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Keynesian Resurgence: Financial Stimulus And Contingent Claims Modelling

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Abstract

Since the commencement of the Global Financial Crisis, a worldwide resurgence in applying Keynesian modelling has occurred, and has been cited as a major factor in averting a worldwide Economic Depression. A key aspect of Keynesian modelling is that Governments gain contingent claims on firms in exchange for financial stimulus. However, there exist few mathematical finance models examining Keynesian modelling, stimulus modelling and the valuation of such Government contingent claims.

In this paper we provide a new mathematical finance framework for modelling firms and financial stimulus under a Keynesian framework; we apply a stochastic differential equation model, rather than the standard time series models. Our model incorporates fundamental concepts of Keynesian modelling and Keynesian stimulus, which is a new characteristic to current financial models. We model the Government's contingent claim on the firm as a real call option, and derive a closed form solution for the value of this option which takes into account firm stimulus. We also derive a solution for the minimum firm value required to exercise the option. We conduct numerical experiments for different firm equilibrium values, firm values, economic cycles and analyse the impact on option and stimulus values.

Keywords: financial crisis; stimulus spending; real options; Keynesian economics; Geometric Ornstein-Uhlenbeck.

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1. Introduction

As the Global Financial Crisis commenced (see [22], [17] and [10]), many countries were facing the possibility of a potential global Economic Depression. Consequently, many policy makers and Academics considered potential actions to avert such a crisis. The typical free market and monetarist approaches, such as substantially lowering interest rates and allowing firms to fail (such as Lehman Brothers [18]) were deemed unsuccessful. This led to a worldwide interest and resurgence in Keynesian economics. This resulted in an unprecedented action of greater Government intervention, in many markets, new spending plans and stimulus packages. The Keynesian approach to modelling and the associated stimulus packages have been widely credited as the key initiatives that prevented the Global Financial Crisis from worsening [39].

The Keynesian approach to modelling economic and financial markets has typically advocated for greater Government intervention, Government spending and that markets should be 'managed' rather than allowed to function autonomously as in the free market approach. Essentially, in classical economics the economy functions at full capacity (such that aggregate demand equals full productive capacity of the economy) however Keynes does not assume this. In the Keynesian view aggregate demand can be influenced by many factors, causing a shortage of demand as well as affecting economic output, employment and inflation. The Keynesian approach has the advantage that many financial and economic aspects (such as inflation, output, etc.) can be determined by decision makers, but potentially enables greater mismanagement. On the other hand the classical economics approach allows economies to autonomously correct themselves and so limits mismanagement, however the self-correcting process may cause too much disturbance.

Keynesian modelling and economics supports the argument that Governments intervene in the economy when there is a disconnect between demand and full productive capacity. However, the Keynesian approach had drastically lost popularity since the 1970s' as Keynesian models and theories were unable to offer solutions to stagnant economies, with prolonged stagflation and rising unemployment.After the commencement of the Global Financial Crisis, the effectiveness of Keynesian modelling and the ideas in dealing with potential worldwide Economic Depressions has re-established Keynesian modelling as an important, credible and practical academic theory for markets. It is also worth pointing out that in [21] the Keynesian approach is reconciled with classical economics by "IS-LM" analysis. Some of the new contributions to classical theory require that macroeconomics be based on the same foundations, such as microeconomic theory, profit-maximizing firms and rational, utility-maximizing consumer [2].

Whilst Governments provide stimulus to firms, they typically obtain some contingent claim in return for the stimulus, for example partial ownership of a firm that is sold at a later point in time. An actual case in point is HBOS bank, where the UK Government gained partial ownership of HBOS in exchange for providing stimulus to the bank. HBOS received approximately £11 billion in exchange for partial ownership by the UK Government, and HBOS received additional benefits from the Government, such as facilities to recapitalise its funds and arrange funds from the private sector. For example, Barclays banks arranged funding for HBOS with assistance from the UK Government. It was also later reported that the UK Government provided further loans to HBOS whilst the bank was still recovering.

The Government's stake or claim on any firm receiving stimulus assistance does not represent a standard ownership in a firm, as we would normally encounter when an investor buys a share. Firstly, the ownership is strictly bound to be sold at some future point in time, although there is no fixed deadline for the sale. Governments are normally committed to selling their stake in firms to avoid their firms having any undue political benefit or liability to the Government. In fact some politicians do not favour giving stimulus assistance because ownership can effectively become indefinite, hence there is a significant incentive to definitely sell.

Secondly, the sale of ownership by Governments typically involves substantial costs (other than transaction costs) which can make sales prohibitively expensive, or not profitable at all. For example, the UK Government needed to investigate and reconcile competition laws and regulations before selling its stake in HBOS. Additionally, the size of ownership and the type of ownership (being a Government) meant that disposal of ownership can lead to a significant drop in value of the firm; such an incident occurred for the firm AIG which received Government assistance. Consequently, the Government ownership stake of firms resembles a contingent claim rather than a standard share in a firm.

Although policy makers and Academics may agree on the need for stimulus spending to encourage growth, a major problem facing Governments is valuation of claims on firms. Firstly, how does the Government value the claim or stake in a firm (which it is bound to sell at some point in the future) whilst taking into account its unique disposal costs? Secondly, what is the optimal condition on which to dispose of its claim? The fact that the UK government has received substantial criticism for obtaining sub-optimal value for money for taxpayers in disposing of its stake in HBOS bank demonstrates the importance of understanding the optimal conditions for sale.

Our discussion therefore leads us to conclude that one needs to examine the incorporation of Keynesian modelling and stimulus within financial models. The current models in relation to Keynesian modelling and stimulus tend to be discrete time and econometric models, sometimes excluding firm specific modelling factors entirely, and do not provide a viable method for continuous time financial modelling. Furthermore, current models analysing Keynesian concepts do not offer a viable mathematical modelling approach. They do not enable valuation and analysis of any contingent claims arising from stimulus funding to a firm, or incorporating the disposal costs.

In this paper we propose a continuous time financial model of firms that incorporate Keynes' model of firms and receive financial stimulus. Our model captures the fundamental aspects of Keynes' model of firms, such as cyclical variations in firm value, as well as incorporating other important aspects such as stochastic movements in firm value. We achieve this using a Geometric Ornstein-Uhlenbeck stochastic differential equation, rather than applying the typical time series and standard discrete time models. We then propose a model for the Government's contingent claim on a firm by applying a real call options approach (options are used for a variety of applications, see for example [4],[35], [41]). Our method takes into account disposal costs by Governments and the indefinite period for sale of the stake (that is no expiry date). We then derive a closed form solution for this option and derive the minimal firm value at which it is beneficial to exercise the option.

This paper is organised as follows: in the next section we provide an introduction to continuous time financial models, Keynesian modelling and stimulus spending, and provide a literature review on models in these areas. In the next section we introduce our model, model the contingent claim in the firm (obtained by the Government) as a real call option on the firm, and derive a closed form solution for the option. We then also derive the minimum firm value that should be attained before the Government exercises its option. We conduct numerical experiments for different firm equilibrium values, firm values, economic cycles (or equivalently the levels of mean reversion) and analyse the impact on the option value and stimulus. We finally we end with a conclusion.

2. Literature Review

In this section we provide a literature review of continuous time financial modelling, and mathematical models that are closest to our research area of Keynesian modelling and stimulus. Whilst a wide range of models exist in finance (see for example [25], [20], [23]) the continuous time financial models are based on stochastic processes, such as in [33]. The advantages of continuous time financial models are that they enable us to derive closed formed and analytic solutions. One can also utilise continuous time financial models so that one can apply well established financial models, such as option pricing.

Let there exist a probability space $\{\Omega, \mathcal{F}, \mathbb{P}\}$ where Ω denotes the sample space, $\mathcal{F}$ denotes a collection of events in Ω with probability measure $\mathbb{P}$, and we have a filtered probability space $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\}$. The set $\{\mathcal{F}_t\}$ denotes the set of information that is available to the observer up to time t and we have

$$\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}_T, \forall s, t \text{ with } s < t < T,$$

and the set $\{\mathcal{F}_t\}, t \in [0, T]$ is also known as a filtration. Furthermore, for a given stochastic process $X(t)$, as more information is revealed to an observer as time t progresses, we introduce the filtration $\mathcal{F}_t^X$ which denotes the information generated by process $X(t)$ on the interval $[0, t]$. Finally, assume we have the probability space $\{\Omega, \mathcal{F}, \mathbb{P}\}$ then we define a change of measure from $\mathbb{P} \sim \mathbb{Q}$ to be the probability space $\{\Omega, \mathcal{F}, \mathbb{Q}\}$.

Despite the resurgence of Keynesian modelling and the importance of Keynesian economics in the post-world war years of the twentieth century, there exist few mathematical finance models in relation Keynesian modelling and Keynesian stimulus, yet there exist a wide range of other mathematical finance models (such as [37], [15], [34], [16]). A significant financial model is the Black and Scholes option pricing model [6] which determined the closed form solution of European call options $C_{BS}(X(t))$ on the assumption of no arbitrage:

$$C_{BS}(X(0), T, r, \sigma, K) = X(0)\Psi(d_1) - Ke^{-rT}\Psi(d_2),$$

where

$$d_1 = \frac{\ln\left(\frac{X(0)}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}},$$
$$d_2 = d_1 - \sigma\sqrt{T}.$$

The variable $X(t)$ is the stock price that follows Geometric Brownian motion

$$X(t) = X(0) + \int_0^t \mu dt + \int_0^t \sigma dW(s),$$

or more conveniently,

$$dX(t)/X(t) = \mu dt + \sigma dW(t),$$

where μ denotes drift, σ denotes volatility and $W(t)$ is a Wiener process. In $C_{BS}(X(t), T, r, \sigma, K)$, T is the expiration date, $\Psi(\cdot)$ is the standard normal cumulative distribution function, r is the riskfree rate of interest and K is the strike price.

The price of a European call option is also determined by risk neutral valuation

$$C_{RN}(X(t), K, T) = e^{-rT}E^{\mathbb{Q}}[X(T) - K]^+,$$

where $\mathbb{Q}$ is the risk neutral probability measure. In terms of Girsanov's Theorem and change of probability measures with respect to stochastic differential equations, let us assume we have a family of information sets $\mathcal{F}_t$ over a period $[0, T]$ where $T < \infty$. We define over $[0, T]$ the random process (also known as the Doleans exponential) χ_t:

$$\chi_t = exp\left\{ -\int_0^t l(u)dW^{\mathbb{P}}(u) - \frac{1}{2}\int_0^t l^2(u)du \right\},$$

where $W^{\mathbb{P}}(t)$ is the Wiener process under probability measure $\mathbb{P}$ and $l(t)$ is an $\mathcal{F}_t$-measurable process that satisfies the Novikov condition

$$E^{\mathbb{P}}\left[exp\left\{ \frac{1}{2}\int_0^t l^2(u)du \right\} \right] < \infty, t \in [0, T].$$

We then have $W^{\mathbb{Q}}(t)$ is a Wiener process with respect to $\mathcal{F}_t$ under probability measure $\mathbb{Q}$, where $W^{\mathbb{Q}}$ is defined by

$$W^{\mathbb{Q}}(t) = W^{\mathbb{P}}(t) + \int_0^t l(u)du, t \in [0, T].$$

Financial models have been also developed to take into account credit risk, for example the model by Leland [29],[28] to model strategic default. This model incorporates debt as an optimal capital structure problem, so that it is consistent with Miller-Modigliani Capital Structure Theory [7]. If we assume the debt is modelled by bond $\tilde{D}(T)$ then the debt is given by

$$\begin{aligned} \tilde{D}(T) &= \frac{\Lambda}{r} + \left(\chi - \frac{\Lambda}{r}\right)\left(\frac{1 - e^{-rT}}{rT} - \frac{1}{T}\int_0^T e^{-rt}\psi(t)dt\right) \\ &+ \left((1 - \iota)J^* - \frac{\Lambda}{r}\right)\frac{1}{T}\int_0^T e^{-rt}\psi'(t)dt, \end{aligned}$$

where Λ is the coupon payment per year, χ total principal value of all bonds, $\psi(t)$ is the cumulative distribution function of the passage time of bankruptcy, $\psi'(t)$ is the associated probability density function, J^* is the asset value that triggers default, ι specifies the fraction of asset values that is distributed to the bondholders in the event of default.

