HELICITY CROSSING RELATIONS BETWEEN THE TWO-BODY SCATTERING AND THREE-BODY DECAY CHANNELS

By J. Jurkiewicz

Institute of Physics, Jagellonian University, Cracow*

(Received February 25, 1970)

The helicity crossing relations between the two-body scattering channels and the three-body decay channel are derived under the assumption that analytic properties of spinor amplitudes allow such a crossing. Each relation contains two or three crossing angles.

1. Crossing relations

The crossing relations of the helicity amplitudes for the scattering reactions have been discussed by a number of authors (*cf.* Refs [1–13]). All these papers considered the scattering processes, including two-body reactions [1–12] and many-body production [13].

In the present paper we discuss an important case of the crossing relations between the three-body decay and the two-body scattering channels.

Among the works studying the crossing between the scattering channels, the most complete is the paper by Cohen-Tannoudji, Morel and Navelet [5]. It contains the rigorous derivation of the crossing relations, including the sign of the crossing matrix. The authors start from the relations between the helicity and spinor amplitudes and derive the crossing relations using the analyticity properties of the spinor amplitudes, proved by Bros, Epstein and Glaser [15].

Here we discuss the crossing relations of the helicity amplitudes between the two-body scattering reactions in the channels

$$s\colon\ 2, 3 \rightarrow 0, \bar{1},$$

$$t\colon\ 3, 1 \rightarrow 0, \bar{2},$$

$$u\colon\ 1, 2 \rightarrow 0, \bar{3}$$

and the three-body decay

$$d\colon\ 0 \rightarrow 1, 2, 3$$

in the case when $m_0 > m_1+m_2+m_3$ (Fig. 1).

* Address: Instytut Fizyki UJ, Kraków, Reymonta 4, Polska.

