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Solvability for a Hadamard-type fractional integral boundary value problem*

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Abstract. In this paper, we study an integral boundary value problem involving a Hadamard-type fractional differential equation. Using fixed point theory and upper-lower solutions, we present some sufficient conditions to obtain existence theorems of positive solutions for the problem. Examples are provided to illustrate our results.

Keywords: Hadamard-type fractional differential equations, integral boundary value problems, positive solutions, fixed point methods, upper-lower solution methods.

1 Introduction

In this paper, we study the existence of positive solutions for the following integral boundary value problem involving the Hadamard-type fractional differential equation:

$$\begin{aligned} &{}^H D^{\mu}_{a+} x(t) + f\bigl(t, x(t)\bigr) = 0, \quad t \in (a, b), \\ &x(a) = x'(a) = 0, \qquad x(b) = \int_a^b h(t)x(t)\frac{\mathrm{d}t}{t}, \end{aligned} \tag{1}$$

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where a, b, μ are real positive numbers with $a < b < +\infty$, $2 < \mu < 3$, ${}^H D^{\mu}_{a^+}$ is the Hadamard fractional derivative of order μ, and f, h satisfy the following conditions.

(H1) $h : [a,b] \to \mathbb{R}^+$ with $h(t) \not\equiv 0$, $t \in [a,b]$, and $\int_a^b h(t)(\ln(t/a))^{\mu-1}/t\,\mathrm{d}t \in [0, (\ln(b/a)^{\mu-1})$.

(H2) $f \in C([a,b] \times \mathbb{R}^+, \mathbb{R}^+)$.

The theory of fractional differential equations plays an important role in biology, physics, medicine, etc., and most of the research has focused on the Riemann–Liouville or the Caputo-type fractional derivative. The Hadamard fractional derivative is another kind of fractional derivative and differs from the usual ones in the sense that the kernel of the integral contains a logarithmic function of arbitrary exponent. For background material of the Hadamard fractional derivative and integral, we refer the reader to the papers [1–10, 14, 15, 17, 19–22, 24, 27–29].

In [14] the authors used the Banach and Schauder fixed point theorems to study the existence and uniqueness of solutions for integral boundary conditions of implicit fractional differential equations involving the Hadamard fractional derivative

$$\begin{aligned} &{}^H D^{\vartheta}_{b+} x(t) = g\big(t, x(t), {}^H D^{\vartheta} x(t)\big), \quad t \in (b, \tau), \\ &x(b) = 0, \qquad x(\tau) = \lambda \int_0^{\sigma} x(s)\,\mathrm{d}s, \quad \sigma \in (b,\tau),\ \lambda \in \mathbb{R}, \end{aligned}$$

where $\vartheta \in (1, 2]$. In [24] the authors used the five functional fixed point theorems to study the multiplicity of positive solutions for the system of Hadamard fractional multipoint boundary value problems

$$\begin{aligned} &{}^H D^{q}_{1+} u(t) + f_1\big(t, u(t), v(t)\big) = 0, \quad 1 < t < \mathrm{e}, \\ &{}^H D^{q}_{1+} v(t) + f_2\big(t, u(t), v(t)\big) = 0, \quad 1 < t < \mathrm{e}, \\ &u(1) = \delta u(1) = 0, \qquad u(\mathrm{e}) = \sum_{i=1}^{m-1} a_i u(\xi_i), \\ &v(1) = \delta v(1) = 0, \qquad v(\mathrm{e}) = \sum_{j=1}^{n-1} b_j v(\eta_j), \end{aligned}$$

where $q \in (2, 3]$, and in [19] the authors used the Schauder fixed point theorem to study the existence of solutions for the following Hadamard-type fractional differential equation:

$$\begin{aligned} &{}^H D^{\alpha}_{1+} u(t) + q(t) f\big(t, u(t), {}^H D^{\beta_1}_{1+} u(t), {}^H D^{\beta_2}_{1+} u(t)\big) = 0, \quad 1 < t < +\infty, \\ &u(1) = 0, \\ &{}^H D^{\alpha-2}_{1+} u(1) = \int_1^{+\infty} g_1(s) u(s) \frac{\mathrm{d}s}{s}, \qquad {}^H D^{\alpha-1}_{1+} u(+\infty) = \int_1^{+\infty} g_2(s) u(s) \frac{\mathrm{d}s}{s}, \end{aligned}$$

where $2 < \alpha \leqslant 3$, $0 < \beta_1 \leqslant \alpha - 2 < \beta_2 \leqslant \alpha - 1$.

The hypothesis characterized by the first eigenvalue of linear operators are important tools to study boundary value problems (see, for example, [12, 13, 23, 25, 26, 30]), and in [30] the authors used the fixed point index to study positive solutions for the following nonlinear fractional differential equation with integral boundary conditions:

$$D^{\alpha}_{0+}u(t) + h(t)f\big(t, u(t)\big) = 0, \quad 0 < t < 1,$$

$$u(0) = u'(0) = u''(0) = 0, \qquad u(1) = \lambda \int_0^{\eta} u(s)\,\mathrm{d}s,$$

where $3 < \alpha \leqslant 4$, $0 < \eta \leqslant 1$, $0 \leqslant \lambda\eta^{\alpha}/\alpha < 1$, and D^{α}_{0+} is the standard Riemann–Liouville derivative.

Motivated by the works mentioned above, in this paper, we use the Krein–Rutman theorem to study the first eigenvalue of the corresponding linear operator, and using the fixed point index, we obtain the existence of positive solutions for (1). When the nonlinearity f is nondecreasing about the second variable, we use the upper-lower solution method to obtain two extremal positive solutions for (1), and we provide two iterative sequences for these solutions. Also, using the Leggett–Williams fixed point theorem, we obtain multiplicity of positive solutions for (1). Finally, some examples are provided to illustrate our results.

2 Preliminaries and lemmas

First, we provide some basic material for Hadamard-type fractional calculus.

Definition 1. (See [3, 15].) Let $a > 0$. Then the Hadamard fractional left integral of order $\mu > 0$ of a function $x : [a, \infty) \to \mathbb{R}$ is defined as

$${}^{H}I^{\mu}_{a+}x(t) = \frac{1}{\Gamma(\mu)} \int_a^t \left(\ln\frac{t}{\tau}\right)^{\mu-1} x(\tau)\frac{\mathrm{d}\tau}{\tau}, \quad t \geqslant a.$$

Definition 2. (See [3, 15].) Let $a > 0$. Then the Hadamard fractional left derivative of a function $x : [a, \infty) \to \mathbb{R}$, $t^{n-1}x^{(n-1)}(t) \in AC[a, \infty)$, $n \in \mathbb{N}$, of order $\mu \in (n-1, n)$ is defined as

$${}^{H}D^{\mu}_{a+}x(t) = \frac{1}{\Gamma(n-\mu)} \left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)^{n} \int_a^t \left(\ln\frac{t}{\tau}\right)^{n-\mu-1} x(\tau)\frac{\mathrm{d}\tau}{\tau}, \quad t > a.$$

Lemma 1. (*See* [3, 7, 15].) *For* $\mu \in (n–1, n)$, $n \in \mathbb{N}$, $y \in L[a, \infty)$, $a > 0$, *the Hadamard fractional differential equation* ${}^{H}D^{\mu}_{a+}x(t) + y(t) = 0$, $t > a$, *has a general solution*

$$x(t) = \sum_{k=1}^{n} c_k \left(\ln\frac{t}{a}\right)^{\mu-k} - \frac{1}{\Gamma(\mu)} \int_a^t \left(\ln\frac{t}{\tau}\right)^{\mu-1} y(\tau)\frac{\mathrm{d}\tau}{\tau}, \quad t \geqslant a,$$

where $c_k \in \mathbb{R}$, $k = 1, 2, \ldots, n$.

Now, we consider the Green's function for our problem.

Lemma 2. *Suppose* (H1) *holds and* $\mu \in (2,3)$, $y \in L[a,b]$. *Then the Hadamard-type fractional boundary value problem*

$$\begin{aligned} &{}^{H}D^{\mu}_{a^+}x(t) + y(t) = 0, \quad t \in (a,b), \\ &x(a) = x'(a) = 0, \qquad x(b) = \int_a^b h(t)x(t)\frac{\mathrm{d}t}{t}, \end{aligned} \tag{2}$$

has a solution

$$x(t) = \int_a^b H(t,\tau)y(\tau)\frac{\mathrm{d}\tau}{\tau}, \quad t \in [a,b],$$

where

$$H(t,\tau) = G(t,\tau) + \frac{(\ln \frac{t}{a})^{\mu-1}}{A}\int_a^b h(t)G(t,\tau)\frac{\mathrm{d}t}{t},$$

$$G(t,\tau) = \frac{1}{\Gamma(\mu)(\ln \frac{b}{a})^{\mu-1}}\begin{cases} (\ln \frac{t}{a})^{\mu-1}l(\ln \frac{b}{\tau})^{\mu-1} - (\ln \frac{t}{\tau})^{\mu-1}(\ln \frac{b}{a})^{\mu-1}, & a \leqslant \tau \leqslant t \leqslant b, \\ (\ln \frac{t}{a})^{\mu-1}(\ln \frac{b}{\tau})^{\mu-1}, & a \leqslant t \leqslant \tau \leqslant b, \end{cases}$$

$$A = \left(\ln \frac{b}{a}\right)^{\mu-1} - \int_a^b h(t)\left(\ln \frac{t}{a}\right)^{\mu-1}\frac{\mathrm{d}t}{t}.$$

Proof. From Lemma 1 we have for $t \in [a,b]$,

$$x(t) = c_1\left(\ln \frac{t}{a}\right)^{\mu-1} + c_2\left(\ln \frac{t}{a}\right)^{\mu-2} + c_3\left(\ln \frac{t}{a}\right)^{\mu-3} - \frac{1}{\Gamma(\mu)}\int_a^t \left(\ln \frac{t}{\tau}\right)^{\mu-1}y(\tau)\frac{\mathrm{d}\tau}{\tau},$$

where $c_i \in \mathbb{R}$, $i = 1,2,3$. Using $x(a) = x'(a) = 0$, we obtain $c_2 = c_3 = 0$. Therefore,

$$x(t) = c_1\left(\ln \frac{t}{a}\right)^{\mu-1} - \frac{1}{\Gamma(\mu)}\int_a^t \left(\ln \frac{t}{\tau}\right)^{\mu-1}y(\tau)\frac{\mathrm{d}\tau}{\tau}, \quad t \in [a,b].$$

