ЛОГИЧЕСКОЕ ПРОЕКТИРОВАНИЕ ДИСКРЕТНЫХ АВТОМАТОВ

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SYNTHESIS OF COMBINATIONAL CIRCUITS BY MEANS OF BI-DECOMPOSITION OF BOOLEAN FUNCTIONS

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The problem of combinational circuits synthesis in the basis of two-input gates is considered. Those gates are AND, OR, NAND and NOR. A method for solving this problem by means of Boolean functions bi-decomposition is suggested. The method reduces the problem to the search for a weighted two-block cover of the orthogonality graph of ternary matrice rows representing the given Boolean function by complete bipartite subgraphs (bi-cliques). Each bi-clique in the obtained cover is assigned in a certain way with a set of variables that are the arguments of the function. This set is the weight of the bi-clique. Each of those bi-cliques defines a Boolean function whose arguments are the variables assigned to it. The functions obtained in such a way constitute the required decomposition. The process of combinational circuit synthesis consists in successively applying bi-decomposition to the functions obtained. The method for two-block covering the orthogonality graph of ternary matrice rows is described.

Keywords: *synthesis of combinational circuits, Boolean function, decomposition of Boolean functions, ternary matrix, complete bipartite subgraph, two-block cover.*

СИНТЕЗ КОМБИНАЦИОННЫХ СХЕМ ПУТЁМ АЛГЕБРАИЧЕСКОЙ ДЕКОМПОЗИЦИИ БУЛЕВЫХ ФУНКЦИЙ

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Рассматривается задача синтеза комбинационных схем в базисе двухвходовых элементов И, ИЛИ, И–НЕ и ИЛИ–НЕ. Предложен метод её решения с помощью применения алгебраической декомпозиции булевых функций. Метод сводит решение задачи к поиску взвешенного двублочного покрытия полными двудольными подграфами (бикликами) графа ортогональности строк троичной матрицы, представляющей заданную булеву функцию. Каждой биклике в полученном покрытии определённым образом приписывается в качестве веса множество переменных, являющихся аргументами заданной функции. Каждая из этих двух биклик определяет булеву функцию с аргументами, приписанными соответствующей биклике. Полученные таким образом функции составляют искомое разложение. Процесс синтеза комбинационной схемы состоит из последовательного применения алгебраической декомпозиции к получаемым функциям. Описан способ получения двублочного покрытия бикликами графа ортогональности строк троичной матрицы.

Ключевые слова: *синтез комбинационных схем, булева функция, декомпозиция булевых функций, троичная матрица, полный двудольный подграф, двублочное покрытие.*

1. Introduction

The problem of bi-decomposition of a Boolean function is set as follows. Given a Boolean function $y = f(\mathbf{x})$, where the components of the vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ are Boolean variables constituting a set X, a superposition $f(\mathbf{x}) = \varphi(g_1(\mathbf{z}_1), g_2(\mathbf{z}_2))$ must be obtained, where the components of the vectors $\mathbf{z}_1$ and $\mathbf{z}_2$ are the variables from the sets $Z_1 \subset X$ and $Z_2 \subset X$ respectively. The kind of the function φ in two variables is given as well. It can be any of the ten Boolean functions which essentially depend on both arguments and are represented by the operations of logic algebra. Usually, the sets Z_1 and Z_2 are given and $Z_1 \cap Z_2 = \varnothing$. Such a decomposition is called *disjoint*, otherwise it is called *non-disjoint*, where the condition $Z_1 \cap Z_2 = \varnothing$ is optional, but some restrictions on the cardinalities of Z_1 and Z_2 can be imposed.

There are known examples of applying methods for bi-decomposition to reduce the delay of combinational circuits [1, 2] and in the synthesis of circuits in the base of FPGA [3]. The problem of bi-decomposition with φ expressed by XOR operation and given partition (Z_1, Z_2) has been considered in [4], where the logical equations are used. The probability of existence of any decomposition of a completely specified Boolean function is very low, but there is another situation with incompletely specified (partial) functions, especially when the domain of their specification is a very small part of Boolean space of arguments. Therefore, the main attention is paid to the decomposition (including bi-decomposition) of partial Boolean functions. Such a case of disjoint bi-decomposition with a given partition (Z_1, Z_2) has been investigated in detail in [5]. A method for bi-decomposition (disjoint or non-disjoint) of partial Boolean functions with non-given partition (Z_1, Z_2) is described in [6], where only the demand is made that the numbers of arguments of g_1 and g_2 be less than the number of arguments of f. This method can be applied also for completely specified functions, but as it was said above, the probability that the mentioned demand can be fulfilled is very low. At the same time, if φ is in the class of non-linear Boolean functions, then the functions g_1 and g_2 turn out to be simpler than f in the sense that the amount of their dependence on some arguments is less than that of f. This parameter has been considered in [7]. The amount of dependence of f on x_i is the number of pairs $(\mathbf{x}^*, \mathbf{x}^{**})$ of adjacent values of the vector $\mathbf{x}$ with different values of x_i, where $f(\mathbf{x}^*) \neq f(\mathbf{x}^{**})$. Moreover, if g_i $(i = 1, 2)$ has the same number of arguments as the completely specified function f, then g_i will be partial in any case. This increases the probability of its decomposability.

In this paper, we propose a method for synthesizing combinational circuits based on two-input gates that implement nonlinear Boolean functions. These gates are NOR, NAND and OR, AND with variable complements available. The method is based on successive application of bi-decomposition to the functions using the approach described in [6].

2. The proposed approach

Let a Boolean function $f(\mathbf{x})$ (completely or partially specified) be given by two sets: M^1 is a domain of Boolean space, where it has value 1, and M^0 is a domain of Boolean space, where it has value 0. We represent these sets by ternary matrices $\mathbf{M}_1$ and $\mathbf{M}_0$, respectively, whose rows represent the intervals in M^1 and M^0, and columns correspond to arguments $x_1, x_2, \ldots, x_n$ of the given function.

Let us consider a complete bipartite graph $G = (V^1, V^0, E)$ whose vertices from V^1 correspond to the rows of $\mathbf{M}_1$ and vertices from V^0 correspond to the rows of $\mathbf{M}_0$. The edges of G are all the pairs of vertices v^1v^0 ($v^1 \in V^1$, $v^0 \in V^0$) corresponding to orthogonal rows of the matrices. Two ternary vectors are orthogonal according to a component x_i if $x_i = 1$ in one of them and $x_i = 0$ in the other [8]. Naturally, any row-vector $\mathbf{m}^1$ of $\mathbf{M}^1$ is orthogonal to any row-vector $\mathbf{m}^0$ of $\mathbf{M}^0$. So the bipartite graph G is complete.

