A CUBIC SPLINE PROJECTION METHOD FOR COMPUTING STATIONARY DENSITY FUNCTIONS OF FROBENIUS -PERRON OPERATOR

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Azzah Ahmed Alshekhi

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A CUBIC SPLINE PROJECTION METHOD FOR COMPUTING STATIONARY DENSITY FUNCTIONS OF FROBENIUS -PERRON OPERATOR

Azzah Alshekhi, Candidate for the Doctor of Philosophy Degree

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ABSTRACT

Stationary density functions of Frobenius-Perron operators have critical applications in many fields of science and engineering. Accordingly, approximating stationary density functions f^* is important and the focus of this dissertation.

Among the computational methods of approximating the smooth f^*, the linear spline and quadratic spline projection methods have been proven effective. However, we intend to improve the convergence rate of the previous methods. We will fulfill this goal by using cubic spline functions since cubic spline functions are twice continuously differentiable on the whole domain.

Theoretically, we prove the existence of a nonzero sequence of cubic spline functions $\{f_n\}$ that converges to the stationary density function f^* of the Frobenius-Perron operator in L^1-norm. The numerical experimental results assure that the cubic spline projection method gives the fastest convergence rate so far. In addition, when the stationary density function f^* lies in the cubic spline space, the cubic spline projection method computes f^* exactly no matter what n may be.

The faculty listed below, appointed by the Dean of the School of Graduate Studies have examined a thesis titled "A Cubic Spline Projection Method for Computing Stationary Density Functions of Frobenius-Perron Operator," presented by Azzah Alshekhi, candidate for the Doctor of Philosophy degree, and hereby certify that in their opinion it is worthy of acceptance.

Supervisory Committee

Noah Rhee, Ph.D.

Department of Mathematics and Statistics

Majid Bani-Yaghoub, Ph.D.

Department of Mathematics and Statistics

Liana Sega, Ph.D.

Department of Mathematics and Statistics

Paul Rulis, Ph.D.

Department of Physics and Astronomy

Elizabeth Stoddard, Ph.D.

Department of Physics and Astronomy

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DEDICATION

To my parents, brothers, and sisters with love and appreciation.

Tell me and I forget,

teach me and I remember,

involve me and I learn.

Benjamin Franklin

CHAPTER 1

INTRODUCTION

The initiative behind this dissertation is to approximate a stationary density function of Frobenius-Perron operator, which can be seen in the following example. Let $S : [0,1] \to [0,1]$ be a transformation that is given by

$$S(x) \quad = \quad \begin{cases} \frac{2x}{1-x} & 0 \leq x < \frac{1}{3} \\ \frac{1-x}{2x} & \frac{1}{3} \leq x < 1 \end{cases}$$

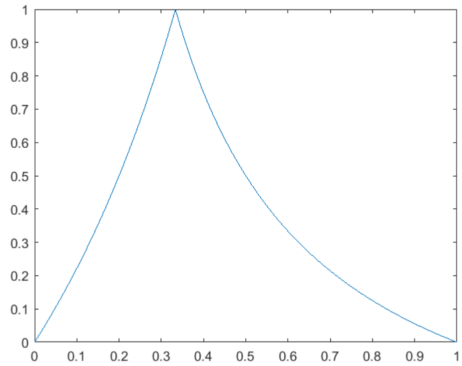
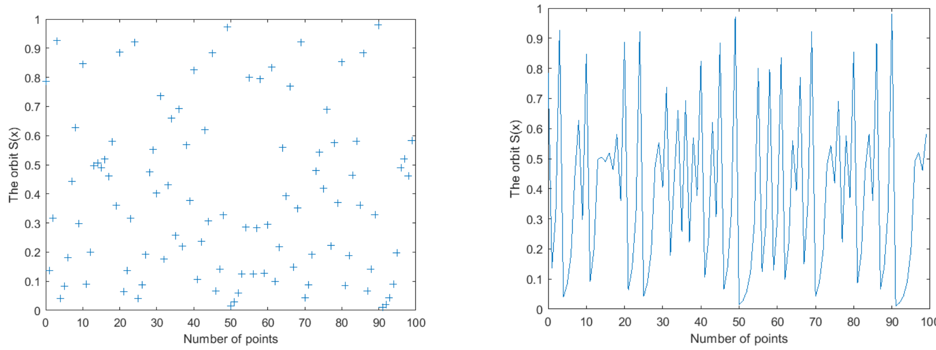


Figure 1: Transformation $S(x)$

whose its graph is shown in Figure 1.

Let $x_0 \in [0,1]$ be an initial point, and $\{S^k(x_0)\}_{k=0}^{\infty}$ be the orbit of S corresponding to the initial point x_0, where $S^k(x_0) = (S \circ S \circ \ldots \circ S)(x_0)$, $k-$times. To study the behavior of the orbit, we use MATLAB to graph the initial part of the orbit $\{S^k(x_0)\}_{k=0}^{99}$ for the initial point $x_0 = \frac{\pi}{4}$, where $S^0\left(\frac{\pi}{4}\right) = 0.785$, $S\left(\frac{\pi}{4}\right) = 0.137$, $S^2\left(\frac{\pi}{4}\right) = S\left(S\left(\frac{\pi}{4}\right)\right) = 0.316$, and so on. As Lasota and Mackey, in their book [44],

"A Cubic Spline Projection Method for Computing Stationary Densities of Dynamical Systems" paper, [1], is a summarized version of this dissertation. The paper was published by the International Journal of Bifurcation and Chaos in 2022.



(a) $n = 100$ (b) $n = 100$

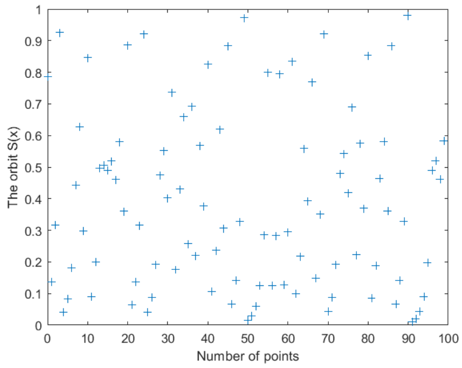
Figure 2: The initial part of orbit $\left\{S^k\left(\frac{\pi}{4}\right)\right\}_{k=0}^{99}$

mentioned and as shown in Figure 2, we can see that the behavior of the orbit is chaotic (the right-hand picture is obtained from the left-hand picture after connecting the points together). So, studying the points in the orbit $\{S(x_0)\}_{k=0}^{\infty}$ does not tell us any meaningful information. However, when we plot more points of the orbit, as Figure 3, we begin to see a pattern in a distribution sense. We observe that more points of the orbit are near 0, and the number of points decrease as we move to 1.

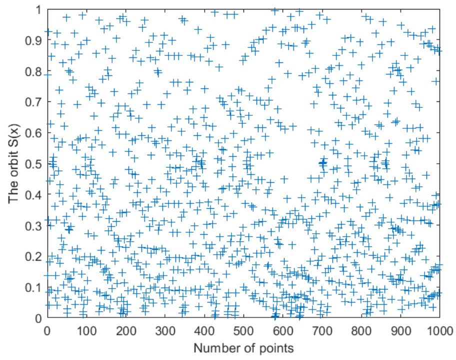
To analyze this situation better, we give the following example.

EXAMPLE 1.1. We divide the domain $[0,1]$ into ten subintervals of equal length, and we experiment the probability of belonging to each subinterval using $n = 10^6$. We compute the probability distributions for four different initial values, $x_0 = \frac{\pi}{4}, \frac{\pi}{8}, \frac{\pi}{16}$, and $\frac{\pi}{32}$, using the following formula

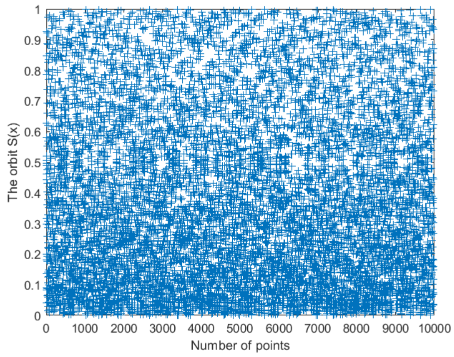
$$\frac{1}{10^6} \sum_{k=0}^{10^6-1} 1_{\left[\frac{i-1}{10}, \frac{i}{10}\right)}(S^k(x_0)) \tag{1.1}$$



(a) $n = 100$



(b) $n = 1000$



(c) $n = 10000$

Figure 3: The initial part of orbit $\left\{S^k\left(\frac{\pi}{4}\right)\right\}_{k=0}^{n-1}$, when n=100, 1000, and 10000, respectively

for $i = 1, 2, \ldots, 10$, where $1_{\left[\frac{i-1}{10}, \frac{i}{10}\right)}$ denotes the indicator for the subinterval $\left[\frac{i-1}{10}, \frac{i}{10}\right)$. We display the results in Table 1, which shows the probability distribution of the orbits in each subinterval with four different initial values.

Table 1: Probability Distributions

	Probability			
x_0	$\frac{\pi}{4}$	$\frac{\pi}{8}$	$\frac{\pi}{16}$	$\frac{\pi}{32}$
$[0, 0.1]$	0.1821	0.1826	0.1824	0.1819
$[0.1, 0.2]$	0.1516	0.1515	0.1515	0.1510
$[0.2, 0.3]$	0.1277	0.1277	0.1283	0.1281
$[0.3, 0.4]$	0.1099	0.1104	0.1100	0.1098
$[0.4, 0.5]$	0.0956	0.0948	0.0947	0.0956
$[0.5, 0.6]$	0.0833	0.0834	0.0829	0.0839
$[0.6, 0.7]$	0.0733	0.0730	0.0736	0.0735
$[0.7, 0.8]$	0.0655	0.0653	0.0654	0.0650
$[0.8, 0.9]$	0.0585	0.0583	0.0582	0.0583
$[0.9, 1]$	0.0524	0.0529	0.0528	0.0528

Our observation from Figure 3, which was the probability distribution decreases as we go to 1, is confirmed by Table 1. We also observe that the distribution is about the same for any initial point x_0; that is, the probability is independent of the initial point x_0. This suggests that there may be a probability measure μ such that

$$\mu(A) \quad = \quad \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} 1_A(S^k(x_0)) \tag{1.2}$$

for any $A \in \mathcal{B}$ and for almost any $x_0 \in [0,1]$. From now on, through out this dissertation, when we say for any $x_0 \in [0,1]$, we mean for almost all $x_0 \in [0,1]$. The class $\mathcal{B}$ is called a Borel σ-field on $[0,1]$, which is defined as follows [28]. First, we need to define a σ-field on $\mathbb{R}$.

A class $\mathcal{C} \in \mathbb{R}$ is called a *σ-field on* $\mathbb{R}$ if:

(1) $\mathbb{R} \in \mathcal{C}$.

(2) $A \in \mathcal{C}$ implies $A^c \in \mathcal{C}$.

(3) $\{A_k\}_{k=1}^{\infty} \in \mathcal{C}$ implies that $\bigcup_{k=1}^{\infty} A_k \in \mathcal{C}$.

Let $\mathcal{O}$ be the collection of all open intervals in $\mathbb{R}$, and let $\sigma(\mathcal{O})$ be the smallest σ-field containing $\mathcal{O}$. Then $\mathcal{B}_{\mathbb{R}} = \sigma(\mathcal{O})$ is called a *Borel σ-field on* $\mathbb{R}$, and $\mathcal{B}$ is defined by $\mathcal{B} = \{A \cap [0,1] : A \in \mathcal{B}_{\mathbb{R}}\}$.

In fact, the probability measure μ in equation (1.2) is constructed as follows. Suppose that f^* is a density function of P_S such that $P_S f^* = f^*$ (such density function of P_S is called a stationary density function of P_S), where P_S is the Frobenius-Perron operator associated with transformation S. We will define P_S, the Frobenius-Perron operator associated with S later in Section 2.2. Let the probability measure μ be defined by

$$\mu(A) \quad = \quad \int_A f^*(x)\, dx$$

for all $A \in \mathcal{B}$. Then we have

$$\mu(A) = \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} 1_A(S^k(x_0)) \tag{1.3}$$

for all $x_0 \in [0,1]$. The justification for (1.3) will be given in chapter 2.

This application shows the importance of computing (or approximating) the stationary density function f^* of the Frobenius-Perron operator P_S associated with S, which is the theme of this dissertation.

The era of computational ergodic theory begun when the first attempt at computing the stationary density was proposed in the book by Ulam [69]. Later, Li in his paper [47] showed the convergence of Ulam's method. But Bose & Murray [8] showed that it only has $O(\ln n/n)$ convergence rate under the L^1-norm. Ulam's method can be understood as a Markov method using piecewise constant functions. Then the paper by Ding and Li [22] has extended Ulam's method to a Markov method using piecewise linear functions, and later in the paper by Ding and Rhee [23] it was proved that it has $O(1/n)$ convergence under the BV-norm.

In addition, Ulam's method can be understood as the projection method using piecewise constant functions, which is illustrated in Section 4.2. A projection method using linear spline functions was proposed and analyzed in the papers by Ding and Rhee [24, 25], where the numerical results show that it has a higher convergence rate than the piecewise linear Markov method. Moreover, the quadratic spline projection method in the paper by Zhou et al. [78], shows a faster convergence rate than the piecewise linear projection method. Even though the results from [24, 78] showed that both the linear spline and quadratic spline projection methods are good options to use to approximate the stationary density function of the Frobenius-Perron operator associated with S, we aim to improve the convergence rate further. Therefore, we will study the cubic spline projection method in this dissertation since cubic spline functions are twice continuously differentiable on the whole domain. So, we expect to obtain the highest convergence rate so far for the computation of stationary density functions of the Frobenius-Perron operators associated with S.

In Chapter 2, we construct the Frobenius-Perron operator, and we cover preliminaries that justify equation (1.3). Then Chapter 3 covers the definition and some properties of cubic spline functions. Chapter 4 explains the general idea of the projection method and then specifies the cubic spline projection method. After that, in chapter 5, we prove the convergence of a sequence of cubic spline functions $\{f_n\}$ to the stationary density function f^*. Then in Chapter 6, we provide numerical experimental results and explain how to get efficient and accurate computations in order to get more accurate results. Finally, we conclude and provide future work in Chapter 7.

CHAPTER 2

PRELIMINARIES

In this chapter, we justify the fact in (1.3) mentioned at the end of Chapter 1. However, we first need to cover some preliminaries, which are crucial to drive the fact. In Section 2.1, we define a Markov operator and present some of its properties. In Section 2.2, we construct a Frobenius-Perron operator. Then in Section 2.3, we define a new operator called a Koopman operator, which is an adjoint operator of a Frobenius-Perron operator. In Section 2.4, we introduce the concept of an invariant measure under a measurable and nonsingular transformation S and show the connection between invariant measure and stationary densities f^* of a Frobenius-Perron operator. Lastly in Section 2.5 we define Ergodic transformation and finally derive the fact in (1.3). (Most of the results in this chapter are known and can be found in the book by Lasota and Mackey [44]).

2.1 Markov Operator

This section defines a Markov operator and states some of its properties. Before we define Markov operators, we provide the definition of Borel Measurable functions.

DEFINITION 2.1. A measurable function $f : [0,1] \to \mathbb{R}$ is called a *Borel measurable* function if $A \in \mathcal{B}_{\mathbb{R}}$ implies $f^{-1}(A) \in \mathcal{B}$ [49].

DEFINITION 2.2. Consider the measure space $(L^1[0,1], \mathcal{B}, \lambda)$, where $L^1[0,1]$ denotes the collection of all Borel measurable functions such that $\int_0^1 |f| \, d\lambda < \infty$ [4] and λ is the Lebesgue measure [32]. A *Markov operator* $P : L^1[0,1] \to L^1[0,1]$ is a linear operator that satisfies the following:

(M1) If $f \geq 0$, then $Pf \geq 0$ for $f \in L^1[0,1]$.

(M2) $\|Pf\|_1 = \|f\|_1$ for $f \geq 0$ and $f \in L^1[0,1]$, where $\|f\|_1 = \int_0^1 |f| \, d\lambda$ [56].

In the following we introduce some crucial properties of a Markov operator.

DEFINITION 2.3. Let $f \in L^1[0,1]$. The positive part of f [66] is denoted by f^+ and is defined by

$$f^+ = \max\{f(x), 0\}.$$

While f^- is the negative part of f and is defined by

$$f^- = \max\{-f(x), 0\} = -\min\{f(x), 0\}.$$

Note that the functions f and $|f|$ can be expressed in terms of the positive and negative parts of f as $f = f^+ - f^-$ and $|f| = f^+ + f^-$.

LEMMA 2.4. Let P be a Markov operator on $L^1[0,1]$. Then for every $f \in L^1[0,1]$:

(1) $(Pf)^+ \leq Pf^+$.

(2) $(Pf)^- \leq Pf^-$.

(3) $|Pf| \leq P|f|$.

(4) $\|Pf\|_1 \leq \|f\|_1$.

PROOF.

(1) Let $f \in L^1[0,1]$. Since $f^+ \geq 0$ and $f^- \geq 0$, $Pf^+ \geq 0$ and $Pf^- \geq 0$. Then by the linearity of P, we have

$$\begin{aligned} (Pf)^+ &= (P(f^+ - f^-))^+ \;=\; (Pf^+ - Pf^-)^+ \\ &= \max\{Pf^+ - Pf^-, 0\} \;\leq\; \max\{Pf^+, 0\} \;=\; Pf^+. \end{aligned}$$

Thus

$$(Pf)^+ \;\leq\; Pf^+.$$

(2) Let $f \in L^1[0,1]$. Since $Pf^+ \geq 0$ and $Pf^- \geq 0$ as above, we have

$$\begin{aligned} (Pf)^- &= (P(f^+ - f^-))^- \;=\; (Pf^+ - Pf^-)^- \\ &= \max\{-(Pf^+ - Pf^-), 0\} \;=\; \max\{Pf^- - Pf^+, 0\} \\ &\leq \max\{Pf^-, 0\} \;=\; Pf^-. \end{aligned}$$

Hence

$$(Pf)^- \;\leq\; Pf^-.$$

(3) Let $f \in L^1[0,1]$. By writing $|Pf| = (Pf)^+ + (Pf)^-$ and using (1) and (2) as well as the linearity of P, we get

$$\begin{aligned} |Pf| &= (Pf)^+ + (Pf)^- \;\leq\; Pf^+ + Pf^- \\ &= P(f^+ + f^-) \;=\; P|f|. \end{aligned}$$

Therefore,

$$|Pf| \leq P|f|.$$

(4) Let $f \in L^1[0,1]$. Using (3) and (M2) of definition 2.2, we have

$$\begin{aligned} \|Pf\|_1 &= \int_0^1 |Pf| \, d\lambda \leq \int_0^1 P|f| \, d\lambda \\ &= \int_0^1 |f| \, d\lambda = \|f\|_1. \end{aligned}$$

Therefore,

$$\|Pf\|_1 \leq \|f\|_1. \qquad \blacksquare$$

DEFINITION 2.5. The *support* of a function f is the set of $x \in [a,b]$ such that $f(x) \neq 0$ [76]; that is,

$$\text{supp } f = \{x \in [a,b] : f(x) \neq 0\}.$$

The concept of a fixed point is essential, and we are going to use it frequently in this dissertation, where a function f is called a fixed point if the income and outcome of f are the same; that is, $f(x) = x$ [71]. So here is the definition and some properties of a fixed point of a Markov operator.

DEFINITION 2.6. For $f \in L^1[0,1]$, we say that f is a *fixed point* of a Markov operator P if $Pf = f$.

THEOREM 2.7. If P is a Markov operator and $Pf = f$, then $Pf^+ = f^+$ and $Pf^- = f^-$.

PROOF. Let $Pf = f$ for some $f \in L^1[0,1]$. By the assumption, and by (1) and

(2) of Lemma 2.4, we have

$$f^+ \;=\; (Pf)^+ \;\leq\; Pf^+ \Longrightarrow 0 \;\leq\; Pf^+ - f^+ \tag{2.1}$$

and

$$f^- \;=\; (Pf)^- \;\leq\; Pf^- \Longrightarrow 0 \;\leq\; Pf^- - f^-. \tag{2.2}$$

Using (2.1), (2.2) and (4) of Lemma 2.4, we get

$$\begin{aligned} 0 \;=\; \int_0^1 0 \; d\lambda \;&\leq\; \int_0^1 (Pf^+ - f^+)\; d\lambda \;+\; \int_0^1 (Pf^- - f^-)\; d\lambda \\ &=\; \int_0^1 (Pf^+ + Pf^-)\; d\lambda \;-\; \int_0^1 (f^+ + f^-)\; d\lambda \\ &=\; \int_0^1 P(f^+ + f^-)\; d\lambda \;-\; \int_0^1 (f^+ + f^-)\; d\lambda \\ &=\; \int_0^1 P|f|\; d\lambda \;-\; \int_0^1 |f|\; d\lambda \\ &=\; \|P\;|f|\;\|_1 - \|\;|f|\;\|_1 \leq\; 0. \end{aligned}$$

Therefore we have,

$$0 \;\leq\; \int_0^1 (Pf^+ - f^+)\; d\lambda \;+\; \int_0^1 (Pf^- - f^-)\; d\lambda \leq 0,$$

Hence

$$\int_0^1 (Pf^+ - f^+)\; d\lambda \;+\; \int_0^1 (Pf^- - f^-)\; d\lambda \;=\; 0.$$

Note that the nonnegativity of $Pf^+ - f^+$ and $Pf^- - f^-$ from (2.1) and (2.2) gives

$$\int_0^1 (Pf^+ - f^+)\; d\lambda \;=\; 0 \qquad \text{and} \qquad \int_0^1 (Pf^- - f^-)\; d\lambda \;=\; 0,$$

which implies

$$Pf^+ \;=\; f^+ \qquad \text{and} \qquad Pf^- \;=\; f^-. \qquad \blacksquare$$

2.2 Construction of Frobenius-Perron Operator

Frobenius-Perron operators form a special class of Markov operators. To define a Frobenius-Perron operator, it is necessary to state the following definitions and theorems.

DEFINITION 2.8. Consider the measure space $(L^1[0,1], \mathcal{B}, \lambda)$. Then $f \in L^1[0,1]$ is called a *density function* [77] if $f \geq 0$ and $\|f\|_1 = \int_0^1 |f| \, d\lambda = \int_0^1 f \, d\lambda = 1$.

DEFINITION 2.9. A real-valued function μ on $\mathcal{B}$ is a *measure* [15] if it satisfies the following:

(1) $\mu(\phi) \;=\; 0$.

(2) $\mu(A) \geq 0$ for all $A \in \mathcal{B}$.

(3) Let $\{A_k\}_{k=1}^{\infty}$ be disjoint subsets of $\mathcal{B}$. Then

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) \;=\; \sum_{k=1}^{\infty} \mu(A_k).$$

LEMMA 2.10. Let $f \in L^1[0,1]$ and $f \geq 0$. If

$$\nu(A) \;=\; \int_A f \, d\lambda$$

for all $A \in \mathcal{B}$, then ν is a measure on $\mathcal{B}$.

PROOF.

(1) Let $f \in L^1[0,1]$. Because the Lebesgue integral value over empty set is 0, we

have

$$\nu(\phi) \;=\; \int_{\phi} f \; d\lambda \;=\; 0.$$

(2) Since by assumption $f \;\geq\; 0$, we have $\int_A f \; d\lambda \;\geq\; 0$. Hence $\nu(A) \;\geq\; 0$.

(3) Let $\{A\}_{i=1}^{\infty}$ be disjoint sets. Then

$$\begin{aligned}
\nu\left(\bigcup_{i=1}^{\infty} A_i\right) &= \int_{\cup_{i=1}^{\infty} A_i} f \; d\lambda \;=\; \int_0^1 f 1_{\cup_{i=1}^{\infty} A_i} \; d\lambda \\
&= \int_0^1 f \sum_{i=1}^{\infty} 1_{A_i} \; d\lambda \;=\; \sum_{i=1}^{\infty} \int_0^1 f 1_{A_i} \; d\lambda \\
&= \sum_{i=1}^{\infty} \int_{A_i} f \; d\lambda \;=\; \sum_{i=1}^{\infty} \nu(A_i).
\end{aligned}$$

Hence,

$$\nu\left(\bigcup_{i=1}^{\infty} A_i\right) \;=\; \sum_{i=1}^{\infty} \nu(A_i).$$

Therefore, by (1), (2), and (3), ν is a measure on $\mathcal{B}$. ■

DEFINITION 2.11. Let ν be a measure on $(L^1[0,1], \mathcal{B}, \lambda)$. We say that ν is an *absolutely continuous* measure with respect to λ [21] if

$$\lambda(A) \;=\; 0 \Longrightarrow \nu(A) \;=\; 0,$$

where $A \in \mathcal{B}$. In this case we write $\nu << \lambda$.

Radon-Nikodym Theorem. Suppose that ν is a finite measure such that $\nu << \lambda$. Then for every $A \in \mathcal{B}$, there exists a unique nonnegative $h \in (L^1[0,1], \mathcal{B}, \lambda)$ such that

$$\nu(A) \;=\; \int_A h \; d\lambda$$

for any $A \in \mathcal{B}$ [72].

PROOF. First we define a functional $\Phi : L^2(\lambda + \nu) \to \mathbb{R}$ by

$$\Phi(f) \;=\; \int_0^1 f \; d\nu$$

for all $f \in L^2(\lambda + \nu)$, where $f \in L^2(\lambda + \nu) \iff f : [0, 1] \to \mathbb{R}$ such that

$$\|f\|^2_{L^2(\lambda+\nu)} \;\equiv\; \int_0^1 |f|^2 \; d(\lambda + \nu) \;<\; \infty$$

and

$$\int_0^1 |f|^2 \; d(\lambda + \nu) \;=\; \int |f|^2 \; (d\lambda + d\nu) = \int_0^1 |f|^2 \; d\lambda + \int_0^1 |f|^2 \; d\nu \;<\; \infty.$$

We establish that $\Phi(f)$ is a real number in Claim 1.

Claim 1. For $f \in L^2(\lambda + \nu)$, we have $|\Phi(f)| < \infty$.

Proof. Suppose that $f \in L^2(\lambda + \nu)$. By Hölder's Inequality [72], we have

$$\begin{aligned}
|\Phi(f)| &= \left| \int_0^1 f \; d\nu \right| \;\leq\; \int_0^1 |f| \; d\nu \\
&\leq \int_0^1 |f| \; d(\lambda + \nu) \;=\; \int_0^1 |f \cdot 1| \; d(\lambda + \nu) \\
&= \|f \cdot 1\|_{L^1(\lambda+\nu)} \;\leq\; \|f\|_{L^2(\lambda+\nu)} \cdot \|1\|_{L^2(\lambda+\nu)} \\
&= \|f\|_{L^2(\lambda+\nu)} \cdot \left[\int_0^1 |1|^2 \; d(\lambda + \nu) \right]^{\frac{1}{2}} \;=\; \|f\|_{L^2(\lambda+\nu)} \cdot \left[\int 1_{[0,1]} \; d(\lambda + \nu) \right]^{\frac{1}{2}} \\
&= \|f\|_{L^2(\lambda+\nu)} \cdot \left[(\lambda + \nu) \left([0, 1] \right) \right]^{\frac{1}{2}} \;=\; \|f\|_{L^2(\lambda+\nu)} \cdot \left[\lambda[0, 1] + \nu[0, 1] \right]^{\frac{1}{2}}
\end{aligned}$$

$< \infty.$ □

In the next claim, we check if Φ is well-defined.

Claim 2. Suppose that both f and $g \in L^2(\lambda+\nu)$ such that $f = g$ almost everywhere on $[0,1]$ with respect to $\lambda+\nu$. Then $\Phi(f) = \Phi(g)$.

Proof. Suppose that $f = g$ almost everywhere on $[0,1]$ with respect to $\lambda+\nu$. Let $A = \{x \in [0,1] : f(x) \neq g(x)\}$. Then $0 = (\lambda+\nu)(A) = \lambda(A) + \nu(A)$. Hence, $\nu(A) = 0$. Therefore, $f = g$ almost everywhere on $[0,1]$ with respect to ν. So,

$$\int_0^1 f \, d\nu \;=\; \int_0^1 g \, d\nu.$$

Hence, $\Phi(f) = \Phi(g)$. Therefore, Φ is well-defined. □

Claim 1 and Claim 2 establish the fact that $\Phi : L^2(\lambda+\nu) \to \mathbb{R}$ is a well-defined functional. Clearly Φ is a linear functional on $L^2(\lambda+\nu)$. Next, we show that Φ is a bounded linear functional on $L^2(\lambda+\nu)$.

Claim 3. Φ is a bounded linear functional on $L^2(\lambda+\nu)$.

Proof. Note that, since ν is a finite measure, we have

$$\begin{aligned}
\|\Phi\| &= \sup\{|\Phi(f)| : \|f\|_{L^2(\lambda+\nu)} \leq 1\} \\
&\leq \sup\{\|f\|_{L^2(\lambda+\nu)}[\lambda[0,1]+\nu[0,1]]^{\frac{1}{2}} : \|f\|_{L^2(\lambda+\nu)} \leq 1\} \\
&\leq \sup\{[\lambda[0,1]+\nu[0,1]]^{\frac{1}{2}}\} \\
&= [\lambda[0,1]+\nu[0,1]]^{\frac{1}{2}} < \infty.
\end{aligned}$$

□

Since Φ is a bounded linear functional on $L^2(\lambda+\nu)$, by Riesz's representation theorem [54], there exists a unique $g \in L^2(\lambda+\nu)$ such that $\Phi(f) = \langle f, g\rangle$ for all $f \in L^2(\lambda+\nu)$; that is, there exists a unique $g \in L^2(\lambda+\nu)$ such that for all $f \in L^2(\lambda+\nu)$

$$\Phi(f) \;=\; \int_0^1 fg\, d(\lambda+\nu). \tag{2.3}$$

Since $\Phi(f) = \int_0^1 f\, d\nu$, for all $f \in L^2(\lambda+\nu)$, (2.3) becomes

$$\int_0^1 f\, d\nu \;=\; \int_0^1 fg\, d(\lambda+\nu) \;=\; \int_0^1 fg\, d\lambda + \int_0^1 fg\, d\nu.$$

So

$$\int_0^1 f(1-g)\, d\nu \;=\; \int_0^1 fg\, d\lambda \;\text{ for all }\; f \in L^2(\lambda+\nu). \tag{2.4}$$

Claim 4. Let $g \in L^2(\lambda+\nu)$. Then $0 \le g < 1$ almost everywhere on $[0,1]$ with respect to $\lambda+\nu$.

Proof. Let $A_1 = \{x \in [0,1] : g(x) \ge 1\}$. Let $f = 1_{A_1}$ in (2.4). Then we have

$$0 \;\le\; \int_0^1 1_{A_1} g\, d\lambda \;=\; \int_0^1 1_{A_1}(1-g)\, d\nu \;\le\; 0.$$

Thus, $1_{A_1}g = 0$ with respect to λ on $[0,1]$. Since $g \ge 1$ on A_1, $\lambda(A_1) = 0$.

Then, let $A_2 = \{x \in [0,1] : g(x) < 0\}$ and $f = 1_{A_2}$. Similarly, we can show that $\lambda(A_2) = 0$. Since $\nu << \lambda$, we have $\nu(A_1) = 0$ and $\nu(A_2) = 0$. Therefore, $(\lambda+\nu)(A_1) = 0$ and $(\lambda+\nu)(A_2) = 0$. So we may assume that $0 \le g < 1$ almost everywhere on $[0,1]$ with respect to $\lambda+\nu$. $\square$

Next, for $x \in [0,1]$, we define $h : [0,1] \to \mathbb{R}$ by

$$h(x) = \frac{g(x)}{1-g(x)}. \tag{2.5}$$

Note that $h \geq 0$ on $[0,1]$ since $0 \leq g < 1$ on $[0,1]$. Let $A \in \mathcal{B}$ and let $f = \frac{1_A}{1-g}$. Suppose $f \in L^2(\lambda+\nu)$. Then (2.4) can be written as

$$\int_0^1 \frac{1_A}{1-g}(1-g)\, d\nu = \int_0^1 \frac{1_A}{1-g} g\, d\lambda$$

or

$$\int_0^1 1_A\, d\nu = \int_0^1 \frac{1_A}{1-g} g\, d\lambda.$$

So

$$\nu(A) = \int_0^1 1_A\, d\nu = \int_0^1 \frac{1_A}{1-g} g\, d\lambda = \int_A \frac{g}{1-g}\, d\lambda = \int_A h\, d\lambda,$$

hence

$$\nu(A) = \int_A h\, d\lambda. \tag{2.6}$$

However, we do not know whether $f = \frac{1_A}{1-g} \in L^2(\lambda+\nu)$. Thus, we justify (2.6) as follows. Let $A \in \mathcal{B}$ and $k \in \mathbb{N}$. We define

$$f_k(x) = \begin{cases} \frac{1_A(x)}{1-g(x)} & \text{if } \frac{1_A(x)}{1-g(x)} \leq k \\ 0 & \text{if } \frac{1_A(x)}{1-g(x)} > k. \end{cases}$$

Note that for any $k \in \mathbb{N}$, we have $0 \leq f_k \leq f$, $0 \leq f_k \leq k$ and $\{f_k\}_{k=1}^{\infty}$ is an increasing sequence. Also note that, $f_k \in L^2(\lambda+\nu)$ for all $k \in \mathbb{N}$. Then, let $f = f_k$ in (2.4), we have

$$\int_0^1 f_k(1-g)\, d\nu = \int_0^1 f_k g\, d\lambda \tag{2.7}$$

for all $k \in \mathbb{N}$. Observe that $\{f_k(1-g)\}_{k=1}^{\infty}$ and $\{f_k g\}_{k=1}^{\infty}$ are increasing sequences of nonnegative functions. Also note that

$$f_k(x)(1-g(x)) \quad = \quad \begin{cases} 1_A(x) & \frac{1_A(x)}{1-g(x)} \leq k \\ 0 & \frac{1_A(x)}{1-g(x)} > k. \end{cases}$$

We have 2 cases for $\frac{1_A(x)}{1-g(x)}$.

