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Complete Integrability and Discretization of Euler Top and Manakov Top

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COMPLETE INTEGRABILITY AND DISCRETIZATION OF EULER TOP AND MANAKOV TOP

A Thesis

by

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ABSTRACT

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The Euler top is a completely integrable system with physical system implications and the Manakov top is its four-dimensional extension. We are concerned about their complete integrability and the preservation of this property under a specific discretization known as the Hirota-Kimura Discretization. Surprisingly, it is not guaranteed that under any discretization the conserved quantities are preserved and therefore they must be discovered. In this work we construct the Poisson bracket and Lax pair for each system and provide the Lie algebra background needed to do such such constructions.

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CHAPTER I

INTRODUCTION

There is a vast amount of literature in the field of "integrable systems" which is a relatively recent field as it gained popularity in the 1960s. There are hundreds of researchers, mathematicians and physicists alike, who have collectively written thousands of papers in this field. To give a full overview of the subject could take hundreds of pages of work which is quite outside the scope of this paper. .

This story will focus on a famous class of differential equations which are completely integrable. These are called *spinning tops* which are rigid bodies with a fixed point in a constant gravitational field. There are many avenues which spinning tops may be investigated which is why this field is so rich in material. Here I will only focus on differential geometry and Lie algebra.

We are specifically interested in two systems of ordinary differential equations set in the ambient space of the Eucliedean space R^n. These systems are generally given by

$$\dot{\mathbf{y}} = \mathbf{f}(y)$$

where $f : U \to R^n$ with an open subset $U \subset R^n$. From our knowledge of ordinary differential equations, we know that if f is smooth (sufficiently enough at least) and initial data is given then the system has a unique local solution.

The Euler Top and Manakov Top, which we will define rigorously later, are *rigid body* systems

which are systems of ordinary differential equations whose solutions are on a submanifold of R^n. We will continue by constructing the tools necessary to properly understand the ever interesting topic of Hamiltonian mechanics which is central to our goal of discussing the *integrability*, a yet to be defined but very interesting property, of these two rigid body tops.

CHAPTER II

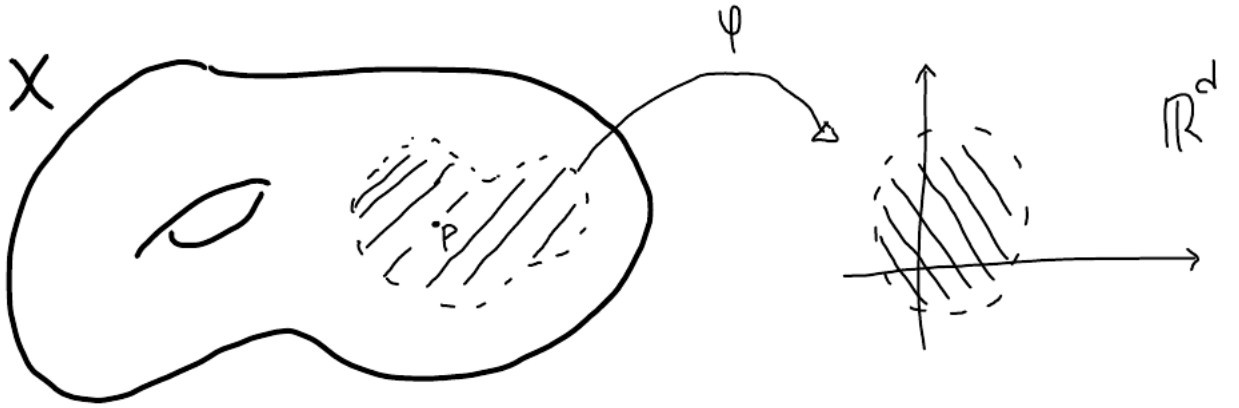
ALL ABOUT MANIFOLDS

2.1 Manifolds

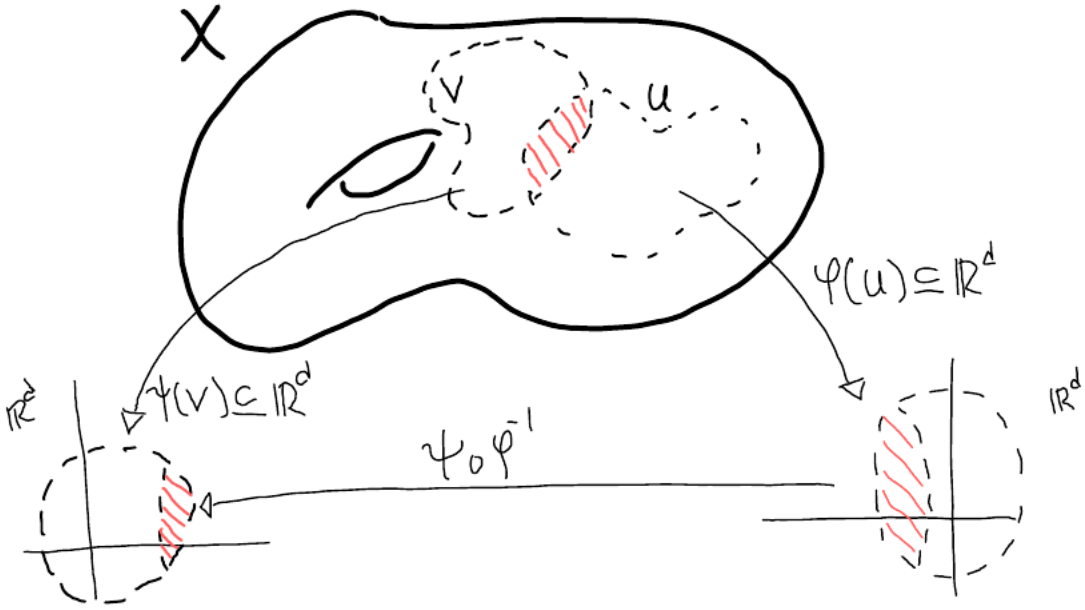
Our work all happens on *smooth manifolds* and the definitions to come are mostly from [5]. Generally, these are spaces that locally look like a Euclidean space R^n and where we can perform calculus. First we must introduce topological manifolds which are the most basic type of manifold. Let M be a topological space such that it satisfies the following conditions

- M is a **Hausdorff space**: for every pair of distinct points $p,q \in M$, there are disjoint open subsets $U,V \subseteq M$ such that $p \in U$ and $q \in V$.
- M is **second-countable**: there exists a countable basis for the topology of M.
- M is **locally Euclidean of dimension n**: each point of M has a neighborhood that is homeomorphic to an open subset of R^n

Note that R^n itself is a topological n-manifold as it is Hausdorff because it is a metric space, and it is second-countable. Now, a **coordinate chart** on M is a pair (U,φ), where U is an open subset of M and $\varphi : U \to \hat{U}$ is a homeomorphism from U to an open subset $\hat{U} = \varphi(U) \subseteq R^n$. The map φ is called a **local coordinate map**, and the component functions $(x^1,\cdots,x^n)$ of φ, defined by $\varphi(p) = (x^1(p),\cdots,x^n(p))$ are called the **local coordinates** on U. See the following figure.


X
φ
p
$\mathbb{R}^d$

Two charts (U,φ) and (V,ψ) are said to be smoothly compatible if either $U \cap V = \emptyset$ or their transition map $\psi \circ \varphi^{-1}$ is a diffeomorphism.



We define an atlas A for X to be a collection of charts whose domains cover X and is called maximal if any chart that is smoothly compatible with every chart in A is already in A. If X is a topological manifold, a **smooth structure on X** is a maximal smooth atlas. A **smooth manifold** is a pair (X,A), where X is a topological manifold and A is a smooth structure on X. Essentially, a topological manifold with the largest possible atlas such that every chart is compatible wish each other.

2.1.1 Curves, Velocities, and the Tangent Space

Consider again a smooth manifold (X,T,A) and let $\varphi : U \to R^d$ be a coordinate chart $(U,\varphi) \in A$ which contains the point p, that is, $p \in U \subset X$. Suppose $\gamma_1, \gamma_2 : C^\infty((-1,1)) \to X$ are curves on X initialized at p: $\gamma_1(0) = p = \gamma_2(0)$. See figure 2.1.

Note that since both curves pass through p it may be that their deritvatives, $\varphi \circ \gamma_1$ and $\varphi \circ \gamma_2$, at $t = 0$ coincide. If this is this the case we call them **equivalent**. That is, $\gamma_1 \equiv \gamma_2 \iff$

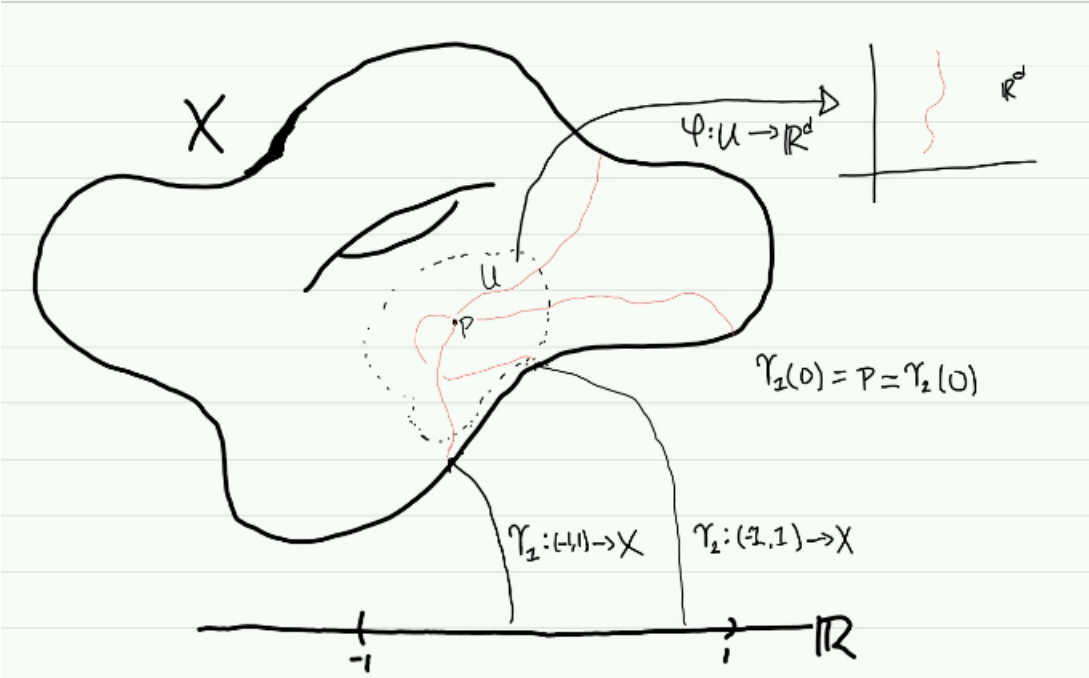


Figure 2.1: Smooth manifold X with smooth curves γ_1, γ_2

$(\varphi \circ \gamma_1(t))\,|_{t=0} = (\varphi \circ \gamma_2(t))\,|_{t=0}$. Hence, we have an equivalence relation on all of the (differentiable) curves intitialized at p and therefore have equivalence classes of curves. These equivalence classes are called *tangent vectors* of the (smooth) manifold X at the point p. We denote these equivalence classes of a curve γ with the recognizable notation of: $\gamma'(0)$. The set of all such tangent vectors at the point p is called the **Tangent space** of X at p and is denoted by T_pX.

If $\Phi \in T_pX$ is some tangent vector at p and $f : X \to R$ is a smooth function defined locally of p, then differentiating f along any curve in Φ at $t = 0$ gives a directional derivative. That is

$$\Phi f(p) := \frac{d}{dt} f(\gamma(t))\,|_{t=0}$$

For a fixed f, we have the linear functional: $df(p) : T_pX \to R$ defined by $df(p) = Xf(p)$ which is called the **differential** of f at the point p.

Let M be a smooth manifold and let p be a point of M. A linear map $v : C^\infty(M) \to R^n$ is

called a **derivation** at p if it satisfies

$$v(fg) = f(p)vg + g(p)vf,\ \forall f,g \in C^{\infty}(M)$$

Generally, a derivation is defined for an algebra A over a field F and then the map $D : A \to A$, $D(ab) = aD(b) + D(a)b$, for all $a,b \in A$ is called a derivation. Since elements of the tangent space satisfy the above property then they are also called derivations. The **tangent bundle of M** is the disjoint union of the tangent spaces of all points of M and is denoted by TM:

$$\begin{aligned} TX &=_{p\in X} T_pX \\ &= \bigcup_{p\in X} \{p\} \times T_pX \\ &= \bigcup_{p\in X} \{(p,q) \mid q \in T_pX\} \\ &= \left\{(p,q) \mid p \in X,\ q \in T_pX\right\} \end{aligned}$$

An element of the above is written as an ordered pair (p,v) where $p \in M$ and $v \in T_pM$. For a smooth manifold M, the tangent bundle TM can be realized as the disjoint union of vector spaces and, very importantly, TM itself is a smooth manifold.

Vector fields on a smooth manifold M are a particular kind of continuous map from the manifold to its tangent bundle . They are essentially the focus of study here. If M is a smooth manifold, a **vector field on M** is a *section* of the map $\pi : TM \to M$. That is, a vector field is a continuous map $X : M \to TM$ such that

$$\pi \circ X = Id_M$$

Equivalently, $X_p \in T_pM$ for every $p \in M$. So this continuous map simly assigns every point in the manifold $p \in M$ to an element (tangent vector) of the corresponding tangent space $X_p \in T_pM$. Naturally, smooth vector fields are those that are smooth as maps from $M \to TM$. The set of all

smooth vector vields on M is denoted by $X(M)$ and it is a vector space under pointwise addition and scalar multiplication. $X(M)$ itself is a module over the ring $C^\infty(M)$. Vector fields define operators on the space of smooth real-valued functions which is an extermely important property. From this propertly we can apply the vector field X to the smooth function f in the following way:

$$(Xf)(p) = X_p f$$

A smooth vector defines a map from $f : C^\infty(M) \to C^\infty(M)$ from $f \mapsto Xf$ which is linear. This is also called a derivation.