We assign the elementary disjunction $x_i \vee x_j \vee \ldots \vee x_k$ of the arguments of the given function to each edge v^1v^0 of G if the row-vectors $\mathbf{m}^1$ and $\mathbf{m}^0$ of $\mathbf{M}^1$ and $\mathbf{M}^0$ corresponding to the vertices v^1 and v^0 are orthogonal according to the components x_i, x_j, $\ldots$, x_k. Each complete bipartite subgraph (*bi-clique*) of the graph G is assigned with conjunctive normal form (CNF) having, as its terms, the elementary disjunctions assigned to the edges from that bi-clique. After removing possibly absorbed terms, we transform the obtained CNF into disjunctive normal form (DNF) and assign a term of minimal rank from the DNF to the corresponding bi-clique.

Let a Boolean function $f(\mathbf{x})$ (completely or partially specified) must be expressed as $f(\mathbf{x}) \preceq \varphi(g_1(\mathbf{z}_1), g_2(\mathbf{z}_2))$, where φ is a Boolean function in two variables, g_1 and g_2, that are a functions of vectorial variables $\mathbf{z}_1$ and $\mathbf{z}_2$ being parts of the vector $\mathbf{x}$, and symbol "$\preceq$" denotes the relation of realization. A Boolean function φ (completely or partially specified) *realizes* a partial Boolean function f if φ takes the same values as f in the entire domain of f [9]. Further, it is convenient to consider the function equality relation as a special case of the realization, and so we use the equality symbol "=" for denoting realization as well.

The functions g_1 and g_2 are constructed in the following way. Choose two bi-cliques, $B_1 = (V_1^1, V_1^0, E_1)$ and $B_2 = (V_2^1, V_2^0, E_2)$, in the graph G so that any edge of G would be at least in one of the sets E_1 or E_2. In other words, the bi-cliques B_1 and B_2 must cover the entire set E with their edges. It is sufficient to define bi-cliques B_1 and B_2 with pairs (V_1^1, V_1^0) and (V_2^1, V_2^0), because any vertex in one part of a bi-clique is connected to all the vertices in the other part with edges.

The arguments of the function g_i, $i \in \{1, 2\}$, are the variables that are assigned to the bi-clique B_i. The set M_i^1 of values of the vectorial variable $\mathbf{z}_i$ for which $g_i = 1$ consists of the parts of the vectors from M^1 or M^0 (depending on the kind of φ) that correspond to the vertices in V_i^1. The parts of these vectors are defined by the variables assigned to the bi-clique B_i, i.e., these variables are the components of the vector $\mathbf{z}_i$. Similarly, the set M_i^0 is formed from parts of the vectors that correspond to the vertices from V_i^0. Thus, each vector from M^1 or from M^0 corresponds to a pair of values of g_1 and g_2. If this pair corresponds to a vector in M^1, then it is an element of the set M_φ^1, where $\varphi = 1$. If it corresponds to a vector in M^0, then it is an element of the set M_φ^0. So the function φ is defined. Note the pairs (V_1^1, V_1^0) and (V_2^1, V_2^0) should be considered as ordered because they are related to the values of g_1 and g_2.

The described method involves the similar decomposition of g_1, g_2 and the next obtained functions until obtaining functions in two variables from the set $X = \{x_1, x_2, \ldots, x_n\}$ of the arguments of the given function.

The method for bi-decomposition of Boolean functions described in [6] is based on the finding of a cover of the graph G with two bi-cliques having the best weight that leads to the minimum sum of arguments of the superposition functions. The cover is searched for among possibly many maximal bi-cliques of G [10] which takes much time without guarantee of obtaining an optimal circuit in our case. The synthesis of combinational circuits by the proposed method involves multiple application of performing the cover task. Therefore, a heuristic method for covering is used, which does not minimize that sum, but takes less time for its realization.

3. Covering the graph G by two bi-cliques

The Table shows the values that g_1 and g_2 must have at the given values of the function φ and at the given kinds of it. It is seen that there must be $V_1^1 = V_2^1 = V^1$ for AND operation, $V_1^0 = V_2^0 = V^0$ for OR operation, $V_1^1 = V_2^1 = V^0$ for NAND operation, and $V_1^0 = V_2^0 = V^1$ for NOR operation.

AND			OR			NAND			NOR		
φ	g_1	g_2	φ	g_1	g_2	φ	g_1	g_2	φ	g_1	g_2
1	1	1	0	0	0	0	1	1	1	0	0
0	–	0	1	–	1	1	–	0	0	–	1
0	0	–	1	1	–	1	0	–	0	1	–

So one of the bi-cliques is always defined by the kind of φ as one of the parts of the complete bipartite graph G and it is one of the parts of both B_1 and B_2. The other parts of B_1 and B_2 are formed as blocks of a partition of the other part of G. For instance, if $V_1^0 = V_2^0 = V^1$, then $B_1 = (V_1^1, V^1)$ and $B_2 = (V_2^1, V^1)$, where $V_1^1 \cup V_2^1 = V^0$ and $V_1^1 \cap V_2^1 = \varnothing$.

The initial information to obtain the desired cover of G is the set of *starred graphs* that are subgraphs of G. A starred graph (or *a star*) is a complete bipartite graph $K_{1,n}$ [11]. Its one-element part is its *center*. In our case, the set of starred graphs is the set of all bi-cliques of G having one part as one-element set with $v \in V^0$ or $v \in V^1$, and the other part as V^1 or V^0, respectively. We call them *starred bi-cliques*.

As it was said above, each bi-clique is assigned with CNF that is transformed into DNF. We choose a term K of minimum rank from DNF and assign the set X_i of variables from K to the corresponding starred bi-clique B_i. Two starred bi-cliques, B_i and B_j, with the intersection $X_i \cap X_j$ of minimal cardinality are chosen among all the pairs of starred bi-cliques under consideration. If there are several variants of such pairs, the preference is given to the sets X_i and X_j of maximal cardinality. Naturally, the variant $X_i \cap X_j = \varnothing$ is desirable. The pair (B_i, B_j) is taken as the initial value of the pair of bi-cliques that must cover the graph G, and we denote it (B_1, B_2).

The subsequent process is successive extending the parts of bi-cliques B_1 and B_2 that were one-element sets by means of adding the vertices which are the centers of the considered starred bi-cliques. The sets X_1 and X_2 change correspondingly. For example, let $B_1 = (V_1^1, V_1^0)$, $B_2 = (V_2^1, V_2^0)$, $V_1^1 \cup V_2^1 = V^0$ and the set V' consists of the vertices of G which do not belong to either V_1^0 or V_2^0. Let the vertex $v_k \in V'$ be the center of a starred bi-clique B_k and V_i^0 $(i = 1, 2)$ be such a set that cardinality of $X_i \cup X_k$ differs from that of X_i or X_k minimally among all bi-cliques B_k corresponding the vertices belonging to V'. The set V_i^0 changes to $V_i^0 \cup \{v_k\}$, and the vertex v_k is removed from V'. The process comes to the end when $V' = \varnothing$. The pair (B_1, B_2) will be the desired cover of G.

