization problem.

2. PLUG-AND-PLAY PRIORS FRAMEWORK

The standard approach for solving $\mathbf{x}$ in (2) is by formulating an optimization problem

$$\min_{\mathbf{x}} \quad \mathcal{D}(\mathbf{x}) + \tau \mathcal{R}(\mathbf{x}), \tag{3}$$

where $\mathcal{D}$ is the data-fidelity term that penalizes the mismatch to the measurements $\mathbf{y}$, $\mathcal{R}$ is the regularizer that imposes prior knowledge of $\mathbf{x}$, and $\tau > 0$ is the regularization parameter. For example, for image reconstruction in the presence of Gaussian noise, least squares, i.e., $\|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2^2$, is often used as the data-fidelity term, and total variation (TV) penalty, i.e., $\|\mathbf{D}\mathbf{x}\|_1$, is a popular regularizer, where $\mathbf{D}$ denotes the discrete gradient operator.

Different variable splitting algorithms, e.g., the iterative shrinkage/thresholding algorithm (ISTA) and the alternating direction method of multipliers (ADMM) algorithm, can be applied to solve the optimization problem in (3). For example, the ISTA has two key steps

$$\mathbf{z}^t = \mathbf{x}^{t-1} - \gamma \nabla \mathcal{D}(\mathbf{x}^{t-1}) \tag{4a}$$

$$\mathbf{x}^t = \mathrm{prox}_{\gamma\tau\mathcal{R}}(\mathbf{z}^t), \tag{4b}$$

where $\gamma > 0$ is the step size. Note that (4a) only depends on the forward model via the gradient of the data-fidelity term, while (4b) depends on the prior via the proximal operator, which is given by

$$\mathrm{prox}_{\gamma\tau\mathcal{R}}(\mathbf{z}) := \arg\min_{\mathbf{x}} \frac{1}{2}\|\mathbf{x} - \mathbf{z}\|_2^2 + \gamma\tau\mathcal{R}(\mathbf{x}). \tag{5}$$

Instead of directly solving (4b) with an explicit handcrafted regularization term, the PPP framework considers to replace the proximal operator by some off-the-shelf and highly engineered denoiser. For example, many image denoisers such as BM3D [2], WNNM [3] and TNRD [4], do not have a regularization objective. Therefore, the PPP is a flexible framework that enables applying various denoising operators without explicitly defining prior models.

The PPP approach was firstly introduced in [5] and has shown to be effective in various tasks, such as image superresolution [6, 7], deblurring [8], tensor completion [9] and Poisson-noisy inverse problems [10]. Recent work has also analysed the theoretical convergence guarantees of PPP algorithms [11, 7, 12, 13, 14, 15, 16, 17]. For example, Sreehari et al. present sufficient mathematical conditions that ensure convergence of the PPP approach [11]. In specific, they show that the denoiser is a proximal mapping if and only if it is non-expansive and is the sub-gradient of a convex function.

3. DEEP PLUG-AND-PLAY PRIORS

Deep neural networks exhibit state-of-the-art results in various linear inverse problems with a fixed known acquisition process. However, they experience a huge performance loss when whenever the specific problem, the noise level, noise type, or desired measure of fidelity changes. The PPP framework exploits modern denoising priors to handle the regularization term in model-based optimization schemes, and does not require the prior to be expressible in the form of a regularization function. Therefore, by leveraging a powerful DL based denoiser, one could achieve superior performance in different linear inverse problems without retraining. This is especially useful when it is difficult to acquire sufficient amount of training data in the same form as the test data.

3.1. Deep Plug-and-Play Priors: Theory

The ideal DL based denoiser should be trained by the current noise level in each iteration. However, the noise level varies and is usually unknown in iterations of the PPP framework. Sommerhoff et al. show an example (as shown in Fig. 2) for a diverging PPP algorithmic scheme that uses a pre-trained DnCNN [19] as the denoiser in the PPP framework [18]. In comparison to the general PPP framework, there is even less progress on the theoretical aspects of deep PPP approaches. The global convergence of PPP is proved in [11] for a denoiser that has a symmetric gradient and is non-expansive. However, it is difficult to be proved, and empirical success of counter example suggests this assumption is too strong. In [15], Ryu et al. theoretically establish convergence of the PPP approach under a certain Lipschitz condition on the denoisers, and use spectral normalization to improve deep denoising networks to satisfy the Lipschitz constraint.

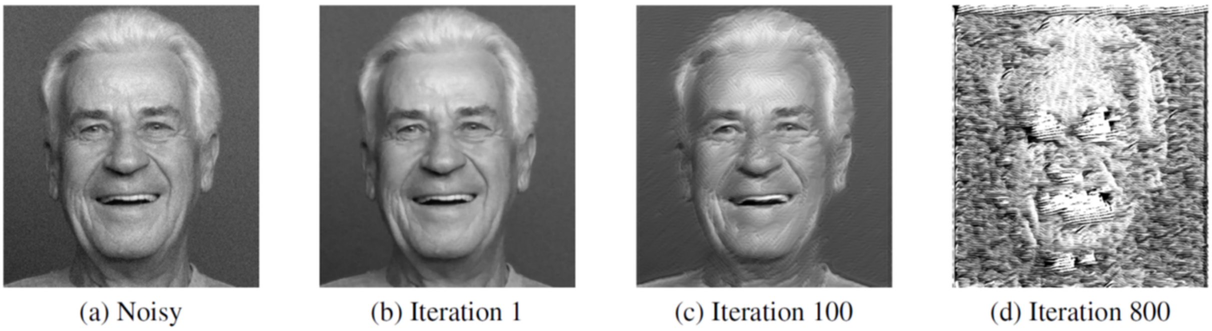

Fig. 2. Example for a diverging PPP algorithmic scheme that uses a pre-trained DnCNN. [18].

