2022

Optimal Impulse Controls With Changing Running Cost And Applications In Mortgage Refinance

Yuchen Cao
University of Central Florida

OPTIMAL IMPULSE CONTROLS WITH CHANGING RUNNING COST AND APPLICATIONS IN MORTGAGE REFINANCE

by

YUCHEN CAO
M.S. University of Central Florida, 2019
B.S. Shandong University of Finance and Economics, 2016

A dissertation submitted in partial fulfilment of the requirements
for the degree of Doctor of Philosophy
in the Department of Mathematics
in the College of Sciences
at the University of Central Florida
Orlando, Florida

Summer Term
2022

Major Professor: Jiongmin Yong

ABSTRACT

Almost all home buyers have mortgages and it is quite common to have mortgage refinanced. There are two main reasons that make people decide to refinance the mortgage: (i) need some cash for urgent purposes, and (ii) lower the monthly payment. In this dissertation, we are not going to discuss (i), and we are investigating problems related to (ii). To begin with, let us intuitively make the following observations: If the interest rate remains the same as the current mortgage interest rate, then the monthly payment will automatically lower if you start a new mortgage with the same term, say, 30-year, because the loan amount is lower than the previous one. It is not hard to see that although the monthly payment is lowered, your overall payment is higher since the overall term is longer. From this, we see that rational people will not refinance the mortgage if the interest rate is not lower than the current one. Now, the subtle question is how much lower the interest rate than the current one, people should start to think about refinance. Actually, besides interest rate, one should also take into account the mortgage size and closing cost. Mathematically, this can be formulated as an optimal impulse control problem, with some interesting features that make this problem significantly from the classical problems.

Let us now make the above a little more precise. We will formulate an optimal impulse control problem for stochastic differential equations with the running cost and the terminal being changed at the time that an impulse of the control is applied. Because of these, unlike the classical impulse control problems, a control with some zero impulses might be optimal. On the other hand, these features bring some technical difficulties to the problem.

Our idea of solving the problem is as follows. First of all, we will prove that the number of impulses must be finite, and optimal impulse control must exist. Second, by using a backward method, we can solve an optimal impulse control problem with given number of impulses. These problems are

parameterized by the number of impulses. Finally, we solve the original problem by optimizing the number of the impulses.

To my grandfather

ACKNOWLEDGMENTS

I would like to express my sincere gratitude to my advisor, Dr. Jiongmin Yong, for the guidance and encouragement he has given me over the past five years. Though my career was full of frustration and struggle, he trusted in me and encouraged me to keep working. This work would not have been possible without his guidance, inspiration, and patience.

My appreciation also goes to Dr. Jason Swanson, Dr. Christian Keller, and Dr. Liqiang Ni for their service on my dissertation committee.

I would like to thank Dr. Qiyu Sun for his support of my career. Sincere gratitude also goes to all my learned and warm-hearted professors, especially Dr. Zuhair Nashed, Dr. Xin Li, Dr. Zhisheng Shuai and Dr. Eduardo Teixeira who have greatly helped me in my study as well as in my life.

I would like to appreciate my wonderful friends: Dr. Jingrui Sun, Dr. Hanxiao Wang, Evan Donald, Dr. Wei Yan, Dr. Chang Li, Jichao Jiang, Seoyun Choe, Xuyan Shen, Lepeng Zhang, Xuanyi Wang, and many more. I have had a wonderful time with their generous help and daily company.

I wish to thank my family, especially my parents, for their endless love. It is your support that encourages me to go further. It is my pity that I cannot accompany my grandfather for the last time in his life. I believe he would be happy if he could witness my graduation. I miss you.

TABLE OF CONTENTS

CHAPTER 1: INTRODUCTION

It is common to receive a letter from the mortgage company to tell us the payment rate will be lower if we refinance now, however this is not really true. Also, many borrowers consider refinancing their mortgage when the market interest rate drops below the contract rate. There are some issues mainly to be discussed when the borrower determines to refinance or not. First is the payment rate, which will be reset depending on the current remaining balance and the market rate. Also, the borrower may modify their mortgage size depending on their financial situation when refinancing their mortgage. Second issue is the closing cost. Usually a closing cost will be paid by the borrower, which mainly depends on the remaining balance. Moreover, the terminal time for the new mortgage usually is not the same as the old one, which implies more numbers of payment periods left. Therefore, the borrower will evaluate whether the potential saving from the lower interest rate counterbalances the cost of refinancing. It is of interest that investigating how to optimally refinance a mortgage from the perspective of the borrower. This has been studied in the literature (see Dunn and McConnell [11, 12], Pliska [35], Mayer, Piskorski, and Tchistyi [27] and Khandani, Lo and Merton [23], etc.).

Mathematically, the valuation of mortgage refinancing can be modeled based on the stochastic optimal control theory. Each refinance can be treated as an impulse to the system and the cost of the refinance is treated as the impulse cost. Since the loan amount, interest rate, payment rate and maturity will be reset, the running cost and terminal cost in the performance index functional depend on the initial pair of the (new) mortgage period when refinance applied. Therefore, an impulse control problem with changing running cost will be developed and we want to optimize the performance index functional. Classical impulse control problem was initiated by Bensoussan–Lions [2]. With the dynamic programming method, under proper conditions, one can show that the value function of the problem is the unique viscosity solution to a Hamilton-Jacobi-Bellman equation of a quasi-variational inequality form. There are a lot of follow-up works, see, for examples, Bensoussan–Lions [3], Barles [1], Tang-Yong [36] and Li-Yong [25] etc.. Also, thre are a quite a few authors discussed the application of impulse control in mathematical finance (see Korn [24], Cadenillas-Zapatero [4], Øksendal-Sulem [30], etc.). In this dissertation, we model the refinance problem. It turns out that such a model is quite different from the classical impulse control problems.

We begin with one-time refinance model to investigate the significant characteristics of refinance. It is similar to an optimal stopping problem, where the obstacle now becomes the expected total payment after refinancing and the time to refinance becomes the optimal stopping time. Following this idea, an optimal impulse control problem with the initial pair depending on running cost will be developed. The running cost rate function and the terminal cost function now depend on the initial pair, which makes our HJB equation significantly different from the classical case. Thanks to the idea from the optimal stopping problem, we introduce a new method to construct a solution for this model piecewisely, and we show this solution is the value function of the performance index functional in our problem. Having done the above, we then can recursively discuss the multi-time refinance situations. We will show that for a given initial loan amount, the optimal number of refinance must be finite, without considering the cash out case. Therefore, after solving a finitely many impulse control problems, parameterized by the number of impulses, we then optimize the family with respect to the impulse number to get the final solution of the original problem.

The dissertation is organized as follows. In Chapter 2 we recall some basics in stochastic optimal control theory, which are mainly tools we will use later. In Chapter 3 we carefully investigate the refinance problem. In Chapter 4 we introduce the optimal impulse control with the initial pair dependent running cost problem and the terminal time. We will provide the clue of constructing the optimal solution. In Chapter 5 we summarize our work and pose some relevant open problems for further research.

CHAPTER 2: MATHEMATICAL PRELIMINARIES

In this chapter, we will review some basic knowledge in stochastic optimal control theory. More details could be found in Yong–Zhou [39].

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space on which a d-dimensional standard Brownian motion $W(\cdot)$ is defined, with $\mathbb{F} \equiv \{\mathcal{F}_t\}_{t\geqslant 0}$ being its natural filtration augmented by all the $\mathbb{P}$-null sets. First, we recall some properties of stochastic differential equations (SDEs, for short). Consider the following stochastic differential equation:

$$\begin{cases} dX(s) = b(s, X(s))ds + \sigma(s, X(s))dW(s), \\ X(0) = x_0 \in \mathbb{R}^n, \end{cases} \tag{2.1}$$

where $b : [0, T] \times \mathbb{R}^n \mapsto \mathbb{R}^n$, $\sigma : [0, T] \times \mathbb{R}^n \mapsto \mathbb{R}^{n\times d}$, and $T \in (0, \infty)$ being fixed. We recall the definition of strong solution of (2.1)

Definition 2.0.1. *Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be given, $W(t)$ be a given d-dimensional standard $\mathbb{F}$-Brownian motion, and $x_0 \in \mathbb{R}^n$. An $\mathbb{F}$-adapted continuous process $X(t),\ t \in [0, T]$, is called a strong solution of* (2.1) *if*

$$X(0) = x_0, \quad \mathbb{P} - \text{a.s.},$$

$$\int_0^t \{|b(s, X(s))| + |\sigma(s, X(s))|^2\}ds < \infty, \quad \forall t \in [0, T], \quad \mathbb{P} - \text{a.s.},$$

$$X(t) = X(0) + \int_0^t b(s, X(s))ds + \int_0^t \sigma(s, X(s))dW(s), \quad \forall t \in [0, T], \quad \mathbb{P} - \text{a.s.}.$$

If for any two strong solutions $X(t)$ and $Y(t)$ of (2.1) *defined on any given $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ along with any standard $\mathbb{F}$-Brownian motion, we have*

$$\mathbb{P}(X(t) = Y(t), 0 \leqslant t \leqslant T) = 1.$$

then we say that the strong solution is unique or that strong uniqueness holds.

We make the following assumption for the coefficients of (2.1).

(S). Maps $b(t,x)$ and $\sigma(t,x)$ are continuous and there exists an $L > 0$ such that for any $t \in [0,T]$, $x, y \in \mathbb{R}^n$

$$\begin{cases} |b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leqslant L|x-y|, \\ |b(t,0)| + |\sigma(t,0)| \leqslant L. \end{cases}$$

Theorem 2.0.2. *Let (S) hold. Then for any $x_0 \in \mathbb{R}^n$,* (2.1) *admits a unique strong solution $X(\cdot) \equiv X(\cdot; x_0)$ such that for any $p \geqslant 1$*

$$\begin{cases} \mathbb{E}\Big[\sup_{s\in[0,T]} |X(s)|^p\Big] \leqslant K(1+|x_0|^p), \\ \mathbb{E}\Big[|X(t) - X(s)|^p\Big] \leqslant K(1+|x_0|^p)|t-s|^{\frac{p}{2}}, \quad \forall s, t \in [0,T]. \end{cases}$$

Moreover, if $\hat{x} \in \mathbb{R}^n$ and $\hat{X}(\cdot) \equiv X(\cdot; \hat{x})$ is the strong solution of (2.1) *corresponding to $\hat{x}$, then*

$$\mathbb{E}\Big[\sup_{s\in[t,T]} |X(s) - \hat{X}(s)|^p\Big] \leqslant K|x - \hat{x}|^p.$$

See Karatzas–Shreve [21], Yong–Zhou [39] for proofs.

Next, we review the main results in stochastic optimal control, optimal impulse control and optimal stopping problems.

2.1 Optimal Control

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and $W(\cdot)$ be as before. Let $T > 0$. Consider the following controlled stochastic differential equation

$$\begin{cases} dX(s) = b(s, X(s), u(s))dt + \sigma(s, X(s), u(s))dW(s), \qquad s \in [t, T] \\ X(t) = x, \end{cases} \tag{2.2}$$

where $(t, x) \in [0, T] \times \mathbb{R}^n$ is called an *initial pair*, $b : [0, T] \times \mathbb{R}^n \times U \mapsto \mathbb{R}^n$, $\sigma : [0, T] \times \mathbb{R}^n \times U \mapsto \mathbb{R}^{n\times d}$ are called *drift* and *diffusion*, respectively, with U being a given separable metric space. Function $u(\cdot)$ is called the an *admissible control process* which is taken from the following set:

$$\mathcal{U}[t, T] = \Big\{ u : [t, T] \times \Omega \mapsto \mathbb{R}^n \mid u(\cdot) \text{ is } \mathbb{F} - \text{progressively measurable}, \mathbb{E} \int_t^T |u(s)|^2 ds < \infty \Big\}.$$

We introduce the following assumption:

(S1). Maps $b : [0, T] \times \mathbb{R}^n \times U \mapsto \mathbb{R}^n$, $\sigma : [0, T] \times \mathbb{R}^n \times U \mapsto \mathbb{R}^{n\times d}$, $f : [0, T] \times \mathbb{R}^n \times U \mapsto \mathbb{R}$, and $h : \mathbb{R}^n \mapsto \mathbb{R}$ are uniformly continuous, and there exists a constant $L > 0$ such that for $\varphi(t, x, u) = b(t, x, u), \sigma(t, x, u), f(t, x, u), h(x)$,

$$\begin{cases} |\varphi(t, x, u) - \varphi(t, y, u)| \leqslant L|x - y|, \quad \forall t \in [0, T], x, y \in \mathbb{R}^n, u \in U, \\ |\varphi(t, 0, u)| \leqslant L, \quad \forall (t, u) \in [0, T]. \end{cases}$$

Then for any $(t, x) \in [0, T] \times \mathbb{R}^n$, and $u(\cdot) \in \mathcal{U}[t, T]$, (2.2) admits a unique strong solution $X(\cdot) \equiv X(\cdot; t, x, u)$ by Theorme 2.0.2. For any $(t, x) \in [0, T] \times \mathbb{R}^n$, and $u(\cdot) \in \mathcal{U}[t, T]$, let $X(\cdot)$

be the corresponding state process, we introduce the cost functional as follows.

$$J(t,x;u(\cdot)) = \mathbb{E}\Big\{\int_t^T f(s,X(s),u(s))ds + h(X(T))\Big\}, \tag{2.3}$$

where $f : [0,T]\times\mathbb{R}^n\times U \mapsto \mathbb{R}^n$ and $g : \mathbb{R}^n \mapsto \mathbb{R}^n$ are two suitable deterministic maps, which are called *running cost* and *terminal cost*, respectively. The classical optimal control problem can be stated as follows:

Problem (C). For given $(t,x) \in [0,T]\times\mathbb{R}^n$, find a $\bar{u}(\cdot) \in \mathcal{U}[t,T]$ such that

$$J(t,x;\bar{u}(\cdot)) = \inf_{u\in\mathcal{U}[0,T]} J(t,x;\bar{u}(\cdot)) = V(t,x). \tag{2.4}$$

Any $\bar{u}(\cdot) \in \mathcal{U}[t,T]$ satisfying (2.4) is called an *optimal control* for the initial pair (t,x). The corresponding state process $\bar{x}(\cdot)$ and the state-control pair $(\bar{x}(\cdot),\bar{u}(\cdot))$ are called an *optimal state process* and an *optimal pair*, respectively. We call $V(\cdot,\cdot)$ the *value function* of Problem (C). With the dynamic programming principle, we have the follow theorem which characterizes the value function (see Yong–Zhou [39]).

Theorem 2.1.1. *Suppose (S1) holds. For any $(t,x) \in [0,T]\times\mathbb{R}^n$, the value function $V(t,x)$ defined in* (2.4) *is the unique viscosity solution of the corresponding Hamilton-Jacobi-Bellman equation (HJB, for short) as follows under proper conditions*

$$\begin{cases} V_t(t,x) + \inf\limits_{u\in U}\Big\{\dfrac{1}{2}\text{tr}\Big[V_{xx}(t,x)\sigma(t,x,u)\sigma(t,x,u)^\top\Big] + \langle V_x(t,x), b(t,x,u)\rangle \\ \qquad\qquad + f(t,x,u)\Big\} = 0, \quad (t,x)\in[0,T]\times\mathbb{R}^n \\ V(T,x) = h(x), \quad x\in\mathbb{R}^n. \end{cases} \tag{2.5}$$

Once the value function is determined, the corresponding optimal control can be constructed, in principle.

2.2 Optimal Impulse Control

In Problem (C) presented in Section 2.1, the state process changes continuously in time t with the influence of the control. However, in some problems, the controller may modify the state instantaneously, that is an impulse applied to the state. The controller chooses the impulse times and the intensity of these impulses in order to optimize a payoff or a cost. We introduce the optimal impulse control problem to model such a situation.

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and $W(\cdot)$ be as before. Consider the following stochastic differential equation:

$$X(s) = x + \int_t^s b(r, X(r))dr + \int_t^s \sigma(r, X(r))dW(r) + \xi(s), \quad s \in [t, T], \tag{2.6}$$

where $b : [0,T] \times \mathbb{R}^n \mapsto \mathbb{R}^n$, $\sigma : [0,T] \times \mathbb{R}^n \mapsto \mathbb{R}^{n\times d}$, $T \in (0, \infty)$ being fixed and $\xi(\cdot)$ is called an *impulse control* of the following form:

$$\xi(s) = \sum_{i\geqslant 1} \xi_i \chi_{[\tau_i, T]}(s), \quad s \in [t, T].$$

Here $\{\tau_i\}_{i\geqslant 1}$ is an increasing sequence of $\mathbb{F}$-stopping times valued in $[t, T]$, and each ξ_i is an $\mathcal{F}_{\tau_i}$-measurable square integrable random variable taking values in K, where $K \subset \mathbb{R}^n$ being a closed convex cone. Moreover, let

$$\mathscr{K}[t, T] = \Big\{\xi(\cdot) = \sum_{i\geqslant 1} \xi_i \chi_{t_i, T}(\cdot) : [t, T] \mapsto K | \tau_i \geqslant t, \tau_i \uparrow, \sum_{i\geqslant 1} \ell(\tau_i, \xi_i) < \infty\Big\}.$$

Now under this state equation with impulse, we consider the cost functional as follows:

$$J(t,x;\xi(\cdot)) = \mathbb{E}\Big\{ \int_t^T f(s,X(s))ds + h(X(T)) + \sum_{i\geqslant 1} \ell(\tau_i,\xi_i)\Big\}, \tag{2.7}$$

where $f:[0,T]\times\mathbb{R}^n \mapsto \mathbb{R}^n$, $g:\mathbb{R}^n\mapsto\mathbb{R}^n$ and $\ell:[0,T]\times K\mapsto\mathbb{R}^n$ are some suitable deterministic maps. The first two terms on the right-hand side have the same meaning as in the cost functional in Section 2.1 and the third term on the right-hand side is called the *impulse cost*. The optimal impulse control problem can be stated as follows:

Problem (IC). For any $(t,x)\in[0,T]\times\mathbb{R}^n$, find a $\bar\xi(\cdot)\in\mathscr{K}[t,T]$ such that

$$J(t,x;\bar\xi(\cdot)) = \inf_{\xi\in\mathcal{K}} J(t,x;\bar\xi(\cdot)) = V(t,x). \tag{2.8}$$

Any $\bar\xi(\cdot)\in\mathscr{K}[t,T]$ satisfies (2.8) is called an *optimal impulse control*, and $\bar x\equiv x(\cdot;t,x,\bar\xi)$ is called the corresponding *optimal state process* and the state-control pair $(\bar x(\cdot),\bar\xi(\cdot))$ is an *optimal pair*. Similarly, we call $V(\cdot,\cdot)$ is the *value function* of Problem (IC) and we introduce the following assumptions:

(S2). Maps $b:[0,T]\times\mathbb{R}^n\mapsto\mathbb{R}^n$, $\sigma:[0,T]\times\mathbb{R}^n\mapsto\mathbb{R}^{n\times d}$, $f:[0,T]\times\mathbb{R}^n\mapsto\mathbb{R}$, and $h:\mathbb{R}^n\mapsto\mathbb{R}$ are continuous, and there exists a constant $L>0$ such that for $\varphi(t,x)=b(t,x),\sigma(t,x),f(t,x),h(x)$,

$$\begin{cases} |\varphi(t,x)-\varphi(t,y)|\leqslant L|x-y|, \quad \forall t\in[0,T], x,y\in\mathbb{R}^n,\\ |\varphi(t,x)|\leqslant L, \quad \forall (t,x)\in[0,T]\times\mathbb{R}^n. \end{cases}$$

(S3). Map ℓ is a continuous and there exists a constant $\ell_0>0$,

$$\ell(t,\xi+\hat\xi) < \ell(t,\xi)+\ell(t,\hat\ell), \quad \forall t\in[0,T], \xi,\hat\xi\in K,$$

$$\inf_{t\in[0,T],\xi\in K}\ell(t,\xi)\equiv\ell_0>0, \qquad \lim_{\xi\in K,|\xi|\to\infty}\inf_{t\in[0,T]}\ell(t,\xi)=\infty.$$

The the following theorem is well-known by the dynamic programming principle (see Tang-Yong [36]).

Theorem 2.2.1. *Let (S2)-(S3) hold. For any* $(t,x) \in [0,T] \times \mathbb{R}^n$*, the value function* $V(t,x)$ *defined in* (2.8) *is the unique viscosity solution of the corresponding HJB equation in the variational inequality form as follows under proper conditions.*

$$\begin{cases} \min\Big\{V_t(t,x) + \frac{1}{2}\text{tr}\Big[V_{xx}(t,x)\sigma(t,x)\sigma(t,x)^\top\Big] + \langle V_x(t,x), b(t,x)\rangle \\ \qquad\qquad + f(t,x), N[V(t,x)] - V(t,x)\Big\} = 0, \quad (t,x) \in [0,T] \times \mathbb{R}^n \\ V(T,x) = h(x), \quad x \in \mathbb{R}^n. \end{cases} \tag{2.9}$$

where

$$N[V(t,x)] = \min_{\xi \in K}\{V(t,x+\xi) + \ell(t,\xi)\}.$$

The optimal impulse control can be constructed once the value function is determined.

2.3 Optimal Stopping

Optimal stopping time problem is a model that controller controls the ending time of the system directly. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and $W(\cdot)$ be as before. In the formulation of such models for given an initial pair (t,x), an admissible control is τ which is an $\mathbb{F}$-stopping time taking values in $[t,T]$. Let $\mathcal{T}[t,T]$ be the set of admissible stopping times valued in $[t,T]$.

We consider the state equation

$$\begin{cases} dX(s) = b(s,X(s))ds + \sigma(s,X(s))dW(s), \\ X(t) = x \in \mathbb{R}^n, \end{cases} \tag{2.10}$$

where $b : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n\times d}$, and $T \in (0,\infty)$ being fixed. For any $(t,x) \in [0,T] \times \mathbb{R}^n$, consider the cost functional as follows:

$$J(t,x;\tau) = \mathbb{E}\Big\{ \int_t^\tau f(s,X(s))ds + h(X(\tau))\Big\}, \tag{2.11}$$

where $f : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^n$ are two suitable deterministic maps, with the same meaning as before. Now the optimal stopping problem can be stated as follow:

Problem (S). For given $(t,x) \in [0,T] \times \mathbb{R}^n$, find $\bar{\tau} \in \mathcal{T}[t,T]$ such that

$$J(t,x;\bar{\tau}) = \inf_{\tau\in\mathcal{T}} J(t,x;\tau) = V(t,x), \tag{2.12}$$

Any $\bar{\tau} \in \mathcal{T}$ satisfying (2.12) is called an *optimal stopping time* and now we call $V(\cdot,\cdot)$ the *value function* of Problem (S).

For the optimal stopping time problem, let (S2) hold, then some routine argument can be applied to prove the following theorem (see Øksendal-Reikvam [29] and Pham [31]).

Theorem 2.3.1. *Let (S2) hold. For any $(t,x) \in [0,T] \times \mathbb{R}^n$, the value function $V(t,x)$ defined in* (2.12) *is the unique viscosity solution of the following HJB equation in the variational inequality form.*

$$\begin{cases} \min\Big\{V_t(t,x) + \frac{1}{2}\text{tr}\Big[V_{xx}(t,x)\sigma(t,x)\sigma(t,x)^\top\Big] + \langle V_x(t,x), b(t,x)\rangle \\ \qquad\qquad +f(t,x), h(x) - V(t,x)\Big\} = 0, \quad (t,x) \in [0,T] \times \mathbb{R}^n \\ V(T,x) = h(x), \quad x \in \mathbb{R}^n. \end{cases} \tag{2.13}$$

CHAPTER 3: Mortgage Refinance Problem

In this chapter, we will discuss mortgage refinancing mathematically.

3.1 One-Time Refinance

We begin with the problem of at most making one time refinance.

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and $W(\cdot)$ be as before. Consider the following interest rate model:

$$d\mathbf{r}(s) = b_0(s, \mathbf{r}(s))ds + \sigma_0(s, \mathbf{r}(s))dW(s), \qquad s \geqslant 0. \tag{3.1}$$

We introduce the following assumption.

(S4). Maps $b_0 : [0, T] \times \mathbb{R}^n \mapsto \mathbb{R}^n$ and $\sigma_0 : [0, T] \times \mathbb{R}^n \mapsto \mathbb{R}^{n \times d}$ are measurable and there exist $L > 0$, for any $t \in [0, T]$ and $x, y \in \mathbb{R}^n$ such that

$$\begin{cases} |b_0(t, x) - b_0(t, y)| + |\sigma_0(t, x) - \sigma_0(t, y)| \leqslant L|x - y|, \\ |b_0(t, x)| + |\sigma_0(t, x)| \leqslant L. \end{cases}$$

For technical convenience, we assume that

$$b_0(s, 0) = 0, \qquad \sigma_0(s, 0) = 0,$$

and (3.1) admits a unique solution, for any given initial condition. This will be true if $r \mapsto (b_0(s, r), \sigma(s, r))$ is uniformly Lipschitz. Therefore, if $\mathbf{r}(t) = 0$ for some $t \geqslant 0$, then $\mathbf{r}(s) = 0$, for all $s \geqslant t$.

The following proposition which can be get from Theorem 2.0.2 directly.

Proposition 3.1.1. *Let (S4) hold. Then for each* $(t, r) \in [0, T] \times \mathbb{R}^+$, *interest rate model* (3.1) *admits a unique(strong) solution* $\mathbf{r}(\cdot) \equiv \mathbf{r}(\cdot; t, r)$. *Moreover,*

$$\mathbb{E}_t \Big\{ \sup_{s \in [t,T]} |\mathbf{r}(s; t, r)| \Big\} \leqslant C(1 + |r|), \quad \forall (t, r) \in [0, T] \times \mathbb{R}^+.$$

Moreover, if both $t, \bar{t} \in [0, T]$, $r, \bar{r} \in \mathbb{R}^+$,

$$\mathbb{E}_t \Big\{ \sup_{s \in [t \vee \bar{t}, T\}} |\mathbf{r}(s; t, r) - \mathbf{r}(s; \bar{t}, r)| \Big\} \leqslant C(1 + |r|)(|t - \bar{t}|^{\frac{1}{2}}),$$

and

$$\mathbb{E}_t \Big\{ \sup_{s \in [t,T]} |\mathbf{r}(s; t, r) - \mathbf{r}(s; t, \bar{r})| \Big\} \leqslant C|r - \bar{r}|.$$

Hereafter, $C > 0$ represents a generic constant which can be different from line to line.

For any $t_0 \geqslant 0$, consider a fixed term $T_0 > 0$ mortgage starting from t_0 with the loan amount $x_0 > 0$, and the mortgage rate $\mathbf{r}(t_0) + \delta(t_0, x_0, T_0)$, where $\delta(\cdot, \cdot, \cdot)$ is a deterministic function depends on the initial time t_0, initial amount x_0 and the loan term T_0. The closing cost (including various fees, such as origination fee, title insurance, etc.) is assumed to be $\kappa_0(t_0) + \kappa_1(t_0)x_0$ for some maps κ_0 and κ_1, depending on the initial time t_0. Next, at any $s \in (t_0, t_0 + T_0)$, the principal balance is denoted by $X(s)$. Suppose there is no prepayment, nor default. Then process $X(\cdot)$ satisfies

$$dX(s) = \big\{ [\mathbf{r}(t_0) + \delta(t_0, x_0, T_0)]X(s) - m(t_0, x_0, T_0) \big\} ds, \qquad s \in [t_0, t_0 + T_0], \tag{3.2}$$

with $X(t_0) = x_0$. Here, $m(t_0, x_0, T_0)$ is the payment rate determined by $X(t_0 + T_0) = 0$. Denote

$r_0 = \mathbf{r}(t_0) + \delta(t_0, x_0, T_0)$ and $m_0 = m(t_0, x_0, T_0)$. Solving (3.2), we obtain

$$X(s) = e^{r_0(s-t_0)}x_0 - m_0 \int_{t_0}^{s} e^{r_0(s-\theta)} d\theta = e^{r_0(s-t_0)}x_0 - m_0 \frac{e^{r_0(s-t_0)} - 1}{r_0}.$$

Hence, for $s = t_0 + T_0$, by the condition $X(t_0 + T_0) = 0$, we have

$$0 = e^{r_0 T_0} x_0 - m_0 \frac{e^{r_0 T_0} - 1}{r_0},$$

which gives the payment rate

$$m(t_0, x_0, T_0) = \frac{r_0 e^{r_0 T_0}}{e^{r_0 T_0} - 1} x_0 \equiv m_0.$$

This implies

$$X(s) = e^{r_0(s-t_0)}x_0 - \frac{r_0 e^{r_0 T_0}}{e^{r_0 T_0} - 1} \frac{e^{r_0(s-t_0)} - 1}{r_0} x_0 = \frac{e^{r_0 T_0} - e^{r_0(s-t_0)}}{e^{r_0 T_0} - 1} x_0$$

Now, let $t \in [t_0, t_0 + T_0)$ be the current time and $s \in [t, t_0 + T_0)$. Then, with $X(t) = x$, one has

$$X(s) = \frac{e^{r_0 T_0} - e^{r_0(s-t_0)}}{e^{r_0 T_0} - e^{r_0(t-t_0)}} x, \qquad s \in [t, t_0 + T_0].$$

Suppose $\tau \in (t, t_0 + T_0)$ at which a refinance has been made, with term $T_i > 0, T_i \in \{T_1 \cdots, T_j\}$, which means different terms for the new mortgage could be chosen. Then the payment rate on $[\tau, \tau + T_i]$ will be

$$m(\tau, X(\tau), T_i) = \frac{[\mathbf{r}(\tau) + \delta(\tau, X(\tau), T_i)] e^{[\mathbf{r}(\tau)+\delta(\tau,X(\tau),T_i)]T_i}}{e^{[\mathbf{r}(\tau)+\delta(\tau,X(\tau),T_i)]T_i} - 1} X(\tau), \tag{3.3}$$

The (closing) cost will be $\kappa_0(\tau) + \kappa_1(\tau)X(\tau)$. Thus the total expected discounted payment is given by

$$J(t,x,r;\tau,T_i) = \mathbb{E}_t\Big\{\int_t^\tau m_0 e^{-\int_t^s \mathbf{r}(\theta)d\theta}ds + \mathbf{1}_{\{\tau<t_0+T_0\}}\Big[\kappa_0(\tau)+\kappa_1(\tau)X(\tau) + \int_\tau^{\tau+T_i} m(\tau,X(\tau),T_i)e^{-\int_t^s \mathbf{r}(\theta)d\theta}ds\Big]\Big\}.$$

where $\mathbb{E}_t$ is the conditional expectation operator. The optimal one-time refinance problem can be stated as follows.

