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On the solvability of the Atangana–Baleanu fractional evolution equations: An integral contractor approach*

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Abstract. We present existence and controllability results for mild solutions to the Atangana–Baleanu fractional evolution equations. We prove our results by applying bounded integral contractors and a sequencing technique. In contrast to the papers available in the literature, in order to establish our controllability results, we need not define the induced inverse of the controllability operator, and the pertinent nonlinear function need not necessarily satisfy a Lipschitz condition. In addition, we also establish trajectory controllability results. Finally, we discuss an application, which illustrates our results.

Keywords: Atangana–Baleanu fractional derivative, controllability, evolution equation, integral contractor.

1 Introduction

The notion of fractional calculus first appeared back in 1695 in the correspondence between l'Hôpital and Leibniz. Since then, many mathematicians, such as Euler, Abel, Fourier, Riemann, and Liouville, have enriched the study of fractional calculus and exhibited its applications to many real world problems [22, 25]. Since the definition of a fractional derivative includes an integral term, it is called a nonlocal derivative. It provides an important tool for studying memory in materials and hereditary characteristics of phenomena. Many real world problems, such as the tautochrone problem, wave propagation in viscoelastic horns, edge detection in image processing, and propagation of sound waves in rigid porous materials, can be modeled more precisely by using a fractional operator rather than a classical operator [19]. Over time, various forms of fractional operators have been introduced by several authors, such as the Grünwald–Letnikov, Riemann–Liouville, Caputo, Weyl, Marchaud, and the Hadamard fractional derivatives [22, 25].

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Of these operators, the Riemann–Liouville derivative and the Caputo derivative are the most frequently applied to model many real world problems. But these operators have a singular kernel, which limits their application to some physical problems. To address this issue, Atangana and Baleanu suggested a new definition of the fractional derivative that uses the Mittag-Leffler function (ML for short) as a nonsingular kernel. Later, many researchers studied the Atangana–Baleanu (AB, for short) fractional derivative [18,20,26] and proved that this nonlocal derivative with a nonsingular kernel plays an important role in many real world applications [4, 8, 17].

In 1976, Altman [2] introduced the notion of a bounded integral contractor. This technique proved to be very useful for solving problems in which nonlinear functions do not satisfy the Lipschitz condition. George [14] applied this technique to show the approximate controllability of semilinear systems. Kumar and Sukavanam [23] extended this concept to fractional-order systems. Later, Zhu et al. [30] applied this concept to Riemann–Liouville fractional evolution equations. Recently, in [12], we have discussed existence and controllability results for Hilfer fractional evolution equations via integral contractors.

Controllability is an important property of a dynamical system, which plays a significant role in its analysis and design [7, 13]. The notion of controllability was introduced in 1963 by Kalman [21]. It led to some important questions concerning the possibility of deciding whether a given linear dynamical control system can be controlled or not and if it is controllable, then what would be the best control so that one can achieve the desired result. Later, several authors extended this idea to nonlinear dynamical systems in infinite dimensional spaces and provided many applications in the fields of aerospace engineering, electrical engineering, biological science, and economics. See, for example, [10, 15, 23, 28]. In some cases, trajectory planning is essential in the process of designing a system so that the system only moves along a prescribed trajectory. This happens, for example, in the cases of industrial robots, launching satellites by rockets and hitting targets by missiles flying in a particular path. In trajectory controllability problems, we intend to find a control using which the system moves along a defined trajectory starting from the initial state to the desired final state. The concept of trajectory controllability is stronger than other controllability concepts. It is also useful for optimizing some of the factors involved in the operation of a system. For more details, see [11, 15, 16].

Recently, controllability results for the AB fractional derivative have been obtained by several authors [1, 5, 6]. In these works, the authors assume that the controllability operator has a bounded inverse operator, which is a very strong assumption. Moreover, given the complexity of the AB fractional derivative, computing the inverse controllability operator can be quite difficult. To address this issue, in this paper, we apply the notion of a bounded integral contractor and investigate existence and controllability of the following AB fractional evolution equations in a Banach space $(Y, \|\cdot\|)$:

$$\begin{aligned} &{}^{AB}D^{\ell}_{0+}p(t) = Gp(t) + Qw(t) + \hbar\big(t, p(t)\big), \quad t \in (0, T], \\ &p(0) = p_0, \end{aligned} \tag{1}$$

where ${}^{AB}D^{\ell}_{0+}$ represents the AB Caputo fractional derivative of order $\ell \in (0, 1)$. The linear operator $G : \mathcal{D}(G) \subset Y \to Y$ is the infinitesimal generator of an ℓ-resolvent family

$(\mathcal{T}_\ell(t))_{t\geqslant 0}$, $(\mathcal{S}_\ell(t))_{t\geqslant 0}$ denotes the solution operator defined on Y, $\mathbb{W}$ is the Banach space of admissible control functions such that the control function $w(\cdot) \in L^2([0,T],\mathbb{W})$, $Q: L^2([0,T],\mathbb{W}) \to L^2([0,T],Y)$ denotes a bounded linear operator, and $\hbar : [0,T]\times Y \to Y$ denotes a nonlinear function.

The novelty and major contributions of this paper are as follows:

1. The concept of an integral contractor with a sequencing technique is employed to prove the existence and controllability of the AB fractional evolution equation (1), a topic which has not yet been treated in the literature.
2. The results proved in this paper are valid for systems with Lipschitz, as well as non-Lipschitz, nonlinear functions.
3. To obtain our controllability results, we do not need to define the inverse of the controllability operator.
4. Trajectory controllability (which is a stronger concept than other controllability concepts) results are also established.
5. An application of our results is presented.

2 Auxiliary definitions and results

Definition 1. (See [3].) The AB fractional integral of order $\ell \in (0,1)$ of a function $p:(0,T]\to\mathbb{R}$ is defined by

$$ {}^{AB}I^{\ell}_{0+}p(t) := \frac{1-\ell}{E(\ell)}p(s) + \frac{\ell}{E(\ell)\Gamma(\ell)}\int_0^t (t-s)^{\ell-1}p(s)\,\mathrm{d}s, $$

where $E(\ell) = (1-\ell)+\ell/\Gamma(\ell)$ denotes a normalization function with $E(0)=E(1)=1$, and Γ denotes the gamma function.

The one-parameter family of ML functions is given by

$$ M_\ell(p) := \sum_{k=0}^{\infty}\frac{p^k}{\Gamma(\ell k+1)}, \quad \operatorname{Re}\ell>0,\ p\in\mathbb{C} \text{ (the set of complex numbers)}, $$

and the two-parameter family is given by

$$ M_{\ell,\varsigma}(p) := \sum_{k=0}^{\infty}\frac{p^k}{\Gamma(\ell k+\varsigma)}, \quad \operatorname{Re}\ell, \operatorname{Re}\varsigma>0,\ p\in\mathbb{C}. $$

Clearly, $M_{\ell,1}(p) = M_\ell(p)$.

Definition 2. (See [3].) Let $p\in H^1(0,T)$, $T>0$. The AB fractional derivative of a function p of order $\ell\in(0,1)$ in the Caputo sense is defined by

$$ {}^{AB}D^{\ell}_{0+}p(t) := \frac{E(\ell)}{1-\ell}\int_0^t p'(s)M_\ell\bigl(-\eta(t-s)^\ell\bigr)\,\mathrm{d}s, \quad 0<t\leqslant T, $$

where M_ℓ denotes the ML function, and $\eta = \ell/(1-\ell)$.