4. Synthesis of a combinational circuit in the NOR basis

Let's build a combinational circuit of NOR gates that implements the completely specified Boolean function $f(x_1, x_2, x_3, x_4, x_5)$. The function is given by the following matrices (through numeration is used):

$$\mathbf{M}^1 = \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{matrix} \\ \left[\begin{matrix} - & - & 1 & 1 & - \\ 0 & - & 1 & - & - \\ - & 1 & - & 1 & - \\ 0 & 0 & - & - & - \\ - & 1 & 0 & - & 0 \\ 1 & 1 & - & - & 1 \\ 1 & - & 0 & 0 & 1 \end{matrix}\right] \end{array} \begin{matrix} \mathit{1} \\ \mathit{2} \\ \mathit{3} \\ \mathit{4} \\ \mathit{5} \\ \mathit{6} \\ \mathit{7} \end{matrix}; \quad \mathbf{M}^0 = \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{matrix} \\ \left[\begin{matrix} 0 & 1 & 0 & 0 & 1 \\ 1 & - & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & - \\ 1 & 0 & 1 & 0 & - \end{matrix}\right] \end{array} \begin{matrix} \mathit{8} \\ \mathit{9} \\ \mathit{10} \\ \mathit{11} \end{matrix}.$$

To reduce the size the bipartite graphs, it would be better to represent the domain of the function by the minimum number of intervals. The complete bipartite graph $G = (V^1, V^0, E)$ is given by matrix $\mathbf{G}$ similar to an adjacency matrix:

$$\mathbf{G} = \begin{array}{c} \begin{matrix} \mathit{8} & \mathit{9} & \mathit{10} & \mathit{11} \end{matrix} \\ \left[\begin{matrix} x_3 \vee x_4 & x_4 & x_3 & x_4 \\ x_3 & x_1 & x_1 \vee x_3 & x_1 \\ x_4 & x_4 & x_2 & x_2 \vee x_4 \\ x_2 & x_1 & x_1 & x_1 \\ x_5 & x_3 & x_2 & x_2 \vee x_3 \\ x_1 & x_5 & x_2 & x_2 \\ x_1 & x_3 \vee x_5 & x_4 & x_3 \end{matrix}\right] \end{array} \begin{matrix} \mathit{1} \\ \mathit{2} \\ \mathit{3} \\ \mathit{4} \\ \mathit{5} \\ \mathit{6} \\ \mathit{7} \end{matrix}.$$

The rows of $\mathbf{G}$ correspond to the vertices in the set $V^1 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ (to the rows of $\mathbf{M}^1$), and the columns — to the vertices in the set $V^0 = \{v_8, v_9, v_{10}, v_{11}\}$ (to the rows of $\mathbf{M}^0$). The elementary disjunction or single variable assigned to the edge $v_i v_j$ is at the i-th row and j-th column of $\mathbf{G}$.

The bi-cliques $B_1 = (V_1^1, V_1^0)$ and $B_2 = (V_2^1, V_2^0)$ covering the graph G have a common part. According to Table on page 98, for the NOR basis, we have $V_1^0 = V_2^0 = V^1$. The starred bi-cliques with the assigned variables are the following:

$$\begin{array}{lcl} (\{v_8\}, \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}) & - & x_1x_2x_3x_4x_5; \\ (\{v_9\}, \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}) & - & x_1x_3x_4x_5; \\ (\{v_{10}\}, \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}) & - & x_1x_2x_3x_4; \\ (\{v_{11}\}, \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}) & - & x_1x_2x_3x_4. \end{array}$$

The first step, decomposition into functions g_1 and g_2, can be easy made because to reduce the total number of arguments one should form bi-cliques $B_1 = (\{v_8, v_9\}, \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\})$ and $B_2 = (\{v_{10}, v_{11}\}, \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\})$ with corresponding sets of variables $\{x_1, x_2, x_3, x_4, x_5\}$ and $\{x_1, x_2, x_3, x_4\}$. The given function $f(x_1, x_2, x_3, x_4, x_5)$ is decomposed into two functions, $g_1(x_1, x_2, x_3, x_4, x_5)$ and $g_2(x_1, x_2, x_3, x_4)$, linked by NOR operation (Pierce function): $f = \varphi = g_1 \uparrow g_2$. They can be given by the matrices $\mathbf{M}_1^1$, $\mathbf{M}_1^0$ and $\mathbf{M}_2^1$, $\mathbf{M}_2^0$, the lower indices of which coincide with the indices of the functions. The matrices look as follows, where $\mathbf{M}_2^0$ represents the minimum number of intervals with value 0 of g_2:

$$\mathbf{M}_1^1 = \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \left[\begin{array}{ccccc} 0 & 1 & 0 & 0 & 1 \\ 1 & - & 1 & 0 & 0 \end{array}\right] \end{array} \begin{array}{l} \\ \mathit{1} \\ \mathit{2} \end{array}; \qquad \mathbf{M}_1^0 = \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \left[\begin{array}{ccccc} - & - & 1 & 1 & - \\ - & 1 & - & 1 & - \\ 0 & 0 & - & - & - \\ - & 1 & 0 & - & 0 \\ 1 & - & 0 & 0 & 1 \\ 0 & - & - & - & 0 \\ - & 1 & 1 & - & 1 \end{array}\right] \end{array} \begin{array}{l} \\ \mathit{3} \\ \mathit{4} \\ \mathit{5} \\ \mathit{6} \\ \mathit{7} \\ \mathit{8} \\ \mathit{9} \end{array};$$

$$\mathbf{M}_2^1 = \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \left[\begin{array}{cccc} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{array}\right] \end{array} \begin{array}{l} \\ \mathit{1} \\ \mathit{2} \end{array}; \qquad \mathbf{M}_2^0 = \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \left[\begin{array}{cccc} - & - & 1 & 1 \\ - & - & 0 & 0 \\ 0 & - & - & - \\ - & 1 & - & - \end{array}\right] \end{array} \begin{array}{l} \\ \mathit{3} \\ \mathit{4} \\ \mathit{5} \\ \mathit{6} \end{array}.$$

