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Steady-state bifurcation of FHN-type oscillator on a square domain

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Abstract. The Turing patterns of reaction-diffusion equations defined over a square region are more complex because of the D_4-symmetry of the spatial region. This leads to the occurrence of multiple equivariant Turing bifurcations. In this paper, taking the FHN model as an example, we give a explicit calculation formula of normal form for the simple and double Turing bifurcation of the reaction-diffusion equation with Dirichlet boundary conditions and defined on a square space, and we also obtain a method for the calculation of the existence of spatially inhomogeneous steady-state solutions. This paper provides a theoretical basis for exploring and predicting the pattern formation of spatial multimode interaction.

Keywords: FitzHugh–Nagumo (FHN) system, reaction-diffusion, steady-state bifurcations, D_4-symmetry, reduced equations.

1 Introduction

The well-known FitzHugh–Nagumo (FHN) system with cubic nonlinearity was derived as a simplified model of the famous Hodgkin–Huxley (HH) model [8] by FitzHugh [5] and Nagumo et al. [14]. We consider the FHN model, which can capture most of the characteristic properties of neuron cells dynamics. The model consists of two equations describing fast and slow dynamics of the system, and it is given as follows:

$$\varepsilon\dot{u} = af(u) - v, \qquad \dot{v} = u - \delta v,$$

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where $\varepsilon > 0$, $\delta > 0$ are small parameters; u represents the membrane potential, v represents the recovery variable, namely, u, v represent the neural neurons, and $f \in C^4$ with $f(0) = 0$, $f'(0) = 1$.

Mathematical models with diffusion have received increasing attention in the pattern formation community. Since Turing [22] famously statement that instability is caused by diffusion, a large number of reaction-diffusion systems have been used to simulate the instability in the formation of biological models known as Turing instability. So far, diffusion-driven instability mechanisms have been widely used in the study of various specific problems in many fields due to the formation of models [1, 16, 18, 20, 23, 25, 26]. It is worth mention that in [23], Wei and his coworkers discussed steady-state bifurcations for a glycolysis model in biochemical reaction based on bifurcation theory, Lyapunov–Schmidt method, and singularity theory. The importance of diffusion versus patterns has also been widely discussed in [3, 6, 9, 15, 17, 19, 27] through theoretical analysis and numerical experiments. In these papers the formations of spatial and temporal patterns are studied under the premise of sufficient nonlinearity of dynamics.

In biological neural network system, due to the inhomogeneity of cell concentration, diffusion exists widely. Therefore, it is necessary to study the diffusion kinetics of FitzHugh–Nagumo model and the resulting Turing instability. In this paper, we consider the effect of diffusion on the FitzHugh–Nagumo model as follows:

$$\begin{aligned}
\varepsilon\frac{\partial \tilde{u}}{\partial t} &= \tilde{d_1}\left(\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}\right) + \tilde{a} f(\tilde{u}) - \tilde{v}, \quad (\tilde{x}, \tilde{y}) \in \tilde{\varOmega},\\
\frac{\partial \tilde{v}}{\partial t} &= d_2\left(\frac{\partial^2 \tilde{v}}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{v}}{\partial \tilde{y}^2}\right) + \tilde{u} - \delta\tilde{v}, \quad (\tilde{x}, \tilde{y}) \in \tilde{\varOmega}.
\end{aligned} \tag{1}$$

Boundary conditions have sophisticated influence on spatial structure of solutions of reaction diffusion equations. In this paper, we consider a square domain $\varOmega$ with homogeneous Dirichilet boundary condition

$$\tilde{u}(\tilde{x}, \tilde{y}, t) = 0, \quad \tilde{v}(\tilde{x}, \tilde{y}, t) = 0, \quad (\tilde{x}, \tilde{y}) \in \partial\tilde{\varOmega}. \tag{2}$$

Writting $a = \tilde{a}/\varepsilon$, $b = 1/\varepsilon$, $d_1 = \tilde{d_1}/\varepsilon$, then system (1) can be rewritten as

$$\begin{aligned}
\frac{\partial \tilde{u}}{\partial t} &= d_1\left(\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}\right) + a f(\tilde{u}) - b\tilde{v}, \quad (\tilde{x}, \tilde{y}) \in \tilde{\varOmega},\\
\frac{\partial \tilde{v}}{\partial t} &= d_2\left(\frac{\partial^2 \tilde{v}}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{v}}{\partial \tilde{y}^2}\right) + \tilde{u} - \delta\tilde{v}, \quad (\tilde{x}, \tilde{y}) \in \tilde{\varOmega}.
\end{aligned} \tag{3}$$

Here $\tilde{\varOmega} = [0, l] \times [0, l]$.

To simplify the discussions, we incorporate explicitly the length l into the unit square $\varOmega = [0, 1] \times [0, 1]$ by the transformation $\tilde{x} = lx$, $\tilde{y} = ly$, and (3) and (2) into

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{d_1}{l^2}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + a f(u) - bv, \quad (x, y) \in \varOmega,\\
\frac{\partial v}{\partial t} &= \frac{d_2}{l^2}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + u - \delta v, \quad (x, y) \in \varOmega,
\end{aligned} \tag{4}$$

with

$$u(x,y,t)=0,\quad v(x,y,t)=0,\quad (x,y)\in\partial\Omega. \tag{5}$$

The symmetric properties of Ω have to be considered when bifurcations of a reaction-diffusion on the two-dimensional space square region; see [12]. The studies of symmetry in influence of boundary conditions upon the solution structure of partial differential equation have been done by many scientists. Z. Mei and his collaborators have done a lot of research in this field. For example, in [11] the authors studied the bifurcations of a semilinear elliptic problem on the unit square with the Dirichlet boundary conditions at corank-2 bifurcation points. They show the existence of bifurcating solution branches and their parameterizations via a nonsingular enlarged problem. We would also like to mention that many kinds of bifurcations of reaction diffusion equation have been investigated in detail by Mei; see [2, 4, 13].

The theory of Lyapunov–Schmidt reduction is an important tool to study nonlinear problems [10, 21, 24, 28]. For example, in [10], Guo and his coworkers obtained the existence of spatially nonhomogeneous steady-state solution by applying Lyapunov–Schmidt reduction method. Moreover, they also considered the stability and nonexistence of Hopf bifurcation at the spatially nonhomogeneous steady-state solution with the changes of a specific parameter. In [28], steady-state bifurcations arising from the reaction-diffusion equations are investigated. Using the Lyapunov–Schmidt reduction on a square domain, a simple and a double steady-state bifurcation caused by the symmetry of spatial region is obtained.

The focus of this work is to describe the dynamic properties for system (4) with homogeneous Dirichilet boundary conditions on a square domain. Using the symmetric theory of bifurcation and the Lyapunov–Schmidt method, we study in this paper how the symmetric properties of domain Ω with homogeneous Dirichilet boundary condition change the nontrivial solution of reaction-diffusion equations. An outline of this paper is as follows. In Section 2, we describes the stability of the constant steady-state solution $(0,0)$ and the symmetry of (4) and (5). In Section 3 the existence of nontrivial solutions is reduced to algebraic equations via the well-known Lyapunov–Schmidt method. We derive the bifurcation scenario at simple and double-bifurcation point. For steady/steady-state mode interactions caused by $b(\lambda_j)=b(\lambda_s)$ for some $j\neq s$, three types steady/steady-state mode interactions are considered, which also caused by the symmetry of Ω in Section 4. We illustrate simple and double bifurcation by some numerical simulation in Section 5. When the homogeneous steady state bifurcates to spatial patterns at a simple eigenvalue, the system supports a pattern such as square. On the other hand, when the bifurcation occurs via a double eigenvalue, more complex patterns arise due to the interaction of different modes (for this reason, they are called mixed mode patterns).