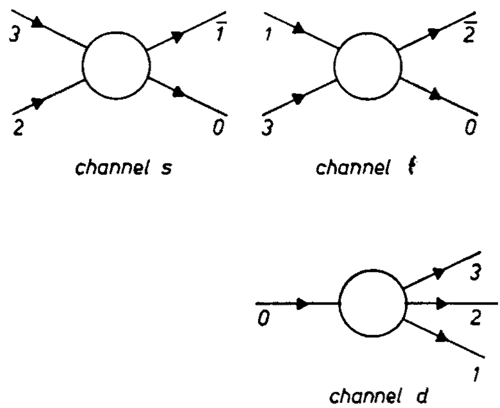

Fig. 1. The channels for two-body scattering and three-body decay

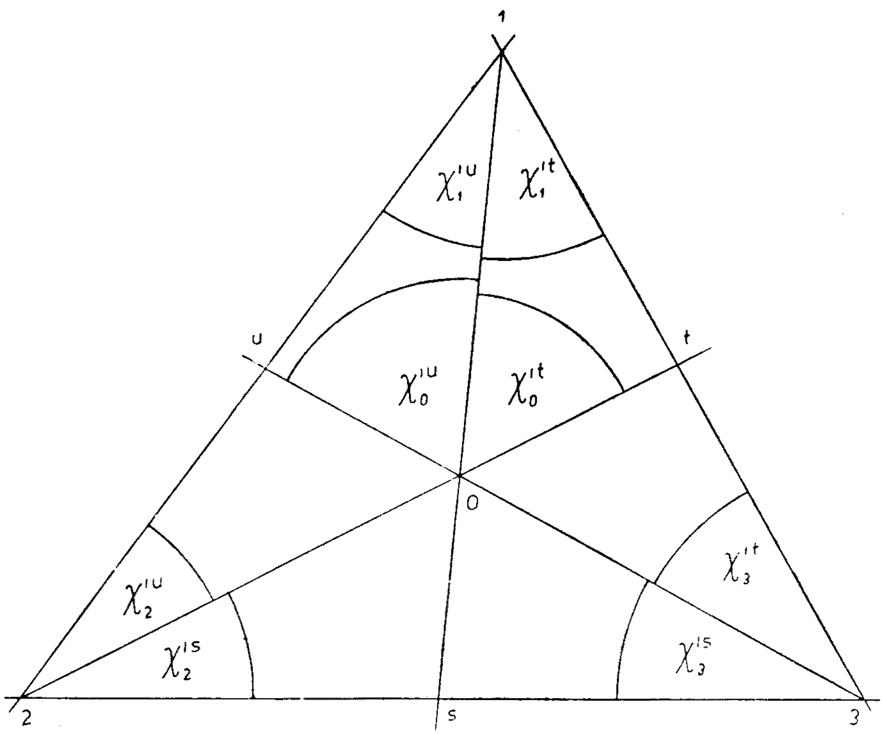

Fig. 2. Crossing angles represented in velocity space

Our argument is a simple generalization of the method of Cohen-Tannoudji *et al.* However, since the crossing properties of the spinor amplitudes were proved only for the two-body scattering reactions [15], we have to assume that corresponding spinor amplitudes are analytic in domains big enough to allow crossing to the d-channel from the s-, t- and

u-channels. Under this assumption and using the methods of Ref. [5] we obtained the crossing relations between the d-channel and the s-, t- and u-channel helicity amplitudes (continued analytically to the d-channel physical region). They read

$$\begin{aligned} M^d_{\lambda_1\lambda_2\lambda_3,\,\lambda_0} &= (-1)^{\Sigma_s}(-1)^{2s_0+2s_2}e^{i\pi(\lambda_2+\lambda_3-\lambda_0)}\sum \delta_{\lambda_0'\lambda_0}\delta_{\lambda_1'\lambda_1}\times \\ &\qquad \times d^{s_2}(\chi_2^s)_{\lambda_2'\lambda_2}\delta^{s_3}(\chi_3^s)_{\lambda_3'\lambda_3}M^s_{\lambda_0'\lambda_1'\lambda_2'\lambda_3'} \\ &= (-1)^{\Sigma_t}(-1)^{2s_0+2s_1}e^{i\pi(\lambda_1+\lambda_3-\lambda_0)}\sum d^{s_0}(\chi_0^t)_{\lambda_0'\lambda_0}\times \\ &\qquad \times d^{s_1}(\chi_1^t)_{\lambda_1'\lambda_1}\delta_{\lambda_2'\lambda_2}d^{s_3}(\chi_3^t)_{\lambda_3'\lambda_3}M^t_{\lambda_0'\lambda_2'\lambda_3'\lambda_1'} \\ &= (-1)^{\Sigma_u}(-1)^{2s_0+2s_3}e^{i\pi(\lambda_1+\lambda_2-\lambda_0)}\sum d^{s_0}(\chi_0^u)_{\lambda_0'\lambda_0}\times \\ &\qquad \times d^{s_1}(\chi_1^u)_{\lambda_1'\lambda_1}d^{s_2}(\chi_2^u)_{\lambda_2'\lambda_2}\delta_{\lambda_3'\lambda_3}M^u_{\lambda_0'\lambda_3'\lambda_1'\lambda_2'}. \end{aligned} \tag{1.1}$$

Our notation follows that of Ref. [5]. The summation goes over helicities λ_0', λ_1', λ_2' and λ_3'. The crossing angles are the functions of the Mandelstam invariants s, t and u. Their explicit form is given in the Appendix.

It is possible to illustrate these angles geometrically on the diagram in the velocity space (Fig. 2) [14].

In the following three Sections we give a sketch of the proof of the formula (1.1). As in Ref. [5] the proof consists of the three elements:

1. Writing down the crossing relations for spinor amplitudes,
2. Construction of the helicity amplitudes from the spinor amplitudes,
3. Analytic continuation of the two-body helicity amplitudes and calculation of the crossing angles.

In the Appendix we collect the values of the crossing angles.