Consequently, we obtain

$$\begin{aligned} x(b) &= c_1\left(\ln \frac{b}{a}\right)^{\mu-1} - \frac{1}{\Gamma(\mu)}\int_a^b \left(\ln \frac{b}{\tau}\right)^{\mu-1}y(\tau)\frac{\mathrm{d}\tau}{\tau} = \int_a^b h(t)x(t)\frac{\mathrm{d}t}{t} \\ &= c_1\int_a^b h(t)\left(\ln \frac{t}{a}\right)^{\mu-1}\frac{\mathrm{d}t}{t} - \frac{1}{\Gamma(\mu)}\int_a^b h(t)\int_a^t \left(\ln \frac{t}{\tau}\right)^{\mu-1}y(\tau)\frac{\mathrm{d}\tau}{\tau}\frac{\mathrm{d}t}{t} \end{aligned}$$

and

$$c_1 = \frac{1}{A\Gamma(\mu)} \int_a^b \left(\ln \frac{b}{\tau}\right)^{\mu-1} y(\tau) \frac{\mathrm{d}\tau}{\tau} - \frac{1}{A\Gamma(\mu)} \int_a^b h(t) \int_a^t \left(\ln \frac{t}{\tau}\right)^{\mu-1} y(\tau) \frac{\mathrm{d}\tau}{\tau} \frac{\mathrm{d}t}{t}.$$

As a result, we have

$$\begin{aligned}
x(t) &= \frac{(\ln \frac{t}{a})^{\mu-1}}{A\Gamma(\mu)} \int_a^b \left(\ln \frac{b}{\tau}\right)^{\mu-1} y(\tau) \frac{\mathrm{d}\tau}{\tau} - \frac{(\ln \frac{t}{a})^{\mu-1}}{A\Gamma(\mu)} \int_a^b h(t) \int_a^t \left(\ln \frac{t}{\tau}\right)^{\mu-1} y(\tau) \frac{\mathrm{d}\tau}{\tau} \frac{\mathrm{d}t}{t} \\
&\quad - \frac{1}{\Gamma(\mu)} \int_a^t \left(\ln \frac{t}{\tau}\right)^{\mu-1} y(\tau) \frac{\mathrm{d}\tau}{\tau} \\
&= \frac{(\ln \frac{t}{a})^{\mu-1}}{A\Gamma(\mu)} \int_a^b \left(\ln \frac{b}{\tau}\right)^{\mu-1} y(\tau) \frac{\mathrm{d}\tau}{\tau} - \frac{(\ln \frac{t}{a})^{\mu-1}}{A\Gamma(\mu)} \int_a^b h(t) \int_a^t \left(\ln \frac{t}{\tau}\right)^{\mu-1} y(\tau) \frac{\mathrm{d}\tau}{\tau} \frac{\mathrm{d}t}{t} \\
&\quad - \frac{1}{\Gamma(\mu)} \int_a^t \left(\ln \frac{t}{\tau}\right)^{\mu-1} y(\tau) \frac{\mathrm{d}\tau}{\tau} + \frac{1}{\Gamma(\mu)(\ln \frac{b}{a})^{\mu-1}} \int_a^b \left(\ln \frac{t}{a}\right)^{\mu-1} \left(\ln \frac{b}{\tau}\right)^{\mu-1} y(\tau) \frac{\mathrm{d}\tau}{\tau} \\
&\quad - \frac{1}{\Gamma(\mu)(\ln \frac{b}{a})^{\mu-1}} \int_a^b \left(\ln \frac{t}{a}\right)^{\mu-1} \left(\ln \frac{b}{\tau}\right)^{\mu-1} y(\tau) \frac{\mathrm{d}\tau}{\tau} \\
&= \int_a^b G(t,\tau) y(\tau) \frac{\mathrm{d}\tau}{\tau} + \frac{\int_a^b h(t)(\ln \frac{t}{a})^{\mu-1} \frac{\mathrm{d}t}{t}}{A\Gamma(\mu)(\ln \frac{b}{a})^{\mu-1}} \int_a^b \left(\ln \frac{t}{a}\right)^{\mu-1} \left(\ln \frac{b}{\tau}\right)^{\mu-1} y(\tau) \frac{\mathrm{d}\tau}{\tau} \\
&\quad - \frac{(\ln \frac{t}{a})^{\mu-1}}{A\Gamma(\mu)} \int_a^b h(t) \int_a^t \left(\ln \frac{t}{\tau}\right)^{\mu-1} y(\tau) \frac{\mathrm{d}\tau}{\tau} \frac{\mathrm{d}t}{t} \\
&= \int_a^b G(t,\tau) y(\tau) \frac{\mathrm{d}\tau}{\tau} + \frac{(\ln \frac{t}{a})^{\mu-1}}{A\Gamma(\mu)(\ln \frac{b}{a})^{\mu-1}} \Bigg[\int_a^b h(t) \Bigg(\int_a^b \left(\ln \frac{t}{a}\right)^{\mu-1} \left(\ln \frac{b}{\tau}\right)^{\mu-1} y(\tau) \frac{\mathrm{d}\tau}{\tau} \\
&\quad - \int_a^t \left(\ln \frac{t}{\tau}\right)^{\mu-1} \left(\ln \frac{b}{a}\right)^{\mu-1} y(\tau) \frac{\mathrm{d}\tau}{\tau} \Bigg) \frac{\mathrm{d}t}{t} \Bigg] \\
&= \int_a^b G(t,\tau) y(\tau) \frac{\mathrm{d}\tau}{\tau} + \frac{(\ln \frac{t}{a})^{\mu-1}}{A} \int_a^b h(t) \int_a^b G(t,\tau) y(\tau) \frac{\mathrm{d}\tau}{\tau} \frac{\mathrm{d}t}{t} \\
&= \int_a^b H(t,\tau) y(\tau) \frac{\mathrm{d}\tau}{\tau}.
\end{aligned}$$

□

Remark 1. In Lemma 2.4 of [7], the following problem was considered:

$$\begin{aligned} &{}^{H}D^{\mu}_{a+}x(t) + y(t) = 0, \quad t \in (a,b),\\ &x(a) = x'(a) = x(b) = 0, \end{aligned} \tag{3}$$

and the solution can take the form $x(t) = \int_a^b G(t,\tau)y(\tau)/\tau \, \mathrm{d}\tau$, where G is defined in Lemma 2.

Although we consider the integral boundary value problem, we do not construct new Green's functions for our problem. Indeed, if (2) is regarded as a perturbation of (3), then our Green's functions can be obtained from (3), and thus we can use the properties of G to study H.

Lemma 3. (*See* [7]). *For* $t, \tau, s \in [a,b]$, *we have*

$$\text{(i)} \quad G(t,\tau) \leqslant \frac{v(\tau)}{\Gamma(\mu)(\ln \frac{b}{a})}, \qquad \text{(ii)} \quad G(t,\tau) \geqslant \frac{u(t)v(\tau)}{\Gamma(\mu)(\ln \frac{b}{a})^{\mu+1}},$$

where $u(t) = (\ln(t/a)^{\mu-1}\ln(b/t)$, $v(t) = \ln(t/a)\ln(b/t)^{\mu-1}$.

Lemma 4. *For* $t, \tau, s \in [a,b]$, *we have*

$$\text{(i)} \quad H(t,\tau) \leqslant \frac{v(\tau)}{\Gamma(\mu)(\ln \frac{b}{a})}\left[1 + \frac{(\ln \frac{b}{a})^{\mu-1}}{A}\int_a^b h(t)\frac{\mathrm{d}t}{t}\right],$$

$$\text{(ii)} \quad H(t,\tau) \geqslant \frac{u(t)v(\tau)}{\Gamma(\mu)(\ln \frac{b}{a})^{\mu+1}}.$$

This is directly from Lemma 3, so we omit its proof.

For $x \in C[a,b]$, let $\|x\| = \max_{t\in[a,b]}|x(t)|$. Then $(C[a,b], \|\cdot\|)$ is a Banach space. For $r > 0$, $B_r := \{x \in C[a,b]\colon \|x\| < r\}$ is a bounded and open subset in $C[a,b]$, and $P := \{x \in C[a,b]\colon x(t) \geqslant 0,\ t \in [a,b]\}$ is a cone of $C[a,b]$. Define an operator $T : C[a,b] \to C[a,b]$ as follows:

$$(Tx)(t) = \int_a^b H(t,\tau)f\bigl(\tau, x(\tau)\bigr)\frac{\mathrm{d}\tau}{\tau}, \quad x \in C[a,b],\ t \in [a,b].$$

Note that the nonnegativity and continuity of H, f imply that $T : P \to P$ is a completely continuous operator, and the existence of positive solutions for (1) is equivalent to that of positive fixed points of T.

Define a linear operator $L : P \to P$ as follows:

$$(Lx)(t) = \int_a^b H(t,\tau)x(\tau)\frac{\mathrm{d}\tau}{\tau}, \quad x \in C[a,b],\ t \in [a,b].$$

Then L is a linear operator, and for all $n \in \mathbb{N}_+$, from Lemma 4(ii) we have

$$\begin{aligned}\|L^n\| &= \max_{t\in[a,b]} \int_a^b \int_a^b \cdots \int_a^b H(t,y_1)H(y_1,y_2)\cdots H(y_{n-1},y_n)\frac{\mathrm{d}y_1}{y_1}\frac{\mathrm{d}y_2}{y_2}\cdots\frac{\mathrm{d}y_n}{y_n} \\ &\geqslant \max_{t\in[a,b]} u(t)\int_a^b \int_a^b \cdots \int_a^b \frac{v(y_1)}{\Gamma(\mu)(\ln\frac{b}{a})^{\mu+1}}\frac{u(y_1)v(y_2)}{\Gamma(\mu)(\ln\frac{b}{a})^{\mu+1}}\cdots \\ &\qquad\qquad \times \frac{u(y_{n-1})v(y_n)}{\Gamma(\mu)(\ln\frac{b}{a})^{\mu+1}}\frac{\mathrm{d}y_1}{y_1}\frac{\mathrm{d}y_2}{y_2}\cdots\frac{\mathrm{d}y_n}{y_n} \\ &= \max_{t\in[a,b]} u(t)\int_a^b \frac{v(\tau)}{\Gamma(\mu)(\ln\frac{b}{a})^{\mu+1}}\frac{\mathrm{d}\tau}{\tau}\left(\int_a^b \frac{u(\tau)v(\tau)}{\Gamma(\mu)(\ln\frac{b}{a})^{\mu+1}}\frac{\mathrm{d}\tau}{\tau}\right)^{n-1}.\end{aligned}$$

By the Gelfand theorem we have

$$r(L) = \lim_{n\to+\infty} \sqrt[n]{\|L^n\|} \geqslant \int_a^b \frac{u(\tau)v(\tau)}{\Gamma(\mu)(\ln\frac{b}{a})^{\mu+1}}\frac{\mathrm{d}\tau}{\tau} > 0,$$

where $r(L)$ denotes the spectral radius of L. Then the Krein–Rutman theorem [16] asserts that L has an eigenfunction $\varphi \in P \setminus \{0\}$ corresponding to its first eigenvalue $\lambda_1 = (r(L))^{-1}$, i.e.,

$$\varphi = \lambda_1 L\varphi. \tag{4}$$

Lemma 5. (*See* [11].) *Let E be a Banach space, P a cone in E, and $\Omega(P)$ a bounded open set in P. Suppose $T : \overline{\Omega(P)} \to P$ is a continuous compact operator. If there exists $u_0 \in P \setminus \{0\}$ such that*

$$u - Tu \neq \mu u_0 \quad \forall u \in \partial\Omega(P),\ \mu \geqslant 0,$$

then the fixed point index $i(T, \Omega(P), P) = 0$.

