Chapter

Stability Estimates for Fractional Hardy-Schrödinger Operators

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Abstract

In this chapter, we derive optimal Hardy-Sobolev type improvements of fractional Hardy inequalities, formally written as $\mathcal{L}_s u \geq \frac{w(x)}{|x|^{\theta}} u^{2_* - 1}$, for the fractional Schrödinger operator $\mathcal{L}_s u = (-\Delta)^s u - k_{n,s} \frac{u}{|x|^{2s}}$ associated with s-th powers of the Laplacian for $s \in (0, 1)$, on bounded domains in $\mathbb{R}^n$. Here, $k_{n,s}$ denotes the optimal constant in the fractional Hardy inequality, and $2_* = \frac{2(n-\theta)}{n-2s}$, for $0 \leq \theta \leq 2s < n$. The optimality refers to the singularity of the logarithmic correction w that has to be involved so that an improvement of this type is possible. It is interesting to note that Hardy inequalities related to two distinct fractional Laplacians on bounded domains admit the same optimal remainder terms of Hardy-Sobolev type. For deriving our results, we also discuss refined trace Hardy inequalities in the upper half space which are rather of independent interest.

Keywords: fractional Laplacian, hardy-Sobolev inequalities, Schrödinger operator

1. Introduction

Fractional Laplacian operators have attracted considerable attention in various areas of pure and applied mathematics, see for instance [1] and the review articles [2–4]. Such non-local operators appear naturally in several branches of the applied sciences to model phenomena where long-range interactions take place, in fluid dynamics, quantum mechanics, biological populations, materials science, finance, image processing, and game theory, to name a few, for example, [5–16]. They have a prominent interest from a mathematical point of view, arising in analysis and partial differential equations (pdes), geometry, probability, and financial mathematics, see for instance [17–22].

For $0 < s < 1$, the fractional Laplacian $(-\Delta)^s$ of a function f in the Schwartz space of rapidly decaying C^∞ functions on $\mathbb{R}^n$, is defined as a pseudodifferential operator (e.g., [1, 23, 24])

$$(-\Delta)^s f = \mathcal{F}^{-1}\left(|\xi|^{2s}(\mathcal{F}f)\right), \qquad \forall \xi \in \mathbb{R}^n, \tag{1}$$

where, $\mathcal{F}f$ denotes the Fourier transform of f defined by

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$$\mathcal{F}f(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi\cdot x} f(x)\,dx.$$

It can be shown that the operator $(-\Delta)^s$ can be equivalently defined as the singular integral operator (see for instance [1], Proposition 3.3])

$$\begin{aligned}(-\Delta)^s f(x) &= c(n,s)\mathrm{P.V.}\int_{\mathbb{R}^n} \frac{f(x)-f(y)}{|x-y|^{n+2s}}\;dy \\ &:= c(n,s)\lim_{\varepsilon\to 0^+}\int_{\{|x-y|>\varepsilon\}} \frac{f(x)-f(y)}{|x-y|^{n+2s}}\;dy, \qquad \forall x\in\mathbb{R}^n,\end{aligned} \tag{2}$$

where

$$c(n,s) = \frac{s4^s}{\pi^{n/2}}\,\frac{\Gamma\left(\frac{n+2s}{2}\right)}{\Gamma(1-s)} \tag{3}$$

and Γ stands for the usual Gamma function defined by $\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}dt$. Notice that, if $s < 1/2$, then the integrand exhibits an integrable singularity, thus the principal value (P.V.) may be dropped. Moreover, by a change of variable, we can avoid the principal value and transform the singular integral in (2) as

$$(-\Delta)^s f(x) = \frac{1}{2}c(n,s)\int_{\mathbb{R}^n} \frac{2f(x)-f(x+y)-f(x-y)}{|y|^{n+2s}}\,dy.$$

We caution the reader to take into account the conventional value imposed for the constant $c(n,s)$ when comparing different definitions for fractional Laplacian. Here, we fix the value (3) so that the singular integral representation (2) accords with the characterization (1) as a Fourier multiplier operator, and notice that $\lim_{s\to 1^-}(-\Delta)^s f = -\Delta f$ and $\lim_{s\to 0^+}(-\Delta)^s f = f$. Note that the definition (1) allows for a wider range of the fractional Laplace's exponents s, while the expression (2) is defined for $s < 1$. We point out that the characterization via Fourier transform is reduced to the standard Laplacian as $s \to 1$, which, however cannot be defined by the pointwise expression (2). Let us also remark that from the definition in the Schwartz space it is possible to extend $(-\Delta)^s$ by duality in a large class of tempered distributions; see, for example [25]. For a further discussion on the fractional Laplacian and the associated fractional Sobolev spaces we refer the readers to ([1], §§2–3]).

In the literature, other characterizations for $(-\Delta)^s$ are also used, that turn out to be equivalent to the definitions (1), (2). A further discussion on the different definitions of the fractional Laplacian on $\mathbb{R}^n$ and a proof of their equivalence can be found in [26]. Each of these equivalent characterizations allows for different approaches for the related problems, and in our context, we exploit a characterization realizing the nonlocal operator via an appropriate extended local problem (see Section 3), where local pdes techniques can be applied.

Regarding the corresponding quadratic form for $(-\Delta)^s$,

$$((-\Delta)^s f, f) := \int_{\mathbb{R}^n} f\,(-\Delta)^s f\;\,dx = \int_{\mathbb{R}^n} |\xi|^{2s}\,(\mathcal{F}f)^2(\xi)\;\,d\xi$$

we have (see Aronszajn-Smith [27], page 402)

$$\int_{\mathbb{R}^n} |\xi|^{2s} (\mathcal{F}f)^2(\xi)\ d\xi = \frac{c(n,s)}{2} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^2}{|x-y|^{n+2s}}\ dx\ dy. \tag{4}$$

We consider the homogeneous fractional Sobolev space $\dot{H}^s(\mathbb{R}^n)$, defined as the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to

$$||f||_{\dot{H}^s(\mathbb{R}^n)} := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^2}{|x-y|^{n+2s}}\ dx\ dy. \tag{5}$$