Problem (RF)$_1$. Find $\tau^* \in [t, t_0+T_0]$ and $T^* \in \{T_1,\cdots,T_j\}$ such that

$$J(t,x,r;\tau^*,T^*) = \inf_{\tau\in[t,t_0+T_0]}\min_{T_i\in\{T_1,\cdots,T_j\}} J(t,x,r;\tau,T_i) = V(t,x,r).$$

Note for any $T>0$,

$$\mathbb{E}_\tau\Big\{\int_\tau^{\tau+T} e^{-\int_t^s \mathbf{r}(\theta)d\theta}ds\Big\} = e^{-\int_t^\tau \mathbf{r}(\theta)d\theta}\mathbb{E}_\tau\Big\{\int_\tau^{\tau+T} e^{-\int_\tau^s \mathbf{r}(\theta)d\theta}ds\Big\}$$

Let

$$\psi(t,r,T) = \mathbb{E}_t\Big\{\int_t^T e^{-\int_t^s \mathbf{r}(\theta)d\theta}ds\Big\}, \qquad \mathbf{r}(t) = r.$$

Then, for any $0<\varepsilon<T-t$,

$$\begin{aligned}\psi(t,r,T) &= \mathbb{E}_t\Big\{\int_t^{t+\varepsilon} e^{-\int_t^s \mathbf{r}(\theta)d\theta}ds + \int_{t+\varepsilon}^T e^{-\int_t^s \mathbf{r}(\theta)d\theta}ds\Big\}\\ &= \mathbb{E}_t\Big\{\int_t^{t+\varepsilon} e^{-\int_t^s \mathbf{r}(\theta)d\theta}ds + e^{-\int_t^{t+\varepsilon} \mathbf{r}(\theta)d\theta}\mathbb{E}_{t+\varepsilon}\Big(\int_{t+\varepsilon}^T e^{-\int_{t+\varepsilon}^s \mathbf{r}(\theta)d\theta}ds\Big)\Big\}\\ &= \mathbb{E}_t\Big\{\int_t^{t+\varepsilon} e^{-\int_t^s \mathbf{r}(\theta)d\theta}ds + e^{-\int_t^{t+\varepsilon} \mathbf{r}(\theta)d\theta}\psi(t+\varepsilon,\mathbf{r}(t+\varepsilon),T)\Big\}.\end{aligned}$$

Hence,

$$
\begin{aligned}
0 &= \frac{1}{\varepsilon}\mathbb{E}_t\Big\{\int_t^{t+\varepsilon} e^{-\int_t^s \mathbf{r}(\theta)d\theta}ds + e^{-\int_t^{t+\varepsilon}\mathbf{r}(\theta)d\theta}\psi(t+\varepsilon,\mathbf{r}(t+\varepsilon),T) - \psi(t,r,T)\Big\} \\
&\to 1 - r\psi(t,r,T) + \psi_t(t,r,T) + \psi_r(t,r,T)b_0(t,r) + \frac{1}{2}\sigma_0(t,r)^2\psi_{rr}(t,r,T),
\end{aligned}
$$

with

$$
\psi(T,r,T) = 0.
$$

Therefore, $\psi(\cdot,\cdot,T)$ is the solution to the following partial differential equation (PDE, for short) with T as a parameter.

$$
\begin{cases}
\psi_t(t,r,T) + \frac{1}{2}\sigma_0(t,r)^2\psi_{rr}(t,r,T) + b_0(t,r)\psi_r(t,r,T) - r\psi(t,r,T) + 1 = 0, \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (t,r) \in [0,T]\times(0,\infty), \\
\psi(T,r,T) = 0, \qquad r \in (0,\infty), \\
\psi(t,0,T) = T - t, \qquad t \in [0,T].
\end{cases}
\tag{3.4}
$$

Next, we let

$$
R(s;t,r) = e^{-\int_t^s \mathbf{r}(\theta)d\theta}, \qquad s \geqslant t.
$$

Then

$$
dR(s;t,r) = -\mathbf{r}(s)R(s;t,r), \qquad R(t;t,r) = 1.
$$

Consequently,

$$
\mathbb{E}_\tau\Big\{\int_\tau^{\tau+T} e^{-\int_t^s \mathbf{r}(\theta)d\theta}ds\Big\} = e^{-\int_t^\tau \mathbf{r}(\theta)d\theta}\mathbb{E}_\tau\Big\{\int_\tau^{\tau+T} e^{-\int_\tau^s \mathbf{r}(\theta)d\theta}ds\Big\} = R(\tau;t,r)\psi(\tau,\mathbf{r}(\tau),\tau+T).
$$

Hence,

$$d\begin{pmatrix}\mathbf{r}(s)\\ R(s;t)\\ X(s)\end{pmatrix}=\begin{pmatrix}b_0(s,\mathbf{r}(s))\\ -\mathbf{r}(s)R(s;t)\\ r_0X(s)-m_0\end{pmatrix}ds+\begin{pmatrix}\sigma_0(s,\mathbf{r}(s))\\ 0\\ 0\end{pmatrix}dW(s)$$

with

$$\begin{aligned}J(t,x,r;\tau,T_i)=\mathbb{E}\Big\{\int_t^\tau m_0R(s;t,r)ds+\mathbf{1}_{\{\tau<t_0+T_0\}}\Big[\kappa_0(\tau)+\kappa_1(\tau)X(\tau)\\ +m(\tau,X(\tau),T_i)R(\tau;t,r)\psi(\tau,\mathbf{r}(\tau),\tau+T_i)\Big]\Big\}.\end{aligned}$$

Unlike with the classical optimal stopping problem, the obstacle in our cost functional above is determined by the solution ($\psi(\tau,\mathbf{r}(\tau),\tau+T_i)$) of a PDE. We should notice that the initial time of this PDE is same as the stopping time τ. This difference makes our HJB equation different from the classical case, which is coupled with a PDE, as follows.

$$\begin{cases}\min\{V_t+\dfrac{\sigma_0(t,r)^2}{2}V_{rr}+b_0(t,r)V_r+(r_0x-m_0)V_x+m_0,\Phi-V\}=0,\\ \qquad\qquad (t,x,r)\in[t_0,t_0+T_0]\times\mathbb{R}^+\times\mathbb{R}^+\equiv\mathcal{D},\\ V(t_0+T_0,x,r)=0,\qquad (x,r)\in\mathbb{R}^+\times\mathbb{R}^+,\\ V(t,0,r)=0,\qquad (t,r)\in[t_0,t_0+T_0]\times\mathbb{R}^+,\\ V(t,x,0)=\kappa_0(t)+\kappa_1(t)x+\dfrac{\delta(t,x,T_0)e^{\delta(t,x,T_0)T_0}}{e^{\delta(t,x,T_0)T_0}-1}xT,\qquad (t,x)\in[t_0,t_0+T_0]\times\mathbb{R}^+.\end{cases}$$

where

$$\Phi(t,x,r)=\min_{T_i\in\{T_1,\cdots,T_j\}}\{\kappa_0(t)+\kappa_1(t)x+m(t,x,T_i)\psi(t,r,t+T_i)\},\quad (t,x,r)\in\mathcal{D},$$

with $\psi(t,r,t+T_i)$ being the solution of (3.4).

3.1.1 Properties of Value Function

Let $t_0, T_0 \in \mathbb{R}^+$, and recall $\mathcal{D} = [t_0, t_0 + T_0] \times \mathbb{R}^+ \times \mathbb{R}^+$. We introduce the following assumption.

(S5). Maps $\kappa_0, \kappa_1 : [0, T] \mapsto \mathbb{R}^+$ satisfy the Lipschitz conditions, $\delta(t, x, T) : \mathcal{D} \mapsto \mathbb{R}^+$ with

$$|\delta(t, x, T)| \leqslant L, \quad \forall (t, x, T) \in [t_0, t_0 + T_0] \times \mathbb{R}^+ \times \mathbb{R}^+,$$

and for any $(t, x, T), (\bar{t}, \bar{x}, \bar{T}) \in [t_0, t_0 + T_0] \times \mathbb{R}^+ \times \mathbb{R}^+$,

$$|\delta(t, x, T) - \delta(\bar{t}, \bar{x}, \bar{T})| \leqslant L(|t - \bar{t}| + |x - \bar{x}| + |T - \bar{T}|).$$

The following result is concerned with some basic properties of the value function.

Proposition 3.1.2. *Let (S4)-(S5) hold. For any* $(t, x, r), (\bar{t}, \bar{x}, \bar{r}) \in \mathcal{D}$,

$$|V(t, x, r) - V(\bar{t}, \bar{x}, \bar{r})| \leqslant C\Big\{(1 + |r| \vee |\bar{r}|)^2 (|x| \vee |\bar{x}|)|t - \bar{t}|^{\frac{1}{2}} + |x - \bar{x}| + |r - \bar{r}|\Big\}.$$

Proof. First we show the value function $V(\cdot, \cdot, \cdot)$ is Lipschitz continuous on x.

For any $(t, x, r), (t, \bar{x}, r) \in \mathcal{D}$ and $\tau \in [t, t_0 + T_0]$, $T_i \in \{T_1, \cdots, T_j\}$ by the definition of $X(\cdot)$, we have

$$\begin{aligned}
&|J(t, x, r; \tau, T_i) - J(t, \bar{x}, r; \tau, T_i)| \\
\leqslant\ & \mathbb{E}_t \Big\{ \mathbf{1}_{\{\tau < t_0 + T_0\}} \Big[|\kappa_1(\tau)| \Big| \frac{e^{r_0 T_0} - e^{r_0(\tau - t_0)}}{e^{r_0 T_0} - e^{r_0(t - t_0)}} \Big| |x - \bar{x}| \\
&+ \Big| \frac{[\mathbf{r}(\tau) + \delta(\tau, X(\tau), T_i)] e^{[\mathbf{r}(\tau) + \delta(\tau, X(\tau), T_i)] T_i}}{e^{[\mathbf{r}(\tau) + \delta(\tau, X(\tau), T_i)] T_i} - 1} \Big| \Big| \frac{e^{r_0 T_0} - e^{r_0(\tau - t_0)}}{e^{r_0 T_0} - e^{r_0(t - t_i)}} \Big| \\
&\times |x - \bar{x}| |R(\tau; t)| |\psi(\tau, \mathbf{r}(\tau); \tau + T_i)| \Big] \Big\} \\
\leqslant\ & C|x - \bar{x}|.
\end{aligned}$$

Similarly, we show the value function $V(\cdot, \cdot, \cdot)$ is Lipschitz continuous on r.

For any $(t,x,r),(t,x,\bar{r})\in\mathcal{D}$ and $T>0$, we have

$$
\begin{aligned}
&\mathbb{E}_t\Big\{\sup_{s\in(t,t_0+T_0)}|m(s,X(s),T;t,x,r)-m(s,X(s),T;t,x,\bar{r})|\Big\}\\
=&\,\mathbb{E}_t\Big\{\sup_{s\in(t,t_0+T_0)}\Big|\frac{[\mathbf{r}(s;t,r)+\delta(s,X(s),T;t,x)]e^{[\mathbf{r}(s;t,r)+\delta(s,X(s),T;t,x)]T}}{e^{[\mathbf{r}(s;t,r)+\delta(s,X(s),T;t,x)]T}-1}X(s;t,x)\\
&-\frac{[\mathbf{r}(s;t,\bar{r})+\delta(s,X(s),T;t,x)]e^{[\mathbf{r}(s;t,\bar{r})+\delta(s,X(s),T;t,x)]T}}{e^{[\mathbf{r}(s;t,\bar{r})+\delta(s,X(s),T;t,x)]T}-1}X(s;t,x)\Big|\Big\}\\
\leqslant&\,\mathbb{E}_t\Big\{\sup_{s\in(t,t_0+T_0)}C(|\mathbf{r}(s;t,r)-\mathbf{r}(s;t,\bar{r})|)\\
&+|\mathbf{r}(s;t,r)+\delta(s,X(s),T;t,x)]e^{[\mathbf{r}(s;t,r)+\delta(s,X(s),T;t,x)]T}\\
&-\mathbf{r}(s;t,\bar{r})+\delta(s,X(s),T;t,x)]e^{[\mathbf{r}(s;t,\bar{r})+\delta(s,X(s),T;t,x)]T}|\Big\}\\
\leqslant&\,\mathbb{E}_t\Big\{\sup_{s\in(t,t_0+T_0)}C(|\mathbf{r}(s;t,r)-\mathbf{r}(s;t,\bar{r})|)\Big\}\\
\leqslant&\,C|r-\bar{r}|.
\end{aligned}
$$

Next,

$$
\mathbb{E}_t\Big\{\sup|R(s;t,r)-R(s;t,\bar{r})|\Big\}\leqslant C\mathbb{E}_t\Big|\int_t^s\mathbf{r}(\theta;t,r)d\theta-\int_t^s\mathbf{r}(\theta;t,\bar{r})d\theta\Big|\leqslant C|r-\bar{r}|.
$$

Therefore,

$$
\begin{aligned}
&|J(t,x,r;\tau,T_i)-J(t,x,\bar{r};\tau,T_i)|\\
\leqslant&\,\mathbb{E}_t\Big\{\int_t^\tau m_0(R(s;t,r)-R(s;t,\bar{r}))ds+\mathbf{1}_{\{\tau<t_0+T_0\}}\Big[(m(\tau,X(\tau),T_i;t,x,r)R(\tau;t,r)\\
&-m(\tau,X(\tau),T_i;t,x,\bar{r})R(\tau;t,\bar{r}))\psi(\tau,\mathbf{r}(\tau);\tau+T_i)\Big]\Big\}\\
\leqslant&\,C|r-\bar{r}|.
\end{aligned}
$$

Finally, we prove the $\frac{1}{2}$-Hölder continuity of $V(\cdot,\cdot,\cdot)$ on t. By Proposition 3.1.1, we obtain

$$
\mathbb{E}_t\Big\{R(s;t,r)\Big\}\leqslant 1\quad\forall s\in[t,T],
$$

and for any $\bar{t} \in (t, T]$,

$$
\begin{aligned}
&\mathbb{E}_t\Big\{ \sup_{s\in[\bar{t},T]} |R(s;t,r) - R(s;\bar{t},r)| \Big\} \\
&\leqslant C\mathbb{E}_t\Big\{ \sup_{s\in[\bar{t},T]} \Big| \int_t^s \mathbf{r}(\theta)d\theta - \int_{\bar{t}}^s \mathbf{r}(\theta)d\theta \Big| \Big\} \\
&\leqslant C\mathbb{E}_t\Big\{ \sup_{s\in[\bar{t},T]} \Big| \int_t^{\bar{t}} \mathbf{r}(\theta;t,r)d\theta + \int_{\bar{t}}^s \mathbf{r}(\theta;t,r)d\theta - \int_{\bar{t}}^s \mathbf{r}(\theta;\bar{t},r)d\theta \Big| \Big\} \\
&\leqslant C\mathbb{E}_t\Big\{ \sup_{s\in[\bar{t},T]} \Big| \int_t^{\bar{t}} \mathbf{r}(\theta;t,r)d\theta \Big| + \sup_{s\in[\bar{t},T]} \Big| \int_{\bar{t}}^s \mathbf{r}(\theta;t,r)d\theta - \int_{\bar{t}}^s \mathbf{r}(\theta;\bar{t},r)d\theta \Big| \Big\} \\
&\leqslant C(1+|r|)(|t-\bar{t}|^{\frac{1}{2}}).
\end{aligned}
$$

Then, for any $(t, x) \in [t_0, t_0 + T_0] \times \mathbb{R}^+$,

$$
\begin{aligned}
|X(s;t,x) - X(s;\bar{t},x)| &\leqslant \Big| \frac{e^{r_0 T_0} - e^{r_0(s-t_0)}}{e^{r_0 T_0} - e^{r_0(t-t_0)}} x - \frac{e^{r_0 T_0} - e^{r_0(s-t_0)}}{e^{r_0 T_0} - e^{r_0(\bar{t}-t_0)}} x \Big| \\
&\leqslant |x| \Big| \frac{e^{r_0 T_0} - e^{r_0(s-t_0)}}{e^{r_0 T_0} - e^{r_0(t-t_0)}} - \frac{e^{r_0 T_0} - e^{r_0(s-t_0)}}{e^{r_0 T_0} - e^{r_0(\bar{t}-t_0)}} \Big| \\
&= |x| \Big| \frac{e^{r_0(t-t_0)} - e^{r_0(\bar{t}-t_0)}}{\big(e^{r_0 T_0} - e^{r_0(t-t_0)}\big)\big(e^{r_0 T_0} - e^{r_0(\bar{t}-t_0)}\big)} \Big| \\
&\leqslant |x| \Big| e^{r_0(t-t_0)} - e^{r_0(\bar{t}-t_0)} \Big| \\
&\leqslant C|x||t-\bar{t}|.
\end{aligned}
$$

We noticed that for any $T \in \mathbb{R}^+$ and any $\tau \in [t_0, t_0 + T_0]$,

$$
\mathbb{E}_t\{\psi(\tau, r; \tau + T)\} \leqslant T
$$

there exist a constant K such that

$$
\mathbb{E}_t\{m(\tau, X(\tau), T)\} < K.
$$

For any $(t,x,r)\in\mathcal{D}$,

$$
\begin{aligned}
&\mathbb{E}_t\Big\{\sup_{s\in(\bar t,t_0+T_0)}|m(s,X(s),T;t,x,r)-m(s,X(s),T;\bar t,x,r)|\Big\}\\
&=\mathbb{E}_t\Big\{\sup_{s\in(\bar t,t_0+T_0)}\Big|\frac{[\mathbf{r}(s;t,r)+\delta(s,X(s),T;t,x)]e^{[\mathbf{r}(s;t,r)+\delta(s,X(s),T;t,x,)]T}}{e^{[\mathbf{r}(s;t,r)+\delta(s,X(s),T;t,x)]T}-1}X(s;t,x)\\
&\quad-\frac{[\mathbf{r}(s;\bar t,r)+\delta(s,X(s),T;\bar t,x)]e^{[\mathbf{r}(s;\bar t,r)+\delta(s,X(s),T;\bar t,x)]T}}{e^{[\mathbf{r}(s;\bar t,r)+\delta(s,X(s),T;\bar t,x)]T}-1}X(s;\bar t,x)\Big|\Big\}\\
&\leqslant\mathbb{E}_t\Big\{C\sup_{s\in(\bar t,t_0+T_0)}\Big[|X(s;t,x)\mathbf{r}(s;t,r)-X(s;\bar t,x)\mathbf{r}(s;\bar t,r)|\\
&\quad+\Big|X(s;t,x)\mathbf{r}(s;t,r)e^{\mathbf{r}(s;\bar t,r)+\delta(s,X(s),T;\bar t,x)]T}-X(s;\bar t,x)\mathbf{r}(s;\bar t,r)e^{\mathbf{r}(s;t,r)+\delta(s,X(s),T;t,x)]T}\Big|\\
&\quad+|X(s;t,x)\delta(s,X(s),T;t,x)-X(s;\bar t,x)\delta(s,X(s),T;\bar t,x)|\\
&\quad+\Big|X(s;t,x)\delta(s,X(s),T;t,x)e^{\mathbf{r}(s;\bar t,r)+\delta(s,X(s),T;\bar t,x)]T}\\
&\quad-X(s;\bar t,x)\delta(s,X(s),T;\bar t,x)e^{\mathbf{r}(s;t,r)+\delta(s,X(s),T;t,x)]T}\Big|\Big]\Big\}\\
&\leqslant\Big\{C(1+|r|)|x||t-\bar t|^{\frac12}+C(1+|r|)^2|x||t-\bar t|^{\frac12}+C|x||t-\bar t|+C(1+|r|)|x||t-\bar t|^{\frac12}\Big\}\\
&\leqslant C(1+|r|)^2|x||t-\bar t|^{\frac12}.
\end{aligned}
$$

Hence for any $(t,x,r),(\bar t,x,r)\in\mathcal{D}$ and $T_i\in\{T_1,\cdots,T_j\}$ suppose $t<\bar t$ and $\tau\in[\bar t,t_0+T_0]$,

$$
\begin{aligned}
&|J(t,x,r;\tau,T_i)-J(\bar t,x,r;\tau,T_i)|\\
&\leqslant\mathbb{E}_t\Big\{\int_t^\tau m_0R(s;t,r)ds-\int_{\bar t}^\tau m_0R(s;\bar t,r)ds+\mathbf{1}_{\{\tau<t_0+T_0\}}\Big[\kappa_1(\tau)|X(\tau;t,x)-X(\tau;\bar t,x)|\\
&\quad+\int_\tau^{\tau+T_i}\Big|m(\tau,X(\tau),T_i;t,x,r)e^{-\int_t^s\mathbf{r}(\theta;t,r)d\theta}-m(\tau,X(\tau),T_i;\bar t,x,r)e^{-\int_t^s\mathbf{r}(\theta;\bar t,r)d\theta}\Big|ds\Big]\Big\}\\
&\leqslant C(1+|r|)^2|x||t-\bar t|^{\frac12}.
\end{aligned}
$$

Above all, we obtain our conclusion. □

3.1.2 Principle of Optimality and HJB Equation

The following proposition gives the existence of optimal stopping time for Problem (RF)$_1$.

Proposition 3.1.3. *Let (S4) and (S5) hold. For any $(t,x,r) \in \mathcal{D}$, and $\tau \in [t, t_0+T_0]$, $T_i \in \{T_1, \cdots, T_j\}$, $J(t,x,r;\tau,T_i)$ is well-defined $\mathcal{F}_t-$measurable random variable. Moreover, there exists $\bar{\tau}(t,x,r) \in [t, t_0+T_0]$ and $T^* \in \{T_1, \cdots, T_j\}$ such that*

$$V(t,x,r) \equiv \inf_{\tau\in[t,t_0+T_0]} \min_{T_i\in\{T_1,\cdots,T_j\}} J(t,x,r;\tau,T_i) = J(t,x,r;\bar{\tau},T^*). \tag{3.5}$$

Consequently, for any $(t,x,r) \in \mathcal{D}$, and $\tau \in [t, t_0+T_0]$, $T_i \in \{T_1, \cdots, T_j\}$, $V(t,x,r)$ is $\mathcal{F}_t-$measurable.

Proof. For any fixed $(t,x,r) \in (t_0, t_0+T_0) \times (0, x_0) \times \mathbb{R}^n$, $\tau \in [t, t_0+T_0]$ and $T_i \in \{T_1, \cdots, T_j\}$,

$$\begin{aligned}
&|J(t,x,r;\tau,T_i)| \\
\leqslant\ & \mathbb{E}_t\Big\{ \int_t^\tau m_0 |e^{-\int_t^s \mathbf{r}(\theta)d\theta}| ds + \mathbf{1}_{\{\tau<t_0+T_0\}} \Big[|\kappa_0(\tau)| + |\kappa_1(\tau)||X(\tau)| \\
&+ \int_\tau^{\tau+T_i} |m(\tau, X(\tau), T_i)||e^{-\int_t^s \mathbf{r}(\theta)d\theta}| ds \Big] \Big\} \\
\leqslant\ & C|x|(1+|r|).
\end{aligned}$$

Hence $J(t,x,r;\tau)$ is well-defined $\mathcal{F}_t-$measurable random variable. Next it is clear that $t \mapsto J(t,x,t;\tau)$ is continuous. Therefore by Theorem 10.1.9 of [28] (see also Theorem D.12 of [22]), we have the existence of an optimal stopping time $\bar{\tau}(t,x,r)$ for Problem (RF)$_1$. □

Next, we will derive the principle of optimality for Problem (RF)$_1$, and we we split it into following steps.