Lemma 6. (*See* [11].) *Let E be a Banach space, P a cone in E, and $\Omega(P)$ a bounded open set in P with $0 \in \Omega(P)$. Suppose $T : \overline{\Omega(P)} \to P$ is a continuous compact operator. If*

$$Tu \neq \mu u \quad \forall u \in \partial\Omega(P),\ \mu \geqslant 1,$$

then the fixed point index $i(T, \Omega(P), P) = 1$.

Let E be a real Banach space with cone P. A map $\beta : P \to [0, +\infty)$ is said to be a nonnegative continuous concave functional on P if β is continuous and

$$\beta\big(tx + (1-t)y\big) \geqslant t\beta(x) + (1-t)\beta(y)$$

for all $x, y \in P$ and $t \in [0,1]$. Let $\widetilde{a}, \widetilde{b}$ be two numbers such that $0 < \widetilde{a} < \widetilde{b}$, and let β be a nonnegative continuous concave functional on P. We define the following convex sets:

$$P_{\widetilde{a}} = \{x \in P\colon\ \|x\| < \widetilde{a}\}, \qquad \partial P_{\widetilde{a}} = \{x \in P\colon\ \|x\| = \widetilde{a}\},$$
$$\overline{P}_{\widetilde{a}} = \{x \in P\colon\ \|x\| \leqslant \widetilde{a}\}, \qquad P(\beta, \widetilde{a}, \widetilde{b}) = \{x \in P\colon\ \widetilde{a} \leqslant \beta(x),\ \|x\| \leqslant \widetilde{b}\}.$$

Lemma 7. (*See* [18].) *Let* $T : \overline{P}_{\widetilde{c}} \to \overline{P}_{\widetilde{c}}$ *be completely continuous, and let* β *a nonnegative continuous concave functional on* P *such that* $\beta(x) \leqslant \|x\|$ *for* $x \in \overline{P}_{\widetilde{c}}$. *Suppose there exist* $0 < \widetilde{d} < \widetilde{a} < \widetilde{b} \leqslant \widetilde{c}$ *such that*

(i) $\{x \in P(\beta, \widetilde{a}, \widetilde{b})\colon\ \beta(x) > \widetilde{a}\} \neq \emptyset$ *and* $\beta(Tx) > \widetilde{a}$ *for* $x \in P(\beta, \widetilde{a}, \widetilde{b})$,

(ii) $\|Tx\| < \widetilde{d}$ *for* $\|x\| \leqslant \widetilde{d}$,

(iii) $\beta(Tx) > \widetilde{a}$ *for* $x \in P(\beta, \widetilde{a}, \widetilde{c})$ *with* $\|Tx\| > \widetilde{b}$.

Then T *has at least three fixed points* x_1, x_2, x_3 *in* $\overline{P}_{\widetilde{c}}$ *such that*

$$\|x_1\| < \widetilde{d}, \quad \widetilde{a} < \beta(x_2), \quad \text{and} \quad \|x_3\| > \widetilde{d}, \quad \beta(x_3) < \widetilde{a}.$$

Definition 3. We say that $x \in E$ is an upper solution of (1) if

$$-{}^H D^{\mu}_{a+} x(t) \geqslant f\big(t, x(t)\big), \quad t \in (a,b),$$
$$x(a) = x'(a) = 0, \qquad x(b) \geqslant \int_a^b h(t)x(t)\frac{\mathrm{d}t}{t}.$$

Definition 4. We say that $x \in E$ is a lower solution of (1) if

$$-{}^H D^{\mu}_{a+} x(t) \leqslant f\big(t, x(t)\big), \quad t \in (a,b),$$
$$x(a) = x'(a) = 0, \qquad x(b) \leqslant \int_a^b h(t)x(t)\frac{\mathrm{d}t}{t}.$$

Lemma 8. *Suppose that* (H1) *holds and* $x \in E$ *satisfies*

$$\begin{aligned} &-{}^H D^{\mu}_{a+} x(t) \geqslant 0, \quad t \in (a,b), \\ &x(a) = x'(a) = 0, \qquad x(b) \geqslant \int_a^b h(t)x(t)\frac{\mathrm{d}t}{t}. \end{aligned} \tag{5}$$

Then $x(t) \geqslant 0$, $t \in [a,b]$.

Proof. Let $-{}^H D^{\mu}_{a+} x(t) = z(t)$ and $M = x(b) - \int_a^b h(t)x(t)/t\,\mathrm{d}t$. From (5) we have $z(t) \geqslant 0$, $M \geqslant 0$, $t \in [a,b]$. Therefore, we obtain the following auxiliary linear boundary value problem:

$${}^H D^{\mu}_{a+} x(t) + z(t) = 0,$$
$$x(a) = x'(a) = 0, \qquad x(b) = \int_a^b h(t)x(t)\frac{\mathrm{d}t}{t} + M.$$

From Lemma 2 we have

$$x(t) = \int_a^b H(t,\tau)z(\tau)\frac{\mathrm{d}\tau}{\tau} + \frac{M}{A}\left(\ln\frac{t}{a}\right)^{\mu-1}, \quad t \in [a,b].$$

Note the nonnegativity of H, z, M, A, we obtain

$$x(t) \geqslant \frac{M}{A}\left(\ln\frac{t}{a}\right)^{\mu-1} \geqslant 0, \quad t \in [a,b].$$

This completes the proof. □

3 Main results

Let $\lambda_1 = r^{-1}(L)$. Now, we present our first main result.

Theorem 1. *Suppose that* (H1)–(H2) *and the following conditions hold*:

(H3) $\liminf_{x\to 0^+} f(t,x)/x > \lambda_1$ *uniformly on* $t \in [a,b]$,
(H4) $\limsup_{x\to +\infty} f(t,x)/x < \lambda_1$ *uniformly on* $t \in [a,b]$.

Then (1) *has at least one positive solution.*

Proof. From (H3) there exists $r_1 > 0$ such that

$$f(t,x) \geqslant \lambda_1 x, \quad x \in [0,r_1],\ t \in [a,b]. \tag{6}$$

For all $x \in \partial B_{r_1} \cap P$, from (6) we have

$$(Tx)(t) \geqslant \int_a^b H(t,\tau)\lambda_1 x(\tau)\frac{\mathrm{d}\tau}{\tau} := (L_1 x)(t), \quad t \in [a,b]. \tag{7}$$

Then $r(L_1) = 1$, and there exists $x^* \in P \setminus \{0\}$ such that

$$L_1 x^* = r(L_1)x^* = x^*. \tag{8}$$

Now, we shall prove that

$$x - Tx \neq \mu x^*, \quad x \in \partial B_{r_1} \cap P,\ \mu \geqslant 0. \tag{9}$$

On the contrary, assume that there exist $x_0 \in \partial B_{r_1} \cap P$, $\mu_0 \geqslant 0$ such that $x_0 - Tx_0 = \mu_0 x^*$. Then $\mu_0 > 0$ and $x_0 = Tx_0 + \mu_0 x^*$. Let $\mu^* = \sup\{\mu\colon x_0 \geqslant \mu x^*\}$. Then $\mu^* \geqslant \mu_0 > 0$, $x_0 \geqslant \mu^* x^*$, and from (7)–(8) we have

$$x_0 = Tx_0 + \mu_0 x^* \geqslant L_1 x_0 + \mu_0 x^* \geqslant L_1 \mu^* x^* + \mu_0 x^* = \mu^* x^* + \mu_0 x^*.$$

This contradicts the definition of μ^*. Therefore, (9) holds, and Lemma 5 implies that

$$i(T,\ B_{r_1} \cap P,\ P) = 0. \tag{10}$$

From (H4) there exist $\delta_1 \in (0, \lambda_1)$ and $c_1 > 0$ such that $f(t, x) \leqslant (\lambda_1 - \delta_1)x + c_1$, $x \geqslant 0$, $t \in [a, b]$. Consequently, we have

$$(Tx)(t) \leqslant \int_a^b H(t,\tau)\big[(\lambda_1 - \delta_1)x(\tau) + c_1\big]\frac{\mathrm{d}\tau}{\tau} = (\lambda_1 - \delta_1)(Lx)(t) + c_1 \int_a^b H(t,\tau)\frac{\mathrm{d}\tau}{\tau}.$$

Let

$$(L_2 x)(t) = (\lambda_1 - \delta_1)(Lx)(t) \quad \text{and} \quad \overline{x}(t) = c_1 \int_a^b H(t,\tau)\frac{\mathrm{d}\tau}{\tau}.$$

Then $r(L_2) = 1 - \delta_1/\lambda_1 < 1$, which implies that $(I - L_2)^{-1}$ exists and

$$(I - L_2)^{-1} = I + L_2 + L_2^2 + \cdots + L_2^n + \cdots. \tag{11}$$

Define a set

$$S = \{x \in P\colon Tx = \mu x,\ \mu \geqslant 1\}.$$

Now, we prove that S is bounded in P. Indeed, if $x \in S$, we have

$$x(t) \leqslant (Tx)(t) \leqslant (L_2 x)(t) + \overline{x}(t), \quad t \in [a, b].$$

This gives $(I - L_2)x \leqslant \overline{x}$. Note that (11) implies that $(I - L_2)^{-1} : P \to P$, and hence we have $\|x\| \leqslant \|(I - L_2)^{-1}\overline{x}\|$. As a result S is bounded. Now, we can take $R_1 > \sup S$ and $R_1 > r_1$ such that $Tx \neq \mu x$, $x \in \partial B_{R_1} \cap P$, $\mu \geqslant 1$. Lemma 6 implies that

$$i(T,\ B_{R_1} \cap P,\ P) = 1. \tag{12}$$

From (10) and (12) we have

$$i\big(T,\ (B_{R_1} \setminus \overline{B_{r_1}}) \cap P,\ P\big) = i(T,\ B_{R_1} \cap P,\ P) - i(T,\ B_{r_1} \cap P,\ P) = 1 - 0 = 1.$$

Then T has a fixed point in $(B_{R_1} \setminus \overline{B_{r_1}}) \cap P$, i.e., (1) has at least one positive solution. This completes the proof. □

To obtain our next main result, we study the conjugate operator of L denoted by L^*. Let P^* be the conjugate cone of P, and let L^* can be expressed as

$$(L^* x)(t) = \int_a^b H(\tau, t)x(\tau)\frac{\mathrm{d}\tau}{\tau}, \quad x \in P,\ t \in [a, b].$$

