Chapter

Stabilization of a Quantum Equation under Boundary Connections with an Elastic Wave Equation

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Abstract

The stability of coupled PDE systems is one of the most important topic because it covers realistic modeling of the most important physical phenomena. In fact, the stabilization of the energy of partial differential equations has been the main goal in solving many structural or microstructural dynamics problems. In this chapter, we investigate the stability of the Schrödinger-like quantum equation in interaction with the mechanical wave equation caused by the vibration of the Euler–Bernoulli beam, to effect stabilization, viscoelastic Kelvin-Voigt dampers are used through weak boundary connection. Firstly, we show that the system is well-posed via the semigroup approach. Then with spectral analysis, it is shown that the system operator of the closed-loop system is not of compact resolvent and the spectrum consists of three branches. Finally, the Riesz basis property and exponential stability of the system are concluded via comparison method in the Riesz basis approach.

Keywords: wave equation, exponential stability, Riesz basis approach, C_0–semigroup, spectral analysis

1. Introduction

There are many coupled systems that have been addressed in the literature, and we can hint here that coupling may be through the association of PDEs with coefficients or via boundary conditions of PDEs. The coupling may be strong or weak as the characteristic is determined based on the results obtained after studying the stability or control. We can divide the coupled systems according to the coupling form. Firstly, the parabolic-hyperbolic coupled systems, such as heat wave system, that arise from the interaction of the fluid structure. See works [1, 2] where stability and control systems are analyzed. Secondly, we can refer heat-beam system through works [3, 4] where the researchers used an effective method for stabilization of the system. Thirdly, in the heat-Schrödinger system, the heat dynamic controller was applied for stabilization and Gevrey regularity property in the paper [5]. Finally, in the case of thermoelastic systems, the exponential stability and Riesz basis property of the coupled heat equation and elastic structure were discussed in reference [6]. The exponential stability of thermoplastic systems with microtemperature in reference [7], for the linear beam system coupled with thermal effect, we refer to the works [8–12]. For the nonlinear beam system with thermal effect, see reference [13].

From general result related to the previously mentioned research works, we can conclude that the heat equation plays the role of dynamic boundary feedback controller of the hyperbolic PDE. Also, for the interconnected system of Euler–Bernoulli beam and heat equation with boundary weak connections where the heat is the dynamic boundary controller to the whole system, which means that this subsystem can be presented as a controller for other subsystems.

Euler–Bernoulli beam equation with boundary energy dissipation is analyzed in the work [14], the problem is given as follows:

$$\begin{cases} \rho y_{tt} + EIy_{xxxx} = 0, & 0 < x < 1, \\ y(0,t) = y_x(0,t) = 0 & \\ -EIy_{xxx}(1,t) = -k_1^2 y_t(1,t), & k_1 \in \mathbb{R}, \\ -EIy_{xx}(1,t) = k_2^2 y_{xt}(1,t), & k_2 \in \mathbb{R}, \\ y(x,0) = y_0(x) \quad y_t(x,0) = y_1(x), & 0 \le x \le 1, \end{cases} \tag{1}$$

where ρ denotes the mass density per unit length, EI is the flexural rigidity coefficient. The authors extract some estimates of the resolvent operator on the imaginary axis by applying Huang's[1] theorem to establish an exponential decay result.

For the asymptotic behavior of the wave equation, we introduce the following problem:

$$\begin{cases} \dfrac{\partial^2 w}{\partial t^2} - \Delta w = 0 \quad \text{in} & \Omega \times (0,\infty), \\ w(x,t) = 0 \quad \text{on} & \Gamma_0 \times [0,\infty), \\ \dfrac{\partial w}{\partial \nu} + a(x)\dfrac{\partial w}{\partial t} = 0 \quad \text{on} & \Gamma_1 \times (0,\infty), \end{cases} \tag{2}$$

where ν is the unit normal of Γ pointing toward exterior of Ω. The function $a \in C^1(\overline{\Gamma_1})$ with $a(x) \ge a_0 > 0$ on Γ_1. Problem (2) has been treated by Lagnese in [17], he used a multiplier method[2] and proved that the energy decay rate is obtained for solutions of wave type equations in a bounded region in $\mathbb{R}^n$ $(n \ge 2)$ whose boundary consists partly of a nontrapping reflecting surface and partly of an energy absorbing surface. We can express this result, as follows:

$$E(t) \le f(t)E(0), \qquad t \ge 0, \tag{3}$$

[1] Huang [15] introduced a frequency domain method to study the exponential decay of such stability problems.

[2] The energy multiplier method [16, 17] has been successfully applied to establish exponential stability, which is a very desirable property for elastic systems.

with energy defined by

$$E(t) = \frac{1}{2}\left(\|w_t\|^2_{L^2(\Omega)} + \|\nabla w\|^2_{L^2(\Omega)}\right). \tag{4}$$

The decay rate of solutions is a function $f(t)$ satisfying $f(t) \to 0$ as $t \to \infty$. However, there are difficulties with some boundary condition problems, which makes the energy multiplier method ineffective in proving the exponential stability property.

Wazwaz [18], used the variational iteration method[3] for the study of both linear and nonlinear Schrödinger equations, these problem is governed by the following equations:

$$\begin{cases} u_t + iu_{xx} = 0, \\ u(x,0) = f(x), \qquad i^2 = -1 \end{cases} \tag{5}$$

and

$$\begin{cases} iu_t + u_{xx} + \gamma|u|^{2r}u = 0, \quad r \geq 1, \\ u(x,0) = f(x), \quad i^2 = -1. \end{cases} \tag{6}$$

The variational iteration method was used to give rapid convergent successive approximations as well as to treat linear and non-linear problems in a uniform manner.

1.1 Statement of the problem

In this work, we consider stabilization for a Schrödinger equation through a boundary feedback dynamic controller interacted by an Euler–Bernoulli beam equation with Kelvin-Voigt damping[4], the system is described by the following coupled partial differential equations:

$$\begin{cases} \partial_t^2 u + \partial_x^4 u + \beta\partial_x^4\partial_t u = 0, \quad 0 < x < 1, t > 0, \\ \partial_t v + i\partial_x^2 v = 0, \qquad\qquad 0 < x < 1, t > 0, \end{cases} \tag{7}$$

boundary conditions are given by

$$\begin{cases} u(1,t) = \partial_x u(0,t) = \partial_x^2 u(1,t) = v(1,t) = 0, \quad t \geq 0, \\ v(0,t) = \alpha\partial_t u(0,t), \qquad t \geq 0, \\ \beta\partial_x^3\partial_t u(0,t) + \partial_x^3 u(0,t) = -\alpha i \partial_x v(0,t), \qquad t \geq 0, \end{cases} \tag{8}$$

the problem is associated with the following initial conditions:

$$u(x,0) = u_0(x), \partial_t u(x,0) = u_1(x), v(x,0) = v_0(x), \qquad 0 \leq x \leq 1. \tag{9}$$

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[3] The variational iteration method is established by He in [19, 20] is thoroughly used by many researchers to handle linear and nonlinear models.

[4] Kelvin-Voigt is one of the most important types of damping and has been used in many works, see for example, [10, 21].