Definition 3. (See [24].) For a linear operator G, the set $\rho(G) := \{\mu \in \mathbb{C}\colon (\mu I - G)$ is invertible$\}$ is called the resolvent set, and the family $\mathcal{R}(\mu, G) := (\mu I - G)^{-1}$ is called the resolvent of G.

Definition 4. (See [24].) A closed linear operator G is called a sectorial operator if for some $\sigma \in \mathbb{R}$ and $\theta \in [\pi/2, \pi]$, there exists $\lambda > 0$ such that

(i) $\rho(G) \subset \sum_{\theta,\sigma} = \{\mu \in \mathbb{C}\colon \mu \neq \sigma, |\arg(\mu - \sigma)| < \theta\}$;
(ii) $\|\mathcal{R}(\mu, G)\| \leqslant \lambda/|\mu - \sigma|$, $\mu \in \sum_{\theta,\sigma}$.

Remark 1. For the resolvent operator $\mathcal{R}(\mu, G)$ of the generator G of a C_0-semigroup, we have the following result, which shows that the resolvent operator is just the Laplace transform of the semigroup operator.

Lemma 1. (*See* [24].) *Let $S(t)$ be a C_0-semigroup with infinitesimal generator G. Then for $\mu \in \rho(G)$ and for all $y \in Y$, the following results hold*:

(i) $\mathcal{R}(\mu, G)y = (\mu I - G)^{-1}y = \int_0^\infty \mathrm{e}^{-\mu t} S(t)y \,\mathrm{d}t$;
(ii) *For all $y \in Y$, $\lim_{\beta\to\infty}(\beta I - G)^{-1}y = y$, where β is constrained to be real.*

Definition 5. (See [1].) For a given sectorial operator G, the mild solution $p(t)$ of Eq. (1) is defined by

$$p(t) := R\mathcal{S}_\ell(t)p_0 + \frac{RV(1-\ell)}{E(\ell)\Gamma(\ell)} \int_0^t (t-s)^{\ell-1}\big[Qw(s) + \hbar\big(s, p(s)\big)\big] \,\mathrm{d}s + \frac{\ell R^2}{E(\ell)} \int_0^t \mathcal{T}_\ell(t-s)\big[Qw(s) + \hbar\big(s, p(s)\big)\big] \,\mathrm{d}s \quad \forall t \in [0, T],$$

where R and V are linear operators given by

$$R = \sigma(\sigma I - G)^{-1}, \qquad V = \eta G(\sigma I - G)^{-1} \quad \text{with } \sigma = \frac{E(\ell)}{1-\ell},$$

$$\mathcal{S}_\ell(t) = M_\ell\big(-Vt^\ell\big) = \frac{1}{2\pi\imath} \int_c \mathrm{e}^{st} s^{\ell-1}\big(s^\ell I - V\big)^{-1} \,\mathrm{d}s$$

and

$$\mathcal{T}_\ell(t) = t^{\ell-1} M_{\ell,\ell}\big(-Vt^\ell\big) = \frac{1}{2\pi\imath} \int_c \mathrm{e}^{st}\big(s^\ell I - V\big)^{-1} \,\mathrm{d}s,$$

where c denotes a certain path lying in $\sum_{\theta,\sigma}$.

Definition 6. (See [9].) The solution operator $\mathcal{S}_\ell(t)$ of Eq. (1) is called analytic if $\mathcal{S}_\ell(t)$ admits an analytic extension to a sector $\sum_{\theta_0} := \{\mu \in \mathbb{C}/\{0\}\colon |\arg \mu| < \theta_0\}$ for some $\theta_0 \in (0, \pi/2]$. An analytic solution operator is said to be of analyticity type (θ_0, σ_0) if for each $\theta < \theta_0$ and $\sigma > \sigma_0$, there is an $M = M(\theta, \sigma)$ such that $\|\mathcal{S}_\ell(t)\| \leqslant M\mathrm{e}^{\sigma t}$ for $t \in \sum_\theta := \{t \in \mathbb{C}/\{0\}\colon |\arg t| < \theta\}$.

Lemma 2. (*See* [27].) *If G generates analytic solution operators $\mathcal{S}_\ell(t)$ of type (θ_0, σ_0), then $\|\mathcal{S}_\ell(t)\| \leqslant M\mathrm{e}^{\sigma t}$ and $\|\mathcal{T}_\ell(t)\| \leqslant C\mathrm{e}^{\sigma t}(1 + t^{\ell-1})$ for every $t > 0$, $\sigma > \sigma_0$.*

Let $D_\mathcal{S} := \sup_{t>0} \|\mathcal{S}_\ell(t)\|$ and $D_\mathcal{T} := \sup_{t>0} C\mathrm{e}^{\sigma t}(1 + t^{\ell-1})$. Then $\|\mathcal{S}_\ell(t)\| \leqslant D_\mathcal{S}$ and $\|\mathcal{T}_\ell(t)\| \leqslant D_\mathcal{T} t^{\ell-1}$.

Consider the reachable set $K_T(\hbar, w) := \{p(T, p_0, w)\colon\ w(\cdot) \in L^2([0,T], \mathbb{W})\}$ of (1). It is the collection of all final states p at terminal time T with initial state p_0 and control w.

Definition 7. Equation (1) is said to be exactly controllable on $[0,T]$ if $K_T(\hbar, w) = Y$.

Definition 8. (See [14].) A bounded linear operator $\Upsilon : [0,T] \times Y \to \mathcal{B}(Y)$ is called a bounded integral contractor of the function $\hbar$ with respect to the operator $\mathcal{T}_\ell(t)$ if

$$\begin{aligned}&\Bigg\|\hbar\Bigg(t,\, p(t) + q(t) + \frac{RV(1-\ell)}{E(\ell)\Gamma(\ell)}\int_0^t (t-s)^{\ell-1}\Upsilon\big(s, p(s)\big)q(s)\,\mathrm{d}s \\ &\quad + \frac{\ell R^2}{E(\ell)}\int_0^t \mathcal{T}_\ell(t-s)\Upsilon\big(s, p(s)\big)q(s)\,\mathrm{d}s\Bigg) - \hbar\big(t, p(t)\big) - \Upsilon\big(t, p(t)\big)q(t)\Bigg\| \\ &\leqslant \tau\big\|q(t)\big\|\end{aligned}$$

for all $t \in (0,T)$ and $p, q \in Y$ with constant $\tau > 0$.

It follows from the boundedness of the operator Υ that $\|\Upsilon(t, p(t))q(t)\| \leqslant \nu\|q(t)\|$ for every $t \in (0,T)$ and $p, q \in Y$ with constant $\nu > 0$.

Definition 9. If for any $p, z \in Y$, the integral equation

$$\begin{aligned}z(t) = q(t) &+ \frac{RV(1-\ell)}{E(\ell)\Gamma(\ell)}\int_0^t (t-s)^{\ell-1}\Upsilon\big(s, p(s)\big)q(s)\,\mathrm{d}s \\ &+ \frac{\ell R^2}{E(\ell)}\int_0^t \mathcal{T}_\ell(t-s)\Upsilon\big(s, p(s)\big)q(s)\,\mathrm{d}s\end{aligned}$$

admits a solution $q \in Y$, then Υ is called a regular integral contractor.

Remark 2. For the case $\Upsilon \equiv 0$, the nonlinear function $\hbar(t, p(t))$ has to satisfy the following Lipschitz-type condition:

$$\big\|\hbar\big(t,\, p(t) + q(t)\big) - \hbar\big(t, p(t)\big)\big\| \leqslant \tau\big\|q(t)\big\|.$$

If $\hbar$ satisfies this condition, then it has the regular integral contractor $\Upsilon \equiv 0$. Thus, the results obtained in the present paper are also valid for those functions, which satisfy this Lipschitz-type condition.