Case 1. $x \in [0,1] - A$. Then $f_k(x) = 0$ for all $k \in \mathbb{N}$. Thus, $f_k(x)(1-g(x)) = 0$ for all $k \in \mathbb{N}$. Hence, $f_k(x)(1-g(x)) \to 1_A(x) \to 0$.

Case 2. $x \in A$. We choose $k \geq \frac{1}{1-g(x)}$. Then

$$\frac{1_A(x)}{1-g(x)} \quad = \quad \frac{1}{1-g(x)} \quad \leq \quad k \implies f_k(x)(1-g(x)) \quad = \quad 1_A(x) \quad = \quad 1,$$

which implies $f_k(x)(1-g(x)) \to 1$. This establishes that

$$f_k(1-g) \text{ increases to } 1_A \text{ on } [0,1]. \tag{2.8}$$

Similarly,

$$f_k g \text{ increases to } \frac{1_A}{1-g} g \text{ on } [0,1]. \tag{2.9}$$

Note that (2.7) gives

$$\lim_{k\to\infty} \int_0^1 f_k(1-g)\, d\nu \quad = \quad \lim_{k\to\infty} \int_0^1 f_k g\, d\lambda,$$

which implies by Monotone Convergence Theorem [73],

$$\int_0^1 \lim_{k\to\infty} f_k(1-g)\, d\nu \quad = \quad \int_0^1 \lim_{k\to\infty} f_k g\, d\lambda.$$

Using (2.8) and (2.9), we get

$$\nu(A) \;=\; \int_A d\nu \;=\; \int_0^1 1_A \, d\nu \;=\; \int_0^1 \frac{1_A}{1-g} g \, d\lambda \;=\; \int_A \frac{g}{1-g} \, d\lambda,$$

where by using (2.5), for all $A \in \mathcal{B}$ we get

$$\nu(A) \;=\; \int_A h \, d\lambda. \tag{2.10}$$

Finally, we need to show that $h \in (L^1[0,1], \mathcal{B}, \lambda)$. Let $A = [0,1]$ in (2.10). Since ν is a finite measure, we have

$$\nu([0,1]) \;=\; \int_0^1 h \, d\lambda \;<\; \infty.$$

Also, since $h \geq 0$, we have

$$\|h\|_1 \;=\; \int_0^1 |h| \, d\lambda \;=\; \int_0^1 h \, d\lambda \;<\; \infty$$

Therefore, $h \in (L^1[0,1], \mathcal{B}, \lambda)$. ■

THEOREM 2.12. If $f \in L^1[0,1]$ and $f \geq 0$, then the measure ν defined by

$$\nu(A) \;=\; \int_A f \, d\lambda$$

for all $A \in \mathcal{B}$ is an absolutely continuous measure with respect to λ.

PROOF. By Lemma 2.10, we know that ν is a measure. So all we need to show is that ν is an absolutely continuous measure with respect to λ. Let $\lambda(A) = 0$. Since the integral over a zero measure set equals zero, we have

$$\nu(A) \;=\; \int_A f \, d\lambda \;=\; 0.$$

Hence, $\lambda(A) = 0 \implies \nu(A) = 0$. Thus ν is an absolutely continuous measure with respect to λ. ■

DEFINITION 2.13. A transformation $S : [0,1] \to [0,1]$ is *measurable* [45] if for all $A \in \mathcal{B}$

$$S^{-1}(A) \in \mathcal{B}.$$

DEFINITION 2.14. A measurable transformation $S : [0,1] \to [0,1]$ is *nonsingular* [45] if for all $A \in \mathcal{B}$

$$\lambda(A) \;=\; 0 \implies \lambda(S^{-1}(A)) \;=\; 0.$$

To construct the Frobenius-Perron operator, we need the following theorem.

THEOREM 2.15. Assume that $S : [0,1] \to [0,1]$ be a measurable and nonsingular transformation. Let $f \in L^1[0,1]$ to be a nonnegative function. Then

$$\nu(A) \;=\; \int_{S^{-1}(A)} f \; d\lambda.$$

is a finite measure such that $\nu << \lambda$.

PROOF. We will start the proof with the following claims.

Claim 1. Let A_n be disjoint sets in $\mathcal{B}$. Then

$$S^{-1}\left(\bigcup_{n=1}^{\infty}(A_n)\right) \;=\; \bigcup_{n=1}^{\infty} S^{-1}(A_n).$$

Proof. Let

$$x \in S^{-1}\left(\bigcup_{n=1}^{\infty}(A_n)\right) \iff S(x) \in \bigcup_{n=1}^{\infty} A_n \iff S(x) \in A_n, \text{ for some } n$$

$$\iff x \in S^{-1}(A_n) \text{ for some } n \iff x \in \bigcup_{n=1}^{\infty} S^{-1}(A_n).$$

Therefore,

$$S^{-1}\left(\bigcup_{n=1}^{\infty}(A_n)\right) \;=\; \bigcup_{n=1}^{\infty} S^{-1}(A_n).$$

$\square$

Claim 2. ν is a finite measure.

Proof. Let $f \in L^1[0,1]$.

(1) Note that

$$\nu(\phi) \;=\; \int_{S^{-1}(\phi)} f \; d\lambda \;=\; \int_{\phi} f \; d\lambda \;=\; 0.$$

Hence, $\nu(\phi) \;=\; 0$.

(2) Since by assumption $f \geq 0$, we have

$$\int_{S^{-1}(A)} f \; d\lambda \;\geq\; 0.$$

Therefore, $\nu(A) \geq 0$.

(3) Let $\{A_i\}_{i=1}^{\infty}$ be disjoint sets. Using Claim 1, we have

$$\begin{aligned}
\nu\left(\bigcup_{i=1}^{\infty} A_i\right) \;&=\; \int_{S^{-1}(\cup_{i=1}^{\infty} A_i)} f \; d\lambda \;=\; \int_{\cup_{i=1}^{\infty} S^{-1}(A_i)} f \; d\lambda \\
&=\; \int_0^1 f 1_{\cup_{i=1}^{\infty} S^{-1}(A_i)} \; d\lambda \;=\; \int_0^1 f \sum_{i=1}^{\infty} 1_{S^{-1}(A_i)} \; d\lambda \\
&=\; \sum_{i=1}^{\infty} \int_{S^{-1}(A_i)} f \; d\lambda \;=\; \sum_{i=1}^{\infty} \nu(A_i).
\end{aligned}$$

Hence,

$$\nu(\cup_{i=1}^{\infty} A_i) \;=\; \sum_{i=1}^{\infty} \nu(A_i).$$

Therefore, by (1), (2), and (3) ν is a measure. In addition, since f is integrable [61], ν is a finite measure. $\square$

Next we prove that ν is an absolutely continuous measure with respect to λ. Assume that $\lambda(A) = 0$ for all $A \in \mathcal{B}$. Since S is a nonsingular transformation, $\lambda(S^{-1}(A)) = 0$. So,

$$\nu(A) = \int_{S^{-1}(A)} f \, d\lambda = 0.$$

Thus, $\nu(A) = 0$ whenever $\lambda(A) = 0$ for all $A \in \mathcal{B}$. Therefore, ν is an absolutely continuous measure with respect to λ. $\blacksquare$

Since the assumption of $\nu << \mu$ in Theorem 2.15 is fulfilled, by Radon-Nikodym Theorem, there exists a unique nonnegative function $h \in L^1[0,1]$ such that

$$\nu(A) = \int_A h \, d\lambda$$

for any $A \in \mathcal{B}$. So we have

$$\int_A h \, d\lambda = \int_{S^{-1}(A)} f \, d\lambda.$$

Since h depends on f, denoting h by $P_S f$, we have

$$\int_A P_S f \, d\lambda = \int_{S^{-1}(A)} f \, d\lambda, \tag{2.11}$$

when $f \in L^1[0,1]$ and $f \geq 0$.

Note that equation (2.11) can be generalize to any $f \in L^1[0,1]$ as follows. Suppose $f \in L^1[0,1]$. We write $f = f^+ - f^-$, and we define $P_S f = P_S f^+ - P_S f^-$ for any $f \in L^1[0,1]$. Then by the linearity of the integral, the fact that $f^+ \geq 0$ and $f^- \geq 0$, and equation (2.11), we get

$$
\begin{aligned}
\int_A P_S f \, d\lambda &= \int_A P_S f^+ \, d\lambda - \int_A P_S f^- \, d\lambda \\
&= \int_{S^{-1}(A)} f^+ \, d\lambda - \int_{S^{-1}(A)} f^- \, d\lambda \\
&= \int_{S^{-1}(A)} (f^+ - f^-) \, d\lambda = \int_{S^{-1}(A)} f \, d\lambda.
\end{aligned}
$$

Hence for any $f \in L^1[0,1]$,

$$\int_A P_S f \, d\lambda = \int_{S^{-1}(A)} f \, d\lambda.$$

Using the above discussion, we make a formal definition of the Frobenius-Perron operator.

DEFINITION 2.16. Let $S : [0,1] \to [0,1]$ be a measurable and nonsingular transformation. Then the operator $P_S : L^1[0,1] \to L^1[0,1]$ defined (implicitly) by

$$\int_A P_S f \, d\lambda = \int_{S^{-1}(A)} f \, d\lambda \tag{2.12}$$

for any $A \in \mathcal{B}$ is the *Frobenius-Perron* operator associated with transformation S.

If A is a subinterval of $[0,1]$, Frobenius-Perron operator can be defined explicitly as the following remark shows.

REMARK 2.17. If $A = [0,x] \subseteq [0,1]$, then (2.12) becomes

$$\int_{[0,x]} P_S f \, d\lambda = \int_{S^{-1}([0,x])} f \, d\lambda.$$

By differentiating both sides with respect to x, we get

$$\frac{d}{dx} \int_{[0,x]} P_S f \, d\lambda = \frac{d}{dx} \int_{S^{-1}([0,x])} f \, d\lambda.$$

Then by using the Fundamental Theorem of Calculus [67], $P_S f(x)$ can be written explicitly as

$$P_S f(x) \;=\; \frac{d}{dx}\int_{S^{-1}([0,x])} f(t)\;dt,\;\; x \in [0,1]. \tag{2.13}$$

We summarize some properties of the Frobenius-Perron operator in the next lemma.

LEMMA 2.18. Let P_S be a Frobenius-Perron operator. Then we have the following statements:

(a) P_S is a linear operator; that is,

$$P_S(\alpha_1 f_1 + \alpha_2 f_2) \;=\; \alpha_1 P_S f_1 + \alpha_2 P_S f_2$$

for all $f_1, f_2 \in L^1[0,1]$ and $\alpha_1, \alpha_2 \in \mathbb{R}$.

(b) P_S is a positive operator; that is, if $f \geq 0$, then $P_S f \geq 0$.

(c) P_S preserves the integral; that is, $\int_0^1 P_S f \; d\lambda \;=\; \int_0^1 f \; d\lambda$ for any $f \in L^1[0,1]$.

PROOF.

(a) Let $f_1, f_2 \in L^1[0,1]$ and $\alpha_1, \alpha_2 \in \mathbb{R}$. Then by the linearity of the integral and the definiton of P_S, we have

$$\begin{aligned}
\int_A P_S(\alpha_1 f_1 + \alpha_2 f_2)\; d\lambda &= \int_{S^{-1}(A)} (\alpha_1 f_1 + \alpha_2 f_2)\; d\lambda \\
&= \int_{S^{-1}(A)} \alpha_1 f_1 \; d\lambda \;+\; \int_{S^{-1}(A)} \alpha_2 f_2 \; d\lambda \\
&= \alpha_1 \int_{S^{-1}(A)} f_1 \; d\lambda \;+\; \alpha_2 \int_{S^{-1}(A)} f_2 \; d\lambda
\end{aligned}$$

$$= \alpha_1 \int_A P_S f_1 \, d\lambda \; + \; \alpha_2 \int_A P_S f_2 \, d\lambda$$

$$= \int_A (\alpha_1 P_S f_1 \; + \; \alpha_2 P_S f_2) \, d\lambda.$$

Hence, for all $f_1, f_2 \in L^1[0,1]$ and $\alpha_1, \alpha_2 \in \mathbb{R}$,

$$P_S(\alpha_1 f_1 + \alpha_2 f_2) \;\; = \;\; \alpha_1 P_S f_1 + \alpha_2 P_S f_2.$$

(b) Let $f \in L^1[0,1]$ and $f \geq 0$. Then

$$\int_A P_S f \, d\lambda \;\; = \;\; \int_{S^{-1}(A)} f \, d\lambda \; \geq \; 0$$

for any $A \in \mathcal{B}$. It follows that $f \geq 0$ implies $P_S f \geq 0$.

(c) For any $f \in L^1[0,1]$. Since $S^{-1}([0,1]) = [0,1]$, because $[0,1]$ is the whole set, we have

$$\int_0^1 P_S f \, d\lambda \;\; = \;\; \int_{S^{-1}([0,1])} f \, d\lambda \;\; = \;\; \int_0^1 f \, d\lambda.$$

■

REMARK 2.19. Note that Lemma 2.18 tells us that Frobenius-Perron operator is a Markov operator. Therefore, all the properties of Markov operators apply to Frobenius-Perron operators.

DEFINITION 2.20. Consider the measure space $(L^1[0,1], \mathcal{B}, \lambda)$. Let P_S be the Frobenius-Perron operator associated with a measurable and nonsingular transformation $S : [0,1] \to [0,1]$. The function f is a *stationary density* function if f is a density function which is also a fixed point of P_S; that is, f is a density function that satisfies $P_S f = f$.

DEFINITION 2.21. Let T be a linear operator from $L^1[0,1]$ to $L^1[0,1]$. Then 1-norm of T [57] is defined by

$$\|T\|_1 = \sup\left\{\frac{\|Tf\|_1}{\|f\|_1} : f \in L^1[0,1] \text{ and } f \neq 0\right\}.$$

LEMMA 2.22. If P_S is a Frobenius-Perron operator, then $\|P_S\|_1 = 1$.

PROOF. Note that P_S is a Markov operator. Then by (4) of Lemma 2.4, we have

$$\|P_S f\|_1 \leq \|f\|_1$$

for any $f \in L^1[0,1]$. In particular, if $f \geq 0$, then by (M2) of Definition 2.2, we have $\|P_S f\|_1 = \|f\|_1$. Thus, for any $f \in L^1[0,1]$

$$\|P_S\|_1 = \sup\left\{\frac{\|P_S f\|_1}{\|f\|_1} : f \in L^1[0,1] \text{ and } f \neq 0\right\} = 1.$$

■

2.3 Koopman Operator

To justify (1.3), we utilize another operator that is called a Koopman operator defined as follows:

DEFINITION 2.23. Let $(L^1[0,1], \mathcal{B}, \lambda)$ be a measure space, $S : [0,1] \to [0,1]$ be a measurable and nonsingular transformation, and $f \in L^\infty([0,1])$, where $L^\infty[0,1]$ is the space of bounded continuous functions [12]. The operator $K_S : L^\infty[0,1] \to L^\infty[0,1]$ defined by

$$K_S f(x) = f(S(x)) \tag{2.14}$$

for any $x \in [0,1]$ is called a *Koopman* operator associated with S.

Note that the nonsingularity of transformation S assures that Koopman operator is well-defined. To see this, we let $f_1 = f_2$ almost everywhere with respect to λ. If we define a set $A = \{x : f_1(S(x)) \neq f_2(S(x))\}$, then $S(A) = \{S(x) : f_1(S(x)) \neq f_2(S(x))\}$. By letting $y = S(x)$, we get $S(A) = \{y : f_1(y) \neq f_2(y)\}$. Since by assumption we have $f_1 = f_2$, then $\lambda(S(A)) = 0$. By the nonsingularity of S, we have $\lambda(A) = \lambda(S^{-1}(S(A))) = 0$. Therefore, $f_1(S(x)) = f_2(S(x))$ almost everywhere with respect to λ. That is, $K_S f_1(x) = K_S f_2(x)$. Thus, Koopman operator is well-defined.

LEMMA 2.24. Let K_S be a Koopman operator associated with S, and P_S be the Frobenius-Perron operator associated with S. Then for all $f \in L^1[0,1]$ and $g \in L^\infty[0,1]$, Koopman operator is the adjoint of the the Frobenius-Perron operator; that is,

$$\langle P_S f, g \rangle = \langle f, K_S g \rangle.$$

PROOF. First let $g = 1_A$, where $A \in \mathcal{B}$. Note that

$$\begin{aligned} 1_A(S(x)) &= \begin{cases} 1 & \text{if } S(x) \in A \\ 0 & \text{if } S(x) \notin A \end{cases} \\ &= \begin{cases} 1 & \text{if } x \in S^{-1}(A) \\ 0 & \text{if } x \notin S^{-1}(A) \end{cases} \\ &= 1_{S^{-1}(A)}(x). \end{aligned}$$

Thus

$$1_A(S(x)) = 1_{S^{-1}(A)}(x). \tag{2.15}$$

So, using the definition of Frobenius-Perron operator (2.12), the definition of Koopman operator (2.14), and (2.15), we have

$$
\begin{aligned}
\langle P_S f, 1_A \rangle &= \int_0^1 1_A P_S f \ d\lambda \ = \ \int_A P_S f \ d\lambda \\
&= \int_{S^{-1}(A)} f \ d\lambda \ = \ \int_0^1 1_{S^{-1}(A)} f \ d\lambda \\
&= \int_0^1 1_A(S) f \ d\lambda \ = \ \int_0^1 K_S 1_A f \ d\lambda \\
&= \langle f, K_S 1_A \rangle.
\end{aligned}
$$

It follows that by the linearity of the Lebesgue integral, the statement is true for any simple function, $g = \sum_{k=1}^{n} a_k I_{A_k}$, and hence for all $g \in L^\infty[0,1]$ [59]. ■

2.4 Invariant Measure

Another concept we need is an invariant measure under a measurable and nonsingular transformation S.

DEFINITION 2.25. Consider the measure space $(L^1[0,1], \mathcal{B}, \mu)$. Let $S : [0,1] \to [0,1]$ be a measurable and nonsingular transformation. Then S is *μ-measure preserving* if for all $A \in \mathcal{B}$

$$
\mu(S^{-1}(A)) \ = \ \mu(A). \tag{2.16}
$$

If S is μ-measure preserving, then we also say μ is an *invariant* measure under S [11].

The following theorem gives the connection between an invariant measure and a stationary density function of P_S.

THEOREM 2.26. Consider the measure space $(L^1[0,1], \mathcal{B}, \lambda)$. Let $S : [0,1] \to [0,1]$ be a measurable and nonsingluar transformation, and P_S be the Frobenius-Perron operator associated with S. Then the probability measure defined by

$$\mu(A) \;=\; \int_A f^* \, d\lambda \tag{2.17}$$

is an invariant probability measure under S if and only if $f^* \in L^1[0,1]$ is a stationary density function of P_S.

PROOF. Let $f^* \in L^1[0,1]$ be a density function and P_S be the Frobenius-Perron operator associated S. First, we assume that μ is an invariant measure under S; that is by (2.16) for all $A \in \mathcal{B}$, $\mu(A) = \mu(S^{-1}(A))$. So by using (2.17) and (2.12), we have

$$\int_A f^* \, d\lambda \;=\; \int_{S^{-1}(A)} f^* d\lambda \;=\; \int_A P_S f^* \, d\lambda.$$

Hence,

$$P_S f^* \;=\; f^*$$

for all $f^* \in L^1[0,1]$, which means f^* is a stationary density function of P_S.

For the other direction, we assume that f^* is a stationary density function of P_S; that is, $P_S f^* = f^*$. Note that by (2.17) and the definition of Frobenius-Perron operator (2.12), we have for any $A \in \mathcal{B}$

$$\mu(A) \;=\; \int_A f^* \, d\lambda \;=\; \int_A P_S f^* \, d\lambda$$

$$= \int_{S^{-1}(A)} f^* \, d\lambda \;\; = \;\; \mu(S^{-1}(A))$$

Hence, for any $A \in \mathcal{B}$

$$\mu(A) \;\; = \;\; \mu(S^{-1}(A)).$$

Therefore, μ is an invariant measure under S. ■

REMARK 2.27. The importance of computing the stationary density function f^* of P_S is shown in Theorem 2.26 because by using f^*, we can generate the invariant measure under S.

In the following three examples, we illustrate Theorem 2.26. We will show that f^* is a stationary density of P_S, and hence if we define measure μ by $\mu(A) = \int_A f^* \, d\lambda$, then μ is an invariant measure under S for all $A \in \mathcal{B}$.

EXAMPLE 2.28. Let

$$S(x) = \begin{cases} \frac{2x}{1-x^2} & 0 \leq x \leq \sqrt{2} - 1 \\ \frac{1-x^2}{2x} & \sqrt{2} - 1 \leq x \leq 1. \end{cases}$$

and

$$f^*(x) = \frac{4}{\pi(1+x^2)}.$$

PROOF. Let

$$S_1(x) \;\; = \;\; \frac{2x}{1-x^2} \text{ for } 0 \leq x \leq \sqrt{2}-1 \quad \text{and} \quad S_2(x) \;\; = \;\; \frac{1-x^2}{2x} \text{ for } \sqrt{2}-1 \leq x \leq 1.$$

Then

$$S_1^{-1}(x) \;=\; \frac{x}{\sqrt{x^2+1}+1} \quad \text{for} \quad 0 \le x \le 1;$$

$$S_2^{-1}(x) \;=\; -x+\sqrt{x^2+1} \quad \text{for} \quad 0 \le x \le 1.$$

Then by the explicit definition of P_S in (2.13) and the Fundamental Theorem of Calculus, we have

$$\begin{aligned}
P_S f^*(x) = \;& \frac{d}{dx}\int_{S^{-1}[0,x]} f^*(t)\;dt \\
= \;& \int_0^{\frac{x}{\sqrt{x^2+1}+1}} f^*(t)\;dt + \int_{-x+\sqrt{x^2+1}}^{1} f^*(t)\;dt \\
= \;& f^*\left(\frac{x}{\sqrt{x^2+1}+1}\right)\frac{d}{dx}\left(\frac{x}{\sqrt{x^2+1}+1}\right) \\
& -\; f^*\left(-x+\sqrt{x^2+1}\right)\frac{d}{dx}\left(-x+\sqrt{x^2+1}\right) \\
= \;& \frac{4}{\pi\left(1+\left(\frac{x}{\sqrt{x^2+1}+1}\right)^2\right)}\frac{\sqrt{x^2+1}+1-x\left(\frac{x}{\sqrt{x^2+1}}\right)}{(\sqrt{x^2+1}+1)^2} \\
& -\; \frac{4}{\pi(1+(-x+\sqrt{x^2+1})^2)}\left(-1+\frac{x}{\sqrt{x^2+1}}\right) \\
= \;& \frac{4}{\pi\left(\frac{(\sqrt{x^2+1}+1)^2+x^2}{(\sqrt{x^2+1}+1)^2}\right)}\cdot\frac{\frac{x^2+1+\sqrt{x^2+1}-x^2}{\sqrt{x^2+1}}}{(\sqrt{x^2+1}+1)^2} \\
& -\; \frac{4}{\pi(1+x^2-2x\sqrt{x^2+1}+x^2+1)}\cdot\frac{-\sqrt{x^2+1}+x}{\sqrt{x^2+1}} \\
= \;& \frac{4(\sqrt{x^2+1}+1)}{\pi\left((\sqrt{x^2+1}+1)^2+x^2\right)\sqrt{x^2+1}}
\end{aligned}$$

$$
\begin{aligned}
& + \quad \frac{2}{\pi(x^2 - x\sqrt{x^2+1}+1)} \cdot \frac{\sqrt{x^2+1}-x}{\sqrt{x^2+1}} \\
= \; & \frac{4(\sqrt{x^2+1}+1)}{\pi\left((x^2+1+2\sqrt{x^2+1}+1+x^2)\sqrt{x^2+1}\right)} \\
& + \quad \frac{2(\sqrt{x^2+1}-x)}{\pi(x^2\sqrt{x^2+1}-x^3-x+\sqrt{x^2+1})} \\
= \; & \frac{2(\sqrt{x^2+1}+1)}{\pi\left((x^2+1)\sqrt{x^2+1}+(x^2+1)\right)} + \frac{2(\sqrt{x^2+1}-x)}{\pi((x^2+1)\sqrt{x^2+1}-x(x^2+1))} \\
= \; & \frac{2(\sqrt{x^2+1}+1)}{\pi(x^2+1)(\sqrt{x^2+1}+1)} + \frac{2(\sqrt{x^2+1}-x)}{\pi(x^2+1)(\sqrt{x^2+1}-x)} \\
= \; & \frac{2}{\pi\,(x^2+1))} + \frac{2}{\pi(x^2+1)} \quad = \quad \frac{4}{\pi\,(x^2+1))} \quad = \quad f^*(x).
\end{aligned}
$$

Therefore, $P_S f^* = f^*$. So f^* is a stationary density function of P_S. Thus $\mu(A) = \int_A f^*\, d\lambda$ is an invariant measure under S for all $A \in \mathcal{B}$. ■

EXAMPLE 2.29. Let

$$
S(x) \quad = \quad \begin{cases} \frac{2x}{1-x} & 0 \le x \le \frac{1}{3} \\ \frac{1-x}{2x} & \frac{1}{3} \le x \le 1, \end{cases}
$$

and let

$$
f^*(x) \quad = \quad \frac{2}{(1+x)^2}.
$$

PROOF. Let

$$
S_1(x) \;=\; \frac{2x}{1-x} \;\text{ for }\; 0 \le x \le \frac{1}{3} \quad \text{and} \quad S_2(x) \;=\; \frac{1-x}{2x} \;\text{ for }\; \frac{1}{3} \le x \le 1.
$$

Then the inverse images of S_1 and S_2 are

$$\begin{aligned} S_1^{-1}(x) &= \frac{x}{2+x} \quad \text{for} \;\; 0 \le x \le 1; \\ S_2^{-1}(x) &= \frac{1}{2x+1} \;\text{for}\; 0 \le x \le 1. \end{aligned}$$

By (2.13), we have

$$\begin{aligned} P_S f^*(x) &= \frac{d}{dx} \int_{S^{-1}[0,x]} f^*(t)\; dt \\ &= \int_0^{\frac{x}{2+x}} f^*(t)\; dt \;+\; \int_{\frac{1}{2x+1}}^{1} f^*(t)\; dt \\ &= f^*\left(\frac{x}{2+x}\right) \frac{d}{dx}\left(\frac{x}{2+x}\right) \;-\; f^*\left(\frac{1}{2x+1}\right) \frac{d}{dx}\left(\frac{1}{2x+1}\right) \\ &= \frac{2}{\left(1+\frac{x}{2+x}\right)^2} \frac{2+x-x}{(2+x)^2} \;-\; \frac{2}{\left(1+\frac{1}{2x+1}\right)^2} \frac{-2}{(2x+1)^2} \\ &= \frac{2}{4\left(\frac{1+x}{2+x}\right)^2} \frac{2}{(2+x)^2} \;+\; \frac{2}{4\left(\frac{x+1}{2x+1}\right)^2} \frac{2}{(2x+1)^2} \\ &= \frac{1}{(1+x)^2} \;+\; \frac{1}{(x+1)^2} \;=\; \frac{2}{(1+x)^2} \;=\; f^*(x). \end{aligned}$$

Hence, f^* is a stationary density function of P_S. Therefore, $\mu(A) = \int_A f^* \, d\lambda$ is an invariant measure under S for all $A \in \mathcal{B}$. ■

EXAMPLE 2.30. Let

$$S(x) = \left(\frac{1}{8} - 2\left|x - \frac{1}{2}\right|^3\right)^{1/3} + \frac{1}{2}$$

and let

$$f^*(x) = 12\left(x - \frac{1}{2}\right)^2.$$

PROOF. The inverse images of S is

$$S_1^{-1}(x) \;=\; \frac{1}{2} - \left(\frac{\frac{1}{8} - \left(x - \frac{1}{2}\right)^3}{2}\right)^{1/3} \quad \text{for} \;\; 0 \le x \le \frac{1}{2}$$

and

$$S_2^{-1}(x) \;=\; \frac{1}{2} + \left(\frac{\frac{1}{8} - \left(x - \frac{1}{2}\right)^3}{2}\right)^{1/3} \quad \text{for} \;\; \frac{1}{2} \le x \le 1.$$

By (2.13), we have

$$\begin{aligned}
P_S f^*(x) &= \frac{d}{dx}\int_{S^{-1}[0,x]} f^*(t)\, dt \\
&= \int_0^{\frac{1}{2} - \left(\frac{\frac{1}{8} - \left(x-\frac{1}{2}\right)^3}{2}\right)^{1/3}} f^*(t)\, dt \;+\; \int_{\frac{1}{2} + \left(\frac{\frac{1}{8} - \left(x-\frac{1}{2}\right)^3}{2}\right)^{1/3}}^{1} f^*(t)\, dt \\
&= f^*\left(\frac{1}{2} - \left(\frac{\frac{1}{8} - \left(x - \frac{1}{2}\right)^3}{2}\right)^{1/3}\right) \frac{d}{dx}\left(\frac{1}{2} - \left(\frac{\frac{1}{8} - \left(x - \frac{1}{2}\right)^3}{2}\right)^{1/3}\right) \\
&\quad - f^*\left(\frac{1}{2} + \left(\frac{\frac{1}{8} - \left(x - \frac{1}{2}\right)^3}{2}\right)^{1/3}\right) \frac{d}{dx}\left(\frac{1}{2} + \left(\frac{\frac{1}{8} - \left(x - \frac{1}{2}\right)^3}{2}\right)^{1/3}\right) \\
&= 12\left(\frac{1}{2} - \left(\frac{\frac{1}{8} - \left(x - \frac{1}{2}\right)^3}{2}\right)^{1/3} - \frac{1}{2}\right)^2 \frac{(x-\frac{1}{2})^2}{\left(2\left(\frac{1}{8} - \left(x - \frac{1}{2}\right)^3\right)^2\right)^{1/3}} \\
&\quad - 12\left(\frac{1}{2} + \left(\frac{\frac{1}{8} - \left(x - \frac{1}{2}\right)^3}{2}\right)^{1/3} - \frac{1}{2}\right)^2 \frac{-(x-\frac{1}{2})^2}{\left(2\left(\frac{1}{8} - \left(x - \frac{1}{2}\right)^3\right)^2\right)^{1/3}} \\
&= 6\left(x - \frac{1}{2}\right)^2 + 6\left(x - \frac{1}{2}\right)^2
\end{aligned}$$

$$= 12\left(x-\frac{1}{2}\right)^2 = f^*(x).$$

Hence, f^* is a stationary density function of P_S. Therefore, $\mu(A) = \int_A f^* \, d\lambda$ is an invariant measure under S for all $A \in \mathcal{B}$. ■

2.5 Ergodic Transformation

To use Ergodic theory, which is important in the existence and uniqueness of the stationary density function of P_S, we utilize ergodic transformation. However, before we give the definition of an ergodic transformation, we will start with the definition of an invariant set.

DEFINITION 2.31. Consider the measure space $(L^1[0,1], \mathcal{B}, \lambda)$. A set $A \in \mathcal{B}$ is called an *invariant set* under S [11] if it satisfies

$$S^{-1}(A) = A.$$

DEFINITION 2.32. Consider the measure space $(L^1[0,1], \mathcal{B}, \lambda)$. The measurable and nonsingular transformation $S : [0,1] \to [0,1]$ is *ergodic* [68] if for every invariant set under S, for instance $A \in \mathcal{B}$, we have

$$\lambda(A) = 0 \text{ or } \lambda(A^c) = 0.$$

In other words, S is *ergodic* if all the invariant sets A are trivial elements of $\mathcal{B}$.

The following result can be helpful to check whether the transformation S is ergodic.

LEMMA 2.33. Consider the measure space $(L^1[0,1], \mathcal{B}, \lambda)$. Assume that $S : [0,1] \to$

$[0,1]$ is a measurable and nonsingluar transformation. Then S is ergodic if and only if for every $f \in L^1[0,1]$ that satisfies

$$f(S(x)) \;=\; f(x) \tag{2.18}$$

for all $x \in [0,1]$ implies that f is a constant function.

PROOF. First we will prove the forward direction, which is if S is ergodic, then f that satisfies $f(S(x)) = f(x)$ is constant, by the way of contradiction. So, we assume that S is ergodic, and f that satisfies $f(S(x)) = f(x)$ is not constant. Then there exists some $r \in \mathbb{R}$ such that both

$$A \;=\; \{x : f(x) \geq r\} \;\text{ and }\; A^c \;=\; \{x : f(x) < r\}$$

have positive measures. Therefore, A and A^c are nontrivial sets; that is, $\lambda(A) > 0$ and $\lambda(A^c) > 0$. However, A and A^c are invariant sets under S. This can be seen, using the equation (2.18), as follows:

$$\begin{aligned} S^{-1}(A) \;&=\; \{x : S(x) \in A\} \;=\; \{x : f(S(x)) \geq r\} \\ &=\; \{x : f(x) \geq r\} \;=\; A. \end{aligned}$$

Similarly, $S^{-1}(A^c) \;=\; A^c$. Thus, S is not ergodic, and this contradicts the assumption. Therefore, f is a constant function.