1.2 Energy space

Initial condition (9) is in the following phase space:

$$\mathcal{H} = H^2_*(0,1) \times L^2(0,1) \times L^2(0,1), \tag{10}$$

where

$$H^2_*(0,1) = \{s | s \in H^2(0,1),\, \partial_x s(0) = s(1) = 0\}.$$

1.3 Energies

The energy is the sum of the potential energy and the kinetic energy, given by

$$E(t) = \frac{1}{2}\left(\|u_t\|^2_{L^2(0,1)} + \left\|\partial_x^2 u\right\|^2_{L^2(0,1)} + \|v\|^2_{L^2(0,1)}\right). \tag{11}$$

Then, we have

$$\frac{d}{dt}E(t) = -\beta\left\|\partial_x^2\partial_t u\right\|^2_{L^2(0,1)}. \tag{12}$$

It is clear that $E(t)$ is nonincreasing with time.

1.4 Remark

1. The energy dissipation is related to the wave equation, that is, there are no explicit terms for a part of the Schrödinger subsystem.

2. We note that the weakness of the boundary connections for problems (7)–(9) lead to a complicated problem in stability analysis.

3. If we take the β coefficient equal to zero in Eq. (12), the system becomes conservative.

1.5 Notations

1. $\langle \cdot, \cdot \rangle_{L^2(0,1)}$ is the $L^2(0,1)-$inner product and $\|\cdot\|_{L^2(0,1)}$ is the $L^2(0,1)-$norm.

2. The symbols $\Re(s)$ and (s) indicate the real part and the imaginary of a complex number s.

3. $(s)^T$ represents the transposed vector of (s).

2. Well-posedness

2.1 Setting of the semigroup

Setting $z = (u, \partial_t u = w, v)^T$. Then, we introduce the norm in the Hilbert space $\mathcal{H}$ as follows:

$$\begin{aligned}\|z\|_{\mathcal{H}}^2 &= \|u_t\|_{L^2(0,1)}^2 + \left\|\partial_x^2 u\right\|_{L^2(0,1)}^2 + \|v\|_{L^2(0,1)}^2 \\ &= 2E(t),\end{aligned} \tag{13}$$

for $z_1, z_2 \in \mathcal{H}$, the norm (13) is induced by the following inner product

$$\langle z_1, z_2\rangle_{L^2(0,1)} = \langle w_1, w_2\rangle_{L^2(0,1)} + \left\langle \partial_x^2 u_1, \partial_x^2 u_2\right\rangle_{L^2(0,1)} + \langle v_1, v_2\rangle_{L^2(0,1)}. \tag{14}$$

System (7) can be written as an abstract Cauchy problem in the phase space (10) as follows:

$$\begin{cases} \dfrac{d}{dt} z = \mathcal{A} z, \; t > 0, \\ z(0) = z_0. \end{cases} \tag{15}$$

The solution at time $t > 0$ to problem (15) can be written as:

$$z(t) = S(t) z_0 = e^{t\mathcal{A}} z_0,$$

where the operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is given by

$$\mathcal{A} z = \begin{pmatrix} w \\ -\partial_x^2\left(\partial_x^2 u + \beta \partial_x^2 w\right) \\ -i \partial_x^2 v \end{pmatrix}, \tag{16}$$

with domain

$$\mathcal{D}(\mathcal{A}) = \left\{ z \in \mathcal{H}, \; \mathcal{A} z \in \mathcal{H} \;\middle|\; \begin{array}{c} \partial_x^2 u + \beta \partial_x^2 w \in H^2(0,1), \\ u(1) = \partial_x u(0) = \partial_x^2 u(1) = v(1) = 0, \\ v(0) = \alpha w(0), \\ \beta \partial_x^3 w(0) + \partial_x^3 u(0) = -\alpha i \partial_x v(0), \end{array} \right\}, \tag{17}$$

Theorem 1.1: Let $\mathcal{A}$ defined by (16). Then, $\mathcal{A}^{-1}$ exists and $\mathcal{A}$ generates a C_0-semigroup of contractions on $\mathcal{H}$.

Proof: We use the semigroup method, we shall show that:

1. The operator $\mathcal{A}$ is dissipative.

2. The operator $I_d - \mathcal{A}$ is onto (I_d is the identity operator).

For the proof of (1). Firstly, we have $\mathcal{D}(\mathcal{A})$ is dense in $\mathcal{H}$, that is,

$$\overline{\mathcal{D}(\mathcal{A})} = \mathcal{H}. \tag{18}$$

Secondly, by applying the scalar product in the Hilbert space $\mathcal{H}$, we obtain

$$\begin{aligned}\langle \mathcal{A} z, z\rangle_{\mathcal{H}} &= \left\langle \partial_x^2 w, \partial_x^2 \overline{u}\right\rangle_{L^2(0,1)} - \left\langle \partial_x^2\left(\partial_x^2 u + \beta \partial_x^2 w\right), \overline{w}\right\rangle_{L^2(0,1)} - \left\langle i \partial_x^2 v, \overline{v}\right\rangle_{L^2(0,1)} \\ &= \left\langle \partial_x^2 w, \partial_x^2 \overline{u}\right\rangle_{L^2(0,1)} + \left(\partial_x^3 u(0) + \beta \partial_x^3 w(0)\right)\overline{w}(0) \\ &\quad + i \partial_x v(0) \overline{v}(0) + \langle i \partial_x v, \partial_x \overline{v}\rangle_{L^2(0,1)} - \left\langle \partial_x^2 u + \beta \partial_x^2 w, \partial_x^2 \overline{w}\right\rangle_{L^2(0,1)}.\end{aligned} \tag{19}$$

By using boundary conditions (8), we get

$$\Re\langle \mathcal{A}z, z\rangle_{\mathcal{H}} = -\beta\left\|\partial_x^2 w\right\|^2_{L^2(0,1)} \leq 0. \tag{20}$$

Then, the density property (18) and inequality (20) show that $\mathcal{A}$ is dissipative. For the proof of (2), we shall solve the equation

$$\mathcal{A}z = F$$

for any $F = (f_1, f_2, f_3)^T \in \mathcal{H}$, we can express the equation as follows:

$$\begin{cases} w = f_1, \\ \partial_x^2\left(\partial_x^2 u + \beta\partial_x^2 w\right) = -f_2, \\ i\partial_x^2 v = -f_3 \end{cases} \tag{21}$$

By using the first equation of (21), we get

$$\begin{cases} \partial_x^4 u = -f_2 + \beta\partial_x^4 f_1, \\ \partial_x^2 v = if_3. \end{cases} \tag{22}$$

We solve the following equation for the function v,

$$\begin{cases} \partial_x^2 v = if_3, \\ v(1) = 0, \quad v(0) = \alpha f_1(0), \end{cases} \tag{23}$$

to obtain

$$\begin{cases} v = \partial_x v(0)x + i\displaystyle\int_0^x (x-y)f_3(y)dy + \alpha f_1(0), \\ \partial_x v(0) = -i\displaystyle\int_0^1 (1-y)f_3(y)dy - \alpha f_1(0). \end{cases} \tag{24}$$