2. *Crossing relations for spinor amplitudes*

We construct the spinor amplitudes for a three-body decay in the similar way as for the two-body scattering [6]

$$\begin{aligned} &\mathcal{M}^d_{A_1A_2A_3,\,A_0}(p_1, p_2, p_3, p_0) \\ &= \langle p_1s_1\dot{A}_1, p_2s_2\dot{A}_2, p_3s_3\dot{A}_3|T|p_0s_0A_0\rangle. \end{aligned} \tag{2.1}$$

Assuming that amplitudes $\mathcal{M}^s_{A_0A_1,\,A_2A_3}$, $\mathcal{M}^t_{A_0A_2,\,A_3A_1}$ and $\mathcal{M}^u_{A_0A_3,\,A_1A_2}$ (describing the two-body scattering) are analytic in domains big enough to allow crossing from the corresponding channels to the d-channel, amplitude $\mathcal{M}^d_{A_1A_2A_3,\,A_0}$ is connected with these amplitudes. One has

$$\begin{aligned} \mathcal{M}^d_{A_1A_2A_3,\,A_0}(p_1, p_2, p_3, p_0) &= (-1)^{\Sigma_s}\mathcal{M}^s_{A_0A_1A_2A_3}(p_0, -p_1, p_2, p_3) \\ &= (-1)^{\Sigma_t}\mathcal{M}^t_{A_0A_2,\,A_3A_1}(p_0, -p_2, p_3, p_1,) \\ &= (-1)^{\Sigma_u}\mathcal{M}^u_{A_0A_3,\,A_1A_2}(p_0, -p_1, p_2, p_3). \end{aligned} \tag{2.2}$$

TABLE I

0 1 2 3	Σ_s Σ_t Σ_u
f f f f	1 1 1
f f b b	1 1 1
f b f b	1 1 1
f b b f	1 1 1
b f f b	0 1 0
b f b f	0 1 1
b b f f	0 0 1
b b b b	0 0 0

f - fermion, b - boson

In these formulae factor $(-1)^{\Sigma}(\Sigma = 0, 1)$ arises from the change of order of the spinor indices in the spinor amplitudes and depends on the kind of particles involved in the reaction. This dependence is given in Table I.

3. *Construction of helicity amplitudes*

Our definition of the two-body helicity amplitudes is taken from [5]. We follow their method to define the three-body decay helicity amplitudes. Our definition consists of the following elements:

1. We define the helicity frames for one- and three-particle states being *in*- and *out*-states in the d-channel,

2. Using Lorentz transformations, which transform the standard frame $(\mathring{t}, \mathring{n}_1, \mathring{n}_2, \mathring{n}_3)$ into these frames, we express the helicity amplitudes in terms of the spinor amplitudes.

The Lorentz transformation is completely defined by its action on three four-vectors. Therefore in the definition of the helicity frames we specialize only three basis vectors.

Let p_i and m_i be the four-momentum and mass of the i-th particle. For $i = 0 \ldots 3$ we define basis vectors 0 and 2 in the following way:

$$t(p_i) = p_i/m_i \tag{3.1}$$

parallel to p_i and

$$n_2(p_i) = w_d \tag{3.2}$$

orthogonal to the reaction plane. Here

$$w_{d\mu} = 2\varepsilon_{\mu\nu\varrho\sigma} p_1^{\nu} p_2^{\varrho} p_3^{\sigma} / [\Phi(s, t)]^{1/2}.$$

($\Phi(s, t) = 0$ is a boundary equation of the physical region. Inside this region $\Phi(s, t) > 0$, we choose here, as one usually does, the positive determination for $[\Phi(s, t)]^{1/2}$.)

The helicity four-vectors for three-particle state [16] are $(i = 1, 2, 3)$

$$n_3(p_i) = h(i) = -\frac{m_i^2 p_0 - (p_0 \cdot p_i) p_i}{m_i [(p_0 \cdot p_i)^2 - m_i^2 m_0^2]^{1/2}}, \tag{3.3}$$

$$p_0 = p_1 + p_2 + p_3.$$

For one-particle state (particle 0) we choose $h(0)$ to be

$$n_3(p_0) = h(0) = \frac{m_0^2 p_1 - (p_0 \cdot p_1) p_0}{m_0[(p_0 \cdot p_1)^2 - m_1^2 m_0^2]^{1/2}} \,. \tag{3.4}$$