Proposition 3.1.4. *Let (S4) and (S5) hold. For any* $(t, x, r) \in \mathcal{D}$,

$$V(t, x, r) \leqslant \Phi(t, x, r) \quad \text{a.s.},$$

where we recall that

$$\Phi(t, x, r) = \min_{T_i \in \{T_1, \cdots, T_j\}} \{\kappa_0(t) + \kappa_1(t)x + m(t, x, T_i)\psi(t, r, t + T_i)\}, \quad (t, x, r) \in \mathcal{D},$$

with $\psi(t, r; t_0 + T_0)$ *being the solution of* (3.4) *and*

$$V(t, x, r) \leqslant \inf_{\tau \in (t_0, t_0 + T_0)} \mathbb{E}_t \Big\{ \int_t^\tau m_0 R(s; t, r) ds + V(\tau, X(\tau), r(\tau)) \Big\} \quad \text{a.s..}$$

Proof. Let $V(t, x, r) = \inf_\tau \min_{T_i} J(t, x, r; \tau, T_i)$. For any stopping time $\tau \in [t, t_0 + T_0]$ and $T_i \in \{T_1, \cdots, T_j\}$, we have

$$\begin{aligned} V(t, x, r) \leqslant J(t, x, r; \tau, T_i) = \mathbb{E}_t \Big\{ \int_t^\tau m_0 R(s; t, r) ds + \mathbf{1}_{\{\tau < t_0 + T_0\}} \Big[\kappa_0(\tau) + \kappa_1(\tau) X(\tau) \\ + m(\tau, X(\tau), T_i) R(\tau; t, r) \psi(\tau, \mathbf{r}(\tau); \tau + T_i) \Big] \Big\}. \end{aligned}$$

First, if taking $\tau = t$,

$$V(t, x, r) \leqslant \kappa_0(t) + \kappa_1(t)x + m(t, x, T_i)\psi(t, r; t + T_i).$$

For any stopping time $\tau \in (t, t_0 + T_0)$ and $T_i \in \{T_1, \cdots, T_j\}$, take another stopping time $\theta \in$

$(\tau, t_0 + T_0)$,

$$
\begin{aligned}
& V(t,x,r) \\
\leqslant\, & J(t,x,r;\theta,T_i) \\
=\, & \mathbb{E}_t\Big\{ \int_t^\theta m_0 R(s;t,r)ds + \mathbf{1}_{\{\theta<t_0+T_0\}}\Big[\kappa_0(\theta)+\kappa_1(\theta)X(\theta) \\
& +m(\theta,X(\theta),T_i)R(\theta;t,r)\psi(\theta,\mathbf{r}(\theta);\theta+T_i)\Big]\Big\} \\
=\, & \mathbb{E}_t\Big\{ \int_t^\tau m_0 R(s;t,r)ds + \int_\tau^\theta m_0 R(s;t,r)ds + \mathbf{1}_{\{\theta<t_0+T_0\}}\Big[\kappa_0(\theta)+\kappa_1(\theta)X(\theta) \\
& +m(\theta,X(\theta),T_i)R(\theta;t,r)\psi(\theta,\mathbf{r}(\theta);\theta+T_i)\Big]\Big\} \\
=\, & \mathbb{E}_t\Big\{ \int_t^\tau m_0 R(s;t,r)Rds + \int_\tau^\theta m_0 R(s;\tau,\mathbf{r}(\tau))R(\tau;t,r)ds + \mathbf{1}_{\{\theta<t_0+T_0\}}\Big[\kappa_0(\theta)+\kappa_1(\theta)X(\theta) \\
& +m(\theta,X(\theta),T_i)R(\theta;\tau,\mathbf{r}(\tau))R(\tau;t,r)\psi(\theta,\mathbf{r}(\theta);\theta+T_i)\Big]\Big\} \\
\leqslant\, & \mathbb{E}_t\Big\{ \int_t^\tau m_0 R(s;t,r)ds + \int_\tau^\theta m_0 R(s;\tau,\mathbf{r}(\tau))ds + \mathbf{1}_{\{\theta<t_0+T_0\}}\Big[\kappa_0(\theta)+\kappa_1(\theta)X(\theta;t,x) \\
& +m(\theta,X(\theta),T_i)R(\theta;\tau,\mathbf{r}(\tau))\psi(\theta,\mathbf{r}(\theta);\theta+T_i)\Big]\Big\} \\
=\, & \mathbb{E}_t\Big\{ \int_t^\tau m_0 R(s;t,r)ds + J(\tau,X(\tau),\mathbf{r}(\tau);\theta,T_i)\Big\}.
\end{aligned}
$$

Taking infimum with respect to $\theta \in (\tau, T)$ yields

$$
V(t,x,r) \leqslant \mathbb{E}_t\Big\{ \int_t^\tau m_0 R(s;t,r)ds + V(\tau,X(\tau),r(\tau))\Big\}.
$$

□

Lemma 3.1.5. *Let (S4)-(S5) hold. For any $(t,x,r)\in\mathcal{D}$, if $\bar\tau \in [t, t_0+T_0]$ is an optimal stopping time of Problem (RF)$_1$ for the initial triple (t,x,r), then*

$$
V(\bar\tau, X(\bar\tau), \mathbf{r}(\bar\tau)) = \min_{T_i\in\{T_1,\cdots,T_j\}}\{\kappa_0(\bar\tau)+\kappa_1(\bar\tau)X(\bar\tau)+m(\bar\tau,X(\bar\tau),T_i)R(\bar\tau;t,r)\psi(\bar\tau,\mathbf{r}(\bar\tau);\bar\tau+T_i)\} \quad \text{a.s.}
$$

Proof. Take a stopping time $\tau \in (t, t_0 + T_0)$, suppose $\bar{\tau} \in [\tau, t_0 + T_0]$ and $T^* \in \{T_1, \cdots, T_j\}$ are optimal for initial triple $(t, x, r) \in \mathcal{D}$. Then

$$\begin{aligned}
V(t,x,r) &= J(t,x,r;\bar{\tau},T^*) \\
&= \mathbb{E}_t\Big\{ \int_t^{\bar{\tau},T^*} m_0 R(s;t,r)ds + \mathbf{1}_{\{\bar{\tau}<t_0+T_0\}}\Big[\kappa_0(\bar{\tau}) + \kappa_1(\bar{\tau})X(\bar{\tau}) \\
&\quad + m(\bar{\tau}, X(\bar{\tau}), T^*)R(\bar{\tau};t,r)\psi(\bar{\tau}, \mathbf{r}(\bar{\tau}), \bar{\tau}+T^*)\Big]\Big\} \\
&\geqslant \mathbb{E}_t\Big\{ \int_t^{\bar{\tau}} m_0 R(s;t,r)ds + V(\bar{\tau}, X(\bar{\tau}), r(\bar{\tau}))\Big\} \\
&\geqslant \inf_{\tau\in[t_0,t_0+T_0]} \mathbb{E}_t\Big\{ \int_t^{\tau} m_0 R(s;t,r)ds + V(\tau, X(\tau), r(\tau))\Big\} = V(t,x,r).
\end{aligned}$$

Thus the above equalities must hold, which implies

$$\begin{aligned}
&\mathbb{E}_t\Big\{V(\bar{\tau}, X(\bar{\tau}), \mathbf{r}(\bar{\tau}))\Big\} \\
= &\mathbb{E}_t\Big\{\min_{T_i\in\{T_1,\cdots,T_j\}}\{\kappa_0(\bar{\tau}) + \kappa_1(\bar{\tau})X(\bar{\tau}) + m(\bar{\tau}, X(\bar{\tau}), T_i)R(\bar{\tau};t,r)\psi(\bar{\tau}, \mathbf{r}(\bar{\tau}), \bar{\tau}+T_i)\}\Big\}.
\end{aligned}$$

Combining the fact

$$\begin{aligned}
&V(\bar{\tau}, X(\bar{\tau}), \mathbf{r}(\bar{\tau})) \\
\leqslant &\min_{T_i\in\{T_1,\cdots,T_j\}}\{\kappa_0(\bar{\tau}) + \kappa_1(\bar{\tau})X(\bar{\tau}) + m(\bar{\tau}, X(\bar{\tau}), T_i)R(\bar{\tau};t,r)\psi(\bar{\tau}, \mathbf{r}(\bar{\tau}), \bar{\tau}+T_i)\} \quad \text{a.s.}
\end{aligned}$$

the equality is desired. □

Lemma 3.1.6. *Let (S4)-(S5) hold. For any $(t, x, r) \in \mathcal{D}$, the following is the optimal stopping time of Problem (RF)$_1$ corresponding to (t, x, r) :*

$$\begin{aligned}
\bar{\tau} = \inf\Big\{s \in [t, t_0+T_0] | V(s, X(s), \mathbf{r}(s)) = &\min_{T_i\in\{T_1,\cdots,T_j\}}\{\kappa_0(s) + \kappa_1(s)X(s) \\
&+ m(s, X(s), T_i)R(s;t,r)\psi(s, \mathbf{r}(s), s+T_i)\}\Big\}.
\end{aligned} \tag{3.6}$$

Further, it holds that

$$\mathbb{P}\Big(\{\bar{\tau} > t\} \triangle \{V(t,x,r) < \min_{T_i \in \{T_1,\cdots,T_j\}} \{\kappa_0(t) + \kappa_1(t)X(t) + m(t,x,T_i)\psi(t,r;t+T_i)\}\}\Big) = 0. \tag{3.7}$$

where $A \triangle B = (A \setminus B) \cup (B \setminus A)$, *for any* $A, B \in \mathcal{F}$.

Proof. Let $(t,x,r) \in \mathcal{D}$. If there exists a $\Omega_0 \subseteq \{V(t,x,r) < \min_{T_i \in \{T_1,\cdots,T_j\}} \{\kappa_0(t) + \kappa_1(t)x + m(t,x,T_i)\psi(t,r;t+T_i)\}\}$, with $\mathbb{P}(\Omega_0) > 0$ such that

$$\bar{\tau} = t, \quad \text{on} \quad \Omega_0. \tag{3.8}$$

Then for $(t,x,r) \in \Omega_0$,

$$V(t,x,r) = \min_{T_i \in \{T_1,\cdots,T_j\}} \{\kappa_0(t) + \kappa_1(t)X(t) + m(t,x,T_i)\psi(t,r;t+T_i)\}. \tag{3.9}$$

which is contradicts the choice of Ω_0. Conversely, if $\Omega_0 \subseteq \{\bar{\tau}(t,x,r) > t\}$ with $\mathbb{P}(\Omega_0) > 0$ such that (3.9) holds, then by the definition of $\bar{\tau}$, (3.8) has to be true, a contradiction to the choice of Ω_0, Hence the equation holds. □

The following theorem gives the principle of optimal for Problem $(\text{RF})_1$.

Theorem 3.1.7. *Let (S4)-(S5) hold. For any* $(t,x,r) \in \mathcal{D}$, *for all stopping time* $\theta \in [t,\bar{\tau}]$, $\tau \in [\theta,\bar{\tau}]$:

$$V(\theta, X(\theta), \mathbf{r}(\theta)) = \mathbb{E}_\theta\Big\{\int_\theta^\tau m_0 R(s;t,r)ds + V(\tau, X(\tau), \mathbf{r}(\tau))\Big\}. \tag{3.10}$$

Proof. Let $(t,x,r) \in \mathcal{D}$. Define $\bar{\tau}$ by Lemma 3.1.6 and suppose $\mathbb{P}\{t < \bar{\tau}\} > 0$. The case $\theta = \bar{\tau}$ is trivial. Thus, fix $\theta \in [t,\bar{\tau})$, and let $\tau \in [\theta,\bar{\tau})$. From (3.6), we know that any $\mu \in [\theta, t_0 + T_0]$ with

$\mathbb{P}\{\mu < \tau\} > 0$ is not optimal for point $(\theta, X(\theta), \mathbf{r}(\theta))$. Therefore

$$\begin{aligned} V(\theta, X(\theta), \mathbf{r}(\theta)) &= \inf_{\mu \in [\tau, t_0+T_0]} \min_{T_i \in \{T_1, \cdots, T_j\}} \mathbb{E}_\theta \Big\{ \int_\theta^\tau m_0 R(s; t, r) ds + J(\tau, X(\tau), \mathbf{r}(\tau); \mu, T_i) \Big\} \\ &= \mathbb{E}_\theta \Big\{ \int_\theta^\tau m_0 R(s; t, r) ds + V(\tau, X(\tau), \mathbf{r}(\tau)) \Big\}. \end{aligned}$$

proving (4.4.1).

Finally, by taking $\theta = t$ and $\tau = \bar{\tau}$, we see that

$$\begin{aligned} V(t, x, r) = \mathbb{E}_t \Big\{ \int_t^{\bar{\tau}} m_0 R(s; t) ds + \mathbf{1}_{\{\bar{\tau} < t_0 + T_0\}} \Big[\min_{T_i \in \{T_1, \cdots, T_j\}} \{ \kappa_0(\bar{\tau}) + \kappa_1(\bar{\tau}) X(\bar{\tau}) \\ + m(\bar{\tau}, X(\bar{\tau}), T_i) R(\bar{\tau}; t, r) \psi(\bar{\tau}, \mathbf{r}(\bar{\tau}), \bar{\tau} + T_i) \} \Big] \Big\} = J(t, x, r; \bar{\tau}, T^*), \end{aligned}$$

which means that $\bar{\tau}$ is an optimal stopping time of Problem (RF)$_1$ for the initial triple (t, x, r), and it must be the smallest one. □

Then we can derive the variational inequality as follows.

Theorem 3.1.8. *For any $(t, x, r) \in \mathcal{D}$, let value function $V(t, x, r) = \inf_\tau \min_{T_i} J(t, x, r; \tau, T_i)$, and suppose $V(\cdot, \cdot, \cdot) \in C^{1,1,2}(\mathcal{D})$, where $C^{1,1,2}(\mathcal{D})$ is the space of all real-valued functions on $\mathcal{D}$ whose first order continuous partial derivatives with respect to first two variables and second order continuous partial derivative with respect to third variable exist. Then V is the solution of* *the following variational inequality:*

$$\begin{cases} \min\{V_t + \dfrac{1}{2}V_{rr}\sigma_0(t,r)^2 + V_r b_0(t,r) + V_x(r_0 x - m_0) + m_0, \Phi - V\} = 0, \\ \hfill (t,x,r) \in \mathcal{D}, \\ V(t_0 + T_0, x, r) = 0, \quad (x,r) \in \mathbb{R}^+ \times \mathbb{R}^+, \\ V(t,0,r) = 0, \quad (t,r) \in [t_0, t_0 + T_0] \times \mathbb{R}^+, \\ V(t,x,0) = \kappa_0(t) + \kappa_1(t)x + \dfrac{\delta(t,x,T_0)e^{\delta(t,x,T_0)T_0}}{e^{\delta(t,x,T_0)T_0} - 1} x T_0, \\ \hfill (t,x) \in [t_0, t_0 + T_0] \times \mathbb{R}^+, \end{cases} \tag{3.11}$$

where

$$\Phi(t,x,r) = \min_{T_i \in \{T_1, \cdots, T_j\}} \{\kappa_0(t) + \kappa_1(t)x + m(t,x,T_i)\psi(t,r;t+T_i)\},$$

with $\psi(t,r;t+T_i)$ *being the solution of* (3.4).

Proof. Let $(t,x,r) \in \mathcal{D}$, and for any stopping time $\tau \in [t,T]$, from Proposition 3.1.4 we know that if taking $\tau = t$,

$$V(t,x,r) \leqslant \Phi(t,x,r).$$

Otherwise,

$$V(t,x,r) \leqslant \mathbb{E}\Big\{ \int_t^\tau m_0 R(s;t) ds + V(\tau, X(\tau), r(\tau)) \Big\}.$$

By Itô's formula,

$$0 \leqslant \int_t^\tau \Big\{ V_t + \frac{1}{2}V_{rr}\sigma_0(t,r)^2 + V_r b_0(t,r) + V_x(r_0 x - m_0) + m_0 \Big\} ds.$$

Since τ is arbitrary,

$$V_t + \frac{1}{2}V_{rr}\sigma_0(t,r)^2 + V_r b_0(t,r) + V_x(r_0 x - m_0) + m_0 \geqslant 0.$$

Thus the variational inequality holds. □

We define the closed set S

$$S = \{(t,x,r) : V(t,x,r) = \min_{T_i \in \{T_1,\cdots,T_j\}} \{\kappa_0(t) + \kappa_1(t)x + m(t,x,T_i)\psi(t,r;t+T_i)\}\}$$

which is called the *stopping set*, and the open set

$$D = \{(t,x,r) : V(t,x,r) < \min_{T_i \in \{T_1,\cdots,T_j\}} \{\kappa_0(t) + \kappa_1(t)x + m(t,x,T_i)\psi(t,r;t+T_i)\}\}$$

which is called the *continuation set*.

3.1.3 Optimal Strategy

First we notice that for any triple $(t,x,r) \in \mathcal{D}$, if

$$m_0\psi(t,r;t+T_0) \geqslant \Phi(t,x,r),$$

which means that if the cash flow of the original mortgage plan is larger than the refinancing cost and cash flow of the new mortgage plan, then refinancing should be taken at once.

Generally, for any $(t,x,r) \in \mathcal{D}$, define the process,

$$Z_u = \int_t^u m_0 R(s;t)ds + V(u, X(u), \mathbf{r}(u)),$$

and the first exit time of the continuation region $\bar{\tau}$ as we defined in (3.6), i.e.,

$$\bar{\tau} = \inf \Big\{ s \in [t, t_0 + T_0] | V(s, X(s), \mathbf{r}(s)) = \min_{T_i \in \{T_1, \cdots, T_j\}} \{\kappa_0(s) + \kappa_1(s) X(s) \\ + m(s, X(s), T_i) R(s; t, r) \psi(s, \mathbf{r}(s); s + T_i)\} \Big\}.$$

From the dynamic programming principle, for any stopping time $\theta \in [u, t_0 + T_0]$, since

$$V_t + \frac{1}{2} V_{rr} \sigma_0(t, r)^2 + V_r b_0(t, r) + V_x(r_0 x - m_0) + m_0 \geqslant 0,$$

we have

$$Z_u \leqslant \mathbb{E}_u[Z_\theta],$$

which means that Z is a submartingale. Moreover, since

$$V_t + \frac{1}{2} V_{rr} \sigma_0(s, \mathbf{r}(s))^2 + V_r b_0(t, r) + V_x(r_0 X(s) - m_0) + m_0 = 0$$

for any $s \in [t, \bar{\tau})$, the process Z is a martingale on $[t, \bar{\tau})$, and so

$$V(t, x, r) = \mathbb{E}_t \Big\{ \int_t^{\bar{\tau}} m_0 R(s; t, r) ds + \mathbf{1}_{\{\bar{\tau}\} < t_0 + T_0\}} \Big[\min_{T_i \in \{T_1, \cdots, T_j\}} \{\kappa_0(\bar{\tau}) + \kappa_1(\bar{\tau}) X(\bar{\tau}) \\ + m(\bar{\tau}, X(\bar{\tau}), T_i) R(\bar{\tau}; t) \psi(\bar{\tau}, \mathbf{r}(\bar{\tau}); \bar{\tau} + T_i)\} \Big] \Big\},$$

which implies $\bar{\tau}$ is an optimal stopping strategy.

3.1.4 Viscosity Solutions

As we mentioned in Section 2.3, we know that the value function $V(\cdot, \cdot, \cdot)$ is the unique viscosity solution of a HJB equation of the variational inequality form for the classical optimal stopping problem. Before we introduce the viscosity solution, first we rewrite the problem equivalently as following:

$$\begin{aligned}\mathscr{J}(t,x,r;\tau) = \mathbb{E}\Big\{\int_t^\tau -m_0 R(s;t,r)ds + \mathbf{1}_{\{\tau<t_0+T_0\}}\Big[-\kappa_0(\tau)-\kappa_1(\tau)X(\tau)\\ - \min_{T_i\in\{T_1,\cdots,T_j\}} m(\tau,X(\tau),T_i)R(\tau;t,r)\psi(\tau,\mathbf{r}(\tau);\tau+T_i)\Big]\Big\}.\end{aligned}$$

Then Problem $(\mathrm{RF})_1$ can be equivalently stated as follows.

Problem (RF)$_1^*$. Find $\tau^* \in [t, t_0 + T_0]$ such that

$$\mathscr{J}(r,x,r;\tau^*,T^*) = \max_\tau \min_{T_i} \mathscr{J}(t,x,r;\tau,T_i) = \mathscr{V}(t,x,r).$$

With the same argument as above, the HJB equation looks like

$$\begin{cases} \min\{-\mathscr{V}_t - \dfrac{\sigma_0(t,r)^2}{2}\mathscr{V}_{rr} - b_0(t,r)\mathscr{V}_r - (r_0x-m_0)\mathscr{V}_x + m_0, \mathscr{V}+\Phi\} = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (t,x,r)\in\mathcal{D}, \\ \mathscr{V}(t_0+T_0,x,r) = 0, \quad (x,r)\in\mathbb{R}^+\times\mathbb{R}^+, \\ \mathscr{V}(t,0,r) = 0, \quad (t,r)\in[t_0,t_0+T_0]\times\mathbb{R}^+, \\ \mathscr{V}(t,x,0) = -\kappa_0(t)-\kappa_1(t)x - \dfrac{\delta(t,x,T_0)e^{\delta(t,x,T_0)T_0}}{e^{\delta(t,x,T_0)T_0}-1}xT_0, \\ \qquad\qquad\qquad\qquad\qquad\qquad (t,x)\in[t_0,t_0+T_0]\times\mathbb{R}^+, \end{cases} \tag{3.12}$$

where

$$\Phi(t,x,r) = \min_{T_i\in\{T_1,\cdots,T_j\}}\{\kappa_0(t)+\kappa_1(t)x + m(t,x,T_i)\psi(t,r;t+T_i)\},$$

with $\psi(t,r;t+T_i)$ being the solution of (3.4). Thus we have for any $(t,s,r)\in\mathcal{D}$, $V(t,x,r) = -\mathscr{V}(t,x,r)$.

We now introduce the definition of the viscosity solution as follows.

Definition 3.1.9. *(a). $\mathscr{V}$ is a viscosity subsolution of* (3.12) *if for each $\varphi \in C^{1,1,2}(\mathcal{D})$ and each $(s_0, y_0, r_0) \in \mathcal{D}$ such that $\varphi \geqslant \mathscr{V}$ and $\varphi(s_0, y_0, r_0) = \mathscr{V}(s_0, y_0, r_0)$ we have,*

$$\min\{-\varphi_t - \frac{1}{2}\varphi_{rr}\sigma_0(s_0, y_0, r_0)^2 - \varphi_r b_0(s_0, y_0, r_0) - \varphi_x(r_0 y_0 - m_0) + m_0,$$
$$\varphi(s_0, y_0, r_0) + \Phi(s_0, y_0, r_0)\} \leqslant 0.$$

(b). $\mathscr{V}$ is a viscosity supersolution of (3.12) *if for each $\phi \in C^{1,1,2}(\mathcal{D})$ and each $(s_0, y_0, r_0) \in \mathcal{D}$ such that $\phi \leqslant \mathscr{V}$ and $\varphi(s_0, y_0, r_0) = \mathscr{V}(s_0, y_0, r_0)$ we have,*

$$\min\{-\phi_t - \frac{1}{2}\phi_{rr}\sigma_0(s_0, y_0, r_0)^2 - \phi_r b_0(s_0, y_0, r_0) - \phi_x(r_0 y_0 - m_0) + m_0,$$
$$\phi(s_0, y_0, r_0) + \Phi(s_0, y_0, r_0)\} \geqslant 0.$$

(c). $\mathscr{V}$ is a viscosity solution of (3.12) *if it is both a viscosity subsolution and a viscosity supersolution of* (3.12)*.*

The main result of this section is stated as follows.

Theorem 3.1.10. *Let $(t, x, r) \in \mathcal{D}$, $V(t, x, r) = \inf_\tau \min_{T_i} J(t, x, r; \tau, T_i)$ is the unique viscosity* *solution of the equation*

$$
\begin{cases}
\min\{V_t + \dfrac{1}{2}V_{rr}\sigma_0(t,r)^2 + V_r b_0(t,r) + V_x(r_0 x - m_0) + m_0, \Phi - V\} = 0, \\
\hfill (t,x,r) \in \mathcal{D}, \\
V(t_0+T_0,x,r) = 0, \quad (x,r) \in \mathbb{R}^+ \times \mathbb{R}^+, \\
V(t,0,r) = 0, \quad (t,r) \in [t_0, t_0+T_0] \times \mathbb{R}^+, \\
V(t,x,0) = \kappa_0(t) + \kappa_1(t)x + \dfrac{\delta(t,x,T_0)e^{\delta(t,x,T_0)T_0}}{e^{\delta(t,x,T_0)T_0}-1} x T_0, \\
\hfill (t,x) \in [t_0, t_0+T_0] \times \mathbb{R}^+,
\end{cases}
$$

where

$$
\Phi(t,x,r) = \min_{T_i \in \{T_1,\cdots,T_j\}} \{\kappa_0(t) + \kappa_1(t)x + m(t,x,T_i)\psi(t,r;t+T_i)\},
$$

with $\psi(t,r;t+T_i)$ *being the solution of* (3.4).

Proof. Note that $V(t_0+T_0,x,r)=0$ and $V(t,0,r)=0$ follow immediately from the definition of V. Then to prove for any $(t,x,r) \in \mathcal{D}$, $V(t,x,r) = \inf_\tau \min_{T_i} J(t,x,r;\tau,T_i)$ is the unique viscosity solution of the 3.11, it suffices to prove that $\mathscr{V}$ is the unique viscosity solution of (3.12). First for the viscosity subsolution case, let $\varphi \in C^{1,1,2}$ and $(s_0,y_0,r_0) \in \mathcal{D}$ such that $\varphi \geqslant \mathscr{V}$ on $\mathcal{D}$ and $\varphi(s_0,y_0,r_0) = \mathscr{V}(s_0,y_0,r_0)$.

Since $\mathscr{V}(s_0,y_0,r_0) \geqslant -\Phi(s_0,y_0,r_0)$, so $\varphi(s_0,y_0,r_0) \geqslant -\Phi(s_0,y_0,r_0)$. If $(s_0,y_0,r_0) \in S$, then $\varphi(s_0,y_0,r_0) = -\Phi(s_0,y_0,r_0)$. Next suppose $(s_0,y_0,r_0) \in \mathcal{D}$. For $\tau \in [t_0,t_0+T_0]$, by Dynkin's formula we obtain

$$\begin{aligned}\mathscr{V}(s_0,y_0,r_0) &= \mathbb{E}_{s_0}\Big\{\int_{s_0}^{\tau} -m_0R(s;s_0,r_0)ds + \mathscr{V}(\tau,X(\tau),\mathbf{r}(\tau))\Big\} \\ &\leqslant \mathbb{E}_{s_0}\Big\{\int_{s_0}^{\tau} -m_0R(s;s_0,r_0)ds + \varphi(\tau,X(\tau),\mathbf{r}(\tau))\Big\} \\ &= \mathbb{E}_{s_0}\Big\{\int_{s_0}^{\tau} -m_0R(s;s_0,r_0)ds + \varphi(s_0,y_0,r_0) \\ &\qquad + \int_{s_0}^{\tau} \varphi_t + \frac{1}{2}\varphi_{rr}\sigma_0(t,r)^2 + \varphi_r b_0(t,r) + \varphi_x(r_0x-m_0)ds\Big\} \\ &= \mathbb{E}_{s_0}\Big\{\int_{s_0}^{\tau} -m_0R(s;s_0,r_0) + \varphi_t + \frac{1}{2}\varphi_{rr}\sigma_0(t,r)^2 + \varphi_r b_0(t,r) \\ &\qquad +\varphi_x(r_0x-m_0)ds\Big\} + \varphi(s_0,y_0,r_0)\end{aligned}$$

or

$$\mathbb{E}_{s_0}\Big\{\int_{s_0}^{\tau} -m_0R(s;s_0,r_0) + \varphi_t + \frac{1}{2}\varphi_{rr}\sigma_0(t,r)^2 + \varphi_r b_0(t,r) + \varphi_x(r_0x-m_0)ds\Big\} \geqslant 0.$$

Letting $\tau \to s_0$, we get

$$\varphi_t + \frac{1}{2}\varphi_{rr}\sigma_0(t,r)^2 + \varphi_r b_0(t,r) + \varphi_x(r_0x-m_0) - m_0 \geqslant 0.$$

Thus

$$\begin{aligned}\min\Big\{ -\varphi_t - \frac{1}{2}\varphi_{rr}\sigma_0(s_0,r_0)^2 - \varphi_r b_0(s_0,r_0) - \varphi_x(r_0y_0-m_0) + m_0, \\ \varphi(s_0,y_0,r_0) + \Phi(s_0,y_0,r_0)\Big\} \leqslant 0.\end{aligned} \tag{3.13}$$

This shows that $\mathscr{V}$ is a viscosity subsolution.

For the supersolution case, let $\phi \in C^{1,1,2}$ and $(s_0,y_0,r_0) \in \mathcal{D}$ such that $\phi \leqslant \mathscr{V}$ on $\mathcal{D}$ and $\phi(s_0,y_0,r_0) = \mathscr{V}(s_0,y_0,r_0)$.

For all bounded stopping time $\tau \leqslant t_0 + T_0$, with the same argument as above, we have

$$\begin{aligned}
\mathscr{V}(s_0, y_0, r_0) &= \mathbb{E}_{s_0}\Big\{ \int_{s_0}^{\tau} -m_0 R(s; s_0, r_0) ds + \mathscr{V}(\tau, X(\tau), \mathbf{r}(\tau)) \Big\} \\
&\geqslant \mathbb{E}_{s_0}\Big\{ \int_{s_0}^{\tau} -m_0 R(s; s_0, r_0) ds + \phi(\tau, X(\tau), \mathbf{r}(\tau)) \Big\} \\
&= \mathbb{E}_{s_0}\Big\{ \int_{s_0}^{\tau} -m_0 R(s; s_0, r_0) ds + \phi(s_0, y_0, r_0) \\
&\qquad + \int_{s_0}^{\tau} \phi_t + \frac{1}{2}\phi_{rr}\sigma_0(t,r)^2 + \phi_r b_0(t,r) + \phi_x(r_0 x - m_0) ds \Big\} \\
&= \mathbb{E}_{s_0}\Big\{ \int_{s_0}^{\tau} -m_0 R(s; s_0, r_0) + \phi_t + \frac{1}{2}\phi_{rr}\sigma_0(t,r)^2 + \phi_r b_0(t,r) \\
&\qquad + \phi_x(r_0 x - m_0) ds \Big\} + \phi(s_0, y_0, r_0)
\end{aligned}$$

or

$$\mathbb{E}_{s_0}\Big\{ \int_{s_0}^{\tau} -m_0 R(s; t) + \phi_t + \frac{1}{2}\phi_{rr}\sigma_0(t,r)^2 + \phi_r b_0(t,r) + \phi_x(r_0 x - m_0) ds \Big\} \leqslant 0.$$

Letting $\tau \to s_0$, we get

$$\phi_t + \frac{1}{2}\phi_{rr}\sigma_0(t,r)^2 + \phi_r b_0(t,r) + \phi_x(r_0 x - m_0) - m_0 \leqslant 0.$$

Thus

$$\begin{aligned}
\min \Big\{ -\phi_t - \frac{1}{2}\phi_{rr}\sigma_0(s_0, r_0)^2 - \phi_r b_0(s_0, r_0) - \phi_x(r_0 y_0 - m_0) + m_0, \\
\phi(s_0, y_0, r_0) + \Phi(s_0, y_0, r_0) \Big\} \geqslant 0.
\end{aligned} \tag{3.14}$$

This shows that $\mathscr{V}$ is a viscosity supersolution. Combining (3.13) and (3.14), we get $\mathscr{V}$ is a viscosity solution.

