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Quantum Phenomena Inside a Black Hole: Quantization of the Scalar Field Iniside Horizon in Schwarzschild Spacetime

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Abstract: We discuss the problem of the quantization and dynamic evolution of a scalar free field in the interior of a Schwarzschild black hole. A unitary approach to the dynamics of the quantized field is proposed: a time-dependent Hamiltonian governing the Heisenberg equations is derived. It is found that the system is represented by a set of harmonic oscillators coupled via terms corresponding to the creation and annihilation of pairs of particles and that the symmetry properties of the spacetime, homogeneity and isotropy are obeyed by the coupling terms in the Hamiltonian. It is shown that Heisenberg equations for annihilation and creation operators are transformed into ordinary differential equations for appropriate Bogolyubov coefficients. Such a formulation leads to a general question concerning the possibility of gravitationally driven instability, that is however excluded in this case.

Keywords: Schwarzschild black hole interior; scalar field; time-dependent Hamiltonian

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1. Introduction

The horizon of a black hole (BH) may be regarded as a geometrical singularity ("fake geometrical singularity"). Indeed, considering a Schwarzschild BH in Schwarzschild coordinates one finds the metric tensor exhibiting an on-horizon singularity that is absent in other, singularity-free coordinate systems. There are a variety of singularity-free coordinate systems in this case, e.g., Kruskal-Szekers, Eddington-Finkelstein, Novikov and others [1,2]. Two interesting observations might be made here. The first is to notice that the presence of the event horizon is manifested both in coordinates revealing the horizon's singularity as well as in the singularity-free systems. The second one is to note the surprising similarities and/or analogies for phenomena taking place outside and inside black holes. A rather well-known example of such a property is the so-called BSW effect [3]. Two-particle collisions occuring in the vicinity of the black hole's horizon may lead to a high-energy outcome according to two scenarios [4,5]. These two scenarios turn out to be the same in the exterior as well as in the interior of BH. A variety of other aspects of the Exterior vs. Interior (a)symmetry have been discussed in Ref. [6].

It was shown by Doran et al. [7] that the interior of a Schwarzschild BH's, which is a dynamically changing spacetime, may be regarded as a solution of Einstein's equation. This interior spacetime, also called "T-sphere" (see [8]) which is globally hyperbolic, gains then the status of a cosmological model. Its 3D spatial-like section is a hypercylinder

$\mathbf{R}^1 \times S^2$, expanding longitudinally, along the homogeneity direction $\mathbf{R}^1$, (see also [6–8]) and contracting transversally, perpendicularly to this direction in the angular coordinates of the sphere S^2. However, as shown in Ref. [7], such a process may be preceded by a process of expansion of the sphere and collapsing of the cylinder to its base sphere of radius r_S. Such an expansion followed by a contraction constitutes the full cycle for the cosmological model introduced in [7].

Various phenomena and processess have been considered both in the interior of the Schwarzschild BH [8–12] and in its extension [7] to which we will hereafter refer to as the "T-model", an anisotropic cosmological model. In particular the Yang-Mills and Higgs fields in the Kantowski-Sachs anisotropic, cigar-like—referred to above as a hypercylinder—cosmological model were discussed in [13] (see also [14]). Canonical quantization of the scalar field inside a Schwarzschild BH was presented by Yajnik and Narayan [15], where a so-called tortoise coordinate was used, in consequence leading to a Hamiltonian of diagonal form and, as claimed by the authors, to "QFT set up by the freely falling observer". Other studies of the quantum properties of scalar field were given for instance in Refs. [16,17] and the investigations of the interior of the Schwarzschild BH were presented in Refs. [18,19]. The most recent results have been given by Almeida and Rodrigues in Ref. [20] where the quantization of the BH gravity was discussed and by Giddings and Perkins in Ref. [21], in which the quantum evolution of the Hawking state in Schwarzschild spacetime was investigated.

