Retrial Queue MMPP/M/N Under Heavy Load Condition

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Abstract. In the paper, a multi-server retrial queueing system with MMPP arrivals is considered. The service and retrial times are exponentially distributed. The two-dimension stochastic process of number of calls in the orbit and states of service unit is analyzed. The system of Kolmogorov differential equations is composed. The matrix form of the equations in steady-state regime for partial characteristic functions is written. The method of asymptotic analysis under the heavy load condition for its solving is proposed. It is proved that the asymptotic characteristic function of the number of calls in the orbit has the gamma distribution with obtained parameters. Some numerical examples of comparison asymptotic and simulate distributions are presented.

Keywords: Retrial queue · MMPP · Heavy load · Asymptotic analysis

1 Introduction

Retrial queueing systems are mathematical models widely used in telecommunication networks, computer systems, call centers, etc. [1–5]. The distinguishing feature of such models is that an arriving call, which can not be served, does not join a queue and does not leave the system immediately (as in classical queueing systems). It joins to an orbit (virtual place), where a call waits some random time and then it tries to be served. Now a large number of publications are devoted to retrial queues. The most detailed description, the comparison of classical queueing systems and retrial queues and detailed overviews up to 2008 are contained in monographs of J. Artalejo and A. Gómez-Corral [6], G. Falin and J. Templeton [7].

In most papers devoted retrial queues with MAP (or MMPP), authors use truncation methods [6,8–11] or matrix methods [12–14] and further numerical analysis. While explicit formulas for probability distributions or performance characteristic of complex retrial queues (e.g. with MMPP arrivals, several orbits, non-exponential retrial or service times) cannot be usually obtained. But some approximations or asymptotic solutions can be proposed. One of approximate

The reported study was funded by RFBR and Tomsk region according to the research project № 19-41-703002.

A. Dudin et al. (Eds.): ITMM 2021, CCIS 1605, pp. 252–265, 2022.
https://doi.org/10.1007/978-3-031-09331-9_20

methods is the method of diffusion approximation of retrial queue proposed in [15,16], etc.

In this paper, the asymptotic analysis method [17,18] is used for the multi-server retrial queue with MMPP arrivals. This method is developed in Tomsk and has different modifications for different types of queueing models and queueing networks. It consists of a derivation of some asymptotic equations determining models characteristics and further getting formulas for asymptotic functions under some limit condition. In previous papers [19,20], we have obtained asymptotic solutions under the heavy load condition for different types of single-server retrial queues: $M/M/1$, $M/GI/1$ and even $MMPP/M/1$, $MMPP/GI/1$. So here, we are going to generalize our results to more complex RQ: the multi-server system with MMPP arrivals. Retrial queues with non-Poisson arrival processes are also studied in [1,10,12,21].

The paper is organized as follows. In Sect. 2, the considered mathematical model is described and the stochastic process under study is defined. Section 3 is devoted to method of asymptotic analysis and study of the retrial queue under a limit condition of heavy load. The theorem about the gamma form of the asymptotic characteristic function is proved and parameters of the distribution are obtained. In Sect. 4, numerical examples of the comparison of the asymptotic distributions with simulation ones are shown. The last section contains conclusions.

2 Mathematical Model

Let us describe the model under study. We consider a multi-server retrial queueing system $MMPP/M/N$. Primary calls arrive at the system according to Markovian Modulated Poisson Process (MMPP) defined by matrices $\mathbf{D_0}$ and $\mathbf{D_1}$ [22,23]. If a primary call finds a server free, it starts service with exponentially distributed service time with rate μ'. If all servers is busy, the call goes to an orbit, where it stays during random time distributed by the exponential law with rate σ. After the delay, the call makes an attempt to get service again. If any server is free, the call gets the service, otherwise, the call instantly returns to the orbit. The arrival process, the service times, the retrial times are assumed to be mutually independent. The system structure is presented in Fig. 1.

The MMPP underlying process $n(t)$ is a Markov chain with continuous time and finite set of states $n = 1, 2, \ldots, W$. Matrix $\mathbf{Q} = \mathbf{D_0} + \mathbf{D_1} = (q_{mv})$ is a generator of the process $n(t)$, where $m, v = 1, 2, \ldots, W$. Matrix $\mathbf{D_1}$ is diagonal with elements λ_n $(n = 1, 2, \ldots, W)$. Further, we will use denotation $\mathbf{D_1} = \mathbf{\Lambda} = \mathtt{diag}\{\lambda_n\}$.

Let us denote a stationary probability distribution of $n(t)$ by $\mathbf{r}$, which is row-vector uniquely determined by the following system

$$\begin{cases} \mathbf{rQ} = \mathbf{0}, \\ \mathbf{re} = 1, \end{cases} \tag{1}$$

where $\mathbf{e} = \{1, 1, \ldots, 1\}^T$ and $\mathbf{0} = \{0, 0, \ldots, 0\}$.