2 Stability of the constant steady-state solution $(0,0)$

Let Ω be spatial region, and let $C^{2,s}(\Omega)$ be the space of 2-times differentiable functions u on the closure of Ω such that u and its derivatives are Hölder continuous with the exponent $s\in(0,1)$. We define $X=\{u\in C^2(\Omega);\ u|_{\partial\Omega}=0\}$ and $Y=C^{0,s}(\Omega)$ endowed with the Hölder norms $\|\cdot\|_{2,s}$ and $\|\cdot\|_{0,s}$, respectively. We rewrite (4) as an operator equation

$$\frac{\partial U}{\partial t} = \varPhi(U, b),$$

where $U = (u, v)$, and the mapping $\varPhi : X \times \mathbb{R} \to Y$ is defined by

$$\varPhi(U) = \begin{pmatrix} \frac{d_1}{l^2}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \\ \frac{d_2}{l^2}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \end{pmatrix} + \begin{pmatrix} af(u) - bv \\ u - \delta v \end{pmatrix}. \tag{6}$$

It is clear that $\varPhi(0) = 0$. Differentiating $\varPhi$ with respect to U at $U_0 = (0, 0)$, we obtain the linearization $\mathcal{L}$ of $\varPhi$,

$$\mathcal{L} = \begin{pmatrix} \frac{d_1}{l^2}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) & 0 \\ 0 & \frac{d_2}{l^2}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \end{pmatrix} + \begin{pmatrix} af'(0) & -b \\ 1 & -\delta \end{pmatrix}.$$

To examine the spectrum of $\mathcal{L}$, we observe the direct sum

$$X = \sum_{m,n=1}^{\infty} X_{m,n}, \quad X_{m,n} = \left\{ \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \sin(m\pi x)\sin(n\pi y);\ c_1, c_2 \in R \right\},$$

and the $\mathcal{L}$ maps $X_{m,n}$ into itself. Further more, the restriction of $\mathcal{L}$ in the subspace $X_{m,n}$ is a 2×2 matrix

$$M_{m,n} = \mathcal{L}|_{X_{m,n}} = \begin{pmatrix} -\frac{d_1}{l^2}(m^2+n^2)\pi^2 + af'(0) & -b \\ 1 & -\frac{d_2}{l^2}(m^2+n^2)\pi^2 - \delta \end{pmatrix}, \tag{7}$$

where $m, n = 1, 2, \dots$.

The eigenvalues of $\mathcal{L}$ consist of those of $M_{m,n} \in \mathbb{R}^{2\times 2}$, $m, n = 1, 2, \dots$. Then the characteristic equations of $\mathcal{L}$ are the following sequence of quadratic equations:

$$\Gamma(m^2, n^2) = \nu^2 + T(m^2, n^2)\nu + D(m^2, n^2) = 0 \tag{8}$$

with

$$T(m^2, n^2) = -a + \delta + \frac{\pi^2(d_1 + d_2)}{l^2}(m^2 + n^2) \tag{9}$$

and

$$D(m^2, n^2) = \frac{\pi^4 d_1 d_2}{l^4}(m^2 + n^2)^2 + \frac{(-ad_2 + \delta d_1)\pi^2}{l^2}(m^2 + n^2) + b - a\delta. \tag{10}$$

Lemma 1. *Assume that $a > 0$, $\delta > 0$, and $b > 0$. Then for system* (4) *without diffusion* ($d_1 = d_2 = 0$)*, the equilibrium $U_0 = (0, 0)$ is asymptotically stable for $\{b \mid a < \delta,\ b > a\delta\}$ and unstable for $\{b \mid b - a\delta < 0\}$ or $\{(a, \delta) \mid a > \delta\}$.*

Proof. For (4), if $d_1 = d_2 = 0$, then we have $\mathcal{L} = \left(\begin{smallmatrix} af'(0) & -b \\ 1 & -\delta \end{smallmatrix}\right)$. It is clear that $U_0 = (0, 0)$ is asymptotically stable for $\{b \mid a < \delta,\ b > a\delta\}$ and unstable for $\{b \mid b - a\delta < 0\}$ or $\{(a, \delta) \mid a > \delta\}$. □

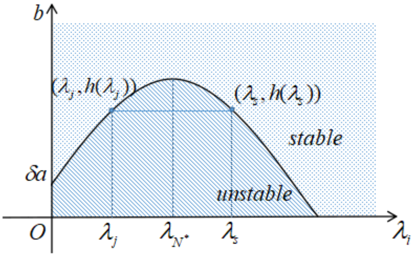

Figure 1. Bifurcation curve of steady-state solution.

Through this paper, we always assume that

(H1) $b \in \{b \mid b > a\delta,\ a < \delta,\ a > 0,\ \delta > 0\}$, which implies that system (4) is diffusion-free stable.

We now turn to the stability of steady state $(0,0)$ of system (4) with diffusion. For the sake of a further discussion, we need to give some notations, which will be used later. Let

$$\lambda_i = \pi^2\left(m_i^2 + n_i^2\right), \quad i = 1, 2, \dots,$$

be the eigenvalues for the Laplacian operator $-\Delta = -\partial^2/\partial x^2 - \partial^2/\partial y^2$ in Ω with the homogeneous Dirichlet boundary condition (5). Denote

$$b_i = h(\lambda_i) = -\frac{d_1 d_2}{l^4}\lambda_i^2 + \frac{a d_2 - \delta d_1}{l^2}\lambda_i + a\delta.$$

Theorem 1. *Assume that* (H1) *holds for system* (4). *Choosing* b *as the bifurcating parameter, we have that the equilibrium* $U_0 = (0,0)$ *is unstable if the following equation holds*:

$$Q(\lambda_i, b_i) \in \left\{(\lambda_i, b_i) \;\middle|\; b_i = h(\lambda_i),\ \delta a < b_i < \frac{4 d_1 d_2 + (a d_2 - \delta d_1)^2}{4 d_1 d_2},\ \delta d_1 < a d_2\right\}. \tag{11}$$

Proof. If (H1) holds, then (9) becomes

$$T(\lambda_i) = -a + \delta + \frac{d_1 + d_2}{l^2}\lambda_i > 0.$$

Consider (10), we have

$$D(\lambda_i) = \frac{d_1 d_2}{l^4}\lambda_i^2 + \frac{-a d_2 + \delta d_1}{l^2}\lambda_i + b - a\delta = 0.$$

Suppose (11) holds, then we find that at least one root of Eq. (8) has the positive real part. Combining with the conclusion of Lemma 1, we get that the solution $(0,0)$ is Turing unstable. Hence, (11) is the region of Turing unstability, and $b_i = h(\lambda_i)$ is Turing bifurcation curve; see Fig. 1. □

3 Steady-state bifurcation caused by $b_i = h(\lambda_i)$ for $\lambda_i \leqslant \lambda_{N^*}$

In this section a weakly nonlinear analysis is carried out to obtain the reduced equations describing the dynamics near the critical bifurcation values. Lyapunov–Schmidt method is employed to determine the near-critical bifurcation structure of the patterns.

Let $\lambda_{N^*} = [(ad_2 - \delta d_1)/(2d_1 d_2)]$. From Fig. 1 we note that if $\lambda_i \leqslant \lambda_{N_*}$, then b_i is in one-to-one correspondence with λ_i. In this case, zero will be a simple or double eigenvalue of $\mathcal{L}$. We will elaborate on why.

Let $\mu = b - b_i$. From (6) and (7) we know that

$$\varPhi(U) = \mathcal{L}U + F(U), \quad F(U) = \begin{pmatrix} u^2/2 + u^3/6 + O\|u^3\| \\ 0 \end{pmatrix}. \tag{12}$$

Therefore, the steady states of (4) are corresponding to the solution of the elliptic problem (12) with the boundary condition $U = 0$.

For discussing the reduced equation, we give the decompositions of space

$$Y = \operatorname{Ran}\mathcal{L} \oplus Y_1, \qquad X = \operatorname{Ker}\mathcal{L} \oplus X_1.$$

Since $\mathcal{L} : X \to Y$ is Fredom with index zero, then $\mathcal{L}|_{X_1} \to \operatorname{Ran}\mathcal{L}$ is invertible and has bounded inverse. In the following, we will use Lyapunov–Schmidt method to obtain the reduced equation and spatially nonhomogeneous solution of (12).

$\varOmega$ has obviously the D_4-symmetry of the unit square, i.e., it is D_4-equivariant. The classical theory of elliptic partial differential equations shows that

$$\mathcal{L} = D_U \varPhi_0 : X \mapsto Y.$$

In D_4-invariant domain $\varOmega$, under the homogeneous Dirichet boundary conditions, the eigenpairs of the Laplacian $-\Delta$ are of the form

$$\lambda_i = \left(m_i^2 + n_i^2\right)\pi^2, \qquad \varphi_i(x, y) = 2\sin(m_i \pi x)\sin(n_i \pi y). \tag{13}$$

These mean that λ_i is an eigenvalue of Laplacian $-\Delta$, while the corresponding eigenfunctions $\varphi_i = 2\sin(m_i\pi x)\sin(n_i\pi y)$ are called modes, and m_i and n_i are called wave numbers.