For the other direction, suppose that f that satisfies $f(S(x)) = f(x)$ is constant, and S is not ergodic. Then, there exists an invariant set $A \in \mathcal{B}$ such that $\lambda(A)$ and $\lambda(A^c)$ are both positive. Let $f = 1_A$. Since A is invariant under S, using (2.15) we have

$$f(S(x)) \;=\; 1_A(S(x)) \;=\; 1_{S^{-1}(A)}(x) \;=\; 1_A(x) \;=\; f(x).$$

Since $\lambda(A)$ and $\lambda(A^c)$ are both positive, f that satisfies $f(S(x)) = f(x)$ is not constant, which is a contradiction. It follows that S is ergodic. ■

In Theorem 2.35, which will be stated later, we will prove that the ergodicity of S assures that if there is a stationary density f^* of P_S, then f^* is unique. This result will be used in Section 6.4 to show the uniqueness of the given f^*. In order to prove this, we need first to prove the following lemma.

LEMMA 2.34. Let $S : [0,1] \to [0,1]$ be a measurable and nonsingular transformation, and P_S be the Frobenius-Perron operator associated with S. Assume that $f \in L^1[0,1]$, $f \geq 0$. Then

$$\text{supp } f \;\subseteq\; S^{-1}(\text{supp } P_S f).$$

PROOF. Note that since $f \geq 0$, by using the definition of Frobenius-Perron operator (2.12), we have

$$\begin{aligned} P_S f(x) \;=\; 0 \text{ for all } x \in A \iff 0 \;&=\; \int_A P_S f \; d\lambda \;=\; \int_{S^{-1}(A)} f \; d\lambda \\ &\iff f(x) \;=\; 0 \text{ for all } x \in S^{-1}(A). \end{aligned}$$

So, in summary, for every $A \in \mathcal{B}$,

$$P_S f \;=\; 0 \text{ on } A \text{ if and only if } f \;=\; 0 \text{ on } S^{-1}(A) \text{ for any A} \in \mathcal{B}. \tag{2.19}$$

We specialize this result with

$$A \;=\; (\text{supp } P_S f)^c.$$

Note that

$$S^{-1}(A) \;=\; S^{-1}((\text{supp } P_S f))^c = (S^{-1}(\text{supp } P_S f))^c. \tag{2.20}$$

Suppose $x \in A = (\text{supp } P_S f)^c$, then $x \notin \text{supp } P_S f$, and hence $P_S f(x) = 0$. This means that $P_S f = 0$ on A. By (2.19), $f = 0$ on $S^{-1}(A)$. So, $x \in S^{-1}(A)$, implies $f(x) = 0$, and hence $x \notin \text{supp } f$ which implies $x \in (\text{supp } f)^c$. So we have $S^{-1}(A) \subseteq (\text{supp } f)^c$. Hence $\text{supp } f \subseteq (S^{-1}(A))^c$, which is, by using (2.20),

$$\text{supp } f \;\subseteq\; S^{-1}(\text{supp } P_S f). \qquad \blacksquare$$

THEOREM 2.35. Consider the measure space $(L^1[0,1], \mathcal{B}, \lambda)$. Let $S : [0,1] \to [0,1]$ be a measurable and nonsingluar transformation and P_S be the Frobenius-Perron operator associated with S. If S is ergodic, then there is at most one stationary density f^* of P_S. Further, if there is a unique stationary density f^* of P_S and $f^* > 0$, then S is ergodic.

PROOF. Assume that S is ergodic. Suppose that there exist two different stationary densities f_1 and f_2 of P_S; that is, $P_S f_1 = f_1$, and $P_S f_2 = f_2$. Let $g = f_1 - f_2$. Then by the linearity of P_S, we have

$$P_S g \;=\; P_S(f_1 - f_2) \;=\; P_S f_1 - P_S f_2 \;=\; f_1 - f_2 \;=\; g.$$

Hence, $P_S g = g$. Then by theorem 2.7, $P_S g^+ = g^+$ and $P g^- = g^-$. Note that the supports of g^+ and g^- are disjoint by the definition of positive part and negative part of g. Hence, if we define sets A and B by

$$A \;=\; \text{supp}\; g^+ \;=\; \{x : g^+(x) > 0\} \quad \text{and} \quad B \;=\; \text{supp}\; g^- \;=\; \{x : g^-(x) > 0\},$$

then A and B are both disjoint and both have positive measures for the following reason. Suppose that $g \geq 0$ or $g \leq 0$. Since $g = f_1 - f_2$, and $f_1 \neq f_2$, then $g > 0$ or $g < 0$. Without loss of generality, we assume that $g > 0$. So $f_1 - f_2 > 0$, then since f_1 and f_2 are both densities, $0 = \int_0^1 f_1 - \int_0^1 f_2 > 0$, which is a contradiction. Therefore, A and B are both disjoint and both have positive measures. By using lemma 2.34, it follows that,

$$A \;=\; \text{supp}\; g^+ \;\subseteq\; S^{-1}(\text{supp}\; P_S g^+) \;=\; S^{-1}(\text{supp}\; g^+) \;=\; S^{-1}(A). \tag{2.21}$$

Similarly, we have

$$B \;\subseteq\; S^{-1}(B).$$

Therefore,

$$A \;\subseteq\; S^{-1}(A) \quad \text{and} \quad B \;\subseteq\; S^{-1}(B).$$

Note that $S^{-1}(A)$ and $S^{-1}(B)$ are disjoint sets as it stated and shown in the following claim.

Claim 1. $S^{-1}(A)$ and $S^{-1}(B)$ are disjoint.

Proof. Assume that $S^{-1}(A)$ and $S^{-1}(B)$ are not disjoint. Then there exists

$$\begin{aligned} x \in S^{-1}(A) \cap S^{-1}(B) &\implies x \in S^{-1}(A) \;\text{ and }\; x \in S^{-1}(B) \\ &\implies S(x) \in A \;\text{ and }\; S(x) \in B \\ &\implies S(x) \in A \cap B. \end{aligned}$$

However, A and B were proven to be disjoint, which is a contradiction. Hence, $S^{-1}(A)$ and $S^{-1}(B)$ are disjoint. $\square$

By the same process as (2.21), we have

$$S^{-1}(A) \; = \; S^{-1}(\text{supp } g^+) \;\subseteq\; S^{-1}(S^{-1}(\text{supp } P_S g^+)) = S^{-2}(\text{supp } g^+) \; = \; S^{-2}(A),$$

and similarly $S^{-1}(B) \subseteq S^{-2}(B)$, which follows by induction that

$$A \;\subseteq\; S^{-1}(A) \;\subseteq\; S^{-2}(A) \;\subseteq \ldots \subseteq\; S^{-n}(A)$$

and

$$B \;\subseteq\; S^{-1}(B) \;\subseteq\; S^{-2}(B) \;\subseteq \ldots \subseteq\; S^{-n}(B).$$

Note that also by induction on Claim 1, we have

$$S^{-n}(A) \text{ and } S^{-n}(B) \text{ are also disjoint.} \tag{2.22}$$

We define

$$\tilde{A} \;=\; \bigcup_{n=0}^{\infty} S^{-n}(A) \qquad \text{and} \qquad \tilde{B} \;=\; \bigcup_{n=0}^{\infty} S^{-n}(B).$$

Since A and B have positive measures, $\tilde{A}$ and $\tilde{B}$ also have positive measures. Moreover, $\tilde{A}$ and $\tilde{B}$ are disjoint as the following claim stated and shown.

Claim 2. $\tilde{A}$ and $\tilde{B}$ are disjoint.

Proof. Assume that $\tilde{A}$ and $\tilde{B}$ are not disjoint. Then there exists x such that $x \in \tilde{A} \cap \tilde{B}$. Therefore,

$$x \in \tilde{A} \text{ and } x \in \tilde{B} \implies x \in \bigcup_{n=0}^{\infty} S^{-n}(A) \text{ and } x \in \bigcup_{n=0}^{\infty} S^{-n}(B)$$

$$\implies x \in S^{-p}(A) \text{ for some } p \text{ in } \mathbb{Z}^+ \text{ and } \; x \in S^{-q}(B) \text{ for some } q \text{ in } \mathbb{Z}^+.$$

Let $r = \max\{p, q\}$. Then $x \in S^{-r}(A) \cap S^{-r}(B)$, which contradicts (2.22). Therefore, $\tilde{A}$ and $\tilde{B}$ are disjoint. $\square$

Claim 3. $\tilde{A}$ and $\tilde{B}$ are invariant sets under S.

Proof. Note that

$$S^{-1}(\tilde{A}) \;=\; S^{-1}\left(\bigcup_{n=0}^{\infty} S^{-n}(A)\right) \;=\; \bigcup_{n=0}^{\infty} S^{-n-1}(A) \;=\; \bigcup_{n=1}^{\infty} S^{-n}(A).$$

Also, note that

$$\bigcup_{n=1}^{\infty} S^{-n}(A) \;=\; \bigcup_{n=0}^{\infty} S^{-n}(A) \text{ since } S^{0}(A) \;=\; A \subseteq \bigcup_{n=1}^{\infty} S^{-n}(A).$$

Hence

$$S^{-1}(\tilde{A}) \;=\; \bigcup_{n=1}^{\infty} S^{-n}(A) \;=\; \bigcup_{n=0}^{\infty} S^{-n}(A) \;=\; \tilde{A}.$$

So $S^{-1}(\tilde{A}) = \tilde{A}$, and similarly, $S^{-1}(\tilde{B}) = \tilde{B}$. Thus $\tilde{A}$ and $\tilde{B}$ are invariant sets under S. $\square$

So $\tilde{A}$ and $\tilde{B}$ are disjoint, invariant sets under S, and both have positive measures, which contradicts the assumption that S is ergodic. Therefore, there is at most one stationary density f^* of P_S.

For the second part, suppose that there is a unique stationary density f^* of P_S and $f^*(x) > 0$. Assume that S is not ergodic. Then there is a nontrivial set, for instance A, that is invariant under S, where nontrivial set means that $\lambda(A) > 0$ and $\lambda(A^c) > 0$. Let $B = A^c$. Then $\lambda(B) > 0$ and B is invariant under S because $S^{-1}(B) = S^{-1}(A^c) = (S^{-1}(A))^c = A^c = B$. Note that f^* can be written as $f^* =$

$1_A f^* + 1_B f^*$. Since f^* is a stationary density of P_S and by the linearity of P_S, we have

$$1_A f^* + 1_B f^* \;=\; f^* \;=\; P_S f^* \;=\; P_S(1_A f^* + 1_B f^*) \;=\; P_S(1_A f^*) + P_S(1_B f^*).$$

Hence,

$$1_A f^* + 1_B f^* \;=\; P_S(1_A f^*) + P_S(1_B f^*).$$

Since $1_A f^*$ is nonnegative, by (2.19) we have

$$1_A f^* = 0 \text{ on } S^{-1}(B) \iff P_S(1_A f^*) = 0 \text{ on } B.$$

However, $S^{-1}(B) = B$ and $1_A f^* = 0$ on B, then $P_S(1_A f^*) = 0$ on B. Thus

$$1_A f^* \;=\; P_S(1_A f^*) \text{ on } B.$$

Also, by (2.19), we have

$$1_A f^* \;=\; P_S(1_A f^*) \text{ on } A.$$

Therefore,

$$1_A f^* \;=\; P_S(1_A f^*) \text{ on } [0,1],$$

which means that $1_A f^*$ is a stationary function of P_S. Similarly, we have $1_B f^*$ is a stationary function of P_S. We normalize $1_A f^*$ and $1_B f^*$ to make them densities of P_S, where normalization a function, for example g, means that we multiply g by a constant so that $\|g\|_1 = 1$ [7, 51]. So we let

$$f_A \;=\; \frac{1_A f^*}{\|1_A f^*\|} \text{ and } f_B \;=\; \frac{1_B f^*}{\|1_B f^*\|}.$$

Wherefore, f_A and f_B are stationary densities of P_S; that is, $f_A = P_S f_A$ and $f_B =$

$P_S f_B$. Accordingly, there are two stationary densities of P_S, which contradicts the assumption of the unique existence of f^*. Therefore, S is ergodic. ■

LEMMA 2.36. Suppose that $(L^1[0,1], \mathcal{B}, \mu)$ is a measure space and $S : [0,1] \to [0,1]$ is a measurable and nonsingular transformation. Assume that μ is an invariant measure under S, and $f \in L^1[0,1]$. Then for every $A \in \mathcal{B}$

$$\int_{S^{-1}(A)} f(S)\, d\mu \;=\; \int_A f\, d\mu.$$

PROOF. We start by letting $f = 1_B$. Note that by (2.15), we have $1_B(S) = 1_{S^{-1}(B)}$. Since μ is an invariant measure under S, we have

$$\begin{aligned}\int_{S^{-1}(A)} 1_B(S)\, d\mu &= \int_0^1 1_{S^{-1}(A)} 1_{S^{-1}(B)}\, d\mu = \int_0^1 1_{S^{-1}(A\cap B)}\, d\mu \\ &= \int_{S^{-1}(A\cap B)} d\mu = \mu(S^{-1}(A\cap B)) \\ &= \mu(A\cap B).\end{aligned}$$

On the other hand,

$$\begin{aligned}\int_A 1_B\, d\mu &= \int_0^1 1_A 1_B\, d\mu = \int_0^1 1_{A\cap B}\, d\mu \\ &= \int_{A\cap B} d\mu = \mu(A\cap B).\end{aligned}$$

Therefore, when $f = 1_B$, the theorem is true. Clearly, the result is true for any simple function by the linearity of the integral. Let $f \in L^1[0,1]$. Since f can be written as $f = f^+ - f^-$, the integral of f^+ and f^- is given by

$$\int_0^1 f^+\, d\mu \;=\; \sup_{0\le s_1 \le f^+} \left\{ \int_0^1 s_1\, d\mu : s_1 \text{ is a simple function} \right\},$$

and

$$\int_0^1 f^- \, d\mu \;=\; \sup_{0 \le s_2 \, d\mu \le f^-} \left\{ \int_0^1 s_2 : s_2 \text{ is a simple function} \right\}.$$

So again by the linearity of integral the theorem is true for any $f \in L^1[0,1]$. ■

REMARK 2.37. Under the same assumption of Lemma 2.36, when the integral is taking over the whole domain $[0,1]$, the result of the theorem becomes

$$\int_0^1 f(S) \, d\mu \;=\; \int_0^1 f \, d\mu.$$

DEFINITION 2.38. Let $S : [0,1] \to [0,1]$ be a measurable and nonsingular transformation. The set $\{S^k(x_0)\}_{k=0}^{\infty}$ is called an *orbit* of S with an initial point x_0 [64].

The following two theorems tell the importance of S being an ergodic transformation.

Birkhoff's Pointwise Ergodic Theorem. Consider the measure space $(L^1[0,1], \mathcal{B}, \mu)$. Let $S : [0,1] \to [0,1]$ be a measurable and nonsingular transformation. Let the measure μ be an invariant measure under S. If $f \in L^1[0,1]$, then there exists $\tilde{f} \in L^1[0,1]$ such that $\tilde{f}(S(x)) = \tilde{f}(x)$ and

$$\tilde{f}(x) \;=\; \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(S^k(x)).$$

Furthermore, if μ is finite measure; that is, $\mu([0,1]) < \infty$, then

$$\int_0^1 \tilde{f} \, d\mu \;=\; \int_0^1 f \, d\mu.$$

(Birkhoff's Pointwise Ergodic Theorem can be found in [2].)

PROOF. We will only prove the last statement under further assumption that f is bounded. Let $f_n(x) = \frac{1}{n}\sum\limits_{k=0}^{n-1} f(S^k(x))$. Remark 2.37 gives

$$\begin{aligned}
\int_0^1 f_n \, d\mu &= \int_0^1 \frac{1}{n}\sum_{k=0}^{n-1} f(S^k) \, d\mu \\
&= \frac{1}{n}\sum_{k=0}^{n-1}\int_0^1 f(S^k) \, d\mu \;=\; \frac{1}{n}\sum_{k=0}^{n-1}\int_0^1 f \, d\mu \\
&= \frac{1}{n}\, n \int_0^1 f \, d\mu \;=\; \int_0^1 f \, d\mu.
\end{aligned}$$

Hence,

$$\int_0^1 f_n \, d\mu = \int_0^1 f \, d\mu. \tag{2.23}$$

Since f is bounded, then there exists $M > 0$ such that $|f| \leq M$. Consequently,

$$\begin{aligned}
|f_n| &= \left|\frac{1}{n}\sum_{k=0}^{n-1} f(S^k(x))\right| \\
&\leq \frac{1}{n}\sum_{k=0}^{n-1}|f(S^k(x))| \;\leq\; \frac{1}{n}\sum_{k=0}^{n-1} M \\
&= \frac{1}{n}\, n\, M \;=\; M
\end{aligned}$$

for all n. Note that since μ is finite,

$$\int_0^1 M \, d\mu = M\mu([0,1]) < \infty.$$

So, M is integrable. Therefore, f_n is bounded by an integrable function. Then we can apply the Dominated Convergence Theorem [53] and equation (2.23) to get,

$$\int_0^1 \tilde{f}\ d\mu \;=\; \int_0^1 \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} f(S^k)\ d\mu \;=\; \int_0^1 \lim_{n\to\infty} f_n\ d\mu$$

$$= \lim_{n\to\infty}\int_0^1 f_n\ d\mu \;=\; \lim_{n\to\infty}\int_0^1 f\ d\mu \;=\; \int_0^1 f\ d\mu.$$

Hence,

$$\int_0^1 \tilde{f}\ d\mu \;=\; \int_0^1 f\ d\mu. \qquad \blacksquare$$

THEOREM 2.39. Assume that $S : [0,1] \to [0,1]$ is measurable, nonsingular, and ergodic. Suppose that the corresponding Frobenius-Perron operator has a unique stationary density f^*. Define the probability measure μ such that for every $A \in \mathcal{B}$

$$\mu(A) \;=\; \int_A f^*\ d\lambda. \tag{2.24}$$

Then for any integrable f, the average of f along the orbits of S with initial point x_0 equals the average of f over the space $[0,1]$. That is,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} f(S^k(x)) \;=\; \int_0^1 f\ d\mu. \tag{2.25}$$

PROOF. First, since μ is defined as in (2.24), then by Theorem 2.26 μ is an invariant measure under S. So it follows that, by Birkhoff's Pointwise Ergodic Theorem, there exists $\tilde{f} \in L^1[0,1]$ such that $\tilde{f}(S(x)) = \tilde{f}(x)$, and

$$\tilde{f}(x) \;=\; \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} f(S^k(x)). \tag{2.26}$$

Furthermore, since μ is a probability measure, it is a finite measure, so Birkhoff's Pointwise Ergodic Theorem gives

$$\int_0^1 \tilde{f}\ d\mu \;=\; \int_0^1 f\ d\mu. \tag{2.27}$$

Also, since S is ergodic, by lemma 2.33, $\tilde{f}$ is constant, and we write this constant as α. So (2.26) can be written as

$$\alpha = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(S^k(x))$$

for any x. Therefore, equation (2.27) becomes,

$$\int_0^1 f\, d\mu = \int_0^1 \tilde{f}\, d\mu = \int_0^1 \alpha\, d\mu = \alpha \int_0^1 d\mu = \alpha\mu([0,1]) = \alpha.$$

So

$$\alpha = \int_0^1 f\, d\mu.$$

Thus, for any $x \in [0,1]$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(S^k(x)) = \int_0^1 f\, d\mu. \qquad \blacksquare$$

The next corollary verifies equation (1.3), which was observed and mentioned at the end of chapter 1.

COROLLARY 2.40. Assume that $S : [0,1] \to [0,1]$ is measurable, nonsingular and ergodic. Suppose that μ is an invariant probability measure under S defined in (2.24). Then for all $x \in [0,1]$, and for any set $A \in \mathcal{B}$, the fraction of the number of the points $\{S^k(x)\}$ in A as $k \to \infty$ is given by $\mu(A)$. That is,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} 1_A(S^k(x)) = \mu(A).$$

PROOF. Let $f = 1_A$ in (2.25). Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} 1_A(S^k(x)) = \int_0^1 1_A\, d\mu = \int_A d\mu = \mu(A).$$

■

Corollary 2.40 is a promised justification of (1.3), mentioned in Chapter 1. In the following example, we strengthen Example 1.1 given in Chapter 1 using Corollary 2.40 by adding the true values.

EXAMPLE 2.41. Let the transformation S be from Example 2.29, Then $f^*(x) = \frac{2}{(1+x)^2}$ is the stationary density function of P_S. Let $I_j = [\frac{j-1}{10}, \frac{j}{10}]$ for $j = 1, 2, \ldots, 10$, and $n = 10^6$. Then

$$\begin{aligned}
\frac{1}{n}\sum_{k=0}^{n-1} 1_{I_j}(S^k(x)) \approx \mu(I_j) &= \int_{I_j} f^*\, d\lambda = \int_{I_j} \frac{2}{(1+x)^2}\, d\lambda \\
&= -\frac{2}{1+x}\Bigg|_{I_j} = -\frac{2}{1+\frac{j}{10}} + \frac{2}{1+\frac{j-1}{10}} = \frac{20}{(j+10)(j+9)}.
\end{aligned}$$

We can see from Table 2 that the approximated probability distribution values of Table 1 in chapter 1 are very close to the true values.

Moreover, we compute the probability distribution using the formula in equation (1.1) for the initial point $x_0 = \frac{\pi}{8}$ when $n = 10^3,\ 10^4,\ 10^5$, and 10^6. We present the results in Table 3, which shows that as n gets bigger, the approximated probability distribution values get closer to the true values.

Table 2: Independence of approximated probability values from initial values

	Probability				True Value
I_j	$x_0 = \frac{\pi}{4}$	$x_0 = \frac{\pi}{8}$	$x_0 = \frac{\pi}{16}$	$x_0 = \frac{\pi}{32}$	$\mu(I_j)$
[0,0.1]	0.1821	0.1826	0.1824	0.1819	0.1818
[0.1,0.2]	0.1516	0.1515	0.1515	0.1510	0.1515
[0.2,0.3]	0.1277	0.1277	0.1283	0.1281	0.1282
[0.3,0.4]	0.1099	0.1104	0.1100	0.1098	0.1099
[0.4,0.5]	0.0956	0.0948	0.0947	0.0956	0.0952
[0.5,0.6]	0.0833	0.0834	0.0829	0.0839	0.0833
[0.6,0.7]	0.0733	0.0730	0.0736	0.0735	0.0735
[0.7,0.8]	0.0655	0.0653	0.0654	0.0650	0.0654
[0.8,0.9]	0.0585	0.0583	0.0582	0.0583	0.0585
[0.9,1]	0.0524	0.0529	0.0528	0.0528	0.0526

Table 3: Convergence of approximated probability values

	Probability				True Value
I_j	$n = 10^3$	$1n = 0^4$	$n = 10^5$	$n = 10^6$	$\mu(I_j)$
[0,0.1]	0.2150	0.1865	0.1835	0.1826	0.1818
[0.1,0.2]	0.1570	0.1487	0.1533	0.1515	0.1515
[0.2,0.3]	0.1120	0.1280	0.1259	0.1277	0.1282
[0.3,0.4]	0.1190	0.1110	0.1114	0.1104	0.1099
[0.4,0.5]	0.0830	0.0934	0.0943	0.0948	0.0952
[0.5,0.6]	0.0770	0.0830	0.0818	0.0834	0.0833
[0.6,0.7]	0.0600	0.0731	0.0722	0.0730	0.0735
[0.7,0.8]	0.0640	0.0667	0.0659	0.0653	0.0654
[0.8,0.9]	0.0510	0.0565	0.0589	0.0583	0.0585
[0.9,1]	0.0620	0.0531	0.0529	0.0529	0.0526

CHAPTER 3

SPLINE SPACE

Since this dissertation aims to approximate the unique stationary density function f^* of the Frobenius-Perron operator associated with S with faster convergence rate than before, we are using a sequence of cubic spline functions. This method works when f^* is smooth. The reason we are using the cubic spline functions is that they are twice continuously differentiable on $[0,1]$, where a function g is twice continuously differentiable if g' and g'' exist and g'' is continuous [30]. So, in this chapter, we explain cubic spline functions, but first we explain the constant, linear, and quadratic spline functions. More details about spline functions can be found in the book by Kincaid and Cheney [40].

3.1 Spline Functions

A *spline function* of degree k [75] is a function defined on $\mathbb{R}$ that is a polynomial of degree at most k locally and of class $C^{k-1}(\mathbb{R})$ globally; that is, $k-1$-times continuously differentiable in the whole domain [33]. We can generate spline functions recursively starting from constant spline functions, which are denoted by B_i^0, $i \in \mathbb{Z}$, where $B_i^0 = 1_{[x_i, x_{i+1})}$. The recursive relation of spline functions of degree k from spline functions of degree $k-1$ is given by

$$B_i^k(x) = \frac{x - x_i}{x_{i+k} - x_i} B_i^{k-1}(x) + \frac{x_{i+k+1} - x}{x_{i+k+1} - x_{i+1}} B_{i+1}^{k-1}(x), \quad k \geq 1. \tag{3.1}$$

We call B_i^k as spline function of degree k.

Some properties of spline functions are stated in the following lemma.

LEMMA 3.1. (1) B_i^k is a nonnegative function and the support of $B_i^k(x)$ is (x_i, x_{i+k+1}) for $k \geq 0$.

(2) On each subinterval (x_i, x_{i+1}), B_i^k is a polynomial of degree at most k for $k \geq 0$.

(3) $B_i^k \in C^{k-1}(\mathbb{R})$ for $k \geq 1$.

(4) $\sum\limits_{i=-k}^{n-1} B_i^k(x) = 1$ for all $x \in [0,1]$.

DEFINITION 3.2. Let $[0,1]$ be a unit interval. A finite set $P = \{0 = x_0 < x_1 < \cdots < x_n = 1\}$ is called a *partition* of $[0,1]$ [42].

DEFINITION 3.3. A set $\{g_1, g_2, \ldots, g_n\}$ is called *linearly independent* set if the linear combination of $\{g_1, g_2, \ldots, g_n\}$ has only a trivial solution [34]; that is,

$$a_1 g_1 + a_2 g_2 + \cdots + a_n g_n \;=\; 0 \iff a_1 = a_2 = \cdots = a_n = 0.$$

For a regular partition $\{0 = x_0 < x_1 < \cdots < x_n = 1\}$ of $[0,1]$, where a regular partition means an equally spaced partition, we define $S^k[x_0, \ldots, x_n]$ to be the set of all spline functions f of degree k whose domain is the unit interval $[0,1]$. Note that, the space $S^k[x_0, \ldots, x_n]$ is a subspace of $L^1[0,1]$ and

$$\left\{B_i^k|_{[0,1]} \colon i = -k, -k+1, \ldots, n-1\right\} \tag{3.2}$$

is a basis for $S^k[x_0, \ldots, x_n]$; that is, B_i^k is a linearly independent subset of $S^k[x_0, \ldots, x_n]$ such that each function in $S^k[x_0, \ldots, x_n]$ can be written as a linear combination of elements of B_i^k [35]. Consequently, the dimension of $S^k\{x_0, \ldots, x_n\}$ is $k + n$.

3.2 Linear Spline Functions

Linear spline functions B_i^1, which used in [24, 25] to approximate f^* of P_S, are spline functions of degree one. They are continuous piecewise linear functions. To drive B_i^1, we use the recursive formula (3.1) with $B_i^0 = 1_{[x_i, x_{i+1})}$ and $B_{i+1}^0 = 1_{[x_{i+1}, x_{i+2})}$ as follows:

$$
\begin{aligned}
B_i^1(x) &= \left(\frac{x - x_i}{x_{i+1} - x_i}\right) B_i^0(x) + \left(\frac{x_{i+2} - x}{x_{i+2} - x_{i+1}}\right) B_{i+1}^0(x) \\
&= \left(\frac{x - x_i}{h}\right) B_i^0(x) + \left(\frac{(2+i)h - x}{h}\right) B_{i+1}^0(x) \\
&= \left(\frac{x - x_i}{h}\right) 1_{[x_i, x_{i+1})}(x) + \left(2 - \left(\frac{x - x_i}{h}\right)\right) 1_{[x_{i+1}, x_{i+2})}(x) \\
&= \begin{cases} \frac{x-x_i}{h} & x_i \le x < x_{i+1} \\ 2 - \frac{x-x_i}{h} & x_{i+1} \le x < x_{i+2} \\ 0 & x \notin (x_i, x_{i+2}). \end{cases}
\end{aligned}
$$

Let $y = \dfrac{x - x_i}{h}$. Then

$$
B_i^1(x) = \begin{cases} y & 0 \le y < 1 \\ 2 - y & 1 \le y < 2 \\ 0 & y \notin (0, 2). \end{cases}
$$

So the linear spline function is represented by:

$$
B_i^1(x) = L\left(\frac{x - x_i}{h}\right),
$$

where

$$L(x) = \begin{cases} x & 0 \le x < 1 \\ 2-x & 1 \le x < 2 \\ 0 & x \notin (0,2). \end{cases} \tag{3.3}$$

3.3 Quadratic Spline Functions

Quadratic spline functions are degree two spline functions, denoted by B_i^2. They are continuous piecewise quadratic functions. In [78], B_i^2 were used to approximate the stationary density f^* of P_S. We drive B_i^2 by using equation (3.1) as follows:

$$\begin{aligned} B_i^2(x) &= \left(\frac{x-x_i}{x_{i+2}-x_i}\right) B_i^1(x) + \left(\frac{x_{i+3}-x}{x_{i+3}-x_{i+1}}\right) B_{i+1}^1(x) \\ &= \left(\frac{x-x_i}{2h}\right) B_i^1(x) + \left(\frac{(3+i)h-x}{2h}\right) B_{i+1}^1(x) \\ &= \frac{1}{2}\left(\frac{x-x_i}{h}\right) B_i^1(x) + \frac{1}{2}\left(3-\left(\frac{x-x_i}{h}\right)\right) B_{i+1}^1(x). \end{aligned}$$

Let $y = \dfrac{x-x_i}{h}$. By using equation (3.3) for $B_i^1(x)$ and $B_{i+1}^1(x)$, we have

$$B_i^2(x) = \begin{cases} \frac{1}{2}y^2 & 0 \le y < 1 \\ \frac{1}{2}y(2-y) + \frac{1}{2}(3-y)(y-1) & 1 \le y < 2 \\ \frac{1}{2}(3-y)(3-y) & 2 \le y < 3 \\ 0 & y \notin (0,3) \end{cases}$$

$$
= \begin{cases} \frac{1}{2}y^2 & 0 \le y < 1 \\ \frac{3}{4} - \left(y - \frac{3}{2}\right)^2 & 1 \le y < 2 \\ \frac{1}{2}\left(y-3\right)^2 & 2 \le y < 3 \\ 0 & y \notin (0,3). \end{cases}
$$

Therefore,

$$
B_i^2(x) \;=\; Q\left(\frac{x - x_i}{h}\right),
$$

where

$$
Q(x) \;=\; \begin{cases} \frac{1}{2}x^2 & 0 \le x < 1 \\ \frac{3}{4} - \left(x - \frac{3}{2}\right)^2 & 1 \le x < 2 \\ \frac{1}{2}\left(x-3\right)^2 & 2 \le x < 3 \\ 0 & x \notin (0,3). \end{cases} \tag{3.4}
$$

3.4 Cubic Spline Functions

In this section, we define the cubic spline functions that we use to approximate f^*, the stationary density function of P_S.

DEFINITION 3.4. A *cubic spline* function s for a regular partition $\{0 = x_0 < x_1 < \cdots < x_n = 1\}$ of $[0,1]$ is a degree three spline function [63], so it satisfies the following:

(1) s is at most a cubic polynomial on $[x_i, x_{i+1}]$ for $i = 0, 1, \ldots, n-1$.

(2) $s \in C^2[0,1]$.

Recall that the collection of all cubic spline functions for the regular partition

$\{0 = x_0 < x_1 < \cdots < x_n = 1\}$ is denoted by $S^3[x_0, x_1, \ldots, x_n]$. Among all the cubic spline functions, B_i^3, $i = -3, -2, \ldots, n-1$ are the important ones. In Figure 4, we graph B_i^3 for $i = -3, 2, 8,$ and 9 when $n = 10$ to see the reason why i starts at -3 and ends at 9, which is, in general, $n - 1$.

