REMARKS ON THE MULTIPLICITY MEASURE FOR e^+e^- ANNIHILATION*

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(*Received May 5, 1989*)

A recent proposal of the Lund group suggesting a multiplicity measure for e^+e^- annihilation is analyzed. It is shown that the results obtained for the multiplicity distributions in full phase space do not change if one removes a questionable assumption of exact localization of branching products in rapidity (thus avoiding troubles for small rapidity bins). The predicted asymptotic form of the distribution is shown to differ from the negative binomial distribution.

PACS numbers: 15.65.+i

In a recent paper of the Lund group [1] a measure of multiplicity in e^+e^- annihilation events has been proposed. Generalizing the available phase space from 2-jet events to multigluon events one finds an infrared stable measure which is suggested to represent the hadronic multiplicity. Its probability distribution may be found from a differential equation resulting from simple probabilistic considerations.

The authors present two results which may be obtained without solving numerically the differential equation:

a) For very large energies, the asymptotic formulae for average multiplicity and dispersion are found. They reproduce (with a minor modification) the QCD results for gluon multiplicity [2] and the Malaza-Webber results for dispersion [3].

b) For small bins in rapidity, the scaled moments are shown to grow as negative powers of the bin size, suggesting the "intermittent" behaviour of the type considered by Białas and Peschanski [4].

However, one should note that the power-like growth of moments obtained in Ref. [1] is in fact much too strong. The exponents for consecutive moments are negative integers, whereas all the existing data suggest values smaller by almost two orders of magnitude [5]. Thus the model in the presented version is not realistic for small rapidity bins.

* Research supported in part by the Polish Ministry of Education, programs CPBP 01.03. and 01.09.

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In this note we trace the origin of the unwanted result and show that it may be quite easily removed without losing the promising results for multiplicities in full phase space. We comment also on the asymptotic shape of the multiplicity distributions.

Let us repeat first the main assumptions adopted in Ref. [1]. Using for variables CM rapidity y and the logarithm of transverse momentum

$$\kappa = \ln(k_T^2/\Lambda^2) \tag{1}$$

and denoting by L the logarithmic energy measure

$$L = \ln(s/\Lambda^2), \tag{2}$$

we consider the approximate phase space for the gluon emission from a $3\bar{3}$ colour dipole as a triangular region given by

$$2|y|+\kappa \leqslant L, \quad \kappa \geqslant 0. \tag{3}$$

If one gluon is emitted with rapidity y_1 and transverse momentum logarithm κ_1, the phase space for the second gluon is increased by an extra triangle of the vertical and horizontal side length κ_1. For strong ordering of gluons in transverse momenta the rapidity of a second gluon emitted from this extra part of phase space is approximately equal to y_1. Phase space available after the emission of further gluons will be similarly increased by adding new triangular surfaces.

Now one notes that without any gluon emission one expects hadronic production with flat rapidity distribution and average multiplicity proportional to L, the baseline of the original triangular phase space. For one gluon emission one can split the system into two $3\bar{3}$ systems with energies given by

$$s_{1,2} = \sqrt{s}\, k_{T1} \exp(\pm y_1), \tag{4}$$

thus the multiplicity should be proportional to

$$\ln(s_1/\Lambda^2)+\ln(s_2/\Lambda^2) = L+\kappa_1, \tag{5}$$

the total baseline length of two triangles forming phase space. Thus in general one may expect that the hadronic multiplicity from a state with n gluons should be related to a measure

$$\lambda = L+\sum_1^n \kappa_i, \tag{6}$$

which is the total baseline length of the multi-triangle phase space.

To specify the predictions one needs the probability distribution of λ for given L. In Ref. [1] this is done by considering first analoguous distribution of a piece of baseline length within a limited region Δ in rapidity. One assumes here that any two regions are completely independent, i.e.

$$P_{\Delta_1+\Delta_2}(\lambda) = \int P_{\Delta_1}(\lambda')P_{\Delta_2}(\lambda'')d\lambda' d\lambda''\delta(\lambda-\lambda'-\lambda''). \tag{7}$$

This assumptions is very strong and seems to be a rough approximation, as it requires e.g. that all the "later" (softer) gluons emitted from the "earlier" (harder) gluon have exactly the same value of rapidity. Moreover, this should also hold for the hadronization products of these gluons.

Let us show now that it is exactly assumption (7) which leads to the power-like increase of scaled moments for decreasing length of rapidity interval Δ. Let us first consider two lowest moments

$$\overline{\lambda_\Delta} = \int \lambda P(\lambda) d\lambda, \quad \overline{\lambda_\Delta^2} = \int \lambda^2 P(\lambda) d\lambda. \tag{8}$$

Dividing the region Δ into two of length $\Delta' = \Delta/2$ we find easily

$$\overline{\lambda_\Delta} = 2\overline{\lambda_{\Delta'}}, \quad \overline{\lambda_\Delta^2} = 2\overline{\lambda_{\Delta'}^2} + 2\overline{\lambda_\Delta}^2. \tag{9}$$

and for scaled moment we get

$$F_2^\Delta \equiv \overline{\lambda_\Delta^2}/\overline{\lambda_\Delta}^2 = (F_2^{\Delta'}+1)/2. \tag{10}$$