Other financial models include modelling volatility as a function of stock price (local volatility), that is $\sigma(X, t)$, and a number of local volatility based option pricing models exist. Dupire's local volatility modelling is able to obtain a unique risk neutral measure $\mathbb{Q}$ from empirical option data. Dupire applies Breeden and Litzenberger equation [8] and the Fokker-Planck equation, to obtain the Dupire equation

[12]:

$$\frac{\partial C}{\partial T} = \sigma^2(X,T).\frac{X^2}{2}.\frac{\partial^2 C}{\partial X^2} - (r-D)X.\frac{\partial C}{\partial X} - DC, \quad (1)$$

where D is the dividend. If we rearrannge equation (1) then we obtain:

$$\sigma(X,T) = \sqrt{\frac{\frac{\partial C}{\partial T} + (r-D)X\frac{\partial C}{\partial X} + DC}{\frac{X^2}{2}\frac{\partial^2 C}{\partial X^2}}}.$$

Therefore the local volatility $\sigma(X,T)$ can be fully extracted from option data.

Despite the different models that have been developed for financial modelling, the issues of Keynesian stimulus and modelling have not been heavily examined in continuous time finance models. For instance, many stock price models exist but do not explicitly take into account Keynesian modelling or Keynesian stimulus. For example, in [1] a stochastic differential equation is proposed for modelling the stock value movements, the stock value of asset i is given by

$$dX_i = \mu_i dt + \sigma_i dW_i(t) + Z_i(t)dN_i(t), i = 1,2,...,k, \quad (2)$$

where k correlated Wiener processes exist, $[Z_1(t), Z_2(t), ..., Z_k(t)]$ is a vector of jump sizes, and $dN_i(t)$ is another stochastic counting process. Although an individual asset model is provided in continuous time, there is no explicit modelling of stimulus or Keynesian firm value dynamics.

The research literature that examines Keynesian stimulus and Keynesian based modelling of firms tend to include econometric, time series or discrete time models, rather than employing stochastic differential equations as in standard financial mathematics models. For example the DSGE model (dynamic stochastic general equilibrium) is a popular econometric model, which has been used to analyse stimulus (see for example [30]) and has been significantly developed due to its importance (for example [3] devises an optimisation method for it). However the DSGE model is a discrete time model and is therefore limited in its financial modelling applications and mathematical analysis. Similarly, in [31] a Keynesian model is examined using a maximum likelihood approach, however this also employs a discrete time model to analyse macroeconomic effects.

In addition to many models adopting a discrete time modelling approach, the majority of Keynesian and stimulus models tend to model sectors or entire economies, rather than model individual firms. Whilst this may be useful to understand the impact on particular sectors or economies, it does not assist in understanding individual firms and most financial modelling is with respect to an individual firm. This enables one to develop mathematical analysis and derive solutions to derivatives related to a specific firm. Moreover, we can understand the impact of Keynesian modelling and stimulus on individual firms.

One paper that is closest to our work is [19], which uses stochastic differential equations to model stimulus and other fiscal policies. Specifically, [19] combines the DSGE model with a stochastic volatility model, and other economic factors are modelled using standard continuous time financial models. For example, bond prices $B(t,T)$ at time t with maturity T are modelled using the Heath-Jarrow-Morton framework. The Heath-Jarrow-Morton framework is given by [5]

$$dB(t,T) = r(t)B(t,T)dt - B(t,T)\left(\int_t^T \sigma(t,s)ds\right)dW^{\mathbb{Q}}(t). \tag{3}$$

In another example, the paper [19] models new wages $\Gamma(t)$ in continuous time as

$$d\Gamma(t) = \left[\nu_h(\Gamma(t)f(\pi(t),\pi,t_h))^{\left(\frac{1}{1-\iota_n}\right)} + (1-\nu_h)d\overline{\Gamma}(t)^{\left(\frac{1}{1-\iota_n}\right)}\right]^{(1-\iota_n)} dt,$$

where $\overline{\Gamma}(t)$ is an optimised wage, $0 \le h \le 1$ is a habit formation variable, $f(\pi(t),\pi,t_h)$ is an economic variable and function, $(1-\nu_h)$ relates to a probability of changes to nominal wages, $1 \le \iota_n \le \infty$ is the wage markup for index n; the reader is referred to [19] for more information. A model is also given to replicate stock returns based on macroeconomic factors, contagion and other spillover effects.

The paper [19] takes into account economic factors such as budget constraints, inflation, capital and consumption, and some of these factors are modelled with continuous time equations. The paper also focuses on the interaction of different economic monetary policies, fiscal policies and their impact on key macroeconomic variables. Additionally, stabilisation policies are discussed, in terms of the model and the impact on different sectors, such as the household and the bond market sectors. However, the paper does not model the dynamics of individual firms and so we do not know the impact of Keynesian modelling and stimulus upon each firm.

Another paper that is close to our work is [9], which uses stochastic differential equations. In [9] they model a number of economic variables using stochastic differential equations, for example capital k is modelled as

$$dk = (\Phi(i) - \delta)kdt + \sigma_k k dW(t),$$

where i is the investment rate, the function $\Phi(.)$ represents investment costs and δ is the capital depreciation rate. The paper studies the equilibrium dynamics of an economy in the presence of financial frictions that limit capital flows. Moreover, they find that financial frictions amplify different risks.

In [9] they model various agents in continuous time, for example, the household sector is assumed to have a utility

$$E\left[\int_0^\infty e^{-rt}dc(t)\right],$$

where $c(t)$ is consumption at time t. They also find that securitisation and derivatives reduce some particular risks but also increase systemic risks. However, [9] does not explicitly take into account Keynesian models or Keynesian stimulus, thus it may have continuous time modelling but it does not model the individual firms or groups of firms that behave in a Keynesian model, or receive any stimulus.

A key problem with all the economic models that include stimulus modelling is that they are generally not continuous time models. Consequently, this limits the amount of mathematical analysis that can be applied, for example we cannot determine closed form solutions for any contingent claims, or determine the optimal point to sell such claims. In fact, models such as [19] are typically not formulated for the purposes of mathematical analysis or deriving any useful analytical solutions. This is evident given that, for instance, the stock return model involves a stochastic differential equation involving over 20 different terms, and so would be analytically intractable to derive any useful solutions.

Another key problem with current models is that they tend not to model individual firms or the firm's specific stimulus, rather the models tend to examine particular sectors or economies. Consequently, we cannot understand the impact of stimulus or Keynesian modelling in relation to a specific firm. This is particularly important to understand so that Governments can analyse the impact of stimulus packages on specific firms, or any contingent claims associated with the stimulus package.

Finally, the economic models generally do not examine any value of any contingent claims that Governments typically obtain in relation to stimulus packages. For example, during the Global Financial Crisis the UK Government provided stimulus assistance to HBOS and gained partial ownership of the firm, and the sale of ownership was to be at some indefinite time in the future. If such claims are not correctly modelled then one cannot correctly value such claims. Hence one cannot determine the optimal point to sell such claims, or examine how such claims are affected by the time dynamics of a firm.

3. Keynesian Financial Model: Firm Value and Stimulus Processes

In this section we introduce our Keynesian financial model for firm value, and the associated firm stimulus.

3.1. Keynesian Model of Firms

Keynes' approach to modelling firms is concerned with the dynamics of the changing value of firms over time. Keynes was concerned about the fluctuating value of firms because this could cause significant economic problems such as recessions or depressions. Whilst fluctuations in firm value may occur due to appropriate variations in firm performance, sometimes the variations were excessively high which would cause significant economic problems. Moreover, Keynes considered such fluctuations as unnecessarily high and as a result of systemic problems in the market, rather than purely the fault of firm performance.

Keynes was particularly concerned with the variation in the intrinsic or fundamental value of firms, as opposed to the market value of firms. The market value of a firm is essentially the value attributed by traders in a market, for example the stock price multiplied by the total number of shares for firms listed on the stock market. Whilst market value can be misleading and may greatly fluctuate each day, it does not pose a significant problem provided the intrinsic value of firms does not change. The intrinsic value of firms is concerned with the fundamental or "true" value of firms, rather than the daily market based valuations. Moreover, Keynesian stimulus is concerned with improving the intrinsic value of firms, rather than the market value. Hence it is important in our modelling that we address intrinsic value based modelling.

The intrinsic value of firms represent the fundamental value of the company and this has been widely studied in finance (see for instance [27] and [40]). Intrinsic value has also been studied to understand asset and stock market bubbles and crashes [38], in that substantial deviations in intrinsic value and market value account for such phenomena. Intrinsic value models typically relate to measuring fundamental aspects of the company itself, for example its assets, liabilities and profits etc.. Hence intrinsic value is strongly related to economic as well as industry specific metrics.

A number of models or measures have been proposed for intrinsic value, for example [27]. However such models are generally not useful for continuous time financial modelling. Firstly, there is no overall consensus on any specific model for intrinsic value and it can be shown that different models have major deficiencies in measuring intrinsic value. Secondly, there is generally no continuous time or stochastic differential equation to model the evolution of the intrinsic value over time. This poses a significant problem in financial modelling because most financial models are formulated in stochastic differential equations (or at least continuous time). If we do not use stochastic differential equations then we cannot apply mathematical techniques and analyses to obtain closed form solutions, or determine optimal points to sell contingent claims etc..

Thirdly, the discrete time formulation of intrinsic value models limits the modelling of the firm value dynamics over time. In fact most intrinsic value models are formulated in discrete time steps that are frequently of time intervals of months or longer. Hence we cannot capture the complex dynamics of any process over time. This is important because the specific dynamics of individual firms are a fundamental aspect of Keynes' model of firms (to be discussed later). Consequently, we cannot ignore such time dynamics if we wish to incorporate Keynes' ideas in modelling firm values. Moreover, Yu [42], Forster and Hayo [19] point out that continuous time models tend to offer better approximations of actual assets and markets. Finally, a correct model of firm value is required if we want to correctly model the associated Keynesian stimulus.

In order to model intrinsic value of firms $V(t)$ in continuous time we model it as a stochastic differential equation, that is $dV(t)$ will be of the form

$$dV(t) = \mu(V(t), t)V(t)dt + \sigma(V, t)V(t)dW(t).$$

The advantage of modelling intrinsic firm value with stochastic differential equations is that firstly, it allows one to model the random fluctuations of firm value more realistically. The Wiener process captures random movements, multiplied by the volatility σ, and the drift μ provides a monotonically increasing trend with time t. This is consistent with the Efficient Markets Hypothesis [14] which implies that firm values should not be predictable. Secondly, a stochastic differential equation also has a predictable component (that is the drift μ) which relates to the long term value of the firm. As Keynes' stimulus method wishes to address the long term value of firms, such a component in a model is essential (to be addressed later).

In order for a stochastic differential equation to model (intrinsic) firm value in the framework of a Keynesian model, we must understand Keynes' model of firms. Keynes assumed firms follow a cyclical process [24]; firm value was subject to cycles of growth and decline and there also exists significant empirical evidence to support this theory. Keynes also gave theoretical justifications for such cycles, namely that differences in demand and output led to cyclical firm values. The cyclical nature of firms is a fundamental property of Keynesian modelling because it is this time dynamic that necessitates Government intervention (namely stimulus) to smooth out such damaging cycles. Thus if firms did not exhibit cyclical behaviour then Keynesian stimulus would not be required. Hence our stochastic differential equation for firm value must include a cyclical process if we are also to model Keynesian stimulus.