To prove the uniqueness, we use the Theorem 3.3 in [7]. Let $\mathscr{U}$ (*resp.* $\mathscr{V}$) be a upper-semicontinuous function (u.s.c.) viscosity subsolution (*resp.* lower-semicontinuous function (l.s.c.) viscosity supersolution) of (3.12), then we claim that $\mathscr{U} \leqslant \mathscr{V}$ on $\mathcal{D}$. To prove this we argue by the contradiction. Assuming that $\sup(\mathscr{U}-\mathscr{V})>0$, which implies there exists $(\bar{t},\bar{x},\bar{r})\in\mathcal{D}$ such that $(\mathscr{U}-\mathscr{V})(\bar{t},\bar{x},\bar{r})=\sup(\mathscr{U}-\mathscr{V})=\delta>0$. For all $\epsilon>0$, consider $\mathscr{U}_\epsilon=\mathscr{U}-\epsilon(|x|^2+|r|^2)$ and $\mathscr{V}_\epsilon=\mathscr{V}+\epsilon(|x|^2+|r|^2)$, then

$$\lim_{|x|\vee|r|\to\infty}\sup_{t\in[t_0,t_0+T_0]}(\mathscr{U}_\epsilon-\mathscr{V}_\epsilon)(t,x,r)\to-\infty.$$

Define for all $k\in\mathbb{N}$,

$$\begin{aligned}\Gamma_k(t,x,r;s,y,z)&=\mathscr{U}(t,x,r)-\mathscr{V}(s,y,z)-\gamma_k(t,x,r;s,y,z)\\&\qquad-\frac{1}{k}(|x|^2+|r|^2+|y|^2+|z|^2)-\frac{t_0+T_0-t-s}{k},\\\gamma_k(t,x,r;s,y,z)&=k[|t-s|^2+|x-y|^2+|r-z|^2].\end{aligned}\tag{3.15}$$

Thus the u.s.c. function Γ_k attains its maximum at $(t_k,x_k,r_k,s_k,y_k,z_k)\in\mathcal{D}$, that is there exist $k_0>0$, for $k>k_0$,

$$\begin{aligned}&\Gamma_k(t_k,x_k,r_k;s_k,y_k,z_k)=\sup_{\mathcal{D}\times\mathcal{D}}\Gamma_k(t,x,r;s,y,z)\\&\geqslant\Gamma_k(\bar{t},\bar{x},\bar{r};\bar{t},\bar{x},\bar{r})=\mathscr{U}(\bar{t},\bar{x},\bar{r})-\mathscr{V}(\bar{t},\bar{x},\bar{r})-\gamma_k(\bar{t},\bar{x},\bar{r};\bar{t},\bar{x},\bar{r})-\frac{2}{k}(|\bar{x}|^2+|\bar{r}|^2)\\&\qquad-\frac{t_0+T_0-2t}{k}\\&\geqslant\delta-\frac{2}{k}(|\bar{x}|^2+|\bar{r}|^2)-\frac{t_0+T_0-2t}{k}\geqslant\frac{\delta}{2}.\end{aligned}\tag{3.16}$$

Therefore,

$$\frac{\delta}{2}\leqslant\mathscr{U}(t_k,x_k,r_k)-\mathscr{V}(s_k,y_k,z_k).\tag{3.17}$$

Next by the definition of $(t_k,x_k,r_k;s_k,y_k,z_k)$, we get

$$2\Gamma_k(t_k,x_k,r_k;s_k,y_k,z_k)\geqslant\Gamma_k(t_k,x_k,r_k;t_k,x_k,r_k)+\Gamma_k(s_k,y_k,z_k;s_k,y_k,z_k),$$

equivalently,

$$
\begin{aligned}
&2\Big[\mathscr{U}(t_k,x_k,r_k)-\mathscr{V}(s_k,y_k,z_k)-\gamma_k(t_k,x_k,r_k;s_k,y_k,z_k)\\
&\quad-\frac{1}{k}(|x_k|^2+|r_k|^2+|y_k|^2+|z_k|^2)-\frac{t_0+T_0-t_k-s_k}{k}\Big]\\
&\geqslant \mathscr{U}(t_k,x_k,r_k)-\mathscr{V}(t_k,x_k,r_k)-\frac{2}{k}(|x_k|^2+|r_k|^2)-\frac{t_0+T_0-2t_k}{k}\\
&\quad+\mathscr{U}(s_k,y_k,z_k)-\mathscr{V}(s_k,y_k,z_k)-\frac{2}{k}(|y_k|^2+|z_k|^2)-\frac{t_0+T_0-2s_k}{k}.
\end{aligned}
$$

Thus there exist $C>0$,

$$
2\gamma_k(t_k,x_k,r_k;s_k,y_k,z_k)\leqslant \mathscr{U}(t_k,x_k,r_k)-\mathscr{U}(s_k,y_k,z_k)+\mathscr{V}(t_k,x_k,r_k)-\mathscr{V}(s_k,y_k,z_k)\leqslant C.
$$

Hence,

$$
|t_k-s_k|+|x_k-y_k|+|r_k-s_k|\leqslant\Big(\frac{C}{k}\Big)^{\frac{1}{2}}. \tag{3.18}
$$

Therefore, as $k\to\infty$,

$$
\begin{aligned}
&\gamma_k(t_k,x_k,r_k;s_k,y_k,z_k)\\
\leqslant &\sup_{|t_k-s_k|+|x_k-y_k|+|r_k-s_k|\leq\left(\frac{C_0}{k}\right)^{\frac{1}{2}}}(|\mathscr{U}(t_k,x_k,r_k)-\mathscr{U}(s_k,y_k,z_k)|\\
&-|\mathscr{V}(t_k,x_k,r_k)-\mathscr{V}(s_k,y_k,z_k)|)\\
\to &\, 0.
\end{aligned} \tag{3.19}
$$

Notice that as $k\to\infty$, the bounded sequence $(t_k,x_k,r_k;s_k,y_k,z_k)_k$ converges, along a subsequence, to some point $(\hat{t},\hat{x},\hat{r};\hat{s},\hat{y},\hat{z})\in[t_0,t_0+T_0]\times\bar{\mathcal{O}}\times\bar{\mathcal{O}}$, for some open bounded set $\mathcal{O}\in\mathbb{R}^+$. By (3.18) we get $\hat{t}=\hat{s},\hat{x}=\hat{y},\hat{r}=\hat{z}$.

If $\mathscr{U}$ is u.s.c., $\varphi\in C^{1,1,2}(\mathcal{D})$, and $(\bar{t},\bar{x},\bar{r})\in\mathcal{D}$ is a maximum point of $\mathscr{U}-\varphi$, then a second

Taylor expansion of φ yields

$$\begin{aligned}\mathscr{U}(t,x,r) &\leqslant \mathscr{U}(\bar{t},\bar{x},\bar{r}) + \varphi(t,x,r) - \varphi(\bar{t},\bar{x},\bar{r}) \\ &= \mathscr{U}(\bar{t},\bar{x},\bar{r}) + \varphi_t(\bar{t},\bar{x},\bar{r})(t-\bar{t}) + \varphi_x(\bar{t},\bar{x},\bar{r})(x-\bar{x}) + \varphi_r(\bar{t},\bar{x},\bar{r})(r-\bar{r}) \\ &\quad + \frac{1}{2}\varphi_{rr}(\bar{t},\bar{x},\bar{r})(r-\bar{r})^2 + o(|t-\bar{t}| + |x-\bar{x}| + |r-\bar{r}|^2).\end{aligned} \tag{3.20}$$

Thus we can define $\mathcal{P}^{2,+}\mathscr{U}(t,x,r)$ as the set of elements $(\bar{q},\bar{p},\bar{h},\bar{M}) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$, satisfying

$$\mathscr{U}(t,x,r) \leqslant \mathscr{U}(\bar{t},\bar{x},\bar{r}) + \bar{q}(t-\bar{t}) + \bar{p}(x-\bar{x}) + \bar{h}(r-\bar{r}) + \frac{1}{2}\bar{M}(r-\bar{r})^2 + o(|t-\bar{t}|+|x-\bar{x}|+|r-\bar{r}|^2).$$

The inequality (3.20) shows that for a given point $(t,x,r) \in \mathcal{D}$, if $\varphi \in C^{1,1,2}(\mathcal{D})$ is such that (t,x,r) is maximum of $\mathscr{U} - \varphi$, then

$$(q,p,h,M) = (\varphi_t(t,x,r), \varphi_x(t,x,r), \varphi_r(t,x,r), \varphi_{rr}(t,x,r)) \in \mathcal{P}^{2,+}\mathscr{U}(t,x,r).$$

Similarly, we can define $\mathcal{P}^{2,-}\mathscr{V}(t,x,r)$ of a l.s.c. function $\mathscr{V}$ as the set of elements $(\bar{q},\bar{p},\bar{h},\bar{M}) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$, satisfying

$$\mathscr{V}(t,x,r) \geqslant \mathscr{V}(\bar{t},\bar{x},\bar{r}) + \bar{q}(t-\bar{t}) + \bar{p}(x-\bar{x}) + \bar{h}(r-\bar{r}) + \frac{1}{2}\bar{M}(r-\bar{r})^2 + o(|t-\bar{t}|+|x-\bar{x}|+|r-\bar{r}|^2),$$

and for a given point $(t,x,r) \in \mathcal{D}$, if $\varphi \in C^{1,1,2}(\mathcal{D})$ is such that (t,x,r) is minimum of $\mathscr{V} - \varphi$, then

$$(q,p,h,M) = (\varphi_t(t,x,r), \varphi_x(t,x,r), \varphi_r(t,x,r), \varphi_{rr}(t,x,r)) \in \mathcal{P}^{2,-}\mathscr{V}(t,x,r).$$

More precisely, define $\bar{\mathcal{P}}^{2,+}\mathscr{U}(t,x,r)$ as the set of element $(q,p,h,M) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$ for which there exists a sequence $(t_\epsilon, x_\epsilon, r_\epsilon, q_\epsilon, p_\epsilon, h_\epsilon, M_\epsilon)_\epsilon$ in $\mathcal{D}\times\mathcal{P}^{2,+}\mathscr{U}(t_\epsilon,x_\epsilon,r_\epsilon)$ such that

$(t_\epsilon, x_\epsilon, r_\epsilon, \mathscr{U}(t_\epsilon, x_\epsilon, r_\epsilon), q_\epsilon, p_\epsilon, h_\epsilon, M_\epsilon) \to (t, x, r, \mathscr{U}(t, x, r), q, p, h, M)$, and define the set $\bar{\mathcal{P}}^{2,-}\mathscr{V}(t, x, r)$ similarly.

Form Ishii's lemma, there exist M and $N \in \mathbb{R}$ such that

$$(2k(t_k - s_k), 2k(x_k - y_k) + \frac{2}{k}(x_k), 2k(r_k - z_k) + \frac{2}{k}(r_k), M) \in \bar{\mathcal{P}}^{2,+}\mathscr{U}(t, x, r),$$

$$(2k(t_k - s_k), 2k(x_k - y_k) - \frac{2}{k}(y_k), 2k(r_k - z_k) - \frac{2}{k}(z_k), N) \in \bar{\mathcal{P}}^{2,-}\mathscr{V}(s, y, z),$$

and

$$(\sigma_0(t_k, r_k)^2 M - \sigma_0(s_k, z_k)^2 N) \leqslant 3k|\sigma_0(t_k, r_k) - \sigma_0(s_k, z_k)|^2.$$

From the viscosity subsolution (*resp.* supersolution) property of $\mathscr{U}$ (*resp.* $\mathscr{V}$) at (t_k, x_k, r_k) (*resp.* (s_k, y_k, z_k)). we have

$$\begin{aligned} \min\Big\{ &- 2k(t_k - s_k) - \frac{\sigma_0(t_k, r_k)^2}{2} M - b_0(t_k, r_k)(2k(r_k - z_k) + \frac{2}{k}(r_k)) \\ &-(r_0 x_k - m_0)(2k(x_k - y_k) + \frac{2}{k}(x_k)) + m_0, \mathscr{U} + \Phi \Big\} \leqslant 0, \end{aligned} \tag{3.21}$$

and

$$\begin{aligned} \min\Big\{ &- 2k(t_k - s_k) - \frac{\sigma_0(s_k, z_k)^2}{2} N - b_0(s_k, z_k)(2k(r_k - z_k) - \frac{2}{k}(z_k)) \\ &-(r_0 y_k - m_0)(2k(x_k - y_k) - \frac{2}{k}(y_k)) + m_0, \mathscr{V} + \Phi \Big\} \geqslant 0. \end{aligned} \tag{3.22}$$

The first term of the left-hand side of (3.22) is nonnegative, therefore, from the Lipschitz condition on b_0, σ_0, continuity of $X, \mathbf{r}$ and for $k \to \infty$, we get

$$\Big(\frac{\sigma_0(s_k, z_k)^2}{2}N - \frac{\sigma_0(t_k, r_k)^2}{2}M\Big) + 2k(r_k - z_k)(b_0(s_k, z_k) - b_0(t_k, r_k))$$
$$-\frac{2}{k}(z_k b_0(s_k, z_k) + r_k b_0(t_k, r_k)) + 2k(x_k - y_k)r_0(y_k - x_k)$$
$$-\frac{2}{k}(y_k(r_0 y_k - m_0) + x_k(r_0 x_k - m_0)) \to 0.$$

which implies for all k, $\mathscr{U}(t_k, x_k, r_k) + \Phi(t_k, x_k, r_k) \leqslant 0$ and $\mathscr{V}(s_k, y_k, z_k) + \Phi(s_k, y_k, z_k) \geqslant 0$, therefore, $\mathscr{U}(t_k, x_k, r_k) - \mathscr{V}(s_k, y_k, z_k) \leqslant -\Phi(t_k, x_k, r_k) - (-\Phi(s_k, y_k, z_k))$. Letting $k \to \infty$, and by the continuity of Φ, we get the contradiction with 3.17. Then by $V(t, x, r) = -\mathscr{V}(t, x, r)$, the theorem is proved. □

3.1.5 Example

In this section, we present an example to illustrate our result in this section. For any $t_0 \geqslant 0$, let the market interest rate $r_0 > 0$ be fixed. Consider a fixed term mortgage starting from t_0 with the loan amount $x_0 > 0$, the fixed term $T_0 > 0$, and the mortgage rate is r_0. Suppose there is no closing cost (including various fees) in this example. Next, at any $s \in (t_0, t_0 + T_0)$, the principal balance is denoted by $X(s)$. Then by above discussion, the process $X(\cdot)$ satisfies

$$\begin{cases} dX(s) = \big\{r_0 X(s) - m(t_0, x_0, T_0)\big\}ds, \qquad s \in [t_0, t_0 + T_0], \\ X(t_0) = x_0, \end{cases} \tag{3.23}$$

and the payment rate

$$m(t_0, x_0, T_0) = \frac{r_0 e^{r_0 T_0}}{e^{r_0 T_0} - 1}x_0 \equiv m_0.$$

Now, one has

$$X(s) = \frac{e^{r_0 T_0} - e^{r_0(s - t_0)}}{e^{r_0 T_0} - 1}x_0, \qquad s \in [t_0, t_0 + T_0].$$

The total discounted payment is given by

$$J_0 = \int_{t_0}^{t_0+T_0} m_0 e^{-r_0(s-t_0)} ds = x_0.$$

Now we consider refinancing the mortgage at $\tau \in (t, t_0+T_0)$ with the market interest rate changing to r_1 where $r_0 > r_1 > 0$. By the discussion above, we know the mortgage balance now is

$$X(\tau) = \frac{e^{r_0 T_0} - e^{r_0(\tau - t_0)}}{e^{r_0 T_0} - 1} x_0.$$

and the payment rate on $[\tau, \tau + T_0]$ will be

$$m(\tau, X(\tau), T_0) = \frac{r_1 e^{r_1 T_0}}{e^{r_1 T_0} - 1} X(\tau) = \frac{r_1 e^{r_1 T_0}}{e^{r_1 T_0} - 1} \times \frac{e^{r_0 T_0} - e^{r_0(\tau - t_0)}}{e^{r_0 T_0} - 1} x_0. \tag{3.24}$$

Now the total expected discounted payment is given by

$$\begin{aligned} J_1 &= \int_{t_0}^{\tau} m_0 e^{-r_0(s-t_0)} ds + \int_{\tau}^{\tau+T_0} m(\tau, X(\tau), T_0) e^{-r_1(s-\tau)} ds \\ &= \frac{e^{r_0 T_0}}{e^{r_0 T_0} - 1} x_0 \Big[1 - e^{-r_0(\tau - t_0)}\Big] + \frac{e^{r_1 T_0}}{e^{r_1 T_0} - 1} \times \frac{e^{r_0 T_0} - e^{r_0(\tau - t_0)}}{e^{r_0 T_0} - 1} x_0 \Big[1 - e^{-r_0 T_0}\Big]. \end{aligned}$$

It is easy to show that

$$m(\tau, X(\tau), T_0) = \frac{r_1 e^{r_1 T_0}}{e^{r_1 T_0} - 1} \times \frac{e^{r_0 T_0} - e^{r_0(\tau - t_0)}}{e^{r_0 T_0} - 1} x_0 < \frac{r_0 e^{r_0 T_0}}{e^{r_0 T_0} - 1} x_0 = m(t_0, x_0, T_0),$$

since $\frac{e^{r_0 T_0} - e^{r_0(\tau - t_0)}}{e^{r_0 T_0} - 1} x_0 < x_0$ and the function $\frac{r e^{r T_0}}{e^{r T_0} - 1}$ is monotone increasing respect to r. This means that the payment rate is lower.

Now we compare J_0 and J_1. This is equivalent to compare the discounted payment after τ, therefore, let

$$J_2 = \int_{\tau}^{t_0+T_0} m_0 e^{-r_0(s-t_0)} ds = \frac{e^{r_0(T_0+t_0-\tau)} - 1}{e^{r_0T_0} - 1} x_0,$$

and

$$J_4 = \int_{\tau}^{\tau+T_0} m(\tau, X(\tau), T_0) e^{-r_1(s-\tau)} ds = \frac{e^{r_1T_0}}{e^{r_1T_0} - 1} \times \frac{e^{r_0T_0} - e^{r_0(\tau-t_0)}}{e^{r_0T_0} - 1} x_0 \Big[1 - e^{-r_0T_0}\Big].$$

Then we have the following relationship:

(1) If

$$1 - \frac{1}{e^{r_1T_0}} = \frac{e^{r_0T_0} + e^{-r_0(t_0+T_0-\tau)} - e^{r_0(\tau-t_0)} - 1}{e^{r_0(T_0+t_0-\tau)} - 1},$$

which implies $J_0 = J_1$. Hence it is no advantage to refinance at this time.

(2) If

$$1 - \frac{1}{e^{r_1T_0}} < \frac{e^{r_0T_0} + e^{-r_0(t_0+T_0-\tau)} - e^{r_0(\tau-t_0)} - 1}{e^{r_0(T_0+t_0-\tau)} - 1},$$

which implies that the discounted payment after refinancing is larger than original mortgage, i.e., $J_1 > J_0$, even the payment rate is lower than original mortgage since r_1 is not lower enough than r_0 and a longer payment period after refinancing than before. Clearly, in this case, it is wise that not refinance the mortgage.

(3) If

$$1 - \frac{1}{e^{r_1T_0}} > \frac{e^{r_0T_0} + e^{-r_0(t_0+T_0-\tau)} - e^{r_0(\tau-t_0)} - 1}{e^{r_0(T_0+t_0-\tau)} - 1},$$

which implies that when r_1 is lower enough, one might want to refinance the mortgage.

Remark 3.1.11. Inspired by this example, an impulse control problem with changing running cost could be established and two features are different from the classical impulse control problem. First zero impulse is meaningful, since the running cost is changed when impulse applied. Second, because of the changing on the terminal time when impulse applied, the lower running cost is not equivalent to the lower total expected discounted payment.

3.2 Multi-Times Refinances

In this section we consider multiple times refinancing and investigate the features by a general two times refinance model.

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and $W(\cdot)$ be as before. Consider the following (market) interest rate model:

$$\begin{cases} d\mathbf{r}(s) = b_0(s, \mathbf{r}(s))ds + \sigma_0(s, \mathbf{r}(s))dW(s), \qquad s \in [t, T], \\ \mathbf{r}(t) = r. \end{cases} \tag{3.25}$$

where $b_0 : [0, T] \times \mathbb{R} \to \mathbb{R}$ and $\sigma_0 : [0, T] \times \mathbb{R} \to \mathbb{R}^{1\times d}$ are some suitable deterministic maps, $\mathbf{r}(\cdot)$ is the *state process* with $t \in [0, T]$ being the *initial time* and $r \in \mathbb{R}$ being the *initial state*.

Consider a mortgage started at $t_0 \geqslant 0$ with the loan amount x_{t_0}, mortgage interest rate $r(t_0) = \mathbf{r}(t_0) + \delta(t_0, x_{t_0}, T_0)$, where $\delta(\cdot, \cdot, \cdot)$ is a deterministic function and with fixed term T_0 as we defined in Section 3.1. If there is no pre-payment, refinance nor default, the remaining balance process $X(\cdot)$ should satisfy:

$$X(s) = x_{t_0} + \int_{t_0}^{s} r(t_0)X(t)d\tau - \sum_{k=1}^{\ell} m(t_0)\chi_{[t_0+k\mu,\infty)}(s), \quad s \in [t_0, t_0 + T_0],$$

with $m(t_0)$ is determined by the terminal constraint

$$X(t_0 + T_0) = 0,$$

and μ is the unit period length, at the end of each $(t_0 + k\mu, t_0 + (k+1)\mu]$, a payment is made. Thus, $\ell\mu = T_0$.

Note that

$$X(s) = e^{[r(t_0)+\delta(t_0,x_{t_0},T_0)](s-t_0)}x_{t_0} - \sum_{k=1}^{\ell}\int_{t_0}^{s} e^{[r(t_0)+\delta(t_0,x_{t_0},T_0)](s-\tau)}m(t_0)\chi_{[t_0+k\mu,\infty)}(\tau)d\tau.$$

Thus, $X(t_0 + T_0) = 0$ leads to

$$\begin{aligned}
e^{[r(t_0)+\delta(t_0,x_{t_0},T_0)]T_0}\eta &= \sum_{k=1}^{\ell}\int_{t_0+k\mu}^{t_0+T_0} m(t_0)e^{[r(t_0)+\delta(t_0,x_{t_0},T_0)](T_0+t_0-\tau)}d\tau \\
&= m(t_0)\sum_{k=1}^{\ell}\frac{e^{[r(t_0)+\delta(t_0,x_{t_0},T_0)](\ell-k)\mu} - 1}{r(t_0) + \delta(t_0, x_{t_0}, T_0)} \\
&= \frac{m(t_0)}{r(t_0)+\delta}\Big[\frac{e^{[r(t_0)+\delta(t_0,x_{t_0},T_0)]T_0} - 1}{e^{[r(t_0)+\delta(t_0,x_{t_0},T_0)]\mu} - 1} - \ell\Big].
\end{aligned}$$

Therefore,

$$m(t_0) = [r(t_0) + \delta(t_0, x_{t_0}, T_0)]e^{[r(t_0)+\delta(t_0,x_{t_0},T_0)]T_0}\Big[\frac{e^{[r(t_0)+\delta(t_0,x_{t_0},T_0)]T_0} - 1}{e^{[r(t_0)+\delta(t_0,x_{t_0},T_0)]\mu} - 1} - \frac{T_0}{\mu}\Big]^{-1}\eta.$$

Thus,

$$m(t_0) = m(T_0, r(t_0), x_{t_0}).$$

For this case, the cost functional is (discount all the payments made at $\tau_k = t_0 + k\mu$ to the time t_0, using continuous compound interest)

$$\begin{aligned}
J(\theta_0, x_{t_0}) &= \sum_{k=1}^{\ell}\int_{\theta}^{\tau_k} m(t_0)e^{-r(t_0)(s-t_0)}ds = \sum_{k=1}^{\ell} m(t_0)\frac{1 - e^{-r(t_0)k\mu}}{r(t_0)} \\
&= \frac{m(t_0)}{r(t_0)}\Big[\frac{T_0}{\mu} - e^{-r(t_0)\mu}\frac{1 - e^{-r(t_0)T_0}}{1 - e^{-r(t_0)\mu}}\Big].
\end{aligned}$$

Now we consider a refinance which is happened at the current situation $(t, X(t))$ and $r(t)$. Suppose some additional impulse control $\xi(\cdot)$ as the following form:

$$\xi(s) = \sum_{k \geqslant 1} \xi_k \chi_{[t_0 + k\mu, \infty)}(s), \qquad s \geqslant t_0$$

applies, with $\xi_k \geqslant 0$. Let at $(t_j, X(t_j))$, the mortgage is refinanced with new term T_j, the loan amount is $X(t_j) = X(t_j + 0)$ and the rate $r(t_j) + \delta$. Then the state between t_j and t_{j+1} is

$$\begin{aligned} X(s) &= e^{[r(t_j)+\delta(t_0, x_{t_0}, T_0)](s-t_j)} X(t_j) \\ &\quad - \sum_{k \geqslant 1} \int_{t_j}^{s} e^{[r(t_j)+\delta(t_0, x_{t_0}, T_0)](s-t_j)} [m(T_j, r(t_j), X(t_j)) + \xi_k] \chi_{[t_j + k\mu, \infty)}(\tau) d\tau. \end{aligned}$$

The monthly payment $m(T_j, r(t_j), X(t_j))$ is determined similarly as $m(T, r(t_0), x_{t_0})$. Expected cost on $[t_j, t_{j+1}]$ can be defined. For technical convenience, we assume that the term for every new refinanced mortgage is fixed T_0.

Now we rewrite above in a general form. Consider the following stochastic differential equations.

$$X(s) = x + \int_t^s b_1(\tau, X(\tau), u(\tau)) d\tau + \int_t^s \sigma_1(\tau, X(\tau), u(\tau)) dW(\tau) + \xi(s), \quad s \in [t, T], \tag{3.26}$$

and

$$r(s) = r + \int_t^s b_2(\tau, r(\tau)) d\tau + \int_t^s \sigma_2(\tau, r(\tau)) dW(\tau), \quad s \in [t, T], \tag{3.27}$$

where $b_1 : [0, T] \times \mathbb{R} \times U \to \mathbb{R}$, $b_2 : [0, T] \times \mathbb{R} \to \mathbb{R}$, $\sigma_1, \sigma_2 : [0, T] \times \mathbb{R} \to \mathbb{R}^{1 \times d}$ are some suitable deterministic maps and U is a metric space. $X(\cdot)$ is the *mortgage balance* and $r(\cdot)$ is the *mortgage rate* with $t \in [0, T]$ being the *initial time* and $x \in \mathbb{R}$ and $r \in \mathbb{R}$ being the *initial states* respectively,

$u(\cdot)$ is called a *control process* which is taken from the following set:

$$\mathcal{U}[t,T] = \Big\{ u : [t,T] \times \Omega \mapsto \mathbb{R} \mid u(\cdot) \text{ is } \mathbb{F} - \text{progressively measurable}, \mathbb{E} \int_t^T |u(s)|^2 ds < \infty \Big\}.$$

$\xi(\cdot)$ is called an *impulse control* of the following term:

$$\xi(s) = \sum_{i \geqslant 1} \xi_i \chi_{[t_i, T]}(s), \qquad s \geqslant t, \tag{3.28}$$

where $\{\tau_i\}_{i\geqslant 1}$ is an increasing sequence of $\mathbb{F}$-stopping times valued in $[t,T]$, and each ξ_i is an $\mathcal{F}_{\tau_i}$-measurable square integrable random variable taking values in K, with $K \subset \mathbb{R}$ being a closed convex cone. Let $\mathscr{K}[t,T]$ be the set of all impulse controls of the form (4.3). Under proper conditions, for any $(t,x,r) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n$, and $\xi(\cdot) \in \mathscr{K}[t,T]$, state equations (4.1) and (4.2) admit unique solutions $X(\cdot) \equiv X(\cdot; t, x, \xi(\cdot))$ and $r(\cdot) \equiv r(\cdot; t, r)$ respectively (see Theorem 2.0.2).

We start from the case that no refinance happened and introduce the following cost functional for $(t,x,r) \in [0,T] \times \mathbb{R} \times \mathbb{R}$ with the given initial triple $(t_0, x_{t_0}, r_{t_0}) \in [0,T] \times \mathbb{R} \times \mathbb{R}$:

$$\begin{aligned} J^0(t_0, x_{t_0}, r_{t_0}; t, x, r; u(\cdot)) = \mathbb{E}\Big\{ \int_t^{t_0+T} & g(t_0, x_{t_0}, r_{t_0}; s, X(s), r(s); u(s)) ds \\ & + h(t_0, x_{t_0}, r_{t_0}; X(t_0+T_0), r(t_0+T_0)) \Big\}, \end{aligned} \tag{3.29}$$

for some suitable deterministic maps $g(t_0, x_{t_0}, r_{t_0}; \cdot)$ and $h(t_0, x_{t_0}, r_{t_0}; \cdot)$, which depends on the initial triples $(t_0, x_{t_0}, r_{t_0}) \in [0,T] \times \mathbb{R} \times \mathbb{R}$. The terms on the right-hand side are the *running cost* and the *terminal cost*, respectively. For any $(t,x,r) \in [t_0, t_0+T_0] \times \mathbb{R} \times \mathbb{R}$, with the classical optimal control theory, we can find a $\bar{u}(\cdot) \in \mathcal{U}[t,T]$ such that

$$J^0(t_0, x_{t_0}, r_{t_0}; t, x, r; \bar{u}(\cdot)) = \min_{u \in \mathcal{U}[t,T]} J^0(t_0, x_{t_0}, r_{t_0}; t, x, r; u(\cdot)) = V^0(t_0, x_{t_0}, r_{t_0}; t, x, r). \tag{3.30}$$

Under proper conditions, the value function $V^0(t_0, x_{t_0}, r_{t_0}; \cdot, \cdot, \cdot)$ is a the unique solution of the following HJB equation:

$$\begin{cases} V_t^0(t_0, x_{t_0}, r_{t_0}; t, x, r) + \inf_{u \in U} \Big\{ \frac{1}{2} V_{xx}^0(t_0, x_{t_0}, r_{t_0}; t, x, r) \sigma_1(t, x, u)^2 \\ \qquad + \frac{1}{2} V_{rr}^0(t_0, x_{t_0}, r_{t_0}; t, x, r) \sigma_2(t, r)^2 + V_x^0(t_0, x_{t_0}, r_{t_0}; t, x, r) b_1(t, x, u) \\ \qquad + V_r^0(t_0, x_{t_0}, r_{t_0}; t, x, r) b_2(t, r) + g(t_0, x_{t_0}, r_{t_0}; t, x, r, u) \Big\} = 0, \\ \qquad\qquad\qquad\qquad\qquad (t, x, r) \in [t_0, t_0 + T_0] \times \mathbb{R} \times \mathbb{R}, \\ V^0(t_0, x_{t_0}, r_{t_0}; t_0 + T_0, x, r) = h(t_0, x_{t_0}, r_{t_0}; x, r), \quad (x, r) \in \mathbb{R} \times \mathbb{R}. \end{cases} \tag{3.31}$$

Any $\bar{u}(\cdot) \in \mathcal{U}[t, t_0 + T_0]$ satisfying (3.30) is called an *optimal control* and $\bar{X}(\cdot) \equiv (\cdot; t, x, r; \bar{u}(\cdot))$ is called the corresponding *optimal state process*. We call $V^0(\theta_0, x_{\theta_0}, r_{\theta_0}; \cdot, \cdot, \cdot)$ the *value function* of (3.29).

