

Can the energy bound $E \geq 0$ imply supersymmetry?

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We utilize the integrality conjecture to show that the torus partition function of a fermionic rational conformal theory in the Ramond-Ramond sector becomes a constant when the bound $h^R \geq \frac{c}{24}$ is satisfied, where h^R denotes the conformal weights of Ramond states and c is the central charge. The constant-valued Ramond-Ramond partition function strongly suggests the presence of supersymmetry unless a given theory has free fermions. The lower bound $h^R \geq \frac{c}{24}$ can then be identified with the unitarity bound of $\mathcal{N} = 1$ supersymmetry. We thus propose that, for rational CFTs without free fermions, $(h^R - c/24) \geq 0$ can imply supersymmetry.

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I. INTRODUCTION

The energy spectrum of a supersymmetric theory is constrained to be non-negative, $E \geq 0$. In contrast, does the lower bound on the energy spectrum imply supersymmetry? The answer is no, in general. To see this, let us consider a nonsupersymmetric quantum system with a potential that allows a normalizable ground state. By adding to the Lagrangian a suitable constant energy, one can always make the energy spectrum positive while maintaining the nonsupersymmetric nature of the system. However, we propose that the answer to the above converse question becomes positive for fermionic rational conformal field theories (RCFTs) in two dimensions.

We start with a two-dimensional fermionic CFT on a circle. To define a fermionic CFT on a manifold X , we need to choose a spin structure on X by specifying a boundary condition of fermions around each nontrivial cycle. In the present work, we suppose that a given fermionic CFT has nontrivial Hilbert space \mathcal{H}_R in the Ramond sector. The dimension Δ and the spin s of a state in \mathcal{H}_R can be determined by its left and right (Ramond) conformal weights h^R and \bar{h}^R , $\Delta = h^R + \bar{h}^R$ and $s = |h^R - \bar{h}^R|$. The rational CFTs refer to CFTs whose partition function can be written as a finite sum of products of holomorphic and antiholomorphic functions on $(\tau, \bar{\tau})$. For example, let us discuss the Ramond-Ramond boundary condition where fermions are periodic around any cycles on \mathbb{T}^2 .

The partition function associated with the Ramond-Ramond (RR) boundary condition can be expressed as follows

$$Z_{RR}(\tau, \bar{\tau}) = \text{Tr}_{\mathcal{H}_R} [(-1)^F q^{h^R - \frac{c}{24}} \bar{q}^{\bar{h}^R - \frac{c}{24}}] = \sum_{i,j=0}^{N-1} f_i(\tau) \mathcal{M}_{ij} \bar{f}_j(\bar{\tau}). \quad (1)$$

Here each holomorphic function $f_i(\tau)$ represents the RR conformal character for a primary state of conformal weight h_i^R . Without loss of generality, we can assume that $h_0^R < h_1^R < \dots < h_{N-1}^R$. We also define the leading exponent of each $f_i(\tau)$ in q -expansion as $m_i = h_i^R - \frac{c}{24}$ in what follows,

$$f_i(\tau) = q^{m_i} \sum_{a=0}^{\infty} F_i(a) q^a. \quad (2)$$

Here $F_i(a)$ are integers that stand for the degeneracy of the state weighted by $(-1)^F$.

A fermionic RCFT having supersymmetry is characterized by the presence of supercurrent $G(z)$, a primary of conformal weight $h = 3/2$ and satisfying the following operator product expansion,

$$\begin{aligned} T(z)G(0) &\sim \frac{3}{z^2}G(0) + \frac{1}{z}\partial G(0), \\ G(z)G(0) &\sim \frac{2c}{3z^3} + \frac{1}{z}T(0), \end{aligned} \quad (3)$$

where $T(z)$ denotes the stress-energy tensor. Therefore, the partition function of the Neveu-Schwarz (NS) sector

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should involve a state of weight $h = 3/2$ as the vacuum descendant.