In order to capture Keynes' idea of cyclical fluctuations in firm value, one cannot use the standard model of Geometric Brownian motion:

$$dV(t)/V(t) = \mu dt + \sigma dW(t),$$

as no cyclical component exists in the equation. A stochastic differential equation that would capture cyclical behaviour is the standard Ornstein-Uhlenbeck process

$$dV(t) = \alpha(V_{eq} - V(t))dt + \sigma dW(t), \tag{4}$$

where α is the speed or rate of mean reversion and V_{eq} is the equilibrium or long run value of $V(t)$. The term α determines the frequency of cyclicity, with higher α leading to more frequent cycles over time. The term V_{eq} represents the long term value of $V(t)$: as $V(t)$ undergoes cycles moving from one extreme value to another extreme value the value $V(t)$ fluctuates around its equilibrium value V_{eq}. For the convenience of writing we replace V_{eq} with $\bar{V}$, that is $V_{eq} = \bar{V}$.

The ability of the Onstein-Uhlenbeck models to capture the cyclical behaviour of $V(t)$ has enabled it to be used in modelling a range of cyclical variables in finance and economics. For example, interest rates $r(t)$ are modelled by the well known Vasicek model:

$$dr(t) = a(b - r(t))dt + \sigma dW(t),$$

or

$$r(t) = r(0)e^{-at} + b(1 - e^{-at}) + \sigma^{-at} \int_0^t e^{as} dW(s),$$

where a is the speed of reversion and b is the long term value. Another example is the modelling of commodity prices [11] such as oil prices.

A key problem with the Ornstein-Uhlenbeck model (and consequently one of the reasons that it is not as widely adopted) is that the model takes on negative values with probability one. This is a significant disadvantage for firm value modelling because we do not assume negative firm values, hence this poses a fundamental problem in our model.

In order to overcome the problem of negative values in the Ornstein-Uhlenbeck process we use the Geometric Ornstein-Uhlenbeck process to model intrinsic firm value

$$dV(t) = \alpha(\bar{V} - V(t))V(t)dt + \sigma V(t)dW(t). \tag{5}$$

The Geometric Ornstein-Uhlenbeck model has been applied in a number of financial and economic applications for mathematical modelling (see [32] and [13] for examples). It can be shown that if $V(t) > \bar{V}$ then equation (5) will tend to push future $V(t)$ values back towards $\bar{V}$; similarly if $V(t) < \bar{V}$ then there will be a tendency for equation (5) to push future $V(t)$ values back up to $\bar{V}$. Consequently $V(t)$ tends to revert around $\bar{V}$ and the rate of reversion is determined by α.

The Expectation of equation (5) is given by

$$E[dV(t)] = (\alpha \bar{V} V(t) - \alpha V^2(t))dt,$$

and so $E[dV(t)]$ is a quadratic in $V(t)$, therefore $E[dV(t)]$ is a parabola with respect to $V(t)$, and $E[dV(t)] = 0$ at $V(t) = 0$ and $V(t) = \bar{V}$. Additionally, $E[dV(t)]$ is maximised by

$$\frac{\partial(E[dV(t)])}{\partial V(t)} = (\alpha \bar{V} - 2\alpha V(t))dt,$$

we therefore have

$$\frac{\partial(E[dV(t)])}{\partial V(t)} = 0 \text{ at } V(t) = \frac{\bar{V}}{2}.$$

Hence $E[dV(t)]$ is maximised at $V(t) = \frac{\bar{V}}{2}$.

We now have a model of firm value under a Keynesian framework (equation (5)). Our model follows a random process, hence it takes on unpredictable changes in value, which we would expect for a realistic model of firm value. Additionally, the unpredictability is consistent with the Efficient Markets Hypothesis. The firm value exhibits cyclical values and the rate of reversion is reflected in α, hence we have incorporated Keynes' idea of cycle theory [24] which is also essential to stimulus modelling.

3.2. Keynesian Model Of Stimulus

As mentioned previously, the cyclical dynamics of firm value can cause substantial problems (such as depressions and unemployment) and it is these cyclical dynamics that necessitate the intervention of Governments in markets. The aim of Government intervention is to stabilise firms so that such cyclical time dynamics no longer occur in firms. In Keynesian modelling stimulus spending is recommended (although a range of alternative measures exist the stimulus spending approach is a popular method and the most publicised method during the Global Financial Crisis). We now state the model for our stimulus with the following theorem.

Theorem 1. *For a firm whose intrinsic value V(t) follows*

$$dV(t) = \alpha(\bar{V} - V(t))V(t)dt + \sigma V(t)dW(t),$$

the associated stimulus is given by

$$\lambda V(t)dt = (\gamma - \alpha(\bar{V} - V(t)))V(t)dt.$$

Proof. The purpose of stimulus spending is to stabilise firm value so that it is no longer exhibiting cyclical swings in firm value. If one were to stabilise the firm value then the drift term should no longer exhibit mean reversion, that is the drift should be constant. If we denote a firm with firm value $V(t)$ that receives stimulus funding as $V_S(t)$ then under Keynesian stimulus $dV_S(t)$ would follow

$$dV_S(t) = \gamma V(t)dt + \sigma V(t)dW(t), \tag{6}$$

where γ is a constant. In other words, the stimulus removes the mean reverting drift of the stochastic differential equation and we have a constant drift γ. We can also interpret γ as the target drift that we aim to give to the firm after receiving stimulus. It is worth also noting that the drift of a stochastic differential equation can be taken to represent the long term and fundamental value of the firm, hence changing the drift from mean reverting to a constant directly incorporates Keynes' purpose of stimulus, that is to remove cyclical variations in the fundamental value of the firm.

As $dV_S(t)$ equals the firm value that receives stimulus then this should be equal to the firm value $dV(t)$ and the stimulus itself. We can therefore write

$$\begin{aligned} dV_S(t) &= dV(t) + \lambda V(t)dt, \\ &= \alpha(\bar{V} - V(t))V(t)dt + \sigma V(t)dW(t) + \lambda V(t)dt, \end{aligned}$$

where $\lambda V(t)dt$ denotes the stimulus, and λ is a constant. The stimulus contribution to $dV_S(t)$ (that is $\lambda V(t)$) increases with $V(t)$ because larger firm values require larger amounts of funding in order to stabilise them. We can now derive an equation for the stimulus because the previous equation and equation (6) are identical. Therefore

$$\begin{aligned} dV_S(t) = \gamma V(t)dt + \sigma V(t)dW(t) &= \alpha(\bar{V} - V(t))V(t)dt + \sigma V(t)dW(t) + \lambda V(t)dt \\ \Rightarrow \gamma dt &= \alpha(\bar{V} - V(t))dt + \lambda dt, \\ \lambda &= \gamma - \alpha(\bar{V} - V(t)). \end{aligned}$$

The total stimulus is therefore

$$\lambda V(t)dt = (\gamma - \alpha(\bar{V} - V(t)))V(t)dt. \blacksquare$$

Hence our stimulus provides a counter-cyclical fund, which is exactly the property that Keynes desired a stimulus should possess.

The stimulus model can be understood by examining the contributions of growth from each component of the model. If we assumed $dV(t)$ followed

$$\frac{dV(t)}{V(t)} = \alpha(\bar{V} - V(t))dt + \sigma dW(t),$$

then an increase in $V(t)$ would cause the growth $dV(t)/V(t)$ to decrease. This reflects the mean reverting nature of Geometric Ornstein-Uhlenbeck to restore $V(t)$ towards its equilibrium value V_{eq} or $\bar{V}$. This time dynamic leads to the cyclical behaviour that Keynes proposed to eliminate by stimulus funding. With stimulus funding we eliminate the cyclical growth so that a drop in intrinsic firm value growth (due to the above equation) is countered this with stimulus. The second component of the firm's growth, stimulus λ, is also a function of $V(t)$. Therefore, as $V(t)$ increases then λ also increases, hence the stimulus restores the growth of the firm to a constant rate of γ. Hence we remove cyclical growth, as we would expect under Keynesian modelling.

4. Options Model For Stimulus Related Contingent Claim

In this section we derive a closed form solution for the value of the contingent claim obtained in relation to the stimulus funding. Specifically, we model the contingent claim as a real call option, whose underlying is a firm value that follows the Keynesian model and receives Keynesian stimulus. We derive a closed form solution for the call option and the minimum firm value for exercise of the option to be profitable.

4.1. Options Model For Contingent Claims

It has been observed during the Global Financial Crisis (and in other periods) that Governments providing stimulus to firms frequently receive some contingent claim in exchange for the stimulus assistance. In particular, the Government receives some contract whose value is typically related to the intrinsic value $V(t)$. Additionally, the Government typically has the right, but not the obligation, to exercise its contingent claim on the firm and there is also no fixed deadline associated with exercising it. For example, during the Global Financial Crisis the UK Government provided funds and assistance to HBOS bank in exchange for part ownership of the firm, which was later sold by the Government.

In addition to the Government receiving some contingent claim on a firm, Governments typically incur substantial disposal costs K when exercising the contract. This is because the Government faces unique and significant costs when selling its stake in a firm. For example, the Government may need to consult legal regulations and make employee provisions for transferring a firm from partial public ownership to private ownership. The size of Government ownership also means that the size of the sale would put significant selling pressure on the sale price of the firm. Finally, the fact that the Government sells part ownership in a firm may prompt firm value to decrease because the firm no longer has the support of the Government (either through direct or indirect assistance). Hence there is a cost associated with the Government selling its stake.

The Government's contingent claims can therefore be modelled as a real call option $C(V)$ (or to be more precise a perpetual call option), whose underlying asset is the intrinsic firm value $V(t)$ that receives stimulus λ. The call option would have no expiration date to reflect that the Government typically has the right but not the obligation to sell its stake at some indefinite time in the future. Additionally, the strike K would model the disposal costs incurred in the sale of a stake in a firm, when it is exercised.

For the call option $C(V)$ we model the payoff as

$$\begin{aligned} C(V) &= V(t) - K, \text{ for } V(t) \geq K, \\ &= 0, \text{ for } V(t) < K. \end{aligned}$$

This payoff reflects the fact that the Government incurs significant disposal costs K on sale of its stake, hence the Government will not receive $V(t)$ but the income net of disposal costs K. Additionally, the Government has a right but no obligation to exercise the option. Hence the Government will not exercise its option unless $V(t) > K$, otherwise the Government will make a loss, thus the payoff is 0 for $V(t) < K$.

The strike K can also be understood in terms of real options theory. Firstly, K represents a 'sunk' cost or an irreversible cost, that cannot be recovered once it is expended. Secondly, the timing of the expenditure of the sunk cost K can be delayed, we are not forced to spend K until $V(t) > K$. This flexibility to spend K (or exercise the option) is extremely valuable, as we can avert negative payoffs when $V(t) < K$.

The call option $C(V)$ must have the following boundary condition from standard option pricing theory. We have

$$V(t) = 0 \Rightarrow C(V(t)) = 0, \ \forall t. \tag{7}$$

This boundary condition tells us that an underlying asset that is worthless ($V(t) = 0$) should also have a call option with 0 value.

4.2. Partial Differential Equation For The Call Option

In order to develop an option pricing equation for $C(V)$ we require a partial differential equation in $C(V)$. In order to obtain this partial differential equation we need to derive dC when the underlying asset $V(t)$ receives stimulus λ. We now give this in our Theorem.

Theorem 2. *For an option $C(V)$ whose underlying asset $V(t)$ receives stimulus $\lambda V(t)dt$, then $dC(V)$ follows*

$$dC(V) = \frac{\partial C}{\partial V}dV + \left(C(V) - \frac{\partial C}{\partial V}V\right) rdt + \frac{\partial C}{\partial V}V\lambda dt.$$

Proof. We assume we can model option $C(V)$ by a standard replicating portfolio or hedging argument. Economically, a replicating portfolio for $C(V)$ corresponds to assuming that the market for $C(V)$ is complete, that is $C(V)$ can be perfectly hedged by the underlying asset $V(t)$ and riskless bonds $B(t)$. The assumption that a market is complete is a standard economic assumption in financial models and is commonly used in option pricing (for example the Black-Scholes option pricing model [6]).