Now consider a refinance will be applied for this mortgage, that is find $t_1 \in (t_0, t_0 + T_0)$ where the first impulse will be applied. For any $(t, x, r) \in [t_0, t_0 + T_0] \times \mathbb{R} \times \mathbb{R}$ and $\xi_i \in K$, let

$$N[V](t_0, x_{t_0}, r_{t_0}; t, x, r) = \inf_{\xi_i \in K} \{ V(t_0, x_{t_0}, r_{t_0}; t, x + \xi_i, r) + \ell(t, x, \xi_i) \} \tag{3.32}$$

where $\ell(t, x, \xi_i) > 0$ is called the *impulse cost*. We state with two times refinance in following two cases.

3.2.1 Case 1: Refinancing One Time After Another

In this case, the first refinance will be applied at $t_1 \in (t_0, t_0 + T_0)$. Hence for any $(t, x, r) \in [t_0, t_0 + T_0] \times \mathbb{R} \times \mathbb{R}$, the cost functional (3.29) now should be:

$$\begin{aligned}
&J^1(t_0, x_{t_0}, r_{t_0}; t, x, r; u(\cdot), \xi(\cdot)) \\
=& \mathbb{E}\Big\{ \int_t^{t_1} g(t_0, x_{t_0}, r_{t_0}; s, X(s), r(s); u(s))ds \\
&+\mathbf{1}_{\{t_1 < t_0 + T_0\}} \Big[\int_{t_1}^{t_1+T_1} g(t_1, X(t_1 + 0), r(t_1); s, X(s), r(s); u(s))ds \\
&+\ell(t_1, X(t_1 - 0), \xi_1) + h(t_1, x_{t_1}, r_{t_1}; X(t_1 + T_1), r(t_1 + T_1))\Big] \\
&+\mathbf{1}_{\{t_1 = t_0 + T_0\}} \Big[h(t_0, x_{t_0}, r_{t_0}; X(t_0 + T_0), r(t_0 + T_0))\Big] \Big\},
\end{aligned} \tag{3.33}$$

and find $(\bar{u}(\cdot), \bar{\xi}(\cdot)) \in \mathcal{U}[t, T] \times \mathscr{K}_1[t, T]$ such that:

$$\begin{aligned}
&J^1(t_0, x_{t_0}, r_{t_0}; t, x, r; \bar{u}(\cdot), \bar{\xi}(\cdot)) \\
=& \inf_{(u(\cdot),\xi(\cdot)) \in \mathcal{U}[t,T] \times \mathscr{K}_1[t,T]} J^1(t_0, x_{t_0}, r_{t_0}; t, x, r; u(\cdot), \xi(\cdot)) = V^1(t_0, x_{t_0}, r_{t_0}; t, x, r).
\end{aligned} \tag{3.34}$$

Any $\bar{u}(\cdot) \in \mathcal{U}[t, T]$ satisfying (3.34) is called an *optimal control*, $\bar{\xi}(\cdot) \in \mathscr{K}_1[t, T]$ satisfying (3.34) is called an *optimal impulse control* and $\bar{X}(\cdot) \equiv (\cdot; t, x, r; \bar{u}(\cdot), \bar{\xi}(\cdot))$ is called the corresponding *optimal state process*. We call $V^1(t_0, x_{t_0}, r_{t_0}; \cdot, \cdot, \cdot)$ the *value function* of (3.33).

To find $(\bar{u}(\cdot), \bar{\xi}(\cdot)) \in \mathcal{U}[t, T] \times \mathscr{K}_1[t, T]$ in (3.34), first for any $t_1 \in (t_0, t_0 + T_0)$, define the cost functional for $(t, x, r) \in [t_1, t_1 + T_1] \times \mathbb{R} \times \mathbb{R}$ as follows.

$$\begin{aligned}
J^1(t_1, x_{t_1}, r_{t_1}; t, x, r; u(\cdot)) = \mathbb{E}\Big\{ \int_t^{t_1+T_1} & g(t_1, x_{t_1}, r_{t_1}; s, X(s), r(s); u(s))ds \\
&+h(t_1, x_{t_1}, r_{t_1}; X(t_1 + T_1), r(t_1 + T_1))\Big\},
\end{aligned} \tag{3.35}$$

and find $\bar{u}(\cdot) \in \mathcal{U}[t,T]$ such that:

$$\begin{aligned} J^1(t_1, x_{t_1}, r_{t_1}; t, x, r; \bar{u}(\cdot)) &= \inf_{u(\cdot)\in\mathcal{U}[t,T]} J^1(t_1, x_{t_1}, r_{t_1}; t, x, r; u(\cdot)) \\ &= V^1(t_1, x_{t_1}, r_{t_1}; t, x, r). \end{aligned} \tag{3.36}$$

Under proper conditions, the value function (3.36) should satisfies the following HJB equation:

$$\begin{cases} V_t^1(t_1, x_{t_1}, r_{t_1}; t, x, r) + \inf_{u\in U} \Big\{ \frac{1}{2} V_{xx}^1(t_1, x_{t_1}, r_{t_1}; t, x, r)\sigma_1(t, x, u)^2 \\ \qquad + \frac{1}{2} V_{rr}^1(t_1, x_{t_1}, r_{t_1}; t, x, r)\sigma_2(t, r)^2 + V_x^1(t_1, x_{t_1}, r_{t_1}; t, x, r) b_1(t, x, u) \\ \qquad + V_r^1(t_1, x_{t_1}, r_{t_1}; t, x, r) b_2(t, r) + g(t_1, x_{t_1}, r_{t_1}; t, x, r, u) \Big\} = 0, \\ \qquad\qquad\qquad\qquad (t, x, r) \in [t_1, t_1 + T_1] \times \mathbb{R} \times \mathbb{R}, \\ V^1(t_1, x_{t_1}, r_{t_1}; t_1 + T_1, x, r) = h(t_1, x_{t_1}, r_{t_1}; x, r), \quad (x, r) \in \mathbb{R} \times \mathbb{R}. \end{cases} \tag{3.37}$$

Then for any $(t, x, r) \in [t_0, t_0 + T_0] \times \mathbb{R} \times \mathbb{R}$, we can re-write (3.33) as:

$$\begin{aligned} & J^1(t_0, x_{t_0}, r_{t_0}; t, x, r; u(\cdot), \xi(\cdot)) \\ = \ & \mathbb{E}\Big\{ \int_t^{t_1} g(t_0, x_{t_0}, r_{t_0}; s, X(s), r(s); u(s)) ds \\ & + \mathbf{1}_{\{t_1 < t_0 + T_0\}} \Big[[V^1](t_0, x_{t_0}, r_{t_0}; t_1, X(t_1), r(t_1)) \Big] \\ & + \mathbf{1}_{\{t_1 = t_0 + T_0\}} \Big[h(t_0, x_{t_0}, r_{t_0}; X(t_0 + T_0), r(t_0 + T_0)) \Big] \Big\}. \end{aligned} \tag{3.38}$$

Let

$$t_1^* = \inf \Big\{ t_1 \in (t_0, t_0 + T_0) \mid N[V^1](t_0, x_{t_0}, r_{t_0}; t_1, x_{t_1}, r_{t_1}) = V^1(t_0, x_{t_0}, r_{t_0}; t_1, x_{t_1}, r_{t_1}) \Big\}. \tag{3.39}$$

Under proper conditions, the value function(3.36) should satisfy the following quasi-variational inequality:

$$\begin{cases} \min\Big\{V_t^1(t_0,x_{t_0},r_{t_0};t,x,r)+\inf_{u\in U}\Big\{\frac{1}{2}V_{xx}^1(t_0,x_{t_0},r_{t_0};t,x,r)\sigma_1(t,x,u)^2 \\ \quad +\frac{1}{2}V_{rr}^1(t_0,x_{t_0},r_{t_0};t,x,r)\sigma_2(t,r)^2+V_x^1(t_0,x_{t_0},r_{t_0};t,x,r)b_1(t,x,u) \\ \quad +V_r^1(t_0,x_{t_0},r_{t_0};t,x,r)b_2(t,r)+g(t_0,x_{t_0},r_{t_0};t,x,r,u)\Big\}, \\ \quad N[V^1](t_0,x_{t_0},r_{t_0};t,x,r)-V^1(t_0,x_{t_0},r_{t_0};t,x,r)\Big\}=0, \\ \qquad\qquad\qquad\qquad (t,x,r)\in[t_0,t_0+T_0]\times\mathbb{R}\times\mathbb{R}, \\ V^1(t_0,x_{t_0},r_{t_0};t_0+T_0,x,r)=h(t_0,x_{t_0},r_{t_0};x,r), \quad (x,r)\in\mathbb{R}\times\mathbb{R} \end{cases} \tag{3.40}$$

then an optimal refinance time $t_1^* \in [t, t_0+T_0]$ could be found and $(\bar{u}\times\bar{\xi})\in\mathcal{U}[t,T]\times\mathscr{K}_1[t,T]$ can be constructed by the value function (3.34).

Suppose there exists a $t_1^*\in(t,t_0+T_0)$ where the first refinance applied we find by the process above, the cost functional with $V^1(t_1,x_{t_1},r_{t_1};t_1,x_{t_1},r_{t_1})$ of (3.38) with this t_1^* now becomes as follows:

$$\begin{aligned} J^1(t_0,x_{t_0},r_{t_0};t,x,r;u(\cdot),\xi(\cdot)) = \mathbb{E}\Big\{\int_t^{t_1} & g(t_0,x_{t_0},r_{t_0};s,X(s),r(s);u(s))ds \\ & +[V^1](t_0,x_{t_0},r_{t_0};t_1^*,X(t_1^*),r(t_1^*))\Big\}. \end{aligned} \tag{3.41}$$

Next we will find the time where the second refinance applied with the same argument above. For any $(t,x,r)\in[t_1^*,t_1^*+T_1]\times\mathbb{R}^n\times\mathbb{R}^n$, and $t_2\in[t_1^*,t_1^*+T_1]$, consider the following cost functional:

$$
\begin{aligned}
&J^1(t_1^*, x_{t_1^*}, r_{t_1^*}; t, x, r; u(\cdot), \xi(\cdot)) \\
=& \mathbb{E}\Big\{ \int_t^{t_2} g(t_1^*, x_{t_1^*}, r_{t_1^*}; s, X(s), r(s); u(s)) ds \\
&+\mathbf{1}_{\{t_2 < t_1^* + T_1\}} \Big[\int_{t_2}^{t_2+T_2} g(t_2, X(t_2+0), r(t_2); s, X(s), r(s); u(s)) ds \\
&+\ell(t_2, X(t_2-0), \xi_1) + h(t_2, X(t_2+0), r(t_2); X(t_2+T_0), r(t_2+T_0)) \Big] \\
&+\mathbf{1}_{\{t_2 = t_1^* + T_1\}} \Big[h(t_1^*, x_{t_1^*}, r_{t_1^*}; X(t_1^*+T_1), r(t_1^*+T_1)) \Big] \Big\},
\end{aligned}
\tag{3.42}
$$

and find $(\bar{u}(\cdot), \bar{\xi}(\cdot)) \in \mathcal{U}[t, T] \times \mathscr{K}_1[t, T]$ such that:

$$
\begin{aligned}
&J^1(t_1^*, x_{t_1^*}, r_{t_1^*}; t, x, r; \bar{u}(\cdot), \bar{\xi}(\cdot)) \\
=& \inf_{(u,\xi) \in \mathcal{U}[t,T] \times \mathscr{K}_1[t,T]} J^1(t_1^*, x_{t_1^*}, r_{t_1^*}; t, x, r; u(\cdot), \xi(\cdot)) = V^1(t_1^*, x_{t_1^*}, r_{t_1^*}; t, x, r).
\end{aligned}
\tag{3.43}
$$

As the same process for finding $t_1^* \in [t_0, t_0 + T_0]$, above, we may find $t_2^* \in [t_1^*, t_1^* + T_1]$, where

$$
t_2^* = \inf \Big\{ t_2 \in (t_1^*, t_1^* + T_1) \mid N[V^1](t_1^*, x_{t_1^*}, r_{t_1^*}; t_2, x_{t_2}, r_{t_2}) = V^1(t_1^*, x_{t_1^*}, r_{t_1^*}; t_2, x_{t_2}, r_{t_2}) \Big\}. \tag{3.44}
$$

Again, under proper conditions, the value function (3.43) satisfies the following variational inequality:

$$
\begin{cases}
\min \Big\{ V_t^1(t_1^*, x_{t_1^*}, r_{t_1^*}; t, x, r) + \inf_{u \in U} \Big\{ \frac{1}{2} V_{xx}^1(t_1^*, x_{t_1^*}, r_{t_1^*}; t, x, r) \sigma_1(t, x, u)^2 \\
\quad + \frac{1}{2} V_{rr}^1(t_1^*, x_{t_1^*}, r_{t_1^*}; t, x, r) \sigma_2(t, r)^2 + V_x^1(t_1^*, x_{t_1^*}, r_{t_1^*}; t, x, r) b_1(t, x, u) \\
\quad + V_r^1(t_1^*, x_{t_1^*}, r_{t_1^*}; t, x, r) b_2(t, r) + g(t_1^*, x_{t_1^*}, r_{t_1^*}; t, x, r, u) \Big\}, \\
\quad N[V^1](t_1^*, x_{t_1^*}, r_{t_1^*}; t, x, r) - V^1(t_1^*, x_{t_1^*}, r_{t_1^*}; t, x, r) \Big\} = 0, \\
\qquad\qquad\qquad\qquad (t, x, r) \in [t_1^*, t_1^* + T_1] \times \mathbb{R} \times \mathbb{R}, \\
V^1(t_1^*, x_{t_1^*}, r_{t_1^*}; t_1^* + T_1, x, r) = h(t_1^*, x_{t_1^*}, r_{t_1^*}; x, r), \quad (x, r) \in \mathbb{R} \times \mathbb{R}.
\end{cases}
\tag{3.45}
$$

Then we can update $V_t^1(t_1^*, x_{t_1^*}, r_{t_1^*}; t_1^*, x_{t_1^*}, r_{t_1^*})$ in (3.42) and find (3.34) by (3.37). Above all, for any $(t, x, r) \in [t_0, t_0 + T_0] \times \mathbb{R} \times \mathbb{R}$, we find the value function in this case.

3.2.2 Case 2: Two Refinance will be Applied

Unlike the refinancing method we did above, in this case, we fix the refinance times first, that is we are given $t_0 \leqslant t_1 \leqslant t_0 + T_0$, and $t_1 \leqslant t_2 \leqslant t_1 + T_1$. For any $(t, x, r) \in [t_0, t_0 + T_0] \times \mathbb{R} \times \mathbb{R}$, the cost functional (3.29) now should be:

$$\begin{aligned}
&J^2(t_0, x_{t_0}, r_{t_0}; t, x, r; u(\cdot), \xi(\cdot)) \\
&= \mathbb{E}\Big\{ \int_t^{t_1} g(t_0, x_{t_0}, r_{t_0}; s, X(s), r(s); u(s))ds \\
&\quad +\mathbf{1}_{\{t_1 < t_0 + T_0\}} \Big[\int_{t_1}^{t_2} g(t_1, X(t_1 + 0), r(t_1); s, X(s), r(s); u(s))ds \\
&\quad\quad +\ell(t_1, X(t_1 - 0), \xi_1) \\
&\quad\quad +\mathbf{1}_{\{t_2 < t_1 + T_1\}} \Big[\int_{t_2}^{t_2 + T_2} g(t_2, X(t_2 + 0), r(t_2); s, X(s), r(s); u(s))ds \\
&\quad\quad\quad +\ell(t_2, X(t_2 - 0), \xi_2) + h(t_2, X(t_2 + 0), r(t_2); X(t_2 + T_2), r(t_2 + T_2))\Big] \\
&\quad\quad +\mathbf{1}_{\{t_2 = t_1 + T_1\}} \Big[h(t_1, X(t_1 + 0), r(t_1); X(t_1 + T_1), r(t_1 + T_1))\Big]\Big] \\
&\quad +\mathbf{1}_{\{t_1 = t_0 + T_0\}} \Big[h(t_0, x_{t_0}, r_{t_0}; X(t_0 + T_0), r(t_0 + T_0))\Big]\Big\},
\end{aligned} \tag{3.46}$$

and now we need to find $(\bar{u}(\cdot), \bar{\xi}(\cdot)) \in \mathcal{U}[t, T] \times \mathscr{K}_2[t, T]$ such that:

$$\begin{aligned}
&J^2(t_0, x_{t_0}, r_{t_0}; t, x, r; \bar{u}(\cdot), \bar{\xi}(\cdot)) \\
&= \inf_{(u(\cdot), \xi(\cdot)) \in \mathcal{U}[t,T] \times \mathscr{K}_2[t,T]} J^2(t_0, x_{t_0}, r_{t_0}; t, x, r; u(\cdot), \xi(\cdot)) = V^2(t_0, x_{t_0}, r_{t_0}; t, x, r).
\end{aligned} \tag{3.47}$$

Similarly, any $\bar{u}(\cdot) \in \mathcal{U}[t, T]$ satisfying (3.47) is called an *optimal control*, $\bar{\xi}(\cdot) \in \mathscr{K}_2[t, T]$ satisfying (3.47) is called an *optimal impulse control* and $\bar{X}(\cdot) \equiv (\cdot; t, x, r; \bar{u}(\cdot), \bar{\xi}(\cdot))$ is called the corresponding *optimal state process*. We call $V^2(t_0, x_{t_0}, r_{t_0}; \cdot, \cdot, \cdot)$ the Problem *value function* of (3.47).

First we fix a $t_1 \in (t_0, t_0 + T_0)$, for any $t_2 \in (t_1, t_1 + T_1)$ define the cost functional for any $(t, x, r) \in [t_2, t_2 + T_2] \times \mathbb{R}^n \times \mathbb{R}^n$ as following:

$$\begin{aligned} J^2(t_2, x_{t_2}, r_{t_2}; t, x, r; u(\cdot)) = \mathbb{E}\Big[\int_t^{t_2+T_2} & g(t_2, x_{t_2}, r_{t_2}; s, X(s), r(s); u(s)) ds \\ & + h(t_2, x_{t_2}, r_{t_2}; X(t_2 + T_2), r(t_2 + T_2))\Big], \end{aligned} \tag{3.48}$$

and let

$$\begin{aligned} & J^2(t_2, x_{t_2}, r_{t_2}; t, x, r; \bar{u}(\cdot)) \\ = & \inf_{u(\cdot) \in \mathcal{U}[t,T]} J^2(t_2, x_{t_2}, r_{t_2}; t, x, r; u(\cdot)) = V^2(t_2, x_{t_2}, r_{t_2}; t, x, r), \end{aligned} \tag{3.49}$$

which should satisfy the following HJB equation under proper conditions:

$$\begin{cases} V_t^2(t_2, x_{t_2}, r_{t_2}; t, x, r) + \inf_{u \in U} \Big\{ \frac{1}{2} V_{xx}^1(t_2, x_{t_2}, r_{t_2}; t, x, r) \sigma_1(t, x, u)^2 \\ \qquad + \frac{1}{2} V_{rr}^1(t_2, x_{t_2}, r_{t_2}; t, x, r) \sigma_2(t, r)^2 + V_x^2(t_2, x_{t_2}, r_{t_2}; t, x, r) b_1(t, x, u) \\ \qquad + V_r^2(t_2, x_{t_2}, r_{t_2}; t, x, r) b_2(t, r) + g(t_2, x_{t_2}, r_{t_2}; t, x, r, u) \Big\} = 0, \\ \qquad\qquad\qquad (t, x, r) \in [t_2, t_2 + T_2] \times \mathbb{R} \times \mathbb{R}, \\ V^2(t_2, x_{t_2}, r_{t_2}; t_2 + T_2, x, r) = h(t_2, x_{t_2}, r_{t_2}; x, r), \quad (x, r) \in \mathbb{R} \times \mathbb{R}. \end{cases} \tag{3.50}$$

Note that the t_2 we choose above is depending on t_1.

Then consider the cost functional for any $(t,x,r) \in [t_1, t_1+T_1] \times \mathbb{R} \times \mathbb{R}$ as following:

$$
\begin{aligned}
&J^2(t_1, x_{t_1}, r_{t_1}; t, x, r; u(\cdot), \xi(\cdot)) \\
=\ &\mathbb{E}\Big\{ \int_{t_1}^{t_2} g(t_1, x_{t_1}, r_{t_1}; s, X(s), r(s); u(s))ds \\
&+\mathbf{1}_{\{t_2 < t_1+T_1\}} \Big[[V^2]((t_1, x_{t_1}, r_{t_1}; t_2, X(t_2), r(t_2) \Big] \\
&+\mathbf{1}_{\{t_2 = t_1+T_1\}} \Big[h(t_1, x_{t_1}, r_{t_1}; X(t_1+T_1), r(t_1+T_1)) \Big] \Big\},
\end{aligned}
\tag{3.51}
$$

Let

$$
\begin{aligned}
&J^2(t_1, x_{t_1}, r_{t_1}; t, x, r; \bar{u}(\cdot), \bar{\xi}(\cdot)) \\
= &\inf_{(u(\cdot),\xi(\cdot)) \in \mathcal{U}[t,T] \times \mathscr{K}_1[t,T]} J^2(t_1, x_{t_1}, r_{t_1}; t, x, r; u(\cdot), \xi(\cdot)) = V^2(t_1, x_{t_1}, r_{t_1}; t, x, r).
\end{aligned}
\tag{3.52}
$$

which should satisfies the following quasi-variational inequality under proper conditions:

$$
\begin{cases}
\min \Big\{ V_t^2(t_1, x_{t_1}, r_{t_1}; t, x, r) + \inf_{u \in U} \Big\{ \frac{1}{2} V_{xx}^2(t_1, x_{t_1}, r_{t_1}; t, x, r) \sigma_1(t, x, u)^2 \\
\qquad + \frac{1}{2} V_{rr}^2(t_1, x_{t_1}, r_{t_1}; t, x, r) \sigma_2(t, r)^2 + V_x^2(t_1, x_{t_1}, r_{t_1}; t, x, r) b_1(t, x, u) \\
\qquad + V_r^2(t_1, x_{t_1}, r_{t_1}; t, x, r) b_2(t, r) + g(t_1, x_{t_1}, r_{t_1}; t, x, r, u) \Big\}, \\
\qquad N[V^2](t_1, x_{t_1}, r_{t_1}; t, x, r) - V^2(t_1, x_{t_1}, r_{t_1}; t, x, r) \Big\} = 0, \\
\qquad\qquad\qquad\qquad\qquad (t, x, r) \in [t_1, t_1+T_1] \times \mathbb{R} \times \mathbb{R}, \\
V^2(t_1, x_{t_1}, r_{t_1}; t_1+T_1, x, r) = h(t_1, x_{t_1}, r_{t_1}; x, r), \quad (x, r) \in \mathbb{R} \times \mathbb{R}.
\end{cases}
\tag{3.53}
$$

and we can find $t_2^* \in [t_1, t_1+T_1]$, where

$$
t_2^* = \inf \Big\{ t_2 \in (t_1, t_1+T_1) \mid N[V^2](t_1, x_{t_1}, r_{t_1}; t_2, x_{t_2}, r_{t_2}) = V^2(t_1, x_{t_1}, r_{t_1}; t_2, x_{t_2}, r_{t_2}) \Big\}. \tag{3.54}
$$

Again, this t_2^* depends on t_1.

Therefore, for any $(t, x, r) \in [t_0, t_0 + T_0] \times \mathbb{R} \times \mathbb{R}$, we can re-write (3.46) as following:

$$
\begin{aligned}
& J^2(t_0, x_{t_0}, r_{t_0}; t, x, r; u(\cdot), \xi(\cdot)) \\
= \mathbb{E}\Big\{ & \int_t^{t_1} g(t_0, x_{t_0}, r_{t_0}; s, X(s), r(s); u(s)) ds \\
& + \mathbf{1}_{\{t_1 < t_0 + T_0\}} \Big[[V^2](t_0, x_{t_0}, r_{t_0}; t_1, X(t_1), r(t_1)) \Big] \\
& + \mathbf{1}_{\{t_1 = t_0 + T_0\}} \Big[h(t_0, x_{t_0}, r_{t_0}; X(t_0 + T_0), r(t_0 + T_0)) \Big] \Big\}.
\end{aligned}
\tag{3.55}
$$

Then the value function defined in (3.47) satisfies the following quasi-variational inequality under proper conditions:

$$
\begin{cases}
\min \Big\{ V_t^2(t_0, x_{t_0}, r_{t_0}; t, x, r) + \inf_{u \in U} \Big\{ \frac{1}{2} V_{xx}^2(t_0, x_{t_0}, r_{t_0}; t, x, r) \sigma_1(t, x, u)^2 \\
\qquad + \frac{1}{2} V_{rr}^2(t_0, x_{t_0}, r_{t_0}; t, x, r) \sigma_2(t, r)^2 + V_x^2(t_0, x_{t_0} r_{t_0}; t, x, r) b_1(t, x, u) \\
\qquad + V_r^2(t_0, x_{t_0}, r_{t_0}; t, x, r) b_2(t, r) + g(t_0, x_{t_0}, r_{t_0}; t, x, r, u) \Big\}, \\
\qquad N[V^2](t_0, x_{t_0}, r_{t_0}; t, x, r) - V^2(t_0, x_{t_0}, r_{t_0}; t, x, r) \Big\} = 0, \\
\qquad\qquad\qquad\qquad (t, x, r) \in [t_0, t_0 + T_0] \times \mathbb{R} \times \mathbb{R}, \\
V^2(t_0, x_{t_0}, r_{t_0}; t_0 + T_0, x, r) = h(t_0, x_{t_0}, r_{t_0}; x, r), \quad \mathbb{R} \times \mathbb{R}.
\end{cases}
\tag{3.56}
$$

and $t_1^* \in [t_0, t_0 + T_0]$ may be found where

$$
t_1^* = \inf \Big\{ t_1 \in (t_0, t_0 + T_0) \mid N[V^2](t_0, x_{t_0}, r_{t_0}; t_1, x_{t_1}, r_{t_1}) = V^2(t_0, x_{t_0}, r_{t_0}; t_1, x_{t_1}, r_{t_1}) \Big\}. \tag{3.57}
$$

Above all, we find the value function (3.47) and the optimal controls could be constructed.

3.2.3 Comparison

Now we compare two value functions obtained in above two cases, for any $(t,x,r) \in [t_0, t_0 + T_0] \times \mathbb{R} \times \mathbb{R}$ we have

$$V^2(t_0, x_{t_0}, r_{t_0}; t, x, r) \leqslant V^1(t_0, x_{t_0}, r_{t_0}; t, x, r), \tag{3.58}$$

since we optimize t_1 after we find t_2^* for each t_1 in $V^2(t_0, x_{t_0}, r_{t_0}; t, x, r)$ while t_1^* is fixed for $V^1(t_0, x_{t_0}, r_{t_0}; t, x, r)$.