Furthermore, the unitarity of a superconformal CFT leads to a lower bound on the conformal weight in the Ramond sector,

$$\{G_0, G_0\} \propto \left(h^R - \frac{c}{24} \right) \geq 0, \quad (4)$$

where G_0 is the superconformal charge and c is the central charge. The main goal of the present work is to show that $Z_{RR}(\tau, \bar{\tau})$ becomes either constant or zero when the lower bound (4) is obeyed, regardless of the structure of the NS sector partition function. In other words, we will utilize the modular constraint of $f_i(\tau)$ to propose constant $Z_{RR}(\tau, \bar{\tau})$ when all the exponents of $f_i(\tau)$ are non-negative. To have a constant RR partition function, the contributions from bosonic and fermionic states other than vacuum has to be canceled. One can hardly expect the above cancellation unless the given theory has either supersymmetry or free fermions. Note that a free fermion subject to the RR boundary condition on \mathbb{T}^2 has a zero mode, which results in vanishing Z_{RR} . Thus, we propose that a fermionic RCFT satisfying the bound (4) is supersymmetric unless it has free fermions. In this case, one can regard the constant Z_{RR} as the index of supersymmetric RCFTs.

As an illustration, let us consider the fermionic theory for the tricritical Ising model. The theory has nontrivial Hilbert spaces in both the NS sector and the R sector. The former consists of the vacuum and the primary of $(h^{\text{NS}}, \bar{h}^{\text{NS}}) = (1/10, 1/10)$ along with their descendants. On the other hand, the latter comprises two primaries $(h^R, \bar{h}^R) = (3/80, 3/80)$, $(7/16, 7/16)$ and their descendants. Since the Ramond spectrum satisfies the bound $h^R \geq c/24 = 7/240$, our proposal predicts that the theory preserves the supersymmetry. Indeed, the tricritical Ising model is a well-known example where its fermionic description has the $\mathcal{N} = 1$ supersymmetry.

To support the proposal, we make use of two essential properties of the RR characters $f_i(\tau)$ of (1). First, the conformal characters should transform under $\text{SL}(2, \mathbb{Z})$ as the vector-valued modular functions; otherwise, the RR partition function cannot be modular invariant. Second, each conformal character has an integral q -expansion whose coefficients count state degeneracies weighted by $(-1)^F$. In [1], such characters are referred to as quasicharacters. In what follows, we provide proof relying on those properties of our claim that Z_{RR} becomes either constant or 0 when (4) is satisfied for fermionic RCFTs.

II. INTEGRALITY CONJECTURE

It has been known that any vector-valued modular form with N -components $f_i(\tau)$ can be understood as independent solutions of a modular invariant linear differential

equation (MLDE) of order N [2]. The valence formula applied to the Wronskian of $f_i(\tau)$ then reads

$$\sum_{i=0}^{N-1} m_i + \frac{l}{6} = \frac{N(N-1)}{12}, \quad (5)$$

where l is the Wronskian index, a non-negative integer except 1 capturing the number of zeroes of the Wronskian. The main idea of classifying two-dimensional RCFT is to pick up the solutions of MLDE that satisfy constraints such as having the positive integer Fourier coefficients, uniqueness of the vacuum, and consistent fusion rule algebra. The classification of the bosonic and fermionic RCFTs has been performed in [2–10]. In particular, a recent paper [7] utilized the integrality conjecture [11,12] to provide a new approach to the classification program.

The modular invariance of $Z_{RR}(\tau, \bar{\tau})$ implies that the RR conformal characters are regarded as a vector-valued modular form. In contrast to the conformal characters of the bosonic RCFTs, the RR conformal characters are allowed to have negative integer coefficients in q -expansion. Nonetheless, the integrality of the coefficients enables us to apply the *integrality conjecture*. According to the integrality conjecture, which is proved in [13] for RCFT characters and recently proved in the general case by [14], each RR conformal character should be invariant under the principal congruence subgroup of $\text{SL}(2, \mathbb{Z})$. Such a subgroup is defined with a positive integer n as follows,

$$\Gamma(n) := \left\{ \begin{pmatrix} a & b \\ s & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}), \right. \\ \left. \begin{pmatrix} a & b \\ s & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{n} \right\}. \quad (6)$$

Since we only require the coefficients to be integral, an immediate lesson we learn is that the RR characters also form a representation of the finite group $\text{SL}(2, \mathbb{Z})/\Gamma(n) = \text{SL}(2, \mathbb{Z}_n)$ for an integer n . Also, we need to remember that the minus of the identity matrix in $\text{SL}(2, \mathbb{Z}_n)$ acts trivially on the characters by construction. In other words, we should consider the representation theory of $\text{PSL}(2, \mathbb{Z}_n)$. For the small number of characters, a plausible strategy then is to classify all possible n for given N and extract information from its representation theory.