If we hedge dC in a complete market then we can achieve this using a quantity of $n(t)$ amount of $V(t)$. We therefore have the equation

$$dC(V) - n(t)dV.$$

We now also set $n(t) = \frac{\partial C}{\partial V}$, so that we have

$$dC(V) - \frac{\partial C}{\partial V}dV.$$

Now a perfectly hedged portfolio is riskless, and so should earn the riskless rate of interest r. Therefore we have

$$dC(V) - n(t)dV = B(t)rdt.$$

The riskless bond amount will increase at the riskless rate r over a time period dt. Additionally, the amount invested in a riskless bond $B(t)$ is equal to the difference $C(V) - n(t)V(t)$, and so we have

$$\begin{aligned} dC(V) &= n(t)dV + B(t)rdt, \\ &= \frac{\partial C}{\partial V}dV + \left(C(V) - \frac{\partial C}{\partial V}V\right) rdt. \end{aligned}$$

Whilst the previous equation gives a partial differential equation for dC, it assumes the underlying receives no stimulus, that is the firm has no λ. We now introduce stimulus into our model, therefore dV is replaced by dV_S so that

$$dC(V) = n(t)dV_S + B(t)rdt.$$

If we re-express this equation by using our definition of dV_S in terms of dV and λ, then we have

$$dC(V) = n(t)dV + B(t)rdt + n(t)\lambda V(t)dt.$$

Over a time period dt the Government gives stimulus, additionally the stimulus should be proportional to the amount invested in the firm, hence the total stimulus is proportional to $n(t)$, to give $n(t)\lambda V dt$. Again $B(t)$ will be the same amount as in the case with no stimulus. Hence we can re-express the previous equation as

$$dC(V) \;=\; \frac{\partial C}{\partial V}dV + \left(C(V) - \frac{\partial C}{\partial V}V\right) r dt + \frac{\partial C}{\partial V}V\lambda dt. \;\blacksquare \tag{8}$$

The equation (8) provides the partial differential for our option $C(V)$ in the presence of stimulus. However we cannot solve this equation without obtaining an additional equation for dC. Using Ito's Lemma we obtain an additional partial differential equation and this enables us to finally obtain a partial differential equation for dC which we can eventually solve. We now state this is in the following theorem.

Theorem 3. *For a firm whose intrinsic value V(t) follows*

$$dV(t) = \alpha(\bar{V} - V(t))V(t)dt + \sigma V(t)dW(t),$$

and receives the associated stimulus

$$\lambda V(t)dt = (\gamma - \alpha(\bar{V} - V(t)))V(t)dt,$$

then the associated option $C(V)$ is given by the partial differential equation

$$0 \;=\; \frac{\sigma^2 V^2(t)}{2}\frac{\partial^2 C}{\partial V^2(t)} + (r - \lambda)V(t)\frac{\partial C}{\partial V(t)} - rC. \tag{9}$$

Proof. We first derive an equation for dC using Ito's Lemma. For an Ito process dS where

$$dS = a(S,t)dt + b(S,t)dW(t),$$

then Ito's Lemma asserts that $F(S)$, a function of S, has the differential

$$dF = \left[a(S,t)\frac{\partial F}{\partial S} + \frac{b^2(S,t)}{2}\frac{\partial^2 F}{\partial S^2}\right]dt + b(S,t)\frac{\partial F}{\partial S}dW(t).$$

If we apply Ito's Lemma to our model for dV, where dV is given by

$$dV = \alpha(\bar{V} - V)V dt + \sigma V dW(t),$$

then by Ito's Lemma we therefore have for dC

$$dC \;=\; \left[\alpha(\bar{V} - V)V\frac{\partial C}{\partial V} + \frac{\sigma^2 V^2}{2}\frac{\partial^2 C}{\partial V^2}\right]dt + \sigma V\frac{\partial C}{\partial V}dW(t).$$

Alternatively, by substitution of dV in the previous equation we can re-express it as

$$dC \;=\; \frac{\partial C}{\partial V}dV + \frac{\sigma^2 V^2}{2}\frac{\partial^2 C}{\partial V^2}dt. \tag{10}$$

Hence the change in the option has a deterministic component, and a random component.

If we now equate equations (8) and (10) then we have

$$\frac{\partial C}{\partial V}dV + \frac{\sigma^2 V^2}{2}\frac{\partial^2 C}{\partial V^2}dt = \frac{\partial C}{\partial V}dV + \left(C - \frac{\partial C}{\partial V}V\right) rdt + \frac{\partial C}{\partial V}V\lambda dt.$$

If we now re-arrange this equation then we have

$$0 = \frac{\sigma^2 V^2(t)}{2}\frac{\partial^2 C}{\partial V^2(t)}dt + (r-\lambda)V(t)\frac{\partial C}{\partial V(t)}dt - rCdt,$$

and dividing by dt we have

$$0 = \frac{\sigma^2 V^2(t)}{2}\frac{\partial^2 C}{\partial V^2(t)} + (r-\lambda)V(t)\frac{\partial C}{\partial V(t)} - rC. \blacksquare$$

We have therefore derived our partial differential equation for our option $C(V)$. We note in passing that this is a second order linear homogeneous partial differential equation, with non-constant coefficients. This is similar to the Euler-Cauchy partial differential equation [26] except that the coefficient of the first order derivative is to the power of V^2 instead of V (recall that λ contains a V term).

4.3. Closed Form Solution To The Partial Differential Equation

We now wish to solve the partial differential equation (9), so that we can obtain a closed form solution for $C(V)$. We now state the solution in the following theorem.

Theorem 4. *The option $C(V)$ has closed form solution*

$$C(V) = \beta V(t)^{\kappa} H\left(\frac{2\alpha V(t)}{\sigma^2}, \kappa, m\right),$$

where

$$\begin{aligned}
\kappa &= \frac{-\Gamma + \sqrt{\Gamma^2 + 2\sigma^2 r}}{\sigma^2},\\
\Gamma &= r - \gamma + \alpha\bar{V} - \frac{\sigma^2}{2},\\
m &= 2\left(\kappa + \frac{\Gamma}{\sigma^2}\right),\\
\beta &\in \mathbb{R},
\end{aligned}$$

and $H(.)$ is the confluent hypergeometric function of the first kind (also known as Kummer's function). The function is defined as

$$H(x,\rho,v) = 1 + \frac{\rho}{v}x + \frac{\rho(\rho+1)}{v(v+1)}\frac{x^2}{2!} + \frac{\rho(\rho+1)(\rho+2)}{v(v+1)(v+2)}\frac{x^3}{3!} + \ldots\ldots,$$

where ρ and v are specified constants.

Proof. In order to obtain a solution to our partial differential equation (9), we first substitute our initial definition of λ into equation (9), so that our partial differential equation becomes

$$0 = \frac{\sigma^2 V^2}{2}\frac{\partial^2 C}{\partial V^2} + (r - \gamma + \alpha(\bar{V} - V))V\frac{\partial C}{\partial V} - rC. \tag{11}$$

This partial differential equation does not follow a standard form (unlike the heat equation). Hence there is no standard solution method to this equation, therefore we solve it by noticing that the solution to a similar Euler-Cauchy equation

$$0 = \frac{\sigma^2 V^2}{2}\frac{\partial^2 C}{\partial V^2} + \tilde{a}V\frac{\partial C}{\partial V} - rC.$$

would be of the form

$$C(V) = \tilde{\beta}V(t)^{\tilde{\kappa}},$$

where $\tilde{a}, \tilde{\kappa}, \tilde{\beta}$ are constants. Therefore we try the solution for $C(V)$ as

$$C(V) = \beta V(t)^{\kappa} f(V),$$

where $\beta \in \mathbb{R}$ is a constant. Using this solution, we therefore have

$$\begin{aligned}\frac{\partial C}{\partial V} &= \kappa\beta V^{\kappa-1}f(V) + \beta V^{\kappa}\frac{\partial f(V)}{\partial V},\\ &= \beta V^{\kappa}\left[\kappa f(V)V^{-1} + \frac{\partial f(V)}{\partial V}\right],\end{aligned}$$

and we also have by taking the second order partial differentials

$$\begin{aligned}\frac{\partial^2 C}{\partial V^2} &= (\kappa-1)\kappa\beta V^{\kappa-2}f(V) + \kappa\beta V^{\kappa-1}\frac{\partial f(V)}{\partial V} + \beta V^{\kappa}\frac{\partial^2 f(V)}{\partial V^2} + \kappa\beta V^{\kappa-1}\frac{\partial f(V)}{\partial V},\\ &= \beta V^{\kappa}\left[(\kappa-1)\kappa\beta V^{\kappa-2}f(V) + 2\kappa V^{\kappa-1}\frac{\partial f(V)}{\partial V} + \frac{\partial^2 f(V)}{\partial V^2}\right].\end{aligned}$$

If we now insert these partial derivatives into equation (11) then we have

$$\begin{aligned}0 &= \frac{\sigma^2}{2}V^2\beta V^{\kappa}\left[(\kappa-1)\kappa\beta V^{-2}f(V) + 2\kappa V^{-1}\frac{\partial f(V)}{\partial V} + \frac{\partial^2 f(V)}{\partial V^2}\right] +\\ &\quad (r-\gamma+\alpha(\bar{V}-V))V\beta V^{\kappa}\left[\kappa V^{-1}f(V) + \frac{\partial f(V)}{\partial V}\right] - r\beta V^{\kappa}f(V).\end{aligned}$$

To simplify this equation, we divide through by the constant β and rearranging gives

$$\begin{aligned}0 &= V^{\kappa}f(V)\left[\frac{1}{2}\sigma^2\kappa(\kappa-1) + (r-\gamma+\alpha\bar{V})\kappa - r\right] +\\ &\quad V^{\kappa+1}\left[\frac{1}{2}\sigma^2 V\frac{\partial^2 f(V)}{\partial V^2} + (\sigma^2\kappa + r - \gamma + \alpha(\bar{V}-V))\frac{\partial f(V)}{\partial V} - \alpha\kappa f(V)\right].\end{aligned}$$

Now the previous equation must hold for all values of V, including $V \neq 0$. Therefore, if we examine the top part of the equation, that is

$$V^{\kappa} f(V) \left[\frac{1}{2}\sigma^2\kappa(\kappa - 1) + (r - \gamma + \alpha\bar{V})\kappa - r \right] = 0,$$

then we deduce that

$$\frac{1}{2}\sigma^2\kappa(\kappa - 1) + (r - \gamma + \alpha\bar{V})\kappa - r = 0. \tag{12}$$

Similarly, if we also examine the bottom equation, that is

$$V^{\kappa+1} \left[\frac{1}{2}\sigma^2 V \frac{\partial^2 f(V)}{\partial V^2} + (\sigma^2\kappa + r - \gamma + \alpha(\bar{V} - V)) \frac{\partial f(V)}{\partial V} - \alpha\kappa f(V) \right] = 0,$$

then we can also conclude that

$$\frac{1}{2}\sigma^2 V \frac{\partial^2 f(V)}{\partial V^2} + (\sigma^2\kappa + r - \gamma + \alpha(\bar{V} - V)) \frac{\partial f(V)}{\partial V} - \alpha\kappa f(V) = 0. \tag{13}$$

We will require both equations (12) and (13) to find the option pricing solution $C(V)$.