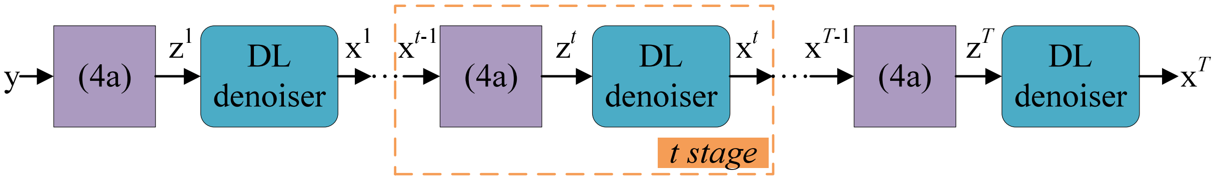

Fig. 3. The architecture of the end-to-end PPP framework [20].

3.2. Deep Plug-and-Play Priors: Architectures

Various DL based denoisers can been applied in the PPP framework, while a standing issue is how to adjust the denoiser to adapt to the varying noise levels in each iteration. In [21], Zhang et al. train a set of neural networks with different noise levels, and apply them to handle various image restoration problems. This method is also used in [22], where a denoiser trained with a suitable noise level is used. It is worthy that the number of the denoisers is much less than that of learning different models for different degradations. However, it is still costly to train and store a number of DL based denoisers.

In [23, 20], the deep PPP framework is trained in an end-to-end fashion as shown in Fig 3. By fixing the number of iterations, the PPP approach is unfolded into a deep neural network composed of multiple denoising networks with the same parameters across iterations. Rather than applying different networks that are trained for decreasing noise variances, a same denoising network learns to deal with various noise/alias patterns with several different statistics, which cannot be fully captured by Gaussian noise process. The end-to-end training approach do not require a recipe for choosing the noise variance at each iteration or regularization parameter. The drawback of the unfolded end-to-end network is that the denoising network is not generic, and can only be applied to the specific problem in the training.

Sommerhoff et al. found that enforcing non-expansiveness of a network as suggested in [11] would drastically decrease the denoising performance [18]. Thus, they propose to guarantee convergence by forcing the neural network to predict a descent direction to a given model-based energy, such that it can be used within a line search algorithm. Intuitively, the descent direction proposed by the network pushes the iteration closer towards the distribution of the training data than a usual gradient descent step. In specific, to avoid divergence, they project the update directions onto the half-space of descent directions. Under weak additional conditions, it guarantees the convergence of the proposed scheme to the minimizer of the original cost function.

In addition to the PPP architecture, there are other DL based approaches that exploit neural networks as the regularization of inverse problems. One such contribution is Regularization by Denoising (RED) [24], which uses the (non-convex) regularizer $\mathbf{x}^T(\mathbf{x} - \mathcal{H}(\mathbf{x}))$ given a denoiser $\mathcal{H}(\mathbf{x})$, and uses denoiser evaluations in its iterations. Another related contribution is the deep image prior (DIP) proposed by Ulyanov, Vedaldi, and Lempitsky [25]. As an unsupervised method, DIP uses a neural network as the regularizer to the inverse image problem. In specific, it replaces the explicit regularization by the assumption that the unknown image should be generated from a neural network, and then learn the networks parameters for the corrupted image. The success of the DIP demonstrates that the structure of the network is natural to capture the image statistics prior with a deep image prior. To enhance the DIP, some modifications are proposed in literature. Van Veen et al. propose to regularize the weights of the network during the optimization process for compressive sensing problems [26]. Ren et al. further consider the case that the sparse dictionary in compressive sensing is uncertain [27]. In [28], Mataev et al. suggest an extra boost to the DIP by returning the explicit regularization, i.e., merge the DIP with the RED. It brings an extra force that does not exist in DIP. The non-locality flavor in the RED complements the DIP architecture regularization effect. There are more works that incorporate model-based optimization with DL methods [1].

3.3. Deep Plug-and-Play Priors: Applications

In the form of the deep PPP, there is a number of inverse problems that can be handled by sequentially applying denoising steps. Empirical results show the success of deep PPP methods in a large variety of imaging applications including demosaicking [29], image deconvolution [29], image super-resolution [21, 30, 31], subsampled Fourier inversion [32], CT scan [33], magnetic resonance imaging (MRI) [23, 34], deblurring [30, 18], compressed sensing MRI [15], single photon imaging [15], image/video/multi-spectral image (MSI) completion [35].

4. CHALLENGES

While the deep PPP scheme looks like a perfect way to leverage the impressive achievement in DL to solve model-based linear inverse problems, there are still some challenges. It has been noticed that the PPP often requires delicate parameter tuning in order to obtain high quality results. For sophisticated denoising methods without provable convergence, parameter tuning becomes a burdensome work. Moreover, as the deep PPP is an iterative method, sometimes a large number of iterations is required. Each iteration involves a denoising DL operation, which could be time-consuming if conducting many iterations and/or using a large neural network. Third, the deep PPP does not have a clear definition of the objective function, which makes theoretical analysis much more challenging.

5. CONCLUSIONS

This overview paper of a Special Session of ICASSP presents the recent achievements in using DL and the PPP framework to solve linear inverse problems. While end-to-end learning approaches require expensive retraining whenever the specific problem, the noise level, noise type, or desired measure of fidelity changes, the deep PPP provides a flexible and effective way in handling various inverse problems with only one neural network learned for denoising. There are still many open challenges in the aspects of theory, algorithm and application, which requires further investigation.

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