Let us state without proof the simplest nontrivial example, i.e., $N = 2$. The finite list of n containing the desired two-dimensional irreducible representation turns out to be

$$n \in \{2, 6, 8, 12, 20, 24, 60\}. \quad (7)$$

A priori, some n in the list does not need to be realized by a rational CFT with two RR characters, but we stress that the converse must be correct. Furthermore, the detail of those

TABLE I. Possible exponents mod 1 for potential RCFTs with two RR characters.

n	Exponents mod 1
2	$\{0, \frac{1}{2}\}$
6	$\{\frac{2}{3}, \frac{1}{6}\}, \{\frac{1}{3}, \frac{5}{6}\}$
8	$\{\frac{1}{8}, \frac{3}{8}\}, \{\frac{5}{8}, \frac{7}{8}\}$
12	$\{\frac{1}{4}, \frac{11}{12}\}, \{\frac{3}{4}, \frac{5}{12}\}, \{\frac{1}{4}, \frac{7}{12}\}, \{\frac{3}{4}, \frac{1}{12}\}, \{\frac{1}{12}, \frac{5}{12}\}, \{\frac{7}{12}, \frac{11}{12}\}$
20	$\{\frac{1}{20}, \frac{9}{20}\}, \{\frac{3}{20}, \frac{7}{20}\}, \{\frac{11}{20}, \frac{19}{20}\}, \{\frac{13}{20}, \frac{17}{20}\}$
24	$\{\frac{11}{24}, \frac{17}{24}\}, \{\frac{5}{24}, \frac{23}{24}\}, \{\frac{1}{24}, \frac{19}{24}\}, \{\frac{7}{24}, \frac{13}{24}\}$
60	$\{\frac{11}{60}, \frac{59}{60}\}, \{\frac{17}{60}, \frac{53}{60}\}, \{\frac{23}{60}, \frac{47}{60}\}, \{\frac{29}{60}, \frac{41}{60}\}, \{\frac{1}{60}, \frac{49}{60}\}, \{\frac{7}{60}, \frac{43}{60}\}, \{\frac{19}{60}, \frac{31}{60}\}, \{\frac{13}{60}, \frac{37}{60}\}$

representations can be easily accessed from computer software such as GAP [15]. In particular, we can extract from the character table of $SL(2, \mathbb{Z}_n)$ the exponents of characters m_i as summarized in Table I. For the detailed discussion, we refer the readers to [7].

Now let us come back to our original problem. Combining the valence formula (9) when $N = 2$ with the exponents in Table I, we easily learn that if RR characters with non-negative exponents are nontrivial, we must have a contradiction. This implies that actually all the characters should vanish, thereby establishing the desired result $Z_{RR} = 0$ for the special case of two RR characters. Moreover, all possible exponents mod 1 up to $N = 5$ are worked out in [7], which cannot satisfy the corresponding valence formula if we assume they are all non-negative. This gives us an affirmative answer to our question.

While being very concrete, this method would not be practical for general N , and we need to look for other approaches. In [16], the authors prove a theorem crucial for our purpose. For the reader's convenience, we restate it below.

Theorem 1. Consider an N -dimensional weakly holomorphic and integral vector-valued modular function $\vec{f}(q)$ with components $f_i(q)$. Define

$$\mathbf{m} := \min(m_0, \dots, m_{N-1}) \quad (8)$$

where $f_i(q) \sim q^{m_i}$ near the cusp $i\infty$. If there exists one component $f_j(q)$ which is a modular function of $\Gamma(n)$ for some integer n with $m_j \neq 0$, then $\mathbf{m} < 0$.