One can solve equation (13) to obtain a solution to $f(V)$ in $C(V) = \beta V^{\kappa} f(V)$, by observing that equation (13) follows Kummer's differential equation. Let w be a function of z, that is $w = f(z)$, then Kummer's differential equation is given by

$$z\frac{d^2w}{dz^2} + (m - z)\frac{dw}{dz} - \rho w = 0, \tag{14}$$

where m, ρ are constants. To transform equation (13) into Kummer's differential equation, we make the substitutions

$$\begin{aligned} z &= \frac{2\alpha V}{\sigma^2} \Rightarrow V = \frac{\sigma^2 z}{2\alpha}, \\ \rho &= \kappa, \end{aligned}$$

and

$$m = 2\kappa + \frac{2(r - \gamma + \alpha\bar{V})}{\sigma^2}. \tag{15}$$

If we set $f(V) = w(z)$ then

$$\begin{aligned} \frac{\partial f(V)}{\partial V} &= \frac{dw}{dz} . \frac{dz}{dV}, \\ &= \frac{dw}{dz} . \left(\frac{2\alpha}{\sigma^2} \right). \end{aligned}$$

Therefore

$$\begin{aligned}(\sigma^2\kappa + r - \gamma + \alpha(\bar{V} - V))\frac{\partial f(V)}{\partial V} &= \frac{dw}{dz}.\left(\frac{2\alpha}{\sigma^2}\right).\left(\sigma^2\kappa + r - \gamma + \alpha\left(\bar{V} - \frac{\sigma^2 z}{2\alpha}\right)\right), \\ &= \alpha(m - z).\frac{dw}{dz}.\end{aligned}$$

We also have

$$\begin{aligned}\frac{\partial^2 f(V)}{\partial V^2} &= \frac{\partial}{\partial V}\left(\frac{\partial f(V)}{\partial V}\right), \\ &= \left(\frac{d}{dz}\left(\frac{dw}{dz}.\frac{2\alpha}{\sigma^2}\right)\right).\frac{dz}{dV}, \\ &= \frac{d^2w}{dz^2}.\left(\frac{2\alpha}{\sigma^2}\right)^2.\end{aligned}$$

Therefore

$$\begin{aligned}\frac{1}{2}\sigma^2 V\frac{\partial^2 f(V)}{\partial V^2} &= \frac{1}{2}\sigma^2\left(\frac{\sigma^2 z}{2\alpha}\right)\left(\frac{d^2w}{dz^2}.\left(\frac{2\alpha}{\sigma^2}\right)^2\right), \\ &= \alpha z\frac{d^2w}{dz^2}.\end{aligned}$$

Therefore equation (13) becomes

$$\alpha z\frac{d^2w}{dz^2} + \alpha(m - z)\frac{dw}{dz} - \alpha\rho w = 0,$$

and given that α is a constant, we then divide by α to obtain Kummer's differential equation

$$z\frac{d^2w}{dz^2} + (m - z)\frac{dw}{dz} - \rho w = 0.$$

The solution to the Kummer differential equation (14) is the confluent hypergeometric function [36], specifically the confluent hypergeometric function of the first kind. The confluent hypergeometric function is denoted by $H(z, \rho, m)$ and defined by

$$\begin{aligned}H(z, \rho, m) &= 1 + \frac{\rho}{m}z + \frac{\rho(\rho+1)}{m(m+1)}\frac{z^2}{2!} + \frac{\rho(\rho+1)(\rho+2)}{m(m+1)(m+2)}\frac{z^3}{3!} + \ldots\ldots \\ &= \sum_{n=0}^{\infty}\frac{z^n\rho^{\tilde{n}}}{m^{\tilde{n}}n!},\end{aligned}$$

where $a^{\tilde{n}} = a(a+1)(a+2)\ldots..$ is the the rising factorial of a to the power n. Hence our solution for $f(V)$ is

$$f(V) = H(z, \kappa, m) = H\left(\frac{2\alpha V}{\sigma^2}, \kappa, m\right).$$

Therefore our final solution for $C(V)$ is

$$C(V) = \beta V^{\kappa} f(V) = \beta V^{\kappa} H\left(\frac{2\alpha V}{\sigma^2}, \kappa, m\right). \tag{16}$$

To determine κ we solve equation (12) in κ. First we observe that equation (12) is a quadratic in κ, that is if we re-arrange equation (12) then we have

$$\frac{1}{2}\sigma^2\kappa^2 + \left(r - \gamma + \alpha\bar{V} - \frac{1}{2}\sigma^2\right)\kappa - r = 0,$$

and this equation is of the form $a\kappa^2 + b\kappa + c = 0$, where $a = \frac{1}{2}\sigma^2$, $b = \left(r - \gamma + \alpha\bar{V} - \frac{1}{2}\sigma^2\right)$, $c = -r$. Therefore we can solve the quadratic in κ with

$$\kappa = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The solutions are $\kappa = \{\phi, \hat{\phi}\}$ where $\kappa \in \mathbb{R}$, therefore we have

$$\begin{aligned}
\phi &= \frac{-\left(r - \gamma + \alpha\bar{V} - \frac{1}{2}\sigma^2\right) + \sqrt{\left(r - \gamma + \alpha\bar{V} - \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2 r}}{\sigma^2}, \\
\hat{\phi} &= \frac{-\left(r - \gamma + \alpha\bar{V} - \frac{1}{2}\sigma^2\right) - \sqrt{\left(r - \gamma + \alpha\bar{V} - \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2 r}}{\sigma^2}.
\end{aligned}$$

If we re-express the solutions $\kappa = \{\phi, \hat{\phi}\}$ with

$$\Gamma = r - \gamma + \alpha\bar{V} - \frac{\sigma^2}{2},$$

then we have

$$\kappa = \frac{-\Gamma \pm \sqrt{\Gamma^2 + 2\sigma^2 r}}{\sigma^2}. \tag{17}$$

Given that $\sigma^2 r \in \mathbb{R}^+$, $\forall\ r, \sigma$ therefore

$$\sqrt{\Gamma^2 + 2\sigma^2 r} > \Gamma,\ \forall \Gamma \in \mathbb{R}.$$

We can therefore conclude that $\phi \in \mathbb{R}^+$ and $\hat{\phi} \in \mathbb{R}^-$. If we now take into account the boundary condition (equation (7)) and assuming $C(V) = \beta V^{\kappa} f(V)$ then κ must be positive to satisfy the boundary condition ($V(t) = 0 \Rightarrow C(0) = 0$). As $\hat{\phi}$ is negative we must therefore discard this solution and so $\kappa = \phi$. Therefore we have

$$\kappa = \frac{-\Gamma + \sqrt{\Gamma^2 + 2\sigma^2 r}}{\sigma^2}. \blacksquare$$

We now have a closed form solution for the option when the underlying asset is a firm that behaves according to Keynesian modelling and receives Keynesian stimulus. The option can be used to model any contingent claims received by the Government in exchange for providing stimulus assistance to a firm, and the disposal costs are incorporated by K.

4.4. Option Exercise Barrier and Beta

We have derived an equation for the value of the option associated with stimulus, so that Governments can value their contingent claims. However, Governments and decision makers would also like to know the conditions that are required for optimal exercise of the option. We already know that $V(t) > K$ (otherwise the option would theoretically give a negative or 0 payoff), however we would like to determine the minimum firm value required to exercise the option.

In this section we derive the minimum firm value V^θ required for option exercise to be profitable. In deriving V^θ we also derive an equation for β in terms of V^θ. We now state this in the following theorem.

Theorem 5. *The option $C(V)$ is subject to the following boundary conditions*

$$\begin{aligned} C(V(t)) &= 0, \forall t, \text{ if } V(t)=0, \\ C(V^\theta) &= V^\theta - K, \\ \frac{\partial C}{\partial V}\Big|_{V=V^\theta} &= 1, \end{aligned}$$

where K is the strike price, $V(t) = V^\theta$ denotes the minimum value to exercise the option $C(V)$. Consequently, β in $C(V)$ is given by

$$\beta = \frac{V^\theta - K}{H\left(\frac{2\alpha V^\theta}{\sigma^2}, \kappa, m\right)(V^\theta)^\kappa}.$$

Proof. First we must recognise that in option theory there exists an exercise barrier or value V^θ, such that if $V(t) \geq V^\theta(t)$ then it is always optimal to exercise the option. This means the holder of the option $C(V)$ should not just exercise the option if $V(t)$ is greater than the disposal costs ($V(t) > K$). Hence the value V^θ has implications for decision makers of the contingent claims associated with stimulus.

According to option theory, options have the following boundary condition (known as the value matching condition) where

$$C(V) = V(t) - K, \text{ for } V(t) \geq V^\theta. \tag{18}$$

This can be understood because at exercise the option payoff is always $V(t) - K$

if we exercise the option, and we have assumed for $V(t) \geq V^\theta$ that the option is exercised. We can therefore write

$$\begin{aligned} C(V) &= V(t) - K = \beta V^\kappa f(V(t)), \text{ for } V(t) \geq V^\theta, \\ \Rightarrow \beta &= \frac{V(t) - K}{f(V(t))V(t)^\kappa}, \end{aligned}$$

or alternatively we can write

$$\beta = \frac{V(t) - K}{H\left(\dfrac{2\alpha V(t)}{\sigma^2}, \kappa, m\right) V(t)^\kappa}. \tag{19}$$

Hence the value matching condition enables us to determine β, assuming that V^θ is known. Typically, V^θ can be calculated for options and so one can obtain β with $V(t) = V^\theta$, so that

$$\beta = \frac{V^\theta - K}{H\left(\dfrac{2\alpha V^\theta}{\sigma^2}, \kappa, m\right) (V^\theta)^\kappa}.$$

If V^θ is not known we can determine V^θ using the third boundary condition, also known as the high-contact or smooth pasting condition. At $V(t) \geq V^\theta$ we have $C(V) = V(t) - K$ and if we take the partial derivative

$$C(V) = V(t) - K \Rightarrow \frac{\partial C}{\partial V} = 1, \text{ for } V(t) \geq V^\theta.$$

Therefore we have for $V(t) > V^\theta$:

$$\begin{aligned} \frac{\partial C}{\partial V} = \frac{\partial(\beta V(t)^\kappa f(V(t)))}{\partial V} &= \beta V(t)^\kappa \left[\frac{\kappa}{V(t)} f(V(t)) + \frac{\partial f(V(t))}{\partial V}\right] = 1, \\ \Rightarrow \beta V(t)^\kappa \left(\frac{\kappa}{V(t)} H + H'\right) &= 1, \end{aligned}$$

where H denotes $H\left(\dfrac{2\alpha V(t)}{\sigma^2}, \kappa, m\right)$, and H' denotes $\dfrac{\partial H\left(\dfrac{2\alpha V(t)}{\sigma^2}, \kappa, m\right)}{\partial V}$. Using equation (19) for β and substituting this into our equation we then have

$$\left(V(t) - K\right) \frac{H'(.)}{H(.)}\Bigg] = 1, \text{ for } V(t) \geq V^\theta. \blacksquare \tag{20}$$

We note in passing that one would find V^θ by numerical solutions as it is non-trivial to obtain by analytical methods. Additionally, to improve computation time we can express our option pricing equation as:

$$\begin{aligned} C(V) &= \beta V(t)^{\kappa} H\left(\frac{2\alpha V(t)}{\sigma^2}, \kappa, m\right), \text{ for } V(t) < V^{\theta}, \\ &= V(t) - K, \text{ for } V(t) \geq V^{\theta}. \end{aligned}$$

The first equation is from Theorem 4, however the second equation is obtained from the value matching condition (that is equation (18)). In using the second equation we significantly improve computation time because we simply subtract a constant (K) from $V(t)$ to obtain $C(V)$ and this can be computed very fast. If we used Theorem 4 to calculate $C(V)$ for $V(t) \geq V^{\theta}$ then we would require computation of many nonlinear functions, which would be more time consuming.

5. Numerical Experiments

In this section we conduct numerical experiments to analyse the option price $C(V)$ and stimulus value λ as key input parameters are varied, such as α and $V(t)$. We calculate option price values $C(V)$ and stimulus λ, provide graphs and analyse the results. The results provide insight into the impact of different input parameters (such as $\bar{V}$ and α) upon stimulus and option price values.