Lemma 3 [Generalized Gronwall's inequality]. (*See* [29].) *Let* $p(t)$ *and* $c(t)$, $t \in [0,T)$, *be two nonnegative locally integrable functions, which satisfy*

$$p(t) \leqslant c(t) + h\int_0^t (t-s)^{\varrho-1} p(s)\,\mathrm{d}s$$

for some $h \geqslant 0$ *and* $\varrho > 0$. *Then*

$$p(t) \leqslant c(t) + \int_0^t \left[\sum_{n=1}^{\infty} \frac{(h\Gamma(\varrho))^n}{\Gamma(n\varrho)} (t-s)^{n\varrho-1} c(s)\right] \mathrm{d}s.$$

3 Existence and uniqueness results

In order to establish the existence of solutions, we consider the following assumptions:

(H_1) G is a sectorial operator.
(H_2) The linear bounded operators R and V satisfy $\|R\| \leqslant k_1$ and $\|V\| \leqslant k_2$, where k_1 and k_2 are positive constants.
(H_3) The nonlinear function $\hbar : [0,T] \times Y \to Y$ satisfies the following conditions:

(i) $\hbar$ has a regular integral contractor Υ;
(ii) $\hbar(\cdot, p) : [0,T] \to Y$ is measurable for each $p \in Y$;
(iii) $\hbar(t, \cdot) : Y \to Y$ is continuous for almost every $t \in [0,T]$.

Theorem 1. *If assumptions* (H_1)–(H_3) *hold true, then the fractional evolution equation* (1) *has a unique mild solution.*

Proof. Consider the two sequences $\{p_n\}$ and $\{q_n\}$ in Y defined as follows:

$$p_0(t) = R\mathcal{S}_\ell(t)p_0 + \frac{RV(1-\ell)}{E(\ell)\Gamma(\ell)} \int_0^t (t-s)^{\ell-1} Qw(s)\,\mathrm{d}s + \frac{\ell R^2}{E(\ell)} \int_0^t \mathcal{T}_\ell(t-s)Qw(s)\,\mathrm{d}s,$$

$$q_n(t) = p_n(t) - \frac{RV(1-\ell)}{E(\ell)\Gamma(\ell)} \int_0^t (t-s)^{\ell-1} \hbar\big(s, p_n(s)\big)\,\mathrm{d}s - \frac{\ell R^2}{E(\ell)} \int_0^t \mathcal{T}_\ell(t-s)\hbar\big(s, p_n(s)\big)\,\mathrm{d}s - p_0(t), \tag{2}$$

$$p_{n+1}(t) = p_n(t) - \Bigg[q_n(t) + \frac{RV(1-\ell)}{E(\ell)\Gamma(\ell)}\int_0^t (t-s)^{\ell-1}\Upsilon\big(s,p_n(s)\big)q_n(s)\,\mathrm{d}s + \frac{\ell R^2}{E(\ell)}\int_0^t \mathcal{T}_\ell(t-s)\Upsilon\big(s,p_n(s)\big)q_n(s)\,\mathrm{d}s\Bigg]. \tag{3}$$

Using Eqs. (2) and (3), we see that

$$\begin{aligned}
p_{n+1}(t) ={}& \frac{RV(1-\ell)}{E(\ell)\Gamma(\ell)}\int_0^t (t-s)^{\ell-1}\Upsilon\big(s,p_n(s)\big)q_n(s)\,\mathrm{d}s \\
&+ \frac{RV(1-\ell)}{E(\ell)\Gamma(\ell)}\int_0^t (t-s)^{\ell-1}\hbar\big(s,p_n(s)\big)\,\mathrm{d}s \\
&- \frac{\ell R^2}{E(\ell)}\int_0^t \mathcal{T}_\ell(t-s)\Upsilon\big(s,p_n(s)\big)q_n(s)\,\mathrm{d}s \\
&+ \frac{\ell R^2}{E(\ell)}\int_0^t \mathcal{T}_\ell(t-s)\hbar\big(s,p_n(s)\big)\,\mathrm{d}s + p_0(t).
\end{aligned}$$

Again, from Eq. (2) it follows that

$$\begin{aligned}
q_{n+1}(t) ={}& p_{n+1}(t) - \frac{RV(1-\ell)}{E(\ell)\Gamma(\ell)}\int_0^t (t-s)^{\ell-1}\hbar\big(s,p_{n+1}(s)\big)\,\mathrm{d}s \\
&- \frac{\ell R^2}{E(\ell)}\int_0^t \mathcal{T}_\ell(t-s)\hbar\big(s,p_{n+1}(s)\big)\,\mathrm{d}s - p_0(t) \\
={}& \frac{RV(1-\ell)}{E(\ell)\Gamma(\ell)}\int_0^t (t-s)^{\ell-1}\big[\Upsilon\big(s,p_n(s)\big)q_n(s) + \hbar\big(s,p_n(s)\big)\big]\,\mathrm{d}s \\
&+ \frac{\ell R^2}{E(\ell)}\int_0^t \mathcal{T}_\ell(t-s)\big[\Upsilon\big(s,p_n(s)\big)q_n(s) + \hbar\big(s,p_n(s)\big)\big]\,\mathrm{d}s \\
&- \frac{RV(1-\ell)}{E(\ell)\Gamma(\ell)}\int_0^t (t-s)^{\ell-1}\hbar\big(s,p_{n+1}(s)\big)\,\mathrm{d}s \\
&- \frac{\ell R^2}{E(\ell)}\int_0^t \mathcal{T}_\ell(t-s)\hbar\big(s,p_{n+1}(s)\big)\,\mathrm{d}s,
\end{aligned}$$

which implies that

$$\begin{aligned}
q_{n+1}(t) = {} & \frac{RV(1-\ell)}{E(\ell)\Gamma(\ell)} \int_0^t (t-s)^{\ell-1} \big[\Upsilon\big(s, p_n(s)\big) q_n(s) + \hbar\big(s, p_n(s)\big)\big] \, \mathrm{d}s \\
& + \frac{\ell R^2}{E(\ell)} \int_0^t \mathcal{T}_\ell(t-s) \big[\Upsilon\big(s, p_n(s)\big) q_n(s) + \hbar\big(s, p_n(s)\big)\big] \, \mathrm{d}s \\
& - \frac{RV(1-\ell)}{E(\ell)\Gamma(\ell)} \int_0^t (t-s)^{\ell-1} \hbar \Bigg[s,\ p_n(s) - q_n(s) - \frac{RV(1-\ell)}{E(\ell)\Gamma(\ell)} \\
& \qquad \times \int_0^s (s-r)^{\ell-1} \Upsilon\big(r, p_n(r)\big) q_n(r) \, \mathrm{d}r \\
& \qquad + \frac{\ell R^2}{E(\ell)} \int_0^s \mathcal{T}_\ell(s-r) \Upsilon\big(r, p_n(r)\big) q_n(r) \, \mathrm{d}r \Bigg] \, \mathrm{d}s \\
& - \frac{\ell R^2}{E(\ell)} \int_0^t \mathcal{T}_\ell(t-s) \hbar \Bigg[s,\ p_n(s) - q_n(s) - \frac{RV(1-\ell)}{E(\ell)\Gamma(\ell)} \\
& \qquad \times \int_0^s (s-r)^{\ell-1} \Upsilon\big(r, p_n(r)\big) q_n(r) \, \mathrm{d}r \\
& \qquad + \frac{\ell R^2}{E(\ell)} \int_0^s \mathcal{T}_\ell(s-r) \Upsilon\big(r, p_n(r)\big) q_n(r) \, \mathrm{d}r \Bigg] \, \mathrm{d}s.
\end{aligned}$$