For u, we solve

$$\begin{cases} \partial_x^4 u = -f_2 + \beta\partial_x^4 f_1, \\ u(1) = \partial_x u(0) = \partial_x^2 u(1) = 0, \\ \beta\partial_x^3 w + \partial_x^3 u(0) = -i\alpha\partial_x v(0), \end{cases} \tag{25}$$

to obtain

$$\begin{cases} u = -\displaystyle\int_0^x (1-x)g(y)dy - \int_x^1 (1-y)g(y)dy, \\ g(x) = \beta\left(\partial_x^2 f_1(1) - \partial_x^2 f_1(x)\right) + \displaystyle\int_0^x (1-x)f_2(y)dy \\ \qquad + \displaystyle\int_x^1 (1-y)f_2(y)dy + i\alpha\partial_x v(0)(1-x). \end{cases} \tag{26}$$

Eqs. (24) and (26) give a unique $z \in \mathcal{D}(\mathcal{A})$ satisfying $\mathcal{A}z = F$.
It is easy to check that $\mathcal{A}^{-1}$ is bounded, that is,

$$0 \in \rho(\mathcal{A}).$$

Therefore, the operator $\mathcal{A}$ generates a C_0-semigroup of contractions on $\mathcal{H}$ by the Lumer–Philips theorem [22].

3. Spectral analysis

We consider the following eigenvalue problem for the system operator $\mathcal{A}$. Let $\mathcal{A}z = \lambda z$. Then, we have

$$\begin{cases} w = \lambda u, \\ \partial_x^2\left(\partial_x^2 u + \beta \partial_x^2 w\right) = -\lambda w, \\ \partial_x^2 v = i\lambda v, \\ u(1) = \partial_x u(0) = \partial_x^2 u(1) = v(1) = 0, \\ \alpha\lambda u(0) = v(0), \\ (1+\beta\lambda)\partial_x^3 u(0) = -i\alpha\partial_x v(0). \end{cases} \tag{27}$$

The first and second equations of system (27) give the following system

$$\begin{cases} (1+\beta\lambda)\partial_x^4 u + \lambda^2 u = 0, \\ \partial_x^2 v = i\lambda v, \\ u(1) = \partial_x u(0) = \partial_x^2 u(1) = v(1) = 0, \\ \alpha\lambda u(0) = v(0), \\ (1+\beta\lambda)\partial_x^3 u(0) = -i\alpha\partial_x v(0). \end{cases} \tag{28}$$

Lemma
For any $\lambda \in \sigma_p(\mathcal{A})$, it holds

$$\Re(\lambda) < 0. \tag{29}$$

Proof: By Theorem 1.1, we have $\Re(\lambda) \leq 0$.[5] Letting $0 \neq \lambda \in \sigma_p(\mathcal{A})$ with $\Re(\lambda) = 0$ and $z \in \mathcal{D}(\mathcal{A})$ satisfying

$$\mathcal{A}z = \lambda z. \tag{30}$$

By using inequality 20, it follows that

$$0 = \Re(\lambda)\|z\|_{\mathcal{H}}^2 = \Re\langle \mathcal{A}z, z\rangle_{\mathcal{H}} = -\beta\left\|\partial_x^2 w\right\|_{L^2(0,1)}^2. \tag{31}$$

From Eq. (31) and boundary conditions $(28)_3$, we have $w = 0$.

[5] $\mathcal{A}$ is dissipative $\Rightarrow \Re(\lambda) \leq 0, \ \forall \lambda \in \sigma_p(\mathcal{A})$.

From $(27)_1$ we have $u = 0$. Moreover, Eq. (30) gives

$$\begin{cases} \partial_x^2 v = i\lambda v, \\ v(0) = v(1) = \partial_x v(0) = 0. \end{cases} \tag{32}$$

It is easy to check that the above equation has only a trivial null solution $v = 0$. Hence, $z = 0$, and all the points that are located on the imaginary axis are not eigenvalues of $\mathcal{A}$. Then the proof is completed.

Setting $\lambda = \rho^2$ in (28), when $1 + \beta\rho^2 \neq 0$, we obtain

$$\begin{cases} \partial_x^4 u = \dfrac{-\rho^4}{1+\beta\rho^2} u, \\ \partial_x^2 v = i\rho^2 v, \\ u(1) = \partial_x u(0) = \partial_x^2 u(1) = v(1) = 0, \\ \alpha\rho^2 u(0) = v(0), \\ \left(1+\beta\rho^2\right)\partial_x^3 u(0) = -i\alpha\partial_x v(0). \end{cases} \tag{33}$$

Let

$$a = \sqrt[4]{\frac{-\lambda^2}{1+\beta\lambda}}.$$

Then, the general solution of system (33) can be expressed as follows:

$$\begin{aligned} &u = c_1 \exp(ax) + c_2 \exp(-ax) + c_3 \exp(iax) + c_4 \exp(-iax), \\ &v = d_1 \exp\left(\sqrt{i}\rho x\right) + d_2 \exp\left(-\sqrt{i}\rho x\right). \end{aligned} \tag{34}$$

By the boundary conditions of (33), we obtain that the constants $c_1, \cdots, c_4$ and d_1, d_2 are not identical to zero if and only if $\det(X) = 0$, where

$$X = \begin{pmatrix} e^a & e^{-a} & e^{ia} & e^{-ia} & 0 & 0 \\ a^2e^a & a^2e^{-a} & -a^2e^{ia} & -a^2e^{-ia} & 0 & 0 \\ a & -a & ia & -ia & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{\sqrt{i}\rho} & -e^{\sqrt{i}\rho} \\ \alpha\rho^2 & \alpha\rho^2 & \alpha\rho^2 & \alpha\rho^2 & -1 & -1 \\ a^3 & -a^3 & -ia^3 & ia^3 & \dfrac{i\sqrt{i}\alpha\rho}{\beta\rho^2+1} & -\dfrac{i\sqrt{i}\alpha\rho}{\beta\rho^2+1} \end{pmatrix}, \tag{35}$$

by using boundary conditions (8), we get

$$c_2 = -e^{2a}c_1, \quad c_4 = -e^{2ia}c_3, \quad d_2 = -e^{2\sqrt{i}\rho}d_1.$$

Then, the solution can be expressed by

$$u = c_1\left(e^{ax} - e^{a(2-x)}\right) + c_3\left(e^{iax} - e^{ia(2-x)}\right), \quad v = d_1\left(e^{\sqrt{i}\rho x} - e^{\sqrt{i}\rho(2-x)}\right),$$

where c_1, c_3, d_1 are determined by the remaining three boundary conditions of (36) that $\det(X) = 0$ if and only if $\det(\tilde{X}) = 0$, where

$$\tilde{X} = \begin{pmatrix} 1 + e^{2a} & i + ie^{2ia} & 0 \\ (1 - e^{2a})\alpha\rho^2 & (1 - e^{2ia})\alpha\rho^2 & -1 + e^{2\sqrt{i}\rho} \\ a^3(1 + e^{2a}) & -ia^3(1 + e^{2ia}) & \dfrac{i\sqrt{i}\alpha\rho}{\beta\rho^2 + 1}\left(1 + e^{2\sqrt{i}\rho}\right) \end{pmatrix}. \tag{36}$$

We recall the result of Lemma (29) and in light of this, we know that all eigenvalues have negative real parts. Thus, we only consider those λ that lie in the second and third quadrants of the complex plane:

$$S := \left\{\rho \in \mathbb{C} | \frac{\pi}{4} \leq \arg\rho \leq \frac{3\pi}{4}\right\}.$$

Denote the region $S := S_1 \cup S_2 \cup S_3$ such that

$$\begin{aligned} S_1 &= \left\{\rho \in \mathbb{C} | \frac{\pi}{4} \leq \arg\rho \leq \frac{3\pi}{8}\right\}, \\ S_2 &= \left\{\rho \in \mathbb{C} | \frac{3\pi}{8} \leq \arg\rho \leq \frac{5\pi}{8}\right\}, \\ S_3 &= \left\{\rho \in \mathbb{C} | \frac{5\pi}{8} \leq \arg\rho \leq \frac{3\pi}{4}\right\}, \end{aligned}$$

the following theorem gives asymptotic distributions of the eigenvalues in S_1, S_2, and S_3.