Remark. One, two, or more pairs (m_i, n_i) may exist such that Eq. (13) is satisfied, and in this case the eigenvalue will have single, double, or higher multiplicity, respectively. In this paper, we shall restrict our analysis to cases where the multiplicity is 1 or 2.

Consider the action of D_4 on the square $\varOmega$, and let

$$S(x, y) = (1 - x,\, y), \qquad R(x, y) = (1 - y,\, x)$$

be the generators of D_4. The function spaces X, Y are obviously D_4-invariant. In the following, we will consider two cases:

(i) If $m_i = n_i$, then $\lambda_i = (m_i^2 + n_i^2)\pi^2$ is a simple eigenvalue of Laplacian $-\Delta$. From Eq. (10) we have

$$\begin{aligned} D(\lambda_i) &= D\bigl(m_i^2, m_i^2\bigr) \\ &= \frac{\pi^4 d_1 d_2}{l^4}\bigl(m_i^2 + m_i^2\bigr)^2 + \frac{(-ad_2 + \delta d_1)\pi^2}{l^2}\bigl(m_i^2 + m_i^2\bigr) + b - a\delta = 0. \end{aligned}$$

Hence, zero is a simple eigenvalue of $\mathcal{L}$. The associated $\operatorname{Ker}\mathcal{L} = E_1 = \operatorname{Span}\{\varphi_1\}$ with

$$\varphi_1 = 2\begin{bmatrix} \frac{2d_2}{l^2}\pi^2 m_i^2 + \delta^2 \\ 1 \end{bmatrix} \sin(m_i\pi x)\sin(m_i\pi y),$$

and $E_1^* = \operatorname{Span}\{\varphi_1^*\}$ with

$$\varphi_1^* = 2\begin{bmatrix} 1 \\ \frac{2d_1}{l^2}\pi^2 m_i^2 + af'(0) \end{bmatrix} \sin(m_i\pi x)\sin(m_i\pi y).$$

In this case the induced action of D_4 in E_1 is

$$S_1 = (-1)^{m_i}, \qquad R_1 = (-1)^{m_i}.$$

(ii) If $m_i \neq n_i$, then $\lambda_i = (m_i^2 + n_i^2)\pi^2$ is double eigenvalue of Laplacian $-\Delta$:

$$\begin{aligned} D(\lambda_i) &= D\bigl(m_i^2, n_i^2\bigr) \\ &= \frac{d_1 d_2 \pi^4}{l^4}\bigl(m_i^2 + n_i^2\bigr)^2 + \frac{(-ad_2 + \delta d_1)\pi^2}{l^2}\bigl(m_i^2 + n_i^2\bigr) + b - a\delta \\ &= D\bigl(n_i^2, m_i^2\bigr) \\ &= \frac{d_1 d_2 \pi^4}{l^4}\bigl(n_i^2 + m_i^2\bigr)^2 + \frac{(-ad_2 + \delta d_1)\pi^2}{l^2}(n_i^2 + m_i^2) + b - a\delta = 0. \end{aligned}$$

Hence, zero is double eigenvalue of $\mathcal{L}$, and the eigenspace is two-dimensional, then $\operatorname{Ker}\mathcal{L} = E_2 = \operatorname{Span}\{\varphi_2, \varphi_3\}$ with

$$\begin{aligned} \varphi_2 &= 2\begin{bmatrix} \frac{d_2\pi^2}{l^2}(m_i^2 + n_i^2) + \delta \\ 1 \end{bmatrix} \sin(m_i\pi x)\sin(n_i\pi y), \\ \varphi_3 &= 2\begin{bmatrix} \frac{d_2\pi^2}{l^2}(m_i^2 + n_i^2) + \delta^2 \\ 1 \end{bmatrix} \sin(n_i\pi x)\sin(m_i\pi y), \end{aligned}$$

and $E_2^* = \operatorname{Span}\{\varphi_2^*, \varphi_3^*\}$ with

$$\begin{aligned} \varphi_2^* &= 2\begin{bmatrix} 1 \\ \frac{d_1\pi^2}{l^2}(m_i^2 + n_i^2) + af'(0) \end{bmatrix} \sin(m_i\pi x)\sin(n_i\pi y), \\ \varphi_3^* &= 2\begin{bmatrix} 1 \\ \frac{d_1\pi^2}{l^2}(m_i^2 + n_i^2) + af'(0) \end{bmatrix} \sin(n_i\pi x)\sin(m_i\pi y). \end{aligned}$$

In this case the representation of D_4 in E_2 is

$$S_2 = \begin{bmatrix} (-1)^{m_i-1} & 0 \\ 0 & (-1)^{n_i-1} \end{bmatrix}, \qquad R_2 = \begin{bmatrix} 0 & (-1)^{m_i-1} \\ (-1)^{n_i-1} & 0 \end{bmatrix}.$$

In Sections 3.1 and 3.2, we use the Lyapunov–Schmidt technique to study reduced equations of system (12).

3.1 Turing instability

Consider the case $m_i = n_i$. Let $\operatorname{Ker}\mathcal{L} = E_1 = \operatorname{Span}\{\varphi_1\}$, and $E - E_1$ denote the projection operators from Y onto $\operatorname{Ran}\mathcal{L}$ and Y_1. Observe that by assumption above $\dim\operatorname{Ker}\mathcal{L} = 1$. The following trivial observation starts the derivation: $U = 0$ iff $E_1 U = 0$ and $(E - E_1)U = 0$. Then use the Lyapunov–Schmidt reduction [7]

$$U = z_1\varphi_1 + w_1,$$

where $z_1 = \langle\varphi_1, U\rangle$, and $w_1 = U - z_1\varphi_1$. Thus, system (12) (i.e., $\Phi(U, \mu) = 0$) may be expanded to an equivalent pairs of equations

$$E_1\Phi(z_1\varphi_1 + w_1,\, \mu) = 0, \tag{14}$$

$$(E - E_1)\Phi(z_1\varphi_1 + w_1,\, \mu) = 0, \tag{15}$$

where $z_1 \in \mathbb{R}$ and $w_1 \in X_1$.

Define a map $G_1 : (\operatorname{Ker}\mathcal{L}) \times X_1 \times \mathbb{R} \to \operatorname{Ran}\mathcal{L}$, where

$$G_1 = E_1\Phi(z_1\varphi_1 + w_1,\, \mu).$$

By the chain rule the differential of (14) with respect to the w_1 variables at the origin is

$$E(d\Phi)_{(0,0)} = E\mathcal{L} = \mathcal{L}.$$

Furthermore, the linear map $\mathcal{L} : X_1 \to \operatorname{Ran}\mathcal{L}$ is invertible. Thus, it follows from the implicit function theorem that (14) is uniquely solvable for w_1 near the origin. Then there exist an open neighborhood N_1 of O in $\mathbb{R}$ and a continuously differentiable map $w_1 = W_1(z_1, \mu) : N_1 \times X_1 \to X_1$ such that

$$W_1(0,0) = 0 \quad \text{and} \quad E_1\Phi\bigl(z_1\varphi_1 + W_1(z_1,\mu), \mu\bigr) = 0.$$

Substituting $w_1 = W_1(z_1, \mu)$ into (15), we obtain the reduced mapping $B : \operatorname{Ker}\mathcal{L} \times \mathbb{R} \to Y_1$:

$$B(z_1, \mu) = (E - E_1)\Phi\bigl(z_1\varphi_1 + W_1(z_1,\mu), \mu\bigr) = 0.$$

Then the zeros of $B(z_1, \mu)$ are in one-to-one correspondence with the zeros of (15), the correspondence being given by

$$B(z_1, \mu) = 0 \quad \text{iff} \quad \Phi\bigl(z_1\varphi_1 + W_1(z_1,\mu), \mu\bigr) = 0.$$

We define $B_1(z_1, \mu)$ by

$$B_1(z_1, \mu) = \big\langle \varphi_1^*, B\big(z_1\varphi_1 + W_1(z_1, \mu), \mu\big)\big\rangle = 0. \tag{16}$$

Since $B(z_1, \mu) \in Y_1$, then $B(z_1, \mu) = 0$ iff $B_1(z_1, \mu) = 0$. Thus, the zeros of $B_1(z_1, \mu) = 0$ are also in one-to-one correspondence with solutions of $\varPhi(z_1, \mu) = 0$. It is worth noting that substituting the definition of $B_1(z_1, \mu)$ into (16), the projection $(E - E_1)$ drops out, i.e.,

$$B_1(z_1, \mu) = \big\langle \varphi_1^*, \varPhi\big(z_1\varphi_1 + W_1(z_1, \mu), \mu\big)\big\rangle = 0. \tag{17}$$

We call (17) the reduced equation. In the following, we consider two cases:

Case I: m_i is an odd number.