The sharp fractional Sobolev inequality, associated to $(-\Delta)^s$, states that

$$S_{n,s}\left(\int_{\mathbb{R}^n} |f|^{2_s^*}(x)\ dx\right)^{2/2_s^*} \le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^2}{|x-y|^{n+2s}}\ dx\ dy, \quad \forall f \in \dot{H}^s(\mathbb{R}^n), \tag{6}$$

where $2_s^* = \frac{2n}{n-2s}$, and the best constant

$$S_{n,s} = \frac{2^{2s}\pi^s\Gamma\left(\frac{n+2s}{2}\right)}{\Gamma\left(\frac{n-2s}{2}\right)}\left(\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma(n)}\right)^{2s/n}$$

is achieved in $\dot{H}^s(\mathbb{R}^n)$, exactly by the multiples, dilates, and translates of the function $\left(1+|x|^2\right)^{(2s-n)/2}$; see [28, 29]. Sobolev inequality (6) yields the continuous embedding $\dot{H}^s(\mathbb{R}^n) \hookrightarrow L^{2_s^*}(\mathbb{R}^n)$, which is sharp within the framework of Lebesgue spaces, in the sense that the embedding fails for any other Lebesgue subspace. In terms of Lorentz spaces, this embedding reads as $\dot{H}^1(\mathbb{R}^n) \hookrightarrow L^{2_s^*,2_s^*}(\mathbb{R}^n)$, which admits an extension within the whole Lorentz space scale $L^{2_s^*,p}(\mathbb{R}^n)$, $p \ge 2$. As a matter of fact, the embeddings for $p > 2$, follow from the continuous inclusions $L^{2_s^*,2}(\mathbb{R}^n) \hookrightarrow L^{2_s^*,p}(\mathbb{R}^n)$, and the continuous embedding

$$\dot{H}^s(\mathbb{R}^n) \hookrightarrow L^{2_s^*,2}(\mathbb{R}^n), \tag{7}$$

which, in turn, follows from the fractional Hardy inequality

$$k_{n,s}\int_{\mathbb{R}^n} \frac{|f(x)|^2}{|x|^{2s}}\ dx \le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^2}{|x-y|^{n+2s}}\ dx\ dy. \tag{8}$$

Indeed, one can derive (7) from (8), by the fact that under radially decreasing rearrangement the $\dot{H}^s(\mathbb{R}^n)$ norm does not increase [30] and the left hand side of (8) does not decrease, while the Lorentz quasinorm $||\cdot||_{L^{2_s^*,2}}$ is invariant and proportional to the left hand side of (8).

In this sense, Hardy's inequality (8) is stronger than Sobolev's inequality (6). The value

$$k_{n,s} = \frac{2\pi^{n/2}\Gamma(1-s)\Gamma^2\left(\frac{n+2s}{4}\right)}{s\Gamma^2\left(\frac{n-2s}{4}\right)\Gamma\left(\frac{n+2s}{2}\right)}$$

is the best possible constant in (8). It is well known that the best constant $k_{n,s}$ in (8) is not attained in $\dot{H}^s(\mathbb{R}^n)$, yet no L^p improvement is possible in $\dot{H}^s(\mathbb{R}^n)$, as demonstrated by testing with suitable perturbations of the solution $|x|^{\frac{2s-n}{2}}$, of the corresponding Euler–Lagrange equation.

An application of Hölder's inequality together with (6) and (8), yield the following Hardy-Sobolev inequality:

$$\Lambda_{n,\theta,s}\int_{\mathbb{R}^n}\frac{|f|^{2_*(\theta)}}{|x|^{\theta}}\ dx \le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f(x)-f\ \ (y)|^2}{|x-y|^{n+2s}}\ dx\ \ dy, \qquad f\in C_0^{\infty}(\mathbb{R}^n), \tag{9}$$

where $2_*(\theta)=\frac{2(n-\theta)}{n-2s}$, $0\le\theta<2s$. The best constant in (9), contrary to the borderline case (8) i.e. $\theta=2s$, is achieved in $\dot{H}^s(\mathbb{R}^n)$; cf. [31].

In view of (3)–(4), inequality (8) is equivalent to

$$h_{n,s}\int_{\mathbb{R}^n}\frac{f^2(x)}{|x|^{2s}}\ dx \le \int_{\mathbb{R}^n}|\xi|^{2s}(\mathcal{F}f)^2(\xi)\ d\xi, \quad \forall f\in\dot{H}^s(\mathbb{R}^n), \tag{10}$$

with the sharp constant

$$h_{n,s}=4^s\Gamma^2\left(\frac{n+2s}{4}\right)/\Gamma^2\left(\frac{n-2s}{4}\right). \tag{11}$$

The dual form of (10), formulated in terms of Riesz integral operator, is a special case of Stein-Weiss inequalities [32], and the best constant $h_{n,s}$ is identified by Herbst [33]; see also Beckner [34], Yafaev [35].

By Hardy-Littlewood and Pólya-Szegö type rearrangement inequalities, it suffices to prove (10) for radial decreasing f; see Almgren and Lieb [30] where it is shown that (4) does not increase if f is replaced by its equimeasurable symmetric decreasing rearrangement. Then, we will show that the inequality is equivalent to a convolution inequality on the multiplicative group $\mathbb{R}_+$ equipped with the Haar measure $\frac{1}{r}dr$.

In particular, (10) is equivalent to the following doubly weighted Hardy-Littlewood-Sobolev inequality of Stein-Weiss [32].

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{f(x)}{|x|^s}\ \frac{1}{|x-y|^{n-2s}}\ \frac{f(y)}{|y|^s}\ dx\,dy\ \le\ C_{n,s}\int_{\mathbb{R}^n}|f(x)|^2\ dx, \tag{12}$$

with sharp constant

$$C_{n,s}=\frac{\pi^{n/2}\Gamma^2\left(\frac{n-2s}{4}\right)\Gamma(s)}{\Gamma^2\left(\frac{n+2s}{4}\right)\Gamma\left(\frac{n-2s}{2}\right)}.$$

Since we can assume that f is radial, we set $f(x)=\mathrm{f}(r)$, and $x=rx'$, $y=\rho y'$ where $|x'|=|y'|=1$. Regarding the convolution integral of the left side in (12), we employ polar coordinates to get

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)\ \frac{1}{|x|^s}\ \frac{1}{|x-y|^{n-2s}}\ \frac{1}{|y|^s}\ f(y)\ dx\,dy\ =$$
$$\int_0^\infty\int_0^\infty \int_{|x'|=1}\int_{|y'|=1} \mathrm{f}(r)\ \frac{r^{n-1}}{r^s}\ \frac{1}{|rx'-\rho y'|^{n-2s}}\ \frac{\rho^{n-1}}{\rho^s}\ \mathrm{f}(\rho)\ d\sigma(x')d\sigma(y')dr\,d\rho\ = \tag{13}$$
$$\int_0^\infty\int_0^\infty \int_{|x'|=1} \left[\mathrm{f}(r)r^{n/2}\right]\ \frac{1}{r^{\frac{2s-n}{2}}}\ K(r,\rho)\ \frac{1}{\rho^{\frac{2s-n}{2}}}\ \left[\mathrm{f}(\rho)\rho^{n/2}\right]\ d\sigma(x')\,\frac{dr}{r}\frac{d\rho}{\rho}$$

where $d\sigma$ denotes $(n-1)$-dimensional Lebesgue integration over the unit sphere $\mathbb{S}^{n-1} = \{x' \in \mathbb{R}^n : |x'| = 1\}$, and we set

$$K(r,\rho) := \int_{|y'|=1} \frac{1}{|rx'-\rho y'|^{n-2s}}\ d\sigma(y'). \tag{14}$$