By the Krein–Rutman theorem [16] there exists $g^*(t) \geqslant 0$ with $g^*(t) \not\equiv 0$, $t \in [a, b]$ ($g^* \in P^*$) such that

$$L^* g^* = r(L) g^*. \tag{13}$$

Lemma 9. *Let* $P_0 = \{x \in P\colon \int_a^b x(t)g^*(t)/t\,\mathrm{d}t \geqslant \omega_0\|x\|\}$. *Then* $L(P) \subset P_0$, *where* $w_0 = \int_a^b \omega(t)g^*(t)/t\,\mathrm{d}t$, *and* w *is given in the proof.*

Proof. From Lemma 4(i)–(ii) we have

$$\begin{aligned} H(t,\tau) &\geqslant \frac{u(t)v(\tau)}{\Gamma(\mu)(\ln\frac{b}{a})^{\mu+1}} \\ &= \frac{u(t)}{(\ln\frac{b}{a})^{\mu}[1 + \frac{(\ln\frac{b}{a})^{\mu-1}}{A}\int_a^b h(t)\frac{\mathrm{d}t}{t}]}\frac{v(\tau)}{\Gamma(\mu)(\ln\frac{b}{a})}\left[1 + \frac{(\ln\frac{b}{a})^{\mu-1}}{A}\int_a^b h(t)\frac{\mathrm{d}t}{t}\right] \\ &\geqslant \omega(t)H(s,\tau), \quad t,s,\tau \in [a,b], \end{aligned}$$

where

$$\omega(t) = \frac{u(t)}{(\ln\frac{b}{a})^{\mu}[1 + \frac{(\ln\frac{b}{a})^{\mu-1}}{A}\int_a^b h(t)\frac{\mathrm{d}t}{t}]}, \quad t \in [a,b].$$

If $x \in P$, we have

$$\begin{aligned} \int_a^b (Lx)(t)g^*(t)\frac{\mathrm{d}t}{t} &= \int_a^b\int_a^b H(t,\tau)x(\tau)\frac{\mathrm{d}\tau}{\tau}g^*(t)\frac{\mathrm{d}t}{t} \geqslant \int_a^b\int_a^b \omega(t)H(s,\tau)x(\tau)\frac{\mathrm{d}\tau}{\tau}g^*(t)\frac{\mathrm{d}t}{t} \\ &= \omega_0(Lx)(s), \quad s \in [a,b]. \end{aligned}$$

This implies that

$$\int_a^b (Lx)(t)g^*(t)\frac{\mathrm{d}t}{t} \geqslant \omega_0\|Lx\|.$$

This completes the proof. □

Theorem 2. *Suppose that* (H1)–(H2) *and the following conditions hold*:

(H5) $\limsup_{x\to 0^+} f(t,x)/x < \lambda_1$ *uniformly on* $t \in [a,b]$,
(H6) $\liminf_{x\to +\infty} f(t,x)/x > \lambda_1$ *uniformly on* $t \in [a,b]$.

Then (1) *has at least one positive solution.*

Proof. From (H5) there exist $r_2 > 0$ and $\delta_2 \in (0,\lambda_1)$ such that

$$f(t,x) \leqslant (\lambda_1 - \delta_2)x, \quad x \in [0,r_2],\ t \in [a,b]. \tag{14}$$

Now, we claim that

$$Tx \neq \mu x, \quad x \in \partial B_{r_2} \cap P,\ \mu \geqslant 1. \tag{15}$$

On the contrary, assume that there exist $x_1 \in \partial B_{r_2} \cap P$, $\mu_1 \geqslant 1$ such that

$$Tx_1 = \mu_1 x_1. \tag{16}$$

Define the Nemytskii operator $F : P \to P$ by $(Fx)(t) := f(t, x(t))$. Then by Lemma 9 we have

$$x_1 = \frac{1}{\mu_1} T x_1 = \frac{1}{\mu_1} L \circ F(x_1) \in P_0. \tag{17}$$

Using (14) and (16), we obtain

$$x_1(t) \leqslant (Tx_1)(t) \leqslant (\lambda_1 - \delta_2) \int_a^b H(t,\tau) x_1(\tau) \frac{\mathrm{d}\tau}{\tau}. \tag{18}$$

Multiply (18) by $g^*(t)$ on both sides and integrate over $[a, b]$, then use (13) to obtain

$$\begin{aligned} \int_a^b x_1(t) g^*(t) \frac{\mathrm{d}t}{t} &\leqslant (\lambda_1 - \delta_2) \int_a^b \int_a^b H(t,\tau) x_1(\tau) \frac{\mathrm{d}\tau}{\tau} g^*(t) \frac{\mathrm{d}t}{t} \\ &= r(L)(\lambda_1 - \delta_2) \int_a^b x_1(\tau) g^*(\tau) \frac{\mathrm{d}\tau}{\tau}. \end{aligned}$$

Note that $\lambda_1 = r^{-1}(L)$, then we have

$$\int_a^b x_1(t) g^*(t) \frac{\mathrm{d}t}{t} = 0.$$

From (17) we have $\|x_1\| = 0$, which contradicts $x_1 \in \partial B_{r_2} \cap P$. As a result, (15) holds. From Lemma 6 we obtain

$$i(T,\, B_{r_2} \cap P,\, P) = 1. \tag{19}$$

From (H6) there exist $\delta_3 > 0$ and $c_2 > 0$ such that

$$f(t,x) \geqslant (\lambda_1 + \delta_3) x - c_2, \quad x \geqslant 0,\ t \in [a,b]. \tag{20}$$

Now, we claim that there exists $R_2 > r_2$ such that

$$x - Tx \neq \mu\varphi, \quad x \in \partial B_{R_2} \cap P,\ \mu \geqslant 0, \tag{21}$$

where φ is defined by (4). If the claim (21) is false, then there exist $x_2 \in \partial B_{R_2} \cap P$, $\mu_2 \geqslant 0$ such that

$$x_2 - Tx_2 = \mu_2 \varphi. \tag{22}$$

Consequently, we have

$$x_2 = Tx_2 + \mu_2\varphi = L \circ F(x_2) + \mu_2 \lambda_1 L\varphi \in P_0.$$

From (20) and (22) we have

$$x_2(t) \geqslant (Tx_2)(t) \geqslant \int_a^b H(t,\tau) \big[(\lambda_1 + \delta_3) x_2(\tau) - c_2\big] \frac{\mathrm{d}\tau}{\tau}. \tag{23}$$

Multiply (23) by $g^*(t)$ on both sides and integrate over $[a,b]$, then use (13) to obtain

$$\int_a^b x_2(t)g^*(t)\frac{\mathrm{d}t}{t} \geqslant \int_a^b\int_a^b H(t,\tau)\big[(\lambda_1+\delta_3)x_2(\tau)-c_2\big]\frac{\mathrm{d}\tau}{\tau}(t)g^*(t)\frac{\mathrm{d}t}{t}$$
$$= r(L)\int_a^b g^*(\tau)\big[(\lambda_1+\delta_3)x_2(\tau)-c_2\big]\frac{\mathrm{d}\tau}{\tau}.$$

Solving this inequality, we obtain

$$\int_a^b x_2(t)g^*(t)\frac{\mathrm{d}t}{t} \leqslant c_2\delta_3^{-1}\int_a^b g^*(\tau)\frac{\mathrm{d}\tau}{\tau}.$$

Note that $x_2 \in P_0$, and then we have

$$\|x_2\| \leqslant c_2(\delta_3\omega_0)^{-1}\int_a^b g^*(\tau)\frac{\mathrm{d}\tau}{\tau}.$$

Now, if we take

$$R_2 > \max\left\{r_2,\ c_2(\delta_3\omega_0)^{-1}\int_a^b g^*(\tau)\frac{\mathrm{d}\tau}{\tau}\right\},$$

then (22) does not hold. This implies that (21) holds, and from Lemma 5 we have

$$i(T,\ B_{R_2}\cap P,\ P) = 0. \tag{24}$$

From (19) and (24) we have

$$i\big(T,\ (B_{R_2}\setminus\overline{B_{r_2}})\cap P,\ P\big) = i(T,\ B_{R_2}\cap P,\ P) - i(T,\ B_{r_2}\cap P,\ P)$$
$$= 0-1 = -1.$$

Then T has a fixed point in $(B_{R_2}\setminus\overline{B_{r_2}})\cap P$, i.e., (1) has at least one positive solution. This completes the proof. □

In what follows, we shall use Lemma 7 to study the existence of three positive solutions for (1). We first provide a useful lemma.

Lemma 10. *Let* $P_1 = \{x\in P\colon \min_{t\in[t_0,t_1]} x(t) \geqslant \gamma_1\|x\|\}$. *Then* $T(P)\subset P_1$, *where* $t_0 \in (a,\ \sqrt[\mu]{ab^{\mu-1}})$, $t_1 \in (\sqrt[\mu]{ab^{\mu-1}}, b)$,

$$\gamma_1 = \frac{\min\{u(t_0),u(t_1)\}}{(\ln\frac{b}{a})^\mu[1+\frac{(\ln\frac{b}{a})^{\mu-1}}{A}\int_a^b h(t)\frac{\mathrm{d}t}{t}]},$$

and u *is defined as in Lemma* 3.

Proof. From Lemma 4, if $x \in P$, we have

$$(Tx)(t) \leqslant \int_a^b \frac{v(\tau)}{\Gamma(\mu)(\ln\frac{b}{a})}\left[1+\frac{(\ln\frac{b}{a})^{\mu-1}}{A}\int_a^b h(t)\frac{\mathrm{d}t}{t}\right] f\big(\tau,x(\tau)\big)\frac{\mathrm{d}\tau}{\tau}, \quad t\in[a,b],$$

and

$$\begin{aligned}(Tx)(t) &\geqslant \int_a^b \frac{u(t)v(\tau)}{\Gamma(\mu)(\ln\frac{b}{a})^{\mu+1}} f\big(\tau,x(\tau)\big)\frac{\mathrm{d}\tau}{\tau} \\ &= \frac{u(t)}{(\ln\frac{b}{a})^{\mu}[1+\frac{(\ln\frac{b}{a})^{\mu-1}}{A}\int_a^b h(t)\frac{\mathrm{d}t}{t}]} \\ &\quad\times \int_a^b \frac{v(\tau)}{\Gamma(\mu)(\ln\frac{b}{a})}\left[1+\frac{(\ln\frac{b}{a})^{\mu-1}}{A}\int_a^b h(t)\frac{\mathrm{d}t}{t}\right] f\big(\tau,x(\tau)\big)\frac{\mathrm{d}\tau}{\tau}.\end{aligned}$$

This implies that

$$(Tx)(t) \geqslant \frac{u(t)}{(\ln\frac{b}{a})^{\mu}[1+\frac{(\ln\frac{b}{a})^{\mu-1}}{A}\int_a^b h(t)\frac{\mathrm{d}t}{t}]}\|Tx\|, \quad t\in[a,b]. \tag{25}$$

Next, we prove that

$$\min_{t\in[t_0,t_1]} u(t) = \min\big\{u(t_0),u(t_1)\big\}. \tag{26}$$

Note that $u(a)=u(b)=0$ and

$$u'(t) = \frac{1}{t}\left(\ln\frac{t}{a}\right)^{\mu-2}\big[(\mu-1)\ln b+\ln a-\mu\ln t\big].$$

