ON PERTURBATIVE SOLUTION OF CLASSICAL YANG-MILLS EQUATIONS WITH EXTERNAL CHARGES*

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(*Received November 28, 1983*)

The method of solving of classical Yang-Mills equations with external charges by means of the perturbative expansion in powers of the external charge $\hat{\varrho}(\vec{x})$ is considered in the $\hat{A}_0 \neq 0$ gauge. The gauge potentials are calculated in the first three orders for generic $\hat{\varrho}$. It is observed that in general the external color charge generates color magnetic field of the monopole type (i.e., behaving like r^{-2} for $r \to \infty$) in the second order, and that in the third order the color electric field behaves like r^{-1} for $r \to \infty$. This gives infinite energy of such a color charge and motivates an argument that nonvanishing total color charge density (resultant for all kinds of colored quarks) can not be produced in any real experiment.

PACS numbers: 03.50.-z, 11.15.-q

1. *Introduction*

In spite of many efforts, as a sample see, e.g., papers [1–8] and numerous references therein, it is still not possible to provide a thorough answer to the simple question "what are classical non-Abelian gauge fields generated by a given, external color charge distribution". There are two reasons for this situation. The first one is the nonlinearity of equations. The second reason is the presence of the local gauge covariance of equations with external sources. For example, the local gauge covariance is the main tool in obtaining an infinite number of gauge-inequivalent solutions for a given external charge, see e.g., [8] for a very good illustration of this point. The lack of the answer to the above question results in rather weak understanding of color interactions of quarks on the relatively simple level of classical Yang-Mills theory.

This paper is devoted to a perturbative approach to solving of classical Yang-Mills equations with a given external charge distribution $q\hat{\varrho}(\vec{x})$. The method consists of expansion of gauge potentials $\hat{A}_\mu$ in powers of the strength q of the external charge. This very appealing approach was proposed in paper [1]. In [7] it was observed that the method in the original

* Paper supported in part by Polish Ministry of Higher Education, Science and Technology, project MR.I.7.

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formulation requires to accompany $q\hat{\varrho}(\vec{x})$ by an external current $\hat{j}_i(\vec{x})$. The current appears already in the order q^2 for generic $\hat{\varrho}$, and in the order q^4 or higher for some particular $\hat{\varrho}$. The perturbative method was proposed in [1] as a tool for obtaining an approximate form of the so called non-Abelian Coulomb (NC) solution of Yang-Mills equations for the external charge $q\hat{\varrho}(\vec{x})$. For a definition and a possible physical interpretation of this solution see [1, 7, 9].

In this paper we reformulate the perturbative method in the $\hat{A}_0 = 0$ gauge (in Section 2). The most important reason for doing this is that in this gauge the method does not require introducing of the spatial currents $\hat{j}_i$. Moreover, the resulting equations are relatively simple. Therefore, we can calculate the gauge potentials $\hat{A}_i(\vec{x}, t)$ of the NC solution up to the third order (q^3) for generic $\hat{\varrho}(\vec{x})$. Next, in Section 3, we estimate the asymptotic behaviour of the obtained potentials and of the corresponding field strengths $\hat{F}_{\mu\nu}$ for $r = |\vec{x}| \to \infty$. In the order q^2 we find that generic external charge develops a color magnetic field $\hat{F}_{ik}$ of the magnetic monopole type, i.e., of order r^{-2} for $r \to \infty$, in addition to the expected color electric field $\hat{F}^{0i}$ of order r^{-2}. Unexpectedly, we also find that in the order q^3 the electric field behaves like r^{-1} for $r \to \infty$. We also find that the solution is in general truly time-dependent, e.g., the gauge invariant density of energy is time-dependent. The paper is concluded by general comments collected in Section 4.

Our solution belongs to the class of the non-Abelian Coulomb solutions. For particular $\hat{\varrho}(\vec{x})$ such that $\hat{\varrho}(\vec{x}) = q\varrho(\vec{x})\omega(\vec{x})\frac{\sigma_3}{2}\omega^{-1}(\vec{x})$, where $\omega(\vec{x})$ is a gauge transformation characterised by the winding number of the π_3 type equal to n, the solution can be interpreted as the gauge field generated by the external charge $q\varrho(\vec{x})\sigma_3/2$ embedded in the sector of the θ-vacuum characterised by the winding number n, see Section II.2.3 in the review paper [9].