Notice that $K(r,\rho)$ in (14) is independent of $x' \in \mathbb{S}^{n-1}$. To show this independence, we may assume $r = 1, \rho = \tau$, or more generally, to use the variable $\tau = \rho/r$ and then it suffices to show that

$$K(\tau) := \int_{|y'|=1} \frac{1}{|x'-\tau y'|^{n-2s}}\,d\sigma(y')$$

is independent of $x' \in \mathbb{S}^{n-1}$. Indeed, take an arbitrary $z' \in \mathbb{S}^{n-1}$. Then there exists a rotation R such that $z' = Rx'$ and we denote by R^T its transpose. Performing the change of variables $w' = R^T y'$, we get

$$\int_{|y'|=1} \frac{1}{|z'-\tau y'|^{n-2s}}\,d\sigma(y') = \int_{|w'|=1} \frac{1}{|x'-\tau w'|^{n-2s}}\,d\sigma(w') = K(\tau),$$

since $|detR| = 1$ and $|R\mathrm{v}_1 - R\mathrm{v}_2| = |\mathrm{v}_1 - \mathrm{v}_2|$, for every $\mathrm{v}_1, \mathrm{v}_2 \in \mathbb{R}^n$. Since $K(r,\rho)$ is independent of $x' \in \mathbb{S}^{n-1}$ we have

$$\int_{\mathbb{S}^{n-1}} K(r,\rho)d\sigma(x')\ =\ K(r,\rho)\int_{\mathbb{S}^{n-1}} 1d\sigma(x')\ =\ K(r,\rho)\,\frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}. \tag{15}$$

Moreover, in (14), we can choose x' to be the first direction unit vector in $\mathbb{R}^n$, that is $\hat{e}_1 = (x_1, x_2, \cdots, x_n)$ with $x_1 = 1, x_2 = x_3 = \cdots = x_n = 0$, hence

$$K(r,\rho) = \int_{|y'|=1} \frac{1}{\left(r^2 - 2r\rho y_1 + \rho^2\right)^{\frac{n-2s}{2}}}\ d\sigma(y')$$

thus

$$\frac{1}{r^{\frac{2s-n}{2}}}\ K(r,\rho)\ \frac{1}{\rho^{\frac{2s-n}{2}}} = \int_{|y'|=1} \frac{1}{\left(\frac{r}{\rho} - 2y_1 + \frac{\rho}{r}\right)^{\frac{n-2s}{2}}}\ d\sigma(y')$$

and substituting (15) into (13), we get

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)\ \frac{1}{|x|^s}\ \frac{1}{|x-y|^{n-2s}}\ \frac{1}{|y|^s}\ f(y)\ dx\,dy = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}\ \int_0^\infty\int_0^\infty h(r)\ \psi\left(\frac{r}{\rho}\right)\ h(\rho)\ \frac{dr}{r}\frac{d\rho}{\rho} \tag{16}$$

where

$$h(r) := \mathrm{f}(r) r^{n/2} \quad \text{and} \quad \psi(\tau) = \psi\left(\frac{1}{\tau}\right) = \int_{|y'|=1} \frac{1}{\left(\tau - 2y_1 + \frac{1}{\tau}\right)^{\frac{n-2s}{2}}}\ d\sigma(y').$$

As for the right side of the fractional integral inequality (12), we use again polar coordinates to get

$$\int_{\mathbb{R}^n} |f(x)|^2\ dx \ = \ \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}\ \int_0^\infty |h(r)|^2\,\frac{dr}{r}. \tag{17}$$

Finally, substituting (16), (17) in (12), we conclude that the fractional Hardy inequality (10) is written equivalently as the convolution inequality

$$\int_0^\infty\int_0^\infty h(r)\ \psi\left(\frac{r}{\rho}\right)\ h(\rho)\ \frac{dr}{r}\frac{d\rho}{\rho} \le C_{n,s}\ \int_0^\infty |h(r)|^2\,\frac{dr}{r}. \tag{18}$$

Inequality (18) is a convolution inequality on the multiplicative group $\mathbb{R}_+$ equipped with the Haar measure $\frac{1}{r}dr$, and using the sharp Young's inequality for convolution on certain noncompact Lie groups, we recover the sharpness of the constant and the non-existence of extremals for the fractional Hardy inequality (10).

2. Fractional hardy-Sobolev inequalities on bounded domains

In the sequel, we will discuss Hardy type inequalities for fractional powers of Laplacian associated with bounded domains, and, more precisely, defined for functions satisfying homogeneous Dirichlet boundary or exterior conditions. So hereafter let us fix a bounded domain $\Omega \subset \mathbb{R}^n$, with $n > 2s$.

In opposition to the case of the whole of $\mathbb{R}^n$, distinct definitions of such non-local operators have been introduced as mathematical models in various applications. In particular, we consider two of the most commonly used operators of this type, which are the so-called spectral Laplacian (see e.g. [36–38] and references therein) and the Dirichlet (also referred to as *restricted* or *regional* or integral, see e.g. [39, 40], and references therein). Both operators are deeply associated with the theory of stochastic processes. They can be characterized as generators of a $(2s)$-stable Lévy process with jumps resulting from two consecutive modifications of Wiener process, the subordination and the stopping (killing the process when leaves the domain), which reflect the homogeneous Dirichlet-type boundary (or exterior) conditions. Depending on which of these modifications is first applied, we take two different stochastic processes and their corresponding infinitesimal generators.

The Dirichlet fractional Laplacian Next, we will discuss improved versions of fractional Hardy inequalities, involving sharp Sobolev-Hardy type correction terms.

We begin with the Dirichlet fractional Laplacian which we again denote by $(-\Delta)^s$. We merely extend any function $f \in C_0^\infty(\Omega)$ in the entire $\mathbb{R}^n$ by defining $f(x) = 0$, for any $x \notin \Omega$, and then we define $(-\Delta)^s f$ as the standard fractional Laplacian on the whole space, acting on the extension of f to $\mathbb{R}^n$. More precisely, we define

$$(-\Delta)^s f = \mathcal{F}^{-1}\Big(|\xi|^{2s}(\mathcal{F}f)\Big), \qquad \forall \xi \in \mathbb{R}^n.$$

The Dirichlet fractional Laplacian can be equivalently characterized as the singular integral operator (2) for the $c(n,s)$ given in (3).

Passing from $\mathbb{R}^n$ to a bounded domain Ω, containing the origin, inequality (8) is still valid with the same best possible constant

$$k_{n,s}\int_\Omega \frac{f^2(x)}{|x|^{2s}}\,dx \leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^2}{|x-y|^{n+2s}}\,dx\,dy, \qquad \forall f \in H_0^s(\Omega), \tag{19}$$

where $H_0^s(\Omega)$ is the homogeneous fractional Sobolev space, defined as the completion of the functions in $C_0^\infty(\Omega)$, extended by zero outside Ω, with respect to the norm (5). Clearly the constant $k_{n,s}$ can not be achieved in $H_0^s(\Omega)$, and various improved versions of (19) have been established by many authors, which amount to adding L^p norms of u or its fractional gradients in the left hand side.