The proof is not sophisticated, so we briefly outline it here. As a modular function for $\Gamma(n)$, the component $f_j(q)$ should obey the valence formula,

$$\sum_{\tau \in \mathbb{H}/\Gamma(n)} \text{Ord}_\tau(f_j) = 0, \quad (9)$$

where \mathbb{H} is the upper half-plane and $\text{Ord}_\tau(f_j)$ denotes the leading order of f_j when Laurent expanded around the

point τ . $\text{Ord}_\tau(f_j)$ is counted as positive and negative for the zeros and poles, respectively. If $m_j < 0$, the theorem is already proved, so we assume $m_j > 0$. Since the order of $f_j(q)$ at $\tau = i\infty$ is $nm_j > 0$, the valence formula (9) says that $f_j(q)$ should diverge at some other cusps, denoted by τ_* , since a physical conformal character is required to be holomorphic inside $\mathbb{H}/\Gamma(n)$. It implies that the $SL(2, \mathbb{Z})$ invariant partition function (1),

$$Z_{RR}(\tau, \bar{\tau}) = \sum f_i(\tau) \mathcal{M}_{ij} \bar{f}_j(\bar{\tau}). \quad (10)$$

also diverges at $\tau = \tau_*$. Invoking an $SL(2, \mathbb{Z})$ transformation that maps τ_* to $i\infty$, we readily see that $Z(\tau, \bar{\tau})$ and hence some other component of $\vec{f}(q)$ diverge at $\tau = i\infty$. In other words, $\mathbf{m} < 0$. This is the end of the proof.

In our present situation, recall that we have the constraint $h_i^R \geq \frac{c}{24}$. If a given theory has the Ramond spectrum satisfying the bound (4) strictly, i.e., $h^R > c/24$, all conformal characters must be trivial and thus the corresponding RR partition function is simply zero according to the above theorem. On the other hand, let us suppose that the bound is saturated, namely $h_\alpha^R = \frac{c}{24}$. The corresponding conformal character can be written as

$$f_\alpha(q) = c_\alpha + \tilde{f}_\alpha(q), \quad (11)$$

with $\tilde{f}_\alpha(q)$ now having strictly positive exponent in the q -expansion. It is clear that \tilde{f}_α is also modular invariant under the congruence subgroup $\Gamma(n)$ indicated in the proof, and the new partition function

$$\tilde{Z}_{RR}(\tau, \bar{\tau}) = Z_{RR}(\tau, \bar{\tau}) - \mathcal{M}_{\alpha\alpha} c_\alpha^2 \quad (12)$$

remains $SL(2, \mathbb{Z})$ invariant. Applying the strategy of the proof to $\tilde{Z}_{RR}(\tau, \bar{\tau})$ entails that it ought to be zero in order to avoid contradiction. Therefore, the original partition function becomes a constant, and the only nonvanishing RR character is $f_\alpha = c_\alpha$. This indicates a perfect cancellation between bosonic and fermionic excited states, which is strong evidence for the presence of supersymmetry.

III. FURTHER REMARK

We conclude with a remark on the Rademacher expansion, which is expected to shed new light on our proposal that the bound $h^R \geq c/24$ implies the presence of supersymmetry. Let us begin with a brief review of the generalization of the Rademacher expansion to a vector-valued modular form of weight w . See [17] for the detailed discussion.

When each component of the weight w vector-valued modular form $f_i(q)$ can be expanded in powers of q as (2), the Fourier coefficient $F_i(a)$ has the contour integral representation below,