Using Definition 8 with $p = p_n$ and $q = -q_n$, we get

$$\begin{aligned}
& \big\|q_{n+1}(t)\big\| \\
& \quad \leqslant \frac{\|R\|\|V\|(1-\ell)}{E(\ell)\Gamma(\ell)} \int_0^t (t-s)^{\ell-1} \tau \big\|q(s)\big\| \, \mathrm{d}s + \frac{\ell \|R\|^2}{E(\ell)} \int_0^t \big\|\mathcal{T}_\ell(t-s)\big\| \tau \big\|q_n(s)\big\| \, \mathrm{d}s \\
& \quad \leqslant \frac{k_1 k_2 \tau (1-\ell)}{E(\ell)\Gamma(\ell)} \int_0^t (t-s)^{\ell-1} \big\|q_n(s)\big\| \, \mathrm{d}s + \frac{\ell \tau (k_1)^2 D_{\mathcal{T}}}{E(\ell)} \int_0^t (t-s)^{\ell-1} \big\|q_n(s)\big\| \, \mathrm{d}s \\
& \quad \leqslant \left[\frac{k_1 k_2 \tau (1-\ell)}{E(\ell)\Gamma(\ell)} + \frac{\ell \tau (k_1)^2 D_{\mathcal{T}}}{E(\ell)} \right] \int_0^t (t-s)^{\ell-1} \big\|q_n(s)\big\| \, \mathrm{d}s.
\end{aligned}$$

Using induction, we obtain

$$\|q_{n+1}\| \leqslant \frac{[\frac{T(k_1k_2\tau(1-\ell)+\Gamma(1+\ell)\tau(k_1)^2D_{\mathcal{T}})}{E(\ell)}]^{n+1}}{\Gamma(1+(n+1)\ell)}\|q_0\|. \tag{4}$$

It follows from the definition of the ML function of order ℓ at the point

$$\frac{T(k_1k_2\tau(1-\ell)+\Gamma(1+\ell)\tau(k_1)^2D_{\mathcal{T}})}{E(\ell)},$$

that is,

$$M_\ell\biggl(\frac{T(k_1k_2\tau(1-\ell)+\Gamma(1+\ell)\tau(k_1)^2D_{\mathcal{T}})}{E(\ell)}\biggr) = \sum_{n=0}^{\infty}\frac{(\frac{T(k_1k_2\tau(1-\ell)+\Gamma(1+\ell)\tau(k_1)^2D_{\mathcal{T}})}{E(\ell)})^n}{\Gamma(1+n\ell)},$$

that $\{q_n\} \to 0$ as $n \to \infty$ in Y.

Our next step is to show that $\{p_n\}$ converges to the mild solution of (1). Using (3) and (4), we have

$$\begin{aligned}
&\bigl\|p_{n+1}(t)-p_n(t)\bigr\| \\
&\quad\leqslant \bigl\|q_n(t)\bigr\| + \left\|\frac{RV(1-\ell)}{E(\ell)\Gamma(\ell)}\int_0^t (t-s)^{\ell-1}\Upsilon\bigl(s,p_n(s)\bigr)q_n(s)\,\mathrm{d}s\right\| \\
&\qquad + \left\|\frac{\ell R^2}{E(\ell)}\int_0^t \mathcal{T}_\ell(t-s)\Upsilon\bigl(s,p_n(s)\bigr)q_n(s)\,\mathrm{d}s\right\| \\
&\quad\leqslant \bigl\|q_n(t)\bigr\| + \frac{\|R\|\|V\|\nu(1-\ell)}{E(\ell)\Gamma(\ell)}\int_0^t (t-s)^{\ell-1}\bigl\|q_n(s)\bigr\|\,\mathrm{d}s \\
&\qquad + \frac{\ell\|R\|^2D_{\mathcal{T}}}{E(\ell)}\int_0^t (t-s)^{\ell-1}\bigl\|q_n(s)\bigr\|\,\mathrm{d}s \\
&\quad\leqslant \bigl\|q_n(t)\bigr\| + \biggl[\frac{\|R\|\|V\|\nu(1-\ell)}{E(\ell)\Gamma(\ell)} + \frac{\ell\|R\|^2D_{\mathcal{T}}}{E(\ell)}\biggr]\int_0^t (t-s)^{\ell-1}\bigl\|q_n(s)\bigr\|\,\mathrm{d}s \\
&\quad\leqslant \|q_n(t)\| + \biggl[\frac{k_1k_2\nu(1-\ell)}{E(\ell)\Gamma(\ell)} + \frac{\ell(k_1)^2D_{\mathcal{T}}}{E(\ell)}\biggr]\int_0^t (t-s)^{\ell-1}\bigl\|q_n(s)\bigr\|\,\mathrm{d}s \\
&\quad\leqslant \biggl[1+\frac{k_1k_2\nu(1-\ell)T^\ell}{E(\ell)\Gamma(1+\ell)} + \frac{T^\ell(k_1)^2D_{\mathcal{T}}}{E(\ell)}\biggr]\frac{[\frac{T(k_1k_2\tau(1-\ell)+\Gamma(1+\ell)\tau(k_1)^2D_{\mathcal{T}})}{E(\ell)}]^n}{\Gamma(1+n\ell)}\|q_0\|.
\end{aligned}$$

Furthermore, for $n > m \geqslant 0$, we have

$$\begin{aligned}\|p_n - p_m\| &\leqslant \sum_{k=m}^{n-1} \|p_{k+1} - p_k\| \\ &\leqslant \left[1 + \frac{k_1 k_2 \nu(1-\ell)T^\ell}{E(\ell)\Gamma(1+\ell)} + \frac{T^\ell (k_1)^2 D_{\mathcal{T}}}{E(\ell)}\right] \|q_0\| \\ &\quad \times \sum_{k=m}^{n-1} \frac{\left[\frac{T(k_1 k_2 \tau(1-\ell) + \Gamma(1+\ell)\tau(k_1)^2 D_{\mathcal{T}})}{E(\ell)}\right]^n}{\Gamma(1+n\ell)}.\end{aligned}$$

Thus, $\{p_n\}$ is a Cauchy sequence in Y, which converges to a point p^* in Y as $n \to \infty$. Using (2), assumption (H_1), and Lebesgue's dominated convergence theorem, we arrive at

$$\begin{aligned}\lim_{n\to\infty} q_n(t) &= \lim_{n\to\infty} p_n(t) - \frac{RV(1-\ell)}{E(\ell)\Gamma(\ell)} \lim_{n\to\infty} \int_0^t (t-s)^{\ell-1} \hbar\bigl(s, p_n(s)\bigr)\,\mathrm{d}s \\ &\quad - \frac{\ell R^2}{E(\ell)} \lim_{n\to\infty} \int_0^t \mathcal{T}_\ell(t-s)\hbar\bigl(s, p_n(s)\bigr)\,\mathrm{d}s - p_0(t)\end{aligned}$$

$$\begin{aligned}0 &= p^*(t) - \frac{RV(1-\ell)}{E(\ell)\Gamma(\ell)} \int_0^t (t-s)^{\ell-1} \hbar\bigl(s, p^*(s)\bigr)\,\mathrm{d}s \\ &\quad - \frac{\ell R^2}{E(\ell)} \int_0^t \mathcal{T}_\ell(t-s)\hbar\bigl(s, p^*(s)\bigr)\,\mathrm{d}s - p_0(t),\end{aligned}$$

which implies that

$$\begin{aligned}p^*(t) &= R\mathcal{S}_\ell(t)p_0 + \frac{RV(1-\ell)}{E(\ell)\Gamma(\ell)} \int_0^t (t-s)^{\ell-1} \bigl[Qw(s) + \hbar\bigl(s, p^*(s)\bigr)\bigr]\,\mathrm{d}s \\ &\quad + \frac{\ell R^2}{E(\ell)} \int_0^t \mathcal{T}_\ell(t-s)\bigl[Qw(s) + \hbar\bigl(s, p^*(s)\bigr)\bigr]\,\mathrm{d}s.\end{aligned}$$

This proves that p^* is a mild solution of Eq. (1).