2.1.2 Poisson Manifolds

Now we will introduce the definition of a Lie algebra and Poisson bracket sometimes called a Poisson structure. The material here is gathered from several sources: [2], [1], [11], and [10]. They will be needed for our discussion of integrability. Let F be a field. A **Lie algebra** over F is an F-vector space L, together with a bilinear map, the **Lie bracket**

$$L \times L \to L,\ (x,y) \mapsto [x,y],$$

satisfying the properties:

- $[x,x] = 0$ for all $x \in L$,
- Jacobi identity: $[x,[y,z]] + [y,[z,x]] + [z[x,y]] = 0$ for all $x,y,z \in L$,

Note that the skew-symmetry property follows from the above: $[x,y] = -[y,z]$. Consider again a smooth manifold X. The ring of functions $C^\infty(X)$ is endowed with a Lie algebra structure $\{,\}$ which is a derivation in both its entries. That is, the bracket is skew-symmetric $\{f,g\} = -\{g,f\}$, satisfies the Jacobi identity

$$\{f,\{g,h\}\} + \{g\{h,f\}\} + \{h\{f,g\}\} = 0,\ \forall f,g,h \in C^\infty(X)$$

and satisfies the Leibniz rule

$$\{f, gh\} = \{f, g\}h + g\{f, h\}.$$

The bracket $\{,\}$ is the **Poisson bracket**. Fix a function H on X in the bracket: $\{H, \cdot\}$. Since the bracket is a derivation in both terms, this is a derivation itself and most importantly defines a vector field. Simply, $\{H, \cdot\}$ takes functions numbers. It behaves just like a derivation so it is a vector field, let's call it V_H. Then what is $V_H f$ $(= df(V_H)$? It is $\{H, f\}$ where $f \in C^\infty(X)$ since that's the number it takes them to. A vector field V_H is famously called the **Hamiltonian vector field** and H the **Hamiltonian**. This is a naturally occurring vector field and so far we have not seen why it is special. From a physics viewpoint, the integral curves are what is interesting as the Hamiltonian equations relate to this. Note that the Hamiltonian vector field V_H defines a defines a a system of differential equations called the **Hamiltonian system**.

By skew-symmetry, $\{H, H\} = 0$ so that $\{H, H\} = dH(V_H) = 0$ and therefore we have the meaning that H is constant along the trajectories of V_H. That is, the solutions of the Hamiltonian system remain in the levels of H. Additionally, if $f \in C^\infty(X)$ is such that the Poisson bracket with a*any* function g on X, $\{f, g\} = 0$ vanishes then it is, by definition earlier, a first integral for any Hamiltonian system on X. This function is called a *Casimir* function [2].

When f is such a function that it remains constant along the trajectories of the hamiltonian vector field V_H, $\{H, f\} = 0 = df(V_H)$ we call it a **first integral**. The name *first integral* may seem unintuitive at first, but this is due to the mesh of physics and mathematics. Conservations laws, the first integrals, and symmetries of our Hamiltonian system are very closely related by the remarkable result of Emmy Noether [10]. Noether's theorem is one of the fundamental theorems of Hamiltonian mechanics and here it will be explained in the context of Hamiltonian mechanics (symplectically). We will see the above definitions explained using a symplectic form.

Etymology: The word 'symplectic' was coined by Hermann Weyl [8] in his book as the

Greek form of 'com-plex'. That is, the word $\mu\pi\lambda\varepsilon\kappa\tau o^{'}\varsigma$ literally translates to as *twined together*. This is, as [8] says, a very agreeable choice of name as the field of symplectic geometry is found in many other fields of math.

A **symplectic form** on X is a closed nondegenerate 2-form. A smooth manifold endowed with a specific choice of symplectic form is called a **symplectic manifold** [5]. A choice of symplectic form is also sometimes called a **symplectic structure** [5]. It is important to note that a symplectic manifold must be even dimension, but that not all even-dimension manifolds are symplectic manifolds. The setting that we will be working is R^{2n}.

With standard coordinates on R^{2n} denoted by $(x^1,\cdots,x^n,y^1,\cdots,y^n)$, the 2-form

$$\omega = \sum_{i=1}^{n} dx^i \wedge dy^i$$

It is closed and non-degenerate so it is the standard symplectic form on R^{2n}.

Noether's theorem: Let H and E be two Hamiltonians on a symplectic phase space (D,ω). Then the following are equivalent [10].

- $\{H,E\} = 0$

- The quantity E is conserved by the Hamiltonian flow of H. That is, E is constant along the integral curves of H

- The quantity H is conserved by the Hamiltonian flow of E.

If any of the above three propositions hold, we say the H and E *Poisson commute*. This theorem gives a very satisfactory link between the symmetries of the Hamiltonian H to the conserved

quantities of the flow. The larger the group of symmetries, the more conserved quantities one obtains [10].

2.2 Hamiltonian Vector Fields

Now we introduce the Hamiltonian vector field in terms of a symplectic form. Suppose (M, ω) is a symplectic manifold. Note the change of variable from using X to using M. For any smooth function $f \in C^\infty(X)$, we define the **Hamiltonian vector field of f** [5] to be the smooth vector field X_f defined by

$$X_f = \hat{\omega}^{-1}(df)$$

where $\hat{\omega} : TM \to T^*M$ is the bundle isomorphism determined by ω. Since we will be working in R^{2n} with the standard symplectic form, the hamiltonian vector field is written as:

$$X_f = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial y^i} \frac{\partial}{\partial x^i} - \frac{\partial f}{\partial x^i} \frac{\partial}{\partial y^i} \right)$$

Important properties of Hamiltonian vector fields [5]:

- $f \in C^\infty(M)$ is constant along each integral curve of X_f
- At each regular point of f, the Hamiltonian vector field X_f is tangent to the level set of f.

These two properties follow immediately from the properties of the symplectic form:

$$X_f f = df(X_f) = \omega(X_f, X_f) = 0$$

2.3 Hamiltonian mechanics

A symplectic manifold (M, ω) together with a smooth function $H \in C^\infty(M)$ is called a Hamiltonian system [5]. The function H is called the **Hamiltonian** of the system; the flow of the Hamiltonian vector field X_H is called its **Hamiltonian flow**, and the integral curves of X_H are called the **trajectories** of the system. Let's derive the famous Hamilton's equations from mechanical

systems following [10].

Consider $M = R^n \times R^n = \{(q_1, \cdots, q_n, p_1, \cdots, p_n) \mid q_i, p_i \in R \forall i \in N \setminus \{0\}\}$. The form $\omega : R^n \times R^n \to R$ is defined using the standard symplectic form we saw earlier:

$$\omega := \sum_{j=1}^{n} dq_j \wedge dp_j$$

or in other words

$$\omega((q_1, \cdots, q_n, p_1, \cdots, p_n), (q'_1, \cdots, q'_n, p'_1, \cdots, p'_n)) := \sum_{j=1}^{n} p'_j q_j - p_j q'_j$$

and so now we see that (M, ω) is symplectic. Consider the two time independent smooth functions $H, E : M \to R$. we have:

$$\begin{aligned} \{H, E\}(u) &= \omega(\nabla_\omega H(u), \nabla_\omega E(u)) \\ &= \sum_{j=1}^{n} \frac{\partial H}{\partial p_j} \frac{\partial E}{\partial q_j} - \frac{\partial H}{\partial q_j} \frac{\partial E}{\partial p_j} \end{aligned}$$

since we have $\nabla_\omega H = \left(\frac{\partial H}{\partial p_1}, \cdots, \frac{\partial H}{\partial p_n}, -\frac{\partial H}{\partial q_1}, \cdots, -\frac{\partial H}{\partial q_n} \right)$. Hence, the Hamiltonian ODEs associated to the Hamiltonian H are given by:

$$\begin{cases} \partial_t q_j(t) = \frac{\partial H}{\partial p_j}(q(t), p(t)), \\ \partial_t p_j(t) = -\frac{\partial H}{\partial q_j}(q(t), p(t)). \end{cases}$$

where $q(t) = (q_1, \cdots, q_n)$ and $p_p = (p_1(t), \cdots, p_n(t))$. These are the familiar Hamilton's equations. Now lets see a less complicated approach for tying Poisson brackets and mechanical systems together. Mechanical systems with N degrees of freedom have a $2N$-dimensional phase space ($n = 2N$). So, $N = 1 \to 2D$, $N = 2 \to 4D$, etc.

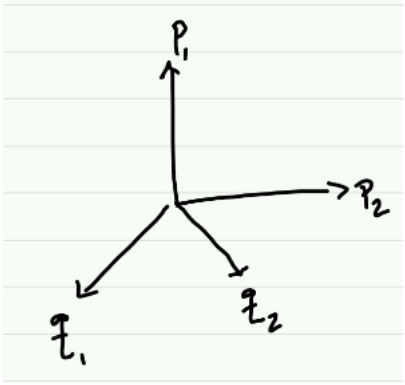


Figure 2.2: 4D phase space

Consider a n-dimensional phase space $(q_1, q_2, , \cdots, q_n p_1, p_2, \cdots, p_n)$ where $q_i, p_i \in R$, $i = 1, 2, \cdots, n$. Consider a function $f(q, p, t)$ along the trajectories of the mechanical system. e.g, $f = q_i$ or $f = q^2\sqrt{p}\sin(tq_2)$. We want to know how f evolves along the trajectories. Naturally, we should take the total derivative of f:

$$
\begin{aligned}
\dot{f} &= \frac{d}{dt} f \\
&= \text{ total derivative of f} \\
&= \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i} \dot{q}_i + \frac{\partial f}{\partial p_i} \dot{p}_i \right) + \frac{\partial f}{\partial t} \text{ by the chain rule} \\
&= \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial H}{\partial q_i} \right) + \frac{\partial f}{\partial t} \text{ from plugging in Hamiltons equations: } \dot{q}_i = \frac{\partial H}{\partial p_i},\ \dot{p}_i = -\frac{\partial H}{\partial q_i}
\end{aligned}
$$

The sum $\sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial H}{\partial q_i} \right)$ is denoted by $\{f, H\}$ and is called the *Poisson bracket* of the function f with the function H. Simply, we have:

$$\dot{f} = \{f, H\} + \frac{\partial f}{\partial t}$$

This is the fundamental governing equation that all Hamiltonian systems must obey for an appropriate H. If f does not depend on time, $\frac{df}{dt} = 0$, then:

$$\dot{f} = \{f, H\}$$

We can easily recover Hamilton's canonical equations. Let $f = q_j$. Then we obtain:

$$\begin{aligned}\dot{q_j} &= \{q_j, H\} \\ &= \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i} \right) \\ &= \frac{\partial H}{\partial p_j}\end{aligned}$$

Similarly, let $f = p_j$. Then $\dot{p}_j = \{p_j, H\} = \cdots = -\frac{\partial H}{\partial q_j}$.

Working with Hamilton's equations have nice advantages:

- Automatically get 1st order ODEs and therefore are easier to implement computationally
- $H(q,p)$ is a constant of motion (if $\frac{\partial H}{\partial t} = 0$)
- In most cases, $H = T + V$ =kinetic plus potential energy, that is, the total energy of the system
- Phase space is "volume preserving" so it acts like a $2n$-dimensional incompressible fluid.

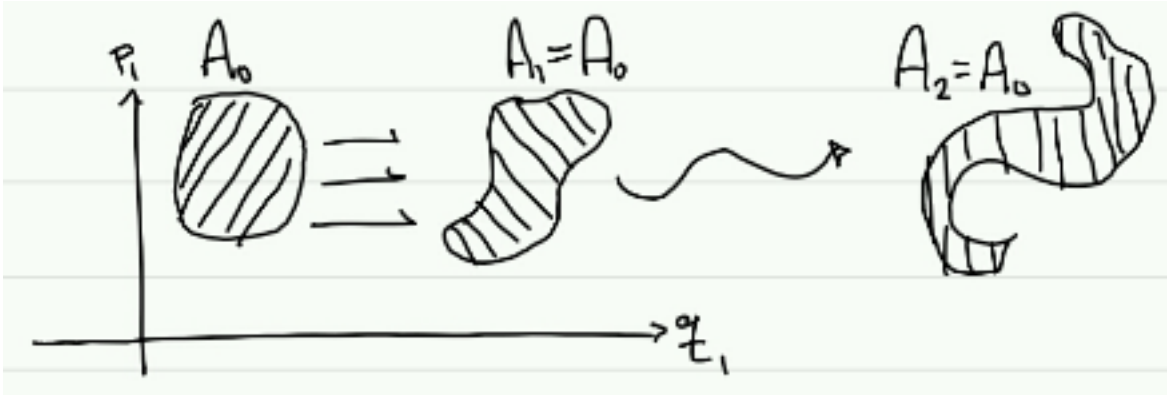


Figure 2.3: Area preserving in 2D

In 2 dimensions we are preservation. For higher degree of freedom systems, smaller subvolumes are also conserved.

2.3.1 Invariant manifold

Consider a ball rolling on a surface so that we have $N = 2$ degrees of freedom ($n = 4$). A

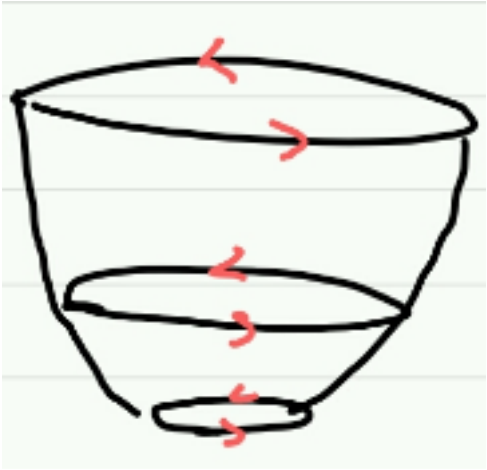


Figure 2.4: 2D curved surface filled with periodic orbits

invariant manifold is a surface which if an initial condition starts on it, it will stay on it for all time under the dynamics of the ODE: $\dot{x} = f(x)$. That is, $\mathscr{M}$ is an invariant manifold if $\forall x(0) \in \mathscr{M}$ we have that $x(t) \in \mathscr{M}$, $\forall t$.