In this paper we will present a particular quantum aspect of the "T-model". Namely the problem of dynamics, i.e., the temporal evolution of the quantized scalar field in the case of such a cosmology will be introduced and briefly discussed within a unitary approach. The Hamiltonian of the system, represented by a set of harmonic oscillators, coupled via creation and annihilation of pairs of particles, revealing interesting symmetry properties, will be derived. The Heisenberg equations of motion for appropriate annihilation and creation operators will be converted into ordinary differential equations for Bogolyubov coefficients and will be shown to reveal the possibility of an instability that is referred to as a gravitationally driven instability.

The paper is organized as follows. In Section 2 we discuss the properties of the Schwarzschild BH and a T-model is formulated. In Section 3 a scalar field and its quantization are discussed. In Section 4. the Hamiltonian of the scalar field is derived and a discussion is presented in the final section, Section 5; Appendix A is devoted for a derivation of explicit form the temporal part of (factorized) Klein-Gordon equation.

2. "T-Sphere" Model—An Anisotropic Cosmological Model

The metric $g_{\mu\nu}$ for the exterior of the Schwarzchild black hole, diagonal in the Schwarzschild coordinates (t, r, θ, φ), reveals the singularity on the horizon:

$$ds^2 = g_t(r)dt^2 - g_r(r)dr^2 - g_2(r)d\Omega^2. \quad (1)$$

where

$$g_t = 1 - \frac{2M}{r} = g_r^{-1} \quad (2)$$

$g_2(r) = r^2$, and $d\Omega^2$ denotes the metric on the two-dimensional unit sphere S^2 with the coordinates (θ, ϕ) :

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2. \quad (3)$$

The geometrical singularity at the horizon, $r_S = 2M$ may be removed by a transformation to a singularity-free coordinate system, such as Kruskal-Szekeres, Eddington-Finkelstein, Novikov, Lemaitre or other systems [1,2].

The coordinate system (1), though ill-defined on the horizon, may be applied inside the horizon (see e.g., [6,7]). The interior of a BH, $r < r_S$ possesses, apart from some well-known, some not so well-known, properties too (see [22]). The Killing vector ∂_t becomes a spatial one that results in momentum conservation instead of energy conservation, as obeyed outside BH (see below). This is accompanied by the interchange of the roles of the coordinates: t and r play the role of the spatial- and temporal-like coordinates, respectively.

The interesting feature of the interior of a Schwarzschild BH is that it may be regarded as a unique spacetime, a cosmological anisotropic model called a "T-sphere" model or simply T-model [8]. It is described by the line element (1) for $r < r_S$ but now expressed in terms of $T(= -r)$ (temporal) and z (spatial) coordinates instead of r and t, coordinates, respectively

$$ds^2_- = g_T dT^2 - g_z dz^2 - g_2(T)\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \tag{4}$$

where, $T \in \langle -r_S, 0\rangle$, $z \in (-\infty, +\infty)$, $g_T = \left(\frac{r_S}{T} - 1\right)^{-1} = g_z^{-1}$. At each instant of T_0 the spatial slice is a hypercylinder $\mathbf{R}^1 \times S^2$, longitudinally expanding and transversally, a two-sphere of radius $|T_0|$, contracting (see e.g., [6]). Along the cylinder axis z the system is homogeneous and that represents the momentum z-component conservation.

Phenomena of a classical nature have been considered in the T-model both within a more traditional approach (see e.g., [9–12]) as well as from other specific perspectives (see [22–25]). Here we will consider a special quantum phenomenon, namely the problem of dynamics of the quantized scalar field in the case T-model will be introduced and briefly discussed within a unitary approach.

3. Scalar Free Field in a T-Model

A scalar free field Φ in a space-time M with a metric $g_{\mu\nu}$ is described in terms of Lagrangian density $\mathcal{L}$:

$$\mathcal{L} = \frac{1}{2}\sqrt{-g}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - \left(\mu^2 + \xi R\right)\Phi^2, \tag{5}$$

where $-g = \det[g_{\alpha\beta}]$, the parameter μ can be interpreted as the mass only in asymptomatically flat space-time, R is the scalar curvature of M and ξ is the field coupling to the spacetime curvature.