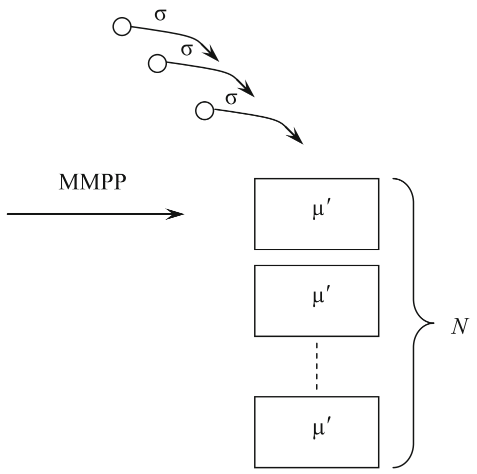

Fig. 1. Retrial queueing system $MMPP/M/N$

Obviously, that the fundamental rate of the arrival process is $\lambda = \mathbf{r} \cdot \mathbf{\Lambda} \cdot \mathbf{e}$.

Let process $i(t)$ define the number of calls in the orbit and $k(t)$ define the service unit state in the following way

$$k(t) = \begin{cases} 0, & \text{if all servers are free,} \\ 1, & \text{if one server is busy,} \\ ..., & \\ N, & \text{if all servers are busy.} \end{cases}$$

The aim of the study is to obtain the stationary probability distribution of the number of calls in the orbit.

Because of process $i(t)$ is not Markovian, we consider the multi-dimensional process $\{k(t), n(t), i(t)\}$, which is a continuous time Markov chain.

Denote $P(k, n, i, t) = P\{k(t) = k, n(t) = n, i(t) = i\}$. The system of Kolmogorov equations is written for $i > 0,\ n = \overline{1, W}$ as follows:

$$\begin{cases} \dfrac{\partial P(0,n,i,t)}{\partial t} = -(\lambda_n + i\sigma - q_{nn})P(0,n,i,t) + \mu' P(1,n,i,t) + \sum_{v \neq n} P(0,v,i,t)q_{vn}, \\ \dfrac{\partial P(k,n,i,t)}{\partial t} = -(\lambda_n + k\mu' + i\sigma - q_{nn})P(k,n,i,t) + \lambda_n P(k,n,i-1,t) \\ + \lambda_n P(k-1,n,i,t) + (i+1)\sigma P(k-1,n,i+1,t) \\ + (k+1)\mu' P(k+1,n,i,t) + \sum_{v \neq n} P(k,v,i,t)q_{vn} \quad \text{for } 1 \leq k \leq N-1, \\ \dfrac{\partial P(N,n,i,t)}{\partial t} = -(\lambda_n + N\mu' - q_{nn})P(N,n,i,t) + \lambda_n P(N,n,i-1,t) \\ + \lambda_n P(N-1,n,i,t) + (i+1)\sigma P(N-1,n,i+1,t) + \sum_{v \neq n} P(N,v,i,t)q_{vn}. \end{cases} \tag{2}$$

In steady-state regime, we have

$$\begin{cases} -(\lambda_n + i\sigma - q_{nn})P(0,n,i) + \mu' P(1,n,i) + \sum_{v \neq n} P(0,v,i)q_{vn} = 0, \\ -(\lambda_n + k\mu' + i\sigma - q_{nn})P(k,n,i) + \lambda_n P(k,n,i-1) \\ + \lambda_n P(k-1,n,i) + (i+1)\sigma P(k-1,n,i+1) \\ + (k+1)\mu' P(k+1,n,i) + \sum_{v \neq n} P(k,v,i)q_{vn} = 0 \quad \text{for } 1 \leq k \leq N-1, \\ -(\lambda_n + N\mu' - q_{nn})P(N,n,i) + \lambda_n P(N,n,i-1) \\ + \lambda_n P(N-1,n,i) + (i+1)\sigma P(N-1,n,i+1) + \sum_{v \neq n} P(N,v,i)q_{vn} = 0, \end{cases} \tag{3}$$

where $P(k,n,i) = \lim_{t \to \infty} P(k,n,i,t)$.

Let us introduce row-vectors $\mathbf{P}_k(i) = \{P(k,1,i), P(k,2,i), \ldots, P(k,W,i)\}$. Then System (3) can be written in matrix form as follows:

$$\begin{cases} -\mathbf{P}_0(i)(\mathbf{\Lambda} + i\sigma\mathbf{I} - \mathbf{Q}) + \mu'\mathbf{P}_1(i) = \mathbf{0}, \\ -\mathbf{P}_k(i)\left(\mathbf{\Lambda} + k\mu'\mathbf{I} + i\sigma\mathbf{I} - \mathbf{Q}\right) + \mathbf{P}_{k-1}(i)\mathbf{\Lambda} \\ + \sigma(i+1)\mathbf{P}_{k-1}(i+1) + (k+1)\mu'\mathbf{P}_{k+1}(i) = \mathbf{0} \quad \text{for } 1 \leq k \leq N-1, \\ -\mathbf{P}_N(i)\left(\mathbf{\Lambda} + N\mu'\mathbf{I} - \mathbf{Q}\right) + \mathbf{P}_{N-1}(i)\mathbf{\Lambda} \\ + \sigma(i+1)\mathbf{P}_{N-1}(i+1) + \mathbf{P}_N(i-1)\mathbf{\Lambda} = \mathbf{0}. \end{cases} \tag{4}$$

where $\mathbf{I}$ is the identity matrix.