Our next aim is to prove the uniqueness of the solution by utilizing the regularity property of the integral contractor. To this end, for a fixed control $w \in L^2([0,T], \mathbb{W})$, let

p_1 and p_2 be two solutions of Eq. (1). Then we have

$$p_2(t)-p_1(t)=\frac{RV(1-\ell)}{E(\ell)\Gamma(\ell)}\int_0^t (t-s)^{\ell-1}\big[\hbar\big(s,p_2(s)\big)-\hbar\big(s,p_1(s)\big)\big]\,\mathrm{d}s + \frac{\ell R^2}{E(\ell)}\int_0^t \mathcal{T}_\ell(t-s)\big[\hbar\big(s,p_2(s)\big)-\hbar\big(s,p_1(s)\big)\big]\,\mathrm{d}s. \tag{5}$$

According to Definition 9 with $p=p_1$ and $z=p_2-p_1$, the equation

$$p_2(t)-p_1(t)=q(t)+\frac{RV(1-\ell)}{E(\ell)\Gamma(\ell)}\int_0^t (t-s)^{\ell-1}\varUpsilon\big(s,p_1(s)\big)q(s)\,\mathrm{d}s + \frac{\ell R^2}{E(\ell)}\int_0^t \mathcal{T}_\ell(t-s)\varUpsilon\big(s,p_1(s)\big)q(s)\,\mathrm{d}s \tag{6}$$

admits a solution $q\in Y$. Combining Eq. (5) with (6), we obtain

$$\begin{aligned}
&p_2(t)-p_1(t)\\
&\quad=\frac{RV(1-\ell)}{E(\ell)\Gamma(\ell)}\int_0^t (t-s)^{\ell-1}\Bigg[\hbar\Bigg(s,\,p_1(s)+q(s)+\frac{RV(1-\ell)}{E(\ell)\Gamma(\ell)}\\
&\qquad\times\int_0^s (s-\xi)^{\ell-1}\varUpsilon\big(\xi,p_1(\xi)\big)q(\xi)\,\mathrm{d}\xi\Bigg)-\hbar\big(s,p_1(s)\big)\\
&\qquad-\varUpsilon\big(s,p_1(s)\big)q(s)\Bigg]\,\mathrm{d}s\\
&\qquad+\frac{\ell R^2}{E(\ell)}\int_0^t \mathcal{T}_\ell(t-s)\Bigg[\hbar\Bigg(s,\,p_1(s)+q(s)+\frac{RV(1-\ell)}{E(\ell)\Gamma(\ell)}\\
&\qquad\times\int_0^s (s-\xi)^{\ell-1}\varUpsilon\big(\xi,p_1(\xi)\big)q(\xi)\,\mathrm{d}\xi\Bigg)-\hbar\big(s,p_1(s)\big)\\
&\qquad-\varUpsilon\big(s,p_1(s)\big)q(s)\Bigg]\,\mathrm{d}s\\
&\qquad+\frac{RV(1-\ell)}{E(\ell)\Gamma(\ell)}\int_0^t (t-s)^{\ell-1}\varUpsilon\big(s,p_1(s)\big)q(s)\,\mathrm{d}s\\
&\qquad+\frac{\ell R^2}{E(\ell)}\int_0^t \mathcal{T}_\ell(t-s)\varUpsilon\big(s,p_1(s)\big)q(s)\,\mathrm{d}s.
\end{aligned}$$

Hence

$$q(t) = \frac{RV(1-\ell)}{E(\ell)\Gamma(\ell)} \int_0^t (t-s)^{\ell-1} \Bigg[\hbar\Bigg(s,\, p_1(s) + q(s) + \frac{RV(1-\ell)}{E(\ell)\Gamma(\ell)} \times \int_0^s (s-\xi)^{\ell-1} \Upsilon\big(\xi, p_1(\xi)\big) q(\xi)\,\mathrm{d}\xi \Bigg) - \hbar\big(s, p_1(s)\big) - \Upsilon\big(s, p_1(s)\big) q(s) \Bigg]\,\mathrm{d}s$$
$$+ \frac{\ell R^2}{E(\ell)} \int_0^t \mathcal{T}_\ell(t-s) \Bigg[\hbar\Bigg(s,\, p_1(s) + q(s) + \frac{RV(1-\ell)}{E(\ell)\Gamma(\ell)} \times \int_0^s (s-\xi)^{\ell-1} \Upsilon\big(\xi, p_1(\xi)\big) q(\xi)\,\mathrm{d}\xi \Bigg) - \hbar\big(s, p_1(s)\big) - \Upsilon\big(s, p_1(s)\big) q(s) \Bigg]\,\mathrm{d}s.$$

Using Definition 8, we have

$$\big\|q(t)\big\| = \Bigg\| \frac{RV(1-\ell)}{E(\ell)\Gamma(\ell)} \int_0^t (t-s)^{\ell-1} \Bigg[\hbar\Bigg(s,\, p_1(s) + q(s) + \frac{RV(1-\ell)}{E(\ell)\Gamma(\ell)} \times \int_0^s (s-\xi)^{\ell-1} \Upsilon\big(\xi, p_1(\xi)\big) q(\xi)\,\mathrm{d}\xi \Bigg) - \hbar\big(s, p_1(s)\big) - \Upsilon\big(s, p_1(s)\big) q(s) \Bigg]\,\mathrm{d}s$$
$$+ \frac{\ell R^2}{E(\ell)} \int_0^t \mathcal{T}_\ell(t-s) \Bigg[\hbar\Bigg(s,\, p_1(s) + q(s) + \frac{RV(1-\ell)}{E(\ell)\Gamma(\ell)} \times \int_0^s (s-\xi)^{\ell-1} \Upsilon\big(\xi, p_1(\xi)\big) q(\xi)\,\mathrm{d}\xi \Bigg) - \hbar\big(s, p_1(s)\big) - \Upsilon\big(s, p_1(s)\big) q(s) \Bigg]\,\mathrm{d}s \Bigg\|$$
$$\leqslant \frac{\|R\|\|V\|(1-\ell)}{E(\ell)\Gamma(\ell)} \int_0^t (t-s)^{\ell-1} \Bigg\| \hbar\Bigg(s,\, p_1(s) + q(s) + \frac{RV(1-\ell)}{E(\ell)\Gamma(\ell)}$$