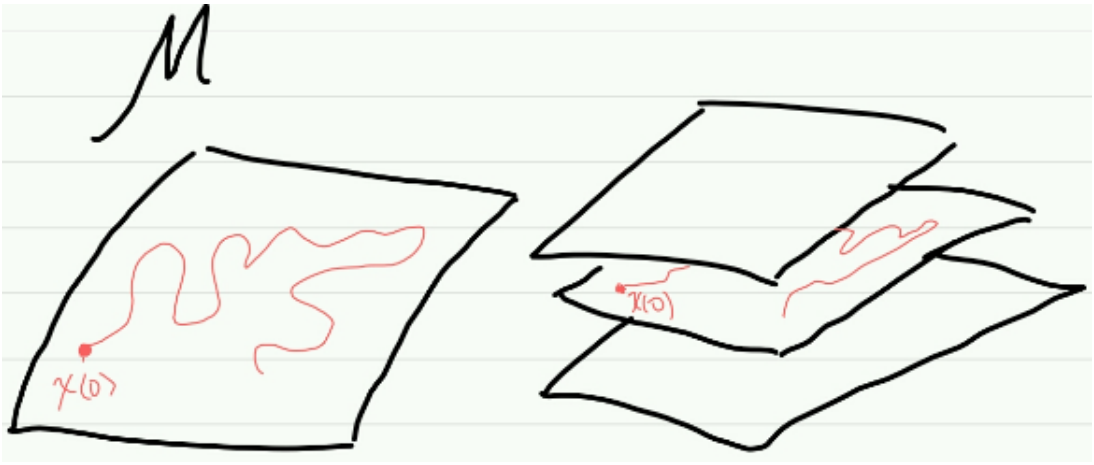


Figure 2.5: Invarient manifold M - "Foliation" of phase space into invariant manifolds

In mechanical system we have tori as seen in the following figure. For example, with 1

degree of freedom we have the circle S^1 and with 2 degrees of freedom we have the 2-torus $S^1 \times S^1$.

Figure 2.6: Circle and 2-torus

The $n-torus$ will show the quasiperiodic motion (in frequencies).

2.4 Action-angle variables

In Hamiltonian mechanics, it is often desired to perform a change of variables to simplify analysis of a system. The advantage advantage of "action-angle" variables are that they help simplify what the phase space looks like. They are a type of **canonical transformation**. Canonincal transformations are the type of transformations that transform a Hamiltonian system from an old set of coordinates to a new set of coordinates and it preserves the form of Hamilton's equations. In a mechanical system our coordinates are denoted as an n-tuple of q's and p's: $(q_1, \cdots, q_n, p_1, \cdots, p_n)$. Under a canonincal transformation we go from the old q's and p's to the new coordinates denoted as θ's and I's called *angles* and *actions* respectively. The motivation for using these two words will be explained later. Let's view an example showcasing such a coordinate change.

Consider a 1 degree of freedom (DOF) system that is a mass attached to a spring on a wall (oscillator) where $p = m\dot{q}$. See the following figure.

The hamiltonian represents the total energy of the system where the kinetic and potential energy are: $V = \frac{1}{2}kq^2$ and $T = \frac{1}{2}m\dot{q}^2 = \frac{1}{2m}p^2$. Therefore, the hamiltonian is written as:

$$H = T + V = \frac{1}{2m}\left(p^2 + m^2\omega^2 q^2\right)$$

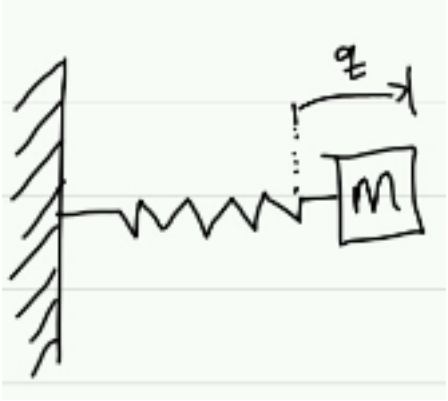


Figure 2.7: Basic spring with attached mass

where $\omega^2 = \frac{k}{m}$. Hamilton's equations are:

$$\begin{cases} \dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m} \\ \dot{p} = -\frac{\partial H}{\partial q} = -m\omega^2 q \end{cases}$$

We could solve the an initial value problem with initial conditions $(q(0), p(0)) \mapsto (q(t), p(t))$, but that may turn out to be really hard. Is there a change of variables that could make it easier? Yes!

We wish to change out variables to $(\overline{q}, \overline{p})$ to simplify our ODEs. I will use the θ and I notation in a later example. Our strategy: Pick $(\overline{q}, \overline{p})$ to simplify the Hamiltonian function. Pick $p\alpha\cos(\overline{q})$ and $q\alpha\sin(\overline{q})$ where the α means "proportional". In particular, pick

$$p = f(\overline{p})\cos(\overline{q}), \text{ and }, q = \frac{f(\overline{p})}{m\omega}\sin(\overline{q})$$

This yields for the Hamiltonian:

$$\begin{aligned} H(q,p) &= \frac{1}{2m}\left(p^2 + m^2\omega^2 q^2\right) \\ &= \frac{1}{2m}\left(f^2(\overline{p})\cos^2(\overline{q} + m^2\omega^2\frac{f^2(\overline{p})}{m^2\omega^2})\sin^2(\overline{q})\right) \\ &= \frac{f^2(\overline{p})}{2m}(\cos^2(\overline{q}) + \sin^2(\overline{q})) \\ &= \frac{f^2(\overline{p})}{2m} \end{aligned}$$

Note that the Hamiltonian is dependent only on $\overline{p}$: $H(\cdot,\overline{p}) = \frac{f^2(\overline{p})}{2m}$. In the new variables $(\overline{q},\overline{p})$, the Hamilton's equations are:

$$\begin{cases} \dot{\overline{q}} = \frac{\partial H}{\partial \overline{p}} = \Omega(\overline{p}) = \text{ constant }, \\ \dot{\overline{p}} = -\frac{\partial H}{\partial \overline{q}} = 0 \end{cases}$$

Hence, we have the following solution (constant of motion): $\overline{q}(t) = q(0) + \Omega(\overline{p})t$ and $\overline{p}(t) = \overline{p}(0)$. This is a simple solution. Let's keep looking at the simple harmonic oscillator, but now I will phrase the situation in terms of "angles" and "actions" which is really what we just did, but with a different viewpoint. Consider again the original Hamiltonian for the simple harmonic oscillator:

$$h(q,p) = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 = \text{ constant } = E$$

It conserves energy so we can write the momentum in terms of the energy E. This yields:

$$p = p(q,E) = \pm\sqrt{2(E - \frac{1}{2}\omega^2 q^2)}$$

The positive part, the $+$ branch, corresponds to when p is positive and the negative part, the $-$ branch, corresponds to when p is negative. Note that the curve $C(E)$ is a trajectory and is called the energy contour of $H = E$. Now invent a new variable that we call the action:

$$I = \frac{1}{2pi}\oint_{C(E)} pdq$$

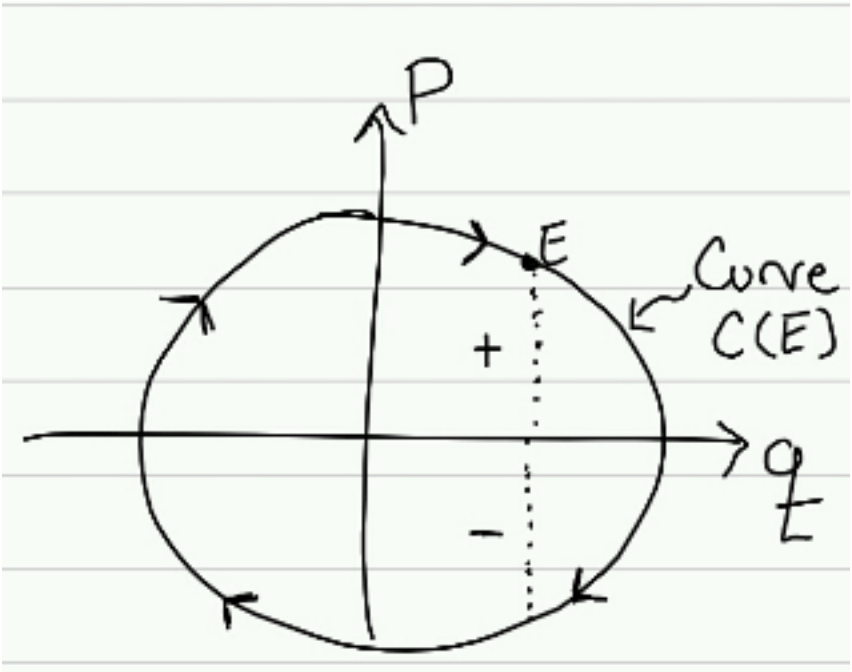


Figure 2.8: Simple harmonic oscillator

which denotes the area enclosed by the curve $C(E)$. From this integral we will end up getting that the action will be a function of the energy $I = I(E)$. We can invert this to get the energy E as a function of the action I. Since $H = E$ and $E = E(I)$ then $H = H(I)$. So instead of having the Hamiltonian be a function of both q and p, $H(q,p)$, we have the Hamiltonian be a function of one phase space variable: $H(I)$, the "momentum" I. H is not a function of the new conjugate coordinate θ, the "angle", so θ is an *ignorable* coordinate.

This is exactly constructing a canonical transform from (q,p) to the action-angle variables θ, I:

$$(q,p) \mapsto (\theta, I)$$

such that $H(q,p) \mapsto H(\underbrace{I}_{\text{new momentum}})$. Then the Hamilton's canonical equations in the new variables are:

$$\begin{cases} \dot{\theta} = \frac{\partial H}{\partial I} = \omega(I) = \text{ constant along the dynamics} \\ \dot{I} = -\frac{\partial H}{\partial \theta} = 0 \end{cases}$$

Here is a graphical interpretation of some 1 degree of freedom system. Consider some 1 degree of freedom system with periodic trajectories as shown below.

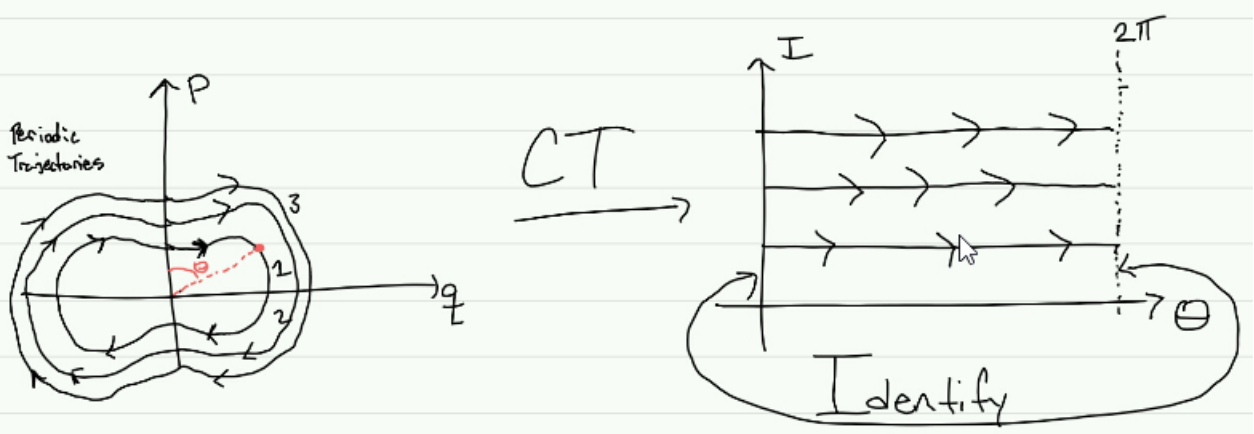


Figure 2.9: periodic trajectories in phase space with a canonical transform into action-angle space

The interpretation of θ as an angle has this motivation: as θ goes from 0 to 2π, that takes us, on the trajectory, through the entire curve of the trajectory. So θ is a measure of how far along the trajectory we have travelled. This is seen in the phase space of p and q in figure 2.9. Similarly, the action variable I is a proxy for amplitude. This is seen in figure 2.9 by the lowest trajectory having a smaller amplitude than two trajectories above it.

Since the value of θ loops from 0 to 2π we can identify the sides of the phase space of θ and I. This is simply the act of gluing the sides together to form a cylinder. If we wanted to really see what the new phase space looks like we get:

So our canonical transformation to action-angle variables reveals that the phase space is topologically a cylinder, $\theta \in S^1$ (circle) and $I \in R$ (real) which gives us $S^1 \times R$ =cylinder. You can think about this way: take the real line extending in both directions vertically and if at each point we attach a circle and glue them all together, the whole continuum, we form a cylinder. We can form other shapes as well in this manner. Consider $S^1 \times S^1$. This is simply a circle with with, at every point, another circle attached. The name of this shape is the 2-torus. Now let's go back to

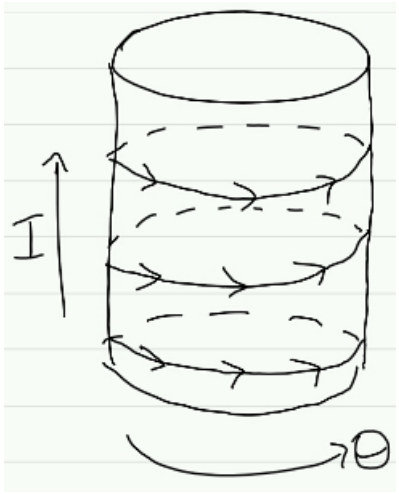


Figure 2.10: Phase space after canonical transformation

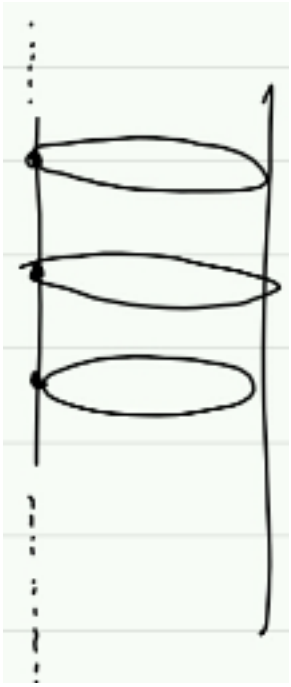


Figure 2.11: Forming $S^1 \times R$

calculating I from earlier for the simple harmonic oscillator.

$$I = \frac{1}{2\pi} \oint_{C(E)} pdq;\ p = p(q;E) = \pm\sqrt{2(E - \frac{1}{2}\omega^2 q^2)}$$

We have $q_{max} = -q_{min} = \frac{\sqrt{2E}}{\omega}$ which represents where the momentum, p, goes to zero. Now the

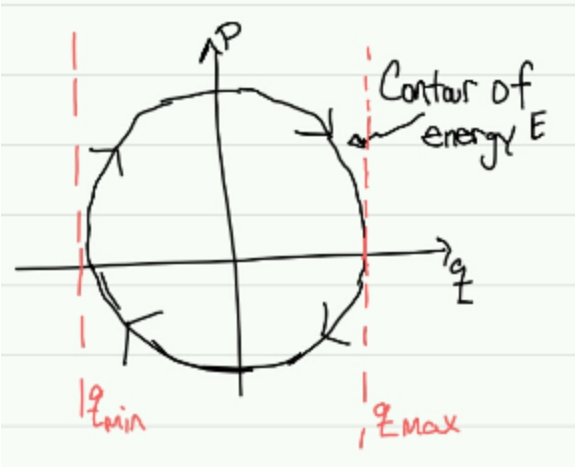


Figure 2.12: Contour of energy E, $C(E)$

integral I becomes:

$$I = \frac{1}{2\pi} \cdot 2 \int_{q_{min}}^{q_{max}} p dq \qquad \text{Calculating the area from } q_{min} \text{ to } q_{max} \text{ and then doubled.}$$

$$= \frac{1}{2\pi} \cdot 2 \int_{q_{min}}^{q_{max}} \sqrt{2E - \omega^2 q^2} d$$

$$= \frac{1}{\pi} \frac{1}{\omega} \int_{-x_{max}}^{x_{max}} \sqrt{a^2 - x^2} \qquad \text{Define } x = \omega q \text{ and get } dx = \omega dq. \text{ Let } a^2 = 2E.$$

$$= \frac{1}{\pi\omega} \left[\frac{x\sqrt{a^2 - x^2}}{2} \Big|_{-a}^{a} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) \Big|_{-a}^{a} \right]$$

$$= \cdots = \frac{E}{\omega}$$

So $I = \frac{E}{\omega}$ and we can invert this to get $E = \omega \cdot I$. Since $E = H$, then we obtain

$$H = H(I) = \omega I$$

where ω represents "frequency". Our Hamiltonian is dependent only on the new momentum!