In the case of the spacetime (4) the coupling with gravitational field vanishes (as $R = 0$) and the action of the scalar free field (5) takes the form

$$S = \frac{1}{2}\int dT \int_{\Sigma} dz d\Omega T^2 \left[\frac{1}{g_T}(\partial_T\Phi)^2 - \frac{1}{g_z}(\partial_z\Phi)^2 + \frac{1}{T^2}\Phi\Delta_{S^2}\Phi - \mu^2\Phi^2\right], \tag{6}$$

where $\Sigma = \mathbf{R}^1 \times S^2$, $d\Omega = \sin\theta d\varphi d\theta$ and we have integrated by parts in the sector S^2 which resulted in the Laplace operator Δ_{S^2} on S^2:

$$\Delta_{S^2}\Phi = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2}. \tag{7}$$

The Klein-Gordon (or Euler-Lagrange) equation

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\Phi\right) + \mu^2\Phi = 0, \tag{8}$$

takes in this case the following form:

$$\partial_T\left(T^2 g_z \partial_T\Phi\right) - \frac{T^2}{g_z}\partial_z^2\Phi - \Delta_{S^2}\Phi + \mu^2 T^2\Phi = 0. \tag{9}$$

Taking the field Φ in the form of a product:

$$\Phi(T, z, \theta, \phi) = R(T)u(z)Y(\theta, \phi). \tag{10}$$

it follows that the wave Equation (9) separates into the following equations:

$$\Delta_{S^2} Y = -l(l+1)Y, \tag{11}$$

$$\frac{d^2 u_\varepsilon}{dz^2} = -\varepsilon^2 u_\varepsilon, \tag{12}$$

$$\frac{d}{dT}\left(T^2 g_z \frac{dR_{\varepsilon l}}{dT}\right) + T^2\left(\frac{\varepsilon^2}{g_z} + \mu^2 + \frac{l(l+1)}{T^2}\right)R_{\varepsilon l} = 0, \tag{13}$$

where ε is a (separation) constant. The solution of Equation (11) is given by the spherical harmonics $Y_{lm}(\theta, \phi)$,

$$\int_{S^2} d\Omega Y_{lm}(\theta,\varphi) Y^*_{l'm'}(\theta,\varphi) = \delta_{ll'}\delta_{mm'}, \tag{14}$$

$$\int_{S^2} d\Omega Y_{lm}(\theta,\varphi) Y_{l'-m'}(\theta,\varphi) = \delta_{ll'}\delta_{m,-m'} \tag{15}$$

where $m = -l, -(l-1), \ldots 0 \ldots, l$. The solution of Equation (12) is

$$u(z) = e^{\pm i\varepsilon z}. \tag{16}$$

One can decompose the field Φ into the complete system of functions on $\mathbf{R}^1$ and S^2. Thus, the real field $\Phi = \Phi^*$ is represented as:

$$\Phi(T,z,\theta,\varphi) = \sum_{\varepsilon,l,m}\left[R_{\varepsilon l}(T)e^{i\varepsilon z}Y_{lm}(\theta,\varphi)A_{\varepsilon lm} + R^*_{\varepsilon l}(T)e^{-i\varepsilon z}Y^*_{lm}(\theta,\varphi)A^*_{\varepsilon lm}\right], \tag{17}$$

where $R_{\varepsilon l}(T)$ are the functions of the temporal variable T satisfying second order differential Equation (13) and $A_{\varepsilon lm}$ are Fourier-like coefficients.

The scalar product $(\cdot,\cdot)$ (Klein-Gordon) is in general defined as:

$$(\Phi,\Psi) = i\int_{\Sigma_t} (\Phi^*\partial_\mu\Psi - \Psi\partial_\mu\Phi^*)n^\mu dvol(\Sigma_t), \tag{18}$$

where $n = n^\mu\partial_\mu$ denotes the unit time-like vector field orthogonal to a space-like hypersurface (slice) Σ_t and Φ, Ψ are the solutions of the Klein-Gordon equation. In this case $\Sigma_t \simeq A \times S^2$ and the scalar product takes the form (see [16,26]):

$$(\Phi,\Psi) = iT^2 g_z \int_{S^2} \sin\theta d\theta d\phi \int_A (\Phi^*\partial_T\Psi - \Psi\partial_T\Phi^*)dz. \tag{19}$$

There is the following normalization condition

$$A_{\varepsilon lm} = \left(R_{\varepsilon l}(T)e^{i\varepsilon z}Y_{lm}(\theta,\varphi), \Phi\right) \tag{20}$$

where Φ is given by (10), which is equivalent to the claim of the canonical commutation relations (see also below).