$$
\begin{aligned}
&\quad\times \int_0^s (s-\xi)^{\ell-1}\Upsilon\big(\xi, p_1(\xi)\big)q(\xi)\,\mathrm{d}\xi\Bigg) - \hbar\big(s, p_1(s)\big) \\
&\quad - \Upsilon\big(s, p_1(s)\big)q(s)\Bigg\|\,\mathrm{d}s \\
&+ \frac{\ell\|R\|^2}{E(\ell)}\int_0^t \big\|\mathcal{T}_\ell(t-s)\big\|\Bigg\|\hbar(s,\, p_1(s)+q(s) + \frac{RV(1-\ell)}{E(\ell)\Gamma(\ell)} \\
&\quad\times \int_0^s (s-\xi)^{\ell-1}\Upsilon\big(\xi, p_1(\xi)\big)q(\xi)\,\mathrm{d}\xi\Bigg) - \hbar\big(s, p_1(s)\big) \\
&\quad - \Upsilon\big(s, p_1(s)\big)q(s)\Bigg\|\,\mathrm{d}s \\
\leqslant{}& \frac{k_1k_2\tau(1-\ell)}{E(\ell)\Gamma(\ell)}\int_0^t (t-s)^{\ell-1}\tau\big\|q(s)\big\|\,\mathrm{d}s \\
&+ \frac{\ell\tau D_{\mathcal{T}}(k_1)^2}{E(\ell)}\int_0^t (t-s)^{\ell-1}\big\|q(s)\big\|\,\mathrm{d}s \\
\leqslant{}& \left[\frac{k_1k_2\tau(1-\ell)}{E(\ell)\Gamma(\ell)} + \frac{\ell\tau D_{\mathcal{T}}(k_1)^2}{E(\ell)}\right]\int_0^t (t-s)^{\ell-1}\big\|q(s)\big\|\,\mathrm{d}s.
\end{aligned}
$$

Lemma 3 now implies that $\|q(t)\| = 0$ for all $t \in [0, T]$, that is, $\|q\| = 0$. Moreover, from (6) we obtain that $\|p_2(t) - p_1(t)\| = 0$, which shows that $p_1 = p_2$. Hence the solution to Eq. (1) is unique as claimed. □

4 Controllability results

To establish exact controllability results, we use the following assumptions:

(H_4) The linear equation corresponding to Eq. (1)

$$
\begin{aligned}
{}^{AB}D^{\ell}_{0+}p(t) &= Gp(t) + Qr(t), \quad t \in (0, T], \\
p(0) &= p_0
\end{aligned}
$$

is exactly controllable with control r.

(H_5) $\mathcal{R}(\hbar) \subseteq \mathcal{R}(Q)$.

Theorem 2. *If assumptions* (H_1)–(H_5) *hold true, then the fractional evolution equation* (1) *is exactly controllable.*

Proof. Consider the linear AB fractional evolution equation

$$\begin{aligned} {}^{AB}D^{\ell}_{0+}q(t) &= Pq(t) + Qr(t), \quad t \in (0,T], \\ q(0) &= q_0 = \varphi. \end{aligned} \tag{7}$$

Using Definition 5, we obtain

$$q(t) = R\mathcal{S}_\ell(t)\varphi + \frac{RV(1-\ell)}{E(\ell)\Gamma(\ell)} \int_0^t (t-s)^{\ell-1} Qr(s)\,\mathrm{d}s + \frac{\ell^2}{E(\ell)} \int_0^t \mathcal{T}_\ell(t-s)Qr(s)\,\mathrm{d}s. \tag{8}$$

Also, consider the perturbed equation

$$\begin{aligned} {}^{AB}D^{\ell}_{0+}p(t) &= Gp(t) + Qr(t) + \hbar\big(t, p(t)\big) \\ &\quad - \hbar\Bigg(t,\, q(t) + \frac{RV(1-\ell)}{E(\ell)\Gamma(\ell)} \int_0^t (t-s)^{\ell-1} \varUpsilon\big(s, p(s)\big)(q-p)(s)\,\mathrm{d}s \\ &\quad + \frac{\ell R^2}{E(\ell)} \int_0^t \mathcal{T}_\ell(t-s)\varUpsilon\big(s, p(s)\big)(q-p)(s)\,\mathrm{d}s\Bigg), \quad t \in (0,T], \\ p(0) &= p_0 = \varphi \end{aligned}$$

with the mild solution

$$\begin{aligned} p(t) &= R\mathcal{S}_\ell(t)\varphi \\ &\quad + \frac{RV(1-\ell)}{E(\ell)\Gamma(\ell)} \int_0^t (t-s)^{\ell-1} \Bigg[Qr(s) + \hbar\big(s,\, p(s)\big) - \hbar\Bigg(s,\, q(s) + \frac{RV(1-\ell)}{E(\ell)\Gamma(\ell)} \\ &\qquad \times \int_0^s (s-\xi)^{\ell-1} \varUpsilon\big(\xi, p(\xi)\big)(q-p)(\xi)\,\mathrm{d}\xi \\ &\qquad + \frac{\ell R^2}{E(\ell)} \int_0^s \mathcal{T}_\ell(s-\xi)\varUpsilon\big(\xi, p(\xi)\big)(q-p)(\xi)\,\mathrm{d}\xi\Bigg)\Bigg]\,\mathrm{d}s \\ &\quad + \frac{\ell R^2}{E(\ell)} \int_0^t \mathcal{T}_\ell(t-s)\Bigg[Qr(s) + \hbar\big(s, p(s)\big) - \hbar\Bigg(s,\, q(s) + \frac{RV(1-\ell)}{E(\ell)\Gamma(\ell)} \\ &\qquad \times \int_0^s (s-\xi)^{\ell-1} \varUpsilon\big(\xi, p(\xi)\big)(q-p)(\xi)\,\mathrm{d}\xi \\ &\qquad + \frac{\ell R^2}{E(\ell)} \int_0^s \mathcal{T}_\ell(s-\xi)\varUpsilon\big(\xi, p(\xi)\big)(q-p)(\xi)\,\mathrm{d}\xi\Bigg)\Bigg]\,\mathrm{d}s. \end{aligned} \tag{9}$$

Equations (1) and (9) imply that

$$Qw(t) = Qr(t) - \hbar\Biggl(t,\, q(t) + \frac{RV(1-\ell)}{E(\ell)\Gamma(\ell)}\int_0^t (t-s)^{\ell-1}\Upsilon\bigl(s,p(s)\bigr)(q-p)(s)\,\mathrm{d}s + \frac{\ell R^2}{E(\ell)}\int_0^t \mathcal{T}_\ell(t-s)\Upsilon\bigl(s,p(s)\bigr)(q-p)(s)\,\mathrm{d}s\Biggr),$$

which holds due to assumption (H_4).

Also, subtracting Eq. (9) from (8), we find that

$$\begin{aligned} q(t)-p(t) = {}& \frac{RV(1-\ell)}{E(\ell)\Gamma(\ell)}\int_0^t (t-s)^{\ell-1}\Biggl[\hbar\Biggl(s,\, q(s) + \frac{RV(1-\ell)}{E(\ell)\Gamma(\ell)} \\ &\times \int_0^s (s-\xi)^{\ell-1}\Upsilon\bigl(\xi,p(\xi)\bigr)(q-p)(\xi)\,\mathrm{d}\xi \\ &+ \frac{\ell R^2}{E(\ell)}\int_0^s \mathcal{T}_\ell(s-\xi)\Upsilon\bigl(\xi,p(\xi)\bigr)(q-p)(\xi)\,\mathrm{d}\xi\Biggr) \\ &- \hbar(s,p(s) - \Upsilon\bigl(s,p(s)\bigr)(q-p)(s)\Biggr]\,\mathrm{d}s \\ &+ \frac{\ell R^2}{E(\ell)}\int_0^t \mathcal{T}_\ell(t-s)\Biggl[\hbar\Biggl(s,\, q(s) + \frac{RV(1-\ell)}{E(\ell)\Gamma(\ell)} \\ &\times \int_0^s (s-\xi)^{\ell-1}\Upsilon\bigl(\xi,p(\xi)\bigr)(q-p)(\xi)\,\mathrm{d}\xi \\ &+ \frac{\ell R^2}{E(\ell)}\int_0^s \mathcal{T}_\ell(s-\xi)\Upsilon\bigl(\xi,p(\xi)\bigr)(q-p)(\xi)\,\mathrm{d}\xi\Biggr) \\ &- \hbar\bigl(s,p(s)\bigr) - \Upsilon\bigl(s,p(s)\bigr) + (q-p)(s)\Biggr]\,\mathrm{d}s \\ &\times \frac{RV(1-\ell)}{E(\ell)\Gamma(\ell)}\int_0^t (t-s)^{\ell-1}\Upsilon\bigl(s,p(s)\bigr)(q-p)(s)\,\mathrm{d}s \\ &+ \frac{\ell R^2}{E(\ell)}\int_0^t \mathcal{T}_\ell(t-s)\Upsilon\bigl(s,p(s)\bigr)(q-p)(s)\,\mathrm{d}s. \end{aligned}$$