Getting the frequency of the motion as a function of the action variable, I, tells us the frequencies and (periods) of periodic orbits in the system. We now have $\omega(I) = \frac{\partial H}{\partial I} = \omega =$ constant .

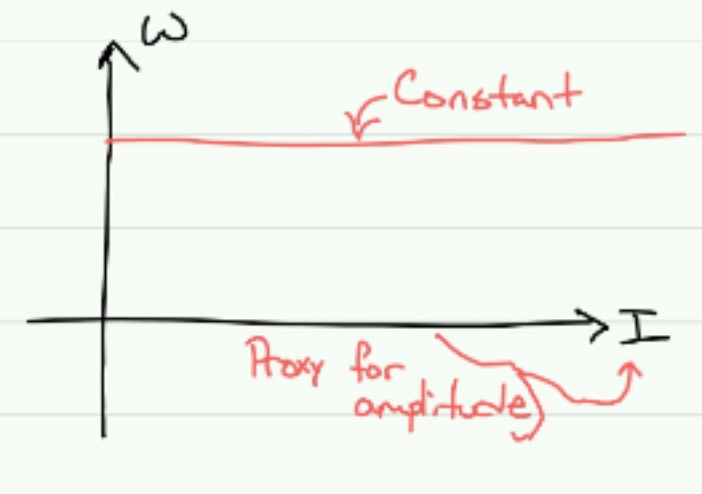


Figure 2.13: Plotting the frequencies

In higher degree of freedom systems $n \geq 2$, the action-angle approach still works. Let's work with an example. Namely, two harmonic oscillators (uncoupled). In this system there will be 2 degrees of freedom. So we have 4 variables which are the two variables and their two conjugate momenta. Following the action angle procedure we wish to apply a canonical transform:

$$(q_1, q_2, p_1, p_2) \mapsto (\theta_1, \theta_2, I_1, I_2)$$

The Hamiltonian is written as the total energy of the system:

$$\begin{aligned} H(q_1, q_2, p_1, p_2) &= H_1(q_1, p_1) + H_2(q_2, p_2) \\ &= \underbrace{\frac{1}{2}p_1^2 + \frac{1}{2}\omega_1^2 q_1^2}_{\mathrm{E}_1} + \underbrace{\frac{1}{2}p_2^2 + \frac{1}{2}\omega_2 q_2^2}_{\mathrm{E}_2} \end{aligned}$$

We can do a transformation to action-angle coordinate:

$$H(q_1, q_2, p_1, p_2) \mapsto H(I_1, I_2) = \omega_1 I_1 + \omega_2 I_2$$

Now, what does this mean? In these new variables, for each of the actions we've got an angle. The dynamics of the first angle, for the first harmonic oscillator, are: $\dot{\theta}_1 = \frac{\partial H}{\partial I_1} = \omega_1$. Similarly, for the

dynamics for the second angle are: $\dot{\theta}_2 = \frac{\partial H}{\partial I_2} = \omega_2$. Of course, the actions (Is) are not changing since $\dot{I}_1 = -\frac{\partial H}{\partial \theta_1} = 0$ and $\dot{I}_2 = -\frac{\partial H}{\partial \theta_2} = 0$. This is an example of a system where we cannot actually draw the phase space as we did in the 1 degree of freedom example. Let's view the action space: (I_1, I_2).

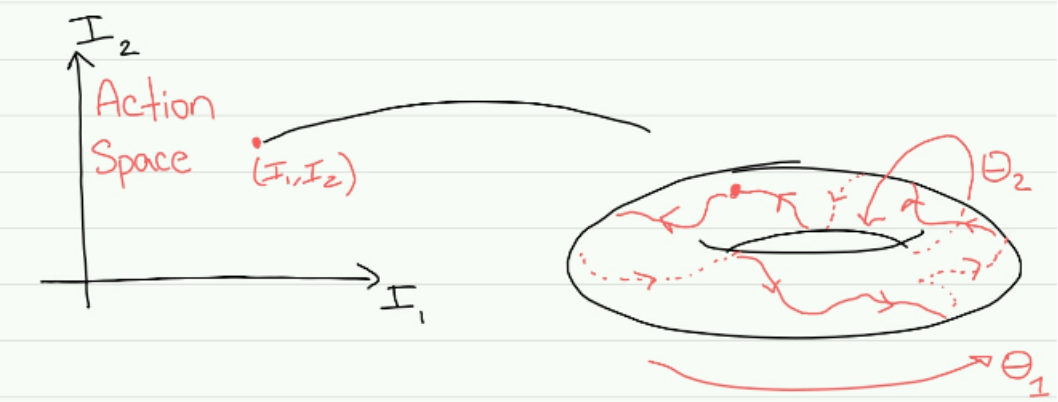


Figure 2.14: Point in action space corresponding to a 2-torus

For each point in action space (I_1, I_2) there corresponds a torus in the phase space. There will be an initial condition on the action space so there will also be an initial condition on the torus depending on what $\omega_1, 2$ are. Therefore, there will be a trajectory that evolves on the torus. For a different choice of action variable we will obtain a whole new tori in the phase space. Hence, the 4D phase space is filled with a 2-parameter family of 2-tori (the two parameters are (I_1, I_2)).

Consider that the ratio of ω_1 and ω_2 are rational. Suppose $\frac{\omega_1}{\omega_2} = 1$. Then that means wherever the trajectory starts on the tori, it will evolve once in the θ_1 direction and then once in the θ_2 direction. Thereby creating a closed curve. Similarly so if $\frac{\omega_1}{\omega_2} = \frac{3}{2}$. Three times in the θ_1 direction and twice in the θ_2 direction. However, if the ratio is irrational, say π. then the trajectory will never actually create a closed loop. It will never intersect itself and eventually fill the tori upon infinite evolution. This is called quasiperiodicity.

If we set $H(q_1, q_2, p_1, p_2) = E$, some constant, this defines all initial conditions that start with energy. Because this is a Hamiltonian system, energy is conserved, trajectories that begin

with a certain energy will stay with that energy for all time. So $H(q_1q_2, p_1, p_2) = E$ defines a 3-dimensional energy surface (manifold) in the 4D phase space. Let's see this explicitly for the double harmonic uncoupled oscillator.

Recall that we have:

$$\frac{1}{2}p_1^2 + \frac{1}{2}\omega_1^2 q_1^2 + \frac{1}{2}p_2^2 + \frac{1}{2}\omega_2^2 q_2^2 = E$$

Define $x_1 = \omega_1 q_1$, $x_2 = p_1$, $x_3 = \omega_2 q_2$, $x_4 = p_2$, and $r^2 = 2E$. This yields:

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = r^2$$

This is the equation for all points equidistant from the origin in R^4. The name for this is the 3-sphere of radius r. To aid with visualization, let $r = 1$ and consider $n = 0$. This is the 0-sphere which is the set $S^0 \subset R^1$. So $x_1 \in R$ and then $x_1^2 = 1 \implies x_1 = \pm 1$. Therefore, the 0-sphere is just two points: $\{+1, -1\}$. Now, let $n = 1$. This is the 1-sphere which is the set $S^1 \subset R^2$, $(x_1, x_2) \in R^2$. This is: $x_1^2 + x_2^2 = 1$ which is simple a circle centered at the origin. Notice that if we draw a line which intersects the circle twice we obtain a 0-sphere. If we do this throughout the circle we obtain a family of 0-spheres. Hence, a 1-sphere is made up of a whole family of 0-spheres. This holds in higher dimensions as well. Consider $n = 2$. This is the 2-sphere $S^2 \subset R^3, x_1^2 + x_2^2 + x_3^2 = 1$. Plane intersection gives a circle. Hence, 2-sphere is made from a family of 1-spheres. Lastly, let $n = 3$ which yields the 3-sphere, $S^3 \subset R^4, (x_1, x_2, x_3, x_4) \in R^4$ which gives us $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$. The 3-sphere looks like a family of 2-spheres.

All energy surfaces for our double (uncoupled) harmonic oscillators is a 3-sphere. On each 3-sphere there are tori. Our formula for the energy

$$\frac{1}{2}p_1^2 + \frac{1}{2}\omega_1^2 q_1^2 + \frac{1}{2}p_2^2 + \frac{1}{2}\omega_2^2 q_2^2 = E$$

describes a topological 3-sphere energy manifold that is filled with 2-tori. Generally, the actions are set of n parameters that identify the n-torus of the angles.

2.5 Integrable Hamiltonian Systems

We have been introduced to action-angle variables as a canonical transform. Now we will see how they fit into the bigger picture of our work. Essentially. we can split Hamiltonian systems into two categories: integrable and non-integrable Hamiltonian systems. Consider a N degree of freedom system:

$$(q_1, q_2 \cdots, q_N, p_1, p_2, \cdots, p_N) \mapsto (\theta_1, \theta_2, \cdots, \theta_N, I_1, I_2, \cdots, I_N)$$

It is always the case that we can construct a canonical transform to take our old variables q_i, p_i to action-angle variables θ_i, I_i. What separates integrable from non-integrable Hamiltonian systems is what happens to the Hamiltonian itself.

We call the system **integrable** if you can take the Hamiltonian in q and p via a canonical transformation so that it becomes a function of only the actions. That is,

$$H(q,p) \mapsto H(I)$$

which is half of the phase space variables. On the other hand, we say the system is non-integrable if you cannot get rid of the dependence on all of the θs via the canonical transformation:

$$H(q,p) \mapsto H(\theta, I)$$

Integrable systems are simplier and easier to handle because we have N ignorable coordinates. Let's connect the discussion about tori in the action-angle introduction to these two categories of systems. In the integrable case, this means the phase space is filled with tori. In particular, the phase space is an N-parameter family of N-tori. The n-parameters are the actions $(I_1, \cdots, I_N)$ and the $N-tori$ are parametrized with the angles $(\theta_1, \cdots, \theta_N)$. The N-tori are denoted by T^N where T^1 is the 1-torus

(circle) which is S^1, $T^2 = 2$-torus which is $S^1 \times S^1$, and generally

$$T^N = \underbrace{S^1 \times \cdots \times S^1}_{N}$$

In N-DOF, by assumption $\frac{\partial H}{\partial t} = 0$ (time independent) which implies H is a constant of motion. H is a function of $2N$ phase variables and defines a $(2N-1)$-dimensional energy manifold (energy surface). For example, when $N = 2$ we have a 3-dimensional energy surface

We have energy conservation and these N-tori throughout the phase space. This has implications for how the trajectories can move around the phase space. For example, in 1-DOF system, the energy manifold is $(2 \cdot 1 - 1)$-dimensional. This is a curve. The phase space is filled with 1-tori and that means each energy manifold has really just one 1-torus. Now consider when $N = 2$. This yields a 3-dimensional energy manifold. The 3D phase space is filled with 2-tori. A 2-torus is two dimensional and so it has enough dimension to block motion on the 3D energy manifold.

$N = 2$ is the highest degree of freedom where this sort of blocking happens because when $N = 3$ DOF, the energy manifold is 5-dimensional. The phase space has 3-tori which are 3-dimensional. Hence, the 3D tori don't have enough dimension to block motion in 5D. Note that to "block" motion, the tori must have co-dimension one.

Integrable systems do not show chaotic motion. In N degree of freedom, all motion is quasi-periodic, periodic, or simply static. This means we have at most N periods. Now let's shortly discuss non-integrable systems.