After some (lengthy but simple) algebra one finds that condition (20) is satisfied iff

$$T^2 g_z\left[R^*_{\varepsilon l}\dot{R}_{\varepsilon l} - \dot{R}^*_{\varepsilon l}R_{\varepsilon l}\right] = -i, \tag{21}$$

$$R^*_{\varepsilon l}\dot{R}^*_{-\varepsilon l} - R^*_{-\varepsilon l}\dot{R}^*_{\varepsilon l} = 0. \tag{22}$$

The condition (21) is derived from the differential Equation (13). First, one writes Equation (13) for the complex conjugated function $R^*_{\varepsilon l}$; then one multiplies it by $R_{\varepsilon l}$ and Equation (13) by $R^*_{\varepsilon l}$; finally one subtracts the former from the latter obtaining

$$d_T\left(T^2 g_z\left[R^*_{\varepsilon l}\dot{R}_{\varepsilon l} - \dot{R}^*_{\varepsilon l}R_{\varepsilon l}\right]\right) = 0. \tag{23}$$

Therefore, Equation (21) turns out to be a normalization condition for $R_{\varepsilon l}$ i.e., the Wronskian in this case, as it should be. On the other hand Equation (22) is just an equivalence.

Quantization

Quantization of the field (5) and (6) is performed in a canonical way. Namely, one introduces the momentum field as the field canonically conjugated to $\Phi(T, z, \theta, \varphi)$, i.e.,

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_T \Phi)} = \frac{T^2}{g_T}\partial_T \Phi. \tag{24}$$

Then one imposes canonical commutation relations

$$\begin{aligned} \left[\widehat{\Phi}(t,\mathbf{x}), \widehat{\pi}(t,\mathbf{y})\right] &= i\delta(\mathbf{x},\mathbf{y}), \\ \left[\widehat{\Phi}(t,\mathbf{x}), \widehat{\Phi}(t,\mathbf{y})\right] &= [\widehat{\pi}(t,\mathbf{x}), \widehat{\pi}(t,\mathbf{y})] = 0, \end{aligned} \tag{25}$$

where $\mathbf{x}, \mathbf{y} \in \Sigma_t$. In our case the slice Σ_t has the topology of the product space of the set $A \subset \mathbf{R}^1$ and the two-dimensional sphere S^2. The momentum field given in its Fourier decomposed form is:

$$\widehat{\pi}(t,r,\theta,\phi) = \frac{T^2}{g_T}\sum_{\varepsilon,l,m}\left[\widehat{A}_{\varepsilon lm}\dot{R}_{\varepsilon l}(T)e^{i\varepsilon z}Y_{lm}(\theta,\phi) + \widehat{A}^\dagger_{\varepsilon lm}\dot{R}^*_{\varepsilon l}(T)e^{-i\varepsilon z}Y^*_{lm}(\theta,\phi)\right] \tag{26}$$

The canonical commutation relations Equations (25) turn out to be satisfied under the following conditions:

(a) $\widehat{A}_{\varepsilon lm}$, $\widehat{A}^\dagger_{\varepsilon lm}$, are the annihilation and creation operators, respectively, i.e., the only nonvanishing commutator is

$$\left[\widehat{A}_{\varepsilon lm}, \widehat{A}^\dagger_{\varepsilon' l' m'}\right] = \delta_{\varepsilon\varepsilon'}\delta_{ll'}\delta_{mm'} \tag{27}$$

(b) the Wronskian (21) must hold.