Denoting partial characteristic functions by $\mathbf{H}_k(u) = \sum_i e^{jui}\mathbf{P}_k(i)$, where $k = 0, 1, ..., N$ and $j = \sqrt{-1}$, System (4) is rewritten as follows

$$\begin{cases} \mathbf{H}_0(u)(\mathbf{Q} - \mathbf{\Lambda}) + j\sigma\mathbf{H}_0'(u) + \mu'\mathbf{H}_1(u) = \mathbf{0}, \\ \mathbf{H}_k(u)\left(\mathbf{Q} - \mathbf{\Lambda} - k\mu'\mathbf{I}\right) + j\sigma\mathbf{H}_0'(u) + \mathbf{H}_{k-1}(u)\mathbf{\Lambda} \\ -j\sigma e^{-ju}\mathbf{H}_{k-1}'(u) + (k+1)\mu'\mathbf{H}_{k+1}(u) = \mathbf{0} \quad \text{for } 1 \leq k \leq N-1, \\ \mathbf{H}_N(u)\left(\mathbf{Q} - \mathbf{\Lambda} - N\mu'\mathbf{I}\right) + \mathbf{H}_{N-1}(u)\mathbf{\Lambda} + \mathbf{H}_N(u)\mathbf{\Lambda}e^{ju} \\ -j\sigma e^{-ju}\mathbf{H}_{N-1}'(u) = \mathbf{0}, \end{cases} \tag{5}$$

System (5) can not be exactly solved. Thus, we propose the method of asymptotic analysis under the heavy load condition [19,20] for its solution.

3 Asymptotic Analysis Under Heavy Load Condition

Let us introduce load parameter $\rho = (\mathbf{r\Lambda e})/(N\mu')$. Denoting $\mu = \rho N \mu'$ (such as $\mu = \mathbf{r\Lambda e}$), System (5) is rewritten as

$$\begin{cases} \mathbf{H}_0(u)(\mathbf{Q}-\mathbf{\Lambda}) + j\sigma \mathbf{H}_0'(u) + \dfrac{\mu}{\rho N}\mathbf{H}_1(u) = \mathbf{0}, \\ \mathbf{H}_k(u)\left(\mathbf{Q}-\mathbf{\Lambda}-\dfrac{k\mu}{\rho N}\mathbf{I}\right) + j\sigma \mathbf{H}_0'(u) + \mathbf{H}_{k-1}(u)\mathbf{\Lambda} \\ -j\sigma e^{-ju}\mathbf{H}_{k-1}'(u) + \dfrac{(k+1)\mu}{\rho N}\mathbf{H}_{k+1}(u) = \mathbf{0} \quad \text{for } 1 \le k \le N-1, \\ \mathbf{H}_N(u)\left(\mathbf{Q}-\mathbf{\Lambda}-\dfrac{N\mu}{\rho N}\mathbf{I}\right) + \mathbf{H}_{N-1}(u)\mathbf{\Lambda} + \mathbf{H}_N(u)\mathbf{\Lambda}e^{ju} \\ -j\sigma e^{-ju}\mathbf{H}_{N-1}'(u) = \mathbf{0}, \end{cases} \tag{6}$$

Let us prove the following theorem.

Theorem 1. *The limit characteristic function $h(u)$ of the process of the number of calls in the orbit in the $MMPP/M/N$ retrial queueing system in the steady-state regime under the heavy load condition has the gamma distribution form*

$$h(u) = \lim_{\rho\to 1} E\left\{e^{jw(1-\rho)i(t)}\right\} = \left(1 - \frac{ju}{(1-\rho)\beta}\right)^{-\gamma},$$

with parameters

$$\beta = \frac{\mu}{\mathbf{v\Lambda e}+\mu}, \quad \gamma = 1 + \frac{\mu}{N\sigma}\beta, \tag{7}$$

where vector $\mathbf{v}$ is a solution of the following system

$$\begin{cases} \mathbf{vQ} = \mathbf{r}(\mu\mathbf{I}-\mathbf{\Lambda}), \\ \mathbf{ve} = 0. \end{cases}$$

Proof. The proof consists of two parts: deriving of asymptotic equations and its solving.

Derivation of Asymptotic Equations
First of all, we introduce the notations:

$$\varepsilon = 1-\rho, u = \varepsilon w,$$
$$\mathbf{H}_0(u) = \varepsilon^N \mathbf{F}_0(w,\varepsilon), \mathbf{H}_1(u) = \varepsilon^{N-1}\mathbf{F}_1(w,\varepsilon), ..., \mathbf{H}_N(u) = \mathbf{F}_N(w,\varepsilon). \tag{8}$$

The condition of heavy load is defined as $\rho \uparrow 1$ (or $\varepsilon \downarrow 0$).