5.1. Introduction

In our numerical experiments we calculate the option price $C(V)$ and stimulus value λ, over different reversion values α, firm values $V(t)$ and equilibrium values $\bar{V}$. The option price $C(V)$ was calculated using equation (16), κ was calculated using equation (17), m was calculated using equation (15) and stimulus λ was calculated using the equation

$$\lambda = \gamma - \alpha(\bar{V} - V(t)). \tag{21}$$

Whilst any numerical values could be chosen for our input parameters, for the benefit of exposition we provide the details here. We set $K = 1$ for convenience, although any value of K could be chosen without affecting our results a convenient choice aids analysis. We set $r = 0.04$ as interest rates were set at low levels during the initiation of stimulus. Typically interest rates are low if the economy is not performing well and this is also most likely to be the time when stimulus will be introduced, hence we reflect this in the models.

We set $\gamma = 0.08$ as γ represents the target growth rate of firms; given that stock market indexes grow approximately at 10% a year, we would want firms under stimulus to perform almost as well as the average firm on the stock market. Hence we expect γ to be below 10% but close to it, and so we chose 8%. Finally we set

$\sigma = 0.2$ as firms requiring stimulus typically suffer from higher risk (and risk is typically related to firm volatility). Consequently we choose a higher figure of σ.

The values of $\bar{V}$ were chosen to be varied from 0.5-1.5, so that the equilibrium value of firm value gave at-the-money, in-the-money and out-of-the-money call options when strike $K = 1$. This enabled us to analyse the relation between equilibrium value $\bar{V}$ and the strike K. The value of α reflects the level of cyclicity in the firm and cyclicity is an important characteristic in Keynesian modelling. Hence it is important to see the impact of cyclicity or reversion upon option pricing and stimulus. The value of α was varied from 0.05-0.5 so that α was increased by a magnitude of 10, to examine the impact of α on stimulus and option prices.

5.2. Results

We now present our results on option pricing $C(V)$ and stimulus λ, for different values of α, $V(t)$ and $\bar{V}$.

5.2.1. Results For α=0.05

Figure 1: Graph Of Option Prices $C(V)$ For Different Values V And $\bar{V}$ (V_{eq})

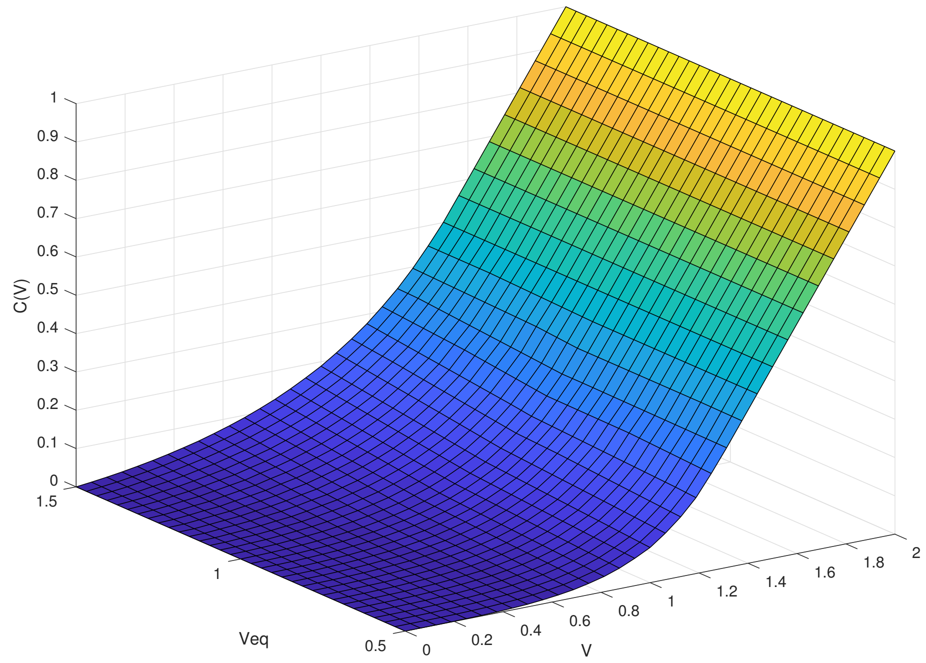


Figure 2: Graph Of Option Prices $C(V)$ For Different Values V And $\bar{V}$ (V_{eq})

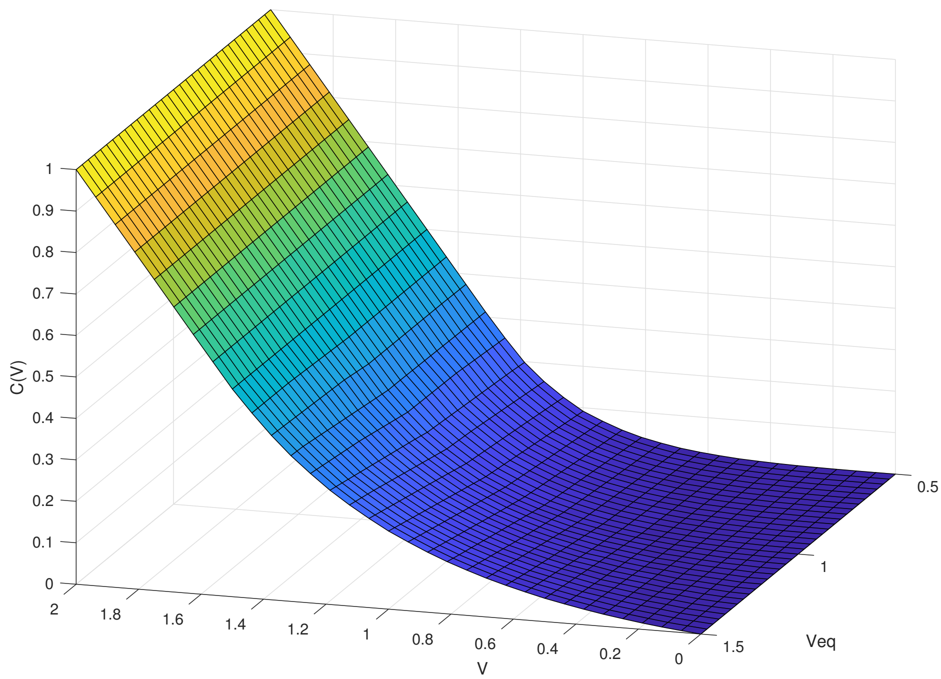
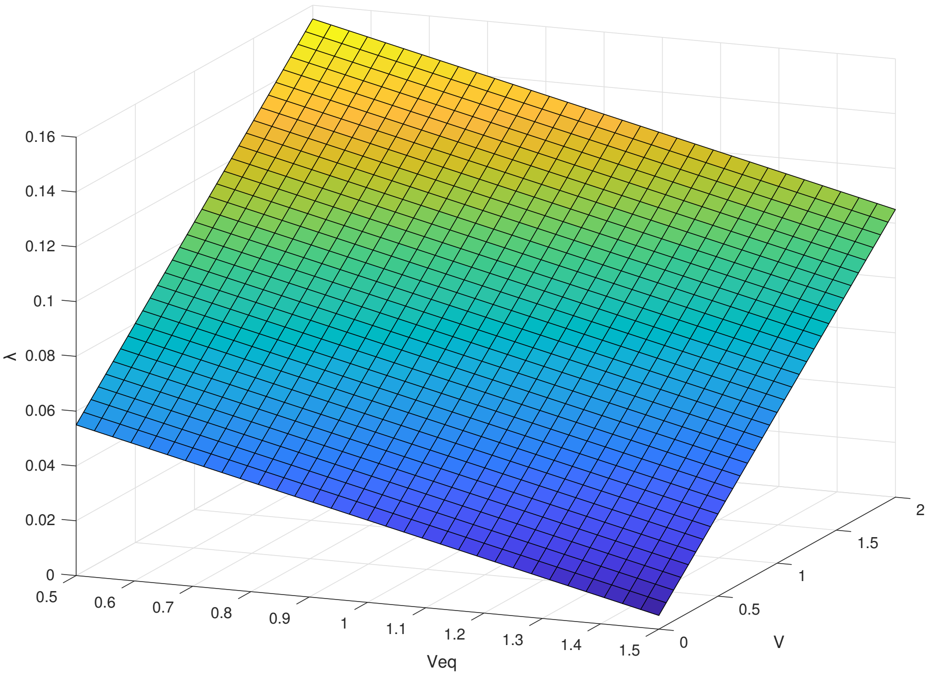


Figure 3: Graph Of Stimulus λ For Different Values V And $\bar{V}$ (V_{eq})



5.2.2. Results For α=0.1

Figure 4: Graph Of Option Prices $C(V)$ For Different Values V And $\bar{V}$ (V_{eq})

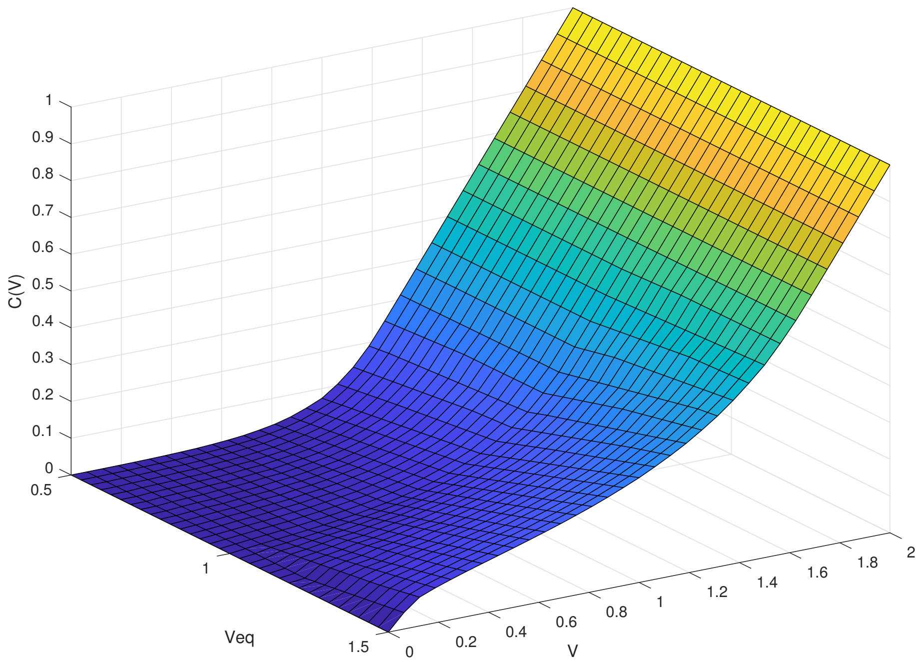


Figure 5: Graph Of Option Prices $C(V)$ For Different Values V And $\bar{V}$ (V_{eq})