2. *The perturbative expansion in the $\hat{A}_0 = 0$ gauge*

We consider the Yang-Mills equations

$$\partial_k \hat{F}^{k0} + ig[\hat{A}_k, \hat{F}^{k0}] = gq\hat{\varrho}(\vec{x}, t), \tag{1}$$

$$\partial_0 \hat{F}^{0i} + \partial_k \hat{F}^{ki} + ig[\hat{A}_0, \hat{F}^{0i}] + ig[\hat{A}_k, \hat{F}^{ki}] = 0, \tag{2}$$

where

$$\hat{F}^{k0} = \partial_0 \hat{A}_k - \partial_k \hat{A}_0 - ig[\hat{A}_k, \hat{A}_0], \tag{3}$$

$$\hat{F}^{ki} = \partial_k \hat{A}_i - \partial_i \hat{A}_k + ig[\hat{A}_k, \hat{A}_i], \tag{4}$$

and $q\hat{\varrho}$ is the density of the external color charge, q is the expansion parameter. We assume that $\hat{\varrho}$ vanishes quickly for $r \to \infty$. We use the matrix notation, e.g. $\hat{A}_\mu = T^a A^a_\mu$, where T^a are the generators of the SU(n) gauge group. We use the Minkowski metric tensor, i.e. $\hat{A}_i = -\hat{A}^i$, $\partial^k = -\partial/\partial x^k$, etc. It is well-known that from (1), (2) it follows that if $\hat{A}_\mu$ is a solution of (1), (2) then

$$\partial_0 \hat{\varrho} + ig[\hat{A}_0, \hat{\varrho}] = 0. \tag{5}$$

In the following we shall consider Eqs (1), (2) in the $\hat{A}_0 = 0$ gauge. This gauge can always be obtained for any $\hat{A}'_0 \neq 0$ by the gauge transformation

$$\hat{A}_\mu = \omega^{-1}\hat{A}'_\mu\omega - \frac{i}{g}\,\omega^{-1}\partial_\mu\omega, \tag{6}$$

where

$$\omega = \mathrm{T}\exp\left[-ig\int_0^t \hat{A}'_0(\vec{x}, t')dt'\right], \tag{7}$$

and T denotes the usual time-ordering ($t' = t$ to the left, $t' = 0$ to the right).

After fixing of the $\hat{A}_0 = 0$ gauge there still remains the freedom of performing of time independent gauge transformations. This has the consequence that the boundary conditions at spatial infinity do not select a unique solution of Eqs (1), (2). Therefore, we fix the gauge further, namely we require that at $t = 0$ $\hat{A}_k$ is in the Coulomb gauge,

$$\partial_k\hat{A}_k|_{t=0} = 0. \tag{8}$$

For the perturbative solution this condition can be easily satisfied in each order.

In the $\hat{A}_0 = 0$ gauge $\hat{\varrho}$ must be constant in time when the spatial currents $\hat{j}^i = 0$ — otherwise Yang-Mills equations have no solutions, as it follows from (5).

Equations (1), (2) have the following form in the $\hat{A}_0 = 0$ gauge:

$$\partial_0(\partial_k\hat{A}_k) + ig[\hat{A}_k, \partial_0\hat{A}_k] = gq\hat{\varrho}(\vec{x}), \tag{9}$$

$$-\partial_0^2\hat{A}_i + \partial_k\partial_k\hat{A}_i - \partial_i\partial_k\hat{A}_k + ig\partial_k[\hat{A}_k, \hat{A}_i] + ig[\hat{A}_k, \hat{F}^{ki}] = 0. \tag{10}$$

In the following we shall put the Yang-Mills constant $g = 1$. We assume that the solution $\hat{A}_i$ of (9), (10) can be sought in the form of the perturbative expansion

$$\hat{A}_k(\vec{x}, t) = q\hat{A}_k^{(1)} + q^2\hat{A}_k^{(2)} + \ldots . \tag{11}$$

where

$$\hat{A}_i^{(n)}(\vec{x}, t) \to 0 \quad \text{when} \quad r \to \infty. \tag{11'}$$

The purpose of the assumption (11′) is the following. As it will be shown below, in n-th order we obtain for $\hat{A}_i^{(n)}$ the unhomogeneous wave equation such that the r.h.s. of it vanishes when $r \to \infty$. The solution of this equation is not unique because to any given solution one can add a wave (i.e., a solution of the sourceless wave equation). In order to eliminate this freedom we assume that $\hat{A}_i^{(n)} \to 0$ when $r \to \infty$, for all t. Of course, we also assume that the solution is regular for all $\vec{x}$, t. In this manner we pin the "core" solution which exists entirely due to nonzero external charge. This core solution vanishes for $\hat{\varrho} = 0$.

From (11) it follows that $\hat{A}_k \to 0$ when $q \to 0$. Let us notice that for non-Abelian Coulomb solution, according to the definition given in [1],

$$\hat{A}'_\mu = q\hat{A}'^{(1)}_\mu + q^2\hat{A}'^{(2)}_\mu + \ldots .$$

Thus, when we transform this solution to the $\hat{A}_0 = 0$ gauge, we obtain $\hat{A}_k$ of the form (11), because the pure gauge term $-i\omega^{-1}\partial_\mu\omega$ with ω given by (7) also has the form $\sum q^n\omega^{(n)}$.

Thus, the lack of the term q^0 in (11) in the $\hat{A}_0 = 0$ gauge does not imply loss of generality when investigating the NC solution.