Now we suppose two refinances are applied in both two cases. For any $(t,x,r) \in [t_0, t_0 + T_0] \times \mathbb{R}^n \times \mathbb{R}^n$, consider the second case with the t_1^* we found for $V^1(t_0, x_{t_0}, r_{t_0}; t, x, r)$, we have as following:

$$\begin{aligned} &\inf_{(u(\cdot),\xi(\cdot)) \in \mathcal{U}[t,T] \times \mathscr{K}_2[t,T]} J^2(t_0, x_{t_0}, r_{t_0}; t, x, r; u(\cdot), \xi(\cdot)) \\ = &\inf_{(u(\cdot),\xi(\cdot)) \in \mathcal{U}[t,T] \times \mathscr{K}_2[t,T]} \mathbb{E}\Big\{ \int_{t_0}^{t_1^*} g(t_0, x_{t_0}, r_{t_0}; s, X(s), r(s); u(s)) ds \\ &\qquad + [V^2](t_0, x_{t_0}, r_{t_0}; t_1^*, X(t_1^*), r(t_1^*)) \Big\}. \end{aligned} \tag{3.59}$$

If

$$\inf_{(u(\cdot),\xi(\cdot)) \in \mathcal{U}[t,T] \times \mathscr{K}_2[t,T]} J^2(t_0, x_{t_0}, r_{t_0}; t, x, r; u(\cdot), \xi(\cdot)) = [V^2](t_0, x_{t_0}, r_{t_0}; t, x, r), \tag{3.60}$$

the refinance should be applied at t_1^* and we have

$$\begin{aligned}
&\inf_{(u(\cdot),\xi(\cdot))\in\mathcal{U}[t,T]\times\mathscr{K}_2[t,T]} J^2(t_0,x_{t_0},r_{t_0};t_1^*,X(t_1^*),r(t_1^*);u(\cdot),\xi(\cdot))\\
&=N[V^1](t_0,x_{t_0},r_{t_0};t_1^*,X(t_1^*),r(t_1^*))\\
&=N[V^2](t_0,x_{t_0},r_{t_0};t_1^*,X(t_1^*),r(t_1^*))\\
&=V^2(t_0,x_{t_0},r_{t_0};t_1^*,X(t_1^*),r(t_1^*)).
\end{aligned} \tag{3.61}$$

Moreover, if this t_1^* is as same as we find in $V^2(t_0,x_{t_0},r_{t_0};t,x,r)$, it should satisfies (3.57). If these two conditions above are satisfied, we can conclude that now

$$V^1(t_0,x_{t_0},r_{t_0};t,x,r)=V^2(t_0,x_{t_0},r_{t_0};t,x,r), \tag{3.62}$$

and the refinancing time t_1^* we find in Case 1 is also the refinancing time we find in Case 2. To conclude the result of this section, we introduce the following proposition.

Proposition 3.2.1. *For any* $(t,x,r)\in[t_0,t_0+T_0]\times\mathbb{R}\times\mathbb{R}$ *and* $(u(\cdot),\xi(\cdot))\in\mathcal{U}[t,T]\times\mathscr{K}_2[t,T]$, *the refinancing strategy we obtained backwardly is optimal.*

Proof. Consider m times refinance applied, and for each $(t,x,r)\in[t_0,t_0+T_0]\times\mathbb{R}\times\mathbb{R}$, on the last time period (t_m,t_m+T_m) the cost functional defined as

$$\begin{aligned}
J(t_m,x_{t_m},r_{t_m};t,x,r;u^m(\cdot))=\mathbb{E}\Big[\int_t^{t_m+T_m} & g(t_m,x_{t_m},r_{t_m};s,X(s),r(s);u^m(s))ds\\
&+h(t_m,x_{t_m},r_{t_m};X(t_m+T_m),r(t_m+T_m))\Big],
\end{aligned}$$

and the corresponding value function

$$V(t_m,x_{t_m},r_{t_m};t,x,r)=\inf_{u^m(\cdot)\in\mathcal{U}[t,T]} J(t_m,x_{t_m},r_{t_m};t,x,r;u^m(\cdot)).$$

Notice that no more refinance will be applied when

$$V(t_m, x_{t_m}, r_{t_m}; t, x, r) < \ell(t, x, \xi_{m+1}) \tag{3.63}$$

For any $(t, x, r) \in [t_0, t_0 + T_0] \times \mathbb{R} \times \mathbb{R}$ and $(u(\cdot), \xi(\cdot)) \in \mathcal{U}[t, T] \times \mathscr{K}_2[t, T]$, let $l < \infty$ times refinancing applied, then we have

$$\begin{aligned}
&J(t_0, x_{t_0}, r_{t_0}; t, x, r; u(\cdot), \xi(\cdot)) \\
\geqslant\; & \mathbb{E}\Big\{ \int_t^{t_1} g(t_0, x_{t_0}, r_{t_0}; s, X(s), r(s); u(s)) ds \\
&+ \sum_{i \geqslant 1}^{l} \Big(\int_{t_i}^{t_{i+1}} g(t_i, X(t_i + 0), r(t_i); s, X(s), r(s); u(s)) ds + \ell(t_i, X(t_i + 0), \xi_i) \Big) \\
&+ h(t_l, X(t_l), r(t_l); X(t_{l+1}), r(t_{l+1})) \Big\} \\
\geqslant\; & \mathbb{E}\Big\{ \int_t^{t_1} g(t_0, x_{t_0}, r_{t_0}; s, X(s), r(s); u(s)) ds \\
&+ \sum_{i \geqslant 1}^{l-1} \Big(\int_{t_i}^{t_{i+1}} g(t_i, X(t_i + 0), r(t_i); s, X(s), r(s); u(s)) ds + \ell(t_i, X(t_i + 0), \xi_i) \Big) \\
&+ N[V](t_{l-1}, X(t_{l-1}), r(t_{l-1}); t_l, X(t_l), r(t_l)) \Big\} \\
\geqslant\; & \mathbb{E}\Big[\int_t^{t_1} g(t_0, x_{t_0}, r_{t_0}; s, X(s), r(s); u(s)) ds \\
&+ \sum_{i \geqslant 1}^{l-2} \Big(\int_{t_i}^{t_{i+1}} g(t_i, X(t_i + 0), r(t_i); s, X(s), r(s); u(s)) ds + \ell(t_i, X(t_i + 0), \xi_i) \Big) \\
&+ N[V](t_{l-2}, X(t_{l-2}), r(t_{l-2}); t_{l-1}, X(t_{l-1}), r(t_{l-1})) \Big] \\
\geqslant\; & \cdots \\
\geqslant\; & \mathbb{E}\Big[\int_t^{t_1} g(t_0, x_{t_0}, r_{t_0}; s, X(s), r(s); u(s)) ds + N[V](t_1, X(t_1), r(t_1); t_1, X(t_1), r(t_1)) \Big] \\
\geqslant\; & V(t_0, x_{t_0}, r_{t_0}; t, x, r),
\end{aligned}$$

which implies the value function $V(t_0, x_{t_0}, r_{t_0}; t, x, r)$ we found by above method is optimal. $\square$

CHAPTER 4: OPTIMAL IMPULSE CONTROL WITH INITIAL PAIR DEPENDENT RUNNING COST

In Chapter 3, we notice that the mortgage size, mortgage interest rate and mortgage terminal time will be changed once the refinance applied. Also, a closing cost will be added to the payoff function. Inspired by this, we now study an impulse control problem with the running cost rate depending on the initial pair.

4.1 Problem Formulation

Consider the following stochastic differential equations:

$$X(s) = x + \int_t^s b_1(\tau, X(\tau), u(\tau))d\tau + \int_t^s \sigma_1(\tau, X(\tau), u(\tau))dW(\tau) + \xi(s), \quad s \in [t, T], \tag{4.1}$$

and

$$r(s) = r + \int_t^s b_2(\tau, r(\tau))d\tau + \int_t^s \sigma_2(\tau, r(\tau))dW(\tau), \quad s \in [t, T], \tag{4.2}$$

where $b_1 : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^n$, $b_2 : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$, $\sigma_1, \sigma_2 : [0, T] \times \mathbb{R}^n \to \mathbb{R}^{n \times d}$ are some suitable deterministic maps and U is a metric space. $X(\cdot)$ and $r(\cdot)$ are the *state processes* with $t \in [0, T]$ being the *initial time* and $x \in \mathbb{R}^n$ and $r \in \mathbb{R}^n$ being the *initial states* respectively, $u(\cdot)$ is called a *control process* as we defined in Chapter 2. $\xi(\cdot)$ is called an *impulse control* of the following term:

$$\xi(s) = \sum_{i \geqslant 1} \xi_i \chi_{[t_i, T]}(s), \qquad s \geqslant t, \tag{4.3}$$

where $\{\tau_i\}_{i \geqslant 1}$ is an increasing sequence of $\mathbb{F}$-stopping times valued in $[t, T]$, and each ξ_i is an $\mathcal{F}_{\tau_i}$-measurable square integrable random variable taking values in K, with $K \subset \mathbb{R}^n$ being a closed convex cone. Let $\mathscr{K}[t,T]$ be the set of all impulse controls of the form (4.3). Under proper conditions, for any $(t,x,r) \in [t,T] \times \mathbb{R}^n \times \mathbb{R}^n$, and $\xi(\cdot) \in \mathscr{K}[0,T]$, state equations (4.1) and (4.2) admit unique solutions $X(\cdot) \equiv X(\cdot;t,x,\xi(\cdot))$ and $r(\cdot) \equiv r(\cdot;t,r)$ respectively (see Theorem 2.0.2).

Under proper conditions, for any $(t,x,r) \in [t_0, t_0+T_0] \times \mathbb{R}^n \times \mathbb{R}^n$ and $(u(\cdot),\xi(\cdot)) \in \mathcal{U}[t,T] \times \mathscr{K}[t,T]$, we have the cost functional as follows,

$$\begin{aligned} & J(t_0, x_{t_0}, r_{t_0}; t, x, r; u(\cdot), \xi(\cdot)) \\ = \mathbb{E}\Big[& \int_t^{t_1} g(t_0, X(t_0), r(t_0); s, X(s), r(s); u(s))ds \\ & + \sum_{i \geqslant 1} \Big(\mathbf{1}_{\{t_i < t_{i-1}+T_{i-1}\}} \Big(\int_{t_i}^{t_{i+1}} g(t_i, X(t_i+0), r(t_i); s, X(s), r(s); u(s))ds \\ & + \ell(t_i, X(t_i-0), \xi_i) \Big) + \mathbf{1}_{\{t_i = t_{i-1}+T_{i-1}\}} (h(t_{i-1}, X(t_{i-1}), r(t_{i-1}); X(t_i), r(t_i))) \Big) \Big], \end{aligned} \tag{4.4}$$

where $g(t_i, x_{t_i}, r_{t_i}; \cdot, \cdot, \cdot)$, and $h(t_i, x_{t_i}, r_{t_i}; \cdot, \cdot, \cdot)$ for $i \geqslant 0$ are some suitable deterministic maps, which depends on the initial triples $(t_0, x_{t_0}, r_{t_0}) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n$, and $\ell(\cdot,\cdot,\cdot)$ is a positive suitable deterministic map. Then the optimal impulse control with initial triple dependent running cost can be stated as follows.

Problem (OC). For any $(t,x,r) \in [t_0, t_0+T_0] \times \mathbb{R}^n \times \mathbb{R}^n$, find $(\bar{u}(\cdot), \bar{\xi}(\cdot)) \in \mathcal{U}[t,T] \times \mathscr{K}[t,T]$ such that

$$\begin{aligned} J(t_0, x_{t_0}, r_{t_0}; t, x, r; \bar{u}(\cdot), \bar{\xi}(\cdot)) &= \inf_{(u(\cdot),\xi(\cdot)) \in \mathcal{U}[t,T] \times \mathscr{K}[t,T]} J(t_0, x_{t_0}, r_{t_0}; t, x, r; u(\cdot), \xi(\cdot)) \\ &\equiv V(t_0, x_{t_0}, r_{t_0}; t, x, r). \end{aligned} \tag{4.5}$$

It should be pointed out that, unlike the classical optimal control problems, we have paid a special attention on the initial triple since the running cost is changed with the given different initial pairs. Because of that, our value function is of form $V(t_0, x_{t_0}, r_{t_0}; t, x, r)$, depends on the initial triple, which make our HJB equation significantly different from the classical case. Moreover, the terminal time of (4.4) depends on the initial triple at the last impulse. Due to this, the impulse $\xi_i = 0$ is also can make the value function changed.

We now will show that the number of impulses is finite and the optimal impulse control exists.

Proposition 4.1.1. *For each* $(t, x, r) \in [t_0, t_0 + T_0] \times \mathbb{R}^n \times \mathbb{R}^n$, *there exists a optimal impulse control* $\bar{\xi}(\cdot) \in \mathscr{K}[t, T]$ *such that*

$$\inf_{u(\cdot)\in\mathcal{U}[t,T]} J(t_0, x_{t_0}, r_{t_0}; t, x, r; ; u(\cdot), \bar{\xi}(\cdot)) = \inf_{u(\cdot)\in\mathcal{U}[t,T], \xi(\cdot)\in\mathscr{K}[t,T]} J(t_0, x_{t_0}, r_{t_0}; t, x, r; ; u(\cdot), \xi(\cdot)),$$

which implies the optimal number ℓ *of impulse exist and finite.*

Proof. First we will show that there exists a maximum times of impulse, $m < \infty$, for (4.4). Suppose m times impulses applied and the last impulse applied at t_m with $x_{t_m}, r_{t_m} \in \mathbb{R}^n \times \mathbb{R}^n$. For any $(t, x, r) \in [t_m, t_m + T_m] \times \mathbb{R}^n \times \mathbb{R}^n$, no more impulses will be applied after time m when

$$V(t_m, x_{t_m}, r_{t_m}; t, x, r) < \ell(t, x, \xi_{m+1}). \tag{4.6}$$

For any $\xi(\cdot) \in \mathscr{K}[t, T]$, there exist $k < \infty$ such that for any $(t, x, r) \in [t_0, t_0 + T_0] \times \mathbb{R}^n \times \mathbb{R}^n$,

$$\begin{aligned}
0 \leqslant & \inf_{u(\cdot)\in\mathcal{U}[t,T]} J(t_0, x_{t_0}, r_{t_0}; t, x, r; u(\cdot), \xi(\cdot)) \leqslant J(t_0, x_{t_0}, r_{t_0}; t, x, r; u(\cdot), \xi(\cdot)) \\
&= \mathbb{E}\Big[\int_t^{t_1} g(t_0, x_{t_0}, r_{t_0}; s, X(s), r(s); u(s)) ds \\
&\quad + \sum_{i\geqslant 1}^{k} \Big(\int_{t_i}^{t_{i+1}} g(t_i, X(t_i + 0), r(t_i); s, X(s), r(s); u(s)) ds + \ell(t_i, X(t_i + 0), t_i) \Big) \\
&\quad + h(t_k, x_{t_k}, r_{t_k}; X(t_k + T_k), r(t_k + T_k)) \Big],
\end{aligned}$$

Hence, there exist a sequence $u_k(\cdot) \in \mathcal{U}[t,T]$, such that

$$\lim_{k\to\infty} J(t_0, x_{t_0}, r_{t_0}; t, x, r; u_k(\cdot), \xi(\cdot)) = \inf_{u(\cdot)\in\mathcal{U}[t,T]} J(t_0, x_{t_0}, r_{t_0}; t, x, r; u(\cdot), \xi(\cdot)).$$

Next suppose there exists a subsequence $\{\xi_i(\cdot)\} \in \mathscr{K}[t,T]$ with infinite time impulse, since the $\ell(\cdot,\cdot,\cdot)$ is bounded below by a strictly positive constant,

$$\lim_{i\to\infty} J(t_0, x_{t_0}, r_{t_0}; t, x, r; u(\cdot), \xi_i(\cdot)) = \infty.$$

Therefore, only finite time impulses will be applied and the optimal number ℓ of impulse exist. $\square$

4.2 Solution Formulation

Now we will introduce the process to solve the Problem (OC), that is for each $(t,x,r) \in [t_0, t_0 + T_0] \times \mathbb{R}^n \times \mathbb{R}^n$ and $(u(\cdot), \xi(\cdot)) \in \mathcal{U}[t,T] \times \mathscr{K}[t,T]$, find (4.21) for (4.4). We will carry out this by several steps.

Step 1. For each $(t,x,r) \in [t_0, t_0+T_0] \times \mathbb{R}^n \times \mathbb{R}^n$, fix $k \in \{1, \cdots, m\}$ where m is the upper bound of number of impulses which a candidate of optimal impulse could have. Then for a minimizing sequence $(u^k(\cdot), \xi^k(\cdot)) \in \mathcal{U}[t,T] \times \mathscr{K}_k[t,T]$, we pick $\{t_{i+1}\}_{0\leqslant i\leqslant k-1}$ to be an increasing sequence of $\mathbb{F}$-stopping times valued in $[t_i, t_i + T_i]$ and the cost functional (4.4) becomes:

$$
\begin{aligned}
&J^k(t_0, x_{t_0}, r_{t_0}; t, x, r; u^k(\cdot), \xi^k(\cdot)) \\
=\ &\mathbb{E}\Big\{ \int_t^{t_1} g(t_0, X(t_0), r(t_0); s, X(s), r(s); u^k(s))ds \\
&+ \sum_{i\geqslant 1}^{k} \Big(\mathbf{1}_{\{t_i < t_{i-1}+T_{i-1}\}} \Big(\int_{t_i}^{t_{i+1}} g(t_i, X(t_i + 0), r(t_i); s, X(s), r(s); u(s))ds \\
&+\ell(t_i, X(t_i - 0), \xi_i) \Big) + \mathbf{1}_{\{t_i = t_{i-1}+T_{i-1}\}} (h(t_{i-1}, X(t_{i-1}), r(t_{i-1}); X(t_i), r(t_i))) \Big) \\
&+h(t_k, X(t_k), r(t_k); X(t_k + T_k), r(t_k + T_k)) \Big\},
\end{aligned}
\tag{4.7}
$$

and the corresponding value function

$$
V^k(t_0, x_{t_0}, r_{t_0}; t, x, r) = \inf_{(u^k(\cdot), \xi^k(\cdot)) \in \mathcal{U}[t,T] \times \mathscr{K}_k[t,T]} J(t_0, x_{t_0}, r_{t_0}; t, x, r; u^k(\cdot), \xi^k(\cdot)). \tag{4.8}
$$

Here we use upper index k to indicate k times impulse supposed to be applied. Similar as before, we try to find the value function (4.8) backwardly.

Step 2. We start from the last time period. On the last time period $[t_k, t_k + T_k]$, for each $(t, x, r) \in [t_k, t_k + T_k] \times \mathbb{R}^n \times \mathbb{R}^n$ and $u^k(\cdot) \in \mathcal{U}$, the cost functional is given as follows:

$$
\begin{aligned}
J^k(t_k, x_{t_k}, r_{t_k}; t, x, r; u^k(\cdot)) = \mathbb{E}\Big[\int_t^{t_k+T_k} & g(t_k, x_{t_k}, r_{t_k}; s, X(s), r(s); u^k(s))ds \\
& +h(t_k, x_{t_k}, r_{t_k}; X(t_k + T_k), r(t_k + T_k)) \Big],
\end{aligned}
\tag{4.9}
$$

and the corresponding value function is

$$
V^k(t_k, x_{t_k}, r_{t_k}; t, x, r) = \inf_{u^k(\cdot) \in \mathcal{U}[t, t_k+T_k]} J^k(t_k, x_{t_k}, r_{t_k}; t, x, r; u^k(\cdot)) \tag{4.10}
$$

should satisfies the following HJB equation under proper conditions:

$$
\begin{cases}
V_t^k(t_k, x_{t_k}, r_{t_k}; t, x, r) \\
\qquad + \inf_{u^k \in U} \Big\{ \frac{1}{2}\mathrm{tr}\left[V_{xx}^k(t_k, x_{t_k}, r_{t_k}; t, x, r)\sigma_1(t, x, u^k)\sigma_1(t, x, u^k)^\top \right] \\
\qquad + \frac{1}{2}\mathrm{tr}\left[V_{rr}^k(\tau_k, x_{t_k}, r_{t_k}; t, x, r)\sigma_2(t, r)\sigma_2(t, r)^\top \right] \\
\qquad + \langle V_x^k(t_k, x_{t_k}, r_{t_k}; t, x, r), b_1(t, x, u^k)\rangle \\
\qquad + \langle V_r^k(t_k, x_{t_k}, r_{t_k}; t, x, r), b_2(t, r)\rangle \\
\qquad + g(t_k, x_{t_k}, r_{t_k}; t, x, r, u^k) \Big\} = 0, \quad (t, x, r) \in [t_k, t_k + T_k] \times \mathbb{R}^n \times \mathbb{R}^n, \\
V^k(t_k, x_{t_k}, r_{t_k}; t_k + T_k, x, r) = h(t_k, x_{t_k}, r_{t_k}; x, r), \quad (x, r) \in \mathbb{R}^n \times \mathbb{R}^n.
\end{cases}
\tag{4.11}
$$

Then the optimal strategy $\bar{u} \in \mathcal{U}[t, t_k + T_k]$ could be constructed.

Step 3. Next consider $[t_{k-1}, t_k]$. The cost functional for each $(t, x, r) \in [t_{k-1}, t_k + T_k] \times \mathbb{R}^n \times \mathbb{R}^n$ and $(u^k(\cdot), \xi^k(\cdot)) \in \mathcal{U}[t, T] \times \mathscr{K}_1[t, T]$, on the last second time period $[t_{k-1}, t_k]$ as following:

$$
\begin{aligned}
&J^k(t_{k-1}, x_{t_{k-1}}, r_{t_{k-1}}; t, x, r; u^k(\cdot), \xi^k(\cdot)) \\
&= \mathbb{E}\Big[\int_{t_{k-1}}^{t_k} g(t_{k-1}, t_{k-1}, t_{k-1}; s, X(s), r(s); u^k(s))ds \\
&\quad + \mathbf{1}_{\{t_k < t_{k-1} + T_{k-1}\}} \Big(N[V^k](t_{k-1}, X(t_{k-1}), r(t_{k-1}); t_k, X(t_k), r(t_k)) \Big) \\
&\quad + \mathbf{1}_{\{t_k = t_{k-1} + T_{k-1}\}} (h(t_{k-1}, X(t_{k-1}), r(t_{k-1}); X(t_k), r(t_k)) \Big],
\end{aligned}
\tag{4.12}
$$

and the value function

$$
V^k(t_{k-1}, x_{t_{k-1}}, r_{t_{k-1}}; t, x, r) = \inf_{(u^k(\cdot), \xi^k(\cdot)) \in \mathcal{U}[t,T] \times \mathscr{K}_1[t,T]} J^k(t_{k-1}, x_{t_{k-1}}, r_{t_{k-1}}; t, x, r; u^k(\cdot), \xi^k(\cdot))
\tag{4.13}
$$

should satisfy the following variational inequality under proper conditions:

$$\begin{cases} \min\Big\{V_t^k(t_{k-1},x_{t_{k-1}},r_{t_{k-1}};t,x,r) \\ \qquad +\inf_{u^k\in U}\Big\{\frac{1}{2}\mathrm{tr}\Big[V_{xx}^k(t_{k-1},x_{t_{k-1}},r_{t_{k-1}};t,x,r)\sigma_1(t,x,u^k)\sigma_1(t,x,u^k)^\top\Big] \\ \qquad +\frac{1}{2}\mathrm{tr}\Big[V_{rr}^k(t_{k-1},x_{t_{k-1}},r_{t_{k-1}};t,x,r)\sigma_2(t,r)\sigma_2(t,r)^\top\Big] \\ \qquad +\langle V_x^k(t_{k-1},x_{t_{k-1}},r_{t_{k-1}};t,x,r),b_1(t,x,u^k)\rangle \\ \qquad +\langle V_r^k(t_{k-1},x_{t_{k-1}},r_{t_{k-1}};t,x,r),b_2(t,r)\rangle \\ \qquad +g(t_{k-1},x_{t_{k-1}},r_{t_{k-1}};t,x,r,u^k)\Big\}, \\ \qquad N[V^k](t_{k-1},x_{t_{k-1}},r_{t_{k-1}};t,x,r)-V^k(t_{k-1},x_{t_{k-1}},r_{t_{k-1}};t,x,r)\Big\}=0, \\ \qquad\qquad (t,x,r)\in[t_{k-1},t_{k-1}+T_{k-1}]\times\mathbb{R}^n\times\mathbb{R}^n, \\ V^k(t_{k-1},x_{t_{k-1}},r_{t_{k-1}};t_{k-1}+T_0,x,r)=h(t_{k-1},x_{t_{k-1}},r_{t_{k-1}};x,r),\quad (x,r)\mathbb{R}^n\times\mathbb{R}^n. \end{cases} \tag{4.14}$$

Step 4. Continuously, consider sub-periods $[t_i, t_{i+1}]$ for $0 \leqslant i \leqslant k-2$. The cost functional for each $(t,x,r) \in [t_i, t_i+T_i] \times \mathbb{R}^n \times \mathbb{R}^n$ and $(u^k(\cdot), \xi^k(\cdot)) \in \mathcal{U}[t,T] \times \mathscr{K}_{k-1}[t,T]$, on each sub-periods $[t_i, t_i+1]$ as following:

$$\begin{aligned} &J^k(t_i,x_{t_i},r_{t_i};t,x,r;u^k(\cdot),\xi^k(\cdot)) \\ =&\,\mathbb{E}\Big[\int_{t_i}^{t_{i+1}} g(t_i,x_{t_i},r_{t_i};s,X(s),r(s);u^k(s))ds \\ &+\mathbf{1}_{\{t_{i+1}<t_i+T_i\}}\Big(N[V^k](t_i,x_{t_i},r_{t_i};t_{i+1},X(t_{i+1}),r(t_{i+1}))\Big) \\ &+\mathbf{1}_{\{t_{i+1}=t_i+T_i\}}(h(t_i,x_{t_i},r_{t_i};X(t_{i+1}),r(t_{i+1}))\Big], \end{aligned} \tag{4.15}$$

and the value function

$$V^k(t_i,x_{t_i},r_{t_i};t,x,r)=\inf_{(u^k(\cdot),\xi^k(\cdot))\in\mathcal{U}[t,T]\times\mathscr{K}_{k-1}[t,T]} J^k(t_i,x_{t_i},r_{t_i};t,x,r;u^k(\cdot),\xi^k(\cdot)) \tag{4.16}$$

should satisfy the following quasi-variational inequality under proper conditions:

$$
\begin{cases}
\min \Big\{ V_t^k(t_i, x_{t_i}, r_{t_i}; t, x, r) \\
\qquad + \inf_{u^k \in U} \Big\{ \frac{1}{2} \text{tr} \Big[V_{xx}^k(t_i, x_{t_i}, r_{t_i}; t, x, r) \sigma_1(t, x, u^k) \sigma_1(t, x, u^k)^\top \Big] \\
\qquad + \frac{1}{2} \text{tr} \Big[V_{rr}^k(t_i, x_{t_i}, r_{t_i}; t, x, r) \sigma_2(t, r) \sigma_2(t, r)^\top \Big] \\
\qquad + \langle V_x^k(t_i, x_{t_i}, r_{t_i}; t, x, r), b_1(t, x, u^k) \rangle \\
\qquad + \langle V_r^k(t_i, x_{t_i}, r_{t_i}; t, x, r), b_2(t, r) \rangle \\
\qquad + g(t_i, x_{t_i}, r_{t_i}; t, x, r, u^k) \Big\}, \\
\qquad N[V^k](t_i, x_{t_i}, r_{t_i}; t, x, r) - V^k(t_i, x_{t_i}, r_{t_i}; t, x, r) \Big\} = 0, \\
\qquad\qquad\qquad\qquad (t, x, r) \in [t_i, t_i + T_i] \times \mathbb{R}^n \times \mathbb{R}^n, \\
V^k(t_i, x_{t_i}, r_{t_i}; t_i + T_i, x, r) = h(t_i, x_{t_i}, r_{t_i}; x, r), \quad (x, r) \mathbb{R}^n \times \mathbb{R}^n.
\end{cases}
\tag{4.17}
$$

Step 5. Since we have shown that the optimal $\bar{\xi}(\cdot) \in \mathscr{K}[t, T]$ exists and there exists an optimal times of impulse $l < \infty$, by the continuous and bounded of $V(t_0, x_{t_0}, r_{t_0}; t, x, r)$, for each $(t, x, r) \in [t_i, t_i + T_i] \times \mathbb{R}^n \times \mathbb{R}^n$ we obtain

$$
\min_{k \in \{0, \cdots, m\}} V^k(t_0, x_{t_0}, r_{t_0}; t, x, r) = V(t_0, x_{t_0}, r_{t_0}; t, x, r) = V^l(t_0, x_{t_0}, r_{t_0}; t, x, r). \tag{4.18}
$$

Remark 4.2.1. *We should notice here for* $0 \leqslant i \leqslant m - 1$, *each* t_{i+1} *which we choose above depends on* t_i.