Non-integrable case: These systems can show chaotic motion. In fact, phase space could be "mixed", showing quasi-periodic motion for some initial conditions and chaotic motion for other

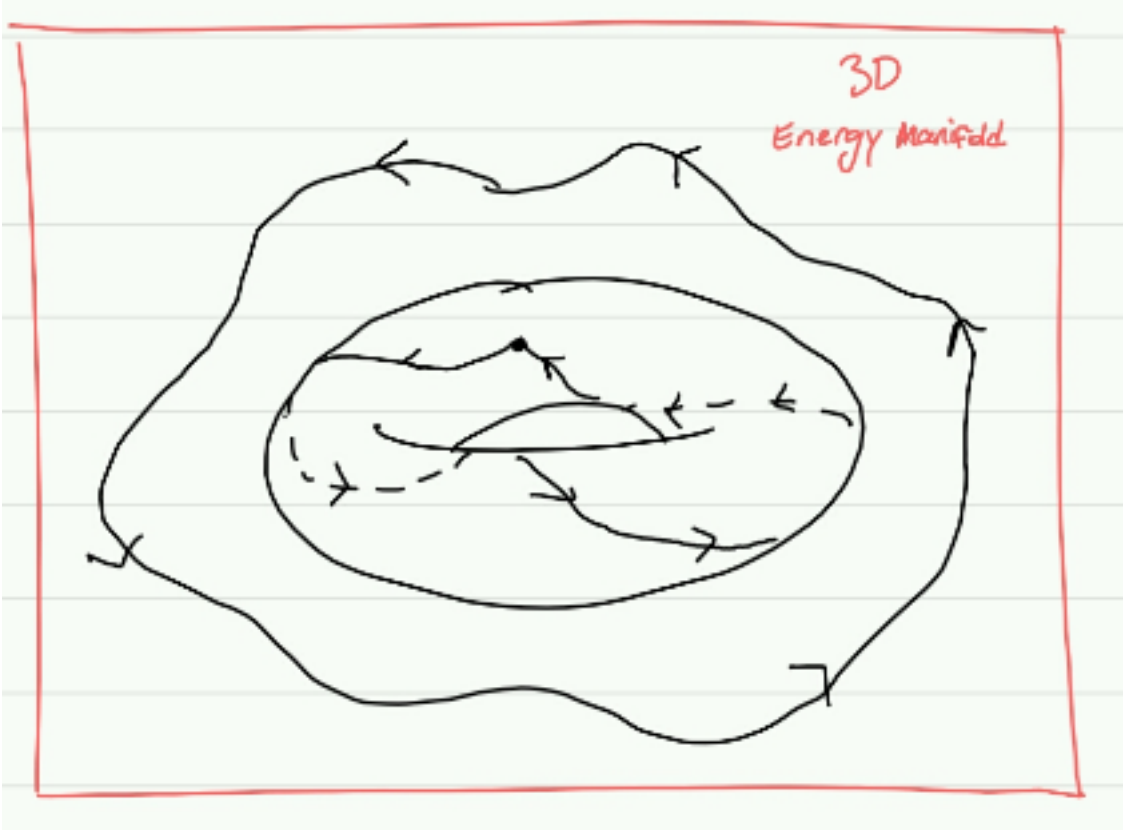


Figure 2.15: Partition of phase space by 2-tori

initial conditions. That is,

$$H(\theta,I) \to \begin{cases} \dot{\theta}_i = \frac{\partial H}{\partial I_i}(\theta,I) \\ \dot{I}_i = -\dfrac{\partial H}{\partial \theta_i}(\theta,I) \neq 0 \end{cases}$$

Note that θ is not constant as it would be in the integrable case.

Special attention has been paid to "near integrable" Hamiltonian systems. These are Hamiltonian systems where you can write the Hamiltonian as the sum an integrable and non-integrable part:

$$H(\theta,I) = \underbrace{H_0(I)}_{\text{Integrable part}} + \varepsilon \cdot H_1(\theta,I)$$

where ε is sufficiently small. A well known example of such a Hamiltonian system is the restricted 3-body problem.

There is a well known result for near integrable Hamiltonian systems: The KAM Theorem. Here *KAM* denotes the letters of the following mathematicans: Kolmogorov, Arnold, and Moser. They studied the case of near-integrable systems and they proved that some tori, called KAM tori, may persist in a near-integrable system. When $\varepsilon = 0$ then all of phase space is foliated by tori.

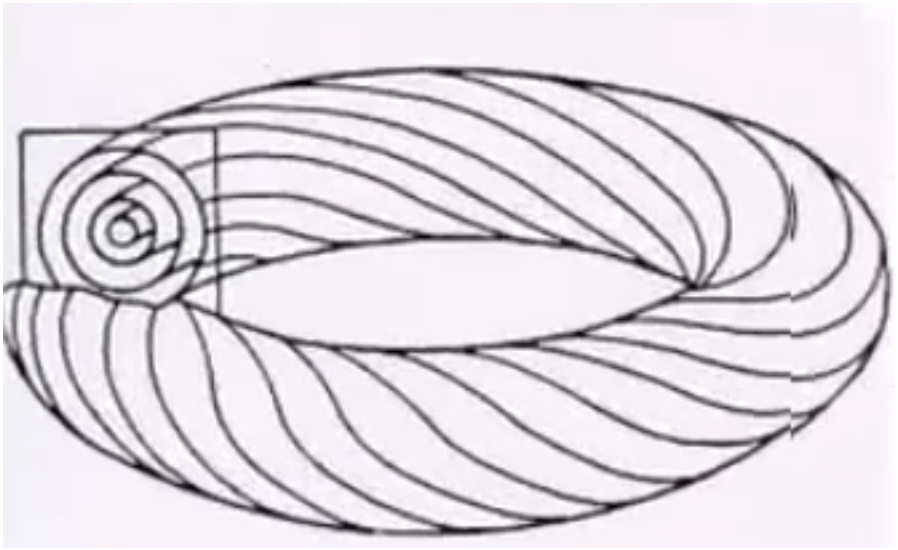


Figure 2.16: tori within tori within tori !

Otherwise, when $\varepsilon \neq 0$, we have some tori that persist in the phase space called KAM-tori. A common tool to see these tori are with Poincaré sections. Poincaré was a french mathematician who did a lot great work at the end of the 1800's particularly within chaotic dynamics. His work was largely forgotten until Lorenz in the 1960's picked his work back up.

Poincaré surface-of-section (aka Poincaré section): Now, what are these? If we are trying to understand what happens in a space and there are trajectories which we wish to analyze, use a Poincaré section would be useful. Instead of look at the spaghetti of all these trajectories in a potentially high dimensional space, taking a section of the phase space reduces the dimension by at least 1. This is often visualized by take a cut of a 3-dimensional phase space:

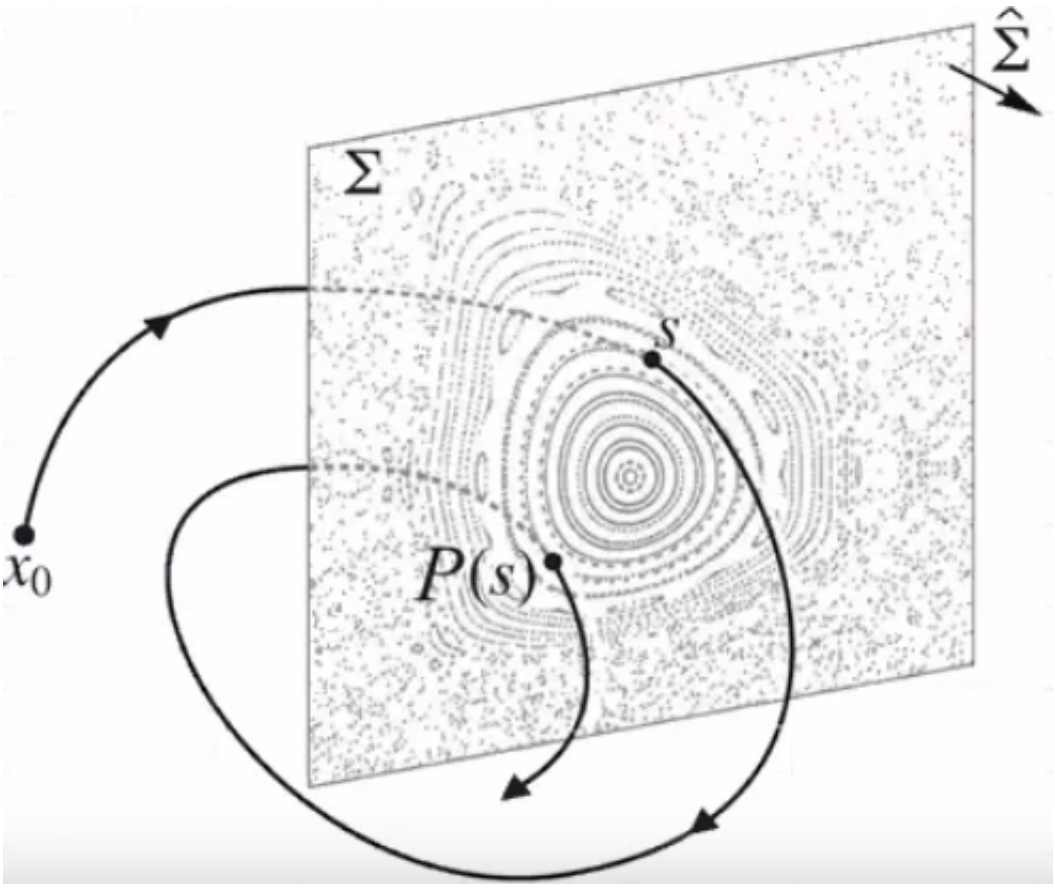


Figure 2.17: An example of a Poincaré section

2.6 Poisson brackets and non-canonical Hamiltonian systems

The fundamental aspect of the Poisson bracket approach is that you are looking at scalar functions, f, that are a function of your phase space coordinates $(q_1, \cdots, q_N, p_1, \cdots, p_N)$ and possibly time t. It is a common desire to want to know how a scalar function evolves along trajectories. Recall from earlier that we took the total derivative of f, applied chain rule and Hamilton's canonical equations, and obtain the following:

$$\dot{f} = \{f, H\} + \frac{\partial f}{\partial t}$$

If my function f does not explicitly depend on time t, $\frac{\partial f}{\partial t} = 0$

$$\dot{f} = \{f, H\}$$

As we saw earlier, letting $f = q_j$ and $f = p_j$ yields us Hamilton's canonical equations. Now we will introduce the symplectic notation ("x" notation) to describe the Poisson bracket as it will be useful. Consider the system $\dot{x} = f(x)$ where this f is a different f in the Poisson bracket above. Now write $x \in R^{2N}$ as:

$$x = \begin{bmatrix} \mathbf{q} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_N \\ p_1 \\ p_2 \\ \vdots \\ p_N \end{bmatrix}$$

and write the gradient of H:

$$\nabla H = DH = \begin{bmatrix} \frac{\partial H}{\partial x_1} \\ \frac{\partial H}{\partial x_2} \\ \vdots \\ \frac{\partial H}{\partial x_N} \end{bmatrix} = \begin{bmatrix} \frac{\partial H}{\partial q_1} \\ \frac{\partial H}{\partial q_2} \\ \vdots \\ \frac{\partial H}{\partial p_N} \end{bmatrix}$$

Define $J = \begin{bmatrix} 0 & \mathscr{I}_N \\ -\mathscr{I}_N & 0 \end{bmatrix}$ and call it the **canonical matrix** or **symplectic matrix**. In this formulation, Hamilton's equations are:

$$\dot{x} = \underbrace{JDH(x)}_{f(x)}$$

Note that determining whether a system is Hamiltonian is if we can find some function $H(x)$ s.t. $f(x) = JDH(x)$. Now back to the Poisson bracket setting:

$$x = \begin{bmatrix} q \\ p \end{bmatrix}$$

and

$$J = \begin{bmatrix} 0 & \mathscr{I}_n \\ -\mathscr{I}_n & 0 \end{bmatrix}$$

as before. Then we have:

$$\{f, H\} = \nabla f^T J \nabla H$$

where we have the following: $\nabla f = \nabla_x f$, $f(q,p) = f(x)$, $\nabla H = \nabla_x H$, $H = H(x)$. Writing it this way we have $\nabla f \cdot \nabla H = \nabla f^T \nabla H$. Here we see that the right-hand side of $\{f, H\}$ looks really quite similar to a strange dot-product of two gradients.

One could take the Poisson bracket of any two functions $u(q,p,t) = u(x,t)$ and $v(q,p,t) = v(x,t)$. As we know from earlier, the Poisson bracket operation defines a type of algebra called the Lie algebra which means it has the usual properties we showed earlier. The Poisson bracket is an operation on two scalar functions that yields another scalar function of phase space.

Let $\mathscr{F}(R^n)$ be the space of all scalar function in $R^n \to R$ (e.g, $u : R^n \to R$). Let $u, v, w \in \mathscr{F}(R^n)$. Then a Poisson bracket on $\mathscr{F}(R^n)$ is an operation which takes two elements of $\mathscr{F}(R^n)$ and gives another, i.e,

$$\{\cdot,\cdot\} : \mathscr{F}(R^n) \times R^n \to \mathscr{F}(R^n)$$
$$(u, v) \mapsto \{u, v\}$$

with the properties we saw in the beginning of our story:

- $\{u,u\} = 0$

- $\{u,v\} = -\{v,y\}$ anti-symmetry

- $a,b \in R, \{au+bv,w\} = a\{u,w\} + b\{v,w\}$ bilinearity

- $\{uv,w\} = \{u,w\}v + u\{v,w\}$ product rule for Poisson bracket

- $\{u,\{v,w\}\} = \{\{u,v\},w\} - \{v,\{w,u\}\}$ non-associtativity / jacobi identity

A common Lie algebra most likely seen by the reader is the cross product $(\times)$ on R^3. Let $a,b \in R^3$ then we have that $a \times b = -b \times a$, $a \times a = 0$, and surprisingly the Jacobi identity is also satisfied.

For Hamiltonian systems, these properties are useful. Given our Hamiltonian function $H(x,t)$ we have $\{H,H\} = 0$ by the first property of the Poisson bracket. If $\frac{\partial H}{\partial t} = 0$, $H = H(x)$, then $\dot{H} = 0$ and therefore H, the energy of the system, is a constant of the motion. Though, we already knew that H was a constant of the motion. However, the Poisson bracket gives us a pleasant way to write any constant of the motion.

They are the time-independent functions $f(x)$ such that $\{f,H\} = 0$ since $\dot{f} = \{f,H\}$. Consider that f and g, functions of the phase space, are constants of motion. That is, $\{f,H\} = 0 = \{g,H\}$. Then by the Jacobi identity $h = \{f,g\}$ is also a constant of motion.

$$\{\underbrace{\{f,h\}}_{0},g\} + \{\underbrace{\{g,H\}}_{0},f\} + \{\{f,g\},H\} = 0$$

which implies $h = \{f,g\}$ is a constant of motion. We can build up a sort of library in this way.

Worth mentioning are the fundamental Poisson brackets.

The fundamental Poisson brackets are just for canonical Hamiltonian systems. Simply put, we have

$$\{q_i, q_j\} = 0 = \{p_i, p_j\},\ \forall i, j$$

However, when we take the Poisson bracket of q and p together we have

$$\{q_i, p_j\} = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Most Hamiltonian systems are non-canonical Hamiltonian systems which are also known as Poisson systems. We can use the properties of the Poisson bracket to generalize to non-canonical Hamiltonian systems. Consider functions of the phase space $f(x), g(x)$ where x is n-dimensional phase space. Then,

$$\{f, g\} = (\nabla f)^T \omega(x) \nabla g$$

where $\omega(x)$ is an $n \times n$ matrix called the **Poisson matrix**. Expanding the Poisson bracket we wrote above yields

$$= \sum_{i,j} \left[\frac{\partial f}{\partial x_i} \omega_{ij} \frac{\partial g}{\partial x_j} \right]$$

where $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$. A special case for even dimensional systems is

$$\omega = J = \begin{pmatrix} 0 & I_{\frac{n}{2}} \\ -I_{\frac{n}{2}} & 0 \end{pmatrix}$$

which is called the **canonical symplectic matrix**. The non-canonical Hamiltonian equation of motion is, for any $f(x)$ and appropriate $H(x)$, written as:

$$\dot{f} = \{f, H\}$$

Let $f = x_i$. Then $\dot{x}_i = \{x_i, H\} = \sum_{j=1}^{n} \omega_{jj} \frac{\partial H}{\partial x_j}$. We can summarize this equation of motion by noting that $\nabla_x x = I_n$. This yields

$$\dot{x} = \omega(x) \nabla H$$

Note that in general ω depends on x, but for a canonical system ω does not depend x. In this case, $\omega = J$ and we get $\dot{x} = J \nabla H$.