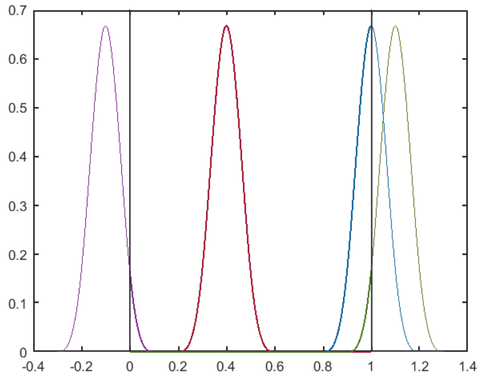


Figure 4: B_i^3 for $i = -3, 2, 8$ and 9 when $n = 10$

Using the same approach as the linear spline and quadratic spline functions, we can drive B_i^3. We use equation (3.1) as follows:

$$\begin{aligned}
B_i^3(x) &= \left(\frac{x - x_i}{x_{i+3} - x_i}\right) B_i^2(x) + \left(\frac{x_{i+4} - x}{x_{i+4} - x_{i+1}}\right) B_{i+1}^2(x) \\
&= \left(\frac{x - x_i}{3h}\right) B_i^2(x) + \left(\frac{(4+i)h - x}{3h}\right) B_{i+1}^2(x) \\
&= \frac{1}{3}\left(\frac{x - x_i}{h}\right) B_i^2(x) + \frac{1}{3}\left(4 - \left(\frac{x - x_i}{h}\right)\right) B_{i+1}^2(x).
\end{aligned}$$

Let $y = \dfrac{x - x_i}{h}$ and use equation (3.4) for B_i^2, and B_{i+1}^2, we have

$$B_i^3(x) = \begin{cases} \frac{1}{3}y\frac{1}{2}y^2 & 0 \le y < 1 \\ \frac{1}{3}y\left[\frac{3}{4} - \left(y - \frac{3}{2}\right)^2\right] + \frac{1}{3}(4-y)\left[\frac{1}{2}\left(y-1\right)^2\right] & 1 \le y < 2 \\ \frac{1}{3}y\left[\frac{1}{2}(y-3)^2\right] + \frac{1}{3}(4-y)\left[\frac{3}{4} - \left(y - \frac{5}{2}\right)^2\right] & 2 \le y < 3 \\ \frac{1}{3}(4-y)\left[\frac{1}{2}(y-4)^2\right] & 3 \le y < 4 \\ 0 & y \notin (0,4) \end{cases}$$

$$= \begin{cases} \frac{1}{6}y^3 & 0 \le y < 1 \\ \frac{1}{6}\left[1 + 3(y-1) + 3(y-1)^2 - 3(y-1)^3\right] & 1 \le y < 2 \\ \frac{1}{6}\left[1 + 3(3-y) + 3(3-y)^2 - 3(3-y)^3\right] & 2 \le y < 3 \\ \frac{1}{6}(4-y)^3 & 3 \le y < 4 \\ 0 & y \notin (0,4). \end{cases}$$

Hence, the cubic spline is represented by

$$B_i^3(x) = C\left(\frac{x - x_i}{h}\right),$$

where

$$C(x) = \begin{cases} \frac{1}{6}x^3 & 0 \le x < 1 \\ \frac{1}{6}\left[1 + 3(x-1) + 3(x-1)^2 - 3(x-1)^3\right] & 1 \le x < 2 \\ \frac{1}{6}\left[1 + 3(3-x) + 3(3-x)^2 - 3(3-x)^3\right] & 2 \le x < 3 \\ \frac{1}{6}(4-x)^3 & 3 \le x < 4 \\ 0 & x \notin (0,4). \end{cases} \tag{3.5}$$

The next lemma states some important properties about the cubic spline functions that is a restating of Lemma 3.1 after being modified to the cubic spline functions.

LEMMA 3.5. (1) $\{B_i^3\}_{i=-3}^{n-1}$ is a basis of $S^3[x_0, x_1, \ldots, x_n]$.

(2) B_i^3 is a nonnegative function and the support of $B_i^3(x) = (x_i, x_{i+4})$.

(3) $\sum_{i=-3}^{n-1} B_i^3(x) = 1$ for any $x \in [0,1]$ (see Figure 5 for clarification).

In Figure 5, B_i^3 was graphed for $i = -3, -2, \ldots, 9$ when $n = 10$, and the point $x = 0.415$ was chosen randomly in $[0,1]$. It turns out there are only four nonzero cubic spline functions B_i^3 at $x = 0.415$ with $i = 1, 2, 3, 4$, and

$$
\begin{aligned}
\sum_{i=-3}^{9} B_i^3(0.415) &= \sum_{i=1}^{4} B_i^3(0.415) \\
&= 0.10235417 + 0.64585417 + 0.25122917 + 0.0005625 \\
&= 1.
\end{aligned}
$$

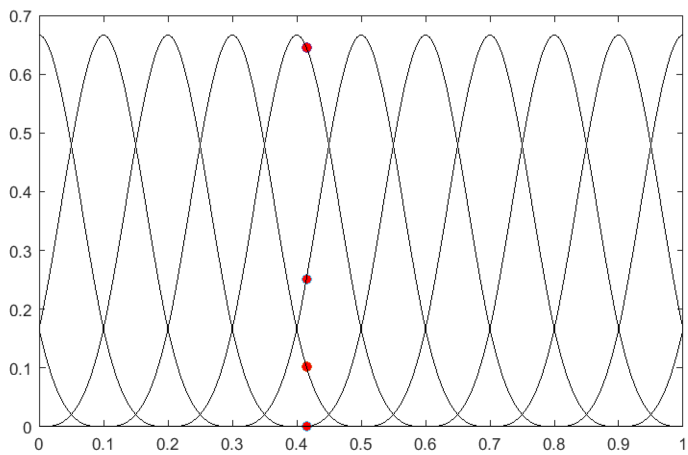


Figure 5: Illustration of Lemma 3.5 (3)

CHAPTER 4

PROJECTION METHOD

This chapter explains the projection method we are using to approximate f^*, the unique stationary density function of Frobenius-Perron operator P_S. Since the projection method uses orthogonal property, we will start with the definition of orthogonal function.

DEFINITION 4.1. Suppose that $f \in L^1[0,1]$ and $g \in L^\infty[0,1]$, where $L^\infty[0,1]$ is the space of bounded functions. Then f is *orthogonal* to g if $\langle f, g\rangle \equiv \int_0^1 f(x)g(x)\,dx = 0$, and it is written as $f \perp g$ [27].

4.1 General Projection Method

A projection method is a numerical method that is used to approximate functions using simpler functions [60]. To explain the projection method further, we let W be a finite dimensional subspace, for example $\dim(W) = m$, of $L^1[0,1]$. Let $\{\phi_i\}_{i=1}^m$ be a basis of W such that each $\phi_i \in L^\infty[0,1]$. We define a linear operator $Q_W : L^1[0,1] \to W$ such that $f - Q_W f$ is orthogonal to W. If $f \in L^2[0,1]$, $Q_W f$ is the least squares projection of f onto W [10]. Since $f - Q_W f$ satisfies the orthogonal property, for all $i = 1, 2, \ldots, m$, we have

$$
\begin{aligned}
f - Q_W f \perp W \iff & f - Q_W f \perp \phi_1, \phi_2, \ldots, \phi_m \\
\iff & \langle f - Q_W f, \phi_i\rangle \;=\; 0
\end{aligned}
$$

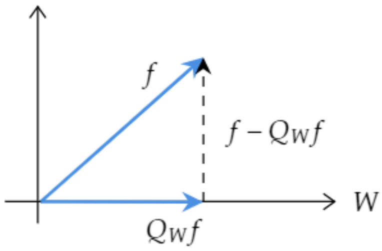


Figure 6: $f - Q_W f$ is orthogonal to W

$$
\begin{aligned}
\iff \langle f, \phi_i \rangle - \langle Q_W f, \phi_i \rangle &= 0 \\
\iff \langle f, \phi_i \rangle &= \langle Q_W f, \phi_i \rangle \qquad (4.1)
\end{aligned}
$$

because $\langle f - g, h \rangle = \langle f, h \rangle - \langle g, h \rangle$ [20].

Moreover, since $Q_W f \in W$, $Q_W f$ can be written as a linear combination of a basis of W; that is,

$$Q_W f = \sum_{j=1}^{m} c_j \phi_j. \qquad (4.2)$$

Using equation (4.1) and (4.2), we have

$$\langle f, \phi_i \rangle = \left\langle \sum_{j=1}^{m} c_j \phi_j, \phi_i \right\rangle = \sum_{j=1}^{m} c_j \langle \phi_j, \phi_i \rangle \qquad (4.3)$$

for $i = 1, 2, \ldots, m$. Let a matrix $B \in \mathbb{R}^{m \times m}$ and vector $b \in \mathbb{R}^m$ be such that

$$b_{ij} = \langle \phi_j, \phi_i \rangle \quad \text{and} \quad b_i = \langle f, \phi_i \rangle. \qquad (4.4)$$

Then we have

$$b_i = \sum_{j=1}^{m} c_j b_{ij} = \sum_{j=1}^{m} b_{ij} c_j$$

for $i = 1, 2, \ldots, m$, which can be written as

$$Bc \quad = \quad b. \tag{4.5}$$

So, $Q_W f = \sum_{j=1}^{m} c_j \phi_j$ is determined by solving the invertible linear system (4.5).

The fact that B is invertible is justified in Claim 1 and Claim 2; nevertheless, before stating and proving the claims, we define the following terms. We say that a matrix $A \in \mathbb{R}^{n \times n}$ is a *symmetric* matrix if $A^T = A$ [18]; that is $a_{ij} = a_{ji}$ for $1 \leq i, j \leq n$. Moreover, a matrix $A \in \mathbb{R}^{n \times n}$ is called a *positive definite* matrix if A is a symmetric matrix and $x^T A x > 0$ for all $x \neq 0$ [41].

Claim 1. The matrix B defined in (4.4) is a positive definite matrix.

Proof. (1) Note that B is a symmetric matrix because $\langle \phi_j, \phi_i \rangle = \langle \phi_i, \phi_j \rangle$ for all $1 \leq i, j \leq m$.

(2) Let $x \neq 0$. Then for all $x \in \mathbb{R}^m$, we have

$$\begin{aligned}
x^T B x &= x_1 [Bx]_1 + x_2 [Bx]_2 + \ldots x_m [Bx]_m \\
&= \sum_{i=1}^{m} x_i [Bx]_i \;=\; \sum_{i=1}^{m} x_i \left(\sum_{j=1}^{m} b_{ij} x_j \right) \\
&= \sum_{i=1}^{m} \sum_{j=1}^{m} x_i x_j \langle \phi_j, \phi_i \rangle \;=\; \sum_{i=1}^{m} \sum_{j=1}^{m} x_i x_j \langle \phi_i, \phi_j \rangle \\
&= \langle \sum_{i=1}^{m} x_i \phi_i, \sum_{j=1}^{m} x_j \phi_j \rangle > 0.
\end{aligned}$$

By (1) and (2), B is a positive definite matrix. $\square$

Claim 2. The matrix B is invertible.

Proof. Assume that B is not an invertible matrix. That is, B is a singular matrix (a square matrix whose determine equals zero [70]). Accordingly, B has 0 as its eigenvalue. Therefore, there exists $x \neq 0 \in \mathbb{R}^m$ such that $Bx = 0$. Then $x^T Bx = x^T 0 = 0$. However, this contradicts Claim 1 that B is a positive definite matrix. Hence, B is an invertible matrix. □

The operator Q_W can be used to approximate the stationary density f^* of the Frobenius-Perron operator P_S, where $Q_W P_S$ is defined from $L^1[0,1]$ to W. If W is chosen properly, see Lemma 4.2, then $Q_W P_S$ has a unique nonzero function f_W in W such that

$$(Q_W P_S) f_W = f_W. \tag{4.6}$$

By normalization, we may assume that $\|f_W\|_1 = 1$.

We expect that f_W to be a good approximation to f^* in $L^1[0,1]$ for the following reason. By (4.6) we have

$$f_W = (Q_W P_S) f_W = Q_W (P_S f_W) \approx P_S f_W$$

because $Q_W P_S f_W$ is the least squares approximation of $P_S f_W$ from W, providing $P_S f_W \in L^2[0,1]$. Hence

$$P_S f_W \approx f_W.$$

This means that f_W is nearly a stationary function of P_S. Since f^* is the exact stationary density of P_S; that is, $P_S f^* = f^*$, we expect that

$$f_W \approx f^*.$$

We call this method of approximating f^* as a (least squares) projection method using W. The idea is that a nonzero stationary function f_W of $Q_W P_S$ may approximate a stationary density of P_S.

Suppose that $\{W_n\}_{n=1}^{\infty}$ is a sequence of subspaces in $L^1[0,1]$ such that $W_1 \subset W_2 \subset \ldots \subset W_n \subset \ldots$. Suppose that there is a sequence of nonzero stationary functions of $Q_{W_n} P_S$; that is, $Q_{W_n} P_S f_{W_n} = f_{W_n}$ with $\|f_{W_n}\|_1 = 1$ for $n = 1, 2, \ldots$. Then we may expect that $f_{W_n} \to f^*$ in $L^1[0,1]$.

4.2 Constant Spline Projection Method

The famous Ulam's method in the paper by Ulam [69], proposed as Markov method, turns out to be the constant spline projection method using $\{S_n^0[x_0, \ldots, x_n]\}_{n=1}^{\infty}$. Ulam considered a sequence of operators $\{U_n P_S\}_{n=1}^{\infty}$, where each $U_n : L^1[0,1] \to S_n^0[x_0, \ldots, x_n]$ is defined by

$$U_n f \;=\; \sum_{i=1}^{n} \left[\frac{1}{\lambda(I_i)} \int_{I_i} f \right] 1_{I_i},$$

where $I_i = \left[\frac{i-1}{n}, \frac{i}{n}\right]$. Then Ulam showed that each $U_n P_S$ has a stationary density f_n. Especially, $U_n P_S f_n = f_n$. Later, Li [47] showed that $f_n \to f^*$ in $L^1[0,1]$ as $n \to \infty$.

Recall that from (3.2), for each $n \in \mathbb{N}$, $\{B_i^0\}_{i=0}^{n-1} = \{1_{I_i}\}$ is a basis of $S_n^0[x_0, \ldots, x_n]$. Let $Q_n^0 : L^1[0,1] \to S_n^0[x_0, \ldots, x_n]$ be the projection operator onto $S_n^0[x_0, \ldots, x_n]$. To show that Ulam's method is a constant spline projection method, it is enough to show that $U_n = Q_n^0$ for each n. If we write $Q_n^0 f = \sum_{j=0}^{n-1} c_j 1_{I_j}$, then by (4.3) for each $i \in \{0, 2, \ldots, n-1\}$ and $f \in L^1[0,1]$, we have

$$\sum_{j=0}^{n-1} c_j \langle 1_{I_j}, 1_{I_i} \rangle \;=\; \langle f, 1_{I_i} \rangle \iff \frac{1}{n} c_i \;=\; \langle f, 1_{I_i} \rangle \iff \lambda(I_i) c_i \;=\; \langle f, 1_{I_i} \rangle$$

$$\iff c_i = \frac{1}{\lambda(I_i)} \int_0^1 f 1_{I_i} \, d\lambda \iff c_i = \frac{1}{\lambda(I_i)} \int_{I_i} f \, d\lambda.$$

So for any $f \in L^1[0,1]$, we have

$$Q_n^0 f = \sum_{i=0}^{n-1} c_i 1_{I_i} = \sum_{i=0}^{n-1} \left[\frac{1}{\lambda(I_i)} \int_{I_i} f \right] 1_{I_i} = U_n f.$$

Consequently, $Q_n^0 f = U_n f$ for any $f \in L^1[0,1]$. Since this is true for any $f \in L^1[0,1]$, we conclude that $U_n = Q_n^0$, as desired.

4.3 Linear Spline Projection Method

The linear spline projection method was studied in the papers by Ding and Rhee [24, 25] using sequence of subspaces $\{S_n^1[x_0, \ldots, x_n]\}_{n=1}^{\infty}$. Note that for each $n \in \mathbb{N}$, $\{B_i^1\}_{i=-1}^{n-1}$ is a basis of $S_n^1[x_0, \ldots, x_n]$. Let $Q_n^1 : L^1[0,1] \to S_n^1[x_0, \ldots, x_n]$ be the projection operator onto $S_n^1[x_0, \ldots, x_n]$. It was shown in [25] that the operator $Q_n^1 P_S$ has a nonzero stationary function f_n (by normalization, it was assumed that $\|f_n\|_1 = 1$) such that $f_n \to f^*$ in $L^1[0,1]$, and the speed of convergence for linear spline projection method is faster than the constant spline projection method.

4.4 Quadratic Spline Projection Method

In Zhou, et al. paper [78], the quadratic spline projection method was studied using the sequence of subspaces $\{S_n^2[x_0, \ldots, x_n]\}_{n=1}^{\infty}$. Note that $\{B_i^2\}_{i=-2}^{n-1}$ is a basis of $S_n^2[x_0, \ldots, x_n]$ for each $n \in \mathbb{N}$. The operator $Q_n^2 : L^1[0,1] \to S_n^2[x_0, \ldots, x_n]$ was defined to be the projection operator onto $S_n^2[x_0, \ldots, x_n]$. It was shown in [78] that the operator $Q_n^2 P_S$ has a nonzero stationary function f_n such that $f_n \to f^*$ in $L^1[0,1]$, and the speed of convergence for quadratic spline projection method is faster than the constant spline and linear spline projection methods.

4.5 Cubic Spline Projection Method

As the tittle of this section indicates, we study the cubic spline projection method. For simplicity we write B_i^3 as ϕ_i, where $i = -3, -2, \ldots, n-1$, so that $\{\phi_i\}_{i=-3}^{n-1}$ is a basis for $S^3[x_0, \ldots, n]$, and each $\phi_i \in L^\infty[0,1]$.

We specialize the general projection method, which was mentioned in Section 4.1, to the cubic spline projection method as follows. We use $S^3[x_0, \ldots, x_n]$ for W in Section 4.1. Then Q_W becomes $Q_{S^3[x_0,\ldots,x_n]}$ and f_W becomes $f_{S^3[x_0,\ldots,x_n]}$. For simplicity of the notation, we will write $Q_{S^3[x_0,\ldots,x_n]}$ as Q_n and $f_{S^3[x_0,\ldots,x_n]}$ as f_n.

We define $Q_n : L^1[0,1] \to S^3[x_0, \ldots, x_n]$ such that $f - Q_n f$ is orthogonal to $S^3[x_0, \ldots, x_n]$. Then (4.1) becomes

$$\langle f, \phi_i \rangle \quad = \quad \langle Q_n f, \phi_i \rangle \tag{4.7}$$

for $i = -3, -2, \ldots, n-1$. Since $Q_n f \in S^3[x_0, \ldots, x_n]$, $Q_n f$ can be written as linear combination of a basis of $S^3[x_0, \ldots, x_n]$; that is,

$$Q_n f \quad = \quad \sum_{j=-3}^{n-1} c_j \phi_j. \tag{4.8}$$

By combining (4.7) and (4.8) we have, for $i = -3, -2, \ldots, n-1$,

$$\langle f, \phi_i \rangle \quad = \quad \sum_{j=-3}^{n-1} c_j \langle \phi_j, \phi_i \rangle. \tag{4.9}$$

Let $B \in \mathbb{R}^{(n+3)\times(n+3)}$ and a vector $b \in \mathbb{R}^{n+3}$ be such that

$$b_{ij} \quad = \quad \langle \phi_j, \phi_i \rangle \qquad \text{and} \qquad b_i \quad = \quad \langle f, \phi_i \rangle. \tag{4.10}$$

Then

$$b_i \;=\; \sum_{j=-3}^{n-1} b_{ij} c_j$$

for $i = -3, 2, \ldots, n-1$, which can be written as

$$Bc \;=\; b. \tag{4.11}$$

Therefore, $Q_n f = \sum_{j=-3}^{n-1} c_j \phi_j$ is determined by solving the invertible linear system (4.11).

By specializing the general projection method in Section 4.1 to the Cubic spline projection method, we have the following:

(1) Using the operator $Q_n : L^1[0,1] \to S^3[x_0, \ldots, x_n]$, we have

$$Q_n P_S \Big|_{S^3[x_0,\ldots,x_n]} : S^3[x_0, \ldots, x_n] \to S^3[x_0, \ldots, x_n].$$

(2) Let a nonzero function $f_n \in S^3[x_0, \ldots, n]$ fulfils

$$Q_n P_S f_n = f_n. \tag{4.12}$$

The existence of such f_n will be proven in Lemma 4.2.

(3) By normalization, we may assume that $\|f_n\|_1 = 1$.

(4) We expect that f_n to be a good approximation to f^* in $L^1[0,1]$.

After specializing the projection method to the cubic spline projection method, we establish (2) from above in next lemma.

LEMMA 4.2. There is a nonzero function $f_n \in S^3[x_0, \ldots, x_n]$ such that $Q_n P_S f_n = f_n$.

PROOF. First note that $Q_n P_S|_{S^3[x_0,\ldots,x_n]}: S^3[x_0, \ldots, x_n] \to S^3[x_0, \ldots, x_n]$. For easy notation we will write Q_nP_S instead of $Q_nP_S|_{S^3[x_0,\ldots,x_n]}$. Let $\beta = \{\phi_i\}_{i=-3}^{n-1}$ be a basis of $S^3[x_0,\ldots,x_n]$. Note that we can write

$$(Q_nP_S)\phi_j = \sum_{k=-3}^{n-1} m_{kj}\phi_k$$

for all $j = -3,\ldots,n-1$. If we let $M = [m_{kj}]$, then M is the matrix representation of Q_nP_S with respect to β [74], so we write $M = [Q_nP_S]_\beta$. Recall that matrix B is defined by $b_{ij} = \langle \phi_j, \phi_i\rangle$. Then

$$\begin{aligned}\langle (Q_nP_S)\phi_j, \phi_i\rangle &= \left\langle \sum_{k=-3}^{n-1} m_{kj}\phi_k, \phi_i \right\rangle = \sum_{k=-3}^{n-1} m_{kj}\langle \phi_k, \phi_i\rangle \\ &= \sum_{k=-3}^{n-1} m_{kj}b_{ik} = \sum_{k=-3}^{n-1} b_{ik}m_{kj} = [BM]_{ij}.\end{aligned}$$

On the other hand, using (4.7), we have

$$\langle (Q_nP_S)\phi_j, \phi_i\rangle = \langle Q_n(P_S\phi_j), \phi_i\rangle = \langle P_S\phi_j, \phi_i\rangle$$

If we define a matrix $A \in \mathbb{R}^{(n+3)\times(n+3)}$ by

$$a_{ij} = \langle P_S\phi_j, \phi_i\rangle, \tag{4.13}$$

then we have

$$\langle (Q_nP_S)\phi_j, \phi_i\rangle = \langle P_S\phi_j, \phi_i\rangle = a_{ij}.$$

So $BM = A$, and hence $M = B^{-1}A$.

Suppose that there exists a nonzero $f_n \in S^3[x_0,\ldots,x_n]$ such that $Q_nP_Sf_n = f_n$. We write $f_n = \sum_{j=-3}^{n-1} d_j\phi_j$, where $d = [d_{-3},\ldots,d_{n-1}]^T$. So d is the coordinate vector of f_n with respect to β [17]. Note that

$$(Q_nP_S)f_n = f_n \iff [(Q_nP_S)f_n]_\beta = [f_n]_\beta \iff [Q_nP_S]_\beta\,[f_n]_\beta = [f_n]_\beta$$

$$\iff Md = d, \quad \text{where } d \neq 0$$

$$\iff (B^{-1}A)d = d \iff Ad = Bd$$

$$\iff (A-B)d = 0.$$

(The equality $[(Q_nP_S)f_n]_\beta = [Q_nP_S]_\beta\,[f_n]_\beta$ can be found in [29]). So showing that the existence of nonzero f_n such that $Q_nP_Sf_n = f_n$ is equivalent to showing $(A-B)d = 0$ with $d \neq 0$, which is again equivalent to showing that $A-B$ has 0 as eigenvalue. So, we need to prove that 0 is an eigenvalue of $A-B$.

Let $e^T = [1,\ldots,1]$ and $e_j = [0,\ldots,0,1,0,\ldots,0]^T$, where 1 is in the j^{th} position. The sum of entries of column j, for $j=-3,\ldots,n-1$, of $A-B$ can be obtained by

$$\begin{aligned}
[e^T(A-B)]e_j &= \sum_{i=-3}^{n-1}(a_{ij}-b_{ij}) = \sum_{i=-3}^{n-1}[\langle P_S\phi_j,\phi_i\rangle - \langle\phi_j,\phi_i\rangle] \\
&= \langle P_S\phi_j, \sum_{i=-3}^{n-1}\phi_i\rangle - \langle\phi_j, \sum_{i=-3}^{n-1}\phi_i\rangle = \langle P_S\phi_j,1\rangle - \langle\phi_j,1\rangle \\
&= \int_{[0,1]} P_S\phi_j\cdot 1\; d\lambda - \langle\phi_j,1\rangle = \int_{[0,1]} P_S\phi_j\; d\lambda - \langle\phi_j,1\rangle \\
&= \int_{S^{-1}[0,1]} \phi_j\; d\lambda - \langle\phi_j,1\rangle = \int_{[0,1]} \phi_j\cdot 1\; d\lambda - \langle\phi_j,1\rangle \\
&= \langle\phi_j,1\rangle - \langle\phi_j,1\rangle = 0.
\end{aligned}$$

(We use the fact that $\sum_{i=-3}^{n-1}\phi_i = 1$ for all $x\in[0,1]$). So

$$e^T(A-B) = 0^T \implies (A^T-B^T)e = 0.$$

Therefore, 0 is the eigenvalue of $A^T - B^T$, and hence 0 is the eigenvalue of $A - B$. Thus there is a nonzero vector $d \in \mathbb{R}^{n+3}$ such that $(A - B)d = 0$. So, using that nonzero vector d, $f_n = \sum_{j=-3}^{n-1} d_j \phi_j$ is a nonzero function in $S^3[x_0, \ldots, x_n]$ such that $Q_n P_S f_n = f_n$. ■

Lemma 4.2 assures that $f_n = \sum_{j=-3}^{n-1} d_j \phi_j$ is a nonzero stationary function of $Q_n P_S$. However, we can not prove the nonnegativity of f_n, and here is the argument behind this assertion. If Q_n is a Markov operator, then by assuming that the dimension of eigenspace of $A - B$ associated with eigenvalue 0 is one [37], we can show that $f_n \geq 0$ as follows. Suppose that f_n^+ and f_n^- are nonzero functions. Since Q_n is a Markov operator, $Q_n P_S$ is also a Markov operator. Moreover, since $Q_n P_S f_n = f_n$, by Theorem 2.7 we have, $Q_n P_S(f_n^+) = f_n^+$ and $Q_n P_S(f_n^-) = f_n^-$. This implies that the dimension of the eigenspace of $A - B$ associated with the 0 eigenvalue is at least two. This contradicts the assumption that the dimension of the eigenspace of $A - B$ associated with the 0 eigenvalue is one. Therefore, $f_n^+ = 0$ or $f_n^- = 0$, which implies that f_n is nonnegative or $-f_n$ is nonnegative. So we may assume that f_n is nonnegative. However, Q_n is not a Markov operator as the next example shows. Consider $Q_4 : L^1[0,1] \to S^3[x_0, \ldots, x_4]$. We find $Q_4 f$ with $f(x) = x^4$. If we solve the corresponding system, $Bc = b$, where $b_{ij} = \langle \phi_j, \phi_i \rangle$ and $b_i = \langle f, \phi_i \rangle$ for $-3 \leq i, j \leq 3$, it can be shown that

$$Q_4 f = \sum_{j=-3}^{3} c_j \phi_j,$$

where

$$c = [-0.0049,\ 0.0018,\ -0.0026, 0.0328,\ 0.2474,\ 0.8768,\ 2.2451]^T.$$

Figure 7 shows the graph of $f(x) = x^4$ and $Q_4 f(x)$, where $x \in [0, 0.18]$, and Figure 8 shows the zoomed in of Figure 7 into the interval $[0, 0.02]$, which clearly shows that there is a small interval near zero on which $Q_4 f$ is negative. This shows that Q_n is not a Markov operator, and hence, f_n can not be assured to be a nonnegative function.

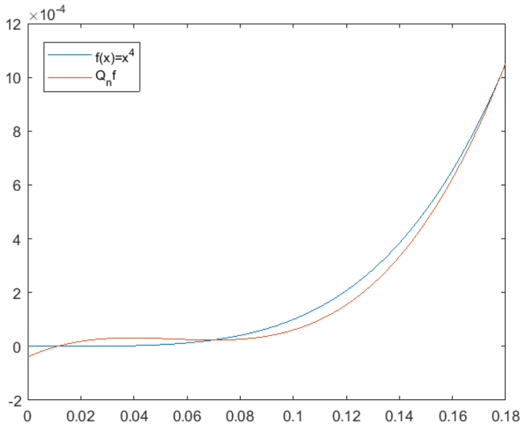


Figure 7: The graphs of $f(x) = x^4$ and $Q_4 f$ on the interval $[0, 0.18]$

From the above example, we note that in general, $Q_n f$ is not necessarily be nonnegative when f is a nonnegative function. Furthermore, we note that every $c_j \geq 0$ is a sufficient condition for $Q_n f$ to be nonnegative but not a necessary condition. For instance, consider $f(x) = x^2$ with $n = 4$. Using MATLAB, we get $Q_4 f$ is nonnegative in the whole domain $[0, 1]$, but not all c_j are nonnegative. In fact, we have

$$c = [0.0417, -0.0208, 0.0417, 0.2292, 0.5417, 0.9792, 1.5417]^T,$$

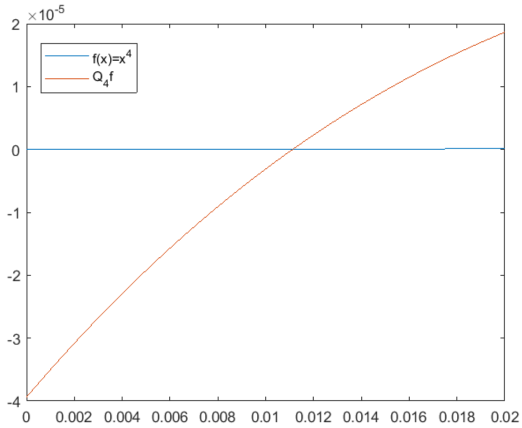


Figure 8: The graphs of $f(x) = x^4$ and $Q_4 f$ on the interval $[0, 0.02]$

where $c_2 < 0$.

Because typically, $f_n \approx f^*$, so if $f^* > 0$, we expect that $f_n > 0$ at least when n is large enough. In fact, in our numerical experiments in Chapter 6, it will be observed that every $d_j \geq 0$, and hence f_n is nonnegative. In that case since $\phi_j \geq 0$ for all j, we can normalize f_n as follows:

$$
\begin{aligned}
\|f_n\|_1 &= \int_0^1 |f_n(x)|\, dx = \int_0^1 \left| \sum_{j=-3}^{n-1} d_j \phi_j(x) \right| dx \\
&= \int_0^1 \sum_{j=-3}^{n-1} d_j \phi_j(x)\, dx = \sum_{j=-3}^{n-1} d_j \int_0^1 \phi_j(x)\, dx = \sum_{j=-3}^{n-1} d_j \, \|\phi_j\|_1 \\
&= d_{-3}\, \|\phi_{-3}\|_1 + d_{-2}\, \|\phi_{-2}\|_1 + \cdots + d_{n-1} \|\phi_{n-1}\|_1 \\
&= \frac{h}{24} d_{-3} + \frac{h}{2} d_{-2} + \frac{23h}{24} d_{-1} + h d_0 + \cdots + h d_{n-4} + \frac{23h}{24} d_{n-3} \\
&\quad + \frac{h}{2} d_{n-2} + \frac{h}{24} d_{n-1} = hs,
\end{aligned}
$$

where

$$s = \frac{d_{-3}}{24} + \frac{d_{-2}}{2} + \frac{23d_{-1}}{24} + d_0 + \cdots + d_{n-4} + \frac{23d_{n-3}}{24} + \frac{d_{n-2}}{2} + \frac{d_{n-1}}{24}.$$

Here we use the facts that

$$\|\phi_{-3}\|_1 = \|\phi_{n-1}\|_1 = \frac{h}{24}, \qquad \|\phi_{-2}\|_1 = \|\phi_{n-2}\|_1 = \frac{h}{2},$$

$$\|\phi_{-1}\|_1 = \|\phi_{n-3}\|_1 = \frac{23h}{24}, \qquad \|\phi_i\|_1 = h \text{ for } i = 0, 1, \ldots, n-4.$$

Since $\|f_n\|_1 = hs$, we have

$$\left\|\frac{f_n}{hs}\right\|_1 = 1.$$

By calling $\dfrac{f_n}{hs}$ again as f_n, we have $\|f_n\|_1 = 1$.

CHAPTER 5

CONVERGENCE ANALYSIS OF CUBIC SPLINE PROJECTION METHOD

The objective of this chapter is to show that $\{f_n\}$ converges to f^* in $L^1[0,1]$; that is, $\lim_{n\to\infty} f_n = f^*$ in $L^1[0,1]$. Recall that $Q_n P_S f_n = f_n$, by (4.12), and f^* is the unique stationary density function of the Frobenius-Perron operator P_S associated with a measurable and nonsingular transformation S.

To achieve our goal of showing $\{f_n\} \to f^*$ in $L^1[0,1]$, we organize this chapter as follows. In Section 5.1, we find a uniform upper bound of $\{\|Q_n\|_1\}$ for $n \geq 8$. In Section 5.2 we show that $\{Q_n f\}$ converges to f for any $f \in L^1[0,1]$ as $n \to \infty$. In Section 5.3 we find that the total variation of $\{Q_n f\}$ is bounded by some positive constant times the total variation of f uniformly for $n \geq 8$ and any $f \in L^1[0,1]$. In Section 5.4 we find a uniform bound of $\{\bigvee_0^1 f_n\}$. Finally in section 5.5 we show that $\{f_n\}$ converges to f^* in $L^1[0,1]$.