In the first order in q we obtain from (9), (10) that

$$\partial_0(\partial_k \hat{A}_k^{(1)}) = \hat{\varrho}(\vec{x}), \tag{12}$$

$$-\partial_0^2 \hat{A}_i^{(1)} + \Delta \hat{A}_i^{(1)} - \partial_i(\partial_k \hat{A}_k^{(1)}) = 0. \tag{13}$$

From (12) it follows that

$$\partial_k \hat{A}_k^{(1)} = t\hat{\varrho}(\vec{x}) + \hat{c}(\vec{x}).$$

We take $\hat{c} = 0$, in accordance with the condition (8). From (13) we obtain the wave equation

$$-\partial_0^2 \hat{A}_i^{(i)} + \Delta \hat{A}_i^{(1)} = t\partial_i \hat{\varrho}. \tag{14}$$

Solution of this equation vanishing at spatial infinity is equal to the retarded (and advanced as well) solution of this equation, [10],

$$A_i^{(1)}(\vec{x}, t) = -\frac{1}{4\pi}\int \frac{1}{|\vec{x}-\vec{x}'|}\hat{s}_i(\vec{x}', t-|\vec{x}-\vec{x}'|)d^3\vec{x}',$$

where

$$\hat{s}_i = t\partial_i \hat{\varrho}.$$

Thus

$$\hat{A}_i^{(1)} = t\partial_i \hat{\alpha}, \tag{15}$$

where

$$\hat{\alpha} = -\frac{1}{4\pi}\int \frac{1}{|\vec{x}-\vec{x}'|}\hat{\varrho}(\vec{x}')d^3\vec{x}', \tag{16}$$

because

$$\int \partial_i \hat{\varrho} d^3\vec{x}' = 0.$$

It is easy to see that the solution (15) can also be obtained from any linear combination of the advanced and the retarded solutions of the wave equation with the sum of the coefficients equal to 1. We will present a more detailed discussion of the obtained gauge fields in the next Section. In this Section we shall present the second and third order contributions to $\hat{A}_i$.

In the second order we have

$$\partial_0(\partial_i \hat{A}_i^{(2)}) + i[\hat{A}_i^{(1)}, \partial_0 \hat{A}_i^{(1)}] = 0, \tag{17}$$

$$-\partial_0^2 \hat{A}_i^{(2)} + \Delta \hat{A}_i^{(2)} - \partial_i(\partial_k \hat{A}_k^{(2)}) + i\partial_k[\hat{A}_k^{(1)}, \hat{A}_i^{(1)}] + i[\hat{A}_k^{(1)}, \hat{F}^{ki(1)}] = 0. \tag{18}$$

The vanishing at $r \to \infty$ solution of Eqs (17), (18), with (8), (11′), (15) taken into account, is

$$\hat{A}_i^{(2)} = t^2 \hat{a}_i + \hat{b}_i, \tag{19}$$

where

$$\hat{a}_i = -\frac{1}{4\pi}\int \frac{1}{|\vec{x}-\vec{x}'|}\hat{R}_i(\vec{x}')d^3\vec{x}', \tag{20}$$

$$\hat{b}_i = -\frac{1}{4\pi}\int |\vec{x}-\vec{x}'|\hat{R}_i(\vec{x}')d^3\vec{x}', \tag{21}$$

and

$$\hat{R}_i = -i\partial_p[\partial_p\hat{\alpha}, \partial_i\hat{\alpha}]. \tag{22}$$

In obtaining (19) we have used the fact that $\int \hat{R}_i = 0$. In the next Section we shall show that $\hat{a}_i, \hat{b}_i$ vanish for $r \to \infty$. Let us also mention that

$$\Delta\hat{a}_i = \hat{R}_i, \quad \Delta\hat{b}_i = 2\hat{a}_i, \quad \partial_i\hat{a}_i = \partial_i\hat{b}_i = 0. \tag{23}$$

Finally, in the third order in q we have

$$\partial_0(\partial_k\hat{A}_k^{(3)}) + i[\hat{A}_k^{(1)}, \partial_0\hat{A}_k^{(2)}] + i[\hat{A}_k^{(2)}, \partial_0\hat{A}_k^{(1)}] = 0, \tag{24}$$

$$\begin{aligned} &-\partial_0^2\hat{A}_i^{(3)} + \Delta\hat{A}_i^{(3)} - \partial_i(\partial_k\hat{A}_k^{(3)}) + i\partial_k[\hat{A}_k^{(1)}, \hat{A}_i^{(2)}] \\ &+ i\partial_k[\hat{A}_k^{(2)}, \hat{A}_i^{(1)}] - [\hat{A}_k^{(1)}, [\hat{A}_k^{(1)}, \hat{A}_i^{(1)}]] \\ &+ i[\hat{A}_k^{(2)}, \partial_k\hat{A}_i^{(1)} - \partial_i\hat{A}_k^{(1)}] + i[\hat{A}_k^{(1)}, \partial_k\hat{A}_i^{(2)} - \partial_i\hat{A}_k^{(2)}] = 0. \end{aligned} \tag{25}$$