Using functions $\mathscr{S}_{ij}$, $\mathscr{T}_{ij}$, $\mathscr{U}_{ij}$ defined in Ref. [5] we can rewrite equations (3.3) and (3.4) to obtain

$$h(0) = -2\, \frac{m_0^2 p_1 - (p_0 \cdot p_1) p_0}{m_0 \mathscr{S}_{01}} \,,$$

$$h(1) = 2\, \frac{m_1^2 p_0 - (p_0 \cdot p_1) p_1}{m_1 \mathscr{S}_{01}} \,, \tag{3.4'}$$

$$h(2) = 2\, \frac{m_2^2 p_0 - (p_0 \cdot p_2) p_2}{m_2 \mathscr{T}_{02}} \,,$$

$$h(3) = 2\, \frac{m_3^2 p_0 - (p_0 \cdot p_3) p_3}{m_3 \mathscr{U}_{03}} \,. \tag{3.3'}$$

Let $\mathscr{L}(i)$ be a Lorentz transformation such that

$$\mathscr{L}(i) \begin{Bmatrix} \overset{\circ}{t} \\ \overset{\circ}{n}_2 \\ \overset{\circ}{n}_3 \end{Bmatrix} = \begin{Bmatrix} t(p_i) \\ w_d \\ h(i) \end{Bmatrix} , \quad (i = 0 \ldots 3). \tag{3.5}$$

With each of these transformations one can connect the two-dimensional matrix $L(i)$ such that

$$p' \cdot \sigma = L(i) p \cdot \sigma L^{+}(i), \tag{3.6}$$

where $p' = \mathscr{L}(i) p$.

Using these matrices we express the decay helicity amplitudes in terms of spinor amplitudes ($\varepsilon = i\sigma_2$)

$$M^d_{\lambda_1\lambda_2\lambda_3,\lambda_0}(s,t,u) = D^{s_0}(L(0))_{A_0\lambda_0} D^{s_1}(L(1)\varepsilon)_{A_1\lambda_1} D^{s_2}(L(2)\varepsilon)_{A_2\lambda_2} \times$$
$$\times D^{s_3}(L(3)\varepsilon)_{A_3\lambda_3} \mathscr{M}^d_{A_1A_2A_3,A_0}(p_1, p_2, p_3, p_0). \tag{3.7}$$

4. Analytic continuation

For the two-body scattering helicity amplitudes expressions similar to (3.7) can be easily constructed [5]. For instance in the s-channel

$$M^s_{\lambda_0\lambda_1,\lambda_2\lambda_3}(\mathrm{s}, \mathrm{t}, \mathrm{u}) = D^{s_0}(L_s(0)\varepsilon)_{A_0\lambda_0} D^{s_1}(L_s(1)\varepsilon)_{A_1\lambda_1} D^{s_2}(L_s(2))_{A_2\lambda_2} \times$$
$$\times D^{s_3}(L(3))_{A_3\lambda_3} \mathscr{M}^s_{A_0A_1,A_2A_3}(q_0, q_1, q_2, q_3). \tag{4.1}$$

As it was shown in [5] this expression is valid also after the analytic continuation is performed. From (3.7) and (4.1) and using explicit expressions for $D^s(\varepsilon)$ we get for $M^d_{\lambda_1\lambda_2\lambda_3,\,\lambda_0}$

$$M^d_{\lambda_1\lambda_2\lambda_3,\,\lambda_0} = (-1)^{\Sigma_s}(-1)^{s_2+\lambda_2+s_3+\lambda_3+\lambda_1-\lambda_1'-s_0-\lambda_0'} D^{s_0}(L_s^{c-1}(0)L(0))_{-\lambda_0'\lambda_0} D^{s_1}(L_s^{c-1}(1)L(1))_{-\lambda_1'-\lambda_1} \times$$
$$\times D^{s_2}(L_s^{c-1}(2)L(2))_{\lambda_2'-_2\lambda} D^{s_3}(L_s^{c-1}(3)L(3))_{\lambda_3'-\lambda_3} M^{sc}_{\lambda_0'\lambda_1',\,\lambda_2'\lambda_3'}. \tag{4.2}$$

Index c denotes the crossed quantities (continued analytically to the d-channel physical region).