The function g_2 is specified in the entire Boolean argument space, and g_1 is a partial one. The value of g_1 is not defined at $(x_1, x_2, x_3, x_4, x_5) = (1, 0, 1, 0, 1), (1, 0, 0, 0, 0)$ and the interval represented by vector $(1, 0, 0, 1, -)$. For the decomposition of g_1 and g_2, the complete bipartite graphs G_1 and G_2 are constructed with the parts $V^{11} = \{v_1^1, v_2^1\}$, $V^{01} = \{v_3^1, v_4^1, v_5^1, v_6^1, v_7^1, v_8^1, v_9^1\}$ and $V^{12} = \{v_1^2, v_2^2\}$, $V^{02} = \{v_3^2, v_4^2, v_5^2, v_6^2\}$, respectively. The graphs G_1 and G_2 are given by the matrices $\mathbf{G}_1$ and $\mathbf{G}_2$:

$$\mathbf{G}_1 = \begin{array}{c} \begin{array}{ccccccc} \mathit{3} & \mathit{4} & \mathit{5} & \mathit{6} & \mathit{7} & \mathit{8} & \mathit{9} \end{array} \\ \left[\begin{array}{ccccccc} x_3 \vee x_4 & x_4 & x_2 & x_5 & x_1 & x_5 & x_3 \\ x_4 & x_4 & x_1 & x_3 & x_3 \vee x_5 & x_1 & x_5 \end{array}\right] \end{array} \begin{array}{l} \\ \mathit{1} \\ \mathit{2} \end{array}; \quad \mathbf{G}_2 = \begin{array}{c} \begin{array}{cccc} \mathit{3} & \mathit{4} & \mathit{5} & \mathit{6} \end{array} \\ \left[\begin{array}{cccc} x_3 & x_4 & x_1 & x_2 \\ x_4 & x_3 & x_1 & x_2 \end{array}\right] \end{array} \begin{array}{l} \\ \mathit{1} \\ \mathit{2} \end{array}.$$

The starred bi-cliques of G_1 (left) and G_2 (right) with assigned variables are

$$\begin{array}{lcl} (\{v_3^1\}, \{v_1^1, v_2^1\}) & - & x_4; \\ (\{v_4^1\}, \{v_1^1, v_2^1\}) & - & x_4; \\ (\{v_5^1\}, \{v_1^1, v_2^1\}) & - & x_1x_2; \\ (\{v_6^1\}, \{v_1^1, v_2^1\}) & - & x_3x_5; \\ (\{v_7^1\}, \{v_1^1, v_2^1\}) & - & x_1(x_3 \vee x_5); \\ (\{v_8^1\}, \{v_1^1, v_2^1\}) & - & x_1x_5; \\ (\{v_9^1\}, \{v_1^1, v_2^1\}) & - & x_3x_5; \end{array} \qquad \begin{array}{lcl} (\{v_3^2\}, \{v_1^2, v_2^2\}) & - & x_3x_4; \\ (\{v_4^2\}, \{v_1^2, v_2^2\}) & - & x_3x_4; \\ (\{v_5^2\}, \{v_1^2, v_2^2\}) & - & x_1; \\ (\{v_6^2\}, \{v_1^2, v_2^2\}) & - & x_2. \end{array}$$

The function g_2 is decomposed trivially as the given function f. The bi-cliques $(\{v_3^2, v_4^2\}, \{v_1^2, v_2^2\})$ with variables x_3, x_4 and $(\{v_5^2, v_6^2\}, \{v_1^2, v_2^2\})$ with variables x_1, x_2 cover the graph G_2. So $g_2 = g_3(x_3, x_4) \uparrow g_4(x_1, x_2)$, and g_3 and g_4 are given by the following matrices obtained from $\mathbf{M}_2^1$ and $\mathbf{M}_2^0$:

$$\mathbf{M}_3^1 = \begin{array}{c} \begin{array}{cc} x_3 & x_4 \end{array} \\ \left[\begin{array}{cc} 1 & 1 \\ 0 & 0 \end{array}\right] \end{array}; \quad \mathbf{M}_3^0 = \begin{array}{c} \begin{array}{cc} x_3 & x_4 \end{array} \\ \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right] \end{array}; \quad \mathbf{M}_4^1 = \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \left[\begin{array}{cc} 0 & - \\ - & 1 \end{array}\right] \end{array}; \quad \mathbf{M}_4^0 = \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \left[\begin{array}{cc} 1 & 0 \end{array}\right] \end{array}.$$

Finally, for g_2, we have $g_3 = x_3x_4 \vee \overline{x}_3\overline{x}_4 = (x_3 \uparrow \overline{x_4}) \uparrow (\overline{x}_3 \uparrow x_4)$ and $g_4 = \overline{x}_1 \vee x_2 = (\overline{x}_1 \uparrow x_2) \uparrow (\overline{x}_1 \uparrow x_2)$.

To decompose g_1, the way to cover graph G_1 by two bi-cliques described in Section 3 can be applied. The initial meanings of bi-cliques B_1 and B_2 are $(\{v_5^1\}, \{v_1^1, v_2^1\})$ and $(\{v_6^1\}, \{v_1^1, v_2^1\})$, because the intersection of $X_1 = \{x_1, x_2\}$ and $X_2 = \{x_3, x_5\}$ is empty and these sets have the maximum cardinality. As a result of the next step, we have

$$(\{v_5^1\}, \{v_1^1, v_2^1\}) - x_1x_2, \qquad (\{v_6^1, v_9^1\}, \{v_1^1, v_2^1\}) - x_3x_5.$$

The sequence of transformations of B_1 and B_2 is presented below, where the last row presents the desired cover of G:

$$\begin{array}{llll}
(\{v_3^1, v_5^1\}, \{v_1^1, v_2^1\}) & - \; x_1x_2x_4; & (\{v_6^1, v_9^1\}, \{v_1^1, v_2^1\}) & - \; x_3x_5; \\
(\{v_3^1, v_5^1\}, \{v_1^1, v_2^1\}) & - \; x_1x_2x_4; & (\{v_6^1, v_7^1, v_9^1\}, \{v_1^1, v_2^1\}) & - \; x_1x_3x_5; \\
(\{v_3^1, v_4^1, v_5^1\}, \{v_1^1, v_2^1\}) & - \; x_1x_2x_4; & (\{v_6^1, v_7^1, v_8^1, v_9^1\}, \{v_1^1, v_2^1\}) & - \; x_1x_3x_5.
\end{array}$$