System of Eqs. (6) can be rewritten in Notations (8) as follows

$$\begin{cases} \varepsilon^N \mathbf{F}_0(w,\varepsilon)(\mathbf{Q}-\mathbf{\Lambda}) + j\sigma\varepsilon^{N-1}\dfrac{\partial \mathbf{F}_0(w,\varepsilon)}{\partial w} + \dfrac{\mu}{(1-\varepsilon)N}\varepsilon^{N-1}\mathbf{F}_1(w,\varepsilon) = \mathbf{0}, \\ \varepsilon^{N-k}\mathbf{F}_k(w,\varepsilon)\left(\mathbf{Q}-\mathbf{\Lambda}-\dfrac{k\mu}{(1-\varepsilon)N}\mathbf{I}\right) + j\sigma\varepsilon^{N-k-1}\dfrac{\partial \mathbf{F}_k(w,\varepsilon)}{\partial w} \\ +\,\varepsilon^{N-(k-1)}\mathbf{F}_{k-1}(w,\varepsilon)\mathbf{\Lambda} - j\sigma e^{-j\varepsilon w}\varepsilon^{N-(k-1)-1}\dfrac{\partial \mathbf{F}_{k-1}(w,\varepsilon)}{\partial w} \\ +\,\dfrac{(k+1)\mu}{(1-\varepsilon)N}\varepsilon^{N-(k+1)}\mathbf{F}_{k+1}(w,\varepsilon) = \mathbf{0} \quad \text{for } 1 \le k \le N-1, \\ \mathbf{F}_N(w,\varepsilon)\left(\mathbf{Q}-\mathbf{\Lambda}-\dfrac{\mu}{(1-\varepsilon)}\mathbf{I}\right) + \,\varepsilon\mathbf{F}_{N-1}(w,\varepsilon)\mathbf{\Lambda} \\ +\mathbf{F}_N(w,\varepsilon)\mathbf{\Lambda}e^{j\varepsilon w} - j\sigma e^{-j\varepsilon w}\dfrac{\partial \mathbf{F}_{N-1}(w,\varepsilon)}{\partial w} = \mathbf{0}, \end{cases}$$

After some transformations, we obtain

$$\begin{cases} \varepsilon(1-\varepsilon)\mathbf{F}_0(w,\varepsilon)(\mathbf{Q}-\mathbf{\Lambda}) + j\sigma(1-\varepsilon)\dfrac{\partial \mathbf{F}_0(w,\varepsilon)}{\partial w} + \dfrac{\mu}{N}\mathbf{F}_1(w,\varepsilon) = \mathbf{0}, \\ \varepsilon\mathbf{F}_k(w,\varepsilon)\left((\mathbf{Q}-\mathbf{\Lambda})(1-\varepsilon)-\dfrac{k\mu}{N}\mathbf{I}\right) + j\sigma(1-\varepsilon)\dfrac{\partial \mathbf{F}_k(w,\varepsilon)}{\partial w} \\ +\,\varepsilon^2(1-\varepsilon)\mathbf{F}_{k-1}(w,\varepsilon)\mathbf{\Lambda} - j\sigma e^{-j\varepsilon w}\varepsilon(1-\varepsilon)\dfrac{\partial \mathbf{F}_{k-1}(w,\varepsilon)}{\partial w} \\ +\,\dfrac{(k+1)\mu}{N}\mathbf{F}_{k+1}(w,\varepsilon) = \mathbf{0} \quad \text{for } 1 \le k \le N-1, \\ \mathbf{F}_N(w,\varepsilon)\left((\mathbf{Q}-\mathbf{\Lambda})(1-\varepsilon)-\mu\mathbf{I}\right) + \varepsilon(1-\varepsilon)\mathbf{F}_{N-1}(w,\varepsilon)\mathbf{\Lambda} \\ +\,(1-\varepsilon)e^{j\varepsilon w}\mathbf{F}_N(w,\varepsilon)\mathbf{\Lambda} - j\sigma(1-\varepsilon)e^{-j\varepsilon w}\dfrac{\partial \mathbf{F}_{N-1}(w,\varepsilon)}{\partial w} = \mathbf{0}, \end{cases} \tag{9}$$

First of all, in System (9) we make limit $\varepsilon \to 0$.

$$\begin{cases} j\sigma\mathbf{F}_0'(w) + \dfrac{\mu}{N}\mathbf{F}_1(w) = \mathbf{0}, \\ j\sigma\mathbf{F}_k'(w) + \dfrac{(k+1)\mu}{N}\mathbf{F}_{k+1}(w) = \mathbf{0}, \\ \mathbf{F}_N(w)\left(\mathbf{Q}-\mu\mathbf{I}\right) - j\sigma\mathbf{F}_{N-1}'(w) = \mathbf{0}, \end{cases} \tag{10}$$

where $\mathbf{F}_k(w) = \lim\limits_{\varepsilon\to 0}\mathbf{F}_k(w,\varepsilon)$.