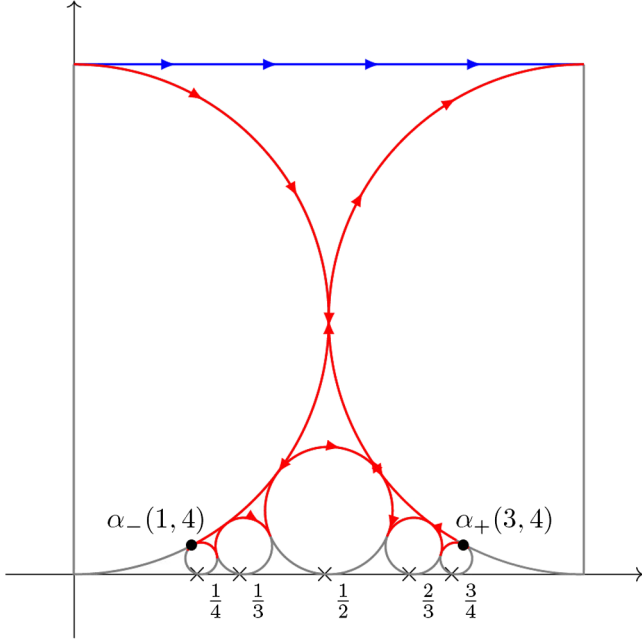


FIG. 1. We deform the original contour \mathcal{C} (blue) to the Rademacher contour $\mathcal{C}_R(M)$ (red) in τ plane. As an example, we present the Rademacher contour with $M = 4$. The number of the Ford circles increases for large M , therefore the Rademacher contour covers the whole arc of the Ford circles except the tangent points on the real axis. $\alpha_+(s, d)$ and $\alpha_-(s, d)$ are intersection points of the Ford circle $C(s, d)$ with its neighboring Ford circles.

$$F_i(a) = \int_{\mathcal{C}} d\tau e^{-2\pi i \tau (a+m_i)} f_i(\tau), \quad (13)$$

where the contour \mathcal{C} is given by a straight line from $\tau = i$ to $\tau = i + 1$. To evaluate the integral (13), we deform \mathcal{C} to the Rademacher contour $\mathcal{C}_R(M)$ for a given integer M . The Rademacher contour is defined in terms of the Ford circle $C(s, d)$, a circle of radius $\frac{1}{2s^2}$ and tangent to the x -axis at a Farey number $d/s \in \mathcal{F}_M$. Here d and s are coprime integers such that d/s is an irreducible fraction, and the Farey numbers \mathcal{F}_M include the fractions between 0 and 1 with denominator less than M . The Rademacher contour can then be described as follows,

$$\mathcal{C}_R(M) = \bigcup_{\frac{d}{s} \in \mathcal{F}_M} \mathcal{C}_{s,d}(M), \quad (14)$$

where $\mathcal{C}_{s,d}(M)$ is the arc between the intersection points $\alpha_-(s, d)$ and $\alpha_+(s, d)$ of the Ford circle $C(s, d)$ with the neighboring Ford circles. To illustrate, we show the Rademacher contour with $M = 4$ in Fig. 1. Note that $\mathcal{C}_R(\infty)$ covers the entire arc of the Ford circles except tangential points on the real line.

Evaluating the contour integral over $\mathcal{C}_R(\infty)$, the Fourier coefficient can be expressed as

$$\begin{aligned} F_i(a) &= 2\pi \sum_{s=1}^{\infty} \left[\sum_{i=0}^{N-1} s^{w-2} K\ell(a, i, b, j; s) \right. \\ &\times \sum_{b+m_j < 0} F_j(b) (2\pi|b+m_j|)^{1-w} \\ &\times \left(\frac{2\pi}{s} \sqrt{(a+m_i)|b+m_j|} \right)^{w-1} \\ &\left. \times I_{1-w} \left(\frac{4\pi}{s} \sqrt{(a+m_i)|b+m_j|} \right) \right] + \tilde{F}_i(a), \quad (15) \end{aligned}$$

where $K\ell(a, i, b, j; s)$ denotes the Kloosterman sum and $I_\alpha(x)$ is the modified Bessel function of the first kind. It was shown in [17] that $\tilde{F}_i(a)$ vanishes when the Fourier coefficients $F_j(b)$ with $(b+m_j) \geq 0$ are all positive for any j and $w \geq 0$. However, since the Fourier coefficients of the RR characters are not necessarily positive, more elaboration is required to show whether $\tilde{F}_i(a) = 0$ in our case. If this is the case, the Rademacher expansion can provide an alternative argument that the RR conformal characters should be trivial when the bound (4) is obeyed. We leave it to future work.

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