The vanishing at spatial infinity solution of Eqs (24), (25) can be written in the following form (obtained with the help of the formulae (23) and by integration by parts)

$$\begin{aligned} \hat{A}_i^{(3)} = &-\frac{1}{4\pi}t^3\int d^3\vec{x}'\frac{1}{|\vec{x}-\vec{x}'|}V_i(\vec{x}') - \frac{1}{4\pi}t\int d^3\vec{x}'\frac{1}{|\vec{x}-\vec{x}'|}W_i(\vec{x}') \\ &-\frac{3}{4\pi}t\int d^3\vec{x}'|\vec{x}-\vec{x}'|V_i(\vec{x}'), \end{aligned} \tag{26}$$

where

$$\begin{aligned} V_i = &\frac{i}{3}\partial_i[\hat{a}_k, \partial_k\hat{\alpha}] - i\partial_k[\partial_k\hat{\alpha}, \hat{a}_i] - i[\hat{a}_k, \partial_k\partial_i\hat{\alpha}] \\ &+ [\partial_k\hat{\alpha}, [\partial_k\hat{\alpha}, \partial_i\hat{\alpha}]] - i[\partial_k\hat{\alpha}, \partial_k\hat{a}_i - \partial_i\hat{a}_k], \end{aligned} \tag{27}$$

$$W_i = -i\partial_i[\hat{b}_k, \partial_k\hat{\alpha}] - i\partial_k[\partial_k\hat{\alpha}, \hat{b}_i] - i[\hat{b}_k, \partial_i\partial_k\hat{\alpha}] - i[\partial_k\hat{\alpha}, \partial_k\hat{b}_i - \partial_i\hat{b}_k]. \tag{28}$$

In the next Section we shall show that $\hat{A}_i^{(3)} \to 0$ when $r \to \infty$.

The next order contribution can be obtained in a similar way. The only difficulty which might appear in higher orders is too slow vanishing at $r \to \infty$ of various terms in the equations. Then, the solutions of the wave equations could not be written in the integral form.

3. *The analysis of the asymptotic behaviour of the perturbative contributions to $\hat{A}_i$*

In this Section we shall find the asymptotic form of the potentials $\hat{A}_i^{(n)}$, $n = 1, 2, 3$, for $r \to \infty$. Incidentally, this will provide a check of convergence of the integrals occurring in the formulae for $\hat{A}_i^{(n)}$. We also calculate the asymptotic values of fields $\hat{E}^i$, $\hat{B}^i$, and the asymptotic values of the energy density.

From the formulae (15), (16) it follows that for $r \to \infty$

$$\hat{A}_i^{(1)} = \frac{t}{4\pi} \frac{x^i}{r^3} \int \hat{\varrho} d^3\vec{x}'. \tag{29}$$

Of course, we assume that $\hat{\varrho}$ is such that the integral

$$\hat{Q} = \frac{1}{4\pi} \int \hat{\varrho} d^3\vec{x}, \tag{30}$$

as well as the higher multipole moments of $\hat{\varrho}$, are convergent. From (29) we obtain that the magnetic field vanishes for all r,

$$\hat{F}_{ik}^{(1)} = 0, \tag{31}$$

and that the electric field for $r \to \infty$ is given by

$$\hat{E}^{(1)i} = \partial_0 \hat{A}_i^{(1)} = \frac{x^i}{r^3} \hat{Q} + O(r^{-3}), \tag{32}$$

i.e., it is of the Coulomb type.

The estimation of the asymptotic behaviour of $\hat{A}_i^{(2)}$ encounters an obstacle. Namely, $\hat{R}_i$ present in the integrand on the r.h.s. of (20), (21) vanishes for $r \to \infty$ as r^{-6} for generic $\hat{\varrho}$ (the monopole term $\hat{Q}/r$ present in $\hat{\alpha}$ cancels out in the commutator). Therefore, higher multipole moments of $\hat{R}_i$ are divergent. In order to obtain the correct asymptotic behaviour of the integrals (20), (21) we apply the theorem which is stated and proved in the Appendix. Let us first consider $\hat{a}_i$. We recall that for $r > |\vec{x}'|$

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{r} + \frac{\vec{x}\vec{x}'}{r^3} + \frac{1}{2}\frac{1}{r^3}\left(-\delta_{ik} + 3\frac{x^i x^k}{r^2}\right) x'^i x'^k + O(r^{-4}). \tag{33}$$