In particular, Frank, Lieb and Seiringer have shown among others in [40], that for any $1 \leq q < 2_s^* := 2n/(n-2s)$ and any bounded domain $\Omega \subset \mathbb{R}^n$ there exists a positive constant $c = c(n,s,q, \ |\Omega|)$ such that

$$k_{s,n}\int_\Omega \frac{f^2(x)}{|x|^{2s}}\,dx + c\left(\int_\Omega |f(x)|^q\,dx\right)^{2/q} \leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^2}{|x-y|^{n+2s}}\,dx\,dy, \quad f \in C_0^\infty(\Omega). \tag{20}$$

Using the Dirichlet to Neumann mapping for the representation of the fractional Laplacian [39] (see Section 3 for details), a partial extension of (20) has been obtained in [41], replacing the remainder term with the $p-$norm of a fractional gradient, $p<2$.

An improvement involving a 2-norm of a fractional gradient, has been obtained in [42], using the following representation of the remainder term ([40], Proposition 4.1),

$$\begin{aligned} &k_{n,s}\int_{\mathbb{R}^n} \frac{f^2(x)}{|x|^{2s}}\,dx - \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\ &= c(n,s)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}} \frac{1}{|x|^{\frac{n-2s}{2}}} \frac{1}{|y|^{\frac{n-2s}{2}}}\,dx\,dy \end{aligned} \tag{21}$$

with the ground state substitution

$$v(x) = f(x)|x|^{\frac{n-2s}{2}}. \tag{22}$$

We point out that the exponent q in (20) is strictly smaller than the critical fractional Sobolev exponent 2_s^* and the inequality fails for $q = 2_s^*$. In [43] we have shown that introducing a logarithmic relaxation we can have a critical Sobolev improvement of (19). More precisely, it has been shown the existence of a positive constant C, depending only on n and s, such that for $f \in H_0^s(\Omega)$,

$$k_{n,s}\int_\Omega \frac{|f(x)|^2}{|x|^{2s}}\ dx + C\left(\int_\Omega X^{\frac{2(n-s)}{n-2s}}\ \left(\frac{|x|}{D}\right) |f(x)|^{\frac{2n}{n-2s}}\ dx\right)^{\frac{n-2s}{n}} \leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f\ \ (y)|^2}{|x-y|^{n+2s}}\ dx\ \ dy, \tag{23}$$

where $D = \sup_{x\in\Omega}|x|$ and

$$X(r) = (1 - \ln r)^{-1}, \quad 0 < r \leq 1.$$

Moreover, the weight $X^{\frac{2(n-s)}{n-2s}}$ cannot be replaced by a smaller power of X. We emphasize that inequality (23) involves the critical exponent but contrary to the subcritical case, that is (20), it has a logarithmic correction. However inequality (23) is sharp in the sense that inequality fails for smaller powers of the logarithmic correction X. This result may be seen as the fractional version of (see [44, 45])

$$\frac{(n-2)^2}{4}\int_\Omega \frac{|f(x)|^2}{|x|^2}\ dx + c_n\left(\int_\Omega |f(x)|^{\frac{2n}{n-2}} X^{\frac{2(n-1)}{n-2}}(|x|/D)\ dx\right)^{\frac{n-2}{n}} \leq \int_\Omega |\nabla f|^2\ dx, \tag{24}$$

in the sense that (23) reduces to (24) when $s \to 1^-$.

Moreover, in [43] we have shown, for some constant $C > 0$,

$$k_{n,s}\int_\Omega \frac{|f(x)|^2}{|x|^{2s}}\ dx + C\int_\Omega X^2\ \left(\frac{|x|}{D}\right) |f(x)|^2 dx \leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^2}{|x-y|^{n+2s}}\ dx\ \ dy, \tag{25}$$

where the weight X^2 cannot be replaced by a smaller power of X.

Let us notice that contrary to the Hardy-Sobolev inequalities obtained in [46], where the Hardy potential entails the distance to the boundary, the Hardy-Sobolev inequalities involving the distance from the origin, miss the critical-Sobolev exponent by a logarithmic correction which cannot be removed. Let us also emphasize that our results cover the full range $s \in (0,1)$, in contrast to the case involving the distance from the boundary, where Hardy inequalities associated with the spectral and Dirichlet fractional Laplacians fail within the range $0 < s < 1/2$.

In view of (23) and (25), we can apply Hölder inequality to get the following Hardy-Sobolev improvement of (19).

Theorem 1. *Let $s \in (0,1)$, $0 \leq \theta \leq 2s$, Ω be a bounded domain in $\mathbb{R}^n$ with $n > 2s$. Then there exists a positive constant $C = C(n,s,\theta)$ such that*

$$h_{n,s}\int_\Omega \frac{|f(x)|^2}{|x|^{2s}}\ dx + C\left(\int_\Omega \frac{X^{p(\theta)}}{|x|^\theta}\ \ |f|^{2_*(\theta)} dx\right)^{\frac{2}{2_*(\theta)}} \leq ((-\Delta)^s f, f),$$

for any $f \in C_0^\infty(\Omega)$, or equivalently,

$$k_{n,s}\int_\Omega \frac{|f(x)|^2}{|x|^{2s}}\ dx + C\left(\int_\Omega \frac{X^{p(\theta)}}{|x|^\theta}\ \ |f|^{2_*(\theta)} dx\right)^{\frac{2}{2_*(\theta)}} \leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^2}{|x-y|^{n+2s}}\ dx\ \ dy, \tag{26}$$

where $2_*(\theta) = \frac{2(n-\theta)}{n-2s}$, $p(\theta) = \frac{2n-\theta-2s}{n-2s}$ and $X = X(|x|/D)$ with $D = \sup_{x\in\Omega}|x|$. The logarithmic weight cannot be replaced by a smaller power of X.

The optimality of the exponent $p := p(\theta) = \frac{2(n-s)-\theta}{n-2s}$ of the logarithmic weight, for the range $\theta \in [0, 2s)$ can be deduced by the optimality of the exponent of the weight X^2, for the case $\theta = 2s$, jointly with Hölder inequality; cf. ([43], Remark), [47].

In view of (21), under the substitution (22) inequality (26) yields sharp limiting cases of certain fractional Caffarelli-Kohn-Nirenberg inequalities established in [48, 49].