Inverting this relation we find

$$F_2^{\Delta'} - 1 = 2(F_2^\Delta - 1) \tag{11}$$

and repeating this division n times we may write for $\delta = 2^{-n}\Delta$

$$F_2^\delta = 2^n(F_2^\Delta - 1) + 1. \tag{12}$$

Thus for large n (small δ) we find

$$\ln F_2^\delta = -\ln(\delta/\Delta) + \ln(F_2^\Delta - 1) + O(\delta/\Delta). \tag{13}$$

Similarly, considering the third moments

$$\overline{\lambda_\Delta^3} = \int \lambda^3 P(\lambda) d\lambda, \quad F_3^\Delta \equiv \overline{\lambda_\Delta^3}/\overline{\lambda_\Delta}^3 \tag{14}$$

we find for small δ

$$\ln F_3^\delta = -2\ln(\delta/\Delta) + \ln(F_3^\Delta - 3F_2^\Delta + 2) + O(\delta/\Delta) \tag{15}$$

and in general, for the i-th scaled moment we find

$$\ln F_i^\delta = -(i-1)\ln(\delta/\Delta) + c_i + O(\delta/\Delta). \tag{16}$$

As already noted, this behaviour is in strong disagreement with data from various multiparticle production processes [5] (including e^+e^- [6]), which seem to show indeed for the rapidity bin length δ between 1 and 0.1 the linear behaviour

$$\ln F_i^\delta = -\alpha_i \ln \delta + \beta_i \tag{17}$$

but the slopes α_i are much smaller than i-1 (by almost two orders of magnitude). This confirms our doubts concerning the validity of assumption (7). On the other hand, this assumption seems to be in Ref. [1] a starting point for the subsequent derivation of the differential equation for $P(\lambda, L)$. Thus a question arises: can one derive the equation for $P(\lambda, L)$ without assuming (7)?

To answer this, let us recall the derivation presented in Ref. [1]. One considers the change in probability $P_\Delta(\lambda, l)$ (where Δ is a small bin around rapidity y and l denotes the maximal value of κ allowed for given L and y) induced by changing l to $l+\varepsilon$. This change allows for the emission of an extra gluon from the phase space region $\kappa \in (l, l+\varepsilon)$, $y \in (y, y+\Delta)$. Since in the leading log approximation the gluon emission probability density is α_0/κ, we find

$$P_\Delta(\lambda, l+\varepsilon) = (\varepsilon\Delta\alpha_0/l) \int d\lambda' P_\Delta(\lambda', l) d\lambda'' P(\lambda'', l)\delta(\lambda-\lambda'-\lambda'')+(1-\varepsilon\Delta\alpha_0/l)P_\Delta(\lambda, l), \quad (18)$$

where $P(\lambda'', l)$ corresponds to the phase space sector open by the extra gluon of $\kappa = l$, equivalent to an isolated $3\bar{3}$ system with $L = l$.

Subtracting $P_\Delta(\lambda, l)$ from both sides and taking the limit $\varepsilon \to 0$, $\Delta \to 0$ and Laplace transforms we find then the equation we are looking for

$$\frac{d^2}{dL^2} \ln \bar{P}(\beta, L) = [\bar{P}(\beta, L)-1]\alpha_0/L, \quad (19)$$

where

$$\bar{P}(\beta, L) = \int_0^\infty d\lambda e^{-\lambda\beta} P(\lambda, L). \quad (20)$$

The boundary conditions supplementing (19) are

$$\bar{P}(\beta, 0) = 1, \quad d/dL[\bar{P}(\beta, L)]_{L=0} = -\beta. \quad (21)$$

One should note the disappearance of interval Δ from the formulae. This is possible because assumption (7) implies the simple multiplication rule for Laplace transforms $P_\Delta(\beta)$ analoguous to (20), and additivity of $\ln \bar{P}_\Delta(\beta)$. Thus we can define the ratio

$$R(\beta, l) = \lim_{\Delta\to 0} [\ln \bar{P}_\Delta(\beta)]/\Delta \quad (22)$$

by which we can express both $\bar{P}(\beta, L)$ and $\bar{P}_\Delta(\beta)$

$$\ln \bar{P}(\beta, L) = \int_{-L/2}^{L/2} dy R(\beta, l = L-2|y|) = \int_0^L dl R(\beta, l) \quad (23)$$

$$\ln \bar{P}_\Delta(\beta) = \int_\Delta dy R[\beta, l(y)]. \quad (24)$$

Thus we can write

$$\lim_{\Delta\to 0, \varepsilon\to 0} [\bar{P}_\Delta(\beta, l+\varepsilon)-\bar{P}_\Delta(\beta, l)]/\varepsilon\Delta = dR(\beta, l)/dl = d^2 \ln \bar{P}(\beta, l)/dl^2. \quad (25)$$

However, the key element of the derivation was only the assumption that creating an extra gluon opens a new sector of phase space, approximately independent of the existing one. The rapidity localization of this gluon and of its hadronization products is irrelevant if we want to derive the equation for the full phase space.