Combining (4.10), (4.16) and (4.18), we can get the optimal times l of impulse and for $i \in$

$\{1, \cdots, l\}$ we have

$$t_i^* = \inf \Big\{ t_i \in (t_{i-1}, t_{i-1} + T_{i-1}) \mid N[V](t_{i-1}, x_{t_{i-1}}, r_{t_{i-1}}; t_i, x_{t_i}, r_{t_i}) \\ = V(t_{i-1}, x_{t_{i-1}}, r_{t_{i-1}}; t_i, x_{t_i}, r_{t_i}) \Big\}. \tag{4.19}$$

where the impulse controls applied. Then we can re-write (4.4) as following: for each $(t, x, r) \in [t_0, t_0 + T_0] \times \mathbb{R}^n \times \mathbb{R}^n$ and $(u(\cdot), \xi(\cdot)) \in \mathcal{U}[t, T] \times \mathscr{K}[t, T]$,

$$\begin{aligned} & J(t_0, x_{t_0}, r_{t_0}; t, x, r; u(\cdot), \xi(\cdot)) \\ & = \mathbb{E}\Big[\int_t^{t_1} g(t_0, x_{t_0}, r_{t_0}; s, X(s), r(s); u(s)) ds \\ & \quad + \sum_{i \geqslant 1}^{l} (N[V]\Big(t_i^*, X(t_i^*), r(t_i^*); t_{i+1}^*, X(t_{i+1}^*), r(t_{i+1}^*)) \\ & \quad + h(t_l^*, X(t_l^*), r(t_l^*); X(t_l^* + T_l), r(t_l^* + T_l))\Big], \end{aligned} \tag{4.20}$$

where l is we find by (4.18) and $N[V](t_i^*, X(t_i^*), r(t_i^*); t_{i+1}^*, X(t_{i+1}^*))$ and $h(t_l^*, X(t_l^*), r(t_l^*); X(t_l^* + T_l), r(t_l^* + T_l))$ are we found by (4.10) and (4.16) corresponding to l. The Problem (OC) is equivalently finding the value function for (4.20)

$$\begin{aligned} & J(t_0, x_{t_0}, r_{t_0}; t, x, r; \bar{u}(\cdot), \bar{\xi}(\cdot)) \\ = & \inf_{(u(\cdot), \xi(\cdot)) \in \mathcal{U}[t,T] \times \mathscr{K}[t,T]} J(t_0, x_{t_0}, r_{t_0}; t, x, r; u(\cdot), \xi(\cdot)) \equiv V(t_0, x_{t_0}, r_{t_0}; t, x, r), \end{aligned} \tag{4.21}$$

which satisfies the following HJB equations in a recursive form backwardly: for any $(t, x, r) \in [t_i^*, t_i^* + T_i] \times \mathbb{R}^n \times \mathbb{R}^n$, the value function $V(t_i^*, x_{t_i^*}, r_{t_i^*}; t, x, r)$ on each time sub-periods should satisfies the following HJB equations under proper conditions:

$$\begin{cases} V_t(t_l^*, x_{t_l^*}, r_{t_l^*}; t, x, r) + \inf_{u \in U} \Big\{ H(t_l^*, x_{t_l^*}, r_{t_l^*}; t, x, r, u, V_x(t_l^*, x_{t_l^*}, r_{t_l^*}; t, x, r), \\ \qquad V_{xx}(t_l^*, x_{t_l^*}, r_{t_l^*}; t, x, r), V_r(t_l^*, x_{t_l^*}, r_{t_l^*}; t, x, r), \\ \qquad V_{rr}(t_l^*, x_{t_l^*}, r_{t_l^*}; t, x, r)) \Big\} = 0, \\ \qquad\qquad (t, x, r) \in [t_l^*, t_l^* + T_l] \times \mathbb{R}^n \times \mathbb{R}^n, \\ V(t_l^*, x_{t_l^*}, r_{t_l^*}; t_l^* + T_l, x, r) = h(t_l^*, x_{t_l^*}, r_{t_l^*}; x, r), \quad (x, r) \in \mathbb{R}^n \times \mathbb{R}^n. \end{cases} \tag{4.22}$$

$$\begin{cases} V_t(t_i^*, x_{t_i^*}, r_{t_i^*}; t, x, r) + \inf_{u \in U} \Big\{ H(t_i^*, x_{t_i^*}, r_{t_i^*}; t, x, r, u, V_x(t_l^*, x_{t_l^*}, r_{t_l^*}; t, x, r), \\ \qquad V_{xx}(tt_i^*, x_{t_i^*}, r_{t_i^*}; t, x, r), V_r(t_i^*, x_{t_i^*}, r_{t_i^*}; t, x, r), \\ \qquad V_{rr}(t_i^*, x_{t_i^*}, r_{t_i^*}; t, x, r)) \Big\} = 0, \\ \qquad\qquad (t, x, r) \in [t_i^*, t_{i+1}^*] \times \mathbb{R}^n \times \mathbb{R}^n, \\ V(t_i^*, x_{t_i^*}, r_{t_i^*}; t_{i+1}^*, x, r) = N[V](t_i^*, x_{t_i^*}, r_{t_i^*}; t_{i+1}^*, x, r), \quad (x, r) \in \mathbb{R}^n \times \mathbb{R}^n. \end{cases} \tag{4.23}$$

Remark 4.2.2. *We should notice that the impulse will only be applied at the specific times t_i^*. Thus there is no impulse between two time sub-periods.*

4.3 The Value Function and Its Properties

In this section, we discuss the properties of the value function. We keep the above setting.

For the coefficients of the state equations (4.1) and (4.2), let b_1, b_2 and σ_1, σ_2 satisfies (S2) and we introduce the following assumption:

(H1). For any $(\hat{t}, \hat{x}, \hat{r}) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n$ and $u \in U$, maps g and h are continuous and any

$x_1, x_2 \in \mathbb{R}^n$, $r_1, r_2 \in \mathbb{R}^n$, there exists a constant $L > 0$,

$$\begin{cases} |g(\hat{t}, \hat{x}, \hat{r}; t, x_1, r_1; u) - g(\hat{t}, \hat{x}, \hat{r}; t, x_2, r_2; u)| \\ \qquad +|h(\hat{t}, \hat{x}, \hat{r}; x_1, r_1) - h(\hat{t}, \hat{x}, \hat{r}; x_2, r_2)| \leqslant L(|x_1 - x_2| + |r_1 - r_2|), \\ |g(\hat{t}, \hat{x}, \hat{r}; t, 0, 0; u)| + |h(\hat{t}, \hat{x}, \hat{r}; 0, 0)| \leqslant L. \end{cases} \tag{4.24}$$

(H2). For any $0 \leqslant \hat{t} \leqslant t \leqslant T$, $x, \hat{x} \in \mathbb{R}^n$, and $\xi, \hat{\xi} \in K$, there exist constants $l_0, \alpha, L > 0$,

$$\begin{aligned} & l_0 \leqslant \ell(\hat{t}, x, \xi) \leqslant \ell(t, x, \xi), \qquad \ell(t, x, \xi) > \alpha|\xi|, \\ & \ell(t, x, \xi + \hat{\xi}) \leqslant \ell(t, x, \xi) + \ell(t, x, \hat{\xi}), \qquad \ell(t, x, \xi) - \ell(t, \hat{x}, \xi)| \leqslant L|x - \hat{x}|. \end{aligned} \tag{4.25}$$

Next, we introduce the following notion of control processes.

Definition 4.3.1. *(1). An $\mathbb{F}-$ adapted process $u(\cdot)$ is called an admissible(continuous) control process on $[t, T]$ if it takes values in U almost surely. (2). An admissible impulse control process on $[t, T]$ is defined to be*

$$\xi(s) = \sum_{i \geqslant 1} \xi_i \chi_{[t_i, T]}(s), \qquad t \leqslant s \leqslant T, \tag{4.26}$$

where each t_i is an $\mathbb{F}-$stopping time with

$$t \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant T, \qquad \text{a.s.} \tag{4.27}$$

each ξ_i is $\mathcal{F}_{t_i}-$measurable with values in K, and

$$\mathbb{E}\Big(\sum_{i \geqslant 1} \ell(t_i, x_i, \xi_i) \Big) < \infty. \tag{4.28}$$

Again, we let $\mathcal{U}[t,T]$ and $\mathscr{K}[t,T]$ be the set of all admissible continuous control processes and impulse control processes on $[t,T]$, respectively.

Remark 4.3.2. *We should note that an impulse control with no impulse and with zero impulses are different due to the condition* (4.25) *for the impulse cost. In the classical impulse control problem, it is clear that any impulse control with some zero impulses are not optimal. However, in our problem, the zero impulses may be optimal due to the change of the running cost of the following time period.*

For notation convenience, we let $t^*_{l+1} = t^*_l + T_l$ for the last time period and $t^*_0 = t_0$ for the first time period. Due to no impulse made between two time sub-periods, therefore, on each $[t^*_i, t^*_{i+1}]$ we can re-write (4.1) equivalently as following:

$$X(s) = x + \xi_{t^*_i} + \int_{t^*_i}^{s} b_1(\tau, X(\tau), u(\tau))d\tau + \int_{t^*_i}^{s} \sigma_1(\tau, X(\tau), u(\tau))dW(\tau), \quad s \in [t^*_i, t^*_{i+1}], \tag{4.29}$$

Theorem 4.3.3. *Let (H1)-(H2) and (S2) hold. Then there exists a constant $K > 0$ such that the value function on each sub-periods $V(t^*_i, x_{t^*_i}, r_{t^*_i};, t, x, r)$ for satisfies the following:*

$$\begin{aligned} &|V(t^*_i, x_{t^*_i}, r_{t^*_i}; t, x, r)| \leqslant K(1 + |x| + |r|), \\ &\qquad\qquad\qquad\qquad \forall (t,x,r) \in (t,x,r) \in [t^*_i, t^*_{i+1}] \times \mathbb{R}^n \times \mathbb{R}^n, \end{aligned} \tag{4.30}$$

$$\begin{aligned} &|V(t^*_i, x_{t^*_i}, r_{t^*_i}; t, x, r) - V(t^*_i, x_{t^*_i}, r_{t^*_i}; \hat{t}, \hat{x}, \hat{r})| \\ &\quad \leqslant K\{|x - \hat{x}| + |r - \hat{r}| + (1 + |x| \vee |\hat{x}| + |r| \vee |\hat{r}|)|t - \hat{t}|^{\frac{1}{2}}\}, \\ &\qquad\qquad\qquad\qquad \forall (t,x,r), (\hat{t}, \hat{x}, \hat{r}) \in [t^*_i, t^*_{i+1}] \times \mathbb{R}^n \times \mathbb{R}^n. \end{aligned} \tag{4.31}$$

We first introduce following useful lemma.

Lemma 4.3.4. *Let (S2) hold. For any initial triple $(t,x,r) \in [t^*_i, t^*_{i+1}] \times \mathbb{R}^n \times \mathbb{R}^n$ and $u(\cdot) \in$*

$\mathcal{U}[t_i^*, t_{i+1}^*]$, *state functions* (4.29) *and* (4.2) *admit a unique strong solution* $X(\cdot) \equiv X(\cdot; t, x, \xi(\cdot))$ *and* $r(\cdot; t, r)$, *respectively. Moreover, for any* $u(\cdot) \in \mathcal{U}[t_i^*, t_{i+1}^*]$ *let* $t_i^* \leqslant t \leqslant \hat{t} \leqslant t_{i+1}^*$ *and* $X(\cdot; t, x)$, $X(\cdot; \hat{t}, \hat{x})$, $r(\cdot; t, r)$, *and* $r(\cdot; \hat{t}, \hat{r})$ *be the states corresponding to* $(t, x, u(\cdot))$, $(\hat{t}, \hat{x}, u(\cdot))$, (t, r), *and* $(\hat{t}, \hat{r})$, *respectively. There exists a constant* $K > 0$ *such that*

$$\mathbb{E}\Big\{\sup_{s \in [t, t_{i+1}^*]} |X(s; t, x)| + \sup_{s' \in [t, t_{i+1}^*]} |r(s'; t, r)|\Big\} \leqslant K\mathbb{E}\Big\{1 + |x| + |r|\Big\},$$

and

$$\begin{aligned} &\mathbb{E}\Big\{\sup_{s \in [\hat{t}, t_{i+1}^*]} |X(s; t, x) - X(s; \hat{t}, \hat{x})| + \sup_{s' \in [\hat{t}, t_{i+1}^*]} |r(s'; t, r) - r(s'; t, \hat{r})|\Big\} \\ &\leqslant K\mathbb{E}\Big\{K\{|x - \hat{x}| + |r - \hat{r}| + (1 + |x| \vee |\hat{x}| + |r| \vee |\hat{r}|)|t - \hat{t}|^{\frac{1}{2}}\}\Big\}. \end{aligned}$$

The proof is standard, see Yong–Zhou [39].

Now we prove Theorem 4.3.3.

Proof. Let $(t, x, r) \in [t_i^*, t_{i+1}^*] \times \mathbb{R}^n \times \mathbb{R}^n$ be fixed. By Lemma 4.3.4 and (H2), we have

$$|V(t_i^*, x_{t_i^*}, r_{t_i^*}; t, x, r)| \leqslant |J(t_i^*, x_{t_i^*}, r_{t_i^*}; t, x, r; u(\cdot))| \leqslant K(1 + |x| + |r|), \quad \forall u(\cdot) \in \mathcal{U}[t_i^*, t_{i+1}^*].$$

Similarly, let $t_i^* \leqslant t \leqslant \hat{t} \leqslant t_{i+1}^*$ and $X(\cdot; t, x)$, $X(\cdot; \hat{t}, \hat{x})$, $r(\cdot; t, r)$, and $r(\cdot; \hat{t}, \hat{r})$ be the states corresponding to $(t, x, u(\cdot))$, $(\hat{t}, \hat{x}, u(\cdot))$, (t, r), and $(\hat{t}, \hat{r})$, respectively. Again by Lemma 4.3.4 and (H2), we have

$$\begin{aligned} &|J(t_i^*, x_{t_i^*}, r_{t_i^*}; t, x, r; u(\cdot)) - J(t_i^*, x_{t_i^*}, r_{t_i^*}; t, \hat{x}, \hat{r}; u(\cdot))| \\ &\leqslant K\mathbb{E}\Big\{K\{|x - \hat{x}| + |r - \hat{r}| + (1 + |x| \vee |\hat{x}| + |r| \vee |\hat{r}|)|t - \hat{t}|^{\frac{1}{2}}\}\Big\}. \end{aligned}$$

Taking the infimum in $u(\cdot) \in \mathcal{U}[t_i^*, t_{i+1}^*]$, we get our result. □

Note that the impulse will applied on the terminal time t_{i+1}^* for $0 \leqslant i \leqslant l-1$.

Next we show the continuous in the state variable (4.1).

Lemma 4.3.5. *Let (S2) hold. For any initial triple* $(t,x,r) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n$ *and* $u(\cdot) \in \mathcal{U}[0,T]$, *state functions* (4.1) *admits a unique strong solution* $X(\cdot) \equiv X(\cdot;t,x,\xi(\cdot))$. *Moreover, there exists a constant* $K > 0$, *such that for any* $t \in [0,T)$, $x, \hat{x} \in \mathbb{R}^n$,

$$\mathbb{E}|X(s;t,x) - X(s;t,\hat{x})| \leqslant K\mathbb{E}|x - \hat{x}|, \quad \forall s \in [t,T].$$

Proof. First of all, for any $(t,x,r) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n$ and $(u(\cdot), \xi(\cdot)) \in \mathcal{U}[0,T] \times \mathscr{K}[[0,T]$, by a standard argument making use of the contraction mapping theorem, we know that the state equations (4.1) admits a unique strong solution $X(\cdot) \equiv X(\cdot;t,x,\xi(\cdot))$. Then for any $\varepsilon > 0$, let $\langle x \rangle_\varepsilon = \sqrt{\varepsilon^2 + |x|^2}$. Then by Itô's formula, we have

$$\mathbb{E}\langle X(s;t,x) - X(s;t,\hat{x})\rangle_\varepsilon \leqslant \mathbb{E}\langle x - \hat{x}\rangle_\varepsilon + (L + L^2) \int_t^s \mathbb{E}\langle X(r;t,x) - X(r;t,\hat{x})\rangle_\varepsilon dr$$

Applying Gronwall's inequality, we get

$$\mathbb{E}\langle X(s;t,x) - X(s;t,\hat{x})\rangle_\varepsilon \leqslant C\mathbb{E}\langle x - \hat{x}\rangle_\varepsilon.$$

Letting $\varepsilon \to 0$ to obtain

$$\mathbb{E}|X(s;t,x) - X(s;t,\hat{x})| \leqslant C|x - \hat{x}|.$$

□

Next theorem will give us the Lipschitz continuous of $N[V](t_i^*, x_{t_i^*}, r_{t_i^*}; t_{i+1}^*, x, r)$ in state variables.

Theorem 4.3.6. *Let (H1)-(H2) and (S2) hold. Then there exists a constant $K > 0$ such that the terminal function on each sub-periods $N[V](t_i^*, x_{t_i^*}, r_{t_i^*}; t_{i+1}^*, x, r)$ satisfies the following:*

$$
\begin{aligned}
&|N[V](t_i^*, x_{t_i^*}, r_{t_i^*}; t_{i+1}^*, x, r) - N[V](t_i^*, x_{t_i^*}, r_{t_i^*}; t_{i+1}^*, \hat{x}, \hat{r})| \\
\leqslant\ & K\{|x-\hat{x}| + |r-\hat{r}|\}, \quad \forall (x,r), (\hat{x}, \hat{r}) \in \times\mathbb{R}^n \times \mathbb{R}^n.
\end{aligned} \tag{4.32}
$$

Proof. Let $(x,r), (\hat{x}, \hat{r}) \in \mathbb{R}^n \times \mathbb{R}^n$. Let

$$
\begin{aligned}
N[V](t_i^*, x_{t_i^*}, r_{t_i^*}; t_{i+1}^*, x, r) &= \inf_{\xi \in K}\{V(t_i^*, x_{t_i^*}, r_{t_i^*}; t_{i+1}^*, x+\xi, r) + \ell(t_{i+1}^*, x, \xi)\} \\
&= V(t_i^*, x_{t_i^*}, r_{t_i^*}; t_{i+1}^*, x+\xi_1, r) + \ell(t_{i+1}^*, x, \xi_1),
\end{aligned}
$$

and

$$
\begin{aligned}
N[V](t_i^*, x_{t_i^*}, r_{t_i^*}; t_{i+1}^*, \hat{x}, \hat{r}) &= \inf_{\xi \in K}\{V(t_i^*, x_{t_i^*}, r_{t_i^*}; t_{i+1}^*, \hat{x}+\xi, \hat{r}) + \ell(t_{i+1}^*, \hat{x}, \xi)\} \\
&= V(t_i^*, x_{t_i^*}, r_{t_i^*}; t_{i+1}^*, \hat{x}+\xi_2, \hat{r}) + \ell(t_{i+1}^*, \hat{x}, \xi_2).
\end{aligned}
$$

Then by Theorem 4.3.3, (S2) and (H1)-(H2), we have

$$
\begin{aligned}
&|N[V](t_i^*, x_{t_i^*}, r_{t_i^*}; t_{i+1}^*, x, r) - N[V](t_i^*, x_{t_i^*}, r_{t_i^*}; t_{i+1}^*, \hat{x}, \hat{r})| \\
=\ & |V(t_i^*, x_{t_i^*}, r_{t_i^*}; t_{i+1}^*, x+\xi_1, r) + \ell(t_{i+1}^*, x, \xi_1) \\
&- V(t_i^*, x_{t_i^*}, r_{t_i^*}; t_{i+1}^*, \hat{x}+\xi_2, \hat{r}) + \ell(t_{i+1}^*, \hat{x}, \xi_2)| \\
\leqslant\ & |V(t_i^*, x_{t_i^*}, r_{t_i^*}; t_{i+1}^*, x+\xi_2, r) + \ell(t_{i+1}^*, x, \xi_2) \\
&- V(t_i^*, x_{t_i^*}, r_{t_i^*}; t_{i+1}^*, \hat{x}+\xi_2, \hat{r}) + \ell(t_{i+1}^*, \hat{x}, \xi_2)| \\
\leqslant\ & K\{|x-\hat{x}| + |r-\hat{r}|\}.
\end{aligned}
$$

□

4.4 Dynamic Programming and HJB Equation

In this section, we first establish a Bellman dynamic programming principle. Then we derive the corresponding HJB equation for the value function. As we show above, although the form of the terminal function on the last time period is different with others, however, since we have shown the Lipschitz continuous of $N[V](t_i^*, x_{t_i^*}, r_{t_i^*}; t_{i+1}^*, x, r)$ for any t_i^* where $0 \leqslant i \leqslant l-1$, we start from the last time period, and then each sub-periods will be established in the same idea.

Theorem 4.4.1. *Let (H1)-(H2) and (S2) hold. Then*

(1). for any $(t, x, r) \in [t_l^*, t_l^* + T_l] \times \mathbb{R}^n \times \mathbb{R}^n$,

$$\begin{aligned}
&V(t_l^*, x_{t_l^*}, r_{t_l^*}; t, x, r) \\
&= \inf_{u(\cdot)\in\mathcal{U}} \mathbb{E}\Big[\int_t^{\hat{t}} g(t_l^*, x_{t_l^*}, r_{t_l^*}; s, X(s; t_l^*, x_{t_l^*}, u(\cdot)), r(s; t_l^*, r_{t_l^*}); u(s))ds \\
&\quad + V(t_l^*, x_{t_l^*}, r_{t_l^*}; \hat{t}, x(\hat{t}; t_l^*, x_{t_l^*}, u(\cdot)), r(\hat{t}; t_l^*, r_{t_l^*}))\Big], \quad \forall t_l^* \leqslant t \leqslant \hat{t} \leqslant t_l^* + T_l.
\end{aligned} \tag{4.33}$$

(2). for any $(t, x, r) \in [t_i^*, t_{i+1}^*] \times \mathbb{R}^n \times \mathbb{R}^n$,

$$\begin{aligned}
&V(t_i^*, x_{t_i^*}, r_{t_i^*}; t, x, r) \\
&= \inf_{(u(\cdot),\xi(\cdot))\in\mathcal{U}\times\mathscr{K}} \mathbb{E}\Big[\int_t^{\hat{t}} g(t_i^*, x_{t_i^*}, r_{t_i^*}; s, X(s; t_i^*, x_{t_i^*}, u(\cdot)), r(s; t_i^*, r_{t_i^*}); u(s))ds \\
&\quad + V(t_i^*, x_{t_i^*}, r_{t_i^*}; \hat{t}, x(\hat{t}; t_i^*, x_{t_i^*}, u(\cdot)), r(\hat{t}; t_i^*, r_{t_i^*}))\Big], \quad \forall t_i^* \leqslant t \leqslant \hat{t} \leqslant t_{i+1}^*.
\end{aligned} \tag{4.34}$$

Proof. (1). Denote the right-hand side of (4.33) by $\bar{V}(t_l^*, x_{t_l^*}, r_{t_l^*}; t, x, r)$. For any $\varepsilon > 0$,

$$
\begin{aligned}
&V(t_l^*, x_{t_l^*}, r_{t_l^*}; t, x, r) + \varepsilon > J(t_l^*, x_{t_l^*}, r_{t_l^*}; t, x, r; u(\cdot)) \\
&= \mathbb{E}\Big[\int_t^{t_l^*+T_l} g(t_l^*, x_{t_l^*}, r_{t_l^*}; s, X(s; t_l^*, x_{t_l^*}, u(\cdot)), r(s; t_l^*, r_{t_l^*}); u(s) ds \\
&\quad + h(t_l^*, x_{t_l^*}, r_{t_l^*}; X(t_l^* + T_0; t_l^*, x_{t_l^*}, u(\cdot)), r(t_l^* + T_0; t_l^*, r_{t_l^*}))\Big] \\
&= \mathbb{E}\Big[\int_t^{\hat{t}} g(t_l^*, x_{t_l^*}, r_{t_l^*}; s, X(s; t_l^*, x_{t_l^*}, u(\cdot)), r(s; t_l^*, r_{t_l^*}); u(s) ds \\
&\quad + \mathbb{E}\Big[\int_{\hat{t}}^{t_l^*+T_l} g(t_l^*, x_{t_l^*}, r_{t_l^*}; t, X(s; t_l^*, x_{t_l^*}, u(\cdot)), r(s; t_l^*, r_{t_l^*}); u(s) ds \\
&\quad + h(t_l^*, x_{t_l^*}, r_{t_l^*}; X(t_l^* + T_0; t_l^*, x_{t_l^*}, u(\cdot)), r(t_l^* + T_0; t_l^*, r_{t_l^*})) | \mathcal{F}_{\hat{t}}\Big]\Big] \\
&\geqslant \mathbb{E}\Big[\int_t^{\hat{t}} g(t_l^*, x_{t_l^*}, r_{t_l^*}; s, X(s; t_l^*, x_{t_l^*}, u(\cdot)), r(s; t_l^*, r_{t_l^*}); u(s)) ds \\
&\qquad + V(t_l^*, x_{t_l^*}, r_{t_l^*}; \hat{t}, x(\hat{t}; t_l^*, x_{t_l^*}, u(\cdot)), r(\hat{t}; t_l^*, r_{t_l^*}))\Big]
\end{aligned}
$$

Sending $\varepsilon \to 0$, we obtain

$$
\begin{aligned}
V(t_l^*, x_{t_l^*}, r_{t_l^*}; t, x, r) \geqslant \mathbb{E}\Big[\int_t^{\hat{t}} & g(t_l^*, x_{t_l^*}, r_{t_l^*}; s, X(s; t_l^*, x_{t_l^*}, u(\cdot)), r(s; t_l^*, r_{t_l^*}); u(s)) ds \\
& + V(t_l^*, x_{t_l^*}, r_{t_l^*}; \hat{t}, x(\hat{t}; t_l^*, x_{t_l^*}, u(\cdot)), r(\hat{t}; t_l^*, r_{t_l^*}))\Big].
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&V(t_l^*, x_{t_l^*}, r_{t_l^*}; t, x, r) \leqslant J(t_l^*, x_{t_l^*}, r_{t_l^*}; t, x, r; u(\cdot)) \\
&= \mathbb{E}\Big[\int_t^{t_l^*+T_l} g(t_l^*, x_{t_l^*}, r_{t_l^*}; s, X(s; t_l^*, x_{t_l^*}, u(\cdot)), r(s; t_l^*, r_{t_l^*}); u(s) ds \\
&\quad + h(t_l^*, x_{t_l^*}, r_{t_l^*}; X(t_l^* + T_0; t_l^*, x_{t_l^*}, u(\cdot)), r(t_l^* + T_0; t_l^*, r_{t_l^*}))\Big] \\
&= \mathbb{E}\Big[\int_t^{\hat{t}} g(t_l^*, x_{t_l^*}, r_{t_l^*}; s, X(s; t_l^*, x_{t_l^*}, u(\cdot)), r(s; t_l^*, r_{t_l^*}); u(s) ds \\
&\quad + J(t_l^*, x_{t_l^*}, r_{t_l^*}; \hat{t}, X(\hat{t}; t_l^*, x_{t_l^*}, u(\cdot)), r(\hat{t}; t_l^*, r_{t_l^*}), r; u(\cdot))]\Big].
\end{aligned}
$$

Thus, we have

$$V(t_l^*, x_{t_l^*}, r_{t_l^*}; t, x, r) \leqslant \mathbb{E}\Big[\int_t^{\hat{t}} g(t_l^*, x_{t_l^*}, r_{t_l^*}; s, X(s; t_l^*, x_{t_l^*}, u(\cdot)), r(s; t_l^*, r_{t_l^*}); u(s))ds$$
$$+V(t_l^*, x_{t_l^*}, r_{t_l^*}; \hat{t}, x(\hat{t}; t_l^*, x_{t_l^*}, u(\cdot)), r(\hat{t}; t_l^*, r_{t_l^*}))\Big].$$

This completes the proof of (4.33).