Let us consider expansions of functions $\mathbf{F}_k(w,\varepsilon)$ in the form

$$\mathbf{F}_k(w,\varepsilon) = \mathbf{F}_k(w) + \varepsilon\mathbf{f}_k(w) + \mathbf{O}(\varepsilon^2), \tag{11}$$

where $\mathbf{O}(\varepsilon^2)$ is an infinitesimal value of order ε^2.

Substituting Expansions (11) into System (6) and making some transformations, we obtain the following system of equations in limit $\varepsilon \to 0$

$$\begin{cases} \mathbf{F}_0(w)(\mathbf{Q}-\mathbf{\Lambda}) - j\sigma\mathbf{F}_0'(w) + j\sigma\mathbf{f}_0'(w) + \dfrac{\mu}{N}\mathbf{f}_1(w) = \mathbf{0}, \\ \mathbf{F}_k(w)\left(\mathbf{Q}-\mathbf{\Lambda}-\dfrac{k\mu}{N}\mathbf{I}\right) - j\sigma\mathbf{F}_k'(w) \\ +\,j\sigma\mathbf{f}_k'(w) - j\sigma\mathbf{F}_{k-1}'(w) + \dfrac{(k+1)\mu}{N}\mathbf{f}_{k+1}(w) = \mathbf{0} \quad \text{for } 1 \le k \le N-1, \\ -\mathbf{F}_N(w)\mathbf{Q} + \mathbf{f}_N(w)\left(\mathbf{Q}-\mu\mathbf{I}\right) + \mathbf{F}_{N-1}(w)\mathbf{\Lambda} \\ +\,jw\mathbf{F}_N(w)\mathbf{\Lambda} + j\sigma(1+jw)\mathbf{F}_{N-1}'(w) - j\sigma\mathbf{f}_{N-1}'(w) = \mathbf{0}. \end{cases} \tag{12}$$

In addition, we sum up all equations of System (6) and multiply the result by vector $\mathbf{e}$.

$$\mathbf{F}_N(w,\varepsilon)e^{jw\varepsilon}\mathbf{\Lambda e}+j\sigma\sum_{k=0}^{N-1}\varepsilon^{N-k-1}\frac{\partial\mathbf{F}_k(w,\varepsilon)}{\partial w}\mathbf{e}=0.$$

Substituting Expansions (11) and writing equalities for members with equal powers of ε, we obtain two additional scalar equations

$$\begin{cases}\mathbf{F}_N(w)\mathbf{\Lambda e}+j\sigma\mathbf{F}'_{N-1}(w)\mathbf{e}=0,\\ jw\mathbf{F}_N(w)\mathbf{\Lambda e}+\mathbf{f}_N(w)\mathbf{\Lambda e}+j\sigma\mathbf{F}'_{N-2}(w)\mathbf{e}+j\sigma\mathbf{f}'_{N-1}(w)\mathbf{e}=0.\end{cases} \tag{13}$$

Thus, we have System (10), (12), (13) of $2(N+1)$ matrix and two scalar differential equations.

Analysis of the Equations

The partial characteristic function of the number of calls in the orbit is calculated as follows

$$H(u)=\mathrm{E}\left\{e^{jui(t)}\right\}=\sum_{k=0}^{N}\mathbf{H}_k(u)\mathbf{e}.$$

Under the heavy load condition, the asymptotic characteristic function $h(u)$ can be written as

$$h(u)=\lim_{\rho\to1}\mathrm{E}\left\{e^{jw(1-\rho)i(t)}\right\}=\mathbf{F}_N\left(\frac{u}{1-\rho}\right)\mathbf{e}+O(\varepsilon). \tag{14}$$

Therefore, it is necessary to find only scalar function $\mathbf{F}_N(w)\mathbf{e}$ from Equations (10), (12), (13). We make it in three steps.

Step 1. By using Equations (10), we obtain that

$$-j\sigma\mathbf{F}'_k(w)=\frac{(k+1)\mu}{N}\mathbf{F}_{k+1}(w)\quad\text{for } k<N. \tag{15}$$

Comparing the equation for $k=N$ in (10) and the equation for $k=N-1$ of (15), we get

$$\mathbf{F}_N(w)\mathbf{Q}=\mathbf{0}.$$

Taking into account (1), function $\mathbf{F}_N(w)$ can be written as the following product:

$$\mathbf{F}_N(w)=\mathbf{r}\cdot\Phi(w), \tag{16}$$

where $\Phi(w)$ is an unknown scalar function.