The $g_1(x_1, x_2, x_3, x_4, x_5)$ is decomposed into two functions, $g_5(x_1, x_2, x_4)$ and $g_6(x_1, x_3, x_5)$, linked by NOR operation: $g_1 = g_5 \uparrow g_6$. They can be given by the matrices $\mathbf{M}_5^1$, $\mathbf{M}_5^0$ and $\mathbf{M}_6^1$, $\mathbf{M}_6^0$:

$$\mathbf{M}_5^1 = \overset{\begin{matrix} x_1 & x_2 & x_4 \end{matrix}}{\begin{bmatrix} - & - & 1 \\ 0 & 0 & - \end{bmatrix}} \begin{matrix} \mathit{1} \\ \mathit{2} \end{matrix}; \quad \mathbf{M}_5^0 = \overset{\begin{matrix} x_1 & x_2 & x_4 \end{matrix}}{\begin{bmatrix} 0 & 1 & 0 \\ 1 & - & 0 \end{bmatrix}} \begin{matrix} \mathit{3} \\ \mathit{4} \end{matrix}; \quad \mathbf{M}_6^1 = \overset{\begin{matrix} x_1 & x_3 & x_5 \end{matrix}}{\begin{bmatrix} 1 & 0 & - \\ 0 & - & 0 \\ - & 1 & 1 \end{bmatrix}} \begin{matrix} \mathit{1} \\ \mathit{2} \\ \mathit{3} \end{matrix}; \quad \mathbf{M}_6^0 = \overset{\begin{matrix} x_1 & x_3 & x_5 \end{matrix}}{\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}} \begin{matrix} \mathit{4} \\ \mathit{5} \end{matrix}.$$

The corresponding graphs G_5 and G_6 are given by the matrices $\mathbf{G}_5$ and $\mathbf{G}_6$:

$$\mathbf{G}_5 = \overset{\begin{matrix} \mathit{3} & \mathit{4} \end{matrix}}{\begin{bmatrix} x_4 & x_4 \\ x_2 & x_1 \end{bmatrix}} \begin{matrix} \mathit{1} \\ \mathit{2} \end{matrix}; \quad \mathbf{G}_6 = \overset{\begin{matrix} \mathit{4} & \mathit{5} \end{matrix}}{\begin{bmatrix} x_1 & x_3 \\ x_5 & x_1 \\ x_3 & x_5 \end{bmatrix}} \begin{matrix} \mathit{1} \\ \mathit{2} \\ \mathit{3} \end{matrix}.$$

Graph G_5 is covered by bi-cliques $(\{v_3^5\}, \{v_1^5, v_2^5\})$ with variables x_2, x_4 and $(\{v_4^5\}, \{v_1^5, v_2^5\})$ with variables x_1, x_4. Then $g_5 = g_7(x_2, x_4) \uparrow g_8(x_1, x_4)$ and g_7 and g_8 are given by the following matrices:

$$\mathbf{M}_7^1 = \overset{\begin{matrix} x_2 & x_4 \end{matrix}}{\begin{bmatrix} 1 & 0 \end{bmatrix}}; \quad \mathbf{M}_7^0 = \overset{\begin{matrix} x_2 & x_4 \end{matrix}}{\begin{bmatrix} - & 1 \\ 0 & - \end{bmatrix}}; \quad \mathbf{M}_8^1 = \overset{\begin{matrix} x_1 & x_4 \end{matrix}}{\begin{bmatrix} 1 & 0 \end{bmatrix}}; \quad \mathbf{M}_8^0 = \overset{\begin{matrix} x_1 & x_4 \end{matrix}}{\begin{bmatrix} - & 1 \\ 0 & - \end{bmatrix}}.$$

Hence, $g_7 = \overline{x}_2 \uparrow x_4$ and $g_8 = \overline{x}_1 \uparrow x_4$.

Graph G_6 is covered by bi-cliques $(\{v_4^6\}, \{v_1^6, v_2^6, v_3^6\})$ with variables x_1, x_3, x_5 and $(\{v_5^6\}, \{v_1^6, v_2^6, v_3^6\})$ with the same variables. Then $g_6 = g_9(x_1, x_3, x_5) \uparrow g_{10}(x_1, x_3, x_5)$ and g_9 and g_{10} are given by the following matrices:

$$\mathbf{M}_9^1 = \overset{\begin{matrix} x_1 & x_3 & x_5 \end{matrix}}{\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}} \mathit{1}; \quad \mathbf{M}_9^0 = \overset{\begin{matrix} x_1 & x_3 & x_5 \end{matrix}}{\begin{bmatrix} 1 & 0 & - \\ 0 & - & 0 \\ - & 1 & 1 \end{bmatrix}} \begin{matrix} \mathit{2} \\ \mathit{3} \\ \mathit{4} \end{matrix}; \quad \mathbf{M}_{10}^1 = \overset{\begin{matrix} x_1 & x_3 & x_5 \end{matrix}}{\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}} \mathit{1}; \quad \mathbf{M}_{10}^0 = \overset{\begin{matrix} x_1 & x_3 & x_5 \end{matrix}}{\begin{bmatrix} 1 & 0 & - \\ 0 & - & 0 \\ - & 1 & 1 \end{bmatrix}} \begin{matrix} \mathit{2} \\ \mathit{3} \\ \mathit{4} \end{matrix}.$$

The corresponding graphs G_9 and G_{10} are given by the matrices

$$\mathbf{G}_9 = \overset{\begin{matrix} \mathit{2} & \mathit{3} & \mathit{4} \end{matrix}}{\begin{bmatrix} x_1 & x_5 & x_3 \end{bmatrix}} \mathit{1}; \quad \mathbf{G}_{10} = \overset{\begin{matrix} \mathit{2} & \mathit{3} & \mathit{4} \end{matrix}}{\begin{bmatrix} x_3 & x_1 & x_5 \end{bmatrix}} \mathit{1}.$$

Graph G_9 is covered by bi-cliques $(\{v_2^9, v_4^9\}, \{v_1^9\})$ with variables x_1, x_3 and $(\{v_3^9\}, \{v_1^9\})$ with x_5 that define $g_9 = g_{11} \uparrow \overline{x}_5$ and $g_{11} = x_1 \vee x_3 = (x_1 \uparrow x_3) \uparrow (x_1 \uparrow x_3)$. Graph G_{10} is covered by bi-cliques $(\{v_2^{10}, v_3^{10}\}, \{v_1^{10}\})$ and $(\{v_4^{10}\}, \{v_1^{10}\})$ with the same variables. Those bi-cliques define $g_{10} = g_{12} \uparrow x_5$ and $g_{12} = \overline{x}_1 \vee \overline{x}_3 = (\overline{x}_1 \uparrow \overline{x}_3) \uparrow (\overline{x}_1 \uparrow \overline{x}_3)$.

Now, the given function f is completely decomposed into superposition of Pierce functions in two variables. The system of functions $f, g_1, \ldots, g_{12}$ gives the circuit with NOR gates and inverters shown in Fig. 1.