Formulae connecting amplitude $M^d_{\lambda_1\lambda_2\lambda_3,\,\lambda_0}$ with amplitudes $M^{tc}_{\lambda_0'\lambda_2',\,\lambda_3'\lambda_1'}$ and $M^{uc}_{\lambda_0'\lambda_3',\,\lambda_1'\lambda_2'}$ are similar.

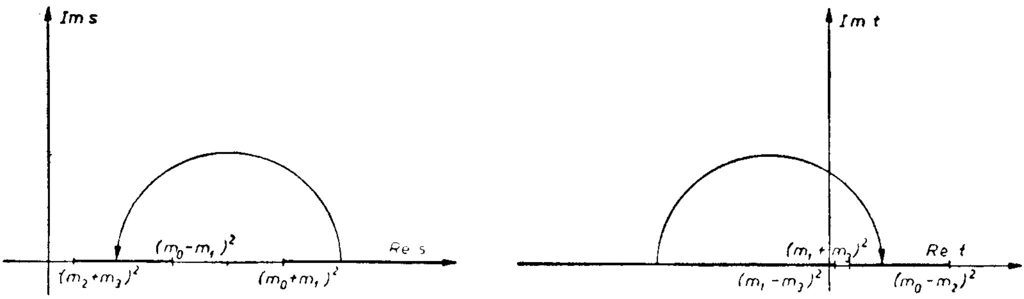

Fig. 3. Path of analytic continuation from the s-channel physical region to the d-channel physical region

In these formulae $D^s(L_r^{c-1}(i)L(i))$ ($r = s, t, u$) can be determined up to a sign by considering the basis vectors of the associated helicity frames.

We choose the following path connecting s- and d-channel physical regions (Fig. 3)

$$s = (2n+1)a + 2nae^{i\varphi},$$
$$t = -(n-1)a - nae^{-i\varphi}, \tag{4.3}$$

where $0 < \varphi < \pi$,

$$a = \frac{1}{3}(m_0^2+m_1^2+m_2^2+m_3^2),$$

n — a big positive real number.

This path connects the point $s = (4n+1)a$, $t = u = -(2n-1)a$ in S^+ with the point $s = t = u = a$ in D. Such a path omits all the singularities of the amplitudes $M^s_{\lambda_0'\lambda_1',\,\lambda_2'\lambda_3'}$ and of the transformation matrices $L_s(i)$ and therefore leads to the unique determination of the crossed amplitudes. One can verify that using such a path one ends with a positive determination of $[\Phi(s, t)]^{1/2}$ (assuming Φ-plane has a cut along the positive real axis). The functions $\mathcal{S}_{ij}, \mathcal{T}_{ij}$ and $\mathcal{U}_{ij}$ do not have left the cut planes, where they have been defined.

Analogous paths can be constructed to join T^+ and U^+ with D.

After the analytic continuation the s, t and u helicity frames are (index c denotes the crossed quantities):

Basis vector 0:

$$t^c(p_i) = \begin{cases} -p_i/m_i & \text{for crossed particles} \\ p_i/m_i & \text{for uncrossed particles.} \end{cases} \tag{4.4}$$

Basis vector 2:

$$n_2^c(p_i) = w_s^c = w_t^c = w_u^c = -w_d. \tag{4.5}$$

(Four-vectors w_s, w_t and w_u are defined as in Ref. [5]),

Basis vector 3:

$$n_3^c(p_i) = h_r^c(i),\ (r = s, t, u) \tag{4.6}$$

where $h_r^c(i)$ is obtained from the helicity four-vector $h_r(i)$ in the corresponding channel by changing p_i/m_i for $-p_i/m_i$ for crossed particles in the definition of this four-vector.