5.1 Uniform Boundedness of $\{Q_n\}_{n=1}^{\infty}$

Our goal in this section is to find an upper bound of $\|Q_n\|_1$ for any $n \geq 8$. Recall that for any $f \in L^1[0,1]$ and for $i = -3, -2, \ldots, n-1$, we have

$$\langle Q_n f, \phi_i \rangle \;=\; \langle f, \phi_i \rangle,$$

which, by (4.9), is equivalent to

$$\sum_{j=-3}^{n-1} \langle \phi_j, \phi_i \rangle c_j \;=\; \langle f, \phi_i \rangle$$

since $Q_n f = \sum_{j=-3}^{n-1} c_j \phi_j$. Also recall that by (4.11), the coefficients $c = [c_{-3}, \ldots, c_{n-1}]^T$ can be found by solving the invertible linear system $B_n c = b$. In (4.11) we wrote B instead of B_n, but due to the dependence of B on n, we write B_n instead of B from now on. For definitions of B_n and b, see (4.10). By calculating the inner products of $B_n = \langle \phi_j, \phi_i \rangle$ for $-3 \leq i, j \leq n-1$, we find matrix B_n as follows:

$$B_n = \frac{h}{5040} \begin{bmatrix} 20 & 129 & 60 & 1 & \ldots & & & & & \\ 129 & 1208 & 1062 & 120 & 1 & \ldots & & & & \\ 60 & 1062 & 2396 & 1191 & 120 & 1 & \ldots & & & \\ 1 & 120 & 1191 & 2416 & 1191 & 120 & 1 & \ldots & & \\ 0 & 1 & 120 & 1191 & 2416 & 1191 & 120 & 1 & \ldots & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & \cdots & 1 & 120 & 1191 & 2416 & 1191 & 120 & 1 & 0 \\ & & \cdots & 1 & 120 & 1191 & 2416 & 1191 & 120 & 1 \\ & & & \cdots & 1 & 120 & 1191 & 2396 & 1062 & 60 \\ & & & & \cdots & 1 & 120 & 1062 & 1208 & 129 \\ & & & & & \cdots & 1 & 60 & 129 & 20 \end{bmatrix},$$

where $h = \dfrac{1}{n}$.

In the following, we give two examples of finding entries of B_n by showing how to find $b_{-3,-3} = \langle \phi_{-3}, \phi_{-3} \rangle$ and $b_{-3,-2} = \langle \phi_{-3}, \phi_{-2} \rangle$. Note that by (3.5),

$$\phi_{-3}(x) = \frac{1}{6}\left(4 - \frac{x-(-3h)}{h}\right)^3.$$

So,

$$
\begin{aligned}
b_{-3,-3} &= \langle \phi_{-3}, \phi_{-3} \rangle = \int_0^1 \left(\phi_{-3}(x)\right)^2 \, dx = \int_0^h \left[\frac{1}{6} \left(4 - \frac{x - (-3h)}{h} \right)^3 \right]^2 \, dx \\
&= \int_0^h \frac{1}{36} \left(1 - \frac{x}{h} \right)^6 \, dx = \frac{h}{36} \int_0^1 (1-t)^6 \, dt = \frac{h}{252} = \frac{h}{5040} \cdot 20.
\end{aligned}
$$

For $b_{-3,-2}$, again by (3.5), we have

$$
\phi_{-2}(x) = \frac{1}{6}\left(1+3\left(3 - \frac{x-(-2h)}{h}\right)+3\left(3 - \frac{x-(-2h)}{h}\right)^2-3\left(3 - \frac{x-(-2h)}{h}\right)^3\right).
$$

So,

$$
\begin{aligned}
b_{-3,-2} &= \langle \phi_{-3}, \phi_{-2} \rangle = \int_0^1 \phi_{-3}(x)\phi_{-2}(x) \, dx \\
&= \int_0^h \left[\frac{1}{6} \left(4 - \frac{x-(-3h)}{h} \right)^3 \right] \left[\frac{1}{6} \left(1 + 3 \left(3 - \frac{x-(-2h)}{h} \right) \right.\right. \\
&\quad \left.\left. + \; 3 \left(3 - \frac{x-(-2h)}{h} \right)^2 - 3 \left(3 - \frac{x-(-2h)}{h} \right)^3 \right) \right] \, dx \\
&= \frac{1}{36} \int_0^h \left(1 - \frac{x}{h}\right)^3 \left(1 + 3\left(1 - \frac{x}{h}\right) + 3\left(1 - \frac{x}{h}\right)^2 - 3\left(1 - \frac{x}{h}\right)^3 \right) \, dx \\
&= \frac{h}{36} \int_0^1 (1-t)^3 \left(1 + 3\,(1-t) + 3\,(1-t)^2 - 3\,(1-t)^3\right) \, dt \\
&= \frac{43h}{1680} = \frac{h}{5040} \cdot 129.
\end{aligned}
$$

For simplicity, we define a matrix $\widehat{B}_n \in \mathbb{R}^{(n+3)\times(n+3)}$ and vector $\hat{b} \in \mathbb{R}^{n+3}$ such that

$$
\hat{B}_n = \frac{5040}{h} B_n \quad \text{and} \quad \hat{b} = \frac{5040}{h} b. \tag{5.1}
$$

Then

$$
B_n c = b \iff \widehat{B}_n c = \hat{b}.
$$

Therefore,

$$c \;=\; \widehat{B}_n^{-1}\hat{b},$$

and $\widehat{B}_n$ is given by

$$\widehat{B}_n = \begin{bmatrix} 20 & 129 & 60 & 1 & \dots & & & & & \\ 129 & 1208 & 1062 & 120 & 1 & \dots & & & & \\ 60 & 1062 & 2396 & 1191 & 120 & 1 & \dots & & & \\ 1 & 120 & 1191 & 2416 & 1191 & 120 & 1 & \dots & & \\ 0 & 1 & 120 & 1191 & 2416 & 1191 & 120 & 1 & \dots & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & \cdots & 1 & 120 & 1191 & 2416 & 1191 & 120 & 1 & 0 \\ & & \cdots & 1 & 120 & 1191 & 2416 & 1191 & 120 & 1 \\ & & & \cdots & 1 & 120 & 1191 & 2396 & 1062 & 60 \\ & & & & \cdots & 1 & 120 & 1062 & 1208 & 129 \\ & & & & & \cdots & 1 & 60 & 129 & 20 \end{bmatrix}.$$

Note that $\widehat{B}_n$ is a band-matrix with band-width three, where bandwidth is the smallest nonnegative integer, for example k, such that $b_{ij} = 0$ for $|i-j| > k$ [48].

Since our goal is to find an upper bound of $\|Q_n\|_1$, first we need to find an upper bound of $\left\|\widehat{B}_n^{-1}\right\|_1$, where $\|\cdot\|_1$ is defined as follows:

DEFINITION 5.1. The 1-norm of a matrix $L \in \mathbb{R}^{m\times m}$ is the maximum of column sum of $|l_{ij}|$ for $1 \le j \le m$ [65]. Namely,

$$\|L\|_1 = \max_{1\le j\le m} \sum_{i=1}^{m} |l_{ij}|.$$

We use MATLAB to find the information given in Table 4. As we can see from

Table 4: 1-norm of $\widehat{B}_n^{-1}$

$n=8$	$\|\widehat{B}_8^{-1}\|_1 = 0.355436$
$n=16$	$\|\widehat{B}_{16}^{-1}\|_1 = 0.351039$
$n=32$	$\|\widehat{B}_{32}^{-1}\|_1 = 0.351003$
$n=64$	$\|\widehat{B}_{64}^{-1}\|_1 = 0.351010$
$n=128$	$\|\widehat{B}_{128}^{-1}\|_1 = 0.351010$

Table 4,

$$\left\|\widehat{B}_n^{-1}\right\|_1 \le 0.36 \quad \text{for all} \quad n \ge 8.$$

In the following lemma, we prove the uniform boundedness of $\|Q_n\|_1$.

THEOREM 5.2. $\|Q_n\|_1 \le 1815$ uniformly for any $n \ge 8$.

PROOF. One can show that $\|\phi_i\|_1 \le h$ for all $i = -3, -2, \ldots, n-1$. Recall that $Q_n f = \sum_{i=-3}^{n-1} c_i \phi_i$. If $n \ge 8$, and by (3) of Lemma 3.5 and properties of 1-norm [36], we have

$$\|Q_n f\|_1 = \left\| \sum_{i=-3}^{n-1} c_i \phi_i \right\|_1 \le \sum_{i=-3}^{n-1} |c_i|\, \|\phi_i\|_1 \le \sum_{i=-3}^{n-1} |c_i|\, h$$

$$= h\,\|c\|_1 \;=\; h\left\|\widehat{B}_n^{-1}\hat{b}\right\|_1 \;=\; h\cdot\frac{5040}{h}\left\|\widehat{B}_n^{-1}\right\|_1\|b\|_1$$

$$\leq 5040(0.36)\|b\|_1 \;=\; 1815\|b\|_1 \;=\; 1815\sum_{i=-3}^{n-1}|\langle f,\phi_i\rangle|$$

$$= 1815\sum_{i=-3}^{n-1}\left|\int_0^1 f\cdot\phi_i(x)\;dx\right| \;\leq\; 1815\sum_{i=-3}^{n-1}\int_0^1|f\cdot\phi_i(x)|\;dx$$

$$= 1815\sum_{i=-3}^{n-1}\int_0^1|f|\cdot|\phi_i(x)|\;dx \;=\; 1815\sum_{i=-3}^{n-1}\int_0^1|f|\cdot\phi_i(x)\;dx$$

$$= 1815\sum_{i=-3}^{n-1}\langle|f|,\phi_i\rangle \;=\; 1815\left\langle|f|,\sum_{i=-3}^{n-1}\phi_i\right\rangle$$

$$= 1815\,\langle|f|,1\rangle \;=\; 1815\,\|f\|_1\,,$$

(since $|\int_a^b f(x)\;dx| \leq \int_a^b|f(x)|\;dx$ [52]). Hence,

$$\|Q_n f\|_1 \;\leq\; 1815\,\|f\|_1 \implies \frac{\|Q_n f\|_1}{\|f\|_1} \;\leq\; 1815.$$

Thus,

$$\|Q_n\|_1 \;\leq\; 1815 \;\text{ for all }\; n\geq 8.$$

■

5.2 Convergence of $\{Q_n f\}_{n=1}^{\infty}$ to f

This section shows that $\{Q_n f\}$ converges to f in $L^1[0,1]$. For this purpose, we first need to define a clamped spline function.

DEFINITION 5.3. A *clamped spline* function s_n in $S^3[x_0,\ldots,x_n]$ of a given function g is a cubic spline function s_n in $S^3[x_0,\ldots,x_n]$ [13] that satisfies the following conditions:

(1) $s(x_i) = g(x_i)$ for $i = 0, 1, \ldots, n$.

(2) $s'(x_0) = g'(x_0)$ and $s'(x_n) = g'(x_n)$.

THEOREM 5.4. For any $f \in L^1[0,1]$, $\lim_{n\to\infty} \|Q_n f - f\|_1 = 0$.

PROOF. Let $f \in L^1[0,1]$. We need to show that for any $\epsilon > 0$, we can find $N \in \mathbb{N}$ such that $\|Q_n f - f\|_1 < \epsilon$ whenever $n > N$ [55]. Since $C^4[0,1]$ is dense in $L^1[0,1]$ [19], then for all $f \in L^1[0,1]$, there exists $g \in C^4[0,1]$ such that

$$\|g - f\|_1 \;\le\; \frac{1}{3632}\epsilon.$$

It is a well-known fact, from [9], that for each n, we have $s_n \in S^3[x_0, \ldots, x_n]$ of g, where s_n is a clamped spline function, such that

$$\max_{x\in[0,1]} |g(x) - s_n(x)| \le \frac{5}{384 n^4} \max_{x\in[0,1]} \left| g^{(4)}(x) \right|.$$

We use Hölder's inequality [72] and the fact that $Q_n g$ is the least squares approximation to g from $S^3[x_0, \ldots, x_n]$ to get

$$\begin{aligned}
\|Q_n g - g\|_1 \;&\le\; \|Q_n g - g\|_2 \|1\|_2 = \|Q_n g - g\|_2 \;\le\; \|s_n - g\|_2 \\
&= \left[\int_0^1 |g(x) - s_n(x)|^2 dx\right]^{1/2} \;\le\; \left[\int_0^1 \left[\max_{x\in[0,1]} |g(x) - s_n(x)|\right]^2 dx\right]^{1/2} \\
&= \max_{x\in[0,1]} |g(x) - s_n(x)| \;\le\; \frac{5}{384 n^4} \max_{x\in[0,1]} \left| g^{(4)}(x) \right|.
\end{aligned}$$

So we can choose N big enough so that

$$\|Q_n g - g\|_1 \;<\; \frac{\epsilon}{2}$$

whenever $n > N$. Accordingly, if $n > N$ we have

$$
\begin{aligned}
\|Q_n f - f\|_1 &= \|Q_n f - Q_n g + Q_n g - g + g - f\|_1 \\
&\le \|Q_n f - Q_n g\|_1 + \|Q_n g - g\|_1 + \|g - f\|_1 \\
&\le \|Q_n\|_1 \, \|f - g\|_1 + \|Q_n g - g\|_1 + \|g - f\|_1 \\
&= (\|Q_n\|_1 + 1)\, \|f - g\|_1 + \|Q_n g - g\|_1 \\
&\le (1815 + 1)\, \|f - g\|_1 + \|Q_n g - g\|_1 \\
&< \frac{1816\epsilon}{3632} + \frac{\epsilon}{2} = \epsilon \, .
\end{aligned}
$$

■

5.3 Uniform Boundedness of the Total Variation of $\{Q_n f\}$ by the Total Variation of f

In this section, we find a uniform bound of the total variation of $\{Q_n f\}$ in terms of the total variation of f when $f \in BV[a,b]$, where the bounded variation space $BV[a,b]$ is the space of functions whose total variation is bounded [31]. We divide the process of finding a uniformly upper bound of $\bigvee_0^1 Q_n f$ in terms of $\bigvee_0^1 f$ into three steps illustrated in Subsections 5.3.2, 5.3.3 and 5.3.4. Nevertheless, first, we start with the definition of total variation of a function f, then provide some important properties of the total variation of functions in Subsection 5.3.1.

DEFINITION 5.5. Let $P : a = x_0 < x_1 < \cdots < x_n = b$ be a partition of $[a,b]$. The *total variation* of a real-valued function on an interval $[a,b] \subset \mathbb{R}$ is defined by

$$\bigvee_a^b f \quad = \quad \sup_P \left\{ \sum_{i=0}^{n} |f(x_{i+1}) - f(x_i)| \right\}, \tag{5.2}$$

where the supremum is taken over all partitions P of $[a, b]$ [16].

5.3.1 Properties of Total Variation of functions

Before we state the properties of total variation of functions, we first need to introduce Riemann sum and an important result about Riemann integral [6].

DEFINITION 5.6. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. If $P : a = x_0 < x_1 < \cdots < x_n = b$ is a partition of $[a, b]$, and if $c_1, c_2, \ldots, c_n$ are numbers such that $x_{i-1} \leq c_i \leq x_i$ for $i = 1, 2, \ldots, n$, then the sum

$$R(P, f) = \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1})$$

is called a *Riemann sum* for f corresponding to the partition P and the intermediate points c_i.

DEFINITION 5.7. Let $P : a = x_0 < x_1 < \cdots < x_n = b$ be a partition of $[a, b]$. Then the *norm of partition* P is the width of the largest subinterval of P [62]; that is, for $i = 1, 2, \ldots, n$

$$\|P\| \quad = \quad \max(x_i - x_{i-1}).$$

LEMMA 5.8. Let $f : [a, b] \to R$ be Riemann integrable on $[a, b]$. Then, if $\epsilon > 0$ is given, then there is a $\delta > 0$ such that if P is any partition of $[a, b]$ such that $\|P\| < \delta$, and if $R(P, f)$ is any Riemann sum for f corresponding to partition P and any intermediate points c_i, then

$$\left|R(P,f) - \int_a^b f(x)\,dx\right| \;<\; \epsilon.$$

PROOF. For the proof and more details, see the book by Bartle and Sherbert [6].

■

The following lemma states some of the important properties of total variation of functions (this lemma is found in [26]).

LEMMA 5.9. (1) If f is a monotonic function on $[a,b]$, then

$$\bigvee_a^b f \;=\; |f(a) - f(b)|,$$

where f is called a *monotonic* function if f is an increasing or decreasing function on $[a,b]$ [58].

(2) Let $f_1, f_2, \ldots, f_n$ be functions of bounded variation on $[a,b]$. Then the summation of f_i is of bounded variation and

$$\bigvee_a^b (f_1 + \cdots + f_n) \;\le\; \bigvee_a^b f_1 + \cdots + \bigvee_a^b f_n.$$

(3) Let $g : [s,t] \to [a,b]$ be a monotonic function on $[s,t]$ and f be a function of bounded variation on $[a,b]$, then the composition function $f \circ g$ is a function of bounded variation on $[s,t]$, and

$$\bigvee_s^t f \circ g \;\le\; \bigvee_a^b f.$$

(4) Let f be a function of bounded variation on $[a,b]$ and let $a = a_0 < a_1 < \cdots < a_m = b$. Then f is a function of bounded variation on $[a_{i-1}, a_i]$ for

$i = 1, \ldots, m$ and

$$\bigvee_{a_0}^{a_1} f + \cdots + \bigvee_{a_{m-1}}^{a_m} f \;\; = \;\; \bigvee_{a}^{b} f.$$

(5) Let f be monotonic and continuously differentiable on $[a, b]$. Then

$$\bigvee_{a}^{b} f \;\; = \;\; \int_a^b |f'(x)| dx.$$

(6) If f is a function of bounded variation on $[a, b]$, and g is continuously differentiable on $[a, b]$, then the product function fg is a function of bounded variation on $[a, b]$, and

$$\bigvee_{a}^{b} fg \;\; \leq \;\; \left(\sup_{x \in [a,b]} |g(x)| \right) \bigvee_{a}^{b} f + \int_a^b |f(x) g'(x)| dx.$$

PROOF. (1) Let f be a monotonic function, and $P : a = x_0 < x_1 < \cdots < x_n = b$ be a partition of $[a, b]$. We divide the proof into two cases.

Case 1. Let f be an increasing function, then

$$f(x_{i-1}) \;\; \leq \;\; f(x_i) \;\; \text{whenever} \;\; x_{i-1} < x_i.$$

By the definition of total variation of f over P in (5.2) , we have

$$\bigvee_{a}^{b} f \;\; = \;\; \sup_P \left\{ \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \right\} \; = \; \sup_P \left\{ \sum_{i=1}^{n} f(x_i) - f(x_{i-1}) \right\}$$

$$= \;\; \sup_P \{ f(b) - f(a) \} = \; f(b) - f(b).$$

Case 2. If f is decreasing, similarly we have

$$\bigvee_a^b f \quad = \quad f(a) - f(b).$$

Hence, if f is a monotonic function, we have

$$\bigvee_a^b f \quad = \quad |f(a) - f(b)|.$$

(2) Suppose that $f_1, f_2, \ldots, f_n$ are functions of bounded variation on $[a, b]$, and P : $a = x_0 < x_1 < \cdots < x_n = b$ is a partition of $[a, b]$. Then

$$\begin{aligned}
\bigvee_a^b (f_1 + \cdots + f_n) &= \sup_P \left\{ \sum_{i=1}^{n} |(f_1 + \cdots + f_n)(x_i) - (f_1 + \cdots + f_n)(x_{i-1})| \right\} \\
&= \sup_P \left\{ \sum_{i=1}^{n} |(f_1(x_i) + \cdots + f_n(x_i)) - (f_1(x_{i-1}) + \cdots + f_n(x_{i-1}))| \right\} \\
&= \sup_P \left\{ \sum_{i=1}^{n} |(f_1(x_i) - f_1(x_{i-1})) + \cdots + (f_n(x_i) - f_n(x_{i-1}))| \right\} \\
&\le \sup_P \left\{ \sum_{i=1}^{n} |f_1(x_i) - f_1(x_{i-1})| + \cdots + |f_n(x_i) - f_n(x_{i-1})| \right\} \\
&\le \sup_P \left\{ \sum_{i=1}^{n} |f_1(x_i) - f_1(x_{i-1})| \right\} + \cdots + \sup_P \left\{ \sum_{i=1}^{n} |f_n(x_i) - f_n(x_{i-1})| \right\} \\
&= \bigvee_a^b f_1 + \ldots \bigvee_a^b f_n.
\end{aligned}$$

Therefore,

$$\bigvee_a^b (f_1 + \cdots + f_n) \quad = \quad \bigvee_a^b f_1 + \ldots \bigvee_a^b f_n.$$

(3) Let f be a function of bounded variation on $[a, b]$, and $g : [s, t] \to [a, b]$ be an increasing function on $[s, t]$. Let $\{s = x_0 < x_1 < \cdots < x_n = t\}$ be a partition of

$[s,t]$. Then $\{a < g(x_0) < g(x_1) < \cdots < g(x_n) < b\}$ is a partition of $[a,b]$. So

$$\begin{aligned}
\sum_{i=1}^{n}|(f\circ g)(x_i)-(f\circ g)(x_{i-1})| \;&\leq\; \sum_{i=1}^{n}|(f\circ g)(x_i))-(f\circ g)(x_{i-1}))| \\
&+\; |(f\circ g)(x_0))-f(a)|+|f(b)-(f\circ g)(x_n))| \\
&\leq\; \bigvee_a^b f.
\end{aligned}$$

Hence,

$$\bigvee_s^t f\circ g \;\leq\; \bigvee_a^b f.$$

Similarly, we can prove the same result when g is decreasing.

(4) Let f be a function of bounded variation on $[a,b]$ and let $a = a_0 < a_1 < \cdots < a_m = b$. We will prove the statement if $a < c < b$, which by induction the statement will be true for any finite number of subintervals.

Let $P : a = x_0 < \cdots < x_n = b$ be a partition of $[a,b]$, and $c \in [a,b]$ such that $c = x_m$ for some $0 < m < n$. Assume that the partition $P' : a = x_0 < \cdots < c = x_m < \cdots < x_n = b$. Note that, $P \subseteq P'$. Then $\sum_{i=1}^{n}|f(x_i)-f(x_{i-1})|$ over partition P is less than or equal $\sum_{i=1}^{n}|f(x_i)-f(x_{i-1})|$ over partition P'. Let $P_1' : a = x_0 < \cdots < x_m = c$ be a partition of $[a,c]$, and $P_2' : c = x_m < \cdots < x_n = b$ be a partition of $[c,b]$. Then we can write

$$\sum_{i=1}^{n}|f(x_i)-f(x_{i-1})| \;=\; \sum_{i=1}^{m}|f(x_i)-f(x_{i-1})|+\sum_{i=m+1}^{n}|f(x_i)-f(x_{i-1})|.$$

Therefore, we have

$$\bigvee_a^b f \;=\; \sup_P\left\{\sum_{i=1}^{n}|f(x_i)-f(x_{i-1})|\right\} \;\leq\; \sup_{P'}\left\{\sum_{i=1}^{n}|f(x_i)-f(x_{i-1})|\right\}$$

$$
\begin{aligned}
&= \sup_{P'}\left\{\sum_{i=1}^{m}|f(x_i)-f(x_{i-1})|+\sum_{i=m+1}^{n}|f(x_i)-f(x_{i-1})|\right\} \\
&\leq \sup_{P_1'}\left\{\sum_{i=1}^{m}|f(x_i)-f(x_{i-1})|\right\}+\sup_{P_2'}\left\{\sum_{i=m+1}^{n}|f(x_i)-f(x_{i-1})|\right\} \\
&= \bigvee_a^c f+\bigvee_c^b f.
\end{aligned}
$$

Thus,

$$
\bigvee_a^b f \;\leq\; \bigvee_a^c f+\bigvee_c^b f. \tag{5.3}
$$

To prove the other inequality, let $P_1 : a = x_0 < \cdots < x_m = c$, and $P_2 : c = x_m < \cdots < x_n = b$ be all the partitions of $[a, c]$ and $[c, b]$, respectively, such that for some $\varepsilon > 0$, we have

$$
\bigvee_a^c f-\frac{\epsilon}{2} \;<\; \sum_{i=1}^{m}|f(x_i)-f(x_{i-1})|
$$

and

$$
\bigvee_c^b f-\frac{\epsilon}{2} \;<\; \sum_{i=m+1}^{n}|f(x_i)-f(x_{i-1})|.
$$

Then

$$
\begin{aligned}
\bigvee_a^c f+\bigvee_c^b f-\epsilon &< \sum_{i=1}^{m}|f(x_i)-f(x_{i-1})|+\sum_{i=m+1}^{n}|f(x_i)-f(x_{i-1})| \\
&= \sum_{i=1}^{n}|f(x_i)-f(x_{i-1})| \leq \sup_{p}\left\{\sum_{i=1}^{n}|f(x_i)-f(x_{i-1})|\right\} \\
&= \bigvee_a^b f.
\end{aligned}
$$

Thus,

$$\bigvee_a^c f + \bigvee_c^b f - \epsilon \;<\; \bigvee_a^b f.$$

Since ϵ was arbitrary,

$$\bigvee_a^c f + \bigvee_c^b f \;\leq\; \bigvee_a^b f. \tag{5.4}$$

Therefore, by (5.3) and (5.4), we have

$$\bigvee_a^c f + \bigvee_c^b f \;=\; \bigvee_a^b f,$$

which follows by induction

$$\bigvee_{a_0}^{a_1} f + \cdots + \bigvee_{a_{n-1}}^{a_n} f \;=\; \bigvee_a^b f.$$

(5) Let g be a monotonic and continuously differentiable function on $[a, b]$, and $P : a = x_0 < x_1 < \cdots < x_n = b$ be a partition of $[a, b]$. Note that, using the Fundamental Theorem of Calculus, we have

$$\begin{aligned}
\bigvee_a^b f &= \sup_P \left\{ \sum_{i=1}^n |f(x_i) - f(x_{i-1})| \right\} = \sup_P \left\{ \sum_{i=1}^n \left| \int_{x_{i-1}}^{x_i} f'(x)\ dx \right| \right\} \\
&\leq \sup_P \left\{ \sum_{i=1}^n \int_{x_{i-1}}^{x_i} |f'(x)|\ dx \right\} = \sup_P \left\{ \int_a^b |f'(x)|\ dx \right\} \\
&= \int_a^b |f'(x)|\, dx.
\end{aligned}$$

Therefore, for any partition of $[a, b]$, we have

$$\bigvee_a^b f \;\leq\; \int_a^b |f'(x)|\,. \tag{5.5}$$

On the other hand, let $\epsilon > 0$. Then by Lemma 5.8, there exists $\delta > 0$ such that if a partition $P: a = x_0 < x_1 < \cdots < x_n = b$ of $[a,b]$ satisfies $\|P\| < \delta$, then

$$\left| \sum_{i=1}^{n} |f'(\tilde{c}_i)|(x_i - x_{i-1}) - \int_a^b |f'(x)|\ dx \right| \ < \ \epsilon, \tag{5.6}$$

where $\tilde{c}_i$ is any point in $[x_{i-1}, x_i]$ for $i = 1, 2, \ldots, n$. Note that since $f \in C^1[a,b]$, by the Mean Value Theorem, which for example can be found in [50], we have

$$|f(x_i) - f(x_{i-1})| \ = \ |f'(c_i)|(x_i - x_{i-1})$$

for some $c_i \in [x_{i-1}, x_i]$. Since (5.6) is valid with $\tilde{c}_i = c_i$, we have

$$\left| \sum_{i=1}^{n} |f'(c_i)|(x_i - x_{i-1}) - \int_a^b |f'(x)|\ dx \right| \ < \ \epsilon. \tag{5.7}$$

Therefore, using (5.7), we have

$$\int_a^b |f'(x)|\ dx \ \leq \ \sum_{k=1}^{n} |f'(c_k)|(x_i - x_{i-1}) + \epsilon,$$

which follows from Mean Value Theorem

$$\int_a^b |f'(x)|\ dx \ \leq \ \sum_{k=1}^{n} |f(x_i) - f(x_{i-1})| + \epsilon.$$

By taking the supremum for both side over partition P, we have,

$$\begin{aligned}
\int_a^b |f'(x)|\ dx \ &\leq \ \sup_{\|P\|<\delta} \left\{ \sum_{k=1}^{n} |f(x_i) - f(x_{i-1})| \right\} + \epsilon \\
&\leq \ \sup_{P} \left\{ \sum_{k=1}^{n} |f(x_i) - f(x_{i-1})| \right\} + \epsilon \\
&= \ \bigvee_a^b f + \epsilon.
\end{aligned}$$

Since ϵ was arbitrary,

$$\int_a^b |f'(x)|\ dx \ \leq\ \bigvee_a^b f. \tag{5.8}$$

Thus by (5.5) and (5.8), it follows that

$$\bigvee_a^b f \ = \ \int_a^b |f'(x)|\ dx.$$

(6) Let f be a function of bounded variation on $[a, b]$, and g be a continuously differentiable function on $[a, b]$. Let f and g be in $C^1[a, b]$. Then using (5), we get

$$\begin{aligned}
\bigvee_a^b fg \ &= \ \int_a^b |(fg)'|\ dx \ = \ \int_a^b |f'g + fg'|\ dx \\
&\leq \ \int_a^b |f'g|\ dx + \int_a^b |fg'|\ dx \\
&\leq \ \sup_{x\in[a,b]} |g(x)| \int_a^b |f'|\ dx + \int_a^b |fg'|\ dx \\
&= \ \sup_{x\in[a,b]} |g(x)| \bigvee_a^b f\ dx + \int_a^b |fg'|\ dx.
\end{aligned}$$

Since the space of C^1-functions is dense in the space of bounded variation functions, the result is still true when f is a function of bounded variation. ■

5.3.2 Finding an Upper Bound of $\bigvee_0^1 Q_n f$ in terms of $\sum_{i=-2}^{n-1} |c_i - c_{i-1}|$.

As the title of this subsection says, we will find an upper bound of $\bigvee_0^1 Q_n f$ in terms of $\sum_{i=-2}^{n-1} |c_i - c_{i-1}|$, which will be provided in Lemma 5.10, where $Q_n f = \sum_{i=-3}^{n-1} c_i \phi_i(x)$. Before we prove Lemma 5.10, we give the idea of the proof.

Since $Q_n f \in S^3[x_0, \ldots, n]$, which is a C^2-function, $Q_n f$ is a C^1-function. So we can compute $\bigvee_0^1 Q_n f$ by using (5) of Lemma 5.9 as follows:

$$\bigvee_0^1 Q_n f \;=\; \int_0^1 \left|(Q_n f)^{'}(x)\right| dx \;=\; \int_0^1 \left|\sum_{i=-3}^{n-1} c_i \phi_i^{'}(x)\right| dx, \tag{5.9}$$

where $\phi_i^{'}$ is given by

$$\phi_i^{'}(x) \;=\; \frac{1}{h} C^{'}\left(\frac{x - x_i}{h}\right)$$

and

$$C^{'}(x) \;=\; \begin{cases} \frac{1}{2}x^2 & 0 \le x \le 1 \\ \frac{1}{6}[3 + 6(x-1) - 9(x-1)^2] & 1 \le x \le 2 \\ \frac{1}{6}[-3 - 6(3-x) + 9(3-x)^2] & 2 \le x \le 3 \\ -\frac{1}{2}(4-x)^2 & 3 \le x \le 4 \\ 0 & x \notin (0,4) \end{cases}$$

$$= \begin{cases} \frac{1}{2}x^2 & 0 \le x \le 1 \\ \frac{1}{2} + (x-1) - \frac{3}{2}(x-1)^2] & 1 \le x \le 2 \\ -\frac{1}{2} + (x-3) + \frac{3}{2}(x-3)^2] & 2 \le x \le 3 \\ -\frac{1}{2}(x-4)^2 & 3 \le x \le 4 \\ 0 & x \notin (0,4). \end{cases}$$

Using (5.9), we are going to prove Lemma 5.10 next.