Hence

$$\begin{aligned}
&\big\|q(t)-p(t)\big\| \\
&\quad\leqslant \frac{k_1k_2(1-\ell)}{E(\ell)\Gamma(\ell)}\int_0^t (t-s)^{\ell-1}\Bigg\|\hbar\Bigg(s,\, q(s)+\frac{RV(1-\ell)}{E(\ell)\Gamma(\ell)} \\
&\qquad\qquad \times\int_0^s (s-\xi)^{\ell-1}\Upsilon\big(\xi,p(\xi)\big)(q-p)(\xi)\,\mathrm{d}\xi \\
&\qquad\qquad +\frac{\ell R^2}{E(\ell)}\int_0^s \mathcal{T}_\ell(s-\xi)\Upsilon\big(\xi,p(\xi)\big)(q-p)(\xi)\,\mathrm{d}\xi\Bigg) \\
&\qquad\qquad -\hbar\big(s,p(s)\big)-\Upsilon\big(s,p(s)\big)(q-p)(s)\Bigg\|\,\mathrm{d}s \\
&\qquad +\frac{\ell(k_1)^2}{E(\ell)}\int_0^t \big\|\mathcal{T}_\ell(t-s)\big\|\Bigg\|\hbar\Bigg(s,\, q(s)+\frac{RV(1-\ell)}{E(\ell)\Gamma(\ell)} \\
&\qquad\qquad \times\int_0^s (s-\xi)^{\ell-1}\Upsilon\big(\xi,p(\xi)\big)(q-p)(\xi)\,\mathrm{d}\xi \\
&\qquad\qquad +\frac{\ell R^2}{E(\ell)}\int_0^s \mathcal{T}_\ell(s-\xi)\Upsilon\big(\xi,p(\xi)\big)(q-p)(\xi)\,\mathrm{d}\xi\Bigg) \\
&\qquad\qquad -\hbar(s,p(s)-\Upsilon\big(s,p(s)\big)(q-p)(s)\Bigg\|\,\mathrm{d}s \\
&\qquad +\frac{k_1k_2(1-\ell)}{E(\ell)\Gamma(\ell)}\int_0^t (t-s)^{\ell-1}\big\|\Upsilon\big(s,p(s)\big)(q-p)(s)\big\|\,\mathrm{d}s \\
&\qquad +\frac{\ell(k_1)^2}{E(\ell)}\int_0^t \big\|\mathcal{T}_\ell(t-s)\big\|\big\|\Upsilon\big(s,p(s)\big)(q-p)(s)\big\|\,\mathrm{d}s \\
&\quad\leqslant \frac{k_1k_2\tau(1-\ell)}{E(\ell)\Gamma(\ell)}\int_0^t (t-s)^{\ell-1}\big\|(q-p)(s)\big\|\,\mathrm{d}s \\
&\qquad +\frac{\ell\tau D_{\mathcal{T}}(k_1)^2}{E(\ell)}\int_0^t (t-s)^{\ell-1}\big\|(q-p)(s)\big\|\,\mathrm{d}s \\
&\qquad +\frac{k_1k_2\nu(1-\ell)}{E(\ell)\Gamma(\ell)}\int_0^t t-s)^{\ell-1}\big\|(q-p)(s)\big\|\,\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{\ell\nu D_{\mathcal{T}}(k_1)^2}{E(\ell)} \int_0^t (t-s)^{\ell-1} \big\|(q-p)(s)\big\| \,\mathrm{d}s \\
&\leqslant \frac{k_1 k_2(1-\ell)}{E(\ell)\Gamma(\ell)}(\tau+\nu) \int_0^t (t-s)^{\ell-1} \big\|(q-p)(s)\big\| \,\mathrm{d}s \\
&+ \frac{\ell D_{\mathcal{T}}(k_1)^2}{E(\ell)}(\tau+\nu) \int_0^t (t-s)^{\ell-1} \big\|(q-p)(s)\big\| \,\mathrm{d}s,
\end{aligned}
$$

which implies that

$$
\big\|q(t)-p(t)\big\| \leqslant (\tau+\nu)\left[\frac{k_1 k_2(1-\ell)}{E(\ell)\Gamma(\ell)} + \frac{\ell D_{\mathcal{T}}(k_1)^2}{E(\ell)}\right] \int_0^t (t-s)^{\ell-1} \big\|(q-p)(s)\big\| \,\mathrm{d}s.
$$

A direct application of Lemma 3 implies that $\|q(t)-p(t)\| = 0$, that is, $\|q-p\| = 0$. Hence $q(t) = p(t)$ for all $t \in \mathcal{I}'$. Hence every mild solution of the linear equation (7) is also a mild solution of the semilinear equation (9). Thus, $K_T(0,r) \subset K_T(\hbar, w)$. Moreover, it follows from assumption (H_4) that $K_T(0,r) = Y$. Hence $K_T(\hbar, w) = Y$, which ensures that Eq. (1) is exactly controllable over $[0,T]$. □

Next, we discuss trajectory controllability results for the following AB fractional evolution equation:

$$
\begin{aligned}
{}^{AB}D_{0+}^{\ell} p(t) &= Gp(t) + w(t) + \hbar\big(t, p(t)\big), \quad t \in (0,T], \\
p(0) &= p_0,
\end{aligned}
\tag{10}
$$

where the linear operator G satisfies assumption (H_1), the control w belongs to the control space $L^2([0,T], \mathbb{W})$, and

(T_C) $\hbar$ is a Lipschitz continuous function, that is, for $\tau > 0$ and $p, q \in Y$,

$$
\big\|\hbar\big(t, p(t)\big) - \hbar\big(t, q(t)\big)\big\| \leqslant \tau \big\|p(t) - q(t)\big\|.
$$

It is clear by Remark 2 that if the function $\hbar$ is Lipschitz continuous, then (H_3) holds with $\Upsilon \equiv 0$. Therefore, using Theorem 1, we may conclude that the control system (10) has a unique mild solution in Y.

Let

$$
\begin{aligned}
\mathbb{U} = \big\{ q \in C([0,T], Y)\colon\ &{}^{AB}D_{0+}^{\ell} q(t) \text{ exists } \forall \ell \in (0,1) \\
&\text{with } q(0) = p_0 \text{ and } q(T) = p_f \big\},
\end{aligned}
$$

where p_f denotes the desired final state. System (10) is called trajectory controllable if for any $q \in \mathbb{U}$, there exists $w \in L^2([0,T], \mathbb{W})$ such that $p(t) = q(t)$ for almost all $t \in [0,T]$.