In this case, we will show that the reduced equation (17) is given in the form

$$B_1(z_1, \mu) = a_1 z_1 \mu + a_2 z_1^2 + \cdots,$$

where $\cdots$ stands for at least cubic terms; and

$$a_1 = \big\langle \varphi_1^*, \varPhi_{U\mu}(\varphi_1, \varphi_1)\big\rangle, \qquad a_2 = \frac{1}{2}\big\langle \varphi_1^*, \varPhi_{UU}(\varphi_1, \varphi_1)\big\rangle.$$

If $a_2 \neq 0$, by using the implicit function theorem we know that there exist a constant δ_{11} and a continuously differentiable map from $(-\delta_{11}, \delta_{11})$ to $\mathbb{R}$ such that $B_1(z_1^{(1)}(\mu), \mu) \equiv 0$ for $\mu \in (-\delta_{11}, \delta_{11})$. In fact, we have $z_1^{(1)}(\mu) = -\mu a_1/a_2 + o(|\mu|)$.

Theorem 2. *Let m_i be odd, and let $a_2 \neq 0$. Then we have*:

(i) *The equivalent forms of reduced equations of system* (12) *up to the second items with the simple bifurcation is*

$$a_1 z_1 \mu + a_2 z_1^2 = 0,$$

and the bifurcation are transcritical.

(ii) *There exist a constant δ_{11} and a continuously differentiable map $\mu \to z_1$ from $(-\delta_{11}, \delta_{11})$ to $\mathbb{R}$ such that system* (12) *has a nonhomogeneous steady-state solution*

$$U_1^\mu = z_1^{(1)}(\mu)\varphi_1 + W_1\big(z_1^{(1)}(\mu), \mu\big) \quad \textit{and} \quad \lim_{\mu \to 0} U_1^\mu = U_0,$$

where $z_1^{(1)}(\mu) = -\mu a_1/a_2 + o|\mu|$.

Proof. According to Eq. (13), we know that $U = z_1\varphi_1 + w_1$. In the following, we give some calculation of Lyapunov–Schmidt reduction of $B_1 = 0$. By calculating the derivatives of (17) we can obtain

$$B_{1UU}(0, 0) = \big\langle \varphi_1^*, \varPhi_{UU}(\varphi_1, \varphi_1)\big\rangle,$$

where $\Phi_{UU}(\varphi_1, \varphi_1)$ can be calculated by

$$\Phi_{t_1 t_2}(U, V) = \frac{\partial^2}{\partial t_1 \partial t_2} \Phi(t_1 U + t_2 V,\, b^*).$$

By Lyapunov–Schmidt reduction we have

$$B_1(z_1, \mu) = z_1 a_1 \mu + a_2 z_1^2 + \text{h.o.t.},$$

where

$$a_1 = \frac{1}{2}\bigl\langle \varphi_1,\, \Phi_{U\mu}(\varphi_1^*, \varphi_1)\bigr\rangle, \qquad a_2 = \frac{1}{2}\bigl\langle \varphi_1^*,\, \Phi_{UU}(\varphi_1, \varphi_1)\bigr\rangle.$$

Hence, the reduced equation of system (12) up to the second items with the simple bifurcation is

$$a_1 z_1 \mu + a_2 z_1^2 = 0.$$

Further more, if $a_2 \neq 0$, then from $B_1(z_1, \mu) = 0$ we can obtain

$$z_1^{(1)}(\mu) = -\frac{\mu a_1}{a_2} + o|\mu|$$

for $\mu \in (-\delta_{11}, \delta_{11})$. So the system has a nonhomogeneous steady-state solution

$$U_1^\mu = z_1^{(1)}(\mu)\varphi_1 + W_1\bigl(z_1^{(1)}(\mu), \mu\bigr). \qquad \square$$

Case II: m_i is an even number, or $F(U)$ is odd function of U.

In this case the reduced equation $B_1(z_1, \mu) = 0$ has the following equivalent form:

$$B_1(z_1, \mu) = a_1 z_1 \mu + a_3 z_1^3 + \text{h.o.t.},$$

where

$$a_1 = \bigl\langle \varphi_1^*, \Phi_{U\mu}(\varphi_1, \varphi_1)\bigr\rangle,$$
$$a_3 = \frac{1}{6}\bigl\langle \varphi_1^*,\, \Phi_{UUU}(\varphi_1, \varphi_1, \varphi_1) + 3\Phi_{UU}\bigl(\varphi_1, W_1^{20}\bigr)\bigr\rangle,$$

and

$$W_2^{20} = \mathfrak{L}^{-1}(E - E_1)\Phi_{UU}(\varphi_1, \varphi_1).$$

For $a_1 a_3 > 0$ (respectively, $a_1 a_3 < 0$), there exist a constant $\delta_{12} > 0$ and two continuously differentiable mappings $\mu \in (-\delta_{12}, 0)$ (respectively, $\mu \in (0, \delta_{12})$) to $\mathbb{R}$ such that $B_1(z_2^\mu, \mu) \equiv 0$ for $\mu \in (-\delta_{12}, 0)$ or $\mu \in (0, \delta_{12})$).

Theorem 3. *Let m_i be even. Then we have*:

(i) *The equivalent forms of reduced equations of system* (12) *up to the third items with the simple bifurcation is*

$$a_1 z_1 \mu + a_3 z_1^3 = 0,$$

and the bifurcation is pitchfork.

(ii) *For* $a_1a_3 > 0$ (*respectively*, $a_1a_3 < 0$), *there exist a constant* $\delta_{12} > 0$ *and continuously differentiable mapping* $\mu \to z_1$ *from* $(-\delta_{12}, 0)$ (*respectively*, $(0, \delta_{12})$) *to* $\mathbb{R}$ *such that system* (12) *has nonhomogeneous solution*

$$U_2^\mu = z_1^{(2)}(\mu)\varphi_1 + W_1\bigl(z_1^{(2)}(\mu), \mu\bigr), \quad \textit{and} \quad \lim_{\mu\to 0} U_2^\mu = U_0.$$

Here $z_1^{(2)}(\mu) = \sqrt{-\mu a_1/a_3}$.

Proof. This proof is similar to that of Theorem 2, we will omit it. ☐

3.2 Double bifurcation

In the following, we consider the double-bifurcation case. From Section 2 we know that if $m_i \neq n_i$, then $E_2 = \operatorname{Span}\{\varphi_2, \varphi_3\}$, where E_2 and $E - E_2$ denote the projection operators from Y onto $\operatorname{Ran}\mathcal{L}$ and Y_1. Observe that by assumption above $\dim\operatorname{Ker}\mathcal{L} = 2$.

Then using the Lyapunov–Schmidt reduction, we have

$$U = z_2\varphi_2 + z_3\varphi_3 + w_2,$$

where $z_2 = \langle\varphi_2, U\rangle$, $z_3 = \langle\varphi_3, U\rangle$, and $w_2 = U - z_2\varphi_2 - z_3\varphi_3$. Thus, system (12) may be expanded to an equivalent pair of equations

$$E_2\varPhi(z_2\varphi_2 + z_3\varphi_3 + w_2,\ \mu) = 0, \tag{18}$$

$$(E - E_2)\varPhi(z_2\varphi_2 + z_3\varphi_3 + w_2,\ \mu) = 0, \tag{19}$$

where $z_2, z_3 \in \mathbb{R}$, and $w_2 \in X_1$.