LEMMA 5.10. For any $f \in L^1[0,1]$,

$$\bigvee_0^1 Q_n f \;\le\; \frac{8}{3} \sum_{i=-2}^{n-1} |c_i - c_{i-1}| \,.$$

PROOF. Note that by (5.9), we have

$$\begin{aligned}
\bigvee_0^1 Q_n f &= \int_0^1 \left|(Q_n f)'(x)\ dx\right| = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \left|(Q_n f)'(x)\right|\ dx \\
&= \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \left| \sum_{j=-3}^{n-1} c_j \phi_j'(x) \right|\ dx \\
&= \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \left| c_{i-3}\phi_{i-3}'(x) + c_{i-2}\phi_{i-2}'(x) + c_{i-1}\phi_{i-1}'(x) + c_i\phi_i'(x) \right|\ dx \\
&= \sum_{i=0}^{n-1} \frac{1}{h} \int_{x_i}^{x_{i+1}} \left| c_{i-3} C'\left(\frac{x-x_{i-3}}{h}\right) + c_{i-2} C'\left(\frac{x-x_{i-2}}{h}\right) \right. \\
&\quad \left. + \ c_{i-1} C'\left(\frac{x-x_{i-1}}{h}\right) + c_i C'\left(\frac{x-x_i}{h}\right) \right|\ dx \\
&= \sum_{i=0}^{n-1} \frac{1}{h} \int_{x_i}^{x_{i+1}} \left| c_{i-3}\left(-\frac{1}{2}\left(\frac{x-x_{i-3}}{h} - 4\right)^2\right) \right. \\
&\quad + \ c_{i-2}\left(-\frac{1}{2} + \left(\frac{x-x_{i-2}}{h} - 3\right) + \frac{3}{2}\left(\frac{x-x_{i-2}}{h} - 3\right)^2\right) \\
&\quad + \ c_{i-1}\left(\frac{1}{2} + \left(\frac{x-x_{i-1}}{h} - 1\right) - \frac{3}{2}\left(\frac{x-x_{i-1}}{h} - 1\right)^2\right) \\
&\quad \left. + \ c_i\left(\frac{1}{2}\left(\frac{x-x_i}{h}\right)^2\right) \right|\ dx.
\end{aligned}$$

Let $t = \dfrac{x-i}{h}$. Then the above integral becomes

$$\begin{aligned}
&= \sum_{i=0}^{n-1} \frac{1}{h} \int_0^1 \left| \left(-\frac{1}{2}(t-1)^2 c_{i-3} - \frac{1}{2}c_{i-2} + (t-1)\, c_{i-2} + \frac{3}{2}(t-1)^2 c_{i-2} \right. \right. \\
&\quad \left. \left. + \ \frac{1}{2}c_{i-1} + t c_{i-1} - \frac{3}{2}t^2 c_{i-1} + \frac{1}{2}t^2 c_i \right) h\ dt \right|
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=0}^{n-1} \int_0^1 \left[\left| \frac{1}{2} \left(c_i - 3c_{i-1} + 3c_{i-2} - c_{i-3} \right) t^2 + \left(c_{i-1} - 2c_{i-2} + c_{i-3} \right) t \right.\right. \\
&\quad + \left.\left. \frac{1}{2} \left(c_{i-1} - c_{i-3} \right) \right| \right] dt \\
&\leq \frac{1}{2} \sum_{i=0}^{n-1} \int_0^1 \Big[\left| c_i - 3c_{i-1} + 3c_{i-2} - c_{i-3} \right| t^2 + 2 \left| c_{i-1} - 2c_{i-2} + c_{i-3} \right| t \\
&\quad + \left| c_{i-1} - c_{i-3} \right| \Big] \, dt \\
&= \frac{1}{2} \sum_{i=0}^{n-1} \left[\frac{1}{3} \left| c_i - 3c_{i-1} + 3c_{i-2} - c_{i-3} \right| + \left| c_{i-1} - 2c_{i-2} + c_{i-3} \right| \right. \\
&\quad + \left. \left| c_{i-1} - c_{i-3} \right| \right] \\
&\leq \frac{1}{2} \sum_{i=0}^{n-1} \left[\frac{1}{3} \Big(\left| c_i - c_{i-1} \right| + 2 \left| c_{i-1} - c_{i-2} \right| + \left| c_{i-2} - c_{i-3} \right| \Big) + \left| c_{i-1} - c_{i-2} \right| \right. \\
&\quad + \left. \left| c_{i-2} - c_{i-3} \right| + \left| c_{i-1} - c_{i-2} \right| + \left| c_{i-2} - c_{i-3} \right| \right] \\
&= \sum_{i=0}^{n-1} \left[\frac{1}{6} \Big| c_i - c_{i-1} \Big| + \frac{4}{3} \Big| c_{i-1} - c_{i-2} \Big| + \frac{7}{6} \Big| c_{i-2} - c_{i-3} \Big| \right] \\
&\leq \sum_{i=-2}^{n-1} \left(\frac{1}{6} + \frac{4}{3} + \frac{7}{6} \right) \Big| c_i - c_{i-1} \Big| \;=\; \sum_{i=0}^{n-1} \frac{16}{6} \Big| c_i - c_{i-1} \Big| \\
&= \frac{8}{3} \sum_{i=-2}^{n-1} \Big| c_i - c_{i-1} \Big|.
\end{aligned}
$$

Hence,

$$\bigvee_0^1 Q_n f \;\; \leq \;\; \frac{8}{3} \sum_{i=-2}^{n-1} \Big| c_i - c_{i-1} \Big|. \qquad \blacksquare$$

The next step is to find an upper bound of $\sum_{i=-2}^{n-1} |c_i - c_{i-1}|$. One way to do so is by constructing a nonsingular matrix $\widetilde{B}_n \in \mathbb{R}^{(n+2)\times(n+2)}$ and vector $\tilde{b} \in \mathbb{R}^{n+2}$ such that

$$\widetilde{B}_n \tilde{c} \;\; = \;\; \tilde{b}, \tag{5.10}$$

where

$$\tilde{c} = [c_{-2} - c_{-3}, c_{-1} - c_{-2}, \ldots, c_{n-1} - c_{n-2}]^T,$$

$\widetilde{B}_n = [\tilde{b}_{ij}]_{i,j=-2}^{n-1}$ is a band-matrix with band-width three, and

$$\tilde{b} \;\; = \;\; \begin{bmatrix} \tilde{b}_{-2} \\ \tilde{b}_{-1} \\ \tilde{b}_0 \\ \tilde{b}_1 \\ \vdots \\ \tilde{b}_{n-4} \\ \tilde{b}_{n-3} \\ \tilde{b}_{n-2} \\ \tilde{b}_{n-1} \end{bmatrix} = \begin{bmatrix} \beta_1 \hat{b}_{-2} + \beta_2 \hat{b}_{-3} \\ \beta_3 \hat{b}_{-1} + \beta_4 \hat{b}_{-2} + \beta_5 \hat{b}_{-3} \\ \beta_6 \hat{b}_0 + \beta_7 \hat{b}_{-1} + \beta_8 \hat{b}_{-2} + \beta_9 \hat{b}_{-3} \\ \gamma_1 \hat{b}_1 + \gamma_0 \hat{b}_0 \\ \vdots \\ \gamma_{n-4} \hat{b}_{n-4} + \gamma_{n-5} \hat{b}_{n-5} \\ \delta_1 \hat{b}_{n-1} + \delta_2 \hat{b}_{n-2} + \delta_3 \hat{b}_{n-3} + \delta_4 \hat{b}_{n-4} \\ \delta_5 \hat{b}_{n-1} + \delta_6 \hat{b}_{n-2} + \delta_7 \hat{b}_{n-3} \\ \delta_8 \hat{b}_{n-1} + \delta_9 \hat{b}_{n-2} \end{bmatrix}.$$

Both $\widetilde{B}_n$ and $\tilde{b}$ are to be determined in the next subsection.

5.3.3 Finding an Upper Bound of $\sum_{i=-2}^{n-1} |c_i - c_{i-1}|$ in terms of $\|\tilde{b}\|_1$

We determine $\widetilde{B}_n$ and $\tilde{b}$ as follows. Note that the first equation of $\widetilde{B}_n\tilde{c} = \tilde{b}$ and the first two equations of $\widehat{B}_n c = \hat{b}$ give

$$\tilde{b}_{-2,-2}(c_{-2} - c_{-3}) + \tilde{b}_{-2,-1}(c_{-1} - c_{-2}) + \tilde{b}_{-2,0}(c_0 - c_{-1}) + \tilde{b}_{-2,1}(c_1 - c_0)$$

$$= \beta_1\hat{b}_{-2} + \beta_2\hat{b}_{-3}$$

$$= \beta_1(129c_{-3} + 1208c_{-2} + 1062c_{-1} + 120c_0 + c_1) + \beta_2(20c_{-3} + 129c_{-2} + 60c_{-1} + c_0)$$

$$= (129\beta_1 + 20\beta_2)c_{-3} + (1208\beta_1 + 129\beta_2)c_{-2} + (1062\beta_1 + 60\beta_2)c_{-1}$$

$$+ \ (120\beta_1 + \beta_2)c_0 + \beta_1 c_1.$$

The left hand side of the above equation can be written as

$$-\tilde{b}_{-2,-2}c_{-3} + (\tilde{b}_{-2,-2} - \tilde{b}_{-2,-1})c_{-2} + (\tilde{b}_{-2,-1} - \tilde{b}_{-2,0})c_{-1} + (\tilde{b}_{-2,0} - \tilde{b}_{-2,1})c_0 + \tilde{b}_{-2,1}c_1,$$

so

$$\begin{aligned}
-\tilde{b}_{-2,-2} &= 129\beta_1 + 20\beta_2, \\
\tilde{b}_{-2,-2} - \tilde{b}_{-2,-1} &= 1208\beta_1 + 129\beta_2, \\
\tilde{b}_{-2,-1} - \tilde{b}_{-2,0} &= 1062\beta_1 + 60\beta_2, \\
\tilde{b}_{-2,0} - \tilde{b}_{-2,1} &= 120\beta_1 + \beta_2, \\
\tilde{b}_{-2,1} &= \beta_1.
\end{aligned}$$

By setting $\tilde{b}_{-2,1} = \beta_1 = 1$, then solving the remaining top four equations for $\tilde{b}_{-2,-2}, \tilde{b}_{-2,-1}, \tilde{b}_{-2,0}$, and β_2, we obtain

$$\tilde{b}_{-2,-2} = 111,\ \tilde{b}_{-2,-1} = 451,\ \tilde{b}_{-2,0} = 109,\ \beta_2 = -12.$$

Continuing to use the similar technique gives us

$$\widetilde{B}_n = \begin{bmatrix}
111 & 451 & 109 & 1 & \ldots & & & & & \\
178 & 1403 & 1071 & 119 & 1 & \ldots & & & & \\
435 & 3956 & 4557 & 1429 & 122 & 1 & \ldots & & & \\
1 & 120 & 1191 & 2416 & 1191 & 120 & 1 & \ldots & & \\
0 & 1 & 120 & 1191 & 2416 & 1191 & 120 & 1 & \ldots & \\
 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & \\
 & \cdots & 1 & 120 & 1191 & 2416 & 1191 & 120 & 1 & 0 \\
 & & \cdots & 1 & 120 & 1191 & 2416 & 1191 & 120 & 1 \\
 & & & \ldots & 1 & 122 & 1429 & 4557 & 3956 & 435 \\
 & & & & \ldots & 1 & 119 & 1071 & 1403 & 178 \\
 & & & & & \ldots & 1 & 109 & 451 & 111
\end{bmatrix}$$

and

$$
\tilde{b} = \begin{bmatrix} \hat{b}_{-2} - 12\hat{b}_{-3} \\ \hat{b}_{-1} - 2\hat{b}_{-2} + \hat{b}_{-3} \\ \hat{b}_0 + \hat{b}_{-1} - 4\hat{b}_{-2} + \hat{b}_{-3} \\ \hat{b}_1 - \hat{b}_0 \\ \vdots \\ \hat{b}_{n-4} - \hat{b}_{n-5} \\ -\hat{b}_{n-1} + 4\hat{b}_{n-2} - \hat{b}_{n-3} - \hat{b}_{n-4} \\ -\hat{b}_{n-1} + 2\hat{b}_{n-2} - \hat{b}_{n-3} \\ 12\hat{b}_{n-1} - \hat{b}_{n-2} \end{bmatrix}. \tag{5.11}
$$

Since our goal is to find an upper bound of $\sum_{i=-2}^{n-1}|c_i - c_{i-1}| = \sum_{i=-2}^{n-1}|\tilde{c}_i|$, we write $\widetilde{B}_n\tilde{c} = \tilde{b}$ as $\tilde{c} = \widetilde{B}_n^{-1}\tilde{b}$. It turns out that $\|\widetilde{B}_n^{-1}\|_1$ is uniformly bounded. In fact, MATLAB gives the following results:

Table 5: 1-norm of $\widetilde{B}_n^{-1}$

$n = 8$	$\|\widetilde{B}_8^{-1}\|_1 = 0.042578$
$n = 16$	$\|\widetilde{B}_{16}^{-1}\|_1 = 0.041892$
$n = 32$	$\|\widetilde{B}_{32}^{-1}\|_1 = 0.041887$
$n = 64$	$\|\widetilde{B}_{64}^{-1}\|_1 = 0.041869$
$n = 128$	$\|\widetilde{B}_{128}^{-1}\|_1 = 0.041869$

Hence, for $n \geq 8$

$$\|\widetilde{B}_n^{-1}\|_1 \;\le\; 0.043. \tag{5.12}$$

So, for $n \ge 8$,

$$\begin{aligned}\sum_{i=-2}^{n-1} |c_i - c_{i-1}| \;&=\; \sum_{i=-2}^{n-1} |\tilde{c}_i| \;=\; \|\tilde{c}\|_1 \\ &=\; \|\widetilde{B}_n^{-1}\tilde{b}\|_1 \;\le\; \|\widetilde{B}_n^{-1}\|_1 \|\tilde{b}\|_1 \\ &\le\; 0.043\|\tilde{b}\|_1.\end{aligned}$$

Therefore,

$$\sum_{i=-2}^{n-1} |c_i - c_{i-1}| \;=\; 0.043\|\tilde{b}\|_1 \quad \text{for} \quad n \ge 8.$$

5.3.4 Finding an Upper Bound of $\|\tilde{b}\|_1$ in terms of $\bigvee_0^1 f$

In this subsection we find an upper bound of $\|\tilde{b}\|_1$ in terms of the total variation of f, which is done in the following lemma, where $\tilde{b} \in \mathbb{R}^{n+2}$ is defined in (5.11) in the previous subsection.

LEMMA 5.11. For $f \in L^1[0,1]$

$$\|\tilde{b}\|_1 \;\le\; 37254 \bigvee_0^1 f.$$

PROOF. Note that by (5.1) and (5.11), $\|\tilde{b}\|_1$ is expressed as follows:

$$\|\tilde{b}\|_1 \;=\; \left|\hat{b}_{-2} - 12\hat{b}_{-3}\right| + \left|\hat{b}_{-1} - 2\hat{b}_{-2} + \hat{b}_{-3}\right| + \left|\hat{b}_0 + \hat{b}_{-1} - 4\hat{b}_{-2} + \hat{b}_{-3}\right|$$

$$
\begin{aligned}
&+ \sum_{i=0}^{n-5} \left| \hat{b}_{i+1} - \hat{b}_i \right| + \left| \hat{b}_{n-1} - 4\hat{b}_{n-2} + \hat{b}_{n-3} + \hat{b}_{n-4} \right| \\
&+ \left| \hat{b}_{n-1} - 2\hat{b}_{n-2} + \hat{b}_{n-3} \right| + \left| 12\hat{b}_{n-1} - \hat{b}_{n-2} \right| \\
= \ & \frac{5040}{h} \Big[\, |b_{-2} - 12b_{-3}| + |b_{-1} - 2b_{-2} + b_{-3}| + |b_0 + b_{-1} - 4b_{-2} + b_{-3}| \\
&+ \sum_{i=0}^{n-5} |b_{i+1} - b_i| + |b_{n-1} - 4b_{n-2} + b_{n-3} + b_{n-4}| \\
&+ |b_{n-1} - 2b_{n-2} + b_{n-3}| + |12b_{n-1} - b_{n-2}| \, \Big].
\end{aligned}
$$

To find an upper bound of $\|\tilde{b}\|_1$, it is sufficient to find an upper bound of each term in the above expression. We first find an upper bound of the general term $\|b_{i+1} - b_i\|_1$ for $i = 0, 1, \ldots, n-5$. Note that (x_i, x_{i+5}) is the support of $\phi_{i+1}(x) - \phi_i(x)$. So $|b_{i+1} - b_i|$ can be written as

$$
\begin{aligned}
|b_{i+1} - b_i| &= \left| \int_0^1 f(x) \left[\phi_{i+1}(x) - \phi_i(x) \right] \, dx \right| \\
&= \left| \int_{ih}^{(i+5)h} f(x) \left[\phi_{i+1}(x) - \phi_i(x) \right] \, dx \right|.
\end{aligned}
$$

Recall that $\|\phi_i\|_1 = h$ for $i = 0, 1, \ldots, n-4$. Then for $i = 0, 1, \ldots, n-5$,

$$
\begin{aligned}
\int_0^1 \left[\phi_{i+1}(x) - \phi_i(x) \right] \, dx &= \int_0^1 \phi_{i+1}(x) \, dx - \int_0^1 \phi_i(x) \, dx \\
&= \|\phi_{i+1}\|_1 - \|\phi_i\|_1 = h - h = 0.
\end{aligned}
$$

If we choose a constant, for example $f(ih)$, in support of $\phi_{i+1}(x) - \phi_i(x)$, then

$$
\int_{ih}^{(i+5)h} f(ih) \left[\phi_{i+1}(x) - \phi_i(x) \right] \, dx = f(ih) \int_{ih}^{(i+5)h} \left[\phi_{i+1}(x) - \phi_i(x) \right] \, dx = 0.
$$

Therefore,

$$
\begin{aligned}
|b_{i+1} - b_i| &= \left| \int_{ih}^{(i+5)h} f(x)\left(\phi_{i+1}(x) - \phi_i(x)\right)\ dx \right. \\
&\qquad \left. - \int_{ih}^{(i+5)h} f(ih)\left(\phi_{i+1}(x) - \phi_i(x)\right)\ dx \right| \\
&= \left| \int_{ih}^{(i+5)h} \left(f(x) - f(ih)\right)\left(\phi_{i+1}(x) - \phi_i(x)\right)\ dx \right| \\
&\leq \int_{ih}^{(i+5)h} |f(x) - f(ih)| \cdot |\phi_{i+1}(x) - \phi_i(x)|\ dx \\
&\leq \int_{ih}^{(i+5)h} \left(\bigvee_{ih}^{(i+5)h} f \right) |\phi_{i+1}(x) - \phi_i(x)|\ dx.
\end{aligned}
$$

By symmetry the zero of $|\phi_{i+1} - \phi_i|$ that lies in $[ih, (i+5)h]$ is $\frac{(2i+5)h}{2}$, which is the intersection point of $\phi_{i+1} - \phi_i$ as Figure 9 shows.

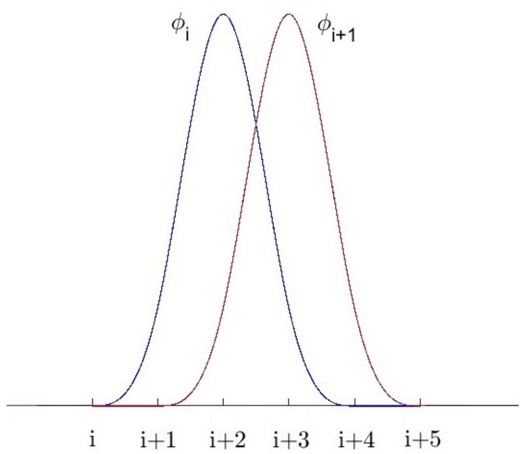


Figure 9: Intersection point between $\phi_{i+1} - \phi_i$ for $i = 0, \ldots, n-5$

Therefore, the expressions of $|\phi_{i+1}(x) - \phi_i(x)|$ are the following:

$$|\phi_{i+1}(x) - \phi_i(x)| \;=\; \begin{cases} \phi_i(x) & [ih, (i+1)h] \\ \phi_i(x) - \phi_{i+1}(x) & [(i+1)h, (i+2)h] \\ \phi_i(x) - \phi_{i+1}(x) & [(i+2)h, (i+\frac{5}{2})h] \\ \phi_{i+1}(x) - \phi_i(x) & [(i+\frac{5}{2})h, (i+3)h] \\ \phi_{i+1}(x) - \phi_i(x) & [(i+3)h, (i+4)h] \\ \phi_{i+1}(x) & [(i+4)h, (i+5)h]. \end{cases}$$

By using (3.5), we calculate $\int_{ih}^{(i+5)h} |\phi_{i+1}(x) - \phi_i(x)| \; dx$ over 6 subintervals specified above as follows:

Over the first and last subintervals, we have

$$\begin{aligned} \int_{ih}^{(i+1)h} \phi_i(x) \; dx \;&=\; \int_{ih}^{(i+1)h} C\left(\frac{x - x_i}{h}\right) \; dx \;=\; h\int_0^1 C(t) \; dt \\ &=\; h\int_0^1 \frac{1}{6}t^3 \; dt \;=\; \frac{h}{24} \end{aligned}$$

and

$$\begin{aligned} \int_{(i+4)h}^{(i+5)h} \phi_{i+1}(x) \; dx \;&=\; \int_{(i+4)h}^{(i+5)h} C\left(\frac{x - x_{i+1}}{h}\right) \; dx \;=\; h\int_4^5 C(t-1) \; dt \\ &= h\int_4^5 \frac{1}{6}(5-t)^3 \; dt \;=\; \frac{h}{24}. \end{aligned}$$

Over the second interval, we have

$$\begin{aligned} \int_{(i+1)h}^{(i+2)h} (\phi_i(x) - \phi_{i+1}(x)) \; dx \;&=\; \int_{(i+1)h}^{(i+2)h} \left(C\left(\frac{x - x_i}{h}\right) - C\left(\frac{x - x_{i+1}}{h}\right)\right) \; dx \\ &=\; h\int_1^2 (C(t) - C(t-1)) \; dt \end{aligned}$$

$$
\begin{aligned}
&= h\int_1^2 \frac{1}{6}\big(1+3(t-1)+3(t-1)^2-3(t-1)^3 \\
&- \ (t-1)^3\big)\ dt \\
&= h\frac{1}{6}\left[t+\frac{3}{2}(t-1)^2+(t-1)^3-(t-1)^4\right]_1^2 \\
&= \frac{5h}{12}.
\end{aligned}
$$

Over the third subinterval, we have

$$
\begin{aligned}
\int_{(i+2)h}^{(i+\frac{5}{2}))h} (\phi_i(x)-\phi_{i+1}(x))\ dx &= \int_{(i+2)h}^{(i+\frac{5}{2}))h} \left(C\left(\frac{x-x_i}{h}\right)-C\left(\frac{x-x_{i+1}}{h}\right)\right)\ dx \\
&= h\int_2^{\frac{5}{2}} \Big[C(t)-C(t-1)\Big]\ dt \\
&= h\int_2^{\frac{5}{2}} \frac{1}{6}\big[1+3(3-t)+3(3-t)^2-3(3-t)^3 \\
&- \ 1-3(t-2)-3(t-2)^2+3(t-2)^3\big]\ dt \\
&= \frac{9h}{64}.
\end{aligned}
$$

Over the fourth subinterval, we have

$$
\begin{aligned}
\int_{(i+\frac{5}{2}))h}^{(i+3)h} (\phi_{i+1}(x)-\phi_i(x))\ dx &= \int_{(i+\frac{5}{2}))h}^{(i+3)h} \left(C\left(\frac{x-x_{i+1}}{h}\right)-C\left(\frac{x-x_i}{h}\right)\right)\ dx \\
&= h\int_{\frac{5}{2}}^{3} (C(t-1)-C(t))\ dt \\
&= h\int_{\frac{5}{2}}^{3} \frac{1}{6}\big[1+3(t-2)+3(t-2)^2-3(t-2)^3
\end{aligned}
$$

$$- \ 1 - 3(3-t) - 3(3-t)^2 + 3(3-t)^3\big] \ dt$$

$$= \frac{9h}{64}.$$

Finally, over the fifth subinterval, we have

$$\begin{aligned}
\int_{(i+3)h}^{(i+4)h} (\phi_{i+1}(x) - \phi_i(x)) \ dx &= \int_{(i+3)h}^{(i+4)h} \left(C\left(\frac{x - x_{i+1}}{h}\right) - C\left(\frac{x - x_i}{h}\right)\right) \ dx \\
&= h \int_3^4 (C(t-1) - C(t)) \ dt \\
&= h \int_3^4 \frac{1}{6} \big[1 + 3(4-t) + 3(4-t)^2 - 3(4-t)^3 \\
&\quad - \ (4-t)^3\big] \ dt = \frac{5h}{12}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|b_{i+1} - b_i| &\leq \bigvee_{ih}^{(i+5)h} f \left[\frac{h}{24} + \frac{5h}{12} + \frac{9h}{64} + \frac{9h}{64} + \frac{5h}{12} + \frac{h}{24}\right] \\
&= \frac{115}{96} h \bigvee_{ih}^{(i+5)h} f \ \text{ for } \ i = 0, 1, \ldots, n-5.
\end{aligned}$$

Using the same technique, one can show that

$$|b_{-2} - 12b_{-3}| \leq \frac{2h}{5} \bigvee_0^{2h} f \quad \text{and} \quad |12b_{n-1} - b_{n-2}| \leq \frac{2h}{5} \bigvee_{1-2h}^{1}$$

$$|b_{-1} - 2b_{-2} + b_{-3}| \leq \frac{19h}{20} \bigvee_0^{3h} f \quad \text{and} \quad |b_{n-1} - 2b_{n-2} + b_{n-3}| \leq \frac{19h}{20} \bigvee_{1-3h}^{1} f$$

$$|b_0 + b_{-1} - 4b_{-2} + b_{-3}| \leq \frac{13h}{5} \bigvee_0^{4h} f \quad \text{and} \quad |b_{n-1} - 4b_{n-2} + b_{n-3} + b_{n-4}| \leq \frac{13h}{5} \bigvee_{1-4h}^{1} f.$$

Therefore,

$$
\begin{aligned}
\|\tilde{b}\|_1 \;\le\; & \frac{5040}{h}\Bigg[\frac{2}{5}h\bigvee_0^{2h} f + \frac{19}{20}h\bigvee_0^{3h} f + \frac{13}{5}h\bigvee_0^{4h} f \\
& + \sum_{i=0}^{n-5}\frac{115}{96}h\bigvee_{ih}^{(i+5)h} f + \frac{13}{5}h\bigvee_{1-4h}^{1} f + \frac{19}{20}h\bigvee_{1-3h}^{1} f + \frac{2}{5}h\bigvee_{1-2h}^{1}\Bigg] \\
\le\; & 2016\bigvee_0^{x_2} f + 4788\bigvee_0^{x_3} f + 13104\bigvee_0^{x_4} f \\
& + \sum_{i=0}^{n-5} 6037.5\bigvee_{ih}^{(i+5)h} f + 13104\bigvee_{x_{n-4}}^{x_n} f + 4788\bigvee_{x_{n-3}}^{x_n} f + 2016\bigvee_{x_{n-2}}^{x_n} f \\
\le\; & 25946\bigvee_0^{x_1} f + 31983\bigvee_{x_1}^{x_2} f + 36005\bigvee_{x_2}^{x_3} f + 37254\bigvee_{x_3}^{x_4} f + \sum_{i=5}^{n-4} 30188\bigvee_{x_{i-1}}^{x_i} f \\
& + 37254\bigvee_{x_{n-4}}^{x_{n-3}} f + 36005\bigvee_{x_{n-3}}^{x_{n-2}} f + 31983\bigvee_{x_{n-2}}^{x_{n-1}} f + 25946\bigvee_{x_{n-1}}^{x_n} \\
\le\; & 37254\bigvee_0^{1} f
\end{aligned}
$$

Hence,

$$\|\tilde{b}\|_1 \;\le\; 37254\bigvee_0^1 f.$$

■

Once we found that $\|\tilde{b}\|_1$ is bounded by $37254\bigvee_0^1 f$, we are ready to find a uniform upper bound of the total variation of $Q_n f$ in terms of the total variation of f.

THEOREM 5.12. Let $f \in BV[0,1]$. Then for $n \ge 8$ we have,

$$\bigvee_0^1 Q_n f \;\leq\; 4272 \bigvee_0^1 f.$$

PROOF. Let $f \in BV[0,1]$. By Lemma 5.10, equation (5.12), and Lemma 5.11, we have

$$\begin{aligned}
\bigvee_0^1 Q_n f \;&\leq\; \frac{8}{3}\sum_{i=-2}^{n-1} |c_i - c_{i-1}| \\
&\leq\; \frac{8}{3} \cdot 0.043 \cdot 37254 \bigvee_0^1 f \;\leq\; 4272 \bigvee_0^1 f.
\end{aligned}$$

Therefore, for $n \geq 8$

$$\bigvee_0^1 Q_n f \;\leq\; 4272 \bigvee_0^1 f.$$

■

5.4 Uniform Boundedness of Total Variation of $\{f_n\}$

This section is about finding a uniform upper bound of the total variation of f_n for $n \geq 8$. To do that, we need Yorke's inequality and Lasota-York inequality; more details can be found in the book by Ding and Zhou [26] and the book by Lasota and Yorke [46].

Yorke's Inequality. Let f be a function of bounded variation on $[a,b]$, which is contained in $[0,1]$. Then we have

$$\bigvee_0^1 f 1_{[a,b]} \;\leq\; 2\bigvee_a^b f + \frac{2}{b-a}\int_a^b |f(x)|dx. \tag{5.13}$$

PROOF. First of all we need to prove the following claim.

Claim. There exists $c \in [a,b]$ such that

$$|f(c)| \;\leq\; \frac{1}{b-a}\int_a^b |f(x)|\; dx.$$

Proof. We prove the claim by way of contradiction. Suppose that for all $x \in [a, b]$, we have

$$|f(x)| \;>\; \frac{1}{b-a}\int_a^b |f(x)|\; dx.$$

Let $v = \displaystyle\int_a^b |f(x)|\; dx$. Then

$$|f(x)| \;>\; \frac{1}{b-a} v.$$

Integrate both side with respect to x, we get

$$\int_a^b |f(x)|\; dx \;\;>\;\; \frac{1}{b-a}\int_a^b v\; dx \;\;=\;\; \frac{1}{b-a}\; v \int_a^b \; dx \;\;=\;\; \frac{1}{b-a}\; v\; (b-a) \;\;=\;\; v.$$

Thus,

$$\int_a^b |f(x)|\; dx \;\;>\;\; \int_a^b |f(x)|\; dx,$$

which is a contradiction. Hence, there exists $c \in [a, b]$ such that

$$|f(c)| \;\leq\; \frac{1}{b-a}\int_a^b |f(x)|\; dx.$$

$\square$

Note that

$$\bigvee_0^1 f1_{[a,b]} \;\;=\;\; \bigvee_a^b f + |f(a)| + |f(b)|.$$

By using (1) and (4) of Lemma 5.9 and the claim, we get

$$\bigvee_0^1 f1_{[a,b]} \;\;=\;\; \bigvee_a^b f + |f(a)| + |f(b)|$$

$$
\begin{aligned}
&= \bigvee_a^b f + |f(a) - f(c) + f(c)| + |f(b) - f(c) + f(c)| \\
&\le \bigvee_a^b f + |f(a) - f(c)| + |f(b) - f(c)| + 2|f(c)| \\
&\le \bigvee_a^b f + \bigvee_a^c f + \bigvee_c^b f + 2\frac{1}{b-a}\int_a^b |f(x)|\, dx \\
&= \bigvee_a^b f + \bigvee_a^b f + 2\frac{1}{b-a}\int_a^b |f(x)|\, dx \\
&= 2\bigvee_a^b f + \frac{2}{b-a}\int_a^b |f(x)|\, dx.
\end{aligned}
$$

■

Lasota-Yorke inequality. Let $f \in BV[0,1]$ be a density function. Let $S : [0,1] \to [0,1]$ be a map that satisfies the following conditions:

(1) There is a partition $0 = a_0 < a_1 < \cdots < a_r = 1$ of the interval $[0,1]$ such that for $i = 1, 2, \ldots, r$, the restriction $S|_{(a_{i-1},a_i)}$ of S to the open interval (a_{i-1}, a_i) can be extended to the closed interval $[a_{i-1}, a_i]$ as a C^2-function.