The simplest non-canonical Hamiltonian system is from rigid body mechanics and is famously known: Euler's rigid body equation (in terms of angular momentum). This system is one of the main topics of this story and we will explain part of it's story now. Later we will expand upon it when we discuss Hirota's discritization. Now we will build euler's rigid body rotational equations.

2.6.1 First look at Euler's rigid body equations

Consider the following rigid body.

In this body fixed frame there is a angular momentum vector $h(t)$, viewed in body-fixed principle axis frame, which evolves.

$$h = \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix} \in R^3$$

Explicitly, we have $h_1 = h \cdot b_1$, etc. The way the mass is distributed is written in terms of the moment

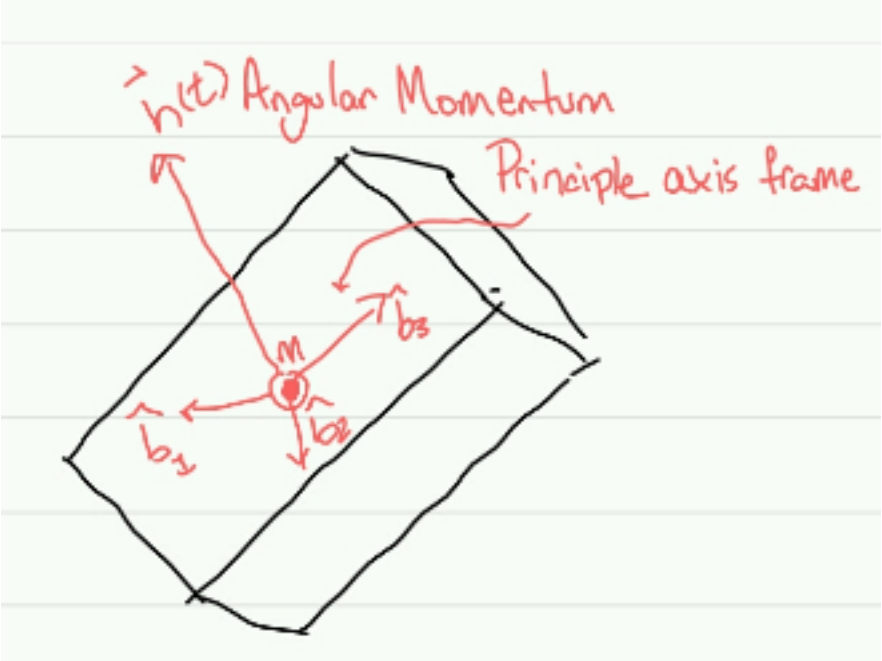


Figure 2.18: Rigid body fixed frame

of inertia matrix:

$$I = \text{ moment of inertia matrix}$$
$$= \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}_{B_p}$$

where the B_p denotes the principle axis. The euler's rigid body equations in terms of angular momentum are:

$$\dot{h} = h \times \left(I^{-1} h\right)$$

This is in $n = 3$ dimensional phase space, $h \in R^3$. Let's demonstrate that this is indeed Hamiltonian. We aim to define a Poisson bracket from a Poisson matrix. By work done by other, and much smarter, mathematicians, it turns out the following Poisson matrix is the matrix we want. Define

$\omega(h)$ as the following:

$$\omega(\mathbf{h}) = \begin{bmatrix} 0 & h_3 & -h_2 \\ -h_3 & 0 & h_1 \\ h_2 & -h_1 & 0 \end{bmatrix} = [\mathbf{h}\times]$$

where $[h\times]$ denotes the cross product in matrix form. Define the Hamiltonian as the total rotational kinetic energy (this is just a free body in ambient space).

$$H(\mathbf{h}) = \frac{1}{2} h \cdot \left(I^{-1} h\right)$$

where $I^{-1} = \begin{bmatrix} \frac{1}{I_1} & 0 & 0 \\ 0 & \frac{1}{I_2} & 0 \\ 0 & 0 & \frac{1}{I_3} \end{bmatrix}$. This is just angular momentum dotted with angular velocity, but because angular velocity is equal to I^{-1} times angular momentum we arrive at what we wrote. Expanded we have:

$$H(\mathbf{h}) = \frac{1}{2}\frac{h_1^2}{I_1} + \frac{1}{2}\frac{h_2^2}{I_2} + \frac{1}{2}\frac{h_3^2}{I_3}$$

The claim is with this Poisson matrix $\omega(\mathbf{h})$ and this Hamiltonian,

$$\dot{f} = \{f, H\}$$

is equivalent to euler's rigid body rotational equations for any scalar function $f(h)$.

$$= (\nabla_{\mathbf{h}} f)^T \, \omega(\mathbf{h}) \nabla_{\mathbf{h}} H$$

If we let $f = \mathbf{h}$, then $\nabla_{\mathbf{h}}\mathbf{h} = I$. This yields:

$$\begin{aligned}\mathbf{h} &= \omega(\mathbf{h})\nabla_{\mathbf{h}}H \\ &= [hx]\left(I^{-1}\mathbf{h}\right) \\ &= \mathbf{h} \times \left(I^{-1}h\right)\end{aligned}$$

which is the same as Euler's rigid body rotational equations as desired. This shows that even though with this odd dimensional space, Euler's rigid body equations are Hamiltonian once defined appropriately. We view it as a non-canonical Hamiltonian system with the Hamiltonian that's equal to the total rotational kinetic energy. The hardest part is finding the correct Poisson matrix, but that was done for us.

The phase space is 3-dimensional and we have two constants of motion. Each constant of motion implies an invariant 2-dimensional surface. The first constant of motion would be the Hamiltonian (total rotational energy) since $\{H,H\} = 0$.

$$H(\mathbf{h}) = \frac{1}{2}\frac{h_1^2}{I_1} + \frac{1}{2}\frac{h_2^2}{I_2} + \frac{1}{2}\frac{h_3^2}{I_3} = \text{ constant}$$

this defines an ellipsoid called the "energy ellipsoid". The second constant of motion is the magnitude of the angular momentum, $||h||^2$, and it can be shown $\{||h||^2,H\} = 0$. In fact, any function $f(||h||^2)$ will satisfy $\{f(||h||^2,H)\} = 0$. We have

$$\{||h||^2,H\} = h_1^2 + h_2^2 + h_3^2 = \text{ constant}$$

defines a 2-sphere (usual sphere). This means the angular momentum vector, $\mathbf{h}$, must evolve along the intersection of the ellipsoid and 2-sphere (energy ellipsoid and energy sphere).

Now we will discuss Casimir functions which are constants of motions arising from the Poisson bracket.

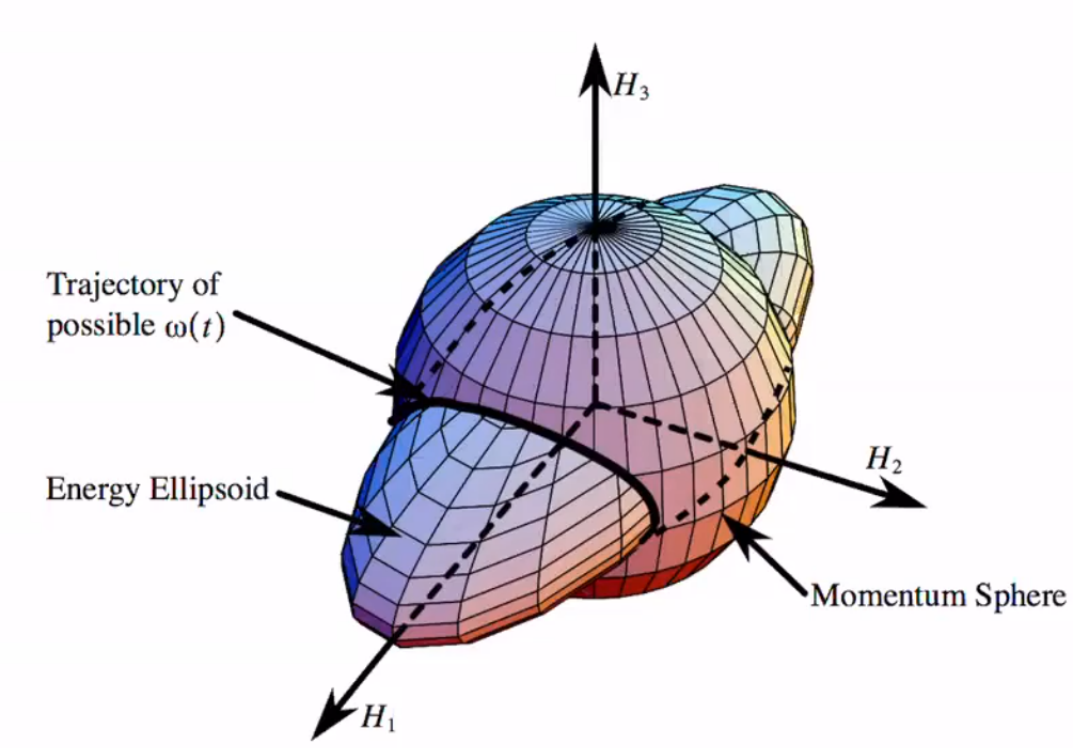


Figure 2.19: Intersection of energy ellipsoid and energy sphere

2.7 Casmir functions and Lie Brackets

Let's review the scope of the story thus far

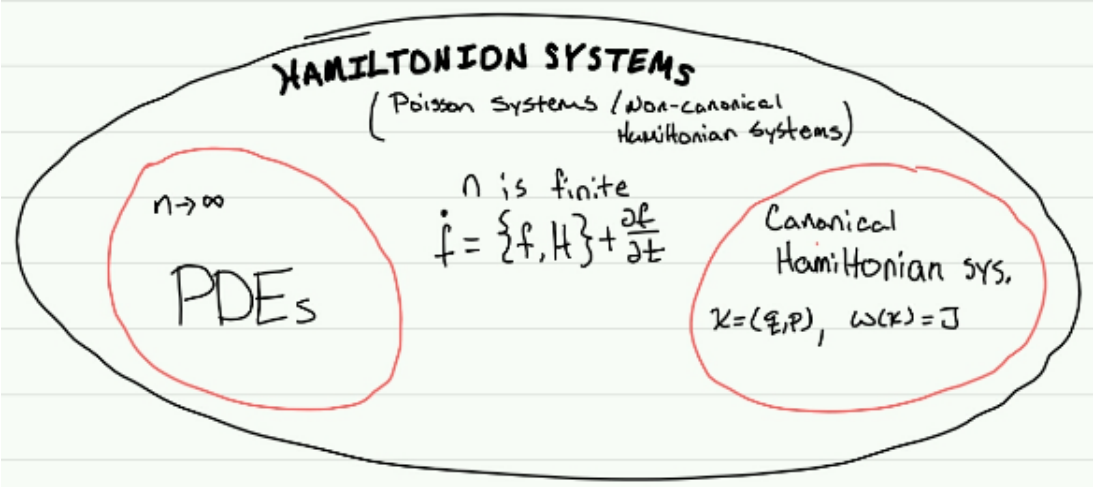


Figure 2.20: Story overview

Any system where you can write the equation(s) of motion as

$$\dot{f} = \{f, H\}$$

where $\frac{\partial f}{\partial t}$ is a Hamiltonian system. Recall that

$$\{f, g\} = (\nabla f)^T \, \omega(x) \nabla g$$

where f, g are scalar functions and $\omega(x)$ is a $n \times n$ matrix called the Poisson matrix. Now let's introduce Casmir functions. When the system being examined is a non-canonical Hamiltonian system, i.e, has non-canonical Hamilton's equations, then

$$\dot{f} = \{f, H\}$$

There is a new category of constants of motion that depend not on the Hamiltonian, but on the Poisson bracket itself, i.e, $\{\cdot, \cdot\}$ depends on the Poisson matrix $\omega(x)$. There are non-trivial functions $C(x)$, called Casmir functions, for which the Poisson bracket with any other function $f(x)$ is zero.

$$\{C(x), f(x)\} = 0 \; \forall f(x)$$

In particular, $f = H$ gives $\{C(x), H(x)\} = 0$ and so $C(x)$ is a constant of motion since he total derivative of C along trajectories is zero, $\dot{C(x)} = 0$. Now, what is the conditions for these Casmir functions to exist? There are a few ways, but a standard condition is to examine the Poisson matrix. If the rank of $\omega(x)$ is not full, then there exist Casmir functions.

$$\{C, f\} = (\nabla C)^T \, \omega(x) \nabla f$$

Let's consider the euler's rigid body rotational equations again. The Poisson matrix was

$$\omega(h) = \begin{bmatrix} 0 & h_3 & -h_2 \\ -h_3 & 0 & h_1 \\ h_2 & -h_1 & 0 \end{bmatrix}$$

Is $\omega(h)$ full rank? Consider the second column and with simple algebraic manipulation we obtain the following:

$$h_2 \begin{pmatrix} h_3 \\ 0 \\ -h_1 \end{pmatrix} + h_3 \begin{pmatrix} -h_2 \\ h_1 \\ 0 \end{pmatrix} = h_1 \begin{pmatrix} 0 \\ -h_3 \\ h_2 \end{pmatrix}$$

$\omega(h)$ has rank 2. Therefore, a Casmir function can be found. Denote the Poisson bracket below with *RB* to denote Rigid Body:

$$\{C,f\}_{\text{RB}} = -\{f,C\}_{\text{RB}} = -(\nabla f) \underbrace{[h\times]}_{\text{Since it looks like cross product}} (\nabla C)$$

If ∇C is in the h direction, then $[h\times]\nabla C = 0$.

$$C(h) = \Phi(||h||^2) \implies \nabla C = a(h)h$$

where Φ is a scalar function. This tells us the form of the Casmir function(s) since

$$\{C,f\}_{RB} = -\{f,C\} = -(\nabla f)^t a(h) \underbrace{h\times h}_{0} = 0$$

Hence, functions of the form any scalar function of the magnitude of h squared, is a Casmir function.

$$C(h) = \Phi(||h||^2)$$

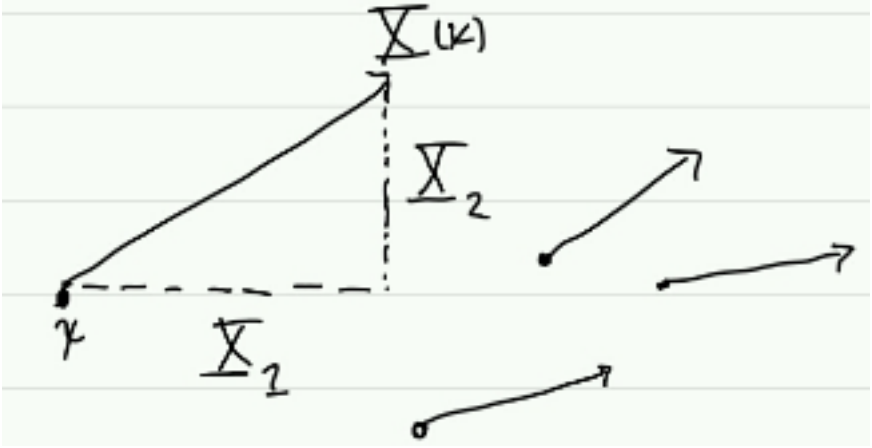


Figure 2.21: vector field

The most trivial Casmir function being

$$C(h) = ||h||^2 = h_1^2 + h_2^2 + h_3^2,$$

and is therefore a constant of motion. Note that if we have a canonical Hamiltonian system we then we

$$\omega(x) = J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

which has full rank.