Theorem 1.2: The eigenvalues of $\mathcal{A}$ have two families:

$$\sigma_p(\mathcal{A}) = \{\lambda_{1n}, n \in \mathbb{N}\} \cup \{\lambda_{2n}^+, \lambda_{2n}^-, n \in \mathbb{N}\},$$

where

$$\begin{aligned} \lambda_{1n} &= in^2\pi^2 + \frac{\sqrt{2}\alpha^2}{\sqrt[4]{\beta}} e^{\frac{5i\pi}{8}} \sqrt{n\pi} - \frac{\alpha^4}{\sqrt{\beta}} e^{\frac{i\pi}{4}} + O\left(n^{\frac{-1}{2}}\right), \\ \lambda_{2n}^+ &= -\beta\left(n\pi - \frac{\pi}{2}\right)^4 + 4\sqrt{i\beta}\alpha^2\left(n\pi - \frac{\pi}{2}\right)^2 - 2\sqrt{2}i\alpha^4\left(n\pi - \frac{\pi}{2}\right) \\ &\quad + \left(6i\pi\alpha^4 - \frac{2\sqrt{i}\alpha^6}{\sqrt{\beta}}\right) + O\left(\frac{1}{n}\right), \\ \lambda_{2n}^- &= -\frac{1}{\beta} - \frac{1}{\beta^3\left(n\pi - \frac{\pi}{2}\right)^4} + O\left(\frac{1}{n^8}\right). \end{aligned} \tag{37}$$

Therefore, we have

$$\Re(\lambda_{1n}), \Re\left(\lambda_{2n}^+\right) \to -\infty, \Re\left(\lambda_{2n}^-\right) \to -\frac{1}{\beta} \quad \text{as} \quad n \to \infty.$$

Proof: When $\rho \in S_1$, it has

$$\Re\left(\sqrt{i}\rho\right) = |\rho| \cos\left(\arg\left(\rho + \frac{\pi}{4}\right)\right) \leq 0.$$

Since

$$a = \sqrt[4]{\frac{-\lambda^2}{1+\beta\lambda}} = \sqrt[4]{\frac{-\rho^4}{1+\beta\rho^2}} = \frac{\sqrt{i\rho}}{\sqrt[4]{\beta}} + O\left(|\rho|^{-\frac{3}{2}}\right) \quad \text{as} \quad |\rho| \to \infty. \tag{38}$$

Based on estimate (38), we can state that there is a positive constant γ_1 such that

$$\begin{aligned} -\Re(a) &= -\frac{\sqrt{|\rho|}}{\sqrt[4]{\beta}} \cos\left(\arg\left(\sqrt{\rho}+\frac{\pi}{4}\right)\right) \leq -\frac{\sqrt{|\rho|}}{\sqrt[4]{\beta}} \sin\left(\frac{\pi}{16}\right) < -\gamma_1\sqrt{|\rho|}, \\ \Re(ia) &= \frac{\sqrt{|\rho|}}{\sqrt[4]{\beta}} \cos\left(\arg\left(\sqrt{\rho}+\frac{3\pi}{4}\right)\right) \leq -\frac{\sqrt{|\rho|}}{\sqrt[4]{\beta}} \cos\left(\frac{\pi}{8}\right) < -\gamma_1\sqrt{|\rho|}. \end{aligned}$$

Therefore, we get the following estimates

$$|e^{-a}| = O\left(e^{-\gamma_1\sqrt{|\rho|}}\right),\ |e^{ia}| = O\left(e^{-\gamma_1\sqrt{|\rho|}}\right),\ |e^{\sqrt{i\rho}}| \leq 1. \tag{39}$$

By multiplying some factors, we make each entry of the $\det(\tilde{X})$ be bounded as $\rho \to \infty$

$$\frac{1}{a^3 e^{2a}} \det(\tilde{X}) = \begin{vmatrix} 1+e^{-2a} & i+ie^{2ia} & 0 \\ \alpha e^{-2a} - \alpha & \alpha - \alpha e^{2ia} & -1+e^{2\sqrt{i\rho}} \\ 1+e^{-2a} & -i\left(1+e^{2ia}\right) & \frac{i\sqrt{i}\alpha\rho^3}{(\beta\rho^2+1)a^3}\left(1+e^{2\sqrt{i\rho}}\right) \end{vmatrix}. \tag{40}$$

By using the expression of a and ρ, and the Taylor expansion, we obtain

$$\frac{i\sqrt{i}\alpha\rho^3}{(\beta\rho^2+1)a^3} = \frac{\alpha}{\sqrt[4]{\beta}}\sqrt{\frac{1}{\rho}} + O\left(|\rho|^{-\frac{5}{2}}\right). \tag{41}$$

By using Eqs. (41) and (39), we get

$$\begin{aligned} \frac{1}{a^3 e^{2a}} \det(\tilde{X}) &= \begin{vmatrix} 1 & i & 0 \\ -\alpha & \alpha & -1+e^{2\sqrt{i\rho}} \\ 1 & -i & \frac{\alpha}{\sqrt[4]{\beta}}\sqrt{\frac{1}{\rho}}\left(1+e^{2\sqrt{i\rho}}\right) \end{vmatrix} + O\left(|\rho|^{\frac{-5}{2}}\right) \\ &= \left(\frac{(1+i)\alpha^2}{\sqrt[4]{\beta}}\sqrt{\frac{1}{\rho}} - 2i\right) + e^{2\sqrt{i\rho}}\left(\frac{(1+i)\alpha^2}{\sqrt[4]{\beta}}\sqrt{\frac{1}{\rho}} + 2i\right) \\ &\quad + O\left(|\rho|^{\frac{-5}{2}}\right). \end{aligned} \tag{42}$$

From the previous equality, we can get $\det(\tilde{X}) = 0$ if and only if

$$e^{2\sqrt{i\rho}} = 1 - \frac{(1-i)\alpha^2}{\sqrt[4]{\beta}}\sqrt{\frac{1}{\rho}} - \frac{i\alpha^4}{\sqrt{\beta}\rho} + O\left(|\rho|^{\frac{-3}{2}}\right). \tag{43}$$