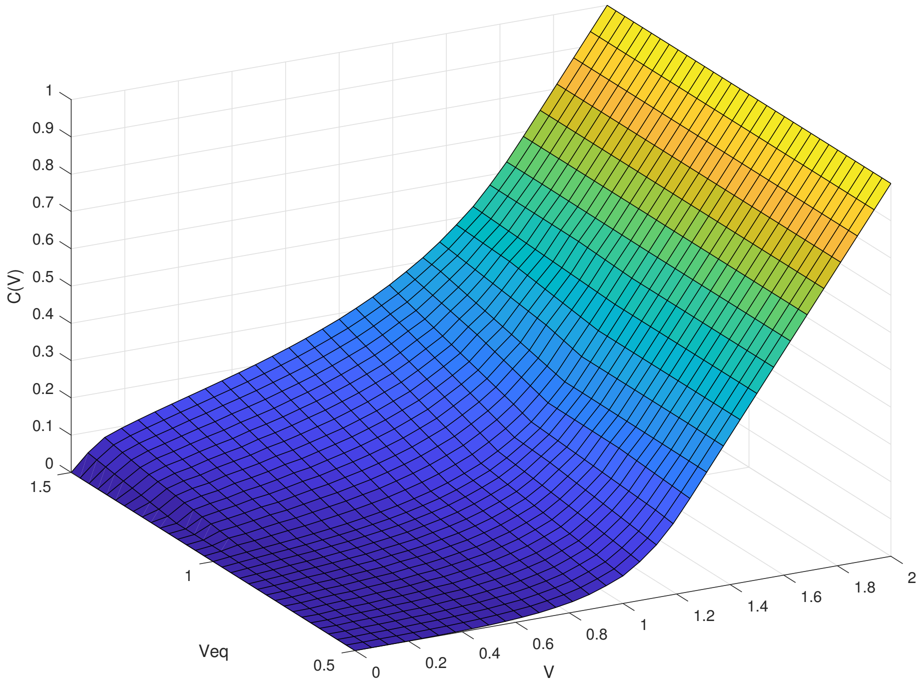
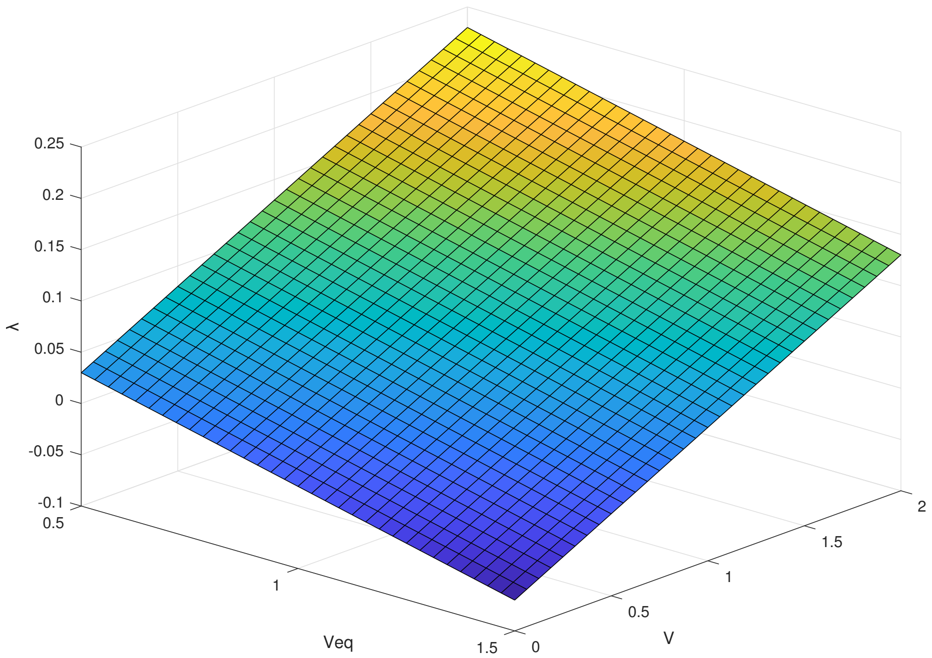


Figure 6: Graph Of Stimulus λ For Different Values V And $\bar{V}$ (V_{eq})



5.2.3. Results For α=0.5

Figure 7: Graph Of Option Prices $C(V)$ For Different Values V And $\bar{V}$ (V_{eq})

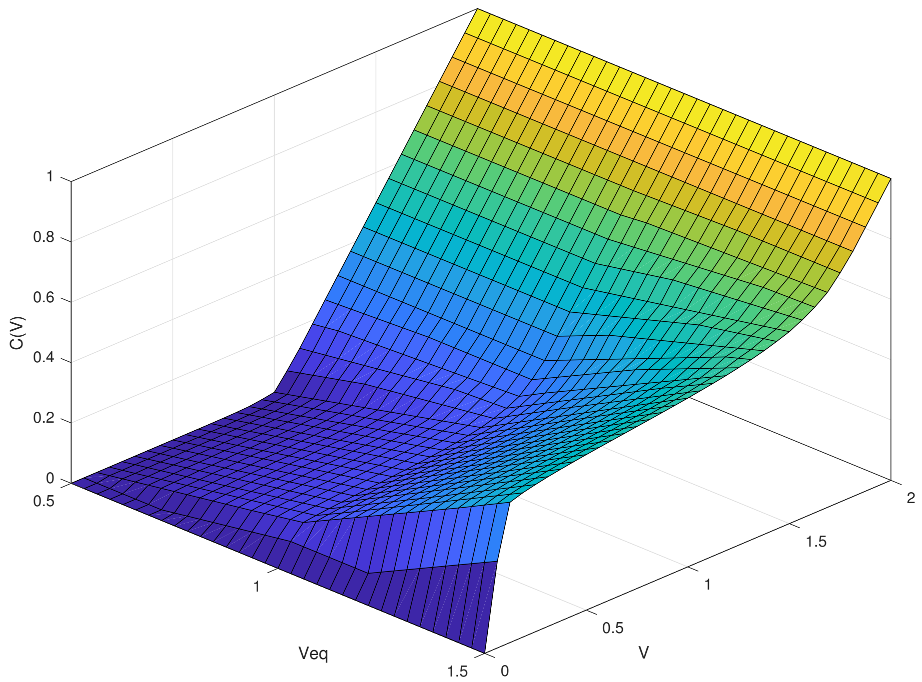


Figure 8: Graph Of Option Prices $C(V)$ For Different Values V And $\bar{V}$ (V_{eq}), for $\alpha = 0.5$

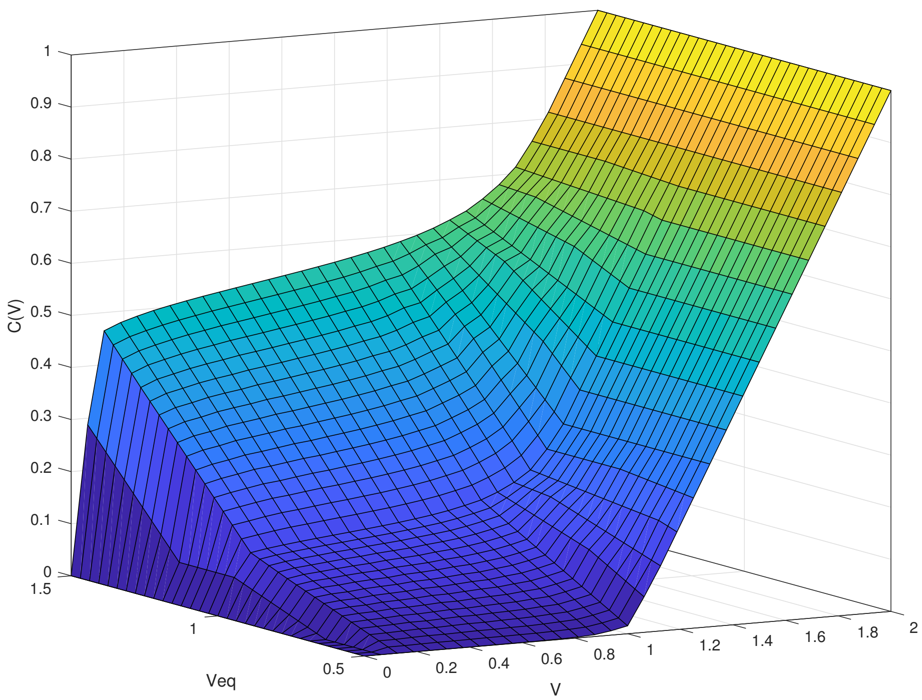


Figure 9: Graph Of Option Prices $C(V)$ For Different Values V And $\bar{V}$ (V_{eq})

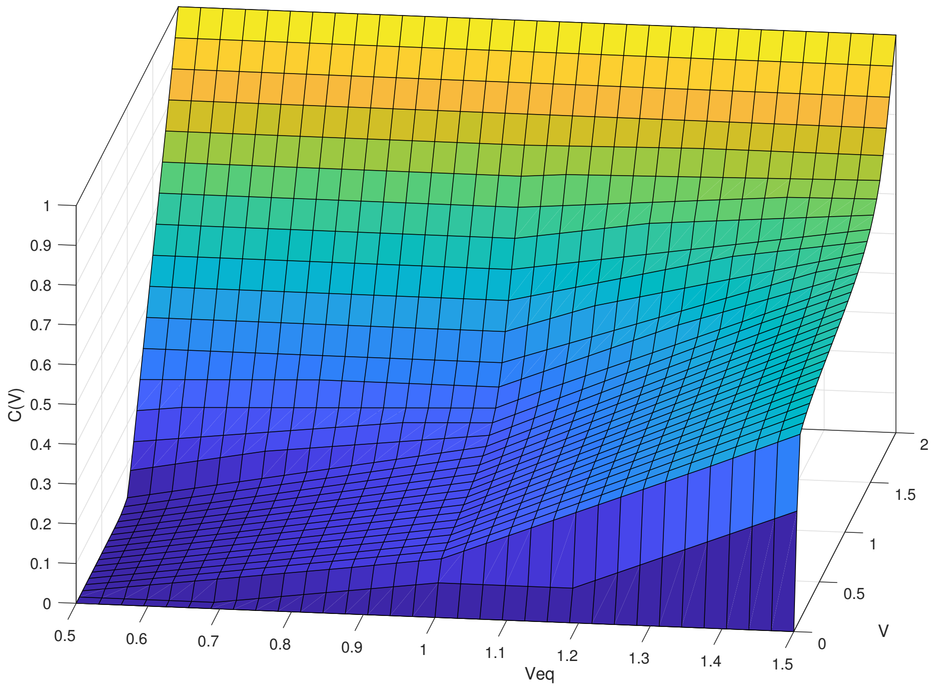
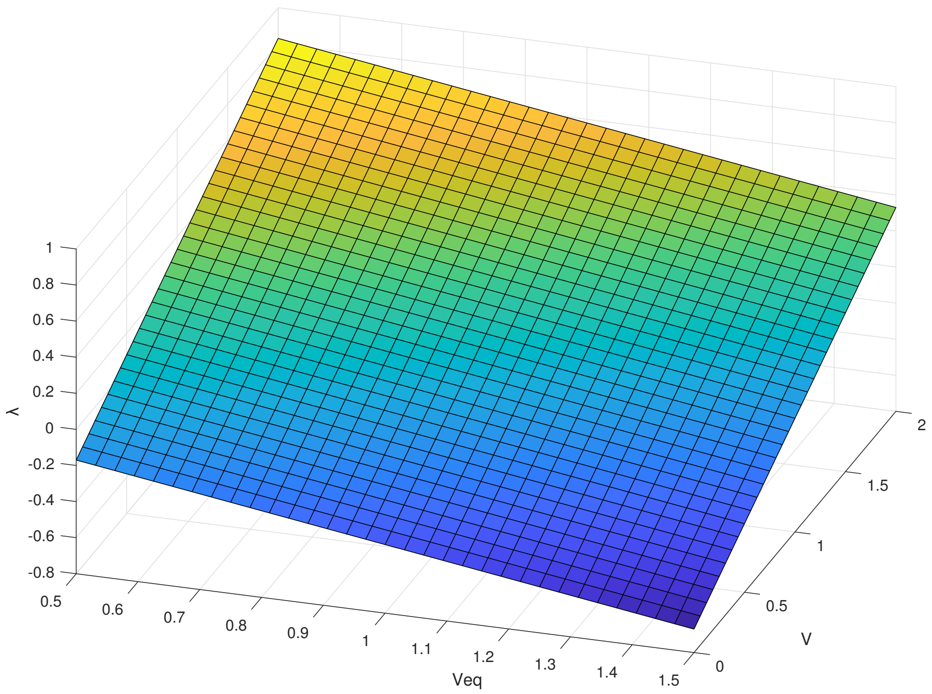


Figure 10: Graph Of Stimulus λ For Different Values V And $\bar{V}$ (V_{eq}), for $\alpha = 0.1$



5.3. Analysis

The Figures 1-10 give the option pricing and stimulus results of our numerical experiments, over different parameter values. In particular we vary α over 0.05, 0.1, 0.5 and observe the relation upon $C(V), V, V_{eq}, \lambda$. The option prices satisfy the boundary condition that $C(V)$ is 0 at V=0, also $C(V)$ increases with V which is expected because $C(V) = \beta V^{\kappa} H\left(\frac{2\alpha V}{\sigma^2}, \kappa, m\right)$. Similarly, we notice that increasing $\bar{V}$ leads to an increase in $C(V)$, over different reversion values α.