The spectral fractional Laplacian We proceed with another reasonable approach in defining a nonlocal operator related to fractional powers of the Laplacian on the bounded domain Ω. We consider an orthonormal basis of $L^2(\Omega)$, consisting of eigenfunctions of $-\Delta$ with homogeneous Dirichlet boundary conditions, say $\phi_1, \dots, \phi_k, \dots,$ with corresponding eigenvalues

$$0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots \qquad \text{with} \qquad \lambda_k \to \infty.$$

More precisely,

$$\begin{cases} -\Delta\phi_k = \lambda_k\phi_k, & \text{in } \ \Omega, \\ \phi_k = 0, & \text{on } \ \partial\Omega. \end{cases}$$

Then we have

$$f = \sum_{k=1}^{\infty} c_k\phi_k \qquad \text{where} \qquad c_k = \int_\Omega f\phi_k \ dx.$$

For any $0 < s < 1$, the spectral fractional Laplacian, denoted hereafter by A_s, is defined, similarly to the spectral decomposition of the standard Laplacian, by

$$A_s f = \sum_{k=1}^{\infty} \lambda_k^s c_k \phi_k, \quad \forall f \in C_0^\infty(\Omega).$$

Notice that the operator A_s can be extended by approximation for functions in the Hilbert space

$$H = \left\{ f = \sum_{k=1}^{\infty} c_k\phi_k \in L^2(\Omega) : ||f||_H = \left(\sum_{k=1}^{\infty} \lambda_k^s c_k^2 \right)^{1/2} < \infty \right\}.$$

The quadratic form corresponding to A_s is given by

$$(A_s f, f) := \int_\Omega f A_s f \ dx = \sum_{k=1}^{\infty} \lambda_k^s c_k^2.$$

Let us point out that, contrary to the case of the whole space $\mathbb{R}^n$, the fractional operators A_s and $(-\Delta)^s$, as they defined above on bounded domains, differ in several aspects. For example, the natural functional domains of their definition are different, as the definition for the Dirichlet Laplacian $(-\Delta)^s$ requires the prescribed zero values of the functions on the whole of the exterior of the domain Ω, while the definition of the spectral Laplacian requires only zero values on boundary (local boundary conditions). They have essential differences even if we consider them as operators on a restricted class of functions, where they are both defined, e.g. in $C_0^\infty(\Omega) \subset C_c^\infty(\mathbb{R}^n)$. For example, the spectral Laplacian depends on the domain Ω through its eigenvalue and eigenfunctions. A further discussion on the differences between the operators A_s and $(-\Delta)^s$ can be found in [50].

The Hardy inequality corresponding to the spectral Laplacian A_s, involving the distance to the origin, reads

$$h_{n,s} \int_\Omega \frac{f^2(x)}{|x|^{2s}} \, dx \le (A_s f, f), \qquad \forall f \in C_0^\infty(\Omega), \tag{27}$$

with the constant $h_{n,s}$ given by (11), and this constant is the best possible in the case of $0 \in \Omega$. Observe that the Hardy inequalities (10), (27) associated with two distinct non-local operators share the same optimal constant. This is not the case when the distance is taken from the boundary, where the optimal constants for the corresponding Hardy inequalities are different, as it was shown among others in [46].

Similarly to Theorem 1, one can show that (27) may be improved by adding a critical Sobolev norm with the same sharp logarithmic corrective weight appearing in (26).

3. Extension problems related to the fractional Laplacians

In the following, we denote a point in $\mathbb{R}^{n+1}$ as (x, y) with $x \in \mathbb{R}^n$, and $y \in \mathbb{R}$, and let us set $\partial\mathbb{R}^{n+1}_+ = \{(x,y) \in \mathbb{R}^{n+1} : x \in \mathbb{R}^n, y = 0\}$. A fundamental property of the fractional Laplacian $(-\Delta)^s$ is its non-local character, which can be expressed as an operator that maps Dirichlet boundary conditions to a Neumann-type condition via an extension problem posed on the upper half space

$$\mathbb{R}^{n+1}_+ = \{(x,y) \in \mathbb{R}^{n+1} : x \in \mathbb{R}^n, y > 0\}.$$

The realization of the fractional Laplacian by a Dirichlet-to-Neumann map is known to Probabilists since the work [51] for any s, while for $s = 1$ we refer to [52]. It is also widely used in the study of PDEs since the work of Caffarelli and Silvestre [39]. The authors in [39] introduced the extended problem

$$\begin{cases} div(y^{1-2s} \nabla u(x,y)) = 0, & x \in \mathbb{R}^n, \ y > 0, \\ u(x,0) = f(x), & x \in \mathbb{R}^n \end{cases} \tag{28}$$

and then showed that

$$(-\Delta)^s f(x) = C_s \lim_{y \to 0^+} y^{1-2s} u_y(x,y),$$

where $C_s > 0$ is a constant depending only on s. The dimensional independence of C_s has been shown in ([39], Section 3.2) and its concrete expression can be found for instance in [38, 53],

$$C_s = -\frac{2^{2s-1}\Gamma(s)}{\Gamma(1-s)}. \tag{29}$$

The partial differential equation in (28) is a linear degenerate elliptic equation with weight $w = y^{1-2s}$. Since $s \in (0,1)$, the weight w belongs to the class of the so-called Muckenhoupt A_2-weights [54], comprising the nonnegative functions w defined in $\mathbb{R}^{n+1}$ such that, for some constant $C > 0$ independent of balls $B \subset \mathbb{R}^{n+1}$,

$$\left(|B|^{-1}\int_B w(x,y)\,dxdy\right)\left(|B|^{-1}\int_B w^{-1}(x,y)\,dxdy\right) < C.$$

Fabes et al. [55, 56] studied systematically differential equations of divergence form with A_2-weights, therefore we can obtain quantitative properties on $(-\Delta)^s f$ from the corresponding properties of solutions of the extension problem (28).

Regarding the operators $A_s, (-\Delta)^s$, which are defined on bounded domains, several authors, motivated by the work in [39], have considered equivalent definitions by means of an extra auxiliary variable. Next we recall the associated extension problems for these two operators.

We start with the Dirichlet Laplacian $(-\Delta)^s$ in Ω, as defined in the introduction, which is plainly the fractional Laplacian $(-\Delta)^s$ in the whole space, of the functions supported in Ω. Then following [39], the fractional Laplacian $(-\Delta)^s$ is connected with the extended problem (cf. (28))

$$\begin{cases} div(y^{1-2s}\nabla u(x,y)) = 0, & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = f(x), & x \in \mathbb{R}^n. \end{cases} \tag{30}$$

In particular, the so-called $2s-$harmonic extension u is related to the fractional Laplacian of the original function f through the pointwise formula

$$(-\Delta)^s f(x) = C_s \lim_{y\to 0^+} y^{1-2s} u_y(x,y), \qquad \forall x \in \mathbb{R}^n, \tag{31}$$

where the constant C_s is given in (29).

A Dirichlet-to-Neumann mapping characterization, similar to (30)–(31), is also available for the spectral fractional Laplacian on Ω (see [36–38]), where the proper extended local problem is posed on the cylinder $\Omega \times (0,\infty)$ in place of the upper-half space. More precisely, for a function $f \in C_0^\infty(\Omega)$, we consider the problem

$$\begin{cases} div(y^{1-2s}\nabla u(x,y)) = 0, & \text{in } \Omega \times (0,\infty), \\ u = 0, & \text{on } \partial\Omega \times [0,\infty), \\ u(x,0) = f(x), & x \in \Omega, \end{cases} \tag{32}$$

with $\int_0^\infty \int_\Omega y^{1-2s} |\nabla u|^2 \,dx\,dy < \infty$. Then the extension function u is related to the spectral Laplacian of the original function f through the pointwise formula

$$(A_s f)(x) = C_s \lim_{y\to 0^+} y^{1-2s} u_y(x,y), \qquad \forall x \in \Omega, \tag{33}$$

where the constant C_s is given by (29).