Suppose

$$2\sqrt{i}\rho = 2n\pi i + O\left(n^{\frac{-1}{2}}\right), \tag{44}$$

where n is a sufficiently large integer. Substituting Eq. (43) into Eq. (42), we arrive at

$$O\left(n^{\frac{-1}{2}}\right) = \frac{(-1)^{\frac{5}{8}}\sqrt{2}\alpha^2}{\sqrt[4]{\beta}\sqrt{n\pi}} - \frac{\sqrt{i}\alpha^4}{n\pi\sqrt{\beta}} + O\left(n^{\frac{-3}{2}}\right). \tag{45}$$

The roots of Eq. (42) have the following asymptotic expressions

$$\rho_{1n} = \sqrt{i}n\pi + \frac{(-1)^{\frac{3}{8}}\alpha^2}{\sqrt[4]{\beta}\sqrt{2n\pi}} - \frac{\alpha^4}{2n\pi\sqrt{\beta}} + O\left(n^{\frac{-3}{2}}\right), n > N_1, \tag{46}$$

where N_1 is a sufficiently large positive integer. By $\lambda = \rho^2$, we have

$$\lambda_{1n} = in^2\pi^2 + \frac{\sqrt{2}\alpha^2}{\sqrt[4]{\beta}}e^{\frac{i\pi}{4}} + O\left(n^{-\frac{1}{2}}\right).$$

By using the value of a given by Eq. (38), we can obtain the expression of a as follows:

$$a_{1n} = \frac{(-1)^{\frac{3}{8}}\sqrt{\pi n}}{\sqrt[4]{\beta}} + O\left(\frac{1}{n}\right). \tag{47}$$

Similarly, when $\rho \in S_2$, it is easier to verify that there exists a $\gamma_2 > 0$ such that

$$\begin{cases} \Re(ia) \leq -\gamma_2\sqrt{|\rho|}, \\ \Re\left(\sqrt{i}\rho\right) = |\rho|\cos\left(\arg\left(\rho + \frac{\pi}{4}\right)\right) \leq |\rho|\cos\left(\frac{5\pi}{8}\right). \end{cases}$$

Hence, we get the following estimations

$$|e^{ia}| = O\left(e^{-\gamma_2\sqrt{|\rho|}}\right), \quad |e^{\sqrt{i}\rho}| = O\left(e^{-\gamma_2|\rho|}\right),$$

by using Eq. (38), we obtain

$$\arg(a) = \arg\left(\sqrt{i\rho}\right) \in \left(\frac{7\pi}{16}, \frac{9\pi}{16}\right] \quad \text{in} \quad S_2.$$

Thus, the sign of a is different under the two conditions:

$$\arg(\rho) \in \left(\frac{7\pi}{16}, \frac{\pi}{2}\right] \quad \text{and} \quad \arg(\rho) \in \left(\frac{\pi}{2}, \frac{9\pi}{16}\right].$$

Therefore, we conclude that

$$\frac{1}{a^3 e^a}\det\left(\tilde{X}\right) = \begin{vmatrix} e^{-a}+e^a & i+ie^{2ia} & 0 \\ e^{-a}\alpha - e^a\alpha & \alpha - \alpha e^{2ia} & -1+e^{2\sqrt{i}\rho} \\ e^{-a}+e^a & -i\left(1+e^{2ia}\right) & \frac{i\sqrt{i}\alpha\rho^3}{(\beta\rho^2+1)a^3}\left(1+e^{2\sqrt{i}\rho}\right) \end{vmatrix}$$

$$= \begin{vmatrix} e^{-a}+e^a & i & 0 \\ e^{-a}\alpha - e^a\alpha & \alpha & -1 \\ e^{-a}+e^a & -i & \frac{i\sqrt{i}\alpha\rho^3}{(\beta\rho^2+1)a^3} \end{vmatrix} + O\left(e^{-\gamma_2\sqrt{|\rho|}}\right)$$

$$= e^a\left(\frac{\sqrt{2}a\alpha^2}{\rho} - 2i\right) - e^{-a}\left(\frac{\sqrt{2}ia\alpha^2}{\rho} + 2i\right) + O\left(e^{-\gamma_2\sqrt{|\rho|}}\right).$$

From the previous equality, it is seen that $\det\left(\tilde{X}\right) = 0$ if and only if

$$e^a\left(\frac{\sqrt{2}a\alpha^2}{\rho} - 2i\right) - e^{-a}\left(\frac{\sqrt{2}ia\alpha^2}{\rho} + 2i\right) + O\left(e^{-\gamma_2\sqrt{|\rho|}}\right) = 0. \tag{48}$$

By using the expression of a and ρ, we obtain

$$\rho = \sqrt{\beta}a^2 - \frac{1}{2\beta^{\frac{3}{2}}a^2} + O\left(\frac{1}{|a|^4}\right), \tag{49}$$

which shows that $|a|, |\rho| \to \infty$ at the same time. Now, substitute the value of ρ given by (48) into equality (47), and we obtain

$$e^a\left(-2i + \frac{\sqrt{2}\alpha^2}{\sqrt{\beta}a} + \frac{\sqrt{2}\alpha^2}{2\beta^{\frac{5}{2}}a^5} + O\left(|a|^{-7}\right)\right) - e^{-a}\left(2i + \frac{i\sqrt{2}\alpha^2}{\sqrt{\beta}a} + \frac{i\sqrt{2}\alpha^2}{2\beta^{\frac{5}{2}}a^5} + O\left(|a|^{-7}\right)\right)$$
$$+O\left(e^{-\gamma_2|a|}\right) = 0.$$

Letting $a = x + iy$, it is easily checked that $\overline{a} = x - iy$ also satisfies the same asymptotic equation above. Hence, we only need to analyze the asymptotic expression of a located in the second quadrant. Given the value of a given by (48), when a is located on the second quadrant, $\Re(-a) \leq 0$ and $|e^{-a}| \leq 1$. Therefore,

$$e^{-2a} = -1 + \frac{(1-i)\alpha^2}{\sqrt{2\beta}a} - \frac{(1-i)\alpha^4}{2a^2\beta} + \frac{(1-i)\alpha^6}{2\sqrt{2}a^3\beta^{\frac{3}{2}}} + O\left(\frac{1}{a^4}\right),$$

and for the quadrant where a is located, we have

$$a_{2n} = i\left(n\pi - \frac{\pi}{2}\right) + \frac{(1+i)\alpha^2}{\sqrt{2\beta}\left(n\pi - \frac{\pi}{2}\right)} - \frac{(1-i)\alpha^4}{2\beta\left(n\pi - \frac{\pi}{2}\right)^2}$$
$$-\frac{(1+i)\alpha^6}{2\sqrt{2}\beta^{\frac{3}{2}}\left(n\pi - \frac{\pi}{2}\right)^3} + O\left(\frac{1}{n^4}\right).$$

Since $a = \sqrt[4]{\frac{\lambda^2}{1+\beta\lambda}}$ or $\lambda^2 - \beta a^4 \lambda - a^4 = 0$, it has

$$\lambda_{2n}^{\pm} = \frac{\beta a^4}{2}\left(1 \pm \sqrt{1 + \frac{4}{\beta^2 a^4}}\right).$$

Using the Taylor expansion, we obtain the expressions of λ_{2n}^{+} and λ_{2n}^{-} given by (37). Moreover, by using $\lambda = \rho^2$, we have the asymptotic expressions of ρ_{2n}^{+} and ρ_{2n}^{-}

$$\begin{cases} \rho_{2n}^{+} = i\sqrt{\beta}\left(n\pi - \frac{\pi}{2}\right)^2 + 2\sqrt{i}\alpha^2 + O(n^{-1}), \\ \rho_{2n}^{-} = \dfrac{i}{\sqrt{\beta}} + \dfrac{i}{2\beta^{\frac{5}{2}}\left(n\pi - \frac{\pi}{2}\right)^4} + O\left(n^{-8}\right). \end{cases} \tag{50}$$

Similarly, in S_3, there exists $\gamma_3 > 0$ such that

$$|e^{a}| = O\left(e^{-\gamma_3\sqrt{|\rho|}}\right), \quad |e^{ia}| = O\left(e^{-\gamma_3\sqrt{|\rho|}}\right), \quad |e^{\sqrt{i}\rho}| = O\left(e^{-\gamma_3|\rho|}\right).$$

It is easy to check that there is no null point of $\det\left(\tilde{X}\right)$, namely, there is no point spectrum in S_3.