Next, it is easy to see that the monopole and dipole moments of $\hat{R}_i$ are zero, and that the quadrupole moments are finite and equal to

$$\hat{P}_{ik}^s \equiv -\frac{1}{4\pi} \int x'^i x'^k \hat{R}_s d^3\vec{x}' = 2\delta_{ik}\hat{d}_s - \delta_{ks}\hat{d}_i - \delta_{is}\hat{d}_k, \tag{34}$$

where

$$\hat{d}_s = \frac{i}{4\pi} \int \alpha \partial_s \alpha d^3\vec{x}.$$

It is easy to show that the integral $\int \alpha \partial_s \alpha$ is convergent due to the integration over angles, in spite of the straightforward power counting analysis. The octupole moments are in general divergent. Therefore, applying the theorem given in the Appendix (in this case $p = 2, k = 1$) we can write that for $r \to \infty$

$$\hat{a}_s(\vec{x}) = \tfrac{1}{2}\frac{1}{r^3}\left(-\delta_{ik}+3\frac{x^i x^k}{r^2}\right)P^s_{ik}+O(r^{-4}\ln r/R)$$
$$= \frac{1}{r^3}\left(\delta_{is}-3\frac{x^i x^s}{r^2}\right)\hat{d}_i+O(r^{-4}\ln r/R). \quad (35)$$

Similarly, for $\hat{b}_s$ we consider the expansion $(r > |\vec{x}'|)$

$$|\vec{x}-\vec{x}'| = r-\frac{\vec{x}\vec{x}'}{r}+\tfrac{1}{2}\frac{1}{r}\left(\delta_{st}-\frac{x^s x^t}{r^2}\right)x'^s x'^t$$
$$+\frac{1}{3!}\left(\delta_{ks}\frac{x^j}{r^3}+\delta_{sj}\frac{x^k}{r^3}+\delta_{kj}\frac{x^s}{r^3}-3\frac{x^s x^k x^j}{r^5}\right)x'^s x'^k x'^j+O(r^{-3}). \quad (36)$$

The theorem implies that (in this case $k = -1, p = 2$) for $r \to \infty$

$$\hat{b}_s(\vec{x}) = \tfrac{1}{2}\frac{1}{r}\left(\delta_{ik}-\frac{x^i x^k}{r^2}\right)P^s_{ik}+O(r^{-2}\ln r/R)$$
$$= -\frac{1}{r}\left(\delta_{is}-\frac{x^i x^s}{r^2}\right)\hat{d}_i+O(r^{-2}\ln r/R). \quad (37)$$

The second order contribution to the magnetic field,

$$\hat{F}^{(2)}_{ik} = \partial_i \hat{A}^{(2)}_k-\partial_k \hat{A}^{(2)}_i+i[\hat{A}^{(1)}_i, \hat{A}^{(1)}_k],$$

for $r \to \infty$ behaves like

$$\hat{F}^{(2)}_{ik} = \frac{2}{r^2}(x^k \hat{d}_i-x^i \hat{d}_k)+O(r^{-3}\ln r/R). \quad (38)$$

From (38) we see that the external charge $\hat{\varrho}$ generates the color magnetic field of the monopole type. For the electric field we have for $r \to \infty$

$$\hat{E}^{(2)s} = \frac{2t}{r^3}\left(\delta_{is}-3\frac{x^i x^s}{r^2}\right)\hat{d}_s+O(r^{-4}\ln r/R). \quad (39)$$

The electric field (39) is time dependent.

Finally, in the third order we have the following asymptotic behaviour of $\hat{A}^{(3)}_i$ for $r \to \infty$

$$\hat{A}^{(3)}_i = \frac{t}{r}\left(\delta_{ik}+\frac{x^i x^k}{r^2}\right)\hat{p}_k+O(r^{-2}\ln r/R), \quad (40)$$

where

$$\hat{p}_i \equiv \frac{i}{2\pi}\int [\hat{\alpha}, \hat{a}_i] d^3\vec{x} + \frac{3}{4\pi}\int \alpha\partial_i\alpha\alpha d^3\vec{x}. \tag{41}$$

Thus, for $r \to \infty$ we obtain the electric field

$$\hat{E}^{(3)i} = \partial_0 \hat{A}_i^{(3)} = \frac{1}{r}\left(\delta_{ik} + \frac{x^i x^k}{r^2}\right)\hat{p}_k + O(r^{-2} \ln r/R) \tag{42}$$

of order r^{-1}. From (40) we also obtain for $r \to \infty$ time-dependent magnetic field of order r^{-2}, namely

$$\hat{F}_{ik}^{(3)} = \frac{2t}{r^3}(x^k \hat{p}_i - x^i \hat{p}_k) + O(r^{-3} \ln r/R). \tag{43}$$

Now, let us calculate the density of energy ε for large r. Because

$$\hat{F}_{ik} = q^2 \hat{F}_{ik}^{(2)} + q^3 \hat{F}_{ik}^{(3)} + \ldots, \qquad \hat{F}^{i0} = q\hat{E}^{(1)i} + q^2 \hat{E}^{(2)i} + q^3 \hat{E}^{(3)i} + \ldots,$$

it is clear that without the knowledge of $\hat{E}^{(4)i}$ we can calculate the energy density

$$\varepsilon = \mathrm{Tr}\,(\vec{\hat{E}}^2) + \tfrac{1}{2}\,\mathrm{Tr}\,(\hat{F}_{ik}\hat{F}_{ik})$$

only up to the order q^4. Taking into account the leading behaviour of the electric and magnetic fields, formulae (31), (32), (38), (39), (42), (43), we obtain for $r \to \infty$

$$\varepsilon = q^2 r^{-4}\,\mathrm{Tr}\,(\hat{Q}^2) + q^3 r^{-5} t f_3(\vartheta, \varphi) + 4q^4 r^{-3} f_4(\vartheta, \varphi) + \text{(nonleading terms in each order in } q), \tag{44}$$

where

$$f_3(\vartheta, \varphi) = -8\frac{x^i}{r}\,\mathrm{Tr}\,(\hat{Q}\hat{d}_i), \tag{45}$$

$$f_4(\vartheta, \varphi) = \frac{x^s}{r}\,\mathrm{Tr}\,(\hat{Q}\hat{p}_s). \tag{46}$$