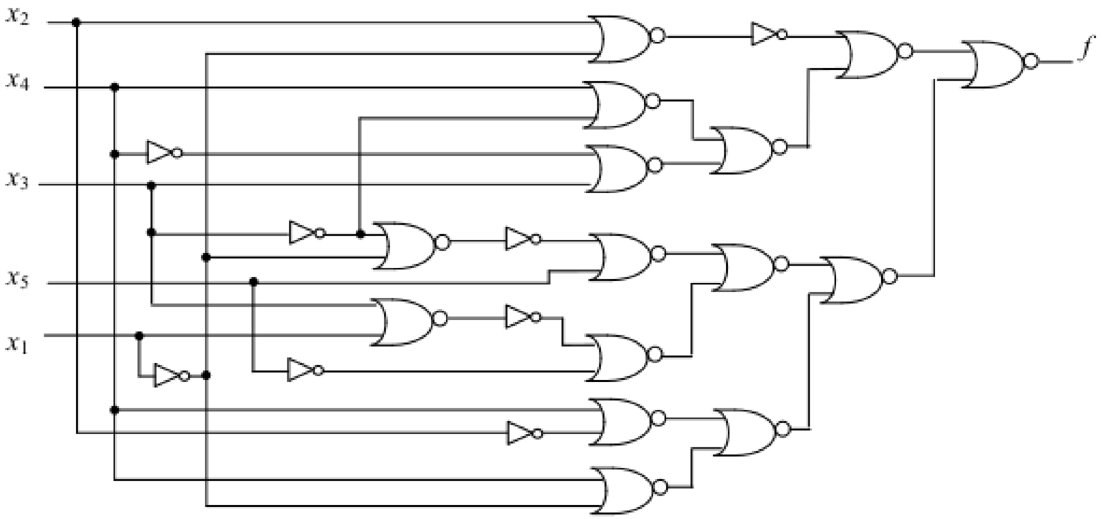

Fig. 1. Circuit with NOR gates and inverters

5. Synthesis of a combinational circuit in the AND, OR basis

We now obtain a combinational circuit that implements the same function in the basis of AND, OR gates with accessible variable complements. According to the Table in Section 3, the bi-cliques $B_1 = (V_1^1, V_1^0)$ and $B_2 = (V_2^1, V_2^0)$ covering the graph G have $V_1^1 = V_2^1 = V^1$ for AND operation and $V_1^0 = V_2^0 = V_0$ for OR operation. It can be noted in the matrix $\mathbf{G}$ that the AND operation for φ is desirable for the best variant of superposition $f = \varphi(g_1, g_2)$ if the matrix $\mathbf{M}^0$ has more rows than the matrix $\mathbf{M}^1$ has. Vice versa, the OR operation is desirable if the matrix $\mathbf{M}^1$ has more rows than the matrix $\mathbf{M}^0$ has. A variant is considered better if g_1 and g_2 have the less number of essential arguments. We choose OR operations for the output function ($f = g_1 \vee g_2$). Then, the starred bi-cliques, from which the initial meanings of B_1 and B_2 must be chosen, look as follows:

$(\{v_1\}, \{v_8, v_9, v_{10}, v_{11}\})$ – x_3x_4; $(\{v_2\}, \{v_8, v_9, v_{10}, v_{11}\})$ – x_1x_3;
$(\{v_3\}, \{v_8, v_9, v_{10}, v_{11}\})$ – x_2x_4; $(\{v_4\}, \{v_8, v_9, v_{10}, v_{11}\})$ – x_1x_2;
$(\{v_5\}, \{v_8, v_9, v_{10}, v_{11}\})$ – $x_2x_3x_5$; $(\{v_6\}, \{v_8, v_9, v_{10}, v_{11}\})$ – $x_1x_2x_5$;
$(\{v_7\}, \{v_8, v_9, v_{10}, v_{11}\})$ – $x_1x_3x_4$.

The bi-cliques $B_1 = (\{v_1, v_2, v_3, v_7\}, \{v_8, v_9, v_{10}, v_{11}\})$ with variables x_1, x_2, x_3, x_4 and $B_2 = (\{v_4, v_5, v_6\}, \{v_8, v_9, v_{10}, v_{11}\})$ with variables x_1, x_2, x_3, x_5 are obtained by the same way as in the case of NOR operation. To decompose the function f in the form $f = g_1 \vee g_2$, the following matrices are used:

$$\mathbf{M}_1^1 = \begin{array}{c} x_1\ x_2\ x_3\ x_4 \\ \left[\begin{array}{cccc} - & - & 1 & 1 \\ 0 & - & 1 & - \\ - & 1 & - & 1 \\ 1 & - & 0 & 0 \end{array}\right] \end{array} \begin{array}{l} \\ \mathit{1} \\ \mathit{2} \\ \mathit{3} \\ \mathit{4} \end{array}; \quad \mathbf{M}_1^0 = \begin{array}{c} x_1\ x_2\ x_3\ x_4 \\ \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 1 & - & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right] \end{array} \begin{array}{l} \\ \mathit{5} \\ \mathit{6} \\ \mathit{7} \end{array};$$

$$\mathbf{M}_2^1 = \begin{array}{c} x_1\ x_2\ x_3\ x_5 \\ \left[\begin{array}{cccc} 0 & 0 & - & - \\ - & 1 & 0 & 0 \\ 1 & 1 & - & 1 \end{array}\right] \end{array} \begin{array}{l} \\ \mathit{1} \\ \mathit{2} \\ \mathit{3} \end{array}; \quad \mathbf{M}_2^0 = \begin{array}{c} x_1\ x_2\ x_3\ x_5 \\ \left[\begin{array}{cccc} 0 & 1 & 0 & 1 \\ 1 & - & 1 & 0 \\ 1 & 0 & - & - \end{array}\right] \end{array} \begin{array}{l} \\ \mathit{4} \\ \mathit{5} \\ \mathit{6} \end{array}.$$

The graphs G_1 and G_2 corresponding to the functions g_1 and g_2 are given by the matrices $\mathbf{G}_1$ and $\mathbf{G}_2$:

$$\mathbf{G}_1=\begin{array}{ccc|c} \mathit{5} & \mathit{6} & \mathit{7} & \\ \hline x_3\vee x_4 & x_4 & x_3 & \mathit{1} \\ x_3 & x_1 & x_1\vee x_3 & \mathit{2} \\ x_4 & x_4 & x_2 & \mathit{3} \\ x_1 & x_3 & x_4 & \mathit{4} \end{array}\ ;\quad \mathbf{G}_2=\begin{array}{ccc|c} \mathit{4} & \mathit{5} & \mathit{6} & \\ \hline x_2 & x_1 & x_1 & \mathit{1} \\ x_5 & x_3 & x_2 & \mathit{2} \\ x_1 & x_5 & x_2 & \mathit{3} \end{array}\ .$$