Define a map $G_2 : (\operatorname{Ker}\mathcal{L}) \times X_1 \times \mathbb{R} \to \operatorname{Ran}\mathcal{L}$, where

$$G_2 = E_2\varPhi(z_2\varphi_2 + z_3\varphi_3 + w_2,\ \mu).$$

By the chain rule the differential of (18) with respect to the w_2 variable at the origin is

$$E(d\varPhi)_{(0,0)} = E\mathcal{L} = \mathcal{L}.$$

Furthermore, the linear map $\mathcal{L} : X_1 \to \operatorname{Ran}\mathcal{L}$ is invertible. Thus, it follows from the implicit function theorem that (18) is uniquely solvable for w_2 near the origin. Then there exist an open neighborhood N_2 of O in $\mathbb{R}$ and a continuously differentiable map

$$w_2 = W_2(z_2, z_3, \mu) = \begin{bmatrix} W_{21}(z_2, z_3, \mu) \\ W_{22}(z_2, z_3, \mu) \end{bmatrix} : N_2 \times X_1 \to X_1$$

such that

$$W_2(0,0,0) = 0 \quad \text{and} \quad E_2\varPhi\bigl(z_2\varphi_2 + z_3\varphi_3 + W_2(z_2, z_3, \mu),\ \mu\bigr) = 0.$$

Substituting $W_2 = W_2(z_2, z_3, \mu)$ into (19), we obtain the reduced mapping $C : \operatorname{Ker}\mathcal{L} \times \mathbb{R} \to Y_1$:

$$C(z_2, z_3, \mu) = (E - E_2)\varPhi\bigl(z_2\varphi_2 + z_3\varphi_3 + W_2(z_2, z_3, \mu),\ \mu\bigr) = 0.$$

Then the zeros of $C(z_2, z_2, \mu)$ are in one-to-one correspondence with the zeros of (19), the correspondence being given by

$$C(z_2, z_3, \mu) = 0 \quad \text{iff} \quad \Phi\big(z_2\varphi_2 + z_3\varphi_3 + W_2(z_2, z_3, \mu)\big) = 0.$$

We define $C_1(z_2, z_3, \mu)$ by

$$C_1(z_2, z_3, \mu) = \begin{bmatrix} C_{11}(z_2, z_3, \mu) \\ C_{21}(z_2, z_3, \mu) \end{bmatrix} = \begin{bmatrix} \langle \varphi_2^*, C(z_2\varphi_2 + z_3\varphi_3 + W_2(z_2, z_3, \mu), \mu)\rangle \\ \langle \varphi_3^*, C(z_2\varphi_2 + z_3\varphi_3 + W_2(z_2, z_3, \mu), \mu)\rangle \end{bmatrix}. \quad (20)$$

Since $C(z_2, z_3, \mu) \in Y_1$, then $C(z_2, z_3, \mu) = 0$ iff $C_1(z_2, z_3, \mu) = 0$. Thus, the zeros of $C_1(z_2, z_3, \mu) = 0$ are also in one-to-one correspondence with solutions of $\Phi(z_2, z_3, \mu) = 0$. It is worth noting that substituting the definition of $C_1(z_2, z_3, \mu)$ in (20) into (19), the projection $E - E_2$ drops out, i.e.,

$$C_1(z_2, z_3, \mu) = \begin{bmatrix} C_{11}(z_2, z_3, \mu) \\ C_{21}(z_2, z_3, \mu) \end{bmatrix} = \begin{bmatrix} \langle \varphi_2^*, \Phi(z_2\varphi_2 + z_3\varphi_3 + W_2(z_2, z_3, \mu), \mu)\rangle \\ \langle \varphi_3^*, \Phi(z_2\varphi_2 + z_3\varphi_3 + W_2(z_2, z_3, \mu), \mu)\rangle \end{bmatrix}. \quad (21)$$

Like in Section 3.1, we should consider the property of m_i, n_i that is caused by the D_4-symmetry of Ω. We also call Eq. (21) the reduced equations.

Case I: m_i and n_i are even numbers, and f is odd with U.

Since the double bifurcation is induced by the D_4-symmetry, m_i and n_i are even numbers, then the generators satisfy

$$S_2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad R_2 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix},$$

then we have

$$\begin{aligned} C_{11}(-z_2, -z_3, \mu) &= -C_{11}(z_2, z_3, \mu), \\ C_{21}(z_2, z_3, \mu) &= C_{11}(z_3, z_2, \mu). \end{aligned}$$

Hence, by some calculations we obtain the reduced equation

$$\begin{aligned} C_{11} &= c_1\mu z_2 + c_2 z_2^3 + c_3 z_2 z_3^2 + o\big(\|z\|^3\big), \\ C_{21} &= c_1\mu z_3 + c_3 z_2^2 z_3 + c_2 z_3^3 + o\big(\|z\|^3\big), \end{aligned} \quad (22)$$

where

$$c_1 = \big\langle \varphi_2^*, \Phi_{U\mu}(\varphi_2)\big\rangle, \quad (23)$$

$$c_2 = \frac{1}{6}\big\langle \varphi_2^*, \Phi_{UUU}(\varphi_2, \varphi_2, \varphi_2)\big\rangle, \quad (24)$$

$$c_3 = \frac{1}{2}\big\langle \varphi_2^*, \Phi_{UUU}(\varphi_2, \varphi_3, \varphi_3)\big\rangle. \quad (25)$$

Using the discusses above, we have the following theorem.

Theorem 4. *Let m_i, n_i be even, and let function f is odd in U. Then there exist the following results*:

(i) *The equivalent forms of reduced equations of system* (12) *up to the third items with the double bifurcation is*

$$\begin{aligned} c_1\mu z_2 + c_2 z_2^3 + c_3 z_2 z_3^2 &= 0, \\ c_1\mu z_3 + c_3 z_2^2 z_3 + c_2 z_3^3 &= 0, \end{aligned}$$

and the bifurcation is pitchfork.

(ii) *If $c_1c_2 < 0$* (*respectively*, $c_1c_2 > 0$)*, there exist four continuously differentiable mappings $\mu \to (z_2, z_3)$, $\mu \in (-\delta_{21}, 0)$* (*respectively*, $\mu \in (0, \delta_{21})$) *to $\mathbb{R}^2$ such that system* (12) *has four nonhomogeneous solutions*:

$$\begin{aligned} u_3^{\mu\pm} &= z_2^{(1)\pm}(\mu)\varphi_2 + W_2\big(z_2^{(1)\pm}(\mu), z_3^{(1)\pm}(\mu), \mu\big), \\ u_4^{\mu\pm} &= z_3^{(1)\pm}(\mu)\varphi_3 + W_2\big(z_2^{(1)\pm}(\mu), z_3^{(1)\pm}(\mu), \mu\big), \end{aligned}$$

where

$$z_2^{(1)\pm}(\mu) = z_3^{(1)\pm}(\mu) = \pm\left(\mu\frac{c_1}{c_3}\right)^{1/2}, \quad \mu \in (-\delta_{21}, 0) \textit{ or } \mu \in (0, \delta_{21}).$$

(iii) *If $c_1/(c_2+c_3) < 0$* (*respectively*, $c_1/(c_2+c_3) > 0$)*, there exist four continuously differentiable mappings $\mu \to (z_2, z_3)$, $\mu \in (-\delta_{22}, 0)$* (*respectively*, $\mu \in (0, \delta_{22})$) *to $\mathbb{R}^2$ such that system* (12) *has four nonhomogeneous solutions*:

$$\begin{aligned} u_5^{\mu\pm} &= z_2^{(2)\pm}(\mu)\varphi_2 + z_3^{(2)+}(\mu)\varphi_3 + W_2\big(z_2^{(2)\pm}(\mu), z_3^{(2)+}(\mu), \mu\big), \\ u_6^{\mu\pm} &= z_2^{(2)\pm}(\mu)\varphi_2 + z_3^{(2)-}(\mu)\varphi_3 + W_2\big(z_2^{(2)\pm}(\mu), z_3^{(2)-}(\mu), \mu\big), \end{aligned}$$

where

$$z_2^{(2)\pm}(\mu) = z_3^{(2)\pm}(\mu) = \pm\left(\frac{\mu c_1}{c_2+c_3}\right)^{1/2}, \quad \mu \in (-\delta_{22}, 0) \textit{ or } \mu \in (-\delta_{22}, 0).$$

Proof. According to Eqs. (22)–(25), the bifurcations of system (12) is pitchfork, and conclusion (i) is obtained immediately. Moreover, we have four nontrivial isolated solutions

$$\pm\left(\left(\mu\frac{c_1}{c_3}\right)^{1/2}, 0\right), \quad \pm\left(0, \left(\mu\frac{c_1}{c_3}\right)^{1/2}\right)$$

depending on the signs of μ and c_1/c_3, and conclusion (ii) is true. Similarly, according to the signs of μ and $c_1/(c_2+c_3)$, there also exist four nontrivial isolated solutions

$$\pm\left(\left(\frac{\mu c_1}{c_2+c_3}\right)^{1/2}\right), \quad \pm\left(\left(\frac{\mu c_1}{c_2+c_3}\right)^{1/2}\right).$$

Hence, conclusion (iii) was obtained immediately. □

Case II: Both m_i and n_i are odd numbers.