(2) Assume that

$$s_1 \equiv \inf\{|S'(x)| : x \in [0,1] \, / \{a_1, a_2, \ldots, a_{r-1}\}\} > 0.$$

(3) Let s_2 be defined by

$$s_2 = \sup\left\{\frac{|S''(x)|}{[S'(x)]^2} : x \in [0,1] \, / \{a_1, a_2, \ldots, a_{r-1}\}\right\}.$$

Then

$$\bigvee_0^1 Pf \;\leq\; \alpha \bigvee_0^1 f + \beta \|f\|_1,$$

where

$$\alpha = \frac{2}{s_1} \quad \text{and} \quad \beta = \max_{i=1,\ldots,n} 2\left[s_2 + \frac{1}{\lambda(I_i)}\right].$$

PROOF. Let $S_i = S|_{(a_{i-1},a_i)}$, $g_i = S_i^{-1}$, and $I_i = S((a_{i-1}, a_i))$ for $i = 1, 2, \ldots, n$. Let's denote

$$\Delta_i(x) \;=\; \begin{cases} (a_{i-1}, g_i(x)) & x \in I_i,\ g_i'(x) > 0 \\ (g_i(x), a_i) & x \in I_i,\ g_i'(x) < 0 \\ (a_{i-1}, a_i) \text{ or } \phi & x \notin I_i. \end{cases}$$

Then for $x \in [0,1]$,

$$S^{-1}((0,x)) \;=\; \bigcup_{i=1}^n \Delta_i(x).$$

Using the explicit definition of Frobenius-Perron operator in equation (2.13) and the fact that Δ_i is disjoint, we have

$$\begin{aligned} Pf \;&=\; \frac{d}{dx}\int_{S^{-1}((0,x)} f(t)\ dt \;=\; \frac{d}{dx}\int_{\bigcup_{i=1}^n \Delta_i(x)} f(t)\ dt \\ &=\; \sum_{i=1}^n \frac{d}{dx}\int_{\Delta_i(x)} f(t)\ dt. \end{aligned}$$

The Fundamental Theorem of Calculus, where $F'(x) = f(x)$, gives

$$\frac{d}{dx}\int_{\Delta_i(x)} f(t)\ dt \;=\; \begin{cases} \frac{d}{dx}\int_{a_{i-1}}^{g_i(x)} f(t)dt & x \in I_i,\ g_i'(x) > 0 \\ \frac{d}{dx}\int_{g_i(x)}^{a_i} f(t)dt & x \in I_i,\ g_i'(x) < 0 \\ \frac{d}{dx}\int_{a_{i-1}}^{a_i} f(t)dt & x \notin I_i \end{cases}$$

$$= \begin{cases} \frac{d}{dx}\left[F(g_i(x)) - F(a_{i-1})\right] & x \in I_i,\ g_i'(x) > 0 \\ \frac{d}{dx}\left[F(a_i) - F(g_i(x))\right] & x \in I_i,\ g_i'(x) < 0 \\ 0 & x \notin I_i \end{cases}$$

$$= \begin{cases} g_i'(x) f(g_i(x)) & x \in I_i,\ g_i'(x) > 0 \\ -g_i'(x) f(g_i(x)) & x \in I_i,\ g_i'(x) < 0 \\ 0 & x \notin I_i. \end{cases}$$

Then

$$Pf = \sum_{i=1}^{n} \sigma_i(x) f(g_i(x)) 1_{I_i}(x), \tag{5.14}$$

where $\sigma_i(x) = |g_i'(x)|$. Note that by (2) for all $x \in I_i$, we have

$$\sigma_i(x) = |g_i'(x)| = \frac{1}{|S'(g_i(x))|} \leq \frac{1}{s_1}. \tag{5.15}$$

Case 1. If $g_i'(x) > 0$, then $\sigma_i(x) = g_i'(x) = \dfrac{1}{S'(g_i(x))}$. So

$$|\sigma_i'(x)| = \left| \left[\frac{1}{S'(g_i(x))} \right]' \right| = \left| \frac{-S''(g_i(x)) g_i'(x)}{[S'(g_i(x))]^2} \right|.$$

Case 2. If $g_i'(x) < 0$, then $\sigma_i(x) = -g_i'(x) = -\dfrac{1}{S'(g_i(x))}$. So

$$|\sigma_i'(x)| = \left| \left[-\frac{1}{S'(g_i(x))} \right]' \right| = \left| \left[\frac{1}{S'(g_i(x))} \right]' \right| = \left| \frac{-S''(g_i(x)) g_i'(x)}{[S'(g_i(x))]^2} \right|.$$

Thus by (3), for all $x \in I_i$,

$$|\sigma_i'(x)| = \left| \frac{-S''(g_i(x)) g_i'(x)}{[S'(g_i(x))]^2} \right| \leq s_2 \sigma_i(x). \tag{5.16}$$

Let $f \in BV[0,1]$ be a density function. Then by (5.13), (5.14), (5.15), (5.16) and (2), (3), (4), and (6) of Lemma 5.9, we have

$$
\begin{aligned}
\bigvee_0^1 Pf &= \bigvee_0^1 \sum_{i=1}^n \sigma_i(x)(f\circ g_i)1_{I_i} \\
&\le \sum_{i=1}^n \bigvee_0^1 \sigma_i(x)(f\circ g_i)1_{I_i} \\
&\le \sum_{i=1}^n \left[2\bigvee_{I_i} \sigma_i(f\circ g_i) + \frac{2}{\lambda(I_i)}\int_{I_i} \sigma_i(f\circ g_i)\ dx \right] \\
&\le 2\sum_{i=1}^n \left[\left(\sup_{x\in I_i} \sigma_i \right) \bigvee_{I_i}(f\circ g_i) + \int_{I_i} |\sigma_i'|(f\circ g_i)\ dx \right] + \sum_{i=1}^n \frac{2}{\lambda(I_i)}\int_{I_i} \sigma_i(f\circ g_i)\ dx \\
&\le 2\sum_{i=1}^n \left[\frac{1}{s_1}\bigvee_{I_i}(f\circ g_i) + \int_{I_i} s_2\sigma_i(f\circ g_i)\ dx \right] + \sum_{i=1}^n \frac{2}{\lambda(I_i)}\int_{I_i} \sigma_i(f\circ g_i)\ dx \\
&= \frac{2}{s_1}\sum_{i=1}^n \bigvee_{I_i}(f\circ g_i) + 2\sum_{i=1}^n \int_{I_i} s_2\sigma_i(f\circ g_i)\ dx + \sum_{i=1}^n \frac{2}{\lambda(I_i)}\int_{I_i} \sigma_i(f\circ g_i)\ dx \\
&= \frac{2}{s_1}\sum_{i=1}^n \bigvee_{I_i}(f\circ g_i) + 2\sum_{i=1}^n \left[s_2 + \frac{1}{\lambda(I_i)} \right] \int_{I_i} \sigma_i(f\circ g_i)\ dx \\
&\le \frac{2}{s_1}\sum_{i=1}^n \bigvee_{a_{i-1}}^{a_i} f + 2\sum_{i=1}^n \left[s_2 + \frac{1}{\lambda(I_i)} \right] \int_{I_i} \sigma_i(f\circ g_i)\ dx \\
&= \frac{2}{s_1}\bigvee_0^1 f + 2\sum_{i=1}^n \left[s_2 + \frac{1}{\lambda(I_i)} \right] \int_{a_{i-1}}^{a_i} f(y)dy \\
&= \alpha\bigvee_0^1 f + \beta\sum_{i=1}^n \int_{a_{i-1}}^{a_i} f(y)\ dy \ \ \le \ \ \alpha\bigvee_0^1 f + \beta\int_0^1 f(y)\ dy \\
&= \alpha\bigvee_0^1 f + \beta\|f\|_1 \ = \ \alpha\bigvee_0^1 f + \beta,
\end{aligned}
$$

where $\alpha = \dfrac{2}{s_1}$ and $\beta = \max_{i=1,\ldots,n} 2\left[s_2 + \frac{1}{\lambda(I_i)}\right]$. ■

From now on, we assume that the map S satisfies all the assumptions in the Lasota-Yorke inequality. Then, there exist two positive constants α and β such that for all density functions f of bounded variation we have,

$$\bigvee_0^1 Pf \;\le\; \alpha \bigvee_0^1 f + \beta. \tag{5.17}$$

THEOREM 5.13. Suppose that $\{f_n\} \in S^3[x_0, \ldots, x_n]$. Then total variation of $\{f_n\}$ is uniformly bounded by $\frac{4272\beta}{1-4272\alpha}$ for $n \ge 8$.

PROOF. we can find the bounded variation of f_n by using (4.12), Theorem 5.12, and the Lasota-Yorke inequality (5.17) as follows:

$$\begin{aligned}
\bigvee_0^1 f_n &= \bigvee_0^1 (Q_n P_S) f_n = \bigvee_0^1 Q_n(P_S f_n) \\
&\le 4272 \bigvee_0^1 P_S f_n \le 4272\left(\alpha \bigvee_0^1 f_n + \beta\right) \\
&= 4272\alpha \bigvee_0^1 f_n + 4272\beta,
\end{aligned}$$

which implies

$$(1 - 4272\alpha)\bigvee_0^1 f_n \;\le\; 4272\beta.$$

So, suppose $\alpha < \dfrac{1}{4272}$. Then, for $n \ge 8$

$$\bigvee_0^1 f_n \;\le\; \frac{4272\beta}{1 - 4272\alpha},$$

which is a uniform bound of the total variation of sequence of $\{f_n\}$. ■

5.5 Convergence of $\{f_n\}$ to f^*

In this section, we gather all the results that have been established to finally show that the sequence $\{f_n\}$ converges to f^*, the unique stationary density function of the Frobenius-Perron operator P_S associated with a measurable and nonsingular transformation S. However, to prove the convergence of $\{f_n\}$ to f^*, we need Helly's Lemma, whose statement and proof are found, for example, in the paper by Kreuzer [43].

Helly's Lemma. Let $\{f_n\} \in BV[0,1]$ be a sequence of functions. If

(1) $\|f_n\|_1 \leq M_1$ for any n

(2) $\bigvee_0^1 f_n \leq M_2$ for any n,

then there exists $g \in L^1[0,1]$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $\{f_{n_k}\} \to g$ as $k \to \infty$ in $L^1[0,1]$ and $\bigvee_0^1 g \leq M_2$.

The following theorem proves the convergence of $\{f_n\}$ to f^* in $L^1[0,1]$ space.

THEOREM 5.14. Let S fulfills the Lasota-Yorke conditions. If $\alpha < \frac{1}{4272}$, then the sequence $\{f_n\}$ converges to f^*, the unique stationary density of P_S.

PROOF. Since $\|f_n\|_1 = 1$ and $\bigvee_0^1 f_n \leq \frac{4272\beta}{1-4272\alpha}$ for $n \geq 8$, we can use Helly's lemma. So the sequence $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ that converges to some $g \in L^1[0,1]$ such that $\|g\|_1 = 1$. We want to show that $g = f^*$; namely, g is a stationary density function of P_S. Note that by (4.12), $f_{n_k} = Q_{n_k} P_S f_{n_k}$, and hence $\|f_{n_k} - Q_{n_k} P_S f_{n_k}\|_1 = 0$. By using Theorem 5.2 and Theorem 5.4, we have the following:

$$\begin{aligned}
\|g - P_S g\|_1 &= \|g - f_{n_k} + f_{n_k} - Q_{n_k} P_S f_{n_k} + Q_{n_k} P_S f_{n_k} - Q_{n_k} P_S g + Q_{n_k} P_S g - P_S g\|_1 \\
&\le \|g - f_{n_k}\|_1 + \|f_{n_k} - Q_{n_k} P_S f_{n_k}\|_1 + \|Q_{n_k} P_S f_{n_k} - Q_{n_k} P_S g\|_1 \\
&\quad + \|Q_{n_k} P_S g - P_S g\|_1 \\
&\le \|g - f_{n_k}\|_1 + \|Q_{n_k}\|_1 \|P_S\|_1 \|f_{n_k} - g\|_1 + \|Q_{n_k} P_S g - P_S g\|_1 \\
&\le \|g - f_{n_k}\|_1 + 1815\,\|f_{n_k} - g\|_1 + \|Q_{n_k} P_S g - P_S g\|_1 \\
&= 1816\,\|g - f_{n_k}\|_1 + \|Q_{n_k} P_S g - P_S g\|_1.
\end{aligned}$$

Therefore as $k \to \infty$, we have $P_S g = g$, then by Theorem 2.7, $P_S g^+ = g^+$ and $P_S g^- = g^-$, which means that g^+ and g^- are two stationary function of P_S unless $g \ge 0$ or $g \le 0$. Since we assumed that P_S has the unique stationary density f^*, it follows that $g \ge 0$ or $g \le 0$. If we choose $\{f_{n_k}\}$ such that

$$\int f_{n_k}^+ \, d\lambda \ \ge \ \int f_{n_k}^- \, d\lambda,$$

since $\{f_{n_k}\} \to g$ as $k \to \infty$ in $L^1[0,1]$, we have $g \ge 0$. It follows that $f^* = g$ and hence $\{f_{n_k}\} \to f^*$ in $L^1[0,1]$.

The next goal is to show that $f_n \to f^*$. Suppose that $\{f_n\}$ does not converge to f^*. Then there exists $\epsilon > 0$ and a subsequence $\{f_{l_k}\}$ of $\{f_n\}$ [5] such that

$$\|f_{l_k} - f^*\|_1 \ \ge \ \epsilon \tag{5.18}$$

for all k. Since $\{f_{l_k}\}$ also fulfils the conditions of Helly's Lemma, $\{f_{l_k}\}$ has a subsequence, for instance $\{f_{l_{k_j}}\}$, such that $f_{l_{k_j}} \to h$ in $L^1[0,1]$ for some $h \in L^1[0,1]$. Therefore, by the same argument as above, we can show that $h = f^*$, so $f_{l_{k_j}} \to f^*$.

However, (5.18) tells us that $f_{l_{k_j}} \not\to f^*$, which is a contradiction. So we conclude that $f_n \to f^*$ in $L^1[0,1]$.

■

CHAPTER 6

NUMERICAL RESULTS

This chapter gives numerical results of the cubic spline projection method we use in this dissertation. We compare the results with the linear spline and quadratic spline projection methods from the papers [25, 78], respectively. In Section 6.1 we cover Gaussian quadrature with three points that we use for numerical integrations. For more accurate results, we use composite Gaussian quadrature with three points iteratively. In Section 6.2 we discuss how to compute the entries of matrix A, where $a_{ij} = \langle P_S\phi_j, \phi_i \rangle$ defined in (4.13). In section 6.3 we show how to compute the errors $e_n = \|f_n - f^*\|_1$. Finally, in Section 6.4 we present numerical experimental results.

6.1 Gaussian Quadrature

For the computation of the entries of matrix A defined in (4.13) and the errors $e_n = \|f_n - f^*\|_1$, we need to compute the integrals of certain functions over certain intervals, which is the core computation problem. We do this numerically using Gaussian quadrature with three points, which is summarized in the following. If $f \in C^6[a, b]$, then

$$\begin{aligned}\int_a^b f(x)\,dx &= \frac{5}{18}(b-a)f\left(\frac{a+b}{2} - \frac{b-a}{2}\sqrt{\frac{3}{5}}\right) + \frac{8}{18}(b-a)f\left(\frac{a+b}{2}\right)\\ &+ \frac{5}{18}(b-a)f\left(\frac{a+b}{2} + \frac{b-a}{2}\sqrt{\frac{3}{5}}\right) + \frac{(b-a)^7}{2016000}f^{(6)}(\xi),\end{aligned}$$

where $a < \xi < b$. More details can be found in [13, 39]. Note that the Gaussian quadrature with three points will be integrated exactly if the integrand is a polynomial of degree at most 5. We denote

$$\begin{aligned} G(f,a,b) &= \frac{5}{18}(b-a)f\left(\frac{a+b}{2}-\frac{b-a}{2}\sqrt{\frac{3}{5}}\right)+\frac{8}{18}(b-a)f\left(\frac{a+b}{2}\right) \\ &+ \frac{5}{18}(b-a)f\left(\frac{a+b}{2}+\frac{b-a}{2}\sqrt{\frac{3}{5}}\right), \end{aligned}$$

so that $G(f,a,b)$ is a basic Gaussian quadrature with three points.

For more accurate results, we consider a composite Gaussian quadrature with three points using 2^m subintervals of $[a,b]$, which is denoted by

$$G(f,a,b,2^m)=\sum_{i=1}^{2^m} G\left(f, a+(i-1)\frac{b-a}{2^m}, a+i\frac{b-a}{2^m}\right).$$

We use composite Gaussian quadratures iteratively with three points as follows. We compute from $m=0$, which is the basic Gaussian quadrature with three points on $[a,b]$. Then we compute for $m=1$, which is a composite Gaussian quadrature with three points in two subintervals, $[a, \frac{a+b}{2}]$ and $[\frac{a+b}{2}, b]$. We keep this process until the difference between the current iteration value, $G(f,a,b,2^m)$, and the previous iteration value, $G(f,a,b,2^{m-1})$, in absolute value, is less than the given tolerance; that is

$$|G(f,a,b,2^m)-G(f,a,b,2^{m-1})| < \text{tolerance}.$$

Then we accept the current iteration value, $G(f,a,b,2^m)$, as accurate enough approximation to $\int_a^b f(x)\,dx$.

6.2 Computing the Entries of Matrix A

We need to compute the entries of matrix A, which are defined by $a_{ij} = \langle P_S\phi_j, \phi_i\rangle$ for $-3 \le i, j, \le n-1$ (see (4.13)). But there are some computational issues in computing a_{ij}, which will be illustrated in the following example. Let

$$S(x) = \begin{cases} \frac{2x}{1-x} & 0 \le x \le \frac{1}{3} \\ \frac{1-x}{2x} & \frac{1}{3} \le x \le 1. \end{cases}$$

We will compute $a_{3,-2}$ when $n = 4$. Note that, by Lemma 2.24, we have

$$\begin{aligned} a_{3,-2} &= \langle P_S\phi_{-2}, \phi_3\rangle = \langle \phi_{-2}, K_S\phi_3\rangle = \langle \phi_{-2}, \phi_3 \circ S\rangle \\ &= \int_0^1 \phi_3(S(x))\phi_{-2}(x)\,dx = \int_{\mathrm{supp}(\phi_{-2})} \phi_3(S(x))\phi_{-2}(x)\,dx \\ &= \int_0^{\frac{1}{2}} \phi_3(S(x))\phi_{-2}(x)\,dx \end{aligned}$$

because support of ϕ_{-2} is $(0, \frac{1}{2})$.

We divide the interval $(0, \frac{1}{2})$ into the subintervals $(0, \frac{1}{4}]$ and $[\frac{1}{4}, \frac{1}{2})$ since on each subinterval, $(0, \frac{1}{4}]$ and $[\frac{1}{4}, \frac{1}{2})$, ϕ_{-2} is a polynomial of degree at most three due to the assumption that $n = 4$. However, there is still an issue of accuracy since the subinterval $[\frac{1}{4}, \frac{1}{2})$ contains the point $\frac{1}{3}$, which is a non-smooth point of the mapping S. Therefore, we carry out the numerical integration by dividing the support of ϕ_{-2} into $(0, \frac{1}{4}]$, $[\frac{1}{4}, \frac{1}{3}]$, and $[\frac{1}{3}, \frac{1}{2})$.

We further improve the efficiency of the integral by looking at $\phi_3(S(x))$ closely. First of all, we write

$$S_1(x) = \frac{2x}{1-x},\ 0 \le x \le \frac{1}{3} \quad \text{and} \quad S_2(x) = \frac{1-x}{2x},\ \frac{1}{3} \le x \le 1.$$

Note first,

$$S\left(\left[0,\frac{1}{4}\right]\right)\cap \operatorname{supp}(\phi_3)=S_1\left(\left[0,\frac{1}{4}\right]\right)\cap \operatorname{supp}(\phi_3)=\left[0,\frac{2}{3}\right]\cap\left[\frac{3}{4},1\right]=\emptyset.$$

Second,

$$S\left(\left[\frac{1}{4},\frac{1}{3}\right]\right)\cap \operatorname{supp}(\phi_3)=S_1\left(\left[\frac{1}{4},\frac{1}{3}\right]\right)\cap \operatorname{supp}(\phi_3)=\left[\frac{2}{3},1\right]\cap\left[\frac{3}{4},1\right]=\left[\frac{3}{4},1\right].$$

Third,

$$S\left(\left[\frac{1}{3},\frac{1}{2}\right]\right)\cap \operatorname{supp}(\phi_3)=S_2\left(\left[\frac{1}{3},\frac{1}{2}\right]\right)\cap \operatorname{supp}(\phi_3)=\left[\frac{1}{2},1\right]\cap\left[\frac{3}{4},1\right]=\left[\frac{3}{4},1\right].$$

After that we refine the subintervals $[\frac{1}{4},\frac{1}{3}]$ and $[\frac{1}{3},\frac{1}{2}]$ as follows:

$$\left[S_1^{-1}\left(\frac{3}{4}\right),S_1^{-1}(1)\right]=\left[\frac{3}{11},\frac{1}{3}\right]$$

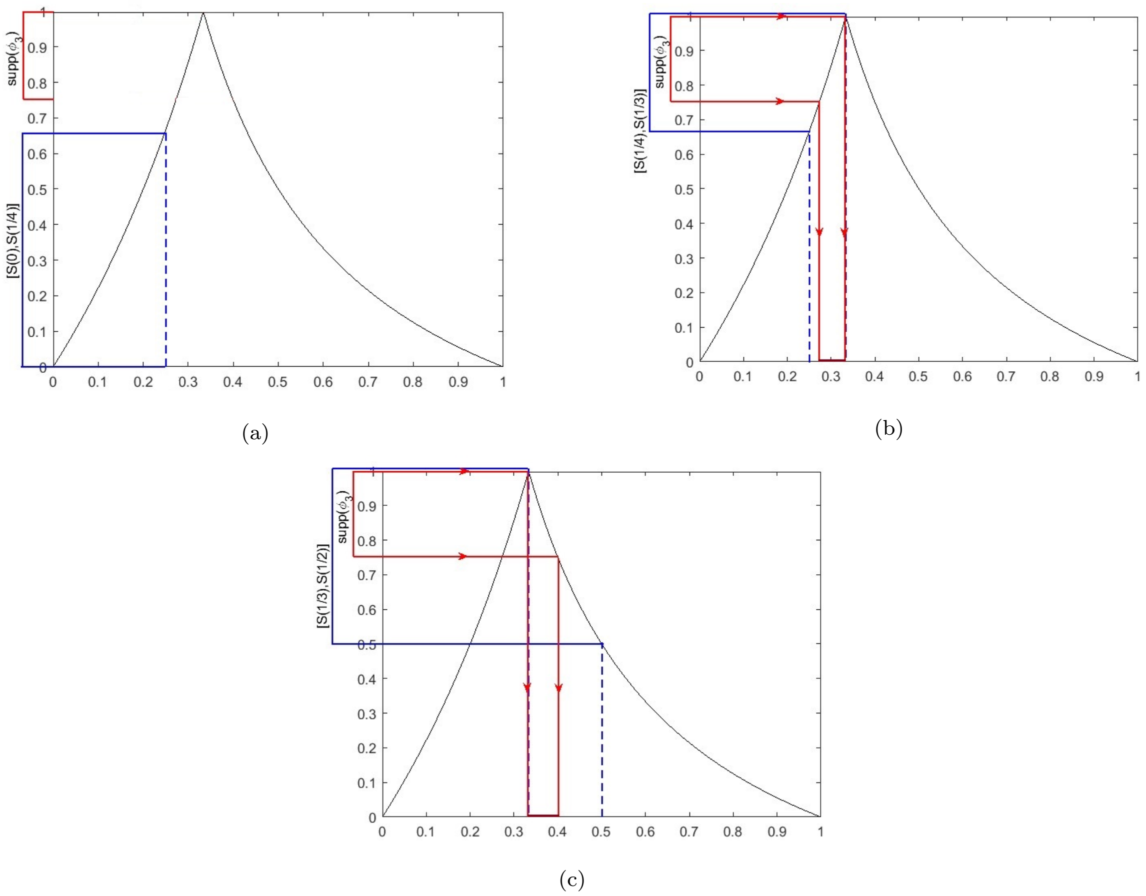
and

$$\left[S_2^{-1}(1),S_2^{-1}\left(\frac{3}{4}\right)\right]=\left[\frac{1}{3},\frac{2}{5}\right],$$

See Figure 10 for more clarification, where (a), (b), (c) show the overlapping area of support of ϕ_3 with the intervals $\left[0,\frac{1}{4}\right]$, $\left[\frac{1}{4},\frac{1}{3}\right]$, and $\left[\frac{1}{3},\frac{1}{2}\right]$. So $a_{3,-2}$ can be computed in the following way:

$$\begin{aligned}a_{3,-2} &= \int_0^{\frac{1}{4}}\phi_3(S(x))\phi_{-2}(x)\,dx+\int_{\frac{1}{4}}^{\frac{1}{3}}\phi_3(S(x))\phi_{-2}(x)\,dx+\int_{\frac{1}{3}}^{\frac{1}{2}}\phi_3(S(x))\phi_{-2}(x)\,dx\\ &= \int_{\frac{3}{11}}^{\frac{1}{3}}\phi_3(S(x))\phi_{-2}(x)\,dx+\int_{\frac{1}{3}}^{\frac{2}{5}}\phi_3(S(x))\phi_{-2}(x)\,dx.\end{aligned}$$

This gives not only more efficient but also more accurate numerical integration because we avoid $\frac{1}{3}$, which is the non-smooth point of S. We compute a_{ij} for every n using this idea.



(a) (b) (c)

Figure 10: Efficient and accurate computations of $a_{3,-2}$

6.3 Computing the errors $e_n = \|f_n - f^*\|_1$

We compute the errors e_n under L^1-norm using the following formula:

$$\begin{aligned} e_n &= ||f_n - f^*||_1 = \int_0^1 |f_n(x) - f^*(x)|\ dx \\ &= \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |f_n(x) - f^*(x)|\ dx. \end{aligned}$$

However, if the equation $f_n(x) = f^*(x)$ has solutions in the interval (x_{i-1}, x_i), a numerical integral $\int_{x_{i-1}}^{x_i} |f_n(x) - f^*(x)|$ may not be accurate. So, if there are $y_1, \ldots, y_k$, which are solutions of the equation $f_n(x) = f^*(x)$ on the interval (x_{i-1}, x_i), then we divide the integral $\int_{x_{i-1}}^{x_i} |f_n(x) - f^*(x)|$ as follows:

$$\begin{aligned} \int_{x_{i-1}}^{x_i} |f_n(x) - f^*(x)|\ dx &= \int_{x_{i-1}}^{y_1} |f_n(x) - f^*(x)|\ dx + \sum_{l=1}^{k-1} \int_{y_l}^{y_{l+1}} |f_n(x) - f^*(x)|\ dx \\ &+ \int_{y_k}^{x_i} |f_n(x) - f^*(x)|\ dx, \end{aligned}$$

where f_n denotes the approximation to f_1^* from $S^3[x_0, \ldots, x_n]$. We apply numerical integration for each integral; nevertheless, the question is how to compute $y_1, \ldots, y_k$. We illustrate this question with two cases: Case 1. $f_1^*(x) = \dfrac{4}{\pi(1+x^2)}$; Case 2. $f_2^*(x) = \dfrac{2}{(1+x)^2}$.

We start with $f_1^*(x) = \dfrac{4}{\pi(1+x^2)}$, and we discuss how to find solutions of $f_n(x) - f^*(x)$ on (x_{i-1}, x_i), $1 \le i \le n$. Note that, on (x_{i-1}, x_i), $1 \le i \le n$, we can write

$$0 = f_n(x) - f_1^*(x) = a_3x^3 + a_2x^2 + a_1x + a_0 - \frac{4}{\pi(1+x^2)}.$$

So we have

$$
\begin{aligned}
a_3x^3 + a_2x^2 + a_1x + a_0 - \frac{4}{\pi(1+x^2)} &= 0 \\
\Longleftrightarrow \quad a_3x^3(1+x^2) + a_2x^2(1+x^2) + a_1x(1+x^2) + a_0(1+x^2) - \frac{4}{\pi} &= 0 \\
\Longleftrightarrow \quad a_3x^5 + a_2x^4 + (a_3+a_1)x^3 + (a_2+a_0)x^2 + a_1x + \left(a_0 - \frac{4}{\pi}\right) &= 0.
\end{aligned}
$$

By assuming $a_3 \neq 0$, which is the generic case, we have

$$
x^5 + \frac{a_2}{a_3}x^4 + \frac{a_3+a_1}{a_3}x^3 + \frac{a_2+a_0}{a_3}x^2 + \frac{a_1}{a_3}x + \frac{a_0 - \frac{4}{\pi}}{a_3} = 0. \tag{6.1}
$$

Therefore, the companion matrix C_1 for (6.1) is

$$
C_1 = \begin{bmatrix}
0 & 0 & 0 & 0 & -\frac{a_0-\frac{4}{\pi}}{a_3} \\
1 & 0 & 0 & 0 & -\frac{a_1}{a_3} \\
0 & 1 & 0 & 0 & -\frac{a_2+a_0}{a_3} \\
0 & 0 & 1 & 0 & -\frac{a_3+a_1}{a_3} \\
0 & 0 & 0 & 1 & -\frac{a_2}{a_3}
\end{bmatrix}.
$$

The definition and more details about companion matrix can be found in [14].

Note that it is well-known that the eigenvalues of C_1 are the zeros of (6.1), the associated monic polynomial, [14]. Suppose that $y_1, \ldots, y_k$ are the eigenvalues of C_1 that are contained in (x_{i-1}, x_i) for some k. Note that the range of k is between 0 and 5.

Next, for $f_2^*(x) = \dfrac{2}{(1+x)^2}$, we have

$$
f_n(x) - f_2^*(x) = b_3x^3 + b_2x^2 + b_1x + b_0 - \frac{2}{(1+x)^2} = 0.
$$

Therefore, we have

$$
\begin{aligned}
b_3x^3 + b_2x^2 + b_1x + b_0 - \frac{2}{(1+x)^2} &= 0 \\
\iff \; b_3x^3(x^2+2x+1) + b_2x^2(x^2+2x+1) + b_1x(x^2+2x+1) & \\
+ \; b_0(x^2+2x+1) - 2 &= 0.
\end{aligned}
$$

By assuming $b_3 \neq 0$,

$$
x^5 + \frac{2b_3+b_2}{b_3}x^4 + \frac{b_3+2b_2+b_1}{b_3}x^3 + \frac{b_2+2b_1+b_0}{b_3}x^2 + \frac{b_1+2b_0}{b_3}x + \frac{b_0-2}{b_3} = 0. \tag{6.2}
$$

Hence, the companion matrix C_2 for (6.2) is

$$
C_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & -\frac{b_0-2}{b_3} \\ 1 & 0 & 0 & 0 & -\frac{b_1+2b_0}{b_3} \\ 0 & 1 & 0 & 0 & -\frac{b_2+2b_1+b_0}{b_3} \\ 0 & 0 & 1 & 0 & -\frac{b_3+2b_2+b_1}{b_3} \\ 0 & 0 & 0 & 1 & -\frac{2b_3+b_2}{b_3} \end{bmatrix}.
$$

To find the companion matrices, we need to fine the expressions of a_0, a_1, a_2 and a_3 in the case of S_1 and b_0, b_1, b_2 and b_3 in the case of S_2. Note that, in the interval (x_{i-1}, x_i), $i = 1, 2, \ldots, n$, f_n can be written as

$$
\begin{aligned}
f_n(x) &= d_{i-4}\phi_{i-4}(x) + d_{i-3}\phi_{i-3}(x) + d_{i-2}\phi_{i-2}(x) + d_{i-1}\phi_{i-1}(x) \\
&= d_{i-4}\frac{1}{6}\left(4 - \frac{x-(i-4)h}{h}\right)^3 + d_{i-3}\frac{1}{6}\Bigg(1 + 3\left(3 - \frac{x-(i-3)h}{h}\right) \\
&\quad + 3\left(3 - \frac{x-(i-3)h}{h}\right)^2 - 3\left(3 - \frac{x-(i-3)h}{h}\right)^3\Bigg)
\end{aligned}
$$

$$+ d_{i-2}\frac{1}{6}\left(1 + 3\left(\frac{x-(i-2)h}{h} - 1\right) + 3\left(\frac{x-(i-2)h}{h} - 1\right)^2\right.$$

$$\left. - 3\left(\frac{x-(i-2)h}{h} - 1\right)^3\right) + d_{i-1}\frac{1}{6}\left(\frac{x-(i-1)h}{h}\right)^3$$

$$= \frac{1}{6}d_{i-4}\left(\frac{-x+ih}{h}\right)^3 + \frac{1}{6}d_{i-3}\left(1 - 3\left(\frac{x-ih}{h}\right) + 3\left(\frac{x-ih}{h}\right)^2 + 3\left(\frac{x-ih}{h}\right)^3\right)$$

$$+ \frac{1}{6}d_{i-2}\left(1 + 3\left(\frac{x-(i-1)h}{h}\right) + 3\left(\frac{x-(i-1)h}{h}\right)^2 - 3\left(\frac{x-(i-1)h}{h}\right)^3\right)$$

$$+ d_{i-1}\frac{1}{6}\left(\frac{x-(i-1)h}{h}\right)^3$$

$$= \frac{1}{6h^3}(-d_{i-4} + 3d_{i-3} - 3d_{i-2} + d_{i-1})x^3$$

$$+ \frac{1}{2h^2}(d_{i-4}i + d_{i-3}(1-3i) + d_{i-2}(3i-2) - d_{i-1}(i-1))x^2$$

$$+ \frac{1}{2h}(-d_{i-4}i^2 + d_{i-3}(i-1)(3i+1) + d_{i-2}(4-3i)i + d_{i-1}(i-1)^2)x$$

$$+ \frac{1}{6}(d_{i-4}i^3 + d_{i-3}(1+3i+3i^2-3i^3) + d_{i-2}(3i^3-6i^2+4) - d_{i-1}(i-1)^3).$$

So

$$\begin{aligned}
a_3 &= \frac{1}{6h^3}(-d_{i-4} + 3d_{i-3} - 3d_{i-2} + d_{i-1}),\\
a_2 &= \frac{1}{2h^2}(d_{i-4}i + d_{i-3}(1-3i) + d_{i-2}(3i-2) - d_{i-1}(i-1)),\\
a_1 &= \frac{1}{2h}(-d_{i-4}i^2 + d_{i-3}(i-1)(3i+1) + d_{i-2}(4-3i)i + d_{i-1}(i-1)^2),\\
a_0 &= \frac{1}{6}(d_{i-4}i^3 + d_{i-3}(1+3i+3i^2-3i^3) + d_{i-2}(3i^3-6i^2+4) - d_{i-1}(i-1)^3).
\end{aligned}$$

Note that, b_0, b_1, b_2 and b_3 allows the same expression as a_0, a_1, a_2 and a_3, but the vector $d = (d_{-3}, d_{-2}, \ldots, d_{n-1})^T$ for f_n has been changed accordingly.

6.4 Numerical Results

In this section, we provide numerical experimental results for the cubic spline projection method, and we compare the results with those of the linear spline projection method [25] and the quadratic spline projection method [78]. The test transformation mappings are given by:

$$S_1(x) = \begin{cases} \frac{2x}{1-x^2} & 0 \le x \le \sqrt{2}-1 \\ \frac{1-x^2}{2x} & \sqrt{2}-1 \le x \le 1, \end{cases}$$

$$S_2(x) = \begin{cases} \frac{2x}{1-x} & 0 \le x \le \frac{1}{3} \\ \frac{1-x}{2x} & \frac{1}{3} \le x \le 1, \end{cases}$$

$$S_3(x) = \left(\frac{1}{8} - 2\left|x - \frac{1}{2}\right|^3\right)^{1/3} + \frac{1}{2}.$$

It is well-known that the unique stationary densities of the above transformations are given by

$$f_1^*(x) = \frac{4}{\pi(1+x^2)},$$

$$f_2^*(x) = \frac{2}{(1+x)^2},$$

$$f_3^*(x) = 12\left(x - \frac{1}{2}\right)^2,$$

respectively. For the numerical experiments, we use $n = 2^k$, where $k = 2, 3, \ldots, 8$.