4. Hamiltonian of the Scalar Field in a T-Model

The Hamiltonian of the field described by the Lagrangian density $\mathcal{L}$ is determined as an integral over the spatial part $\mathbf{\Sigma}$ of the spacetime

$$H = \int_{\mathbf{\Sigma}} d^3x[\pi\partial_T\Phi - \mathcal{L}], \tag{28}$$

and this expression is equivalent to the (integrated) T_{TT} element of the stress-energy tensor. Applying formula (28) for the case (4) and (5) one obtains

$$H = \frac{1}{2}\int_{\mathbf{\Sigma}} dz d\theta d\varphi T^2 \sin\theta\left[\frac{1}{g_T}(\partial_T\Phi)^2 + \frac{1}{g_z}(\partial_z\Phi)^2 - \Phi\Delta_{S^2}\Phi + \mu^2\Phi^2\right]. \tag{29}$$

Using the Fourier decomposition of the quantized field and momentum (see Equations (17) and (27)) one finds the Hamiltonian of the quantized scalar field as expressed in terms of annihilation and creation operators:

$$H = \frac{1}{2}\sum_{\varkappa}\Big[\omega_{\varkappa}\widehat{A}_{\varkappa}\widehat{A}^{\dagger}_{\varkappa} + \gamma_{\varkappa\varkappa'}\widehat{A}_{\varkappa}\widehat{A}_{\varkappa'} + (c.c)\Big] \tag{30}$$

where indices $\varkappa, \varkappa'$ correspond to the appropriate three letter sets $\varepsilon l m$. The parameters $\omega_{\varkappa}, \gamma_{\varkappa\varkappa'}$ are given as

$$\gamma_{\varepsilon lm/\varepsilon' l m'} = \left[T^2 g_z \dot{R}_{\varepsilon l}\dot{R}_{-\varepsilon l} + T^2\left\{\frac{\varepsilon^2}{g_z} + \frac{l(l+1)}{T^2} + \mu^2\right\}R_{\varepsilon l}(T)R_{-\varepsilon l}(T)\right]\delta_{\varepsilon,-\varepsilon'}\delta_{m,-m'} \tag{31}$$

$$\omega_{\varepsilon lm} = \left[T^2 g_z \dot{R}_{\varepsilon l}\dot{R}^{*}_{\varepsilon l} + T^2\left\{\frac{\varepsilon^2}{g_z} + \frac{l(l+1)}{T^2} + \mu^2\right\}R_{\varepsilon l}(T)R^{*}_{\varepsilon l}(T)\right]. \tag{32}$$

Therefore, the Hamiltonian of the scalar field in the T-model, i.e., anisotropic cosmological model representing interior of the Schwarzschild BH, turns out to be

$$H = \frac{1}{2}\sum_{\varepsilon lm}\Big[\omega_{\varepsilon lm}\Big(\widehat{A}_{\varepsilon lm}\widehat{A}^{\dagger}_{\varepsilon lm} + \widehat{A}^{\dagger}_{\varepsilon lm}\widehat{A}_{\varepsilon lm}\Big) + \gamma_{\varepsilon lm/-\varepsilon l-m}\widehat{A}_{\varepsilon lm}\widehat{A}_{-\varepsilon l-m} + \gamma^{*}_{\varepsilon lm/-\varepsilon l-m}\widehat{A}^{\dagger}_{\varepsilon lm}\widehat{A}^{\dagger}_{-\varepsilon l-m}\Big]. \tag{33}$$

representing the set of interacting, time-dependent harmonic oscillators.

On this basis one can study the dynamics of the quantized scalar field. The evolution of the system is described by the Heisenberg equation of motion for the operators $\widehat{A}_{\varepsilon lm}$

$$i\frac{d}{dt}\widehat{A}_{\varepsilon lm} = \Big[\widehat{A}_{\varepsilon lm}, \widehat{H}\Big] = \omega_{\varepsilon lm}(t)\widehat{A}_{\varepsilon lm}(t) + \gamma^{*}_{\varepsilon lm}(t)\widehat{A}^{\dagger}_{-\varepsilon l-m}(t) \tag{34}$$

where, $\gamma_{\varepsilon lm/-\varepsilon l-m} \equiv \gamma_{\varepsilon lm}$. One can search for the solutions of the above equations by using the following ansatz:

$$\widehat{A}_{\varepsilon lm}(t) = \alpha_{\varepsilon lm}(t)\widehat{A}_{\varepsilon lm} + \beta_{\varepsilon lm}(t)\widehat{A}^{\dagger}_{-\varepsilon l-m}, \tag{35}$$

where $\alpha_{\varepsilon lm}(t)$ and $\beta_{\varepsilon lm}(t)$ are some unknown complex functions and $\widehat{A}_{\varepsilon lm}$ and $\widehat{A}^{\dagger}_{-\varepsilon l-m}$ are time independent operators. By definition the relation (35) preserves the commutation relations (27), hence it turns out to be the Bogolyubov transformation,

$$|\alpha_{\varepsilon lm}(t)|^2 - |\beta_{\varepsilon lm}(t)|^2 = 1. \tag{36}$$

Then, the Heisenberg Equation (34) are converted into differential equations for the Bogolyubov coefficients

$$i\frac{d}{dt}\alpha_{\varepsilon lm}(t) = \omega_{\varepsilon lm}(t)\alpha_{\varepsilon lm}(t) + \gamma^{*}_{\varepsilon lm}(t)\beta^{*}_{\varepsilon lm}(t), \tag{37}$$

$$i\frac{d}{dt}\beta_{\varepsilon lm}(t) = \omega_{\varepsilon lm}(t)\beta_{\varepsilon lm}(t) + \gamma^{*}_{\varepsilon lm}(t)\alpha^{*}_{\varepsilon lm}(t). \tag{38}$$

In general, one can't expect exact solutions of the Equations (37) and (38) and approximate schemes would therefore be proposed. Our forthcoming paper will be devoted to the comprehenssive discussion of this problem.

5. Discussion

Considering the interior of a Schwarzschild BH as a unique spacetime, an anisotropic cosmological model, we have performed the quantization of the free (noninteracting) scalar field by imposing the canonical commutation relations. One decomposes the field and momentum in terms of the complete set of solutions of the Klein-Gordon (or in fact Euler-Lagrange equations) with the coefficients of expansion being annihilation and creation operators. This procedure leads to the Hamiltonian of the quantized scalar field taking the form of the set of harmonic, time-dependent oscillators coupled in a special way: there are terms in the Hamiltonian corresponding to creation, $\gamma_{\varepsilon lm}\widehat{A}_{\varepsilon lm}\widehat{A}_{-\varepsilon l-m}$ and annihilation $\gamma^*_{\varepsilon lm}\widehat{A}^\dagger_{\varepsilon lm}\widehat{A}^\dagger_{-\varepsilon l-m}$ particles in pairs.

Such a picture, peculiar at first sight, appears to have a deeper sense. The spacetime considered is a dynamic one—there is no energy conservation there, hence the Hamiltonian contains terms representing spontaneous creation and annihilation pairs of particles. Homogeneity of the spacetime along the z-direction results in the presence of a spatial-like Killing vector representing, z-momentum-component conservation. Hamiltonian (33) reflects this symmetry property: pairs of the particles with opposite z-component momenta may be created, $\widehat{A}^\dagger_{\varepsilon lm}\widehat{A}^\dagger_{-\varepsilon l-m}$ and annihilated $\widehat{A}_{\varepsilon lm}\widehat{A}_{-\varepsilon l-m}$; the Hamiltonian of the system also obeys rotational invariance.

The conservation of the z-momentum component in the terms represented by $\gamma_{\varepsilon lm}$ and $\gamma^*_{\varepsilon lm}$ in the Hamiltonian is an analogue of energy conservation outside the BH, i.e., the particles in a pair carry positive/negative energy; the one with negative energy cannot survive outside the BH but only within the horizon of the BH.