Clearly, u has a unique stationary point $t=\sqrt[\mu]{ab^{\mu-1}}$ in $[t_0,t_1]$, and thus (26) holds. Therefore, by (25) we have

$$\min_{t\in[t_0,t_1]}(Tx)(t) \geqslant \min_{t\in[t_0,t_1]}\frac{u(t)}{(\ln\frac{b}{a})^{\mu}[1+\frac{(\ln\frac{b}{a})^{\mu-1}}{A}\int_a^b h(t)\frac{\mathrm{d}t}{t}]}\|Tx\| = \gamma_1\|Tx\|.$$

This completes the proof. □

Theorem 3. *Let* (H1)–(H2), (H4) *hold with* λ_1 *replaced by* ζ_1^{-1}. *Suppose there exist* $\widetilde{a} > \widetilde{d} > 0$ *such that the following conditions are satisfied*:

(H7) $f(t,x) \leqslant \widetilde{d}/\zeta_1$, $x\in[0,\widetilde{d}]$, $t\in[a,b]$,
(H8) $f(t,x) \geqslant \widetilde{a}/\zeta_2$, $x\in[\widetilde{a},\widetilde{a}/\gamma_1]$, $t\in[a,b]$,

where

$$\zeta_1 > \frac{[1+\frac{(\ln\frac{b}{a})^{\mu-1}}{A}\int_a^b h(t)\frac{\mathrm{d}t}{t}](\ln\frac{b}{a})^{\mu}}{\Gamma(\mu+2)}, \qquad 0<\zeta_2<\frac{\min\{u(t_0),u(t_1)\}}{\Gamma(\mu+2)}.$$

Then (1) *has at least three positive solutions.*

Proof. Let P_1 be defined as in Lemma 10. For $x \in P_1$, define $\beta(x) = \min_{t\in[t_0,t_1]} x(t)$. Then β is a nonnegative continuous concave functional on P_1, and the following inequality holds: $\beta(x) \leqslant \max_{t\in[a,b]} x(t) = \|x\|$, $x \in P_1$. From (H4) there exist $\delta_5 \in (0, \zeta_1^{-1})$ and $c_4 > 0$ such that

$$f(t,x) \leqslant (\zeta_1^{-1} - \delta_5)x + c_4, \quad x \geqslant 0,\ t \in [a,b]. \tag{27}$$

Choose $\widetilde{c} \geqslant \max\{\widetilde{a}/\gamma_1, c_4/\delta_5\}$. When $\|x\| \leqslant \widetilde{c}$, from (27) and Lemma 4(i) we have

$$\begin{aligned}
(Tx)(t) &\leqslant \max_{t\in[a,b]} \int_a^b H(t,\tau) f\bigl(\tau, x(\tau)\bigr)\frac{\mathrm{d}\tau}{\tau} \leqslant \max_{t\in[a,b]} \int_a^b H(t,\tau)\bigl[\bigl(\zeta_1^{-1} - \delta_5\bigr)x(\tau) + c_4\bigr]\frac{\mathrm{d}\tau}{\tau} \\
&\leqslant \int_a^b \frac{v(\tau)}{\Gamma(\mu)(\ln\frac{b}{a})}\left[1 + \frac{(\ln ba)^{\mu-1}}{A}\int_a^b h(t)\frac{\mathrm{d}t}{t}\right]\frac{\mathrm{d}\tau}{\tau}\bigl[\bigl(\zeta_1^{-1} - \delta_5\bigr)\widetilde{c} + c_4\bigr] \\
&= \zeta_1\bigl[\bigl(\zeta_1^{-1} - \delta_5\bigr)\widetilde{c} + c_4\bigr] \leqslant \widetilde{c}.
\end{aligned}$$

This implies that $T : \overline{P}_{\widetilde{c}} \to \overline{P}_{\widetilde{c}}$.

Now, we show that $\{x \in P(\beta, \widetilde{a}, \widetilde{a}/\gamma_1)\colon \beta(x) > \widetilde{a}\} \neq \emptyset$ and $\beta(Tx) > \widetilde{a}$ for all $x \in P(\beta, \widetilde{a}, \widetilde{a}/\gamma_1)$. In fact, choose $x(t) \equiv (\widetilde{a} + \widetilde{a}/\gamma_1)/2 > \widetilde{a}$, so $x \in \{x \in P(\beta, \widetilde{a}, \widetilde{a}/\gamma_1)\colon \beta(x) > \widetilde{a}\}$. Moreover, for $x \in P(\beta, \widetilde{a}, \widetilde{a}/\gamma_1)$, $\beta(x) > \widetilde{a}$, and we have $\widetilde{a}/\gamma_1 \geqslant \|x\| \geqslant \beta(x) > \widetilde{a}$. Therefore, by (H8) we obtain

$$\begin{aligned}
\beta(Tx) &= \min_{t\in[t_0,t_1]} \int_a^b H(t,\tau) f\bigl(\tau, x(\tau)\bigr)\frac{\mathrm{d}\tau}{\tau} \geqslant \min_{t\in[t_0,t_1]} \int_a^b \frac{u(t)v(\tau)}{\Gamma(\mu)(\ln\frac{b}{a})^{\mu+1}} f\bigl(\tau, x(\tau)\bigr)\frac{\mathrm{d}\tau}{\tau} \\
&\geqslant \frac{\min\{u(t_0), u(t_1)\}}{\Gamma(\mu)(\ln\frac{b}{a})^{\mu+1}} \int_a^b v(\tau) f\bigl(\tau, x(\tau)\bigr)\frac{\mathrm{d}\tau}{\tau} \geqslant \frac{\min\{u(t_0), u(t_1)\}}{\Gamma(\mu)(\ln\frac{b}{a})^{\mu+1}} \int_a^b v(\tau)\frac{\widetilde{a}}{\zeta_2}\frac{\mathrm{d}\tau}{\tau} \\
&> \widetilde{a}.
\end{aligned}$$

Next, we assert that $\|Tx\| < \widetilde{d}$ for $\|x\| \leqslant \widetilde{d}$. In fact, if $x \in \overline{P}_{\widetilde{d}}$, from (H7) we have

$$\begin{aligned}
\|Tx\| &\leqslant \max_{t\in[a,b]} \int_a^b H(t,\tau) f\bigl(\tau, x(\tau)\bigr)\frac{\mathrm{d}\tau}{\tau} \\
&\leqslant \max_{t\in[a,b]} \int_a^b \frac{v(\tau)}{\Gamma(\mu)(\ln\frac{b}{a})}\left[1 + \frac{(\ln\frac{b}{a})^{\mu-1}}{A}\int_a^b h(t)\frac{\mathrm{d}t}{t}\right] f\bigl(\tau, x(\tau)\bigr)\frac{\mathrm{d}\tau}{\tau} \\
&\leqslant \int_a^b \frac{v(\tau)}{\Gamma(\mu)(\ln\frac{b}{a})}\left[1 + \frac{(\ln\frac{b}{a})^{\mu-1}}{A}\int_a^b h(t)\frac{\mathrm{d}t}{t}\right]\frac{\widetilde{d}}{\zeta_1}\frac{\mathrm{d}\tau}{\tau} < \widetilde{d}.
\end{aligned}$$

Hence, $T : \overline{P}_{\widetilde{d}} \to P_{\widetilde{d}}$ for $x \in \overline{P}_{\widetilde{d}}$.

Finally, we assert that if $x \in P(\beta, \widetilde{a}, \widetilde{c})$ and $\|Tx\| > \widetilde{a}/\gamma_1$, then $\beta(Tx) > \widetilde{a}$. To see this, if $x \in P(\beta, \widetilde{a}, \widetilde{c})$ and $\|Tx\| > \widetilde{a}/\gamma_1$, then from Lemma 7 we have

$$\begin{aligned}
\beta(Tx) &= \min_{t\in[t_0,t_1]} \int_a^b H(t,\tau) f\bigl(\tau, x(\tau)\bigr)\frac{\mathrm{d}\tau}{\tau} \geqslant \min_{t\in[t_0,t_1]} \int_a^b \frac{u(t)v(\tau)}{\Gamma(\mu)(\ln\frac{b}{a})^{\mu+1}} f\bigl(\tau, x(\tau)\bigr)\frac{\mathrm{d}\tau}{\tau} \\
&= \frac{\min\{u(t_0), u(t_1)\}}{\Gamma(\mu)(\ln\frac{b}{a})^{\mu+1}} \int_a^b v(\tau) f\bigl(\tau, x(\tau)\bigr)\frac{\mathrm{d}\tau}{\tau} \\
&= \frac{\min\{u(t_0), u(t_1)\}\Gamma(\mu)(\ln\frac{b}{a})}{\Gamma(\mu)(\ln\frac{b}{a})^{\mu+1}[1+\frac{(\ln\frac{b}{a})^{\mu-1}}{A}\int_a^b h(t)\frac{\mathrm{d}t}{t}]} \\
&\quad\times \int_a^b \frac{v(\tau)}{\Gamma(\mu)(\ln\frac{b}{a})}\Biggl[1+\frac{(\ln\frac{b}{a})^{\mu-1}}{A}\int_a^b h(t)\frac{\mathrm{d}t}{t}\Biggr] f\bigl(\tau, x(\tau)\bigr)\frac{\mathrm{d}\tau}{\tau} \\
&\geqslant \gamma_1\|Tx\|.
\end{aligned}$$

Consequently, we have

$$\beta(Tx) \geqslant \gamma_1\|Tx\| > \gamma_1 \cdot \frac{\widetilde{a}}{\gamma_1} = \widetilde{a}.$$

As a result, all the conditions of Lemma 7 are satisfied by taking $\widetilde{b} = \widetilde{a}/\gamma_1$. Hence, T has at least three fixed points, i.e., (1) has at least three positive solutions x_1, x_2, and x_3 such that

$$\|x_1\| < \widetilde{d}, \quad \widetilde{a} < \beta(x_2), \quad \text{and} \quad \|x_3\| > \widetilde{d} \quad \text{with} \quad \beta(x_3) < \widetilde{a}.$$

This completes the proof. □

Theorem 4. *Let* (H1)–(H2) *hold. Assume that the following conditions are satisfied*:

- (H9) $w_0, v_0 \in E$ *are, respectively, the upper and lower solutions of problem* (1) *with* $v_0(t) \leqslant w_0(t)$, $t \in [a, b]$,
- (H10) f *is nondecreasing about the second variable, i.e.,* $f(t, x) \geqslant f(t, y)$ *if* $x \geqslant y$ *for* $t \in [a, b]$.