Transformations $\mathscr{L}_r^{c-1}(i)\mathscr{L}(i)$ $(r = s, t, u)$ can be proved [5] to be the real rotations, namely

$$\mathscr{L}_r^{c-1}(i)\mathscr{L}(i) = \begin{cases} R_2(\chi_i'^r) \text{ for crossed particles} \\ R_3(\pi)\, R_2(\chi_i'^r) \text{ for uncrossed particles.} \end{cases} \tag{4.7}$$

We calculate the crossing angles $\chi_i'^r(-\pi < \chi_i'^r < \pi)$ from the following formulae

$$\cos\, \chi_i'^r = \varepsilon_i h_r^c(i) h(i),$$

$$\sin\, \chi_i'^r = n_{1r}^c(i) h(i), \tag{4.8}$$

$$\varepsilon_i = \begin{cases} -1 \text{ for crossed particles} \\ +1 \text{ for uncrossed particles.} \end{cases}$$

It can be easily seen from (3.6) that matrices L and $-L$ correspond to the same Lorentz transformation $\mathscr{L}$. Therefore the determination of the signs η_s, η_t and η_u of the crossing matrices is another problem and must be discussed separately. Using methods analogous as in Ref. [5] we find the signs η_s, η_t and η_u to be

$$\eta_s = (-1)^{2s_0+2s_3},$$

$$\eta_t = (-1)^{2s_0+2s_1}, \tag{4.9}$$

$$\eta_u = (-1)^{2s_0+2s_2}.$$

We redefine the angles $\chi_i'^r$ in order to tidy up the indices in the crossing matrix. Expressions for χ_i^r through the angles $\chi_i'^r$ are given in the Appendix.

This completes the proof of the formulae (1.1).

APPENDIX

In the formulae for the crossing matrices obtained from (4.2) we get expressions of the form $d^s(\chi')_{-\lambda'\lambda}$, $d^s(\chi')_{-\lambda'-\lambda}$ and $d^s(\chi')_{\lambda'-\lambda}$. In order to rearrange the helicity indices in these expressions we redefine the crossing angles, using identities such as

$$d^s(\pi-\chi')_{\lambda'\lambda} = (-1)^{s+\lambda'} d^s(-\chi')_{-\lambda'\lambda}, \tag{A.1}$$

$$d^s(\chi')_{\lambda'\lambda} = (-1)^{\lambda'-\lambda} d^s(\chi')_{-\lambda'-\lambda}.$$

Angles χ_i^r can be expressed through the angles $\chi_i'^r$ (Tab. II).

TABLE II

Angles χ_i^r expressed through the angles $\chi_i'^r$

r \ i	0	1	2	3
s	$\pi-\chi'$	χ'	$-\pi-\chi'$	$-\pi-\chi'$
t	$\pi-\chi'$	$-\pi-\chi'$	χ'	$-\pi-\chi'$
u	$\pi-\chi'$	$-\pi-\chi'$	$-\pi-\chi'$	χ'

The analytic expressions for $\cos\chi_i^r$ and $\sin\chi_i^r$ are given in Table III.