For decomposition $g_1 = g_3 \vee g_4$ and $g_2 = g_5 \vee g_6$, the starred bi-cliques of G_1 and G_2 are

$$\begin{array}{lcl} (\{v_1^1\},\{v_5^1,v_6^1,v_7^1\}) & - & x_3x_4; \\ (\{v_2^1\},\{v_5^1,v_6^1,v_7^1\}) & - & x_1x_3; \\ (\{v_3^1\},\{v_5^1,v_6^1,v_7^1\}) & - & x_2x_4; \\ (\{v_4^1\},\{v_5^1,v_6^1,v_7^1\}) & - & x_1x_3x_4. \end{array}\qquad \begin{array}{lcl} (\{v_1^2\},\{v_5^2,v_6^2,v_7^2\}) & - & x_1x_2; \\ (\{v_2^2\},\{v_5^2,v_6^2,v_7^2\}) & - & x_2x_3x_5; \\ (\{v_3^2\},\{v_5^2,v_6^2,v_7^2\}) & - & x_1x_2x_5; \end{array}$$

From these bi-cliques, the pairs (B_1^1, B_2^1) and (B_1^2, B_2^2) covering G_1 and G_2 are obtained, where

$$\begin{array}{lcl} B_1^1=(\{v_1^1,v_3^1\},\{v_5^1,v_6^1,v_7^1\}) & - & x_2x_3x_4; \\ B_2^1=(\{v_2^1,v_4^1\},\{v_5^1,v_6^1,v_7^1\}) & - & x_1x_3x_4; \end{array}\quad \begin{array}{lcl} B_1^2=(\{v_1^2,v_3^2\},\{v_5^2,v_6^2,v_7^2\}) & - & x_1x_2x_5; \\ B_2^2=(\{v_2^2\},\{v_5^2,v_6^2,v_7^2\}) & - & x_2x_3x_5. \end{array}$$

The incompletely specified functions g_3, g_4, g_5 and g_6 are given by the following matrices:

$$\mathbf{M}_3^1=\begin{array}{ccc|c} x_2 & x_3 & x_4 & \\ \hline - & 1 & 1 & \mathit{1} \\ 1 & - & 1 & \mathit{2} \end{array}\ ;\quad \mathbf{M}_3^0=\begin{array}{ccc|c} x_2 & x_3 & x_4 & \\ \hline 1 & 0 & 0 & \mathit{3} \\ - & 1 & 0 & \mathit{4} \\ 0 & 0 & 1 & \mathit{5} \end{array}\ ;\quad \mathbf{M}_4^1=\begin{array}{ccc|c} x_1 & x_3 & x_4 & \\ \hline 0 & 1 & - & \mathit{1} \\ 1 & 0 & 0 & \mathit{2} \end{array}\ ;\quad \mathbf{M}_4^0=\begin{array}{ccc|c} x_1 & x_3 & x_4 & \\ \hline 0 & 0 & 0 & \mathit{3} \\ 1 & 1 & 0 & \mathit{4} \\ 1 & 0 & 1 & \mathit{5} \end{array}\ ;$$

$$\mathbf{M}_5^1=\begin{array}{ccc|c} x_1 & x_2 & x_5 & \\ \hline 0 & 0 & - & \mathit{1} \\ 1 & 1 & 1 & \mathit{2} \end{array}\ ;\quad \mathbf{M}_5^0=\begin{array}{ccc|c} x_1 & x_2 & x_5 & \\ \hline 0 & 1 & 1 & \mathit{3} \\ 1 & - & 0 & \mathit{4} \\ 1 & 0 & - & \mathit{5} \end{array}\ ;\quad \mathbf{M}_6^1=\begin{array}{ccc|c} x_2 & x_3 & x_5 & \\ \hline 1 & 0 & 0 & \mathit{1} \end{array}\ ;\quad \mathbf{M}_6^0=\begin{array}{ccc|c} x_2 & x_3 & x_5 & \\ \hline 1 & 0 & 1 & \mathit{2} \\ - & 1 & 0 & \mathit{3} \\ 0 & - & - & \mathit{4} \end{array}\ .$$

The corresponding graphs G_3, G_4, G_5 and G_6 are represented by the following matrices:

$$\mathbf{G}_3=\begin{array}{ccc|c} \mathit{3} & \mathit{4} & \mathit{5} & \\ \hline x_3\vee x_4 & x_4 & x_3 & \mathit{1} \\ x_4 & x_4 & x_2 & \mathit{2} \end{array}\ ;\quad \mathbf{G}_4=\begin{array}{ccc|c} \mathit{3} & \mathit{4} & \mathit{5} & \\ \hline x_3 & x_1 & x_1\vee x_3 & \mathit{1} \\ x_1 & x_3 & x_4 & \mathit{2} \end{array}\ ;$$

$$\mathbf{G}_5=\begin{array}{ccc|c} \mathit{3} & \mathit{4} & \mathit{5} & \\ \hline x_2 & x_1 & x_1 & \mathit{1} \\ x_1 & x_5 & x_2 & \mathit{2} \end{array}\ ;\quad \mathbf{G}_6=\begin{array}{ccc|c} \mathit{2} & \mathit{3} & \mathit{4} & \\ \hline x_5 & x_3 & x_2 & \mathit{1} \end{array}\ .$$

To implement the functions g_3, g_4, g_5 and g_6, take the AND operation. Then, the starred bi-cliques of the graphs G_3, G_4, G_5 and G_6 are

$$\begin{array}{llll} (\{v_1^3,v_2^3\},\{v_3^3\})-x_4; & (\{v_1^4,v_2^4\},\{v_3^4\})-x_1x_3; & (\{v_1^5,v_2^5\},\{v_3^5\})-x_1x_2; & (\{v_1^6\},\{v_2^6\})-x_5; \\ (\{v_1^3,v_2^3\},\{v_4^3\})-x_4; & (\{v_1^4,v_2^4\},\{v_4^4\})-x_1x_3; & (\{v_1^5,v_2^5\},\{v_4^5\})-x_1x_5; & (\{v_1^6\},\{v_3^6\})-x_3; \\ (\{v_1^3,v_2^3\},\{v_5^3\})-x_2x_3; & (\{v_1^4,v_2^4\},\{v_5^4\})-x_3x_4; & (\{v_1^5,v_2^5\},\{v_5^5\})-x_1x_5; & (\{v_1^6\},\{v_4^6\})-x_2. \end{array}$$

The covers of G_3, G_4, G_5 and G_6 are

$$\begin{array}{lcl} B_1^3=(\{v_1^3,v_2^3\},\{v_3^3,v_4^3\}) & - & x_4, \\ B_2^3=(\{v_1^3,v_2^3\},\{v_5^3\}) & - & x_2x_3; \end{array}\qquad \begin{array}{lcl} B_1^4=(\{v_1^4,v_2^4\},\{v_3^4,v_2^4\}) & - & x_1x_3, \\ B_2^4=(\{v_1^4,v_2^4\},\{v_5^4\}) & - & x_3x_4; \end{array}$$

$$\begin{array}{lcl} B_1^5=(\{v_1^5,v_2^5\},\{v_3^5,v_5^5\}) & - & x_1x_2, \\ B_2^5=(\{v_1^5,v_2^5\},\{v_4^5\}) & - & x_1x_5; \end{array}\qquad \begin{array}{lcl} B_1^6=(\{v_1^6\},\{v_2^6,v_3^6\}) & - & x_3x_5, \\ B_2^6=(\{v_1^6\},\{v_4^6\}) & - & x_2. \end{array}$$