Step 2. From Eqs. (12) and Equalities (15), it can be written that

$$\begin{cases} j\sigma\mathbf{f}_0'(w) = -\mathbf{F}_0(w)(\mathbf{Q}-\mathbf{\Lambda}) - \dfrac{\mu}{N}\mathbf{F}_1(w) - \dfrac{\mu}{N}\mathbf{f}_1(w), \\ j\sigma\mathbf{f}_k'(w) = -\mathbf{F}_k(w)(\mathbf{Q}-\mathbf{\Lambda}-\dfrac{k\mu}{N}\mathbf{I}) \\ \quad -\dfrac{(k+1)\mu}{N}\mathbf{F}_{k+1}(w) - \dfrac{k\mu}{N}\mathbf{F}_k(w) - \dfrac{(k+1)\mu}{N}\mathbf{f}_{k+1}(w) \text{ for } 1 \le k \le N-2, \\ j\sigma\mathbf{f}_{N-1}'(w) = -\mathbf{F}_{N-1}(w)(\mathbf{Q}-\mathbf{\Lambda}-\dfrac{N-1\mu}{N}\mathbf{I}) \\ \quad -\dfrac{(N)\mu}{N}\mathbf{F}_N(w) - \dfrac{N-1\mu}{N}\mathbf{F}_{(N-1)}(w) - \dfrac{(N)\mu}{N}\mathbf{f}_N(w), \\ j\sigma\mathbf{f}_{N-1}'(w) = -\mathbf{f}_N(w)(\mathbf{Q}-\mu\mathbf{I}) + \mathbf{F}_{N-1}(w)\mathbf{\Lambda} + jw\mathbf{F}_N(w)\mathbf{\Lambda} - (1+jw)\mu\mathbf{F}_N(w). \end{cases} \tag{17}$$

Subtracting the two last equations of System (17), we obtain

$$(\mathbf{F}_{N-1}(w) + \mathbf{f}_N(w))\mathbf{Q} = \mathbf{F}_N(w)\mathbf{Q} + jw\mathbf{F}_N(w)(\mathbf{\Lambda} - \mu\mathbf{I}).$$

Substituting Formula (16), we have the following equation

$$(\mathbf{F}_{N-1}(w) + \mathbf{f}_N(w))\mathbf{Q} = -jw\Phi(w)\mathbf{r}(\mathbf{\Lambda} - \mu\mathbf{I}). \tag{18}$$

Let us introduce the following notation:

$$\mathbf{F}_{N-1}(w) + \mathbf{f}_N(w) = -jw\Phi(w)\mathbf{v}, \tag{19}$$

where vector $\mathbf{v}$ is a solution of the equation

$$\mathbf{v}\mathbf{Q} = \mathbf{r}(\mu\mathbf{I} - \mathbf{\Lambda}). \tag{20}$$

For Eq. (20) solution existence, it is necessary that ranks of the system matrix and augmented one will be equal. Because $\mathbf{r}(\mu\mathbf{I} - \mathbf{\Lambda})\mathbf{e} = 0$, that it is true.

Matrix Eq. (20) has infinitely many solutions. We can present the general solution as follows

$$\mathbf{v} = C\mathbf{r} + \mathbf{v}_0,$$

where $C = const$ and $\mathbf{v}_0$ is a particular solution, for example, $\mathbf{v}_0\mathbf{e} = 0$.

Step 3. Substituting (10), (15), (17) into the last equation of System (13), we obtain the following equation:

$$2jw\mathbf{F}_N(w)\mathbf{\Lambda}\mathbf{e} + \mathbf{f}_N(w)\mathbf{\Lambda}\mathbf{e} - \frac{(N-1)\mu}{N}\mathbf{F}_{N-1}(w)\mathbf{e}$$
$$-\mu\mathbf{f}_N(w)\mathbf{e} + \mathbf{F}_{N-1}(w)\mathbf{\Lambda}\mathbf{e} - (1+jw)\mu\mathbf{F}_N(w)\mathbf{e} = 0.$$

Taking into account Equality (19), we have

$$jw\Phi(w)(2\mathbf{r}\mathbf{\Lambda}\mathbf{e} + \mathbf{v}(\mathbf{\Lambda}\mathbf{e} - \mu\mathbf{e}) - \mu) - \mu\Phi(w) + \frac{\mu}{N}\mathbf{F}_{N-1}(w)\mathbf{e} = 0. \tag{21}$$

The we differentiate this equation. Taking into account (15), we obtain the following differential equation

$$j\Phi(w)\left(\mathbf{v}(\mathbf{\Lambda}\mathbf{e} - \mu\mathbf{e}) + \mu + \frac{\mu^2}{N\sigma}\right) - \Phi'(w)\left(\mu - jw(\mathbf{v}(\mathbf{\Lambda}\mathbf{e} - \mu\mathbf{e}) + \mu)\right) = 0. \tag{22}$$

Let us divide (22) by $(\mathbf{v}\boldsymbol{\Lambda}\mathbf{e} - \mu\mathbf{v}\mathbf{e} + \mu)$ and introduce denotations

$$\beta = \frac{\mu}{\mathbf{v}\boldsymbol{\Lambda}\mathbf{e} - \mu\mathbf{v}\mathbf{e} + \mu}, \quad \gamma = 1 + \frac{\mu}{N\sigma}\beta.$$