2.7.1 Lie derivatives

Here we will discuss the connection of Hamiltonian mechanics with Lie derivatives. Consider phase space $x \in R^n$ and suppose we have a vector field $\mathscr{X}(x)$. $\mathscr{X}(x) = (\mathscr{X}_1(x), \mathscr{X}_2(x), \cdots, \mathscr{X}_n(x))$. Suppose we have a scalar field $f(x)$. The "Lie derivative of f along $\mathscr{X}$" is:

$$\mathscr{L}_{\mathscr{X}} f = \text{ directional derivative of } f \text{ along } \dot{\mathscr{X}} = \nabla f \mathscr{X}$$

If the equation of motion for a system is

$$\dot{x} = \mathscr{X}(x)$$

then we have

$$\begin{aligned}\dot{f} &= \frac{df}{dx} \\ &= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\frac{dx_i}{dt} \\ &= \nabla f \cdot \dot{x} \\ &= \nabla f \cdot \mathscr{X}\end{aligned}$$

So, we write

$$\dot{f} = \mathscr{L}_{\mathscr{X}} f$$

to denote the Lie derivative of f along $\mathscr{X}$. What if $\mathscr{X}$ is a Hamiltonian vector field? Meaning, $\mathscr{X}$ arises from a Hamiltonian function H, $X = \mathscr{X}_H$. In this case, we know something about the form of the vector field. If we have a Hamiltonian vector field

$$\dot{x} = \mathscr{X}_H = \omega(x)\nabla H$$

and

$$\begin{aligned}\dot{f} &= \mathscr{L}_{\mathscr{X}_H} f \\ &= \nabla f \cdot \dot{x} \\ &= \nabla f \cdot (\omega(x)\nabla H) \\ &= \underbrace{(\nabla f)^T \omega(x)\nabla H}_{\{f,H\}}\end{aligned}$$

In short,

$$\mathscr{L}_{\mathscr{X}_H} f = \{f, H\}$$

Hence, there is a connection between the Lie derivative of a scalar field f and the Poisson bracket when the vector field is Hamiltonian. Similarly, we can take the Lie derivative of another vector

field $\mathscr{Y}$ with respect to $\mathscr{X}$.

$$\mathscr{L}_{\mathscr{X}}\mathscr{Y} = \text{ directional derivative of } \mathscr{Y}(x) \text{ with respect to } \mathscr{X}(x)$$

The Lie derivative of two vector fields defines another vector field

$$\mathscr{Z} = \mathscr{L}_{\mathscr{X}}\mathscr{Y}$$

where $\mathscr{Z}(x) = (\mathscr{Z}_1(x), \cdots, \mathscr{Z}_n(x))$. Per term we have

$$\mathscr{Z}_i(x) = \sum_{j=1}^{n} \left(\mathscr{X}_j \frac{\partial \mathscr{Y}_i}{\partial x_j} - \mathscr{Y}_j \frac{\partial \mathscr{X}_i}{\partial x_j} \right)$$

This is the ith component of the Lie bracket $[\mathscr{X}, \mathscr{Y}]_i$ of $\mathscr{X}$ and $\mathscr{Y}$. The Lie bracket, as a remark, also shows up in the field of "control theory". If $\mathscr{X}, \mathscr{Y}$ commute, then

$$[\mathscr{X}, \mathscr{Y}] = 0$$

and $\mathscr{X}$, $\mathscr{Y}$ are said to be "compatible". Here is a geometric interpretation of a Lie bracket. Follow the steps below and refer to the figure.

- **1.** Go forward under $\mathscr{X}$ for the time step dt,
- **2.** Go forward under $\mathscr{Y}$ for the time step dt,
- Go backward under $\mathscr{X}$ for dt, i.e, follow $-\mathscr{X}$,
- Lastly, go backward under $\mathscr{Y}$ for dt, i.e, follow $-\mathscr{Y}$.

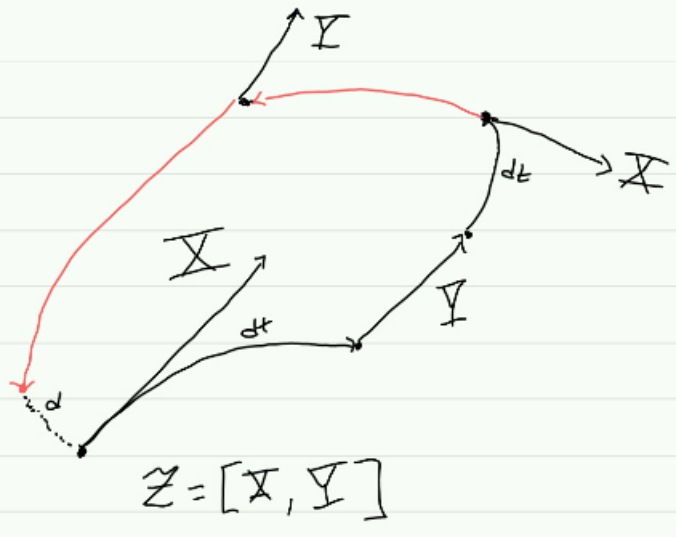


Figure 2.22: $Z = [\mathscr{X}, \mathscr{Y}]$

We will arrive not at the original point and the displacement d is proportional to $\mathscr{Z}(dt)^2$, where $Z = [\mathscr{X}, \mathscr{Y}]$. So if $\mathscr{X}$ and $\mathscr{Y}$ commute, this procedure takes us back to the original point.

2.7.2 Specialize to Hamiltonian vector fields

Consider the Hamiltonian vector fields generated by $f(x), g(x)$ respectively: $\mathscr{X}_f$ and $\mathscr{X}_g$. The Lie bracket of these two vector fields yields:

$$[\mathscr{X}_f, \mathscr{X}_g] = \mathscr{X}_{\{f,g\}}$$

If $\{f,g\} = 0$ then $\mathscr{X}_0 = 0$, so the vector fields $\mathscr{X}_f$ and $\mathscr{X}_g$ commute, i.e, are compatible. They correspond to compatible constants of motion. $f = const.$, a level set of f, describes a codimension-1 surface in phase space that can co-exist with $g =$ codimension-1 surface.

$$\nabla f = \text{grad } f = \text{ vector field normal to level sets of } f$$

Let's look at some canonical Hamiltonian illustrations. Consider a 1 degree of freedom Hamiltonian system (harmonic oscillator) with the Hamiltonian:

$$H(q,p) = \frac{1}{2}(q^2 + p^2)$$

and $\mathbf{x} = \begin{pmatrix} q \\ p \end{pmatrix}$ with the Hamiltonian vector field $\dot{x} = \mathscr{X}_H = \omega \nabla H = \begin{pmatrix} p \\ -q \end{pmatrix}$ and Poisson matrix $\omega(x) = J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. $J\nabla H = sgradH$ to mean skew gradient of H. This is the Hamiltonian

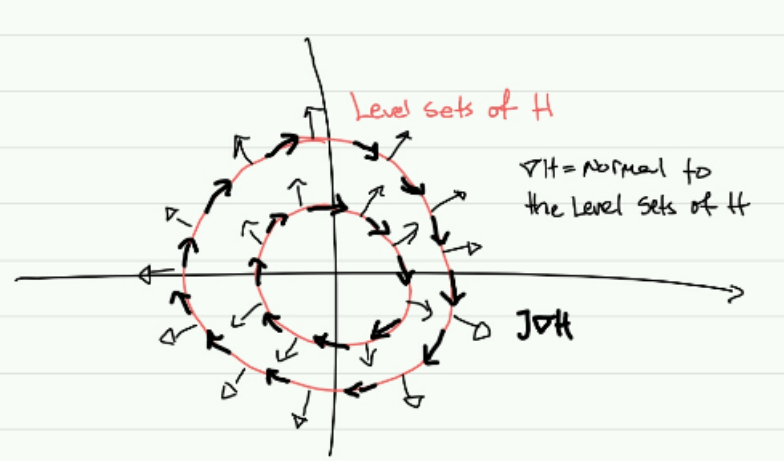


Figure 2.23: Level sets of H and its gradient

vector field along level sets of H. For any scalar function f, we can define a Hamiltonian vector field generated by f, which is the skew gradient of f.

$$\mathscr{X}_f = J\nabla f = sgrad f$$

For $f = H$, this vector field is special because it relates to motion. $\dot{x} = \mathscr{X}_H = J\nabla H = sgradH$. In a 2 degree of freedom system, we now have a 4D phase space: (q_1, q_2, p_1, p_2). Write $H : R^4 \to R$ and $E \in R$. Then $H^{-1}(E)$ is a 3D level set of H called the energy manifold. H is a constant of motion, but there could be others. Let's assume there are two constants of motion (functions which remain

constant along trajectories) for sake of argument and geometric interpretation.

$$f(q_1, q_2, p_1, p_2) \rightarrow \text{ 3D level sets}$$

and

$$g(q_1, q_2, p_1, p_2) \rightarrow \text{ 3D level sets}$$

Now, how can you have motion along two 3D level sets in a 4D space? In 4 dimensions, two 3D surfaces generically intersect as a 2D surface. The gradient is $\nabla = (\frac{\partial}{\partial q_1}, \frac{\partial}{\partial q_2}, \frac{\partial}{\partial p_1}, \frac{\partial}{\partial p_2})$. The skew gradient is $J\nabla = (\frac{\partial}{\partial p_1}, \frac{\partial}{\partial p_2}, -\frac{\partial}{\partial q_2}, -\frac{\partial}{\partial q_2})$. The regular gradient gives the vector field that is orthogonal (normal) to level sets of a scalar function. The skew gradient is one vector field that is tangent to the level set of the scalar function (there are two other independent vector fields). So, along $s\nabla f$, we want g to remain constant. The condition for this is

$$\text{grad} g \cdot sgrad f = 0$$

which means

$$(\nabla g)^T J \nabla f = 0$$

This is the same as saying $\{g, f\} = 0$. So f, g are compatible constants of motion if the Poisson bracket of the two vanish. a.k.a, we say f and g are *in involution*.

2.8 Completely integrable Hamiltonian system

This is usually considered in the canonical Hamiltonian system context. For n degree of freedom system, it is said to be **complete integrable** if there n functions in involution.

$$F_1 = H, F_2 = H, \cdots, F_n$$

where $\{F_i, F_j\}\ \forall i, j$. We have a $2n$-dimensional phase space and each function has a $(2n-1)$-dimensional level sets. There are n dimensional surfaces in which all of these functions remain constant. The surfaces are topologically n-tori. Note that we can utilize Poincaré sections to search for evidence of constant of motion since if there are tori then are constants of motion, However, we may never know exactly what those functions are and instead just know that they exist.

The notion of completely integrable systems also extends to non-canonical Hamiltonian setting. e.g, Euler's Rigid Body equations which had a 3-dimensionsal phase space. There were two (independent) constants of motion:

$$\{F_1, F_2\} = 0$$

where F_1 was the total energy and F_2 was the magnitude squared of the angular momentum. PDEs are also an example. They are an ∞-dimensional system and so ∞ number of constants of motion. e.g, the KdV PDE which is where solitons were first observed.

Shortly, we have the following:

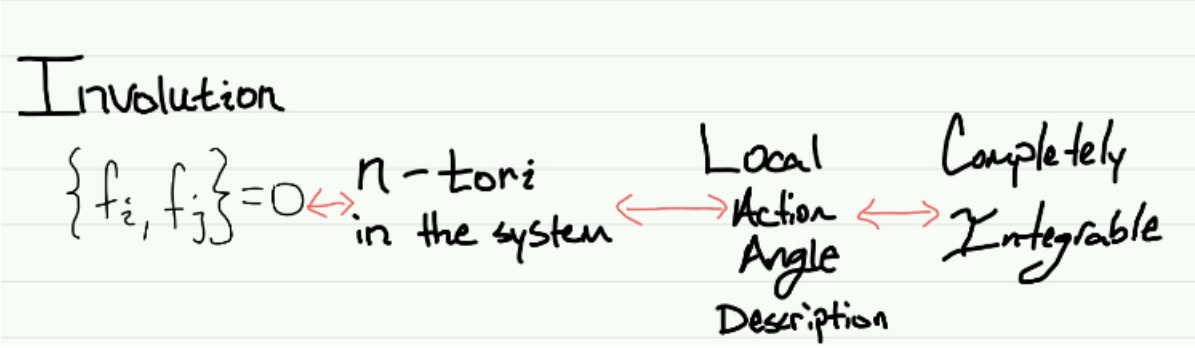


Figure 2.24: summary

CHAPTER III

KAHAN-HIROTA-KIMURA DISCRETIZATION OF EULER TOP

3.1 Introduction

Evidence suggests that integrable discretization is in general difficult to accomplish [3]. However, there is one method attributed to Kahan (1993) and discovered independently by Hirota and Kimura in 2000 that seems surprisingly successful.