Then there exist monotone iterative sequences $\{v_n\}, \{w_n\} \subset [v_0, w_0]$ *such that* $v_n \to v^*$, $w_n \to w^*$ *as* $n \to \infty$ *uniformly in* $[v_0, w_0]$, *and* v^*, w^* *are a minimal and a maximal positive solution of* (1) *in* $[v_0, w_0]$, *respectively.*

Proof. We first define two sequences $\{v_n\}_{n=0}^{\infty}$ and $\{w_n\}_{n=0}^{\infty}$, which satisfy the following Hadamard-type fractional differential equations:

$$\begin{aligned}
&-{}^H D^{\mu}_{a^+} v_n(t) = f\bigl(t, v_{n-1}(t)\bigr), \quad t \in (a, b), \\
&v_n(a) = v_n'(a) = 0, \qquad v_n(b) = \int_a^b h(t) v_n(t)\frac{\mathrm{d}t}{t}
\end{aligned} \tag{28}$$

and

$$
\begin{aligned}
&-{}^{H}D^{\mu}_{a^+} w_n(t) = f\bigl(t, w_{n-1}(t)\bigr), \quad t \in (a,b),\\
&w_n(a) = w_n'(a) = 0, \qquad w_n(b) = \int\limits_a^b h(t) w_n(t) \frac{\mathrm{d}t}{t}.
\end{aligned}
\tag{29}
$$

From Lemma 2, (28)–(29) are, respectively, equivalent to the following integral equations:

$$
\begin{aligned}
v_n(t) &= \int\limits_a^b H(t,\tau) f\bigl(\tau, v_{n-1}(\tau)\bigr) \frac{\mathrm{d}\tau}{\tau} = (T v_{n-1})(t),\\
w_n(t) &= \int\limits_a^b H(t,\tau) f\bigl(\tau, w_{n-1}(\tau)\bigr) \frac{\mathrm{d}\tau}{\tau} = (T w_{n-1})(t).
\end{aligned}
$$

Note that $w_0 \geqslant v_0$, from (H10) we have

$$
w_1(t) = \int\limits_a^b H(t,\tau) f\bigl(\tau, w_0(\tau)\bigr) \frac{\mathrm{d}\tau}{\tau} \geqslant \int\limits_a^b H(t,\tau) f\bigl(\tau, v_0(\tau)\bigr) \frac{\mathrm{d}\tau}{\tau} = v_1(t), \quad t \in [a,b].
$$

Using mathematical induction, it is easy to verify that $w_n(t) \geqslant v_n(t)$, $t \in [a,b]$, $n = 1, 2, \dots$.

Next, we prove that $v_0(t) \leqslant v_1(t) \leqslant w_1(t) \leqslant w_0(t)$, $t \in [a,b]$. Let $z_v(t) = v_1(t) - v_0(t)$, $t \in [a,b]$. Then we have

$$
\begin{aligned}
-{}^{H}D^{\mu}_{a^+} z_v(t) &= -{}^{H}D^{\mu}_{a^+} \bigl[v_1(t) - v_0(t)\bigr] = -{}^{H}D^{\mu}_{a^+} v_1(t) + {}^{H}D^{\mu}_{a^+} v_0(t)\\
&\geqslant f\bigl(t, v_0(t)\bigr) - f\bigl(t, v_0(t)\bigr) = 0,\\
z_v(a) &= v_1(a) - v_0(a) = 0, \qquad z_v'(a) = v_1'(a) - v_0'(a) = 0,\\
z_v(b) &= v_1(b) - v_0(b) \geqslant \int\limits_a^b h(t) v_1(t) \frac{\mathrm{d}t}{t} - \int\limits_a^b h(t) v_0(t) \frac{\mathrm{d}t}{t} = \int\limits_a^b h(t) z_v(t) \frac{\mathrm{d}t}{t},
\end{aligned}
$$

and from Lemma 8 we have $z_v(t) \geqslant 0$, i.e., $v_1(t) \geqslant v_0(t)$, $t \in [a,b]$.

Let $z_w(t) = w_0(t) - w_1(t)$, $t \in [a,b]$. Then we have

$$
\begin{aligned}
-{}^{H}D^{\mu}_{a^+} z_w(t) &= -{}^{H}D^{\mu}_{a^+} \bigl[w_0(t) - w_1(t)\bigr] = -{}^{H}D^{\mu}_{a^+} w_0(t) + {}^{H}D^{\mu}_{a^+} w_1(t)\\
&\geqslant f\bigl(t, w_0(t)\bigr) - f\bigl(t, w_0(t)\bigr) = 0,\\
z_w(a) &= w_0(a) - w_1(a) = 0, \qquad z_w'(a) = w_0'(a) - w_1'(a) = 0,\\
z_w(b) &= w_0(b) - w_1(b) \geqslant \int\limits_a^b h(t) w_0(t) \frac{\mathrm{d}t}{t} - \int\limits_a^b h(t) w_1(t) \frac{\mathrm{d}t}{t} = \int\limits_a^b h(t) z_w(t) \frac{\mathrm{d}t}{t},
\end{aligned}
$$

and from Lemma 8 we have $z_w(t) \geqslant 0$, i.e., $w_0(t) \geqslant w_1(t)$, $t \in [a,b]$.

Now, we prove that w_1, v_1 are upper and lower solutions of problem (1), respectively. In fact, from (28) and (H10) we have

$$-{}^H D^{\mu}_{a+} v_1(t) = f\bigl(t, v_0(t)\bigr) \leqslant f\bigl(t, v_1(t)\bigr), \quad t \in (a, b),$$
$$v_1(a) = v_1'(a) = 0, \qquad v_1(b) = \int_a^b h(t) v_1(t) \frac{\mathrm{d}t}{t},$$

and from Definition 4 v_1 is a lower solution for (1). Furthermore, from (29) and (H10) we have

$$-{}^H D^{\mu}_{a+} w_1(t) = f\bigl(t, w_0(t)\bigr) \geqslant f\bigl(t, w_1(t)\bigr), \quad t \in (a, b),$$
$$w_1(a) = w_1'(a) = 0, \qquad w_1(b) = \int_a^b h(t) w_1(t) \frac{\mathrm{d}t}{t}.$$

Using Definition 3, w_1 is an upper solution for (1).

For w_2, v_2, w_1, v_1, we can repeat the above process. We obtain

$$v_1(t) \leqslant v_2(t) \leqslant w_2(t) \leqslant w_1(t), \quad t \in [a, b],$$

and w_2, v_2 are upper and lower solutions for (1), respectively.

Using mathematical induction, we obtain

$$v_0(t) \leqslant v_1(t) \leqslant \cdots \leqslant v_n(t) \leqslant \cdots \leqslant w_n(t) \leqslant \cdots \leqslant w_1(t) \leqslant w_0(t), \quad t \in [a, b].$$

It is easy to conclude that $\{v_n\}_{n=0}^{\infty}$ and $\{w_n\}_{n=0}^{\infty}$ are uniformly bounded in E, and from the monotone bounded theorem we have

$$\lim_{n\to\infty} v_n(t) = v^*(t), \qquad \lim_{n\to\infty} w_n(t) = w^*(t), \quad t \in [a, b].$$

Note that T is a completely continuous operator, and then

$$v^*(t) = (Tv^*)(t), \quad w^*(t) = (Tw^*)(t), \quad t \in [a, b],$$

i.e., v^*, w^* are solutions for (1).

Next, we need to prove that v^* and w^* are extremal solutions for (1) in $[v_0, w_0]$. Let $u \in [v_0, w_0]$ be any solution for (1). We assume that $v_m(t) \leqslant u(t) \leqslant w_m(t)$, $t \in [a, b]$, for some m. Let $p(t) = u(t) - v_{m+1}(t)$, $q(t) = w_{m+1}(t) - u(t)$. Then from (1), (28), and (H10) we have

$$\begin{aligned} -{}^H D^{\mu}_{a+} p(t) &= -{}^H D^{\mu}_{a+} \bigl[u(t) - v_{m+1}(t)\bigr] = -{}^H D^{\mu}_{a+} u(t) + {}^H D^{\mu}_{a+} v_{m+1}(t) \\ &= f\bigl(t, u(t)\bigr) - f\bigl(t, v_m(t)\bigr) \geqslant 0, \quad t \in (a, b), \end{aligned}$$
$$p(a) = p'(a) = 0, \qquad p(b) = \int_a^b h(t) p(t) \frac{\mathrm{d}t}{t}.$$

Lemma 8 implies that $p(t) \geqslant 0$, i.e., $u(t) \geqslant v_{m+1}(t)$, $t \in [a,b]$.

By (1), (29), and (H10) we have

$$-{}^H D^{\mu}_{a^+} q(t) = -{}^H D^{\mu}_{a^+} \big[w_{m+1}(t) - u(t)\big] = -{}^H D^{\mu}_{a^+} w_{m+1}(t) + {}^H D^{\mu}_{a^+} u(t)$$
$$= f\big(t, w_m(t)\big) - f\big(t, u(t)\big) \geqslant 0, \quad t \in (a,b),$$
$$q(a) = q'(a) = 0, \qquad q(b) = \int\limits_a^b h(t) q(t) \frac{\mathrm{d}t}{t}.$$

Lemma 8 implies that $q(t) \geqslant 0$, i.e., $w_{m+1}(t) \geqslant u(t)$, $t \in [a,b]$.

Therefore, applying mathematical induction, we obtain $v_n(t) \leqslant u(t) \leqslant w_n(t)$ for any $n \in \mathbb{N}$, $t \in [a,b]$. Taking the limit, we conclude $v^*(t) \leqslant u(t) \leqslant w^*(t)$, $t \in [0,1]$. This completes the proof. □

4 Examples

Let $a = 1$, $b = \mathrm{e}$, $\mu = 2.5$, $t_0 = \mathrm{e}^{2/5}$, $t_1 = \mathrm{e}^{4/5}$, $h(t) = \ln t$, $t \in [1, \mathrm{e}]$. Then $(\ln(b/a))^{\mu-1} = 1$ $\int_a^b h(t)(\ln(t/a))^{\mu-1}/t \,\mathrm{d}t = 2/7$, $A = 5/7$, $\sqrt[\mu]{ab^{\mu-1}} = \mathrm{e}^{3/5}$, and

$$\gamma_1 = \frac{\min\{u(t_0), u(t_1)\}}{(\ln \frac{b}{a})^{\mu}[1 + \frac{(\ln \frac{b}{a})^{\mu-1}}{A} \int_a^b h(t) \frac{\mathrm{d}t}{t}]} = \frac{0.143}{1.7} = 0.084.$$

Example 1. Let $f(t,x) = x^{\sigma_1}$, $\sigma_1 \in (0,1)$, $t \in [a,b]$, $x \in \mathbb{R}^+$. Then (H2)–(H4) hold, and from Theorem 1 we have that (1) has at least one positive solution.

Example 2. Let $f(t,x) = x^{\sigma_2}$, $\sigma_2 \in (1,+\infty)$, $t \in [a,b]$, $x \in \mathbb{R}^+$. Then (H2), (H5)–(H6) hold, and from Theorem 2 we have that (1) has at least one positive solution.