According to the conclusion of Theorem 1.2, it is obvious that $-\frac{1}{\beta}$ is an accumulation point of the point spectrum of the operator $\mathcal{A}$. We thus have the following corollary.

Corollary

$$\sigma_c(\mathcal{A}) = -\frac{1}{\beta}. \tag{51}$$

We next analyze the asymptotic expression of eigenfunctions of the operator $\mathcal{A}$.

Theorem 1.3: Let $\sigma_p(\mathcal{A}) = \{\lambda_{1n}, n \in \mathbb{N}\} \cup \left\{\lambda_{2n}^{+}, \lambda_{2n}^{-}, n \in \mathbb{N}\right\}$ be the point spectrum of $\mathcal{A}$. Let $\lambda_{1n} = \rho_{1n}^2$, $\lambda_{2n}^{+} = \left(\rho_{2n}^{+}\right)^2$ and $\lambda_{2n}^{-} = \left(\rho_{2n}^{-}\right)^2$ with ρ_{1n}, ρ_{2n}^{+} and ρ_{2n}^{-} being given by Eqs. (45) and (49), respectively. Then, there are three families of approximated normalized eigenfunctions of $\mathcal{A}$

1. One family $\{z_{1n} = (u_{1n}, \lambda u_{1n}, v_{1n}), n \in \mathbb{N}\}$, where z_{1n} is the eigenfunction of $\mathcal{A}$ corresponding to the eigenvalue λ_{1n}, has the following asymptotic expression:

$$\left(\partial_x^2 u_{1n}, \lambda u_{1n}, v_{1n}\right) = (0,\ 0,\ \sin\left[a_n(1-x)\right]) + O_x\left(n^{\frac{-3}{4}}\right), \tag{52}$$

where

$$a_n = n\pi + \frac{(-1)^{\frac{1}{8}}\alpha^2}{\sqrt[4]{\beta}\sqrt{2n\pi}} + O\left(n^{-1}\right) \tag{53}$$

and $O_x\left(n^{\frac{-3}{4}}\right)$ means that $\left\|O_x\left(n^{\frac{-3}{4}}\right)\right\|_{L^2(0,1)} = O\left(n^{\frac{-3}{4}}\right)$.

2. The second family $\{z_{2n}^+ = (u_{2n}^+, \lambda u_{2n}^+, v_{2n}^+), n \in \mathbb{N}\}$, where z_{2n}^+ is the eigenfunction of $\mathcal{A}$ corresponding to the eigenvalue λ_{2n}^+, has the following asymptotic expression:

$$\left(\partial_x^2 u_{2n}^+, \lambda u_{2n}^+, v_{2n}^+\right) = \left(0, \ \sin\left[\left(n\pi - \frac{\pi}{2}\right)(1-x)\right], 0\right) + O_x\left(n^{-1}\right). \tag{54}$$

3. The third family $\{z_{2n}^- = (u_{2n}^-, \lambda u_{2n}^-, v_{2n}^-), n \in \mathbb{N}\}$, where z_{2n}^- is the eigenfunction of $\mathcal{A}$ corresponding to the eigenvalue λ_{2n}^-, has the following asymptotic expression:

$$\left(\partial_x^2 u_{2n}^-, \lambda u_{2n}^-, v_{2n}^-\right) = \left(\sin\left[\left(n\pi - \frac{\pi}{2}\right)(1-x)\right], 0, 0\right) + O\left(n^{-1}\right). \tag{55}$$

The proof is limited to the first result declared in Theorem 1.3.

Proof: We look for z_{1n} associated with λ_{1n}. From the expression ρ_{1n} given by (45) and a_{1n} given by (46) we have

$$\begin{cases} e^{-a_{1n}y} = e^{\frac{(-1)^{\frac{-11}{8}}\sqrt{\pi x y}}{\sqrt[4]{\beta}} + O\left(n^{-1}\right)}, \quad e^{ia_{1n}y} = e^{\frac{(-1)^{\frac{-7}{8}}\sqrt{\pi x y}}{\sqrt[4]{\beta}} + O\left(n^{-1}\right)}, \\ e^{\pm\sqrt{i}\rho_{1n}(1-x)} = e^{\pm in\pi(1-x) + O\left(n^{-1}\right)}, \end{cases} \tag{56}$$

and the following estimations:

$$\|e^{-a_{1n}y}\| = O\left(n^{\frac{-1}{4}}\right), \ \|e^{ia_{1n}y}\| = O\left(n^{\frac{-1}{4}}\right);$$
$$\left\|e^{\pm\sqrt{i}\rho_{1n}(1-x)}\right\| = O(1),$$

where $y = x$ or $2 - x \in [0, 1]$. According to the matrix $\tilde{X}$ given by (36), for ρ with (45) and a_{1n} given by (46), we obtain

$$u_1 = \frac{1}{e^{2a}e^{\sqrt{i}\rho}} \begin{vmatrix} 1 + e^{2a} & i + ie^{2ia} & 0 \\ (1 - e^{2a})\alpha\rho^2 & \left(1 - e^{2ia}\right)\alpha\rho^2 & -1 + e^{2\sqrt{i}\rho} \\ e^{ax} - e^{a(2-x)} & e^{iax} - e^{ia(2-x)} & 0 \end{vmatrix}$$
$$= \frac{e^{-\sqrt{i}\rho} - e^{\sqrt{i}}\rho}{\rho^2} \begin{vmatrix} 1 + e^{-2a} & i + ie^{2ia} \\ e^{-a(2-x)} - e^{ax} & e^{iax} - e^{ia(2-x)} \end{vmatrix}.$$

By using estimates (39), we can write

$$u_1 = \frac{e^{-\sqrt{i}\rho} - e^{\sqrt{i}}\rho}{\rho^2} \begin{vmatrix} 1 & i \\ e^{-a(2-x)} - e^{ax} & e^{iax} - e^{ia(2-x)} \end{vmatrix} + O\left(e^{-\gamma_1\sqrt{|\rho|}}\right)$$
$$= \frac{1}{\rho^2}\left(e^{-\sqrt{i}\rho} - e^{\sqrt{i}\rho}\right)\left[\left(e^{iax} - e^{ia(2-x)}\right) - i\left(e^{-a(2-x)} - e^{-ax}\right)\right] + O\left(e^{-\gamma_1\sqrt{|\rho|}}\right).$$