The fact that the energy density is time-dependent for the time-independent external charge $\hat{\varrho}$ does not contradict any general theorem about non-Abelian gauge fields. On the other hand, the total energy

$$E = \int d^3\vec{x}\varepsilon$$

should be a constant of motion. For our perturbative solution this can be checked up to the order q^4 with the help of the formulae (15), (19), (26), which are valid for all $\vec{x}$. The fact that the energy density is time-dependent means that the solution is truly time-dependent; because ε is a gauge invariant quantity the time-dependence cannot be eliminated by gauge transformations.

It might seem that the q^4-order contribution to the energy density is nonintegrable at large r. However, it is easy to see that $\int f_4(\vartheta, \varphi) d\cos\theta d\varphi = 0$, so in fact the displayed term does not contribute to the total energy. From (44), (45) it follows that the energy density decreases (increases) in the half-space $\cos\vartheta < 0$ ($\cos\vartheta > 0$), where

$$\cos\vartheta = \vec{x}\vec{n}/r,$$

$$\vec{n} = \mathrm{Tr}\,(\hat{Q}\hat{\vec{d}})/|\mathrm{Tr}\,\hat{Q}\hat{\vec{d}}|. \tag{47}$$

Finally, let us write the total energy E in the lowest order,

$$E = q^2\,\mathrm{Tr}\int \hat{\varrho}(\vec{x})\,\frac{1}{|\vec{x}-\vec{x}'|}\,\hat{\varrho}(\vec{x}')d^3\vec{x}d^3\vec{x}'. \tag{48}$$

This expression was also obtained in the paper [1], as the lowest order contribution to the energy of the non-Abelian Coulomb solution. In [1] it was obtained for a particular $\hat{\varrho}$, namely obeying certain condition. Our approach in the $\hat{A}_0 = 0$ gauge gives (48) for generic $\hat{\varrho}$. From (48) it follows that the energy of the constructed solution is in general smaller than the energy of the so called Abelian Coulomb solution of Yang-Mills equations (see, e.g., arguments given in [1]), provided that the higher order contributions to the energy are finite. It seems that this assumption is not satisfied by our perturbative solution, see paragraph 4° of the next Section.

4. *Concluding remarks*

1° It is easy to see that the expansion in q in the $\hat{A}_0 = 0$ gauge is equivalent to the expansion in g (Yang-Mills coupling constant). Namely, rescaling $\hat{A}_k$ by

$$\hat{A}_k = \sqrt{q}\,\hat{A}'_k$$

we obtain Yang-Mills equations (9), (10) with the rescaled coupling constant $x = \sqrt{q}g$ on both sides of (9) and on the l.h.s. of (10). The coupling constant expansion for $\hat{A}'_k$ is

$$\hat{A}'_k = \sum_{n=0}^{\infty} x^{2n+1}\hat{A}_k^{(n)}.$$

This expansion gives

$$\hat{A}_k = \sum_{n=0}^{\infty} q^{n+1}g^{2n+1}\hat{A}_k^{(n)},$$

which for $g = 1$ becomes the considered expansion (11).

2° In this paper we have not discussed the problem of convergence of the perturbative series. This we leave for another investigation. However, we believe that at worst it is an asymptotic series, like the perturbative expansions in quantum field theory. In fact, we think that for smooth, quickly vanishing at inifinity, not too quickly oscillating, and close to zero $\hat{\varrho}$, the series might be just convergent, at least for small t.

3° Our solutions have the obvious shortcoming that they are approximate. However, for this price we have gained in that the perturbative solution can be constructed for quite generic $\hat{\varrho}$. It seems highly unlikely that one could obtain the explicit, exact solutions of the non-Abelian Coulomb type for generic $\hat{\varrho}$.

4° As the most interesting result of our paper we regard the observation that a generic color charge develops a color magnetic field of the monopole type (in the order q^2) and that the electric field (in the order q^3) behaves like r^{-1} for $r \to \infty$.