(2). First of all, for any $(t, x, r) \in [t_i^*, t_i^* + T_i] \times \mathbb{R}^n \times \mathbb{R}^n$, take any $\xi(\cdot) \in \mathscr{K}[\hat{t}, t_0 + T_0]$, one have

$$V(t_i^*, x_{t_i^*}, r_{t_i^*}; t, x, r) \leqslant \mathbb{E}\Big[\int_t^{\hat{t}} g(t_i^*, x_{t_i^*}, r_{t_i^*}; s, X(s; t_i^*, x_{t_i^*}, u(\cdot)), r(s; t_i^*, r_{t_i^*}); u(s)ds$$
$$+J(t_i^*, x_{t_i^*}, r_{t_i^*}; \hat{t}, X(\hat{t}; t_i^*, x_{t_i^*}, u(\cdot)), r(\hat{t}; t_i^*, r_{t_i^*}), r; u(\cdot))]\Big].$$

Hence, by taking infimum over $(u(\cdot), \xi(\cdot)) \in \mathcal{U}[\hat{t}, t_0 + T_0] \times \mathscr{K}[\hat{t}, t_0 + T_0]$, we obtain

$$V(t_i^*, x_{t_i^*}, r_{t_i^*}; t, x, r) \leqslant \mathbb{E}\Big[\int_t^{\hat{t}} g(t_i^*, x_{t_i^*}, r_{t_i^*}; s, X(s; t_i^*, x_{t_i^*}, u(\cdot)), r(s; t_i^*, r_{t_i^*}); u(s))ds$$
$$+V(t_i^*, x_{t_i^*}, r_{t_i^*}; \hat{t}, x(\hat{t}; t_i^*, x_{t_i^*}, u(\cdot)), r(\hat{t}; t_i^*, r_{t_i^*}))\Big].$$

Moreover, one have

$$V(t_i^*, x_{t_i^*}, r_{t_i^*}; t, x, r) \leqslant V(t_i^*, x_{t_i^*}, r_{t_i^*}; t, x + \xi, r) + \ell(l, x, \xi), \qquad \forall \xi \in K.$$

Suppose above strictly inequality holds for at some points $(t, x, r) \in [t_i^*, t_i^* + T_i] \times \mathbb{R}^n \times \mathbb{R}^n$. We claim that (4.34) holds for some $t_0 \in (t, t_i^* + T_i]$, that is, there exists a minimizing sequence $\xi^\varepsilon(\cdot) \in \mathscr{K}[t, t_i^* + T_i]$ such that the first impulse time $t_{i+1}^\varepsilon \geqslant t_0$. Suppose (4.34) fails, which means that for any minimizing sequence $\xi^\varepsilon(\cdot) \in \mathscr{K}[t, t_i^* + T_i]$, the first impulse time t_{i+1}^ε satisfies

$$\lim_{\varepsilon \to 0} t_{i+1}^\varepsilon = t,$$

and

$$\lim_{\varepsilon\to 0}\min_{u(\cdot)\in\mathscr{U}} J(t_i^*, x_{t_i^*}, r_{t_i^*}; t, x, r; u(\cdot), \xi^\varepsilon(\cdot)) = V(t_i^*, x_{t_i^*}, r_{t_i^*}; t, x, r),$$

Consequently, we may assume that

$$\begin{aligned}
&V(t_i^*, x_{t_i^*}, r_{t_i^*}; t, x, r) + \varepsilon \geqslant J(t_i^*, x_{t_i^*}, r_{t_i^*}; t, x, r; u(\cdot), \xi^\varepsilon(\cdot))\\
&= \mathbb{E}\Big[\int_t^{t_{i+1}^\varepsilon} g(t_l^*, x_{t_l^*}, r_{t_l^*}; s, X(s; t, x, u(\cdot)), r(s; t, x; u(s))ds\\
&\quad + \int_{t_{i+1}^\varepsilon}^{t_{i+1}^\varepsilon + T_{i+1} g(t_{i+1}^\varepsilon, X(t_{i+1}^\varepsilon; t, x), r(t_{i+1}^\varepsilon; t, r); s, X(s; t, x), u(\cdot)), r(s; t, x); u(s))ds}\\
&\quad + h(t_{i+1}^\varepsilon, X(t_{i+1}^\varepsilon; t, x), r(t_{i+1}^\varepsilon; t, r); X(t_{i+1}^\varepsilon + T_0; t, x), r(t_{i+1}^\varepsilon + T_0; t, r))\\
&\quad + \ell(t_{i+1}^\varepsilon, X(t_{i+1}^\varepsilon - 0), \xi_{i+1}^\varepsilon)\Big]\\
&\geqslant \mathbb{E}\Big[\int_t^{t_{i+1}^\varepsilon} g(t_l^*, x_{t_l^*}, r_{t_l^*}; s, X(s; t_l^*, x_{t_l^*}, u(\cdot)), r(s; t_l^*, r_{t_l^*}); u(s))ds\\
&\quad + N[V](t_i^\varepsilon, X(t_i^\varepsilon; t, x), r(t_i^\varepsilon; t, r); t_{i+1}^\varepsilon, X(t_{i+1}^\varepsilon; t, x), r(t_{i+1}^\varepsilon; t, r))\Big].
\end{aligned}$$

Sending $\varepsilon \to 0$ and by the continuity of $(t, x, r) \mapsto N[V](t_i^*, x_{t_i^*}, r_{t_i^*}; t, x, r)$, we obtain

$$V(t_i^*, x_{t_i^*}, r_{t_i^*}; t, x, r) \geqslant N[V](t_i^*, x_{t_i^*}, r_{t_i^*}; t, x, r),$$

which is a contradiction, proving (4.34). □

Now let us introduce the following two Hamiltonian, respectively:

$$\begin{aligned}
&H(t_l^*, x_{t_l^*}, r_{t_l^*}; t, x, r, u, p, P, q, Q)\\
\triangleq\ & \frac{1}{2}\text{tr}\left[P\sigma_1(t, x, u)\sigma_1(t, x, u)^\top + Q\sigma_2(t, r)\sigma_2(t, r)^\top\right] + \langle p, b_1(t, x, u)\rangle\\
&+\langle q, b_2(t, r)\rangle + g(t_l^*, x_{t_l^*}, r_{t_l^*}; t, x, r, u),\\
&\forall (t, x, r, u, p, P, q, Q) \in [t_l^*, t_l^* + T_l] \times \mathbb{R}^n \times \mathbb{R}^n \times U \times \mathbb{R}^n \times \mathcal{S}^n \times \mathbb{R}^n \times \mathcal{S}^n.
\end{aligned} \tag{4.35}$$

and

$$
\begin{aligned}
& H(t_i^*, x_{t_i^*}, r_{t_i^*}; t, x, r, u, p, P, q, Q) \\
\triangleq & \frac{1}{2}\text{tr}\left[P\sigma_1(t,x,u)\sigma_1(t,x,u)^\top + Q\sigma_2(t,r)\sigma_2(t,r)^\top\right] + \langle p, b_1(t,x,u)\rangle \\
& +\langle q, b_2(t,r)\rangle + g(t_i^*, x_{t_i^*}, r_{t_i^*}; t, x, r, u), \\
& \forall (t,x,r,u,p,P,q,Q) \in [t_i^*, t_{i+1}^*] \times \mathbb{R}^n \times \mathbb{R}^n \times U \times \mathbb{R}^n \times \mathcal{S}^n \times \mathbb{R}^n \times \mathcal{S}^n.
\end{aligned}
\tag{4.36}
$$

Then we obtain the Hamilton-Jacobi-Bellman equations for our value function:

Theorem 4.4.2. *Suppose the value function $V(t_i^*, x_{t_i^*}, r_{t_i^*}; \cdot, \cdot, \cdot)$ is smooth on each sub-period for any given $(t_i^*, x_{t_i^*}, r_{t_i^*})$. Then the following system is satisfied:*

$$
\begin{cases}
V_t(t_l^*, x_{t_l^*}, r_{t_l^*}; t, x, r) + \inf_{u \in U} \Big\{ H(t_l^*, x_{t_l^*}, r_{t_l^*}; t, x, r, u, V_x(t_l^*, x_{t_l^*}, r_{t_l^*}; t, x, r), \\
\qquad V_{xx}(t_l^*, x_{t_l^*}, r_{t_l^*}; t, x, r), V_r(t_l^*, x_{t_l^*}, r_{t_l^*}; t, x, r), \\
\qquad V_{rr}(t_l^*, x_{t_l^*}, r_{t_l^*}; t, x, r)) \Big\} = 0, \\
\qquad\qquad (t, x, r) \in [t_l^*, t_l^* + T_l] \times \mathbb{R}^n \times \mathbb{R}^n, \\
V(t_l^*, x_{t_l^*}, r_{t_l^*}; t_l^* + T_l, x, r) = h(t_l^*, x_{t_l^*}, r_{t_l^*}; x, r), \quad (x, r) \in \mathbb{R}^n \times \mathbb{R}^n.
\end{cases}
\tag{4.37}
$$

$$
\begin{cases}
V_t(t_i^*, x_{t_i^*}, r_{t_i^*}; t, x, r) + \inf_{u \in U} \Big\{ H(t_i^*, x_{t_i^*}, r_{t_i^*}; t, x, r, u, V_x(t_l^*, x_{t_l^*}, r_{t_l^*}; t, x, r), \\
\qquad V_{xx}(tt_i^*, x_{t_i^*}, r_{t_i^*}; t, x, r), V_r(t_i^*, x_{t_i^*}, r_{t_i^*}; t, x, r), \\
\qquad V_{rr}(t_i^*, x_{t_i^*}, r_{t_i^*}; t, x, r)) \Big\} = 0, \\
\qquad\qquad (t, x, r) \in [t_i^*, t_{i+1}^*] \times \mathbb{R}^n \times \mathbb{R}^n, \\
V(t_i^*, x_{t_i^*}, r_{t_i^*}; t_{i+1}^*, x, r) = N[V](t_i^*, x_{t_i^*}, r_{t_i^*}; t_{i+1}^*, x, r), \quad (x, r) \in \mathbb{R}^n \times \mathbb{R}^n.
\end{cases}
\tag{4.38}
$$

Proof. Let us first prove that $V(t_l^*, x_{t_l^*}, r_{t_l^*}; \cdot, \cdot, \cdot)$ satisfies (4.37), then for $V(t_i^*, x_{t_i^*}, r_{t_i^*}; \cdot, \cdot, \cdot)$ satisfies (4.38) can be proved in the similar way. Fix $(t, x, r) \in [t_l^*, t_l^* + T_l] \times \mathbb{R}^n \times \mathbb{R}^n$ and $u \in U$.

Let $x(\cdot)$ be the state trajectory corresponding to the control $u(\cdot) \in \mathcal{U}[t, t_l^* + T_l]$ with $u(t) \equiv u$. By (4.33) with $\hat{t} \downarrow t$ and Itô's formula, we obtain

$$\begin{aligned}
0 \leqslant\ & \frac{\mathbb{E}\{V(t_l^*, x_{t_l^*}, r_{t_l^*}; \hat{t}, X(\hat{t}; t, x), r(\hat{t}; t, r)) - V(t_l^*, x_{t_l^*}, r_{t_l^*}; t, x, r)\}}{\hat{t} - t} \\
& + \frac{1}{\hat{t} - t} \int_t^{\hat{t}} g(t_l^*, x_{t_l^*}, r_{t_l^*}; s, X(s; t, x), r(s; t, r); u(s; t, u)) ds \\
= & \frac{1}{\hat{t} - t} \int_t^{\hat{t}} V_t(t_l^*, x_{t_l^*}, r_{t_l^*}; s, X(s; t, x), r(s, t, r)) + H(t_l^*, x_{t_l^*}, r_{t_l^*}; s, X(s; t, x), r(s; t, r), u(s), \\
& V_x(t_l^*, x_{t_l^*}, r_{t_l^*}; s, X(s; t, x), r(s, t, r)), V_{xx}(t_l^*, x_{t_l^*}, r_{t_l^*}; s, X(s; t, x), r(s; t, r)), \\
& V_r(t_l^*, x_{t_l^*}, r_{t_l^*}; s, X(s; t, x), r(s; t, r)), V_{rr}(t_l^*, x_{t_l^*}, r_{t_l^*}; s, X(s; t, x), r(s; t, r))) ds \\
\to & V_t(t_l^*, x_{t_l^*}, r_{t_l^*}; t, x, r) + H(t_l^*, x_{t_l^*}, r_{t_l^*}; t, x, r, u, V_x(t_l^*, x_{t_l^*}, r_{t_l^*}; t, x, r), V_{xx}(t_l^*, x_{t_l^*}, r_{t_l^*}; t, x, r), \\
& V_r(t_l^*, x_{t_l^*}, r_{t_l^*}; t, x, r), V_{rr}(t_l^*, x_{t_l^*}, r_{t_l^*}; t, x, r)), \qquad \forall u \in U.
\end{aligned}$$

This results in

$$\begin{aligned}
0 \leqslant V_t(t_l^*, x_{t_l^*}, r_{t_l^*}; t, x, r) + \inf_{u \in U} \Big\{ & H(t_l^*, x_{t_l^*}, r_{t_l^*}; t, x, r, u, V_x(t_l^*, x_{t_l^*}, r_{t_l^*}; t, x, r), \\
& V_{xx}(t_l^*, x_{t_l^*}, r_{t_l^*}; t, x, r), V_r(t_l^*, x_{t_l^*}, r_{t_l^*}; t, x, r), V_{rr}(t_l^*, x_{t_l^*}, r_{t_l^*}; t, x, r)) \Big\}.
\end{aligned}$$

On the other hand, for any $\varepsilon > 0$, $t_l^* \leqslant t \leqslant \hat{t} \leqslant t_l^* + T_l$ with $\hat{t} - t > 0$ small enough, there exists a $u(\cdot) \equiv u_{\varepsilon, \hat{t}}(\cdot) \in \mathcal{U}[t, t_l^* + T_l]$ such that

$$\begin{aligned}
& V_t(t_l^*, x_{t_l^*}, r_{t_l^*}; t, x, r) + \varepsilon(\hat{t} - t) \\
\geqslant\ & \mathbb{E}\Big\{ \int_t^{\hat{t}} g(t_l^*, x_{t_l^*}, r_{t_l^*}; s, X(s; t, x), r(s; t, r); u(s)) ds + V(t_l^*, x_{t_l^*}, r_{t_l^*}; \hat{t}, X(\hat{t}; t, x), r(\hat{t}; t, r)) \Big\}.
\end{aligned}$$

Thus, it follows from Itô's formula that as $\hat{t} \downarrow t$,

$$
\begin{aligned}
\varepsilon \geqslant\ & \frac{\mathbb{E}\{V(t_l^*, x_{t_l^*}, r_{t_l^*}; \hat{t}, X(\hat{t}; t, x), r(\hat{t}; t, r)) - V(t_l^*, x_{t_l^*}, r_{t_l^*}; t, x, r)\}}{\hat{t} - t} \\
& + \frac{1}{\hat{t} - t} \int_t^{\hat{t}} g(t_l^*, x_{t_l^*}, r_{t_l^*}; s, X(s; t, x), r(s; t, r); u(s; t, u)) ds \\
=\ & \frac{1}{\hat{t} - t} \int_t^{\hat{t}} V_t(t_l^*, x_{t_l^*}, r_{t_l^*}; s, X(s; t, x), r(s, t, r)) + H(t_l^*, x_{t_l^*}, r_{t_l^*}; s, X(s; t, x), r(s; t, r), u(s), \\
& V_x(t_l^*, x_{t_l^*}, r_{t_l^*}; s, X(s; t, x), r(s, t, r)), V_{xx}(t_l^*, x_{t_l^*}, r_{t_l^*}; s, X(s; t, x), r(s; t, r)), \\
& V_r(t_l^*, x_{t_l^*}, r_{t_l^*}; s, X(s; t, x), r(s; t, r)), V_{rr}(t_l^*, x_{t_l^*}, r_{t_l^*}; s, X(s; t, x), r(s; t, r))) ds \\
\geqslant\ & \frac{1}{\hat{t} - t} \int_t^{\hat{t}} V_t(t_l^*, x_{t_l^*}, r_{t_l^*}; s, X(s; t, x), r(s, t, r)) + \inf_{u \in U} H(t_l^*, x_{t_l^*}, r_{t_l^*}; s, X(s; t, x), r(s; t, r), u(s), \\
& V_x(t_l^*, x_{t_l^*}, r_{t_l^*}; s, X(s; t, x), r(s, t, r)), V_{xx}(t_l^*, x_{t_l^*}, r_{t_l^*}; s, X(s; t, x), r(s; t, r)), \\
& V_r(t_l^*, x_{t_l^*}, r_{t_l^*}; s, X(s; t, x), r(s; t, r)), V_{rr}(t_l^*, x_{t_l^*}, r_{t_l^*}; s, X(s; t, x), r(s; t, r))) ds \\
\to\ & V_t(t_l^*, x_{t_l^*}, r_{t_l^*}; t, x, r) + \inf_{u \in U} \Big\{ H(t_l^*, x_{t_l^*}, r_{t_l^*}; t, x, r, u, V_x(t_l^*, x_{t_l^*}, r_{t_l^*}; t, x, r), \\
& V_{xx}(t_l^*, x_{t_l^*}, r_{t_l^*}; t, x, r), V_r(t_l^*, x_{t_l^*}, r_{t_l^*}; t, x, r), V_{rr}(t_l^*, x_{t_l^*}, r_{t_l^*}; t, x, r)) \Big\}.
\end{aligned}
$$

Thus we obtain our conclusion. With same idea, we obtain the same conclusion on each sub-periods. □

4.5 Viscosity Solution

Since the value function $V(t_i^*, x_{t_i^*}, r_{t_i^*}; \cdot, \cdot, \cdot)$ is not necessarily smooth on $[t_i^*, t_{i+1}^*] \times \mathbb{R}^n \times \mathbb{R}^n$, the notion of viscosity solution (see Crandall-Lions [8, 9, 6]) is introduced as an important tool to costrtuct the solution. First we recall the definition of viscosity solution on each sub-periods.

Definition 4.5.1. *A function $V \in C([t_l^*, t_l^* + T_l] \times \mathbb{R}^n \times \mathbb{R}^n)$ is called a viscosity subsolution (resp.*

viscosity supersolution) of (4.37) *if*

$$V(t_l^*, x_{t_l^*}, r_{t_l^*}; t_l^* + T_l, x, r) \leqslant (resp. \geqslant) h(t_l^*, x_{t_l^*}, r_{t_l^*}; x, r), \quad \forall (t, x, r) \in [t_l^*, t_l^* + T_l] \times \mathbb{R}^n \times \mathbb{R}^n, \tag{4.39}$$

and for any $\varphi \in C^{1,2,2}([t_l^*, t_l^* + T_l] \times \mathbb{R}^n \times \mathbb{R}^n)$, *whenever* $V - \varphi$ *attains a local maximum (resp. minimum) at* $(t_0, x_0, r_0) \in [t_l^*, t_l^* + T_l] \times \mathbb{R}^n \times \mathbb{R}^n$, *we have*

$$\begin{aligned}&\varphi_t(t_l^*, x_{t_l^*}, r_{t_l^*}; t_0, x_0, r_0) + \inf_{u \in U} \Big\{ H(t_l^*, x_{t_l^*}, r_{t_l^*}; t, x, r, u, \varphi_x(t_l^*, x_{t_l^*}, r_{t_l^*}; t, x, r), \\ &\varphi_{xx}(t_l^*, x_{t_l^*}, r_{t_l^*}; t, x, r), \varphi_r(t_l^*, x_{t_l^*}, r_{t_l^*}; t, x, r), \varphi_{rr}(t_l^*, x_{t_l^*}, r_{t_l^*}; t, x, r)) \Big\} \geqslant 0 \quad (resp. \leqslant 0).\end{aligned} \tag{4.40}$$

A function $V \in C([t_l^*, t_l^* + T_l] \times \mathbb{R}^n \times \mathbb{R}^n)$ *s called a viscosity solution of* (4.37) *if it is both a viscosity sub- and super-solution of* (4.37).

Similarly, a function $V \in C([t_i^*, t_{i+1}^*] \times \mathbb{R}^n \times \mathbb{R}^n)$ *is called a viscosity subsolution (resp. viscosity supersolution) of* (4.38) *if*

$$V(t_i^*, x_{t_i^*}, r_{t_i^*}; t_i^* + T_i, x, r) \leqslant (resp. \geqslant) N[V](t_i^*, x_{t_i^*}, r_{t_i^*}; t_{i+1}^*, x, r), \quad \forall (t, x, r) \in [t_i^*, t_{i+1}^*] \times \mathbb{R}^n \times \mathbb{R}^n. \tag{4.41}$$

and for any $\varphi \in C^{1,2,2}([t_i^*, t_{i+1}^*] \times \mathbb{R}^n \times \mathbb{R}^n)$, *whenever* $V - \varphi$ *attains a local maximum (resp. minimum) at* $(t_0, x_0, r_0) \in [t_i^*, t_{i+1}^*] \times \mathbb{R}^n \times \mathbb{R}^n$, *we have*

$$\begin{aligned}&\varphi_t(t_i^*, x_{t_i^*}, r_{t_i^*}; t_0, x_0, r_0) + \inf_{u \in U} \Big\{ H(t_i^*, x_{t_i^*}, r_{t_i^*}; t, x, r, u, \varphi_x(t_i^*, x_{t_i^*}, r_{t_i^*}; t, x, r), \\ &\varphi_{xx}(t_i^*, x_{t_i^*}, r_{t_i^*}; t, x, r), \varphi_r(t_i^*, x_{t_i^*}, r_{t_i^*}; t, x, r), \varphi_{rr}(t_i^*, x_{t_i^*}, r_{t_i^*}; t, x, r)) \Big\} \geqslant 0 \quad (resp. \leqslant 0).\end{aligned} \tag{4.42}$$

A function $V \in C([t_i^*, t_{i+1}^*] \times \mathbb{R}^n \times \mathbb{R}^n)$ *s called a viscosity solution of* (4.38) *if it is both a viscosity sub- and super-solution of* (4.38).

The main results of this section is the following.

Theorem 4.5.2. *Let (H1)-(H2) hold. Then the value function V is the unique viscosity solution of* (4.37) *and* (4.38) *on* $[t_i^*, t_{i+1}^*] \times \mathbb{R}^n \times \mathbb{R}^n$, *repectively.*

The proof of this theorem is standard, see Yong-Zhou [39].

CHAPTER 5: CONCLUSION AND FUTURE RESEARCH

In this dissertation, we discussed a new type optimal impulse control problem, in which running cost is dependent on the initial pairs, motivated the refinance of mortgage. Starting from the one-time refinance model, essential features of refinancing are found, and following this model we set up a general model to describe multiple times refinancing. Unlike with classical optimal impulse control problems, a backward method to construct the solution is established piecewisely. Thanks to the classical stochastic optimal control theory, the value function is determined by a set of recursive HJB equations in the sense of viscosity solution. Also, the optimal controls are determined from this solution.

A list of topics can be discussed following our results in the future. First, we expect that the optimal number of impulses and the optimal times to apply each impulses can be characterized in terms of the given coefficients of the state equation and the weighting functions in the cost functional.

Second, our study is from the borrower's point of view. It is equally interesting that how one can study it from the lender side. There is a list of literature discussing the optimal mortgage design, see Piskorski–Tchistyi [32], Piskorski–Tchistyi [33], Piskorski–Tchistyi [34], Guren–Krishnamurthy-Mcquade [16]. Note that mortgage contract is designed by the lender, and the refinance, default, prepayment, etc. strategies are decided by the borrower. Therefore, it will lead to a very challenging differential game problem.

Finally, from practical viewpoint, how one can use nowadays's popular deep learning tools to solve mortgage refinance problems numerically. We have notices that deep learning in stochastic optimal control theory attracted lots of attentions from Han–Jentzen–E [17], which introduced deep learning in solving high dimensional PDEs. Thus, we expect that by adopting newly developed tools, one will be able to solve mortgage refinance type problems more practically.

LIST OF REFERENCES

[1] G. Barles, *Deterministic Impulse Control Problems*. SIAM Journal on Control Optimization, 23(3) (1985), 419-432.

[2] A. Bensoussan and J. L. Lions, *Optimal Impulse And Continuous Control: Method Of Non-linear Quasi-variational Inequalities*. Trudy Mat. Inst. Steklov., 134 (1975), 5-22.

[3] A. Bensoussan and J. L. Lions, *Impulse Control And Quasi-Variational Inequalitie*. Bordes, Paris, 1984.

[4] A. Cadenillas and F. Zapatero, *Classical and Impulse Stochastic Control of the Exchange Rate Using Interest Rates and Reserves*. Mathematical Finance, 10(2) (2001), 141-156.

[5] M. Chang, T. Pang and J. Yong, *Optimal Stopping Problem For Stochastic Differential Equations With Random Coefficients*. SIAM Journal on Control Optimization, 48(2) (2009), 941-971.

[6] M. G. Crandall, L. C. Evans and P. L. Lions, *Some Properties Of Viscosity Solutions Of Hamilton-Jacobi Equations*. Transactions of the American Mathematical Society, 282(2) (1984), 487-502.

[7] M. G. Crandall, H. Ishii and P. L. Lions, *User's Guide to Viscosity Solutions of Second Order Partial Differential Equations*. Bulletin of the American Mathematical Society, 27(1) (1992), 1-67.

[8] M. G. Crandall, and P. L. Lions, *Viscosity Solutions Of Hamilton-Jacobi Equations*. Transactions of the American Mathematical Society, 277(1) (1983), 1-42.

[9] M. G. Crandall, and P. L. Lions, *Two Approximations Of Solutions Of Hamilton-Jacobi Equations*. Mathematics of Computation, 43(167) (1984), 1-19.

[10] A. Davidson and A. Levin, *Mortgage Valuation Models: Embedded Options, Risk and Uncertainties*. Oxford University Press, 2014.

[11] K. B. Dunn and J. J. McConnell, *A Comparison Of Alternative Models For Pricing GNMA Mortgage-Backed Securities*. The Journal of Finance, 36(2) (1981), 471-484.

[12] K. B. Dunn and J. J. McConnell, *Valuation Of GNMA Mortgage-Backed Securities*. The Journal of Finance, 36 (1981), 599-616.

[13] A. Friedman, *Optimal Stopping Problems In Stochastic Control*. SIAM Review, 21(1) (1979), 71-80.

[14] Y. Goncharov, *Mathematical Theory Of Mortgage Modeling*. Ph.D. dissertation, University of Illinois at Chicago, 2003.

[15] Y. Goncharov, *An Intensity-Based Approach To The Valuation Of Mortgage Contracts And Computation Of The Endogenous Mortgage Rate*. International Journal of Theoretical Applied Finance, 9(6) (2006), 889–914.

[16] A. Guren, A. Krishnamurthy and T. Mcquade, *Mortgage Design In An Equilibrium Model Of The Housing Market*. The Journal of Finance, 76(1) (2021), 113–168.

[17] J. Han, A. Jentzen and W. E, *Solving High-Dimensional Partial Differential Equations Using Deep Learning*. National Academy of Sciences, 115(34) (2018), 8505-8510.

[18] S. Hu, *Existence And Uniqueness Of Viscosity Solutions For Nonlinear Variational Inequalities Associated With Mixed Control*. Chinese Annals of Mathematics, Series B, 41(5) (2020), 793-820.

[19] H. Ishii, *Uniqueness Of Unbounded Viscosity Solution Of Hamilton-Jacob Equations*. Indiana University Mathematics Journal, 33(5) (1984), 721-748.

[20] A. Kalotay, D. Yang and F. J. Fabozzi, *An Option-Theoretic Prepayment Model For Mortgages And Mortgage-Backed Securities*. International Journal of Theoretical and Applied Finance, 7(8) (2004), 949–978.

[21] I. Karatzas, and S. Shreve, *Brownian Motion And Stochastic Calculus*. Springer-Verlag, New York, 1998.

[22] I. Karatzas, and S. Shreve, *Methods Of Mathematical Finance*. Springer-Verlag, New York, 1998.

[23] A. Khandani, A. Lo and R. Merton, *Systemic Risk And The Refinancing Ratchet Effect*. Journal of Financial Economics, 108(1) (2013), 29-45.

[24] A. Korn, *Some Applications Of Impulse Control In Mathematical Finance*. Mathematical Methods of Operations Research, 50 (1999), 493-518.

[25] C. Li, J. Yong, *A Finite Horizon Optimal Stochastic Impulse Control Problem With A Decision Lag*. Dynamics of Continuous, Discrete and Impulsive Systems Series B: Applications Algorithms, 28 (2021) , 89-123.

[26] J. Ma and J. Yong, *Dynamic Programming For Multidimensional Stochastic Control Problems*. Acta Mathematica Sinica, 15(4) (1999), 485-506.

[27] C. Mayer, T. Piskorski and A. Tchistyi, *The Inefficiency Of Refinancing: Why Prepayment Penalties Are Good For Risky Borrowers*. Journal of Financial Economics, 107(3) (2013), 694-714.

[28] B. Øksendal, *Stochastic Differential Equation*. Springer-Verlag, Berlin Heidelberg, 2005.

[29] B. Øksendal and K. Reikvam, *Viscosity Solutions Of Optimal Stopping Problems*. Stochastic and Stochastic Report, 62 (1998), 285-301.

[30] B. Øksendal and A. Sulem, *Optimal Consumption And Portfolio With Both Fixed And Proportional Transaction Costs*. SIAM Journal on Control and Optimization, 40(6) (2002), 1765-1790.

[31] H. Pham, *Continuous-Time Stochastic Control And Optimization With Financial Applications*. Springer-Verlag, Berlin Heidelberg, 2009.

[32] T. Piskorski and A. Tchistyi, *Optimal Mortgage Design*. The Review of Financial Studies, 23(8) (2010), 3098–3140.

[33] T. Piskorski and A. Tchistyi, *Stochastic House Appreciation And Optimal Mortgage Lending*. The Review of Financial Studies, 24(5) (2011), 1407-1446.

[34] T. Piskorski and A. Tchistyi, *An Equilibrium Model Of Housing And Mortgage Markets With State-Contingent Lending Contracts*. NBER Working Paper No. 23452.

[35] S. Pliska, *Mortgage Valuation And Optimal Refinancing*. Stochastic Finance, Springer-Verlag, Boston, 1999, 183-196.

[36] S. Tang, and J. Yong, *Finite Horizon Stochastic Optimal Switching And Impulse Controls With A Viscosity Solution Approach*. Stochastics and Stochastic Reports, 45(3-4) (1993), 145-176.

[37] J. Yong, *Systems Governed By Ordinary Differential Equations With Continuous, Switching And Impulse Controls*. Applied Mathematics and Optimization, 20 (1989), 223-235.

[38] J. Yong, *Zero-Sum Differential Games Involving Impulse Controls*. Applied Mathematics and Optimization, 29 (1994), 243-261.

[39] J. Yong, and X. Y. Zhou, *Stochastic Controls:Hamiltonian Systems And HJB Equations*. Springer-Verlag, New York, 1999.