By the expression ρ_{1n} given by (45), we can obtain

$$e^{-\sqrt{i}\rho} - e^{\sqrt{i}\rho} = -2i\sin n\pi + O\left(n^{\frac{-1}{2}}\right) = O\left(n^{\frac{-1}{2}}\right).$$

This together with estimates 77 gives, after a direct computation, that

$$\partial_x^2 u_1 = \frac{a^2}{\rho^2}\left(e^{-\sqrt{i\rho}} - e^{\sqrt{i\rho}}\right)\left[\left(e^{ia(2-x)} - e^{iax}\right) - i\left(e^{-a(2-x)} - e^{-ax}\right)\right] + O\left(e^{-\gamma_1\sqrt{n}}\right) = O_x\left(n^{\frac{-7}{4}}\right),$$

and

$$\lambda u_1 = \left(e^{-\sqrt{i\rho}} - e^{\sqrt{i\rho}}\right)\left[\left(e^{iax} - e^{ia(2-x)}\right) - i\left(e^{-a(2-x)} - e^{-ax}\right)\right] + O\left(e^{-\gamma_1\sqrt{n}}\right) = O_x\left(n^{\frac{-3}{4}}\right).$$

Here,

$O_x\left(n^{\frac{-3}{4}}\right)$ means that $\left\|O_x\left(n^{\frac{-3}{4}}\right)\right\|_{L^2(0,1)} = O\left(n^{\frac{-3}{4}}\right)$ because $\|e^{-ax}\| = \|e^{iax}\| = O\left(n^{\frac{-1}{4}}\right)$.

Similarly, by using estimates (39) and (55), we have

$$\begin{aligned} v_1 &= \frac{1}{e^{2a}e^{\sqrt{i\rho}}\rho^2}\begin{vmatrix} 1+e^{2a} & i+ie^{2ia} & 0 \\ (1-e^{2a})\alpha\rho^2 & \left(1-e^{2ia}\right)\alpha\rho^2 & -1+e^{2\sqrt{i\rho}} \\ 0 & 0 & e^{\sqrt{i\rho}x} - e^{\sqrt{i\rho}(2-x)} \end{vmatrix} \\ &= -2\alpha(1+i)\sin\left[a_n(1-x)\right] + O\left(e^{-\gamma_1\sqrt{n}}\right), \end{aligned}$$

where a_n is given by (52). Let

$$z_{1n} = \frac{-1}{2\alpha(1+i)} z_1,$$

so, we obtain

$$\left(\partial_x^2 u_{1n},\, \lambda u_{1n},\, v_{1n}\right) = \left(0,\, 0,\, \sin\left[a_n(1-x)\right]\right) + O\left(n^{\frac{-3}{4}}\right).$$

The second and third results of Theorem 3 are obtained by the same procedure as before.

Corollary

$$\sigma_r \neq \varnothing. \tag{57}$$

4. Riesz basis property

Lemma. (see [23])

Let $\lambda_n \in \mathbb{C}$, $n = 1, 2, \cdots$, be a sequence that satisfies $\sup_n|\,(\lambda_n)| \leq M$, where M is a positive constant. Then the sine system $\{\sin\lambda_n x,\, n \geq 1\}$ is a Riesz basis for $L^2(0, 1)$ provided that the sequence λ_n satisfies one of the following conditions:

$$\sup_n|\Re(\lambda_n) - n\pi| < \frac{\pi}{4};$$

$$\sup_n|\Re(\lambda_n) - n\pi + \frac{\pi}{2}| < \frac{\pi}{4}.$$

Lemma.(see [24])

Let $\mathcal{A}$ be a densely defined closed linear operator in a Hilbert space $\mathcal{H}$ with isolated eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$. Let $\{\phi_i\}_{i=1}^{i=\infty}$ be a Riesz basis for $\mathcal{H}$. Suppose that there is an integer $N \geq 1$ and a sequence of generalized eigenvectors $\{\psi_i\}_{i=N}^{\infty}$ of $\mathcal{A}$ such that

$$\sum_{i=N}^{\infty} \|\psi_i - \phi_i\|^2 < \infty.$$

Then, there exists M a number of generalized eigenvectors $\{\psi_{i_0}\}_{i=1}^{M}$ of $\mathcal{A}$ such that

$$\{\psi_{i_0}\}_{i=1}^{M} \cup \{\psi_i\}_{i=M+1}^{\infty}$$

forms a Riesz basis for $\mathcal{H}$.

Theorem 1.4: The generalized eigenfunctions of $\mathcal{A}$ forms a Riesz basis for $\mathcal{H}$. As a result, all eigenvalues with large modules must be algebraically simple and, hence, the spectrum-determined growth condition holds for

$$e^{\mathcal{A}t} : \Phi(\mathcal{A}) = S(\mathcal{A})$$

where

$$\Phi(\mathcal{A}) = \inf\left\{\Phi | \text{there exists an} M \text{ such that } \|e^{\mathcal{A}t}\| \leq Me^{\Phi t}\right\},$$

and

$$S(\mathcal{A}) = \sup\{\Re(\lambda) | \lambda \in \sigma(\mathcal{A})\}.$$

Proof: By the bounded invertible mapping:

$$\mathbb{T}(u, w, v) = \left(\partial_x^2 u, w, v\right),$$

the space $\mathcal{H}$ is mapped onto

$$L^2(0, 1) \times L^2(0, 1) \times L^2(0, 1).$$

The value of a_n given by (52) satisfies

$$\sup_n | (a_n)| = \sup\left|\frac{\sin \frac{\pi}{8}\alpha^2}{\sqrt{2n\pi\sqrt[4]{\beta}}}\right|$$

is bounded and its real part satisfies

$$\sup_n |\Re(a_n) - n\pi| = \sup_n \left|\frac{\cos \frac{\pi}{8}\alpha^2}{\sqrt{2n\pi\sqrt[4]{\beta}}}\right| \leq \frac{\pi}{4}.$$

Then, it follows that the sequence

$$\{\sin[a_n(1-x)],\, n = 1, 2, \cdots\},$$

forms a Riesz basis for $L^2(0, 1)$. Similarly, the sequences

$$\left\{ \sin\left[\left(n\pi - \frac{\pi}{2}\right)(1-x)\right], n = 1, 2, \cdots \right\},$$

form a Riesz basis for $L^2(0, 1)$.