We observe that the option price $C(V)$ tends to increase with V_{eq} (or $\bar{V}$), for different levels of V and α. This implies that the contingent claim given to Governments is generally worth more if the long term value of the firm is also higher. This is an economically consistent result, since an option should be worth more if the underlying asset has higher long term fundamental value. Given that financial institutions such as banks generally have high long term fundamental values, our model therefore suggest that an option obtained on the bank should be good value for money.

As we would expect most of the figures exhibit a "hockey stick" shaped graph, which is a common shape for option graphs. Additionally, as V exceeds V^{θ} the option pricing equation follows $C = V - K$ and so the option payoff is linear with respect to V in the graphs. We can also notice that as V_{eq} increases then the linear part of the graph occurs at higher values of V. Therefore this implies that higher V_{eq} leads to higher V^{θ} values, over different α values. Hence firms with higher long term or fundamental firm value will also require higher firm values V before it is profitable for the Government to exercise its option. This may explain the opposition of politicians to give funding to firms because the firm many require many years before V is sufficiently high enough to enable exercising the option.

The most striking feature of all the figures is the impact of α and $\bar{V}$ for higher values, particularly if we compare Figures 7-9 to Figures 1, 2, 4 and 5. We notice that for $\alpha = 0.05$ that C(V) is convex for all V_{eq} and V values. However as the rate of mean reversion increases (that is $\alpha = 0.1$ or $\alpha = 0.5$) the convexity is lost and in fact for low values of V we have a concave relation. This implies there is a non-convex relation between option value $C(V)$ and V for high mean reversions. As the mean reversion tends to be dependent upon the behaviour of the economy, the results imply that different economies will have different option values with V, and minimum exercise values V^{θ}. Hence Keynesian stimulus and contingent claims modelling is not a homogeneous or "one size fits all" approach, the Keynesian approach needs to be adjusted to each economy.

In Figures 3, 6 and 10 stimulus λ is plotted for different α, $\bar{V}$ and V values. Whilst the figures for $C(V)$ change significantly in shape and behaviour as α, $\bar{V}$

and V are varied, the shape and behaviour of λ generally remains consistent. In other words there is a linear relationship between λ, $\bar{V}$ and V, and this is expected given that λ follows equation (21). This is also a useful result because it suggests that Government stimulus funding is quite predictable, even with fluctuations in long term equilibrium value $\bar{V}$, firm value V and cyclical changes α. However, it is the value of the Government's contingent claim that can vary significantly with parameters.

6. Conclusion

Since the commencement of the Global Financial Crisis, Keynesian modelling and Keynesian stimulus have been gaining increasing credibility and interest from Academics, Economists, policy makers as well as the media. In this paper we develop a new framework for modelling firms and Keynesian stimulus, under a Keynesian framework. We model the dynamics in our framework by using stochastic differential equations, rather than standard time series models, which is a new approach to current Keynesian models.

In this paper we devise a contingent claims model for Government stimulus, using a real call option model (specifically a perpetual call option). We have derived the partial differential equation for our option model, where the underlying asset follows a Keynesian model and receives Keynesian stimulus. We have derived a closed form solution for this option and conducted numerical experiments to illustrate the model. We have shown how important parameters such as the long term equilibrium firm value $\bar{V}$, the firm value V and rate of reversion α affect option prices $C(V)$ and stimulus λ. In particular, such results show the importance to manage the stimulus approaches according to each particular economy, rather than applying a generic approach to all economies.

In terms of future work, we would like to explore the impact of finite expirations upon the valuation of options and stimulus and compare these to our current model. We would also like to develop our model to include more realistic factors, for example stochastic interest rates as interest rates have a significant impact on economic dynamics as well as the timing of stimulus spending. Another important aspect in our model to incorporate is credit risk related factors, since stimulus spending is mainly concerned with mitigating default risks. Finally, we would also like to develop our model to incorporate liquidity factors as liquidity factors can have a significant impact on firms. In fact in the Global Financial Crisis the liquidity risk of many firms was heavily underestimated, and led to significant economic problems. Hence liquidity risk management is important in the model.

[1] Y. Ait-Sahalia, J. Cacho-Diaz, and R. J. Laeven. Modeling financial contagion using mutually exciting jump processes. *Journal of Financial Economics*, 117(3):585–606, 2015.

[2] G. A. Akerlof. The missing motivation in macroeconomics. *American Economic Review*, 97(1):5–36, 2007.

[3] M. M. Andreasen. How to maximize the likelihood function for a dsge model. *Computational Economics*, 35(2):127–154, 2009.

[4] L. Benothman and F. Trabelsi. Asymptotic analysis of european and american options with jumps in the underlying. *International Journal of Mathematics in Operational Research*, 4(5):548, 2012.

[5] N. H. Bingham and K. Rudiger. *Risk-neutral valuation: pricing and hedging of financial derivatives.* Springer, 2004.

[6] F. Black and M. Scholes. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81(3):637–654, 1973.

[7] R. A. Brealey, S. C. Myers, and F. Allen. *Principles of corporate finance.* McGraw-Hill Education, 2017.

[8] D. Breeden and R. Litzenberger. Prices of State-Contingent Claims Implicit in Option Prices. *The Journal of Business*, 51(4):621–651, 1978.

[9] M. K. Brunnermeier and Y. Sannikov. A macroeconomic model with a financial sector. *National Bank of Belgium Working Paper*, 236, 2012.

[10] Y.-W. Cheung, R. Fatum, and Y. Yamamoto. The exchange rate effects of macro news after the global financial crisis. *Journal of International Money and Finance*, 95:424–443, 2019.

[11] A. K. Dixit and R. S. Pindyck. *Investment under uncertainty.* Princeton University Press, 1994.

[12] B. Dupire. Pricing with a smile. *Risk*, 7(1):18–20, 1994.

[13] D. Epstein, N. Mayor, P. Schonbucher, A. Whalley, and P. Wilmott. The valuation of a firm advertising optimally. *The Quarterly Review of Economics and Finance*, 38(2):149–166, 1998.

[14] E. Fama. The behaviour of stock market prices. *Journal of Business*, 38(1):34–105, 1965.

[15] L. P. L. Fávero. The zero-inflated negative binomial multilevel model: demonstrated by a brazilian dataset. *International Journal of Mathematics in Operational Research*, 11(1):90, 2017.

[16] R. Goel. Partial backlogging eoq model with trade-credit facility and inflationary environment. *International Journal of Mathematics in Operational Research*, 13(3):303, 2018.

[17] J. L. Gunn, I. K. Khurana, and S. E. Stein. Determinants and consequences of timely asset impairments during the financial crisis. *Journal of Business Finance Accounting*, 45(1-2):3–39, 2017.

[18] R. D. Haas and N. V. Horen. International shock transmission after the lehman brothers collapse: Evidence from syndicated lending. *American Economic Review*, 102(3):231–237, 2012.

[19] B. Hayo and B. Forster. Monetary and fiscal policy in times of crises: A new keynesian perspective in continuous time. *The Manchester School*, (1), 2018.

[20] Z. A. A. Hemyari, H. A. Dabag, and A. A. Humairi. A class of always pooling shrinkage testimators for the weibull model. *International Journal of Mathematics in Operational Research*, 14(3):407, 2019.

[21] J. Hicks. Mr. Keynes and the 'Classics': A Suggested Interpretation. *Econometrica*, 5(2):147–159, 1937.

[22] S. Jayech and N. B. Zina. A copula-based approach to financial contagion in the foreign exchange markets. *International Journal of Mathematics in Operational Research*, 3(6):636, 2011.

[23] Y. Kansal, P. Kapur, U. Kumar, and D. Kumar. Prioritising vulnerabilities using anp and evaluating their optimal discovery and patch release time. *International Journal of Mathematics in Operational Research*, 14(2):236, 2019.

[24] J. Keynes. *The general theory of employment interest and money.* Macmillan London, 1936.

[25] F. Keynia and A. Heydari. A new short-term energy price forecasting method based on wavelet neural network. *International Journal of Mathematics in Operational Research*, 14(1):1, 2019.

[26] E. Kreyszig. *Advanced Engineering Mathematics.* John Wiley & Sons, 1993.

[27] C. M. C. Lee, J. Myers, and B. Swaminathan. What is the intrinsic value of the dow? *The Journal of Finance*, 54(5):1693–1741, 1999.

[28] H. E. Leland. Corporate debt value, bond covenants, and optimal capital structure. *The Journal of Finance*, 49(4):1213, 1994.

[29] H. E. Leland and K. B. Toft. Optimal capital structure, endogenous bankruptcy, and the term structure of credit spreads. *The Journal of Finance*, 51(3):987, 1996.

[30] S. M. Li and A. H. Spencer. Effectiveness of the australian fiscal stimulus package: A dsge analysis. *Economic Record*, 92(296):94–120, Dec 2015.

[31] J. Lindé. Estimating new-keynesian phillips curves: A full information maximum likelihood approach. *Journal of Monetary Economics*, 52(6):1135–1149, 2005.

[32] G. E. Metcalf and K. A. Hassett. Investment under alternative return assumptions comparing random walks and mean reversion. *Journal of Economic Dynamics and Control*, 19(8):1471–1488, 1995.

[33] S. W. Mgobhozi and E. Chikodza. Optimal combined dividend and reinsurance policies under interest rate in lévy markets. *International Journal of Mathematics in Operational Research*, 10(1):69, 2017.

[34] A. Mohtashami, A. Alinezhad, and M. J. N. Sadeghloo. Measuring the productivity changes with double frontiers data envelopment analysis for two-stage processes. *International Journal of Mathematics in Operational Research*, 13(4):513, 2018.

[35] N. J. B. Naouara and F. Trabelsi. Boundary classification and simulation of one-dimensional diffusion processes. *International Journal of Mathematics in Operational Research*, 11(1):107, 2017.

[36] M. Nardin, W. Perger, and A. Bhalla. Numerical evaluation of the confluent hypergeometric function for complex arguments of large magnitudes. *Journal of Computational and Applied Mathematics*, 39(2):193–200, 1992.

[37] N. H. Shah, M. Y. Jani, and U. Chaudhari. Optimal ordering policy for deteriorating items under down-stream trade credit dependent quadratic demand with full up-stream trade credit and partial down-stream trade credit. *International Journal of Mathematics in Operational Research*, 12(3):378, 2018.

[38] V. L. Smith, G. L. Suchanek, and A. W. Williams. Bubbles, crashes, and endogenous expectations in experimental spot asset markets. *Econometrica*, 56(5):1119, 1988.

[39] E. Stockhammer. Post-keynesian economics. *Rethinking Economics*, page 6–18, 2017.

[40] K. R. Subramanyam and M. Venkatachalam. Earnings, cash flows, and ex post intrinsic value of equity. *The Accounting Review*, 82(2):457–481, 2007.

[41] X. Wang and C. Carlsson. Patent-related decision-making with fuzzy real option analysis. *International Journal of Mathematics in Operational Research*, 9(4):467, 2016.

[42] J. Yu. Econometric analysis of continuous time models: A survey of peter phillips's work and some new results. *Econometric Theory*, 30(04):737–774, Jan 2014.