The latter fact probably implies that the total energy of the perturbative solution is infinite. In order to check this one has to calculate the energy in the order q^6. In principle this requires to know $\hat{\vec{E}}^{(4)}$ and $\hat{\vec{E}}^{(5)}$ explicitly. However, one can give a qualitative argument without explicit formulae. Namely, counting of powers of r in the equations for $\hat{A}_i^{(4)}$, $\hat{A}_i^{(5)}$ suggests that $\hat{A}_i^{(4)}$, $\hat{A}_i^{(5)}$ will behave like r^{-1} for $r \to \infty$. Therefore $\vec{E}^{(4)}$ and $\vec{E}^{(5)}$ will behave like r^{-n}, $n \geqslant 1$ for $r \to \infty$. Then, the only linearly divergent term in the total energy in the order q^6 is given by $\mathrm{Tr} \int d^3x(\hat{\vec{E}}^{(3)})^2$, because $\vec{E}^{(1)} \sim r^{-2}$, $\hat{\vec{E}}^{(2)} \sim r^{-3}$. So one should not expect any cancellation of the linearly divergent term — the energy will come out infinite.

The infinite energy of the weak (because $q\hat{\varrho}$ is assumed to be small) external color charge suggests that such a charge can not be created in any real experiment. The total color charge density, resultant for all kinds of colored quarks, has to be zero at each point. Otherwise, the total energy would be infinite. This corresponds rather well with the commonly believed confinement of color charges. Actually, this suggestion should be regarded with a care. The problem is that for a given $q\hat{\varrho}(\vec{x})$ one could construct also solutions (exact) with finite energy (which can be made arbitrarily close to zero), see e.g. [8]. So the question is which solution is the relevant one. Our opinion is that the relevant solution is the non-Abelian Coulomb solution. The reason is that this solution can be regarded as the gauge field created by an external charge $q\varrho(\vec{x})^{\sigma_3}/2$ embedded in the true vacuum of non-Abelian gauge theory. On the level of unquantized gauge theory, in Minkowski space-time, this vacuum should be pictured as slowly fluctuating pure gauge configurations (i.e., classical configurations with zero energy).

Finally, let us notice that a preliminary investigation shows that the above results can be extended to the case $\hat{j}_i \neq 0$. In particular, in order to avoid the infinite energy one has to have $\hat{j}_i = 0$ and $\hat{\varrho} = 0$.

APPENDIX

Here we prove the following theorem:

Theorem. Let $\hat{\sigma}(\vec{x})$ be a function regular for all $\vec{x}$ and such that for $r = |\vec{x}| > R$

$$\hat{\sigma}(\vec{x}) = \hat{c}(\vartheta, \varphi)r^{-p-4} + O(r^{-p-4}), \qquad p \geqslant 0, \tag{A1}$$

where $\hat{c}$ can depend on the spherical angles ϑ, φ. Then, for $r \to \infty$

$$\int \frac{1}{|\vec{x}-\vec{x}'|^k} \hat{\sigma}(\vec{x}')d^3\vec{x}' = \sum_{l=0}^{p} r^{-l-k} T_{i_1 \dots i_l}(\vartheta, \varphi) Q_{i_1 \dots i_l} + O(r^{-p-k-1} \ln r/R) \tag{A2}$$

for integer k such that $2 \geqslant k > -1-p$. Here

$$Q_{i_1 \dots i_l} = \int x'^{i_1} \dots x'^{i_l} \hat{\sigma}(\vec{x}') d^3\vec{x}', \tag{A3}$$

and

$$T_{i_1 \dots i_l}(\vartheta, \varphi) = \frac{1}{l!} r^{l+k} \left. \frac{\partial^l (|\vec{x}-\vec{x}'|^{-k})}{\partial x'^{i_1} \dots \partial x'^{i_l}} \right|_{\vec{x}'=0}. \tag{A4}$$

Before plunging into the proof of this theorem let us remark that the assumption (A1) implies that the first $p+1$ multipole moments of $\hat{\sigma}(\vec{x})$ ($l = 0, 1, \dots, p$ in (A3)) are finite, while some of the next ($l = p+1$) multipole moments are divergent for generic $\hat{\sigma}(\vec{x})$.

In order to prove the theorem we write

$$\frac{1}{|\vec{x}-\vec{x}'|^k} = \sum_{l=0}^{p} r^{-l-k} T_{i_1 \dots i_l}(\vartheta, \varphi) x'^{i_1} \dots x'^{i_l}$$

$$+ \frac{1}{|\vec{x}-\vec{x}'|^k} - \sum_{l=0}^{p} r^{-l-k} T_{i_1 \dots i_l}(\vartheta, \varphi) x'^{i_1} \dots x'^{i_l}. \tag{A5}$$

This identity we insert on the l.h.s. of (A2). The first term on the r.h.s. of (A5) immediately gives the term $\sum_{l=0}^{p} r^{-l-k} TQ$ present on the r.h.s. of (A2). Thus, we have to estimate the asymptotic behaviour for $r \to \infty$ of

$$\int d^3\vec{x}' \left[\frac{1}{|\vec{x}-\vec{x}'|^k} - \sum_{l=0}^{p} r^{-l-k} T_{i_1 \dots i_l}(\vartheta, \varphi) x'^{i_1} \dots x'^{i_l} \right] \hat{\sigma}(\vec{x}').$$

This integral we split into two parts,

$$\int d^3\vec{x}' [\;] \hat{\sigma}(\vec{x}') = \int^{|\vec{x}'|<R} d^3\vec{x}' [\;] \hat{\sigma}(\vec{x}') + \int_{|\vec{x}'|>R} [\;] \hat{\sigma}(\vec{x}') d^3\vec{x}'. \tag{A6}$$

In order to estimate the first term on the r.h.s. of (A6) for $r \to \infty$ we can use the standard multipole expansion, which gives the asymptotic behaviour of order r^{-p-k-1}.