Example 3. Let $\widetilde{a} = 10$, $\widetilde{d} = 1$, $\zeta_1 = 0.156$, $\zeta_2 = 0.02$, and

$$f(t,x) = \begin{cases} t + 3x, & x \in [0,1],\ t \in [1,\mathrm{e}], \\ t + \frac{497}{9}x - \frac{470}{9}, & x \in [1,10],\ t \in [1,\mathrm{e}], \\ t + 500, & x \in [10,120],\ t \in [1,\mathrm{e}], \\ t + 10\sqrt{\frac{125}{6}x}, & x \geqslant 120,\ t \in [1,\mathrm{e}]. \end{cases}$$

Then

$$\limsup_{x \to +\infty} \frac{f(t,x)}{x} = \limsup_{x \to +\infty} \frac{t + 10\sqrt{\frac{125}{6}x}}{x} = 0$$

uniformly on $t \in [1,\mathrm{e}]$, and

$$f(t,x) \leqslant \mathrm{e} + 3 \leqslant \frac{\widetilde{d}}{\zeta_1} = 6.41, \quad x \in [0,1],\ t \in [1,\mathrm{e}],$$
$$f(t,x) \geqslant 1 + 500 \geqslant \frac{\widetilde{a}}{\zeta_2} = 500, \quad x \in [10, 119.05] \subset [10,120],\ t \in [1,\mathrm{e}].$$

Therefore, (H2), (H4), and (H7)–(H8) hold, and from Theorem 3 we have that (1) has at least three positive solutions.

Example 4. We use the Example 1 to illustrate Theorem 4.

Step 1. For $t,\tau \in [a,b]$, we have

$$\text{(i)}\ \ H(t,\tau) \leqslant \frac{\xi_A (\ln\frac{t}{a})^{\mu-1}(\ln\frac{b}{\tau})^{\mu-1}}{\Gamma(\mu)(\ln\frac{b}{a})^{\mu-1}}, \qquad \text{(ii)}\ \ H(t,\tau) \geqslant \frac{\xi_A (\ln\frac{t}{a})^{\mu-1}(\ln\frac{b}{\tau})^{\mu}\ln\frac{\tau}{a}}{\Gamma(\mu)(\ln\frac{b}{a})^{\mu+1}},$$

where

$$\xi_A = \left[1 + \frac{1}{A}\int_a^b h(t)\left(\ln\frac{t}{a}\right)^{\mu-1}\frac{\mathrm{d}t}{t}\right].$$

From the definition of H we have

$$H(t,\tau) \leqslant \frac{(\ln\frac{t}{a})^{\mu-1}(\ln\frac{b}{\tau})^{\mu-1}}{\Gamma(\mu)(\ln\frac{b}{a})^{\mu-1}}\left[1 + \frac{1}{A}\int_a^b h(t)\left(\ln\frac{t}{a}\right)^{\mu-1}\frac{\mathrm{d}t}{t}\right], \quad t,\tau \in [a,b].$$

Furthermore, from [7, pp. 6 and 7] we have

(i) For $a \leqslant \tau \leqslant t \leqslant b$, we obtain

$$\begin{aligned} G(t,\tau) &\geqslant \frac{1}{\Gamma(\mu)(\ln\frac{b}{a})^{\mu+1}}\left(\ln\frac{t}{a}\right)^{\mu-1}\left(\ln\frac{b}{\tau}\right)^{\mu}\left[\ln\frac{t}{a} - \frac{(\ln\frac{b}{a})^{\mu+1}(\ln\frac{t}{\tau})^{\mu-1}}{\ln\frac{b}{\tau}(\ln\frac{t}{a})^{\mu-1}(\ln\frac{b}{\tau})^{\mu}}\right] \\ &\geqslant \frac{1}{\Gamma(\mu)(\ln\frac{b}{a})^{\mu+1}}\left(\ln\frac{t}{a}\right)^{\mu-1}\left(\ln\frac{b}{\tau}\right)^{\mu}\ln\frac{\tau}{a}. \end{aligned}$$

(ii) For $a \leqslant t \leqslant \tau \leqslant b$, we obtain

$$\begin{aligned} G(t,\tau) &= \frac{1}{\Gamma(\mu)(\ln\frac{b}{a})^{\mu-1}}\left(\ln\frac{t}{a}\right)^{\mu-1}\left(\ln\frac{b}{\tau}\right)^{\mu-1} \\ &\geqslant \frac{1}{\Gamma(\mu)(\ln\frac{b}{a})^{\mu-1}\ln\frac{b}{a}\ln\frac{b}{a}}\left(\ln\frac{t}{a}\right)^{\mu-1}\left(\ln\frac{b}{\tau}\right)^{\mu-1}\ln\frac{b}{\tau}\ln\frac{\tau}{a} \\ &= \frac{1}{\Gamma(\mu)(\ln\frac{b}{a})^{\mu+1}}\left(\ln\frac{t}{a}\right)^{\mu-1}\left(\ln\frac{b}{\tau}\right)^{\mu}\ln\frac{\tau}{a}. \end{aligned}$$

Consequently, we have

$$H(t,\tau) \geqslant \frac{(\ln\frac{t}{a})^{\mu-1}(\ln\frac{b}{\tau})^{\mu}\ln\frac{\tau}{a}}{\Gamma(\mu)(\ln\frac{b}{a})^{\mu+1}}\left[1 + \frac{1}{A}\int_a^b h(t)\left(\ln\frac{t}{a}\right)^{\mu-1}\frac{\mathrm{d}t}{t}\right], \quad t,\tau \in [a,b].$$

Step 2. Let $\theta \in P$ with $\theta(\tau) \not\equiv 0$, $\tau \in [a,b]$. Then there exist $0 < \kappa_{1\theta} \leqslant \kappa_{2\theta}$ such that

$$\kappa_{1\theta} \int_a^b H(t,\tau)\frac{\mathrm{d}\tau}{\tau} \leqslant \int_a^b H(t,\tau)\theta(\tau)\frac{\mathrm{d}\tau}{\tau} \leqslant \kappa_{2\theta} \int_a^b H(t,\tau)\frac{\mathrm{d}\tau}{\tau}, \quad t \in [a,b],$$

where

$$\kappa_{1\theta} = \frac{\mu}{(\ln\frac{b}{a})^{\mu+2}} \int_a^b \left(\ln\frac{b}{\tau}\right)^{\mu} \ln\frac{\tau}{a}\,\theta(\tau)\,\frac{\mathrm{d}\tau}{\tau}$$

and

$$\kappa_{2\theta} = \frac{(\mu+1)(\mu+2)}{(\ln\frac{b}{a})^{\mu}} \int_a^b \left(\ln\frac{b}{\tau}\right)^{\mu-1} \theta(\tau)\,\frac{\mathrm{d}\tau}{\tau}.$$

Using Step 1(i)–(ii), we have

$$\int_a^b H(t,\tau)\frac{\mathrm{d}\tau}{\tau} \leqslant \int_a^b \frac{\xi_A(\ln\frac{t}{a})^{\mu-1}(\ln\frac{b}{\tau})^{\mu-1}}{\Gamma(\mu)(\ln\frac{b}{a})^{\mu-1}}\,\frac{\mathrm{d}\tau}{\tau} = \frac{\xi_A\ln\frac{b}{a}}{\Gamma(\mu+1)}\left(\ln\frac{t}{a}\right)^{\mu-1}$$

and

$$\begin{aligned}\int_a^b H(t,\tau)\frac{\mathrm{d}\tau}{\tau} &\geqslant \int_a^b \frac{\xi_A(\ln\frac{t}{a})^{\mu-1}(\ln\frac{b}{\tau})^{\mu}\ln\frac{\tau}{a}}{\Gamma(\mu)(\ln\frac{b}{a})^{\mu+1}}\,\frac{\mathrm{d}\tau}{\tau} \\ &= \frac{\xi_A\ln\frac{b}{a}}{(\mu+1)(\mu+2)\Gamma(\mu)}\left(\ln\frac{t}{a}\right)^{\mu-1}.\end{aligned}$$

Consequently, we have

$$\begin{aligned}&\int_a^b H(t,\tau)\theta(\tau)\frac{\mathrm{d}\tau}{\tau} \\ &\quad\leqslant \int_a^b \frac{\xi_A(\ln\frac{t}{a})^{\mu-1}(\ln\frac{b}{\tau})^{\mu-1}}{\Gamma(\mu)(\ln\frac{b}{a})^{\mu-1}}\theta(\tau)\frac{\mathrm{d}\tau}{\tau} \\ &\quad= \frac{\xi_A\ln\frac{b}{a}}{(\mu+1)(\mu+2)\Gamma(\mu)}\left(\ln\frac{t}{a}\right)^{\mu-1}\frac{(\mu+1)(\mu+2)}{(\ln\frac{b}{a})^{\mu}}\int_a^b\left(\ln\frac{b}{\tau}\right)^{\mu-1}\theta(\tau)\frac{\mathrm{d}\tau}{\tau} \\ &\quad\leqslant \kappa_{2\theta}\int_a^b H(t,\tau)\frac{\mathrm{d}\tau}{\tau}\end{aligned}$$

and

$$\begin{aligned}
&\int_a^b H(t,\tau)\theta(\tau)\frac{\mathrm{d}\tau}{\tau} \\
&\quad \geqslant \int_a^b \frac{\xi_A(\ln\frac{t}{a})^{\mu-1}(\ln\frac{b}{\tau})^{\mu}\ln\frac{\tau}{a}}{\Gamma(\mu)(\ln\frac{b}{a})^{\mu+1}}\theta(\tau)\frac{\mathrm{d}\tau}{\tau} \\
&\quad = \frac{\xi_A\ln\frac{b}{a}}{\Gamma(\mu+1)}\left(\ln\frac{t}{a}\right)^{\mu-1}\frac{\mu}{(\ln\frac{b}{a})^{\mu+2}}\int_a^b \left(\ln\frac{b}{\tau}\right)^{\mu}\ln\frac{\tau}{a}\,\theta(\tau)\frac{\mathrm{d}\tau}{\tau} \\
&\quad \geqslant \kappa_{1\theta}\int_a^b H(t,\tau)\frac{\mathrm{d}\tau}{\tau}.
\end{aligned}$$

Step 3. Establishing the upper and lower solutions for (1) with $f(t,x) = x^{\sigma_1}$, $\sigma_1 \in (0,1)$, $t \in [a,b]$, $x \in \mathbb{R}^+$.