In this case the generators satisfy

$$S_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad R_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then we have

$$C_{21}(z_2, z_3, \mu) = C_{11}(z_3, z_2, \mu).$$

Hence, by some calculations we obtain the reduced equation

$$\begin{aligned} C_{11} &= k_1\mu z_2 + k_2\mu z_3 + k_3 z_2^2 + k_4 z_2 z_3 + k_5 z_3^2, \\ C_{21} &= k_2\mu z_3 + k_1\mu z_2 + k_5 z_3^2 + k_4 z_3 z_2 + k_3 z_2^2, \end{aligned} \tag{26}$$

where

$$k_1 = k_2 = \langle \varphi_3^*, \varPhi_{U\mu}(\varphi_3, \varphi_3) \rangle, \quad k_3 = \left\langle \varphi_3^*, \frac{1}{2}\varPhi_{UU}(\varphi_2, \varphi_2) \right\rangle, \tag{27}$$

$$k_4 = \langle \varphi_3^*, \varPhi_{UU}(\varphi_2, \varphi_3) \rangle, \qquad k_5 = \left\langle \varphi_3^*, \frac{1}{2}\varPhi_{UU}(\varphi_3, \varphi_3) \right\rangle. \tag{28}$$

Using the discusses above, we have

Theorem 5. *Let m_i and n_i be odd. Then there exist the following results*:

(i) *The equivalent forms of reduced equations of system* (12) *up to the second items with the double bifurcation is*

$$\begin{aligned} k_1\mu z_2 + k_2\mu z_3 + k_3 z_2^2 + k_4 z_2 z_3 + k_5 z_3^2 &= 0, \\ k_1\mu z_3 + k_2\mu z_2 + k_3 z_3^2 + k_4 z_2 z_3 + k_5 z_2^2 &= 0, \end{aligned}$$

and the bifurcations are transcritical.

(ii) *If $\sqrt{(k_3 - 5k_5)/(k_3 - k_5)} > 0$, there exist three continuously differentiable mappings $\mu \to (z_2, z_3)$ ($\mu \in (-\delta_{22}, \delta_{22})$) to $\mathbb{R}^2$ such that Eq.* (12) *has three nonhomogeneous solutions*:

$$\begin{aligned} u_7^\mu &= z_2^{(3)}(\mu)\varphi_2 + z_3^{(3)}(\mu)\varphi_3 + W_2\bigl(z_2^{(3)}(\mu), z_3^{(3)}(\mu), \mu\bigr), \\ u_8^\mu &= z_2^{(4)}(\mu)\varphi_2 + z_3^{(4)}(\mu)\varphi_3 + W_2\bigl(z_2^{(4)}(\mu), z_3^{(4)}(\mu), \mu\bigr), \\ u_9^\mu &= z_3^{(4)}(\mu)\varphi_2 + z_2^{(4)}(\mu)\varphi_3 + W_2\bigl(z_2^{(4)}(\mu), z_3^{(4)}(\mu), \mu\bigr), \end{aligned}$$

where

$$\begin{aligned} z_2^{(3)}(\mu) &= z_3^{(3)}(\mu) = \frac{k_1\mu}{k_3 + 3k_5}, \quad \mu \in (-\delta_{22}, \delta_{22}), \\ z_2^{(4)}(\mu) &= \frac{k_1\mu}{2(k_3 - k_5)}\left(-1 - \sqrt{\frac{k_3 - 5k_5}{k_3 - k_5}}\right), \quad \mu \in (-\delta_{22}, \delta_{22}), \\ z_3^{(4)}(\mu) &= \frac{k_1\mu}{2(k_3 - k_5)}\left(-1 + \sqrt{\frac{k_3 - 5k_5}{k_3 - k_5}}\right), \quad \mu \in (-\delta_{22}, \delta_{22}). \end{aligned}$$

Proof. According to Eq. (26)–(28), we have three nontrivial isolated solutions:

$$\left(\frac{k_1\mu}{k_3+3k_5}, \frac{k_1\mu}{k_3+3k_5}\right),$$
$$\left(\frac{k_1\mu}{2(k_3-k_5)}\left(-1-\sqrt{\frac{k_3-5k_5}{k_3-k_5}}\right), \frac{k_1\mu}{2(k_3-k_5)}\left(-1+\sqrt{\frac{k_3-5k_5}{k_3-k_5}}\right)\right),$$
$$\left(\frac{k_1\mu}{2(k_3-k_5)}\left(-1+\sqrt{\frac{k_3-5k_5}{k_3-k_5}}\right), \frac{k_1\mu}{2(k_3-k_5)}\left(-1-\sqrt{\frac{k_3-5k_5}{k_3-k_5}}\right)\right)$$

depending on the signs of $\sqrt{(k_3-5k_5)/(k_3-k_5)} > 0$. Hence, the bifurcations of Eq. (4) are transcritical, and the conclusion is obtained immediately. ☐

Case III: m_i is an even number, n_i an odd number, or n_i is an even number, m_i an odd number.

In this case the generators satisfy

$$S_2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

then we have

$$\begin{aligned} C_{21}(z_2, z_3, \mu) &= C_{11}(z_3, z_2, \mu), \\ C_{11}(-z_2, z_3, \mu) &= -C_{11}(z_2, z_3, \mu), \\ C_{11}(-z_3, z_2, \mu) &= -C_{21}(z_2, z_3, \mu). \end{aligned}$$

Hence, by some calculations, we find that the reduce equation is same as Case I. Therefore, the conclusion of this part is the same as that of the Case I, so it will not be repeated.

4 Steady/steady-state mode interactions caused by $b(\lambda_j) = h(\lambda_j) = b(\lambda_s) = h(\lambda_s)$ for some $j \neq s$

In this section, we remove the restriction $\lambda_{N^*} = [(ad_2 - \delta d_1)/(2d_1d_2)]$. That means that the multiple bifurcations occur when $b_j = h(\lambda_j) = b_s = h(\lambda_s)$ for some

$$\lambda_j = (m_j^2 + n_j^2)\pi^2, \qquad \lambda_s = (m_s^2 + n_s^2)\pi^2, \quad j \neq s.$$

Using Eq. (10), we have

$$\begin{aligned} D(\lambda_j) &= D(m_j^2, n_j^2) \\ &= \frac{\pi^4 d_1 d_2}{l^4}(m_j^2 + n_j^2)^2 + \frac{(-ad_2 + \delta d_1)\pi^2}{l^2}(m_j^2 + n_j^2) + b - a\delta = 0 \end{aligned}$$

and

$$\begin{aligned} D(\lambda_s) &= D(m_s^2, n_s^2) \\ &= \frac{\pi^4 d_1 d_2}{l^4}(m_s^2 + n_s^2)^2 + \frac{(-ad_2 + \delta d_1)\pi^2}{l^2}(m_s^2 + n_s^2) + b - a\delta = 0, \end{aligned}$$

for some $j \neq s$.

In the following, we will consider three cases. Due to the complexity of calculation, we will not calculate the specific forms of Lyapunov reduction in this section. We only give the basic preparations for calculation.

Case I: Steady/steady-state mode interactions of two simple bifurcations.

If there exist $m_j = n_j$ and $m_s = n_s$ such that

$$\begin{aligned} D(\lambda_j) &= D\bigl(m_j^2, m_j^2\bigr) \\ &= \frac{\pi^4 d_1 d_2}{l^4}\bigl(m_j^2 + m_j^2\bigr)^2 + \frac{(-ad_2 + \delta d_1)\pi^2}{l^2}\bigl(m_j^2 + m_j^2\bigr) + b - a\delta = 0 \end{aligned}$$

and

$$\begin{aligned} D(\lambda_s) &= D\bigl(m_s^2, m_s^2\bigr) \\ &= \frac{\pi^4 d_1 d_2}{l^4}\bigl(m_s^2 + m_s^2\bigr)^2 + \frac{(-ad_2 + \delta d_1)\pi^2}{l^2}\bigl(m_s^2 + m_s^2\bigr) + b - a\delta = 0, \end{aligned}$$

then, both λ_j and λ_s are simple eigenvalue of Laplacian $-\Delta$. In this case, we obtain a double-bifurcation point $b(\lambda_j) = b(\lambda_s)$ for some $j \neq s$. Hence, zero is a double eigenvalue of $\mathcal{L}$. The associated eigenspace is $E_3 = \operatorname{Span}\{\varphi_j, \varphi_s\}$ with

$$\varphi_i = 2\begin{bmatrix} \frac{2d_2\pi^2}{l^2} m_i^2 + \delta \\ 1 \end{bmatrix} \sin(m_i\pi x)\sin(m_i\pi y)$$

for $i = j, s$ and $E_3^* = \operatorname{Span}\{\varphi_j^*, \varphi_s^*\}$ with

$$\varphi_i^* = 2\begin{bmatrix} 1 \\ \frac{2d_1\pi^2}{l^2} m_i^2 + af'(0) \end{bmatrix} \sin(m_i\pi x)\sin(m_i\pi y)$$

for $i = j, s$. In this case the induced action of D_4 in E_3 is

$$S_3 = \begin{bmatrix} (-1)^{m_i-1} & 0 \\ 0 & (-1)^{n_i-1} \end{bmatrix}, \qquad R_3 = \begin{bmatrix} 0 & (-1)^{m_i-1} \\ (-1)^{n_i-1} & 0 \end{bmatrix}$$

for $i = j, s$. Hence, by using the Lyapunov–Schmidt reduction we have

$$U = z_4\varphi_j + z_5\varphi_s + w_4,$$

where $z_4 = \langle \varphi_j, U\rangle$, $z_5 = \langle \varphi_s, U\rangle$ and $w_3 = U - z_4\varphi_j - z_5\varphi_s$.