Thus instead of considering a small bin in rapidity we may start directly from full phase space, calculating the change in probability $P(\lambda, L)$ induced by a small change in L. We find

$$P(\lambda, L+\varepsilon) = \int P(\lambda', L)P(\lambda'', L)\varepsilon\varrho(l)dl d\lambda' d\lambda''\delta(\lambda-\lambda'-\lambda'') + [1-\int_0^L \varepsilon\varrho(l)dl]P(\lambda, L). \tag{26}$$

Here $\varrho(l)$ is the probability density for emitting a gluon with $\kappa = l$. Taking the Laplace transforms and subtracting $\bar{P}(\beta, L)$ we find

$$\bar{P}(\beta, L+\varepsilon)-\bar{P}(\beta, L) = \bar{P}(\beta, L)\,[\int_0^L \bar{P}(\beta, l)\varrho(l)dl - \int_0^L \varrho(l)dl]\varepsilon, \tag{27}$$

and in the limit $\varepsilon \to 0$

$$d[\ln \bar{P}(\beta, L)]/dL = \int_0^L \varrho(l)dl[\bar{P}(\beta, l)-1], \tag{28}$$

or

$$d^2[\ln \bar{P}(\beta, L)]/dL^2 = \varrho(L)\,[\bar{P}(\beta, L)-1]. \tag{29}$$

As already noted, for large L we take $\varrho(L) = \alpha_0/L$. Thus we have recovered Eq. (19) without any corrections.

We have shown that to obtain the probability distribution for the multiplicity measure in full phase space we do not need the questionable assumption of the exact rapidity localization of all the branching and hadronization products of an initial hard gluon. It would be very interesting to assume a more realistic distribution for the rapidity spread in the consecutive branching processes and to see if there is a chance of reproducing experimental data for small rapidity bins. This problem is, however, unlikely to be solved analytically and we will not consider it here.

Before concluding, let us comment shortly on the asymptotic properties of the solutions of Eq. (19). We may easily calculate the moments of λ distribution using

$$\overline{\lambda^i} \equiv \int_0^\infty d\lambda\lambda^i P(\lambda, L) = d^i[\bar{P}(\beta, L)]/d(-\beta)^i|_{\beta=0}. \tag{30}$$

Thus expanding $\bar{P}(\beta, L)$ and its logarithm into powers of β we find first the equation considered in Ref. [1]

$$d^2[-\bar{\lambda}(L)]/dL^2 = \alpha_0[-\bar{\lambda}(L)]/L \tag{31}$$

which has a solution

$$\alpha_0\bar{\lambda}(L) = \Sigma\,(\alpha_0 L)^{j+1}/[j!(j+1)!] = \sqrt{\alpha_0 L}\, I_1(2\sqrt{\alpha_0 L}) \tag{32}$$

with well-known asymptotic behaviour for large L

$$\alpha_0\bar{\lambda}(L) = (\alpha_0 L)^{1/4} \exp[2\sqrt{\alpha_0 L}\,(1-0(1/L))]/\sqrt{4\pi}. \tag{33}$$

Further terms of the expansion allow to calculate asymptotic values of scaled moments

$$F_2 = 4/3, \quad F_3 = 9/4, \quad F_4 = 208/45 \tag{34}$$

the first of which was already quoted in Ref. [1]. Here we may note that the values of F_3 and F_4 do not agree with those we could deduce from F_2 if the distribution would be negative binomial (NBD). Indeed, for NBD we have

$$F_i = \prod_{j=0}^{i-1} (1+j/k), \tag{35}$$

and for $k = 1/3$ corresponding to $F_2 = 4/3$ we find $F_3 = 20/9$, $F_4 = 40/9$.

We shall note here that the differences are numerically very small, suggesting that NBD fit may be quite successful. This agrees with observation made in Ref. [1] from numerical studies of solutions for finite energies. Nevertheless, choosing more sensitive parameters, as e.g. the scaled Mueller correlation coefficients

$$f_2 = F_2 - 1, \quad f_3 = F_3 - 3f_2 - 1, \quad f_4 = F_4 - 4f_3 - 3f_2^2 - 6f_2 - 1, \tag{36}$$

we find 11% difference for f_3 and 23% difference for f_4. Thus, although our results confirm the possibility of good NBD fits to data up to the asymptotic energies, we find that the asymptotic distribution predicted by the model of Ref. [1] is significantly different from NBD.

To summarize, we have reanalysed the predictions for multiplicity distributions in e^+e^- collisions derived from a proposal to relate the hadronic multiplicity to a simple measure on partonic states. We find that the unrealistic (and leading to wrong predictions) assumption of the exact localization in rapidity of the branching and hadronization products of a hard gluon may be removed. This does not affect the predictions for the multiplicity distributions in full phase space. We show also that the asymptotic form of multiplicity distribution obtained is not NBD, although the differences in lowest moments are not big. We conclude that the model looks very promising and deserves further studies which should allow, in particular, to formulate reliable predictions also for multiplicity distribution in small rapidity bins.

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