Consider a feedback control $w(t)$ given by

$$w(t) = {}^{AB}D^{\ell}_{0+}q(t) - Gv(t) - \hbar\bigl(t, q(t)\bigr). \tag{11}$$

Theorem 3. *If* (H_1)*,* (H_2)*, and* $(\mathrm{T_C})$ *hold true, then the fractional evolution equation* (10) *is trajectory controllable.*

Proof. Using the feedback control w given by (11), it follows from (10) that

$$\begin{aligned} {}^{AB}D^{\ell}_{0+}\bigl[p(t) - q(t)\bigr] &= G\bigl[p(t) - q(t)\bigr] + \bigl[\hbar\bigl(t, p(t)\bigr) - \hbar\bigl(t, q(t)\bigr)\bigr], \\ p(0) &= p_0. \end{aligned}$$

Setting $y(t) = p(t) - q(t)$ for all $t \in [0, T]$, we obtain

$$\begin{aligned} {}^{AB}D^{\ell}_{0+}y(t) &= Gy(t) + \bigl[\hbar\bigl(t, p(t)\bigr) - \hbar\bigl(t, q(t)\bigr)\bigr], \quad t \in (0, T], \\ y(0) &= 0. \end{aligned}$$

Using Definition 5, the mild solution is given by

$$\begin{aligned} y(t) &= \frac{RV(1-\ell)}{E(\ell)\Gamma(\ell)} \int_0^t (t-s)^{\ell-1}) \bigl[\hbar\bigl(s, p(s)\bigr) - \hbar\bigl(s, q(s)\bigr)\bigr] \,\mathrm{d}s \\ &\quad + \frac{\ell R^2}{E(\ell)} \int_0^t \mathcal{T}_\ell(t-s) \bigl[\hbar\bigl(s, p(s)\bigr) - \hbar\bigl(s, q(s)\bigr)\bigr] \,\mathrm{d}s \\ &\leqslant \frac{\|R\|\|V\|(1-\ell)}{E(\ell)\Gamma(\ell)} \int_0^t (t-s)^{\ell-1} \bigl\|\hbar\bigl(s, p(s)\bigr) - \hbar\bigl(s, q(s)\bigr)\bigr\| \,\mathrm{d}s \\ &\quad + \frac{\ell\|R\|^2}{E(\ell)} \int_0^t \bigl\|\mathcal{T}_\ell(t-s)\bigr\| \bigl\|\hbar\bigl(s, p(s)\bigr) - \hbar\bigl(s, q(s)\bigr)\bigr\| \,\mathrm{d}s. \end{aligned}$$

Hence

$$\begin{aligned} \bigl\|y(t)\bigr\| &\leqslant \frac{k_1 k_2 \tau(1-\ell)}{E(\ell)\Gamma(\ell)} \int_0^t (t-s)^{\ell-1} \bigl\|p(s) - q(s)\bigr\| \,\mathrm{d}s \\ &\quad + \frac{\ell\tau(k_1)^2 D_{\mathcal{T}}}{E(\ell)} \int_0^t (t-s)^{\ell-1} \bigl\|p(s) - q(s)\bigr\| \,\mathrm{d}s \\ &\leqslant \biggl[\frac{k_1 k_2 \tau(1-\ell)}{E(\ell)\Gamma(\ell)} + \frac{\ell\tau(k_1)^2 D_{\mathcal{T}}}{E(\ell)}\biggr] \int_0^t (t-s)^{\ell-1} \bigl\|y(s)\bigr\| \,\mathrm{d}s. \end{aligned}$$

An application of Lemma 3 now yields $\|y(t)\| = 0$. In other words, $p(t) = q(t)$ for almost all $t \in [0, T]$, which confirms the trajectory controllability of Eq. (10). □

5 An application

Consider the following AB fractional evolution equation:

$$\begin{aligned} {}^{AB}D^{\ell}_{0+}v(t,\epsilon) &= \frac{\partial^2}{\partial x^2}v(t,\epsilon)+\omega(t,\epsilon)+\sin\bigl(1+v(t,\epsilon)\bigr), \quad t\in(0,1],\ \epsilon\in[0,\pi], \\ v(0,\epsilon) &= v_0, \qquad v(t,0)=v(t,\pi)=0, \quad t\in[0,1], \end{aligned} \tag{12}$$

where $\ell\in(0,1)$. Let the space $Y=\mathbb{W}=L^2[0,\pi]$ and define the operator $G:\mathcal{D}(G)\subset Y\to Y$ by

$$Gv := \frac{\partial^2}{\partial\epsilon^2}v(t,\epsilon)=v'', \quad v\in\mathcal{D}(G),$$

with $\mathcal{D}(G):=\{v\in Y\colon\ v,v' \text{ are absolutely continuous},\ v''\in Y,\ v(0)=v(1)=0\}$. Then

$$Gv=\sum_{m=1}^{\infty}m^2(v,v_m)v_m, \quad v\in\mathcal{D}(G),$$

where $v_m(s)=(\sqrt{2}/\pi)\sin(ms)$, $m\in\mathbb{N}$, is the orthogonal set of eigenvectors of G. Using [24], we see that G generates an analytic semigroup $(\mathcal{S}(t))_{t\geqslant0}$ in Y given by

$$\mathcal{S}(t)v=\sum_{m=1}^{\infty}\mathrm{e}^{-m^2t}(v,v_m)v_m, \quad v\in Y,\ t>0.$$

Hence $(\mathcal{S}(t))_{t\geqslant0}$ is a uniformly bounded compact semigroup. In other words, $\mathcal{R}(\mu,G):=(\mu I-G)^{-1}$ is a compact operator for all $\mu\in\rho(G)$, that is, G generates an analytic solution operator $(\mathcal{S}_\ell(t))_{t\geqslant0}$ with $\|\mathcal{S}_\ell(t)\|\leqslant D_{\mathcal{S}}$ for $t\in[0,1]$.

Let

$$\begin{aligned} p(t)(\epsilon) &= v(t,\epsilon), \\ \hbar\bigl(t,p(t)\bigr)(\epsilon) &= \hbar\bigl(t,v(t,\epsilon)\bigr) \end{aligned}$$

and define the control operator $Qw:[0,1]\to\mathbb{R}$ by

$$(Qw)(t)(\epsilon)=\omega(t,\epsilon), \quad \epsilon\in[0,\pi].$$

Hence (1) is an abstract formulation of (12) given by

$$\begin{aligned} {}^{AB}D^{\ell}_{0+}p(t) &= Gu(t)+Qw(t)+\hbar\bigl(t,p(t)\bigr), \quad t\in(0,1], \\ p(0) &= p_0. \end{aligned}$$

Here the function $\hbar(t,p(t))=\sin(1+p(t))$ is such that $\mathcal{R}(\hbar)\subseteq\mathcal{R}(Q)$, and the function $\hbar$ has a regular integral contractor $\Upsilon\equiv0$. Hence, using Theorem 1, we conclude that Eq. (12) has a unique integral solution. If the linear equation corresponding to Eq. (12) is exactly controllable, then, using Theorem 2, we see that Eq. (12) is exactly controllable on $[0,1]$. Moreover, from Theorem 3 it follows that Eq. (12) is trajectory controllable.

6 Conclusions

In this paper, we have successfully established sufficient conditions for the existence of a unique solution, exact controllability, and trajectory controllability results for AB fractional evolution equations. The results are established by applying a sequencing technique combined with a bounded integral contractor, the theory of fractional calculus, and the generalized Gronwall inequality. An application is also presented to illustrate the applicability of our results. By making some appropriate assumptions and using the ideas presented in this paper, one can also establish existence and controllability results for Atangana–Baleanu fractional stochastic differential equations.

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