Case II: Steady/steady-state mode interactions of one simple and one double bifurcation.

If there exist $m_j = n_j$ and $m_s \neq n_s$ such that

$$\begin{aligned} D(\lambda_j) &= D\bigl(m_j^2, m_j^2\bigr) \\ &= \frac{\pi^4 d_1 d_2}{l^4}\bigl(m_j^2 + m_j^2\bigr)^2 + (-ad_2 + \delta d_1)\frac{\pi^2}{l^2}\bigl(m_j^2 + m_j^2\bigr) + b - a\delta = 0 \end{aligned}$$

and

$$\begin{aligned} D(\lambda_s) &= D\bigl(m_s^2, n_s^2\bigr) \\ &= \frac{\pi^4 d_1 d_2}{l^4}\bigl(m_s^2 + n_s^2\bigr)^2 + (-ad_2 + \delta d_1)\frac{\pi^2}{l^2}\bigl(m_s^2 + n_s^2\bigr) + b - a\delta = 0, \end{aligned}$$

then, λ_j is a simple eigenvalue and λ_s a double ones of Laplacian $-\Delta$. Hence, zero is a triple eigenvalue of $\mathcal{L}$.

In this case the associated eigenspace is $E_5 = \mathrm{Span}\{\varphi_j, \varphi_{s_1}, \varphi_{s_2}\}$ with

$$\begin{aligned} \varphi_j &= 2\begin{bmatrix} \frac{2d_2\pi^2}{l^2}m_j^2 + \delta \\ 1 \end{bmatrix}\sin(m_j\pi x)\sin(m_j\pi y), \\ \varphi_{s_1} &= 2\begin{bmatrix} \frac{d_2\pi^2}{l^2}(m_s^2 + n_s^2) + \delta \\ 1 \end{bmatrix}\sin(m_s\pi x)\sin(n_s\pi y), \end{aligned}$$

and

$$\varphi_{s_2} = 2\begin{bmatrix} \frac{d_2\pi^2}{l^2}(m_s^2 + n_s^2) + \delta^2 \\ 1 \end{bmatrix}\sin(n_s\pi x)\sin(m_s\pi y).$$

Further more, $E_4^* = \mathrm{Span}\{\varphi_j^*, \varphi_{s_1}^*, \varphi_{s_2}^*\}$ with

$$\varphi_j^* = 2\begin{bmatrix} 1 \\ \frac{2d_1\pi^2}{l^2}m_i^2 + af'(0) \end{bmatrix}\sin(m_j\pi x)\sin(m_j\pi y),$$

and

$$\varphi_{s2}^* = 2\begin{bmatrix} 1 \\ \frac{d_1\pi^2}{l^2}(m_s^2 + n_s^2) + af'(0) \end{bmatrix}\sin(n_s\pi x)\sin(m_s\pi y).$$

The induced action of D_4 in E_4 is

$$S_3 = \begin{bmatrix} (-1)^{m_j-1} & 0 & 0 \\ 0 & (-1)^{m_s-1} & 0 \\ 0 & 0 & (-1)^{n_s-1} \end{bmatrix}, \quad R_3 = \begin{bmatrix} 0 & 0 & (-1)^{m_j-1} \\ 0 & (-1)^{m_s-1} & 0 \\ (-1)^{n_s-1} & 0 & 0 \end{bmatrix}.$$

Hence, by using the Lyapunov–Schmidt reduction, we have

$$U = z_6\varphi_j + z_7\varphi_{s1} + z_8\varphi_{s2} + w_4,$$

where $z_6 = \langle\varphi_j, U\rangle$, $z_7 = \langle\varphi_{s1}, U\rangle$, $z_8 = \langle\varphi_{s2}, U\rangle$ and $w_4 = U - z_6\varphi_j - z_7\varphi_{s1} - z_8\varphi_{s2}$.

Case III: Steady/steady-state mode interactions of two double bifurcations.
If there exist $m_j \neq n_j$ and $m_s \neq n_s$ such that

$$\begin{aligned} D(\lambda_j) &= D\bigl(m_j^2, n_j^2\bigr) \\ &= \frac{\pi^4 d_1 d_2}{l^4}(m_j^2 + n_j^2)^2 + \frac{(-ad_2 + \delta d_1)\pi^2}{l^2}\bigl(m_j^2 + n_j^2\bigr) + b - a\delta = 0 \end{aligned}$$

and

$$\begin{aligned} D(\lambda_j) &= D\big(m_s^2, n_s^2\big) \\ &= \frac{\pi^4 d_1 d_2}{l^4}(m_s^2 + n_s^2)^2 + \frac{(-ad_2 + \delta d_1)\pi^2}{l^2}\big(m_s^2 + n_s^2\big) + b - a\delta = 0, \end{aligned}$$

then, both λ_j and λ_s are double eigenvalues of Laplacian $-\Delta$. Hence, zero is a 4-fold eigenvalue of $\mathcal{L}$.

The associated eigenspace is $E_5 = \operatorname{Span}\{\varphi_{j1}, \varphi_{j2}, \varphi_{s_1}, \varphi_{s_2}\}$ with

$$\begin{aligned} \varphi_{j_1} &= 2\begin{bmatrix} \frac{d_2\pi^2}{l^2}(m_j^2 + n_j^2) + \delta \\ 1 \end{bmatrix} \sin(m_j\pi x)\sin(n_j\pi y), \\ \varphi_{j_2} &= 2\begin{bmatrix} \frac{d_2\pi^2}{l^2}(m_j^2 + n_j^2) + \delta^2 \\ 1 \end{bmatrix} \sin(n_j\pi x)\sin(m_j\pi y), \\ \varphi_{s_1} &= 2\begin{bmatrix} \frac{d_2\pi^2}{l^2}(m_s^2 + n_s^2) + \delta \\ 1 \end{bmatrix} \sin(m_s\pi x)\sin(n_s\pi y), \end{aligned}$$

and

$$\varphi_{s_2} = 2\begin{bmatrix} \frac{d_2\pi^2}{l^2}(m_s^2 + n_s^2) + \delta^2 \\ 1 \end{bmatrix} \sin(n_s\pi x)\sin(m_s\pi y).$$

Further more, $E_5^* = \operatorname{Span}\{\varphi_{j1}^*, \varphi_{j2}^*, \varphi_{s_1}^*, \varphi_{s_2}^*\}$ with

$$\begin{aligned} \varphi_{j1}^* &= 2\begin{bmatrix} 1 \\ \frac{d_1\pi^2}{l^2}(m_j^2 + n_j^2) + af'(0) \end{bmatrix} \sin(m_j\pi x)\sin(n_j\pi y), \\ \varphi_{j2}^* &= 2\begin{bmatrix} 1 \\ \frac{d_1\pi^2}{l^2}(m_j^2 + n_j^2) + af'(0) \end{bmatrix} \sin(n_j\pi x)\sin(m_j\pi y), \\ \varphi_{s1}^* &= 2\begin{bmatrix} 1 \\ \frac{d_1\pi^2}{l^2}(m_s^2 + n_s^2) + af'(0) \end{bmatrix} \sin(m_s\pi x)\sin(n_s\pi y), \end{aligned}$$

and

$$\varphi_{s2}^* = 2\begin{bmatrix} 1 \\ \frac{d_1\pi^2}{l^2}(m_s^2 + n_s^2) + af'(0) \end{bmatrix} \sin(n_s\pi x)\sin(m_s\pi y).$$

In this case the induced action of D_4 in E_5 is

$$S_3 = \begin{bmatrix} (-1)^{m_j-1} & 0 & 0 & 0 \\ 0 & (-1)^{n_j-1} & 0 & 0 \\ 0 & 0 & (-1)^{m_s-1} & 0 \\ 0 & 0 & 0 & (-1)^{n_s-1} \end{bmatrix},$$

$$R_3 = \begin{bmatrix} 0 & 0 & 0 & (-1)^{m_j-1} \\ 0 & 0 & (-1)^{n_j-1} & 0 \\ 0 & (-1)^{m_s-1} & 0 & 0 \\ (-1)^{n_s-1} & 0 & 0 & 0 \end{bmatrix}.$$

Using the Lyapunov–Schmidt reduction, we have

$$U = z_9\varphi_{j1} + z_{10}\varphi_{j2} + z_{11}\varphi_{s1} + z_{12}\varphi_{s2} + w_5,$$

where $z_9 = \langle\varphi_{j1}, U\rangle$, $z_{10} = \langle\varphi_{j2}, U\rangle$, $z_{11} = \langle\varphi_{s1}, U\rangle$, $z_{12} = \langle\varphi_{s2}, U\rangle$, and $w_5 = U - z_9\varphi_{j1} - z_{10}\varphi_{j2} - z_{11}\varphi_{s1} - z_{12}\varphi_{s2}$.