Thus Eq. (22) is rewritten as

$$\Phi'(w)(\beta - jw) = j\gamma\Phi(w).$$

Clearly, the solution of this equation has the form

$$\Phi(w) = C_0 \left(1 - \frac{jw}{\beta}\right)^{-\gamma}.$$

From formula (16), we obtain

$$\mathbf{F}_N(w) = \mathbf{r} \cdot C_0 \left(1 - \frac{jw}{\beta}\right)^{-\gamma}.$$

Taking into account $\mathbf{v} = C\mathbf{r} + \mathbf{v}_0$, it is easy to show that the parameters β and γ do not depend on C. Choosing a solution $\mathbf{v}_0$ such as

$$\begin{cases} \mathbf{v_0}\mathbf{Q} = \mathbf{r}(\mu\mathbf{I} - \boldsymbol{\Lambda}), \\ \mathbf{v_0}\mathbf{e} = 0, \end{cases}$$

we can write that

$$\beta = \frac{\mu}{\mathbf{v_0}\boldsymbol{\Lambda}\mathbf{e} + \mu}, \quad \gamma = 1 + \frac{\mu}{N\sigma}\beta,$$

Returning to characteristic function (14), we can write that

$$h(u) = C_0 \left(1 - \frac{ju}{\beta}\right)^{-\gamma},$$

where $C_0 = 1$ due to the normalisation requirement.

Thus, we have proved that the asymptotic characteristic function of the probability distribution of the number of calls in the orbit under the heavy load condition has the gamma distribution form.

4 Numerical Analysis

In this section, we present some numerical examples and make conclusions about the asymptotic method applicability area. First of all, we denote the probability distribution function of the gamma distribution with parameters (7) as $\Gamma(x)$. We will calculate of the discrete probability distribution of the number of calls in the orbit $p(i)$ as follows

$$p(i) = \Gamma(i + 1) - \Gamma(i).$$

Further, we present the comparison of asymptotic and simulated distributions for different values of the retrial queuing system parameters.

In the first example, let the retrial queue have three server ($N = 3$), and the arrival MMPP have three states and be defined by following matrices

$$\mathbf{\Lambda} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \qquad \mathbf{Q} = \begin{bmatrix} \text{-}0.5 & 0.2 & 0.3 \\ 0.1 & \text{-}0.3 & 0.2 \\ 0.3 & 0.6 & \text{-}0.9 \end{bmatrix}.$$

The retrial rate is $\sigma = 1$, the service rate equals $\mu = \dfrac{\mathbf{r\Lambda e}}{N\rho}$, then the load parameter ρ has values $0 < \rho < 1$.

In Fig. 2, the comparison of the asymptotic and simulated distributions is presented for $\rho = 0.90$ and $\rho = 0.95$, where dashed lines are the asymptotic distributions and solid lines are simulated ones.

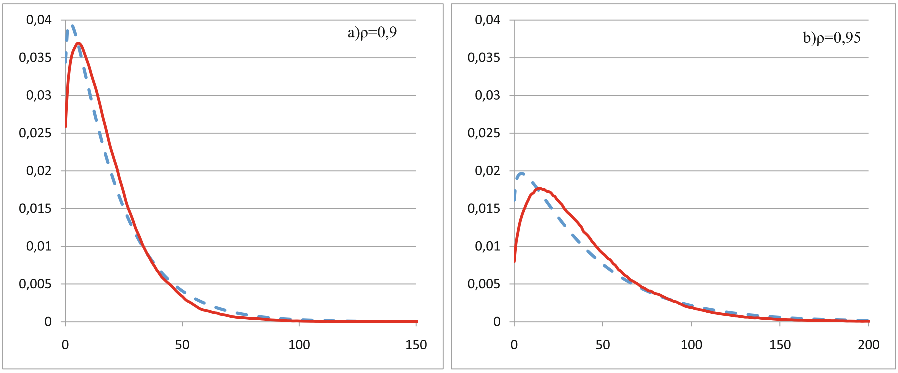

Fig. 2. Comparison of the asymptotic and the simulated distributions for $MMPP/M/3$ with a) $\rho = 0.90$ and b) $\rho = 0.95$

In the second example, let us consider a particular cases of the retrial queue - the single-server retrial queue with following values of parameters

$$N = 1, \qquad \sigma = 1, \qquad \mu = \frac{\mathbf{r\Lambda e}}{\rho},$$

the comparison of the asymptotic and simulated distributions is presented in Fig. 3 and 4.

Also we demonstrate a numerical example for multi-server retrial queue with Poisson arrival process (Fig. 4), where $\lambda = 1, N = 10, \sigma = 1, \mu = \dfrac{\lambda}{N\rho}$ (Fig. 5).