There is a more or less obvious interpretation of the $\beta_{\varepsilon lm}(t)$ coefficient of the Bogolyubov's transformation (35): it is proportional to the number of the particles created during the evolution of the system,

$$\langle 0(t)|\widehat{A}^\dagger_{\varepsilon lm}\widehat{A}_{\varepsilon lm}|0(t)\rangle = \langle 0|\widehat{A}^\dagger_{\varepsilon lm}(t)\widehat{A}_{\varepsilon lm}(t)|0\rangle = |\beta_{\varepsilon lm}(t)|^2, \tag{39}$$

where $|0\rangle$ is the vacuum state for fixed time $t = 0$ and annihilation operators $\widehat{A}_{\varepsilon lm}$ while $|0(t)\rangle$ is the vacuum state for later time t and annihilation operators $\widehat{A}_{\varepsilon lm}(t)$. Due to the violent dynamics of the background spacetime, one may expect the dynamics of the creation and annihilation of the (pairs of) particles to be violent, and conventional adiabatic-like approaches (see e.g., [16,17]) could hardly be regarded as a working scheme. Therefore, attempts to find an approximate solution within a treatment here proposed that might be called a "unitary approach" as based on a unitarity of the evolution of the system, will be discussed in our following paper.

An interesting aspect of the dynamics of the model (5) will be however briefly discussed here. That is the question of the possible instability of the system of interacting harmonic oscillators (33) (see [27,28]). The oscillators interact in pairs, $(\varepsilon lm)/(-\varepsilon l - m)$ and one can consider diagonalization (at an arbitrary instant T') of the Hamiltonian corresponding to such a subsystem. Then the frequency in such a diagonalized case is given as:

$$\Omega^2_{\varepsilon lm} = \omega^2_{\varepsilon lm} - |\gamma_{\varepsilon lm/-\varepsilon l-m}|^2. \tag{40}$$

This expression should be positive, otherwise the system is unstable (see [27]) (this problem will be discussed in detail in our following paper)—this would be named a "gravitationally driven instability". One can check that in this case, Equations (31) and (32) the right hand side of Equation (40)

$$\Omega^2_{\varepsilon lm} = \frac{1}{g_z}\left[\frac{\varepsilon^2}{g_z} + \frac{l(l+1)}{T^2} + \mu^2\right] \tag{41}$$

is positive: there is no gravitational instability in the scalar field quantized in Doran et al. [7] spacetime. An interesting issue is that, apart from the possible instability of type (40), that might be referred to as "a restoring force instability" there is also another possible instability, namely "a friction driven instability" but the problem of its origin and character will be discussed elsewhere.

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Appendix A

Let us briefly analyze the form of the temporal part of Klein-Gordon equation in this case, i.e., Equation (13):

$$\frac{1}{T^2}\frac{d}{dT}\left(T^2 g_z \frac{dR}{dT}\right) + \left(\frac{\varepsilon^2}{g_z} + \mu^2 + \frac{l(l+1)}{T^2}\right)R = 0, \tag{A1}$$

where $g_z = \frac{r_S}{T} - 1$, and lower labels have been omitted here. Making the substitution, $R = f\eta$, one finds

$$\frac{1}{T^2}\frac{d}{dT}\left(T^2 g_z \frac{dR}{dT}\right) = \frac{1}{T^2}\left[(r_S - 2T)(f'\eta + f\eta') + \left(r_S T - T^2\right)(f''\eta + f\eta'' + 2f'\eta')\right] \tag{A2}$$

and prime means differentiation with respect to T. Claiming

$$(r_S - 2T)f + 2\left(r_S T - T^2\right)f' = 0, \tag{A3}$$

one gets $R(T)$ in the form

$$R = \frac{\eta}{\sqrt{T(r_S - T)}}, \tag{A4}$$

and $\eta(T)$ satisfies the following confluent Heun equation

$$\left[\frac{d^2}{dT^2} + \nu^2(T)\right]\eta = 0, \tag{A5}$$

where

$$\nu^2(T) = A + \frac{B}{T} + \frac{C}{(r_S - T)} + \frac{D}{T^2} + \frac{E}{(r_S - T)^2}, \tag{A6}$$

and the five coefficients $A, \ldots, E$ are equal to:

$$A = \left(\varepsilon^2 - \mu^2\right), \quad B = \frac{1}{2r_S}(2l(l+1) + 1),$$
$$C = r_S\left(\mu^2 + 2\varepsilon^2\right) + B, \quad D = \frac{1}{4},$$
$$E = D - 2\left(1 + r_S^2\varepsilon^2\right).$$

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