These pairs define the following decompositions:

$$g_3(x_2, x_3, x_4) = g_7(x_2, x_3) \wedge x_4, \qquad g_4(x_2, x_3, x_4) = g_8(x_1, x_3) \wedge g_9(x_3, x_4),$$
$$g_5(x_1, x_2, x_5) = g_{10}(x_1, x_2) \wedge g_{11}(x_1, x_5), \qquad g_6(x_1, x_3, x_5) = g_{12}(x_3, x_5) \wedge x_2.$$

The functions $g_7, g_8, g_9, g_{10}, g_{11}$ and g_{12} are given by the following matrices:

$$\mathbf{M}_7^1 = \overset{\begin{matrix} x_2 & x_3 \end{matrix}}{\begin{bmatrix} - & 1 \\ 1 & - \end{bmatrix}}; \quad \mathbf{M}_7^0 = \overset{\begin{matrix} x_2 & x_3 \end{matrix}}{\begin{bmatrix} 0 & 0 \end{bmatrix}}; \quad \mathbf{M}_8^1 = \overset{\begin{matrix} x_1 & x_3 \end{matrix}}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}; \quad \mathbf{M}_8^0 = \overset{\begin{matrix} x_1 & x_3 \end{matrix}}{\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}};$$

$$\mathbf{M}_9^1 = \overset{\begin{matrix} x_3 & x_4 \end{matrix}}{\begin{bmatrix} 1 & - \\ - & 0 \end{bmatrix}}; \quad \mathbf{M}_9^0 = \overset{\begin{matrix} x_3 & x_4 \end{matrix}}{\begin{bmatrix} 0 & 1 \end{bmatrix}}; \quad \mathbf{M}_{10}^1 = \overset{\begin{matrix} x_1 & x_2 \end{matrix}}{\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}}; \quad \mathbf{M}_{10}^0 = \overset{\begin{matrix} x_1 & x_2 \end{matrix}}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}};$$

$$\mathbf{M}_{11}^1 = \overset{\begin{matrix} x_1 & x_5 \end{matrix}}{\begin{bmatrix} 0 & - \\ - & 1 \end{bmatrix}}; \quad \mathbf{M}_{11}^0 = \overset{\begin{matrix} x_1 & x_5 \end{matrix}}{\begin{bmatrix} 1 & 0 \end{bmatrix}}; \quad \mathbf{M}_{12}^1 = \overset{\begin{matrix} x_3 & x_5 \end{matrix}}{\begin{bmatrix} 0 & 0 \end{bmatrix}}; \quad \mathbf{M}_{12}^0 = \overset{\begin{matrix} x_3 & x_5 \end{matrix}}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}.$$

These matrices are used to obtain the algebraic representations of the completely specified functions $g_7, g_8, g_9, g_{10}, g_{11}$ and the realization of the partial function g_{12}:

$$g_7 = x_2 \vee x_3, \qquad g_8 = x_1 \oplus x_3 = \overline{x}_1 x_3 \vee x_1 \overline{x}_3, \qquad g_9 = x_3 \vee \overline{x}_4,$$
$$g_{10} = x_1 \sim x_2 = \overline{x}_1 \overline{x}_2 \vee x_1 x_2, \qquad g_{11} = \overline{x}_1 \vee x_5, \qquad g_{12} = \overline{x}_3 \overline{x}_5.$$

The corresponding combinational circuit with AND and OR gates is shown in Fig. 2.

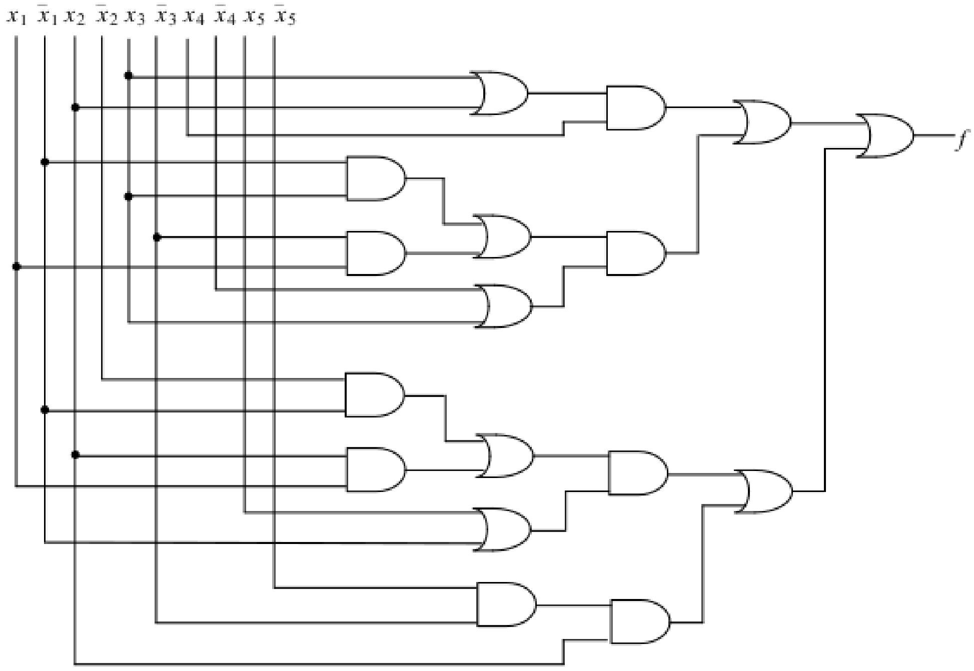

Fig. 2. Circuit with AND and OR gates

6. Conclusion

The paper shows how to apply the method for bi-decomposition in the synthesis of combinational circuits. The advantage of the suggested approach is the possibility of constructing circuits of short delay that is characterized by the number of levels in the circuit. The method is convenient to be applied for incompletely specified Boolean functions, where the functions are given by two domains of Boolean space, as opposed to completely specified functions when the zero domains must be obtained. The "bottle-neck" of the proposed approach is transformation CNF into DNF which is a non-polynomial problem. Thus, the scope of application of the proposed method is limited. It would be established by computer experiment, which is an independent research. The joint implementation of a system of Boolean functions demands reveling the function coincidence at every level of decomposition.

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