4. Weighted trace hardy inequality

An alternative proof of (8) and its improvement (26) may be given following local variational techniques exploiting the characterization of [39]. In particular, using the representation of $(-\Delta)^s$ in terms of a Dirichlet to Neumann map, we consider the proper extended local problem with test functions in $C_0^\infty(\mathbb{R}^{n+1})$. Then we can get (8) by applying, for the solution $u = u(x, y)$ of the extended problem, the following trace Hardy inequality (cf. [57], Proposition 1)

$$H_{n,s}\int_{\mathbb{R}^n} \frac{u^2(x,0)}{|x|^{2s}}\ dx \le \int_0^\infty \int_{\mathbb{R}^n} y^{1-2s}|\nabla u|^2\ dx\ \ dy, \qquad \forall u \in C_0^\infty(\mathbb{R}^{n+1}), \tag{34}$$

where the constant

$$H_{n,s} = \frac{2s\Gamma^2\left(\frac{n+2s}{4}\right)\Gamma(1-s)}{\Gamma(1+s)\Gamma^2\left(\frac{n-2s}{4}\right)} \tag{35}$$

is the best possible. This argumentation has been applied by Filippas, Moschini and Tertikas [46, 58] to obtain fractional Hardy and Hardy-Sobolev inequalities involving the distance to the boundary.

In the case of bounded domains, we have

$$H_{n,s}\int_{\Omega} \frac{u^2(x,0)}{|x|^{2s}} dx \le \int_0^\infty \int_{\mathbb{R}^n} y^{1-2s}|\nabla u|^2\ dx\ \ dy \tag{36}$$

for any $u \in C_0^\infty(\mathbb{R}^{n+1})$ with $u(x,0) = 0, x \notin \Omega$. By a scaling argument it is clear that (34), (36) share the same optimal constant. Then the key estimate in deriving (26) turn out to be the sharpened versions of (34). A proof of (34) is given by the author [57], after identifying the energetic solution $\psi = \psi(x, y)$ of the Euler Lagrange equations (see [57], Proposition 1)

$$\begin{cases} div(y^{1-2s}\nabla\psi) = 0, & \text{in } \mathbb{R}^{n+1}_+, \\ \lim\limits_{y\to 0^+} y^{1-2s}\dfrac{\partial\psi(x,y)}{\partial y} = -H_{n,s}\dfrac{\psi}{|x|^{2s}}, & \text{on } \partial\mathbb{R}^{n+1}_+\backslash\{0\}. \end{cases} \tag{37}$$

In the following, we set

$$\beta := \frac{2s-n}{2}.$$

Noticing the invariant properties of problem (37), we search for solutions of the form

$$\psi(z) = |x|^\beta B(t), \quad x \in \mathbb{R}^n, \quad y \ge 0, \quad z = (x,y) \ne (0,0) \tag{38}$$

where

$$t(x,y) := \frac{y}{|x|}.$$

Then, by direct manipulations and a normalization, we can see that problem (37) has a solution of the form (38) for the solution $B : [0, \infty) \to \mathbb{R}$ of the boundary conditions problem

$$\begin{cases} t(1+t^2)B''(t) + [(3-2s)t^2 + (1-2s)]\ B'(t) + \dfrac{\beta(2s+n-4)}{2} tB(t) = 0, \quad t > 0, & \text{(a)} \\ B(0) = 1, & \text{(b)} \\ \lim\limits_{t\to\infty} t^{-\beta} B(t) \in \mathbb{R}. & \text{(c)} \end{cases} \tag{39}$$

Let us remark that the boundary value (39b) comes from a normalization, and it plays no essential role in our subsequent analysis, contrary to condition (39c) which yields a solution of (39) with the less possible singularity. Note also that the ground state $\psi = \psi(x, y)$ is well defined for $x = 0$ with $y > 0$, by virtue of (39b). Furthermore, it is useful to notice that (39a) is transformed into divergence form, after multiplying by t^{-2s},

$$\left(t^{1-2s}\left(1+t^2\right)B'(t)\right)' + \frac{\beta(2s+n-4)}{2} t^{1-2s} B(t) = 0, \qquad t > 0. \tag{40}$$

Clearly, in the special instance $n = 3$ with $s = 1/2$, problem (39) can be solved directly and more precisely, $B(t) = 1 - \frac{2}{\pi} \arctan(t)$. For the general case, we perform the change of variable $z = -t^2$ and then problem (39) is reduced to the boundary conditions problem for the hypergeometric equation, for the function $\omega(z) = B(t)$,

$$\begin{cases} z(1-z)\dfrac{d^2\omega}{dz^2} + [1 - s - (2-s)z]\dfrac{d\omega}{dz} + \dfrac{\beta(4-n-2s)}{8}\omega(z) = 0, \quad -\infty < z < 0, & \text{(a)} \\ \omega(0) = 1, & \text{(b)} \\ \lim\limits_{z\to-\infty} (-z)^{-\beta/2}\omega(z) \in \mathbb{R}. & \text{(c)} \end{cases} \tag{41}$$

For convenience of the reader, next we just record the properties of B that we shall need, and give their proof in Section 5. See also ([57], Lemma 1) and ([59], (42)–(48)). In the following, we use the notation $g \sim h$ for real functions g, h to denote that $c_1 g \le h \le c_2 g$ on their domain, for some constants $c_1, c_2 > 0$.

It can be shown (see Section 5) that problem (39) has a positive decreasing solution B and

$$B \sim \left(1+t^2\right)^{\beta/2} \qquad \text{and} \qquad B' \sim -t^{2s-1}\left(1+t^2\right)^{\frac{1}{2}}, \qquad \forall t > 0, \tag{42}$$

with

$$tB' - \beta B(t) = O\left(t^{\beta-2}\right), \quad as \quad t \to \infty. \tag{43}$$

Moreover, we have

$$\lim_{t\to 0^+} t^{1-2s} B'(t) = -H_{n,s}, \tag{44}$$

with the constant $H_{n,s}$ given in (35).

Moreover, in view of (38), we can see that

$$\nabla\psi\cdot z=\frac{2s-n}{2}\psi(z),\quad \forall z\in\mathbb{R}^{n+1}_{+}\backslash\{0\}. \tag{45}$$

Using (42)–(44), (45), we obtain the following uniform asymptotic behavior of the ground state ψ; cf. ([57], Lemma 2).

Lemma. There holds

$$\psi\sim\left(|x|^2+y^2\right)^{\frac{2s-n}{4}},\quad in\quad \mathbb{R}^{n+1}_{+}. \tag{46}$$

Moreover, for $s\in[1/2,1),$ there holds

$$|\nabla\psi|\sim\left(|x|^2+y^2\right)^{\frac{2s-n-2}{4}},\qquad in\quad \mathbb{R}^{n+1}_{+}.$$

If $s\in(0,1/2),$ then there holds

$$|\nabla\psi|\sim\left(|x|^2+y^2\right)^{-\frac{n+2s}{4}}y^{2s-1},\qquad in\quad \mathbb{R}^{n+1}_{+}.$$

5. Ground state

In this section we prove the properties of the function B of the ground state ψ given in (38).