In the second term on the r.h.s. of (A6) we can use the asymptotic formula (A1). Then we have ($r' = |\vec{x}'|$)

$$\int_{|\vec{x}'|>R} [\;] \hat{c}(\vartheta', \varphi') r'^{-p-4} d^3\vec{x}' = \int_{r'>R} \frac{1}{|\vec{x}-\vec{x}'|^k} r'^{-p-4} c(\vartheta', \varphi') d^3\vec{x}'$$

$$- \sum_{l=0}^{p} r^{-l-k} T_{i_1 \dots i_l}(\vartheta, \varphi) \int_{|\vec{x}'|>R} \hat{c}(\vartheta', \varphi') r'^{-p-4} x'^{i_1} \dots x'^{i_l} d^3\vec{x}'$$

$$= r^{-p-k-1} \int\limits_{|\vec{z}|>R/r} \hat{c}(\vartheta', \varphi') \frac{1}{|\vec{n}-\vec{z}|^k} |\vec{z}|^{-p-4} d^3\vec{z}$$

$$- \sum_{l=0}^{p} T_{i_1 \ldots i_l}(\vartheta, \varphi) r^{-l-p-1} \int\limits_{|\vec{z}|>R/r} \hat{c}(\vartheta', \varphi') \frac{z^{i_1} \ldots z^{i_l}}{|\vec{z}|^{p+4}} d^3\vec{z},$$

where $\vec{z} = \vec{x}/r$, $\vec{n} = \vec{x}/r$. For $r \to \infty$ we split the above integrals into two parts,

$$\int\limits_{|\vec{z}|>R/r} = \int\limits_{|\vec{z}|>1} + \int\limits_{|\vec{z}|>R/r}^{|\vec{z}|<1} . \tag{A7}$$

The first integral in (A7) is convergent for $2 \geqslant k > -p-1$, yielding the asymptotic behaviour of order r^{-p-k-1}. Let us estimate the second integral in (A7). To this end we consider the expression

$$I = \int\limits_{|\vec{z}|>R/r}^{|\vec{z}|<1} \hat{c}(\vartheta', \varphi') \frac{1}{|\vec{n}-\vec{z}|^k} |\vec{z}|^{-p-4} d^3\vec{z}$$

$$- \sum_{l=0}^{p} T_{i_1 \ldots i_l}(\vartheta, \varphi) \int\limits_{|\vec{z}|>R/r}^{|\vec{z}|<1} \hat{c}(\vartheta', \varphi') \frac{z^{i_1} \ldots z^{i_l}}{|\vec{z}|^{p+4}} d^3\vec{z}.$$

Because now $|\vec{z}| < 1$ we can use the expansion

$$\frac{1}{|\vec{n}-\vec{z}|^k} = 1 + \sum_{l=1}^{\infty} T_{i_1 \ldots i_l}(\vartheta, \varphi) z^{i_1} \ldots z^{i_l}.$$

Thus,

$$I = \sum_{l=p+1}^{\infty} \int\limits_{|\vec{z}|>R/r}^{|\vec{z}|<1} \hat{c}(\vartheta', \varphi') \frac{z^{i_1} \ldots z^{i_l}}{|\vec{z}|^{p+4}} d^3\vec{z}$$

$$= \sum_{l=p+1}^{\infty} \int\limits_{z>R/r}^{1} z^{l-p-2} dz \int d\cos\vartheta' d\varphi' \hat{c}(\vartheta', \varphi') \frac{z^{i_1} \ldots z^{i_l}}{z^l},$$

where $z = |\vec{z}|$. The integral over angles is finite and nonzero for generic $\hat{\varrho}(\vec{x})$ (i.e. for generic $\hat{c}(\vartheta, \varphi)$). The integral over z gives

$$\frac{1}{l-p-1}\left[1-\left(\frac{R}{r}\right)^{l-p-1}\right] \quad \text{for} \quad 1 > p+1,$$

and

$$\ln r/R \text{ for } l = p+1.$$

Thus, the asymptotic behaviour of the integral (A6) is of the order

$$r^{-k-p-1} \ln r/R.$$

This ends the proof of the theorem.

The above theorem can be easily generalized to $\hat{\sigma}$ behaving at $r \to \infty$ in another manner, e.g.,

$$\hat{\sigma}(\vec{x}) = \hat{c}(\vartheta, \varphi) r^{-p-4} \ln r/R.$$

The asymptotic behaviour in this case can be obtained by performing calculations along the lines of the above proof.

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