5 Numerical simulations

The goal of this section is to present the results of numerical simulations, which complement the analytic results in the previous Section 3. Choose for $f = u - u^3/3!$ and fixed values a, δ in all simulations, namely, $a = 3$, $\delta = 5$. We take $l = 1.0$ and $d_1 = 0.001$, $d_2 = 0.01$ satisfying (H1). According to Theorem 1, $\lambda_{N^*} = 1375$. Hence, we know that the constant steady state $(0, 0)$ is Turing unstable, and the simple and double Turing bifurcation occurs when $b = b_j$. From Section 3 a spatially inhomogeneous steady-state structure is characterized by φ_1 or φ_2, φ_3 is generated for $b > a\delta$ and $\lambda \leqslant \lambda_{N^*}$.

Choose $\lambda_1 = 315.8273$, then the bifurcation parameter $b_1 = 30.3756$. In this case a simple bifurcation occurs for $m_1 = 4$, $n_1 = 4$; see Fig. 2. In this case the system supports square patterns.

Choose $\lambda_2 = 986.9604$, then the Bifurcation parameter $b_2 = 49.8010$. In this case a double bifurcation occurs for $m_3 = 11$, $n_3 = 13$ or $m_3 = 13$, $n_3 = 11$; see Fig. 3.

Choose fixed values $a = 18$, $\delta = 20$ and take $l = 1.0$ and $d_1 = 0.001$, $d_2 = 0.01$ satisfying (H1). According to Theorem 1, $\lambda_{N^*} = 8500$. Hence, we know that the constant steady state $(0, 0)$ is Turing unstable, and the simple and double Turing bifurcation occurs when $b = b_j$.

Choose $\lambda_3 = 3480.1$, then the bifurcation parameter $b_3 = 795.7$. In this case a double bifurcation occurs for $m_2 = 8$, $n_2 = 17$ or $m_2 = 17$, $n_2 = 8$; see Fig. 4.

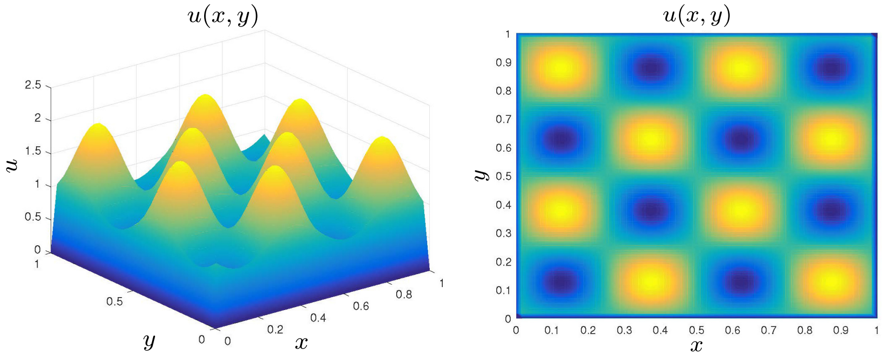

Figure 2. Turing pattern of u when $m = 4$, $n = 4$, $t = 10000$.

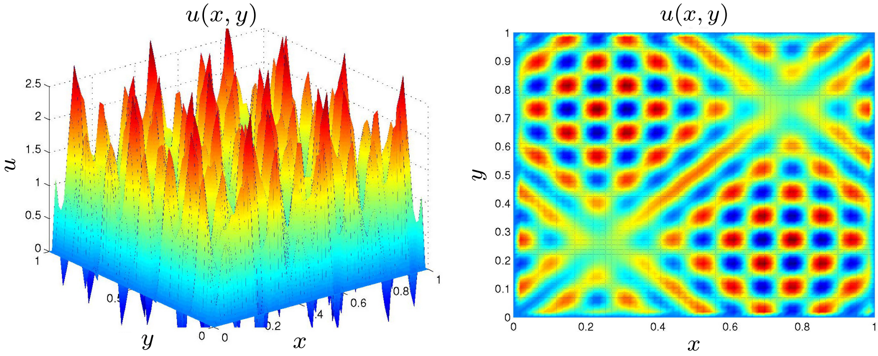

Figure 3. Turing pattern of u when $m = 11$, $n = 13$, $t = 10000$.

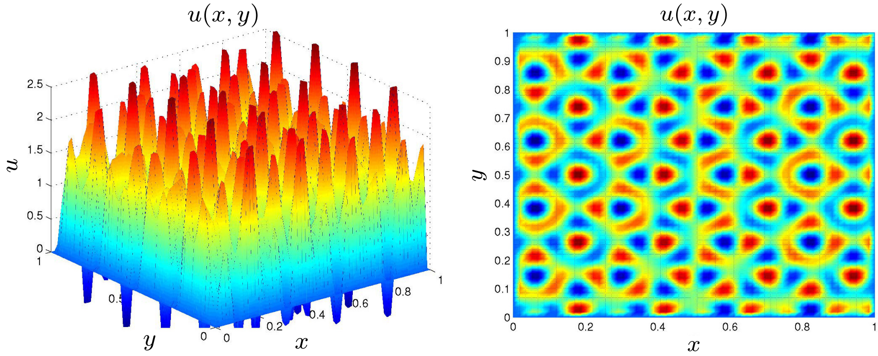

Figure 4. Turing pattern of u when $m = 8$, $n = 17$, $t = 10000$.

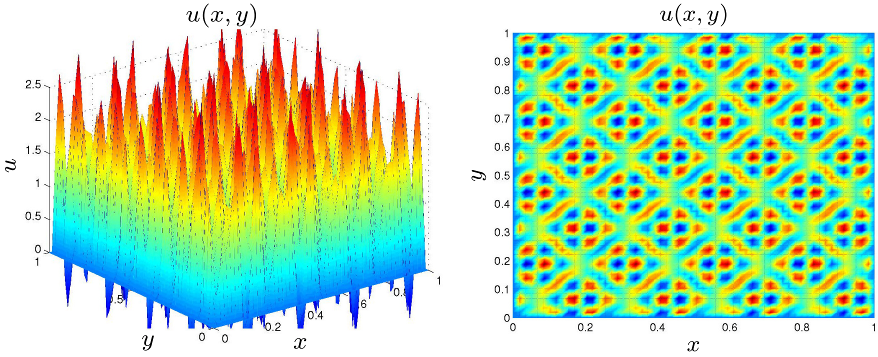

Figure 5. Turing pattern of u when $m = 16$, $n = 24$, $t = 10000$.

Choose $\lambda_4 = 8211.5$, then the bifurcation parameter $b_4 = 1803.3$. In this case a double bifurcation occurs for $m_4 = 16$, $n_4 = 24$ or $m_4 = 24$, $n_4 = 16$; see Fig. 5.

6 Conclusions

Problem (4) has obviously the D_4-symmetry of the unit square, i.e., it is D_4-equivariant. We are interested in the bifurcation structure of solution branches of (4)–(5) of simple and double bifurcation on the trivial solution curve. Using Lyapunov–Schmidt method, we show the existence of nonhomogeneous solutions. After calculating the reduced equations of Eq. (12), we investigate the necessary structure of steady-state bifurcating solutions. Numerical simulations show that the structure of pattern is determined by wave numbers. Through the analysis of the steady/steady-state mode interactions, we found that the model can have highly degenerate branches, which is caused by the symmetry of the spatial region.

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