In this example, the main difference between asymptotic and the simulation distributions is in point $i = 0$.

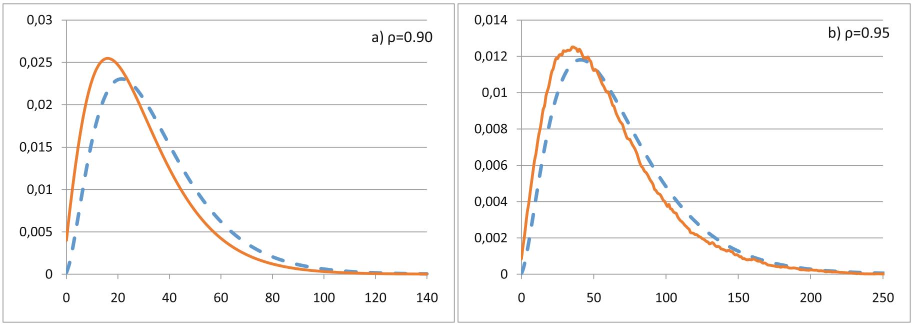

Fig. 3. Comparison of the asymptotic and the simulated distributions for the single-server retrial queue with a) $\rho = 0.90$ and b) $\rho = 0.95$

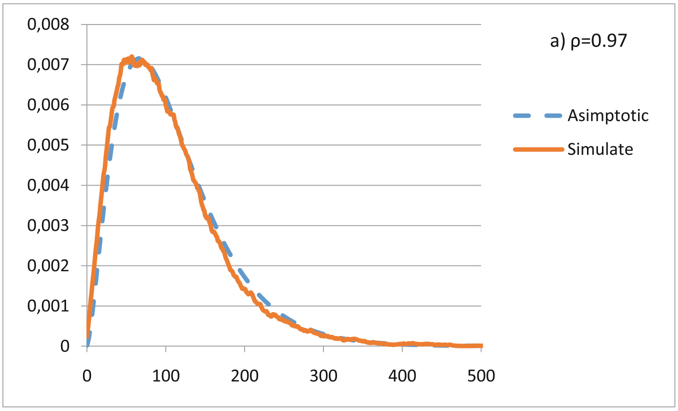

Fig. 4. Comparison of the asymptotic and the simulated distributions for the single-server retrial queue $\rho = 0.97$

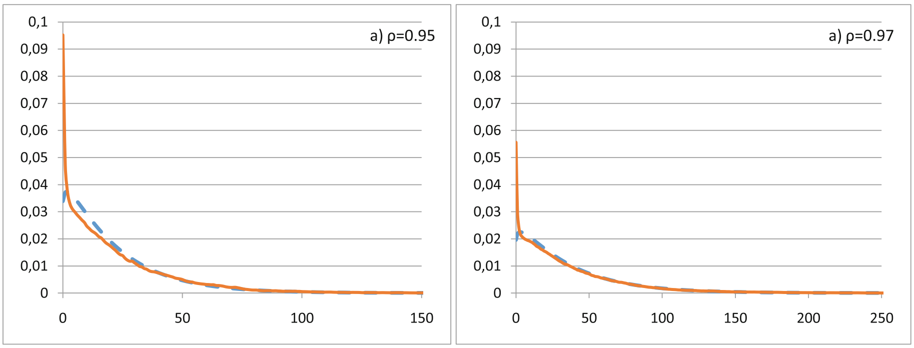

Fig. 5. Comparison of the asymptotic and the simulated distributions for Poisson arrival process with a) $\rho = 0.95$ and b) $\rho = 0.97$

For the method accuracy estimation, we use Kolmogorov distance between respective distribution functions:

$$d = \max_{i \geq 0} \left| \sum_{l=0}^{i} [\tilde{p}(l) - p(l)] \right|,$$

where $p(l)$ is an asymptotic probability distribution and $\tilde{p}(l)$ is a simulated one. In Table 1, there are values of the Kolmogorov distance for all presented numerical examples.

Table 1. Kolmogorov distances d for various values of the parameter ρ

	$N = 1$	$N = 3$	$N = 10, \lambda = 1$
$\rho = 0.90$	0.070	0.070	0.068
$\rho = 0.95$	0.043	0.043	0.040
$\rho = 0.97$	0.036	0.038	0.035

Note, we have obtained the same results of the numerical analysis for different arrivals and number of servers. For our purpose, the asymptotic analysis method under the heavy load condition can be applied for $\rho \geq 0.95$, where the Kolmogorov distance between asymptotic and the simulation distributions $d \leq 0.05$.

5 Conclusions

In the paper, the multi-server retrial queueing system with MMPP arrivals has been studied by the asymptotic analysis method under the heavy load condition. We have proved that the asymptotic characteristic function of the number of calls in the orbit has the gamma distribution form, as for single-server retrial queue. In this way, we generalize our results for more complex model. By means of the numerical analysis, we have shown a good accuracy of the proposed approximation in the applicability area $\rho \geq 0.95$.

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