The differential eq. (41a) is a special instance of the general class of hypergeometric equations and the relevant theory of the subsequent discussion, can be found in ([60], §15), ([61], Chap. II) and ([62], §§2.1.2–2.1.5). In the following, we also refer to ([57], §3) and the Appendix of [59].

We will denote by $F(\mathrm{a},\mathrm{b};\mathrm{c};z)$ the hypergeometric function which is defined in the open unit disk through the series ([60], 15.1.1)

$$F(\mathrm{a},\mathrm{b};\mathrm{c};z)=\sum_{k=0}^{\infty}\frac{(\mathrm{a})_k(\mathrm{b})_k}{(\mathrm{c})_k}\frac{z^k}{k!} \tag{47}$$

and then by analytic continuation into $\mathbb{C}\backslash[1,\infty)$. In (45) we set $(\mathrm{a})_k=\mathrm{a}(\mathrm{a}+1)\cdots(\mathrm{a}+k-1)$ and $(\mathrm{a})_0=1$. It is clear that

$$F(\mathrm{a},\mathrm{b};\mathrm{c};z)=F(\mathrm{b},\mathrm{a};\mathrm{c};z).$$

We consider the hypergeometric differential equation

$$z(1-z)\omega''(z)+[\mathrm{c}-(\mathrm{a}+\mathrm{b}+1)z]\,\omega'(z)-\mathrm{ab}\omega(z)=0 \tag{48}$$

for complex functions $\omega=\omega(z)$ with $z\in\mathbb{C},$ and real parameters $\mathrm{a},\mathrm{b},\mathrm{c}$ satisfying the conditions

$$\mathrm{c}-\mathrm{a}-\mathrm{b}\geq 0,\qquad \mathrm{b}>0,\qquad \mathrm{c}>0. \tag{49}$$

By formulae ([60], 15.5.3, 15.5.4), we have the following expression for the (general) solution of (48), defined in $\mathbb{C}\backslash[1,\infty)$,

$$\omega(z)=C_1F(\mathrm{a},\mathrm{b};\mathrm{c};z)+C_2z^{1-\mathrm{c}}F(\mathrm{a}-\mathrm{c}+1,\mathrm{b}-\mathrm{c}+1;2-\mathrm{c};z) \tag{50}$$

with any $C_1, C_2\in\mathbb{C}$. Let us next derive an explicit formula for the analytic continuation of the series (47) into the domain $\{z\in\mathbb{C}:|z|>1, z\notin(1,\infty)\}$. To this end, we consider $|z|>1$ with $z\notin(1,\infty)$ and we discriminate among four cases, depending on $n, s,$ as follows.

We begin with the case that all of the three numbers a, $\mathrm{c}-\mathrm{b}$, and $\mathrm{a}-\mathrm{b}$ are different from any non-positive integer $m=0,-1,-2,\dots$. Then by expression ([60], 15.3.7) we get

$$\begin{aligned}F(\mathrm{a},\mathrm{b};\mathrm{c};z)=&\frac{\Gamma(\mathrm{c})\Gamma(\mathrm{b}-\mathrm{a})}{\Gamma(\mathrm{b})\Gamma(\mathrm{c}-\mathrm{a})}(-z)^{-\mathrm{a}}F\left(\mathrm{a},\mathrm{a}-\mathrm{c}+1;\mathrm{a}-\mathrm{b}+1;\frac{1}{z}\right)\\&+\frac{\Gamma(\mathrm{c})\Gamma(\mathrm{a}-\mathrm{b})}{\Gamma(\mathrm{a})\Gamma(\mathrm{c}-\mathrm{b})}(-z)^{-\mathrm{b}}F\left(\mathrm{b},\mathrm{b}-\mathrm{c}+1;\mathrm{b}-\mathrm{a}+1;\frac{1}{z}\right).\end{aligned} \tag{51}$$

As for the case of $\mathrm{a}=\mathrm{b}\neq-m$, $\forall m=0,-1,-2,\dots,$ and $\mathrm{c}-\mathrm{a}\neq l$, for any $l=1,2,\dots,$ we have, by ([60], 15.3.13),

$$F(\mathrm{a},\mathrm{a};\mathrm{c};z)=\frac{\Gamma(\mathrm{c})(-z)^{-\mathrm{a}}}{\Gamma(\mathrm{a})\Gamma(\mathrm{c}-\mathrm{a})}\sum_{k=0}^{\infty}\frac{(\mathrm{a})_k(1-\mathrm{c}+\mathrm{a})_k}{(k!)^2}z^{-k}[\ln(-z)+2\Psi(k+1)-\Psi(\mathrm{a}+k)-\Psi(\mathrm{c}-\mathrm{a}-k)] \tag{52}$$

where we set $\Psi(z)=-\gamma-\sum\limits_{k=0}^{\infty}\left(\frac{1}{z+k}-\frac{1}{k+1}\right)$ with the so-called Euler's constant $\gamma\ \approx0.5772156649$.

Let us next proceed with the case where $\mathrm{b}-\mathrm{a}=m$, $m=1,2,\dots,$ and $\mathrm{a}\neq-k$, for any $k=0,1,2,\dots$. Firstly, if $\mathrm{c}-\mathrm{a}\neq l$, for any $l=1,2,\dots,$ then the formula ([60], 15.3.14) yields

$$\begin{aligned}F(\mathrm{a},\mathrm{a}+m;\mathrm{c};z)=&\frac{\Gamma(\mathrm{c})(-z)^{-\mathrm{a}-m}}{\Gamma(\mathrm{a}+m)\Gamma(\mathrm{c}-\mathrm{a})}\sum_{k=0}^{\infty}\frac{(\mathrm{a})_{k+m}(1-\mathrm{c}+\mathrm{a})_{k+m}}{(k+m)!k!}z^{-k}[\ln(-z)+\Psi(1+m+k)+\Psi(1+k)\\&-\Psi(\mathrm{a}+m+k)-\Psi(\mathrm{c}-\mathrm{a}-m-k)]+(-z)^{-\mathrm{a}}\frac{\Gamma(\mathrm{c})}{\Gamma(\mathrm{a}+m)}\sum_{k=0}^{m-1}\frac{\Gamma(m-k)(\mathrm{a})_k}{k!\Gamma(\mathrm{c}-\mathrm{a}-k)}z^{-k}.\end{aligned} \tag{53}$$