The uniqueness of the above stationary densities of Frobenius-Perron associated with transformations S can be shown using Theorem 2.35 as follows. Note that transformations S_1, S_2, and S_3 can be easily proven to be ergodic transformations by showing that $\mu(A) = 0$ or $\mu(A) = 1$ for any invariant $A \in \mathcal{B}$. Recall from Examples 2.28, 2.29, and 2.30 that f_1^*, f_2^*, and f_3^* were proven to be stationary density functions of Frobenius-Perron operator. This implies by Theorem 2.35 that f_1^*, f_2^*, and f_3^* are unique stationary density functions of Frobenius-Perron operator associated with S_1, S_2, and S_3, respectively.

We use MATLAB to test the cubic spline projection method of approximating f^*, and we observe the following. In the case of f_1^* and f_2^*, the numerical results in Table 6 and Table 8, respectively, show that the cubic spline projection method performed well and gave much smaller errors than the linear spline and quadratic spline projection methods for all n-values. In addition, Table 7 and 9 show the ratios of $e_4/e_8, \ldots, e_{128}/e_{256}$ for f_1^* and f_2^*, respectively. We can see from Table 7 and Table 9 that the errors of the cubic spline projection method is of order $O\left(1/n^4\right)$.

On the other hand, in the case of f_3^*, since f_3^* is a polynomial of degree 2, it is in the quadratic and cubic spline space for any n. For that reason, we have $f_n = f_3^*$ for any n when we use the quadratic spline and cubic spline projection methods. However, f_n is not f_3^* when we use constant spline and linear spline projection methods. Table 10 shows errors of f_3^* for the constant spline and linear spline projection methods, and Table 11 shows the ratios of two consecutive errors of constant spline and linear spline projection methods for f_3^*.

Table 6: Error e_n comparison for f_1^*

n	Constant-SPM	Linear-SPM	Quadratic-SPM	Cubic-SPM
4	5.3×10^{-2}	2.7×10^{-3}	5.1×10^{-4}	4.6×10^{-5}
8	2.4×10^{-2}	6.5×10^{-4}	4.9×10^{-5}	3.1×10^{-6}
16	1.2×10^{-2}	1.7×10^{-4}	6.2×10^{-6}	1.9×10^{-7}
32	5.5×10^{-3}	4.3×10^{-5}	7.3×10^{-7}	1.1×10^{-8}
64	2.7×10^{-3}	1.1×10^{-5}	8.4×10^{-8}	7.1×10^{-10}
128	1.3×10^{-3}	2.7×10^{-6}	1.0×10^{-8}	4.5×10^{-11}
256	6.6×10^{-4}	6.4×10^{-7}	1.3×10^{-9}	2.8×10^{-12}

Table 7: Ratio comparison for f_1^*

	Constant-SPM	Linear-SPM	Quadratic-SPM	Cubic-SPM
e_4/e_8	1.26	4.15	10.41	14.83
e_8/e_{16}	3.5	3.82	7.9	16.32
e_{16}/e_{32}	2.18	3.95	8.49	17.27
e_{32}/e_{64}	2.04	3.9	8.69	15.49
e_{64}/e_{128}	2.08	4.07	8.4	15.78
e_{128}/e_{256}	1.97	4.22	7.96	16.07

Table 8: Error e_n comparison for f_2^*

n	Constant-SPM	Linear-SPM	Quadratic-SPM	Cubic-SPM
4	1.0×10^{-1}	7.7×10^{-3}	9.6×10^{-4}	9.7×10^{-5}
8	5.1×10^{-2}	1.9×10^{-3}	1.3×10^{-4}	8.9×10^{-6}
16	2.6×10^{-2}	5.4×10^{-4}	1.9×10^{-5}	7.2×10^{-7}
32	1.3×10^{-2}	1.4×10^{-4}	2.2×10^{-6}	5.1×10^{-8}
64	6.6×10^{-3}	3.6×10^{-5}	3.0×10^{-7}	3.1×10^{-9}
128	3.3×10^{-3}	8.5×10^{-6}	3.8×10^{-8}	2.0×10^{-10}
256	1.6×10^{-3}	2.2×10^{-6}	4.6×10^{-9}	1.3×10^{-11}

Table 9: Ratio comparison for f_2^*

n	Constant-SPM	Linear-SPM	Quadratic-SPM	Cubic-SPM
e_4/e_8	1.96	4.05	7.38	10.90
e_8/e_{16}	1.96	3.52	6.84	12.36
e_{16}/e_{32}	2	3.86	8.63	14.12
e_{32}/e_{64}	1.97	3.89	7.33	16.45
e_{64}/e_{128}	2	4.24	7.89	15.50
e_{128}/e_{256}	2.06	3.86	8.26	16.67

Table 10: Error e_n comparison for f_3^*

n	Constant-SPM	Linear-SPM
4	4.4×10^{-1}	5.3×10^{-2}
8	2.2×10^{-1}	1.4×10^{-2}
16	1.0×10^{-1}	3.2×10^{-3}
32	5.3×10^{-2}	8.9×10^{-4}
64	2.6×10^{-2}	2.1×10^{-4}
128	1.3×10^{-2}	5.2×10^{-5}
256	6.5×10^{-3}	1.3×10^{-5}

Table 11: Ratio comparison for f_3^*

n	Constant-SPM	Linear-SPM
e_4/e_8	2.00	3.79
e_8/e_{16}	2.20	4.38
e_{16}/e_{32}	1.89	3.60
e_{32}/e_{64}	2.04	4.24
e_{64}/e_{128}	2.00	4.04
e_{128}/e_{256}	2.00	4.00

CHAPTER 7

CONCLUSION AND FUTURE WORK

In this dissertation, we used the idea of the projection method to develop a numerical method using cubic spline functions to approximate the unique stationary density functions f^* of the Frobenius-Perron operators P_S associated with a measurable and nonsingular transformation S (when the stationary density function is smooth). We aimed to improve the convergence rate of the linear spline and quadratic spline projection methods from the papers [24, 78], respectively. We succeed in achieving our goal to have a faster convergence rate due to the fact that cubic spline functions are more smooth than linear spline and quadratic spline functions. We proved the existence of a nonzero sequence of cubic spline functions f_n that converges to f^* in L^1-norm.

We also devised ways to compute the entries of matrix A and the errors between f_n and f^* more accurately. That was achieved by dividing the domains of integrals into proper subintervals when we compute $a_{ij} = \langle P_S\phi_j, \phi_i\rangle$, entries of matrix A, and the errors $e_n = \|f_n - f^*\|_1$. This resulted in not only more accurate but also more efficient results.

The numerical experiments showed that the cubic spline projection method has a faster convergence rate compared to the linear spline and quadratic spline projection methods. In fact, the numerical experiments indicated that the cubic spline projection method has a convergence rate of order four compared to order two of the linear spline projection method and order three of the quadratic spline projection method under the L^1-norm. If the stationary density function is a polynomial of degree at most three, then it is in the cubic spline space for any n. So, the method can compute it precisely no matter what n may be.

There are three possible areas for future works related to the cubic spline projection method. First, the proposed projection method can be used for the density computation of more general Markov operators that appear in many applied problems such as stochastic analysis and random maps [38]. The linear spline projection method for random maps is already done in [3]. Meanwhile, we are working on the quadratic spline projection method for random maps and planning to apply the cubic spline projection method to the random maps. Second, one can consider approximating the stationary density functions using even higher-order splines such as quartic or quintic spline functions. Third, one can justify the convergence rate that was observed from the numerical experimental results theoretically.

APPENDIX

A MATLAB CODE FOR f_2^*

```
function CSP=cubic_sp(n,tol)
A=int_csp_A(n,tol);
B=int_csp_B(n);
d=vec_d_csp(n,A,B);
CSP=csp_err(d,tol);
end
%%%% matrix B=< \phi_j, \phi_i > %%%%
function B=int_csp_B(n)
h=1/n;
B=zeros(n+3);
B(1,1)=20*h/5040; B(n+3,n+3)=20*h/5040;
B(2,2)=1208*h/5040; B(n+2,n+2)=1208*h/5040;
B(3,3)=2396*h/5040; B(n+1,n+1)=2396*h/5040;
B(1,2)=(129/5040)*h; B(2,1)=(129/5040)*h;
B(n+2,n+3)=(129/5040)*h; B(n+3,n+2)=(129/5040)*h;
B(1,3)=(60/5040)*h; B(3,1)=(60/5040)*h;
B(n+1,n+3)=(60/5040)*h; B(n+3,n+1)=(60/5040)*h;
B(2,3)=(1062/5040)*h; B(3,2)=(1062/5040)*h;
B(n+1,n+2)=(1062/5040)*h; B(n+2,n+1)=(1062/5040)*h;
for t=2:n
    B(t,t+2)=120*h/5040;
    B(t+2,t)=120*h/5040;
end
for t=3:n
    B(t+1,t)=1191*h/5040;
    B(t,t+1)=1191*h/5040;
end
for t=4:n
    B(t,t)=(2416/5040)*h;
end
for t=1:n
    B(t,t+3)=h/5040;
    B(t+3,t)=h/5040;
end
end
%%%% matrix A=< P_S\phi_j, \phi_i > %%%%%
function A=int_csp_A(n,tol)
A=zeros(n+3);
h= 1/n; p=1/3;
for i=-3:n-1
    for j=-3:n-1
        k=ceil(p/h);
        if j<k-4
            aj=(j)*h; bj=(j+1)*h;
            A(i+4,j+4)=A(i+4,j+4)+int_j_a(aj,bj,i,j,n,tol);
            aj=(j+1)*h; bj=(j+2)*h;
            A(i+4,j+4)=A(i+4,j+4)+int_j_a(aj,bj,i,j,n,tol);
            aj=(j+2)*h; bj=(j+3)*h;
            A(i+4,j+4)=A(i+4,j+4)+int_j_a(aj,bj,i,j,n,tol);
            aj=(j+3)*h; bj=(j+4)*h;
            A(i+4,j+4)=A(i+4,j+4)+int_j_a(aj,bj,i,j,n,tol);
        elseif j==k-4
            aj=(j)*h; bj=(j+1)*h;
            A(i+4,j+4)=A(i+4,j+4)+int_j_a(aj,bj,i,j,n,tol);
            aj=(j+1)*h; bj=(j+2)*h;
            A(i+4,j+4)=A(i+4,j+4)+int_j_a(aj,bj,i,j,n,tol);
            aj=(j+2)*h; bj=(j+3)*h;
```

```
            A(i+4,j+4)=A(i+4,j+4)+int_j_a(aj,bj,i,j,n,tol);
            aj=(j+3)*h; pj=p;
            A(i+4,j+4)=A(i+4,j+4)+int_j_a(aj,pj,i,j,n,tol);
            pj=p; bj=(j+4)*h;
            A(i+4,j+4)=A(i+4,j+4)+int_j_b(pj,bj,i,j,n,tol);
        elseif j==k-3
            aj=(j)*h; bj=(j+1)*h;
            A(i+4,j+4)=A(i+4,j+4)+int_j_a(aj,bj,i,j,n,tol);
            aj=(j+1)*h; bj=(j+2)*h;
            A(i+4,j+4)=A(i+4,j+4)+int_j_a(aj,bj,i,j,n,tol);
            aj=(j+2)*h; pj=p;
            A(i+4,j+4)=A(i+4,j+4)+int_j_a(aj,pj,i,j,n,tol);
            pj=p; bj=(j+3)*h;
            A(i+4,j+4)=A(i+4,j+4)+int_j_b(pj,bj,i,j,n,tol);
            aj=(j+3)*h; bj=(j+4)*h;
            A(i+4,j+4)=A(i+4,j+4)+int_j_b(aj,bj,i,j,n,tol);
        elseif j==k-2
            aj=(j)*h; bj=(j+1)*h;
            A(i+4,j+4)=A(i+4,j+4)+int_j_a(aj,bj,i,j,n,tol);
            aj=(j+1)*h; pj=p;
            A(i+4,j+4)=A(i+4,j+4)+int_j_a(aj,pj,i,j,n,tol);
            pj=p; bj=(j+2)*h;
            A(i+4,j+4)=A(i+4,j+4)+int_j_b(pj,bj,i,j,n,tol);
            aj=(j+2)*h; bj=(j+3)*h;
            A(i+4,j+4)=A(i+4,j+4)+int_j_b(aj,bj,i,j,n,tol);
            aj=(j+3)*h; bj=(j+4)*h;
            A(i+4,j+4)=A(i+4,j+4)+int_j_b(aj,bj,i,j,n,tol);
        elseif j==k-1
            aj=(j)*h; pj=p;
            A(i+4,j+4)=A(i+4,j+4)+int_j_a(aj,pj,i,j,n,tol);
            pj=p; bj=(j+1)*h;
            A(i+4,j+4)=A(i+4,j+4)+int_j_b(pj,bj,i,j,n,tol);
            aj=(j+1)*h; bj=(j+2)*h;
            A(i+4,j+4)=A(i+4,j+4)+int_j_b(aj,bj,i,j,n,tol);
            aj=(j+2)*h; bj=(j+3)*h;
            A(i+4,j+4)=A(i+4,j+4)+int_j_b(aj,bj,i,j,n,tol);
            aj=(j+3)*h; bj=(j+4)*h;
            A(i+4,j+4)=A(i+4,j+4)+int_j_b(aj,bj,i,j,n,tol);
        else
            aj=(j)*h; bj=(j+1)*h;
            A(i+4,j+4)=A(i+4,j+4)+int_j_b(aj,bj,i,j,n,tol);
            aj=(j+1)*h; bj=(j+2)*h;
            A(i+4,j+4)=A(i+4,j+4)+int_j_b(aj,bj,i,j,n,tol);
            aj=(j+2)*h; bj=(j+3)*h;
            A(i+4,j+4)=A(i+4,j+4)+int_j_b(aj,bj,i,j,n,tol);
            aj=(j+3)*h; bj=(j+4)*h;
            A(i+4,j+4)=A(i+4,j+4)+int_j_b(aj,bj,i,j,n,tol);
        end
    end
end
end
%%%% increasing part of S %%%%
function iv=int_j_a(aj,bj,i,j,n,tol)
if aj<0
    iv=0;
    return
```

```
end
r1=2*aj/(1-aj); r2=2*bj/(1-bj);
h=1/n;
if i==-3
    ai=0; bi=h;
    aa=max(r1,ai); bb=min(r2,bi);
    if aa < bb
        inv_s1=aa/(aa+2); inv_s2=bb/(bb+2);
        iv=ite_comp_gq3(inv_s1,inv_s2,i,j,n,tol);
    else
        iv=0;
    end
elseif  i==-2
    ai=0; bi=h;
    aa=max(r1,ai); bb=min(r2,bi);
    if aa < bb
        inv_s1=aa/(aa+2); inv_s2=bb/(bb+2);
        iv1=ite_comp_gq3(inv_s1,inv_s2,i,j,n,tol);
    else
        iv1=0;
    end
    ai=h; bi=2*h;
    aa=max(r1,ai); bb=min(r2,bi);
    if aa < bb
        inv_s1=aa/(aa+2); inv_s2=bb/(bb+2);
        iv2=ite_comp_gq3(inv_s1,inv_s2,i,j,n,tol);
    else
        iv2=0;
    end
    iv=iv1+iv2;
elseif  i==-1
    ai=0; bi=h;
    aa=max(r1,ai); bb=min(r2,bi);
    if aa < bb
        inv_s1=aa/(aa+2); inv_s2=bb/(bb+2);
        iv1=ite_comp_gq3(inv_s1,inv_s2,i,j,n,tol);
    else
        iv1=0;
    end
    ai=h; bi=2*h;
    aa=max(r1,ai); bb=min(r2,bi);
    if aa < bb
        inv_s1=aa/(aa+2); inv_s2=bb/(bb+2);
        iv2=ite_comp_gq3(inv_s1,inv_s2,i,j,n,tol);
    else
        iv2=0;
    end
    ai=2*h; bi=3*h;
    aa=max(r1,ai); bb=min(r2,bi);
    if aa < bb
        inv_s1=aa/(aa+2); inv_s2=bb/(bb+2);
        iv3=ite_comp_gq3(inv_s1,inv_s2,i,j,n,tol);
    else
        iv3=0;
    end
    iv=iv1+iv2+iv3;
```

```
elseif i==n-1
    ai=(n-1)*h; bi=(n)*h;
    aa=max(r1,ai); bb=min(r2,bi);
    if aa < bb
        inv_s1=aa/(aa+2); inv_s2=bb/(bb+2);
        iv=ite_comp_gq3(inv_s1,inv_s2,i,j,n,tol);
    else
        iv=0;
    end
elseif i==n-2
    ai=(n-2)*h; bi=(n-1)*h;
    aa=max(r1,ai); bb=min(r2,bi);
    if aa < bb
        inv_s1=aa/(aa+2); inv_s2=bb/(bb+2);
        iv1=ite_comp_gq3(inv_s1,inv_s2,i,j,n,tol);
    else
        iv1=0;
    end
    ai=(n-1)*h; bi=(n)*h;
    aa=max(r1,ai); bb=min(r2,bi);
    if aa < bb
        inv_s1=aa/(aa+2); inv_s2=bb/(bb+2);
        iv2=ite_comp_gq3(inv_s1,inv_s2,i,j,n,tol);
    else
        iv2=0;
    end
    iv=iv1+iv2;
elseif i==n-3
    ai=(n-3)*h; bi=(n-2)*h;
    aa=max(r1,ai); bb=min(r2,bi);
    if aa < bb
        inv_s1=aa/(aa+2); inv_s2=bb/(bb+2);
        iv1=ite_comp_gq3(inv_s1,inv_s2,i,j,n,tol);
    else
        iv1=0;
    end
    ai=(n-2)*h; bi=(n-1)*h;
    aa=max(r1,ai); bb=min(r2,bi);
    if aa < bb
        inv_s1=aa/(aa+2); inv_s2=bb/(bb+2);
        iv2=ite_comp_gq3(inv_s1,inv_s2,i,j,n,tol);
    else
        iv2=0;
    end
    ai=(n-1)*h; bi=n*h;
    aa=max(r1,ai); bb=min(r2,bi);
    if aa < bb
        inv_s1=aa/(aa+2); inv_s2=bb/(bb+2);
        iv3=ite_comp_gq3(inv_s1,inv_s2,i,j,n,tol);
    else
        iv3=0;
    end
    iv=iv1+iv2+iv3;
else
    ai=(i)*h; bi=(i+1)*h;
    aa=max(r1,ai); bb=min(r2,bi);
```

```
    if aa < bb
        inv_s1=aa/(aa+2); inv_s2=bb/(bb+2);
        iv1=ite_comp_gq3(inv_s1,inv_s2,i,j,n,tol);
    else
        iv1=0;
    end
    ai=(i+1)*h; bi=(i+2)*h;
    aa=max(r1,ai); bb=min(r2,bi);
    if aa < bb
        inv_s1=aa/(aa+2); inv_s2=bb/(bb+2);
        iv2=ite_comp_gq3(inv_s1,inv_s2,i,j,n,tol);
    else
        iv2=0;
    end
    ai=(i+2)*h; bi=(i+3)*h;
    aa=max(r1,ai); bb=min(r2,bi);
    if aa < bb
        inv_s1=aa/(aa+2); inv_s2=bb/(bb+2);
        iv3=ite_comp_gq3(inv_s1,inv_s2,i,j,n,tol);
    else
        iv3=0;
    end
    ai=(i+3)*h; bi=(i+4)*h;
    aa=max(r1,ai); bb=min(r2,bi);
    if aa < bb
        inv_s1=aa/(aa+2); inv_s2=bb/(bb+2);
        iv4=ite_comp_gq3(inv_s1,inv_s2,i,j,n,tol);
    else
        iv4=0;
    end
    iv=iv1+iv2+iv3+iv4;
end
end
%%%% decreasing part of S %%%%
function iv=int_j_b(aj,bj,i,j,n,tol)
if bj>1
    iv=0;
    return
end
r1=(1-bj)/(2*bj); r2=(1-aj)/(2*aj);
h=1/n;
if i==-3
    ai=0; bi=h;
    aa=max(r1,ai); bb=min(r2,bi);
    if aa < bb
        inv_s1=1/(2*bb+1); inv_s2=1/(2*aa+1);
        iv=ite_comp_gq3(inv_s1,inv_s2,i,j,n,tol);
    else
        iv=0;
    end
elseif i==-2
    ai=0; bi=h;
    aa=max(r1,ai); bb=min(r2,bi);
    if aa < bb
        inv_s1=1/(2*bb+1); inv_s2=1/(2*aa+1);
        iv1=ite_comp_gq3(inv_s1,inv_s2,i,j,n,tol);
```

```
    else
        iv1=0;
    end
    ai=h; bi=2*h;
    aa=max(r1,ai); bb=min(r2,bi);
    if aa < bb
        inv_s1=1/(2*bb+1); inv_s2=1/(2*aa+1);
        iv2=ite_comp_gq3(inv_s1,inv_s2,i,j,n,tol);
    else
        iv2=0;
    end
    iv=iv1+iv2;
elseif i==-1
    ai=0; bi=h;
    aa=max(r1,ai); bb=min(r2,bi);
    if aa < bb
        inv_s1=1/(2*bb+1); inv_s2=1/(2*aa+1);
        iv1=ite_comp_gq3(inv_s1,inv_s2,i,j,n,tol);
    else
        iv1=0;
    end
    ai=h; bi=2*h;
    aa=max(r1,ai); bb=min(r2,bi);
    if aa < bb
        inv_s1=1/(2*bb+1); inv_s2=1/(2*aa+1);
        iv2=ite_comp_gq3(inv_s1,inv_s2,i,j,n,tol);
    else
        iv2=0;
    end
    ai=2*h; bi=3*h;
    aa=max(r1,ai); bb=min(r2,bi);
    if aa < bb
        inv_s1=1/(2*bb+1); inv_s2=1/(2*aa+1);
        iv3=ite_comp_gq3(inv_s1,inv_s2,i,j,n,tol);
    else
        iv3=0;
    end
    iv=iv1+iv2+iv3;
elseif i==n-1
    ai=(n-1)*h; bi=n*h;
    aa=max(r1,ai); bb=min(r2,bi);
    if aa < bb
        inv_s1=1/(2*bb+1); inv_s2=1/(2*aa+1);
        iv=ite_comp_gq3(inv_s1,inv_s2,i,j,n,tol);
    else
        iv=0;
    end
elseif i==n-2
    ai=(n-2)*h; bi=(n-1)*h;
    aa=max(r1,ai); bb=min(r2,bi);
    if aa < bb
        inv_s1=1/(2*bb+1); inv_s2=1/(2*aa+1);
        iv1=ite_comp_gq3(inv_s1,inv_s2,i,j,n,tol);
    else
        iv1=0;
    end
```

```
    ai=(n-1)*h; bi=(n)*h;
    aa=max(r1,ai); bb=min(r2,bi);
    if aa < bb
        inv_s1=1/(2*bb+1); inv_s2=1/(2*aa+1);
        iv2=ite_comp_gq3(inv_s1,inv_s2,i,j,n,tol);
    else
        iv2=0;
    end
    iv=iv1+iv2;
elseif i==n-3
    ai=(n-3)*h; bi=(n-2)*h;
    aa=max(r1,ai); bb=min(r2,bi);
    if aa < bb
        inv_s1=1/(2*bb+1); inv_s2=1/(2*aa+1);
        iv1=ite_comp_gq3(inv_s1,inv_s2,i,j,n,tol);
    else
        iv1=0;
    end
    ai=(n-2)*h; bi=(n-1)*h;
    aa=max(r1,ai); bb=min(r2,bi);
    if aa < bb
        inv_s1=1/(2*bb+1); inv_s2=1/(2*aa+1);
        iv2=ite_comp_gq3(inv_s1,inv_s2,i,j,n,tol);
    else
        iv2=0;
    end
    ai=(n-1)*h; bi=(n)*h;
    aa=max(r1,ai); bb=min(r2,bi);
    if aa < bb
        inv_s1=1/(2*bb+1); inv_s2=1/(2*aa+1);
        iv3=ite_comp_gq3(inv_s1,inv_s2,i,j,n,tol);
    else
        iv3=0;
    end
    iv=iv1+iv2+iv3;
else
    ai=(i)*h; bi=(i+1)*h;
    aa=max(r1,ai); bb=min(r2,bi);
    if aa < bb
        inv_s1=1/(2*bb+1); inv_s2=1/(2*aa+1);
        iv1=ite_comp_gq3(inv_s1,inv_s2,i,j,n,tol);
    else
        iv1=0;
    end
    ai=(i+1)*h; bi=(i+2)*h;
    aa=max(r1,ai); bb=min(r2,bi);
    if aa < bb
        inv_s1=1/(2*bb+1); inv_s2=1/(2*aa+1);
        iv2=ite_comp_gq3(inv_s1,inv_s2,i,j,n,tol);
    else
        iv2=0;
    end
    ai=(i+2)*h; bi=(i+3)*h;
    aa=max(r1,ai); bb=min(r2,bi);
    if aa < bb
        inv_s1=1/(2*bb+1); inv_s2=1/(2*aa+1);
```

```
        iv3=ite_comp_gq3(inv_s1,inv_s2,i,j,n,tol);
    else
        iv3=0;
    end
    ai=(i+3)*h; bi=(i+4)*h;
    aa=max(r1,ai); bb=min(r2,bi);
    if aa < bb
        inv_s1=1/(2*bb+1); inv_s2=1/(2*aa+1);
        iv4=ite_comp_gq3(inv_s1,inv_s2,i,j,n,tol);
    else
        iv4=0;
    end
    iv=iv1+iv2+iv3+iv4;
end
end
%%%% Gaussian quadrature with three points %%%%
function iv=ite_comp_gq3(a,b,i,j,n,tol)
iv=comp_gq3(a,b,4,i,j,n);
niv=comp_gq3(a,b,8,i,j,n);
cor=abs(niv-iv);
m=8;
while cor > tol
    iv=niv;
    m=2*m;
    niv=comp_gq3(a,b,m,i,j,n);
    cor=abs(niv-iv);
end
iv=niv;
end
%%%% Composite of Gaussian quadrature with three points %%%%
function iv=comp_gq3(a,b,m,i,j,n)
h=(b-a)/m;
x=a:h:b;
iv=0;
for k=1:m
    iv=iv+gq3(x(k),x(k+1),i,j,n);
end
end
%%%% Find C's and X's of Gaussian quadrature with three points %%%%
function iv=gq3(a,b,i,j,n)
C=zeros(3,1); X=zeros(3,1);
C(1)=(5/18)*(b-a); X(1)=((a+b)/2)-(0.5*(b-a)*sqrt(0.6));
C(2)=(8/18)*(b-a); X(2)=(a+b)/2;
C(3)=C(1);         X(3)=((a+b)/2)+(0.5*(b-a)*sqrt(0.6));
iv=C(1)*g(X(1),i,j,n)+C(2)*g(X(2),i,j,n)+C(3)*g(X(3),i,j,n);
end
%%%% phi_i(sx)*phi_j %%%%
function fx=g(x,i,j,n)
h=1/n;
Sx=sx(x);
fx=csp_fun(Sx,i*h,h)*csp_fun(x,j*h,h);
end
%%%% Transformation S %%%%
function  s=sx(x)
if x<=1/3
    s=(2*x)/(1-x);
else
```

```
    s=(1-x)/(2*x);
end
end
%%%% Finding vector d from (A-B)d=0 %%%%%
function d=vec_d_csp(n,A,B)
h=1/n;
d=ones(n+3,1);
d=(A-B)\d;
d=d/sum(d);
s=h*(d(1)/24+(d(2)/2)+(23*d(3)/24)+sum(d(4:n))+(23*d(n+1)/24)+(d(n+2)/2)
  +d(n+3)/24); % ||fn||
d=d/s;
end
%%%% Computing the error %%%%%
function err=csp_err(d,tol)
n=length(d)-3; h=1/n;
e=zeros(n,1);
for i=1:n
    R=int_det(i,n,d,tol);
    m=length(R);
    if m==0
        e(i)=gq3_err((i-1)*h,(i)*h,i,n,d);
    elseif m==1
        e1(i)=gq3_err((i-1)*h,R(1),i,n,d);
        e2(i)=gq3_err(R(1),(i)*h,i,n,d);
        e(i)=e1(i)+e2(i);
    elseif m==2
        e1(i)=gq3_err((i-1)*h,R(1),i,n,d);
        e2(i)=gq3_err(R(1),R(2),i,n,d);
        e3(i)=gq3_err(R(2),(i)*h,i,n,d);
        e(i)=e1(i)+e2(i)+e3(i);
    elseif m==3
        e1(i)=gq3_err((i-1)*h,R(1),i,n,d);
        e2(i)=gq3_err(R(1),R(2),i,n,d);
        e3(i)=gq3_err(R(2),R(3),i,n,d);
        e4(i)=gq3_err(R(3),(i)*h,i,n,d);
        e(i)=e1(i)+e2(i)+e3(i)+e4(i);
    elseif m==4
        e1(i)=gq3_err((i-1)*h,R(1),i,n,d);
        e2(i)=gq3_err(R(1),R(2),i,n,d);
        e3(i)=gq3_err(R(2),R(3),i,n,d);
        e4(i)=gq3_err(R(3),R(4),i,n,d);
        e5(i)=gq3_err(R(4),(i)*h,i,n,d);
        e(i)=e1(i)+e2(i)+e3(i)+e4(i)+e5(i);
    else
        e1(i)=gq3_err((i-1)*h,R(1),i,n,d);
        e2(i)=gq3_err(R(1),R(2),i,n,d);
        e3(i)=gq3_err(R(2),R(3),i,n,d);
        e4(i)=gq3_err(R(3),R(4),i,n,d);
        e5(i)=gq3_err(R(4),R(5),i,n,d);
        e6(i)=gq3_err(R(5),(i)*h,i,n,d);
        e(i)=e1(i)+e2(i)+e3(i)+e4(i)+e5(i)+e6(i);
    end
end
err=sum(e);
end
```

```
%%%%% Interval determination %%%%%%
function R2=int_det(i,n,d,tol)
h=1/n;
a3=(1/(6*h^3))*(-d(i)+3*d(i+1)-3*d(i+2)+d(i+3));
a2=(1/(2*h^2))*(i*d(i)+(1-3*i)*d(i+1)+(3*i-2)*d(i+2)-(i-1)*d(i+3));
a1=(1/(2*h))*(-i^2*d(i)+(i-1)*(3*i+1)*d(i+1)+(4-3*i)*i*d(i+2)
   +(i-1)^2*d(i+3));
a0=(1/6)*(i^3*d(i)+(1+3*i+3*i^2-3*i^3)*d(i+1)+(3*i^3-6*i^2+4)*d(i+2)
   -(i-1)^3*d(i+3));
C=[0 0 0 0 -(a0-2)/a3;
    1 0 0 0 -(a1+2*a0)/a3;
    0 1 0 0 -(a2+2*a1+a0)/a3;
    0 0 1 0 -(a3+2*a2+a1)/a3;
    0 0 0 1 -(2*a3+a2)/a3];
ev=eig(C); %eigenvalues of the companion matrix C
R1=[];
for k=1:5
    if abs(imag(ev(k)))<tol
        R1=[R1;real(ev(k))];
    end
end
m=length(R1);
R2=[];
for k=1:m
    if R1(k)>(i-1)*h & R1(k)<i*h
        R2=[R2;R1(k)];
    end
    R2=sort(R2);
end
end
%%%% Find C's and X's of Gaussian quadrature of error function %%%%
function ive=gq3_err(a,b,i,n,d)
D=zeros(3,1); XX=zeros(3,1);
D(1)=(5/18)*(b-a); XX(1)=((a+b)/2)-(0.5*(b-a)*sqrt(0.6));
D(2)=(8/18)*(b-a); XX(2)=(a+b)/2;
D(3)=D(1);         XX(3)=((a+b)/2)+(0.5*(b-a)*sqrt(0.6));
ive=D(1)*f_n(XX(1),i,n,d)+D(2)*f_n(XX(2),i,n,d)+D(3)*f_n(XX(3),i,n,d);
end
%%%% fn & |fn-f*| %%%%
function ex=f_n(x,i,n,d)
h=1/n;
fn=d(i)*csp_fun(x,(i-4)*h,h)+d(i+1)*csp_fun(x,(i-3)*h,h)
   +d(i+2)*csp_fun(x,(i-2)*h,h)+d(i+3)*csp_fun(x,(i-1)*h,h);
fx=2/(x+1)^2;
ex=abs(fn-fx);
end
%%%% Cubic spline function %%%%
function fx=csp_fun(x,c,h)
y=(x-c)/h;
if y > 0 & y <= 1
    fx=(1/6)*(y)^3;
elseif y > 1 & y <= 2
    fx=(1/6)*(1+3*((y)-1)+3*((y)-1)^2-3*((y)-1)^3);
elseif y > 2 & y <= 3
    fx=(1/6)*(1+3*(3-(y))+3*(3-(y))^2-3*(3-(y))^3);
elseif y > 3 & y <= 4
    fx=(1/6)*(4-(y))^3;
```

```
else
    fx=0;
end
end
```

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VITA

Azzah Alshekhi was born on April 13, in Jubail City, Saudi Arabia. In 2012, she received her B.S. in Mathematics from King Abdulaziz University, Jeddah City, Saudi Arabia. She received her M.S. from the University of Missouri-Kansas City, USA, in 2017.