Let

$$\xi_\rho(t) = \int_a^b H(t,\tau)f\bigl(\tau,\rho(\tau)\bigr)\frac{\mathrm{d}\tau}{\tau}, \quad \text{where} \quad \rho(t) = \int_a^b H(t,\tau)\frac{\mathrm{d}\tau}{\tau}.$$

Lemma 2 implies that

$$\begin{aligned}
&{}^{H}D^{\mu}_{a^+}\xi_\rho(t) + f\bigl(t,\rho(t)\bigr) = 0, \quad t \in (a,b), \\
&\xi_\rho(a) = \xi'_\rho(a) = 0, \qquad \xi_\rho(b) = \int_a^b h(t)\xi_\rho(t)\frac{\mathrm{d}t}{t},
\end{aligned} \tag{30}$$

and from Step 2 there exist $0 < \kappa_{1\rho} \leqslant \kappa_{2\rho}$ such that

$$\kappa_{1\rho}\rho(t) \leqslant \xi_\rho(t) = \int_a^b H(t,\tau)f\bigl(\tau,\rho(\tau)\bigr)\frac{\mathrm{d}\tau}{\tau} \leqslant \kappa_{2\rho}\rho(t), \quad t \in [a,b].$$

Let $\xi_1(t) = \vartheta_1\xi_\rho(t)$, $\xi_2(t) = \vartheta_2\xi_\rho(t)$, $t \in [a,b]$, where

$$0 < \vartheta_1 < \min\left\{\frac{1}{\kappa_{2\rho}},\, \kappa_{1\rho}^{\sigma_1/(1-\sigma_1)}\right\}, \qquad \vartheta_2 > \max\left\{\frac{1}{\kappa_{1\rho}},\, \kappa_{2\rho}^{\sigma_1/(1-\sigma_1)}\right\}.$$

Then we easily obtain

$$\xi_i(a) = \xi'_i(a) = 0, \quad \xi_i(b) = \int_a^b h(t)\xi_i(t)\frac{\mathrm{d}t}{t}, \quad i = 1,2.$$

Moreover,

$$f\big(t,\xi_1(t)\big) = f\big(t,\vartheta_1\xi_\rho(t)\big) = f\left(t,\vartheta_1\frac{\xi_\rho(t)}{\rho(t)}\rho(t)\right) = \left[\vartheta_1\frac{\xi_\rho(t)}{\rho(t)}\rho(t)\right]^{\sigma_1}$$
$$= \left[\vartheta_1\frac{\xi_\rho(t)}{\rho(t)}\right]^{\sigma_1} f\big(t,\rho(t)\big) \geqslant (\vartheta_1\kappa_{1\rho})^{\sigma_1}\big(t,\rho(t)\big) \geqslant \vartheta_1 f\big(t,\rho(t)\big),$$

and from (30) we have

$$-{}^H D^\mu_{a+}\xi_1(t) = -\vartheta_1 {}^H D^\mu_{a+}\xi_\rho(t) = \vartheta_1 f\big(t,\rho(t)\big) \leqslant f\big(t,\xi_1(t)\big).$$

So Definition 4 implies that ξ_1 is a lower solution for (1).

Also, we have

$$\vartheta_2 f\big(t,\rho(t)\big) = \vartheta_2 f\left(t,\frac{\rho(t)}{\xi_2(t)}\xi_2(t)\right) = \vartheta_2 f\left(t,\frac{\rho(t)}{\vartheta_2\xi_\rho(t)}\xi_2(t)\right)$$
$$= \vartheta_2\left[\frac{\rho(t)}{\vartheta_2\xi_\rho(t)}\right]^{\sigma_1} f\big(t,\xi_2(t)\big) \geqslant \vartheta_2\left(\frac{1}{\vartheta_2\kappa_{2\rho}}\right)^{\sigma_1} f\big(t,\xi_2(t)\big)$$
$$\geqslant f\big(t,\xi_2(t)\big),$$

and we have

$$-{}^H D^\mu_{a+}\xi_2(t) = -\vartheta_2 {}^H D^\mu_{a+}\xi_\rho(t) = \vartheta_2 f\big(t,\rho(t)\big) \geqslant f\big(t,\xi_2(t)\big).$$

So Definition 3 implies that ξ_2 is an upper solution for (1).

References

1. K. Abuasbeh, M. Awadalla, M. Jneid, Nonlinear Hadamard fractional boundary value problems with different orders, *Rocky Mt. J. Math.*, **51**(1):17–29, 2021, `https://doi.org/10.1216/rmj.2021.51.17`.
2. B. Ahmad, A.F. Albideewi, S.K. Ntouyas, A. Alsaedi, Existence results for a multipoint boundary value problem of nonlinear sequential Hadamard fractional differential equations, *Cubo*, **23**(2), 2021, `https://doi.org/4298319`.
3. B. Ahmad, A. Alsaedi, S.K. Ntouyas, J. Tariboon, *Hadamard-type Fractional Differential Equations, Inclusions and Inequalities*, Springer, Cham, 2017, `https://doi.org/10.1007/978-3-319-52141-1`.
4. B. Ahmad, S.K. Ntouyas, A fully Hadamard type integral boundary value problem of a coupled system of fractional differential equations, *Fract. Calc. Appl. Anal.*, **17**(2):348–360, 2014, `https://doi.org/10.2478/s13540-014-0173-5`.
5. A. Ardjouni, Existence results for a Caputo-Hadamard type fractional boundary value problem, *Fract. Differ. Calc.*, **11**(2):241–253, 2021, `https://doi.org/10.7153/fdc-2021-11-16`.

6. Y. Arioua, N. Benhamidouche, Boundary value problem for Caputo-Hadamard fractional differential equations, *Surv. Math. Appl.*, **12**:103–115, 2017.
7. N.A. Asif, Positive solution for a Hadamard fractional singular boundary value problem of order $\mu \in (2,3)$, *Z. Angew. Math. Phys.*, **10**(5):1631–1641, 2022, https://doi.org/10.4236/jamp.2022.105113.
8. W. Benhamida, J.R. Graef, S. Hamani, Boundary value problems for Hadamard fractional differential equations with nonlocal multi-point boundary conditions, *Fract. Differ. Calc.*, **8**(1): 165–176, 2018, https://doi.org/10.7153/fdc-2018-08-10.
9. W. Benhamida, S. Hamani, J. Henderson, A boundary value problem for fractional differential equations with Hadamard derivative and nonlocal conditions, *Panamer. Math. J.*, **26**(3):1–11, 2016.
10. N. Guerraiche, S. Hamani, J. Henderson, Nonlinear boundary value problems for Hadamard fractional differential inclusions with integral boundary conditions, *Adv. Dyn. Syst. Appl.*, **12**(2):107–121, 2017.
11. D.J. Guo, V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Notes Rep. Math. Sci. Eng., Vol. 5, Academic Press, Boston, MA, 1988.
12. J. Jiang, D. O'Regan, J. Xu, Z. Fu, Positive solutions for a system of nonlinear Hadamard fractional differential equations involving coupled integral boundary conditions, *J. Inequal. Appl.*, p. 204, 2019, https://doi.org/10.1186/s13660-019-2156-x.
13. W. Jiang, Eigenvalue interval for multi-point boundary value problems of fractional differential equations, *Appl. Math. Comput.*, **219**:4570–4575, 2013, https://doi.org/10.1016/j.amc.2012.10.068.
14. P. Karthikeyan, R. Arul, Integral boundary value problems for implicit fractional differential equations involving Hadamard and Caputo-Hadamard fractional derivatives, *Kragujevac J. Math.*, **45**(3):331–341, 2021, https://doi.org/10.46793/kgjmat2103.331k.
15. A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, Vol. 204, Elsevier, Amsterdam, 2006.
16. M.G. Kreĭn, M.A. Rutman, Linear operators leaving invariant a cone in a Banach space, *Am. Math. Soc. Translation*, **1950**(26):128, 1950.
17. A. Lachouri, A. Ardjouni, A. Djoudi, Existence and uniqueness of mild solutions of boundary value problems for Caputo-Hadamard fractional differential equations with integral and anti-periodic conditions, *J. Fract. Calc. Appl.*, **12**(2):60–68, 2021.
18. R.W. Leggett, L.R. Williams, Multiple positive solutions of nonlinear operators on ordered Banach spaces, *Indiana Univ. Math. J.*, **28**(4):673–688, 1979.
19. W. Liu, L. Liu, Y. Wu, Existence of solutions for integral boundary value problems of singular Hadamard-type fractional differential equations on infinite interval, *Adv. Difference Equ.*, **2020**: 274, 2020, https://doi.org/10.1186/s13662-020-02726-6.
20. S.K. Ntouyas, J. Tariboon, W. Sudsutad, Boundary value problems for Riemann-Liouville fractional differential inclusions with nonlocal Hadamard fractional integral conditions, *Mediterr. J. Math.*, **13**(3):939–954, 2016, https://doi.org/10.1007/s00009-015-0543-1.

21. S.K. Ntouyas, J. Tariboon, C. Thaiprayoon, Nonlocal boundary value problems for Riemann-Liouville fractional differential inclusions with Hadamard fractional integral boundary conditions, *Taiwanese J. Math.*, **20**(1):91–107, 2016, `https://doi.org/10.11650/tjm.20.2016.5654`.

22. G. Wang, K. Pei, R.P. Agarwal, L. Zhang, B. Ahmad, Nonlocal Hadamard fractional boundary value problem with Hadamard integral and discrete boundary conditions on a half-line, *J. Comput. Appl. Math.*, **343**:230–239, 2018, ISSN 0377-0427, `https://doi.org/10.1016/j.cam.2018.04.062`.

23. Y. Wang, L. Liu, Y. Wu, Positive solutions for a nonlocal fractional differential equation, *Nonlinear Anal., Theory Methods Appl.*, **74**(11):3599–3605, 2011, `https://doi.org/10.1016/j.na.2011.02.043`.

24. J. Xu, L. Liu, S. Bai, Y. Wu, Solvability for a system of Hadamard fractional multi-point boundary value problems, *Nonlinear Anal. Model. Control*, **26**(3):502–521, 2021, ISSN 1392-5113, `https://doi.org/10.15388/namc.2021.26.22538`.

25. J. Xu, Z. Wei, Positive solutions for a class of fractional boundary value problems, *Nonlinear Anal. Model. Control*, **21**(1):1–17, 2016, `https://doi.org/10.15388/NA.2016.1.1`.

26. J. Xu, Z. Yang, Multiple positive solutions of a singular fractional boundary value problem, *Appl. Math. E-Notes*, **10**:259–267, 2010.

27. A. Zahed, S. Hamani, J.R. Graef, Boundary value problems for Hadamard-Caputo implicit fractional differential inclusions with nonlocal conditions, *Arch. Math., Brno*, **57**(5):285–297, 2021, `https://doi.org/10.5817/am2021-5-285`.

28. K. Zhang, Z. Fu, Solutions for a class of Hadamard fractional boundary value problems with sign-changing nonlinearity, *J. Funct. Spaces*, p. 9046472, 2019, `https://doi.org/10.1155/2019/9046472`.

29. W. Zhang, W. Liu, Existence, uniqueness, and multiplicity results on positive solutions for a class of Hadamard-type fractional boundary value problem on an infinite interval, *Math. Methods Appl. Sci.*, **43**(5):2251–2275, 2020, `https://doi.org/10.1002/mma.6038`.

30. X. Zhang, L. Wang, Q. Sun, Existence of positive solutions for a class of nonlinear fractional differential equations with integral boundary conditions and a parameter, *Appl. Math. Comput.*, **226**:708–718, 2014, `https://doi.org/10.1016/j.amc.2013.10.089`.