Otherwise, if $\mathrm{c}-\mathrm{a}=l$, for some $l=1,2,\dots,$ such that $l>m$, then we get from formula ([61], (19) in §2.1.4),

$$\begin{aligned}F(\mathrm{a},\mathrm{a}+m;\mathrm{a}+l;z)=&\frac{\Gamma(\mathrm{a}+l)}{\Gamma(\mathrm{a}+m)}(-z)^{-\mathrm{a}}\Bigg[(-1)^l(-z)^{-m}\sum_{k=l-m}^{\infty}\frac{(\mathrm{a})_{k+m}(k+m-l)!}{(k+m)!k!}z^{-k}\\&+\sum_{k=0}^{m-1}\frac{(m-k-1)!(\mathrm{a})_k}{(l-k-1)!k!}z^{-k}+\frac{(-z)^{-m}}{(l-1)!}\sum_{k=0}^{l-m-1}\frac{(\mathrm{a})_{k+m}(1-l)_{k+m}}{(k+m)!k!}z^{-k}\times\\&\times[\ln(-z)+\Psi(1+m+k)+\Psi(1+k)-\Psi(\mathrm{a}+m+k)-\Psi(l-m-k)]\Bigg].\end{aligned} \tag{54}$$

We conclude with the case that some of the parameters a or c − b equals a nonpositive integer. In this case, $F(\mathrm{a}, \mathrm{b}; \mathrm{c}; z)$ is an elementary function of z. In particular, if $\mathrm{a} = -m$ for some $m = 0,1,2, \dots$ then, ([60], 15.4.1), the hypergeometric series in (47) is the polynomial

$$F(-m, \mathrm{b}; \mathrm{c}; z) = \sum_{k=0}^{m} \frac{(-m)_k (\mathrm{b})_k}{(\mathrm{c})_k} \frac{z^k}{k!}. \tag{55}$$

Otherwise, if $\mathrm{c} - \mathrm{b} = -l$, for some $l = 0,1,2, \dots,$ then from formula ([60], 15.3.3), $F(\mathrm{a}, \mathrm{b}; \mathrm{c}; z)$ is given by

$$F(\mathrm{a}, \mathrm{b}; \mathrm{c}; z) = (1 - z)^{-\mathrm{a}-l} F(\mathrm{c} - \mathrm{a}, -l; \mathrm{c}; z) \tag{56}$$

and notice by (55) that the hypergeometric function of the right side is a polynomial of degree l.

In the following, we will also use the differentiation formula ([60], 15.2.1), that is

$$\frac{d}{dz} F(\mathrm{a}, \mathrm{b}; \mathrm{c}; z) = \frac{\mathrm{ab}}{\mathrm{c}} \; F(\mathrm{a} + 1, \mathrm{b} + 1; \mathrm{c} + 1; z). \tag{57}$$

Let us now proceed to prove that B is positive and monotone, and also derive the asymptotics (42)–(44). To simplify the presentation, we set

$$\begin{aligned} a_1 &= \frac{4 - n - 2s}{4}, a_2 = a_1 - c_1 + 1 = \frac{4 - n + 2s}{4}, c_1 = 1 - s, \\ b_1 &= -\frac{\beta}{2} = \frac{n - 2s}{4}, b_2 = b_1 - c_1 + 1 = \frac{n + 2s}{4}, c_2 = 2 - c_1 = 1 + s. \end{aligned}$$

For these values, and recalling the assumption $n > 2s$ with $0 < s < 1$, it is easily seen that the parameters $\{a_1, b_1, c_1\}$ and $\{a_2, b_2, c_2\}$, satisfy the assumptions (49), so we can apply the aforementioned formulas. The first main step is to get an explicit expression of $B(t) = \omega(z)$. In view of (50) the general solution of (41a) is given by

$$\omega(z) = C_1 F(a_1, b_1; c_1; z) + C_2 (-z)^{1-c_1} F(a_2, b_2; c_2; z), \qquad z \le 0, \tag{58}$$

for certain constants C_1, C_2. We apply (41b) to (58), and take into account that $F(a_1, b_1; c_1; \; 0) = F(a_2, b_2; c_2; \; 0) = 1$, to get that $C_1 = 1$.

The constant C_2 will be determined by the condition at ∞, and to this aim we will get an expression for $\omega(z)$ for $z < -1$. By considering separately the cases for n, s, corresponding to the formulas (51)–(56), which give the explicit expression for the hypergeometric functions in (58), we get, in all instances, that

$$C_2 = -\frac{\Gamma(c_1)\Gamma(b_2)\Gamma(c_2 - a_2)}{\Gamma(c_2)\Gamma(b_1)\Gamma(c_1 - a_1)}, \tag{59}$$

and the asymptotics

$$\omega(z) = O\left((-z)^{-b_1}\right), \quad as \quad z \to -\infty. \tag{60}$$

In order to determine the limit

$$H_{n,s} \coloneqq -\lim_{t \to 0^+} t^{1-2s} B'(t) = 2 \lim_{z \to 0^-} (-z)^{1-s} \omega'(z)$$

we differentiate (58) and using (57) we obtain

$$\omega'(z) = \frac{a_1 b_1}{c_1} F(a_1+1, b_1+1; c_1+1; z) - C_2 s\,(-z)^{s-1} F(a_2, b_2; c_2; z)$$
$$+C_2 \frac{a_2 b_2}{c_2} (-z)^s F(a_2+1, b_2+1; c_2+1; z)$$

and then let $z \to 0^-$ to get

$$H_{n,s} = 2 \lim_{z \to 0^-} (-z)^{1-s} \omega'(z) = -2sC_2$$

and taking into account (59) we obtain (44).

Let us next show that B is decreasing and positive. We first assume that $4 - n - 2s < 0$. In this case, the positivity of B follows from the fact that if there exist $t_0 > 0$ such that $B(t_0) = 0$, then since $\lim_{t\to\infty} B(t) = 0$, there exists $t_m > t_0$ where B attains local non-negative maximum or local non-positive minimum which disagree with the differential eq. (39a). Therefore B is positive and the same argument shows that B is decreasing.

For the case that $4 - n - 2s \geq 0$, we perform the transformation $g(t) = (1+t^2)^{b_1} B(t)$ which reduces (39) to the problem

$$\begin{cases} t(1+t^2)^2 g''(t) + [1 - 2s + (3-n)t^2](1+t^2) g'(t) - \beta^2 t g(t) = 0, \quad t > 0, & \text{(a)} \\ g(0) = 1, & \text{(b)} \\ \lim_{t\to\infty} g(t) \in \mathbb{R}. & \text{(c)} \end{cases} \quad (61)$$

One can verify condition (61c) directly from the explicit formula of $B(t) = \omega(z)$. Then, by a standard minimum principle argumentation for the boundary conditions problem (61), we can verify that g is not negative, and as a consequence B is nonnegative. Then the fact that B is monotone and positive follows from (40) together with the negativity of the derivative of B near the origin.

To show the asymptotics for B in (42), we use conditions (39b)-(39c) taking into account that B is positive, and to show the asymptotics of B' in (42), we differentiate the expression (58) exploiting (57).

To conclude, it is straightforward to show (43) by substituting the concrete expression for $B(t) = \omega(-t^2)$ through the corresponding formulas (depending on the parameters n, s) and the B'.

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