Let

$$\Psi_{1n} = (\sin[a_n(1-x)], 0, 0), \Psi_{2n}^{+} = \left(0, \sin\left[\left(n\pi - \frac{\pi}{2}\right)(1-x)\right], 0\right)$$

and

$$\Psi_{2n}^{-} = \left(0, 0, \sin\left[\left(n\pi - \frac{\pi}{2}\right)(1-x)\right]\right).$$

Then, the sequences

$$\{\Psi_{1n}\}_{n\geq 1} \cup \{\Psi_{2n}^{+}\}_{n\geq 1} \cup \{\Psi_{2n}^{-}\}_{n\geq 1}$$

forms a Riesz basis for the following space

$$L^2(0, 1) \times L^2(0, 1) \times L^2(0, 1).$$

Therefore, by the expression of z_{1n}, z_{2n}^{+}, and z_{2n}^{-} given by (51), (53), and (54), respectively, this implies that there exists $N > 0$ such that

$$\sum_{n\geq N}^{\infty} \left[\|\mathbb{T}z_{1n} - \Psi_{1n}\|^2 + \left\|\mathbb{T}z_{2n}^{+} - \Psi_{2n}^{+}\right\|^2 + \left\|\mathbb{T}z_{2n}^{-} - \Psi_{2n}^{-}\right\|^2\right] \leq \sum_{n\geq N}^{\infty} O\left(n^{-2}\right) < \infty.$$

This shows that there is a sequence of generalized eigenfunctions of $\mathcal{A}$, which forms a Riesz basis for $\mathcal{H}$, and all eigenvalues with large modulus must be algebraically simple.

5. Exponential stability

Theorem 1.5: The C_0−semigroup $S(t)$ generated by the operator $\mathcal{A}$ is exponentially stable, that is,

$$\left\|e^{\mathcal{A}t}\right\| \leq Me^{\omega t},$$

where M and ω are positive constants[6].

Proof: By the asymptotic distribution of eigenvalues given by Theorem 1.2 and the continuous spectrum given by Eq. (50), in addition to the empty residual spectrum set given by Eq. (56), we conclude that $S(\mathcal{A}) = -\frac{1}{\beta}$. The proof is completed by the spectrum-determined growth condition, which is similar to [24–26].

[6] By recalling the eigenvalues of $\mathcal{A}$ given by 44, we deduce that $\omega \geq -\frac{1}{\beta}$.

6. Conclusion

The main results of this work are similar to those mentioned in [27], the results are summarized as follows:

1. The system operator of the closed-loop system is not of compact resolvent and the spectrum consists of three branches.

2. By means of asymptotic analysis, the asymptotic expressions of eigenfunctions are obtained.

3. By the comparison method in the Riesz basis approach, exponential stability is obtained.

Conflict of interest

The authors declare no conflict of interest.

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References

[1] Zhang X, Zuazua E. Polynomial decay and control of a 1–d hyperbolic–parabolic coupled system. Journal of Differential Equations. 2004;**204**(2): 380-438

[2] Zhang X, Zuazua E. Asymptotic behavior of a hyperbolic-parabolic coupled system arising in fluid-structure interaction. In: Free Boundary Problems. Basel: Birkhäuser; 2006. pp. 445-455

[3] Zhang Q, Wang J-M, Guo B-Z. Stabilization of the Euler–Bernoulli equation via boundary connection with heat equation. Mathematics of Control, Signals, and Systems. 2014;**26**(1):77-118

[4] Wang J-M, Krstic M. Stability of an interconnected system of euler–bernoulli beam and heat equation with boundary coupling. ESAIM. 2015;**21**(4): 1029-1052

[5] Wang J-M, Ren B, Krstic M. Stabilization and Gevrey regularity of a Schrödinger equation in boundary feedback with a heat equation. IEEE Transactions on Automatic Control. 2011;**57**(1):179-185

[6] Guo BZ. Further results for a one-dimensional linear thermoelastic equation with Dirichlet-Dirichlet boundary conditions. The ANZIAM Journal. 2002;**43**(3):449-462

[7] Dridi H, Djebabla A. On the stabilization of linear porous elastic materials by microtemperature effect and porous damping. Annali Dell'Universita'di ferrara. 2020;**66**(1): 13-25

[8] Dridi H. Timoshenko system with fractional operator in the memory and spatial fractional thermal effect. Rendiconti del Circolo Matematico di Palermo Series. 2021;**70**(1):593-621

[9] Hanni D, Feng B, Zennir K. Stability of Timoshenko system coupled with thermal law of Gurtin-Pipkin affecting on shear force. Applicable Analysis. 2021;**2021**:1-22

[10] Dridi H, Zennir K. Well-posedness and energy decay for some thermoelastic systems of Timoshenko type with Kelvin–Voigt damping. SeMA Journal. 2021;**78**(3):385-400

[11] Dridi H. Decay rate estimates for a new class of multidimensional nonlinear Bresse systems with time-dependent dissipations. Ricerche di Matematica. 2021;**2021**:1-33

[12] Dridi H, Saci M, Djebabla A. General decay of Bresse system by modified thermoelasticity of type III. Annali Dell'universita'di ferrara. 2022;**68**(1): 203-222

[13] Hanni D, Djebabla A, Tatar N. Well-posedness and exponential stability for the von Karman systems with second sound. Eurasian Journal of Mathematical and Computer Applications. 2019;**7**(4):52-65

[14] West H. The Euler-Bernoulli beam equation with boundary energy dissipation. Operator methods for optimal control problems. 1987

[15] Huang F. Characteristic conditions for exponential stability of linear dynamical systems in Hilbert spaces. Annals of Differential Equations. 1985;**1**: 43-56

[16] Chen G. Energy decay estimates and exact boundary value controllability for the wave equation in a bounded domain. 1979

[17] Lagnese J. Decay of solutions of wave equations in a bounded region with

boundary dissipation. Journal of Differential Equations. 1983;**50**(2): 163-182

[18] Wazwaz A-M. A study on linear and nonlinear Schrodinger equations by the variational iteration method. Chaos, Solitons & Fractals. 2008;**37**(4): 1136-1142

[19] He J-H. Some asymptotic methods for strongly nonlinear equations. International Journal of Modern Physics B. 2006;**20**(10):1141-1199

[20] He J-H. Variational iteration method–a kind of non-linear analytical technique: Some examples. International Journal of Non-Linear Mechanics. 1999; **34**(4):699-708

[21] Hanni D, Khaled Z. New Class of Kirchhoff Type Equations with Kelvin-Voigt Damping and General Nonlinearity: Local Ex-istence and Blow-up in Solutions. Journal of Partial Differential Equations. 2021;**34**(4): 313-347

[22] Pazy A. Semigroups of Linear Operators and Applications to Partial Differential Equations. Vol. 44. Springer Science Business Media. 2012

[23] Bilalov B. Bases of exponentials, cosines, and sines formed by eigenfunctions of differential operators. Differential Equations. 2003;**39**(5):652-657

[24] Guo B-Z, Guo-Dong Z. On Spectrum and Riesz basis property for one-dimensional wave equation with Boltzmann damping. ESAIM: Control Optimisation and Calculus of Variations. 2012;**18**(3):889-913

[25] Guo B-Z, Chan KY. Riesz basis generation, eigenvalues distribution, and exponential stability for a Euler-Bernoulli beam with joint feedback control. Revista Matemática Complutense. 2001;**14**(1):205-229

[26] Guo B-Z. Riesz basis approach to the stabilization of a flexible beam with a tip mass. SIAM Journal on Control and Optimization. 2001;**39**(6):1736-1747

[27] Guo B-Z, Ren H-J. Stabilization and regularity transmission of a Schrödinger equation through boundary connections with a Kelvin-Voigt damped beam equation. ZAMM-Journal of Applied Mathematics and Mechanics/Zeitschrift für Angewandte Mathematik und Mechanik. 2020;**100**(2):e201900013