TABLE III

The analytic expressions for $\cos\chi_i^r$ and $\sin\chi_i^r$

r	i	$\cos\chi_i^r$	$\sin\chi_i^r$
s	0	1	0
s	1	1	0
s	2	$-\dfrac{[s+m_2^2-m_3^2]\,[t+m_2^2-m_0^2]+2m_2^2\,[m_0^2+m_3^2-m_1^2-m_2^2]}{\mathscr{S}_{23}\mathscr{T}_{02}}$	$-\dfrac{2m_2[\Phi]^{1/2}}{\mathscr{S}_{23}\mathscr{T}_{02}}$
s	3	$-\dfrac{[s+m_3^2-m_2^2]\,[u+m_3^2-m_0^2]+2m_3^2\,[m_0^2+m_2^2-m_1^2-m_3^2]}{\mathscr{S}_{23}\mathscr{U}_{03}}$	$\dfrac{2m_3[\Phi]^{1/2}}{\mathscr{S}_{23}\mathscr{U}_{03}}$
t	0	$\dfrac{[s+m_0^2-m_1^2]\,[t+m_0^2-m_2^2]-2m_0^2[m_0^2+m_3^2-m_1^2-m_2^2]}{\mathscr{S}_{01}\mathscr{T}_{02}}$	$-\dfrac{2m_0[\Phi]^{1/2}}{\mathscr{S}_{01}\mathscr{T}_{02}}$
t	1	$-\dfrac{[s+m_1^2-m_0^2]\,[t+m_1^2-m_3^2]+2m_1^2\,[m_0^2+m_3^2-m_1^2-m_2^2]}{\mathscr{S}_{01}\mathscr{T}_{31}}$	$\dfrac{2m_1[\Phi]^{1/2}}{\mathscr{S}_{01}\mathscr{T}_{31}}$
t	2	1	0
t	3	$-\dfrac{[t+m_3^2-m_1^2]\,[u+m_3^2-m_0^2]+2m_3^2\,[m_0^2+m_1^2-m_2^2-m_3^2]}{\mathscr{U}_{03}\mathscr{T}_{31}}$	$-\dfrac{2m_3[\Phi]^{1/2}}{\mathscr{U}_{03}\mathscr{T}_{31}}$
u	0	$\dfrac{[s+m_0^2-m_1^2]\,[u+m_0^2-m_3^2]-2m_0^2\,[m_0^2+m_2^2-m_1^2-m_3^2]}{\mathscr{S}_{01}\mathscr{U}_{03}}$	$\dfrac{2m_0[\Phi]^{1/2}}{\mathscr{S}_{01}\mathscr{U}_{03}}$
u	1	$-\dfrac{[s+m_1^2-m_0^2]\,[u+m_1^2-m_2^2]+2m_1^2[m_0^2+m_2^2-m_1^2-m_3^2]}{\mathscr{S}_{01}\mathscr{U}_{12}}$	$-\dfrac{2m_1[\Phi]^{1/2}}{\mathscr{T}_{01}\mathscr{U}_{12}}$
u	2	$-\dfrac{[t+m_2^2-m_0^2]\,[u+m_2^2-m_1^2]+2m_2^2\,[m_0^2+m_1^2-m_2^2-m_3^2]}{\mathscr{T}_{02}\mathscr{U}_{12}}$	$\dfrac{2m_2[\Phi]^{1/2}}{\mathscr{T}_{02}\mathscr{U}_{12}}$
u	3	1	0

The author would like to thank Dr A. Kotański for suggesting this investigation and many helpful discussions. He is also grateful to Drs A. Białas and K. Zalewski for critical reading of the manuscript and valuable comments.

REFERENCES

[1] T. L. Trueman, G. C. Wick, *Ann. Phys.*, **26**, 322 (1964).
[2] J. Bjørneboe, and Z. Koba, *Nuclear Phys.*, **B7**, 53 (1968).
[3] I. J. Muzinich, *J. Math. Phys.*, **5**, 1481 (1964).
[4] R. C. Brunet, R. W. Childers, *Phys. Rev.*, **150**, 1280 (1966).
[5] G. Cohen-Tannoudji, A. Morel, H. Navelet, *Ann. Phys.*, **46**, 239 (1968).
[6] M. S. Marinov, V. I. Roginsky, *Nuclear Phys.*, **49**, 251 (1963).
[7] G. Immirzi, *Nuovo Cimento*, **58**, 619 (1968).
[8] A. Chakrabarti, preprint Ecole Polyt. A118.11 68/16 (1968).
[9] Y. Hara, preprint Tokyo University TUEP-68-3 (1968).
[10] A. Białas, B. E. Y. Svensson, *Nuovo Cimento*, **42**, 672 (1966).
[11] B. E. Y. Svensson, *Ark. Fys.* **33**, 93 (1966).
[12] E. L. Surkov, *Yadernaya Fizika*, **1**, 1113 (1965).
[13] A. Capella, *Nuovo Cimento*, **56A**, 701 (1968).
[14] G. C. Wick, *Ann. Phys.*, **18**, 65 (1962).
[15] J. Bros, H. Epstein, V. Glaser, *Commun. Math. Phys.*, **1**, 240 (1968).
[16] P. Moussa, R. Stora, in *Methods in Subnuclear Physics*, Vol. II, p. 265–340, Gordon and Breach, N. Y. 1968.