Given a set of equations that are linear in derivatives and at most quadratic in the other terms, the discretization rule is generally given as follows [3]:

- The derivative is discretized by a forward difference

$$\frac{dx_j}{dt} \mapsto \frac{\left[x_j(n+1) - x_j(n)\right]}{h}$$

  where h is the lattice parameter;

- a first-order monomial is discretized by taking average:

$$x_j \mapsto \frac{1}{2}\left[x_j(n+1) + x_j(n)\right](1 - \delta_k \delta_i \tilde{x}_j^2)$$

- a quadratic term is discretized by

$$x_j x_k \mapsto \frac{1}{2}\left[x_j(n+1) + x_j(n)\right]$$

After applying these rules one solves for the $x_j(n+1)$ term and obtains a rational map, which often

is integrable.

3.1.1 Euler's Top

Now we will consider Euler's rigid body (rotational) equations and follow [7]. Euler's rigid body (rotational) equations will be written as follows:

$$\begin{cases} \dot{x}_1 &= \alpha_1 x_2 x_3, \\ \dot{x}_2 &= \alpha_2 x_3 x_1, \\ \dot{x}_3 &= \alpha_3 x_1 x_2 \end{cases}$$

where $x_i \in R, i = 1,2,3$ and $\alpha_i = \frac{1}{I_k} - \frac{1}{I_j}$ for principle momenta. Recall that the Euler Top had the Poisson matrix

$$\omega(x) = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}$$

which is an element of $so(3)$. The Lie algebra $so(3)$ has the basis consisting of:

$$e_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$e_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$e_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Now let's obtain the above 3-dimensional rigid body rotational equations by utilizing the Poisson bracket and basis elements of $so(3)$. Recall that

$$\{f,H\} = (\nabla_x f(x))^T \, \omega(x) \nabla_x H(x)$$

where $H = \frac{1}{2}\frac{x_1^2}{I_1} + \frac{1}{2}\frac{x_2^2}{I_2} + \frac{1}{2}\frac{x_3^2}{I_3}$ is the Hamiltonian of the system and $\omega(x)$ is the Poisson matrix.

Let $f = x_1$. Our claim is that we will recover $\dot{x}_1$. Computing the gradient of f we obtain:

$$\begin{aligned}
\nabla_x f &= \sum_{i=1}^{3} \frac{\partial f}{\partial x_i} e_i \\
&= \frac{\partial x_1}{\partial x_1} e_1 + \frac{\partial x_1}{\partial x_2} e_2 + \frac{\partial x_1}{\partial x_3} e_3 \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}
\end{aligned}$$

Then the transpose of the above is:

$$(\nabla_x f)^T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

Computing the gradient of the Hamiltonian:

$$\begin{aligned}
\nabla_x H &= \sum_{i=1}^{3} \frac{\partial H}{\partial x_i} e_i \\
&= \frac{\partial H}{\partial x_1} e_1 + \frac{\partial H}{\partial x_2} e_2 + \frac{\partial H}{\partial x_3} e_3 \\
&= \frac{1}{2}\frac{x_1}{I_1} e_1 + \frac{1}{2}\frac{x_2}{I_2} e_2 + \frac{1}{2}\frac{x_3}{I_3} e_3 \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{x_1}{I_1} \\ 0 & \frac{x_1}{I_1} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \frac{x_2}{I_2} \\ 0 & 0 & 0 \\ -\frac{x_2}{I_2} & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\frac{x_3}{I_3} & 0 \\ \frac{x_3}{I_3} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -\frac{x_3}{I_3} & \frac{x_2}{I_2} \\ \frac{x_3}{I_3} & 0 & -\frac{x_1}{I_1} \\ -\frac{x_2}{I_2} & \frac{x_1}{I_1} & 0 \end{pmatrix}
\end{aligned}$$

Now we can compute the Poisson bracket:

$$\begin{aligned}
\{x_1, H\} &= \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}}_{(\nabla_x f)^T} \underbrace{\begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}}_{\omega(x)} \underbrace{\begin{pmatrix} 0 & -\frac{x_3}{I_3} & \frac{x_2}{I_2} \\ \frac{x_3}{I_3} & 0 & -\frac{x_1}{I_1} \\ -\frac{x_2}{I_2} & \frac{x_1}{I_1} & 0 \end{pmatrix}}_{\nabla_x H} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ -x_2 & x_2 & 0 \\ -x_3 & 0 & x_1 \end{pmatrix} \begin{pmatrix} 0 & -\frac{x_3}{I_3} & \frac{x_2}{I_2} \\ \frac{x_3}{I_3} & 0 & -\frac{x_1}{I_1} \\ -\frac{x_2}{I_2} & \frac{x_1}{I_1} & 0 \end{pmatrix} \\
&= \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ \frac{x_1 x_3}{I_3} & \frac{x_2 x_3}{I_3} & -\frac{x_2^2}{I_2} - \frac{x_1^2}{I_1} \\ -\frac{x_1 x_2}{I_2} & \frac{x_3^2}{I_3} + \frac{x_1^2}{I_1} & -\frac{x_2 x_3}{I_2} \end{pmatrix}}_{A}
\end{aligned}$$

Taking the trace of the matrix above yields:

$$\begin{aligned} tr(A) &= \frac{x_2x_3}{I_3} - \frac{x_2x_3}{I_2} \\ &= (\frac{1}{I_3} - \frac{1}{I_2})x_2x_3 \end{aligned}$$

Since $\alpha_i = \frac{1}{I_k} - \frac{1}{I_j}$ then $\alpha_1 = \frac{1}{I_3} - \frac{1}{I_2}$. Hence, we recover $\dot{x}_1$ as desired:

$$\dot{x}_1 = \alpha_1 x_2 x_3$$

The other two equations for $\dot{x}_2$ and $\dot{x}_3$ can similarly be recovered.

Below the tilde denotes the shift $t \mapsto t + \varepsilon$ where ε is a small time step. Write x_i for $x_i(n\varepsilon)$ and $\tilde{x}_i$ for $x_i(n\varepsilon + \varepsilon)$, $n \in Z$. The system has two constants of motion as discussed earlier.

3.1.2 The Kahan-Hirota-Kimura Discretization of the Euler Top

The explicit discretization of Euler's equations of motion introduced by Hirota and Kimura are:

$$\tilde{x}_i - x_i = \delta_i \left(\tilde{x}_j x_k + x_j \tilde{x}_k \right)$$

where $\delta_i = \frac{\varepsilon \alpha_i}{2}$. We will write $\delta = (\delta_1, \delta_2, \delta_3) \in R^3$. Explicitly this is given as

$$\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix} = \begin{pmatrix} 1 & -\delta_1 x_3 & -\delta_1 x_2 \\ -\delta_2 x_3 & 1 & \delta_2 x_1 \\ -\delta_3 x_2 & -\delta_3 x_1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Now write

$$A(x, \delta) = \begin{pmatrix} 1 & -\delta_1 x_3 & -\delta_1 x_2 \\ -\delta_2 x_3 & 1 & -\delta_2 x_1 \\ -\delta_3 x_2 & -\delta_3 x_1 & 1 \end{pmatrix}$$

Now we can write the equations as follow:

$$\tilde{x} = A^{-1}(x,\delta)x = A(\tilde{x},-\delta)x$$

where $x = (x_1, x_2, x_3)$, $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$, and $\delta = (\delta_1, \delta_2, \delta_3)$.

Proposition: The quantities

$$F_i = \frac{1 - \delta_k \delta_i x_j^2}{1 - \delta_i \delta_j x_k^2}$$

are conserved quantities of motion for Euler's rigid body equations. Note that only two of them are in involution.

Proof by [7]: Consider the equation $\tilde{F}_i = F_i$.

$$(1 - \delta_k \delta_i \tilde{x}_j^2)(1 - \delta_i \delta_j x_k^2) = (1 - \delta_i \delta_j \tilde{x}_k^2)(1 - \delta_k \delta_i x_j^2)$$

Then by algebraic manipulation we obtain

$$\delta_j(\tilde{x}_k^2 - x_k^2) - \delta_k(\tilde{x}_j^2 - x_j^2) = \delta_i \delta_j \delta_k(\tilde{x}_j^2 x_k^2 - x_j^2 \tilde{x}_k^2)$$

Notice the difference of squares which upon expansion yields:

$$\delta_j(\tilde{x}_k + x_k)(\tilde{x}_k - x_k) - \delta_k(\tilde{x}_j + x_j)(\tilde{x}_j - x_j) = \delta_i \delta_j \delta_k(\tilde{x}_j x_k + x_j \tilde{x}_k)(\tilde{x}_k x_j - x_k \tilde{x}_j)$$

Applying $\tilde{x}_i - x_i = \delta_i(\tilde{x}_j x_k + x_j \tilde{x}_k)$, the discretized equations of motion, we obtain:

$$(\tilde{x}_k + x_k)(\tilde{x}_i x_j + x_j \tilde{x}_j) - (\tilde{x}_j + x_j)(\tilde{x}_k x_i + x_k \tilde{x}_i) = (\tilde{x}_i - x_i)(\tilde{x}_k x_j - x_k \tilde{x}_j)$$

Upon simplification of the left hand side we have equality. Done.

Hence, the discretized euler top using the Hirota-Kimura discretization is completely integrable.

CHAPTER IV

THE MANAKOV TOP

4.1 The conserved quantities

The Manakov Top is the system corresponding to the equations of motion for a rigid body in four-dimensional Euclidean space with a fixed point coinciding with the center of mass otherwise known as the *Manakov Top*. In four-dimensional euclidean space our rotations are elements of $so(4)$

which has basis consisting of:

$$e_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$e_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$e_3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$e_4 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$e_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$e_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

e_1, e_2, e_3 are generalizations of the basis matrices for $so(3)$. In particular, they are the basis matrices of $so(3)$ with a column and row of zeros attached. We have the following commutator relations between the basis matrices where we have the cycle (ijk) and i, j, k can be $1, 2, 3$:

$$\left[e_i, e_j\right] = e_k$$

For example,

$$\begin{aligned}
[e_1, e_2] &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&= e_3
\end{aligned}$$

which is what was claimed. We also have:

$$\left[\tilde{e}_i, \tilde{e}_j\right] = e_k$$

where $\tilde{e}_{1,2,3} := e_4, e_5, e_6$ respectively. The above commutator relations are the same commutator relations of $so(3)$. Now define the following:

$$X_i \equiv \frac{1}{2}[e_i + \tilde{e}_i],\ Y_i \equiv \frac{1}{2}[e_i - \tilde{e}_i]$$

Note that each relation independently satisfies the commutation relations for $so(3)$ and that we have $[X_i, Y_j] = 0$. Since these relations are the relations for $so(4)$, we have shown that $so(4)$ is a direct sum of two (independent) $so(3)$ Lie algebras. That is,

$$so(4) \cong so(3) \oplus so(3)$$

This provides motivation into investigating the structure of the system corresponding to the Manakov Top described in [1], [6], [9]:

$$\begin{aligned}
\dot{x}_1 &= x_2 x_3 A_{32} + x_5 x_6 A_{65} \\
\dot{x}_2 &= x_1 x_3 A_{13} + x_4 x_6 A_{46} \\
\dot{x}_3 &= x_1 x_2 A_{21} + x_4 x_5 A_{54} \\
\dot{x}_4 &= x_3 x_5 A_{35} + x_2 x_6 A_{62} \\
\dot{x}_5 &= x_3 x_4 A_{43} + x_1 x_6 A_{16} \\
\dot{x}_6 &= x_2 x_4 A_{24} + x_1 x_5 A_{51}
\end{aligned}$$

where $A_{ij} = \left(\lambda_i - \lambda_j\right)$. In the Euler Top case it was fairly easy to determine the form of the Poisson matrix. The Poisson matrix for the Manakov Top will require a detailed construction. Following [4] we can construct the matrix known as the *Structure matrix* or *Hamiltonian operator* or *Poisson tensor* and then continue to construct our Poisson bracket. Before the Poisson bracket is constructed we will explain its construction in [1] with additional detail from [9].

Following [1] which follows [6], let $\Lambda : so(4) \to so(4), X \mapsto \Lambda X$ where $X \in so(4)$ be a non-singular symmetric linear map. Also assume that it is *diagonal* and *non-degenerate* in the standard basis. That is, $(\Lambda X)_{ij} = \left(\Lambda_{ij} X_{ij}\right)$. The Manakov top, the geodisic motion on $so(4)$, corresponds to

$$\dot{X} = [X, \Lambda X]$$

and Hamiltonian vector fields written as

$$\dot{X} = [X, \nabla H]$$

The Hamiltonian $H = \langle X, \Lambda X \rangle$ where $\langle X, Y \rangle = -\frac{1}{2} tr(XY)$. The following theorem from [1] is "Note 1" from [6].

Theorem 1. The geodisic flow of $\dot{X} = [X, \Lambda X]$ on $SO(4)$ is algebraically completely integrable with the six abelian functions X_{ij} for $1 \leq i \langle j \leq 4$ if and only if the diagonal metric Λ satisfies the commutation relation

$$[X, \beta] + [\alpha, \Lambda X] = 0$$

as an identity in X, with diagonal matrices $\alpha = \text{diag}(\alpha_1, \cdots, \alpha_4)$ with α_i's all distinct and $beta = \text{diag}(\beta_1, \cdots, \beta_4)$. This identity amounts to:

$$\Lambda_{ij} = \frac{\beta_i - \beta_j}{\alpha_i - \alpha_j}$$

Then the system has besides the orbit invarients $H_1 = x_1^2 + \cdots + x_6^2, H_2 = X_{12}X_{34} - X_{13}X_{24} + X_{23}X_{14}$ and the Hamiltonian

$$\sum_{1 \leq i \langle j \leq 4} \frac{\beta_i - \beta_j}{\alpha_i - \alpha_j} X_{ij}^2,$$

the following invariant

$$\sum_{1 \leq i \langle j \leq 4} \frac{\gamma_i - \gamma_j}{\alpha_i - \alpha_j} X_{ij}^2, \gamma_i (i \leq i \leq 4) \in C$$

The four quadrics obtained by setting these four invariants equal to a constant, generically intersect in the affine part of an Abelian variety of genus 2.

We verify the identity below.

Identity Proof. Consider the commutation relation in the above theorem

$$[X,\beta]+[\alpha,\Lambda X]=0$$

where

$$X=\begin{pmatrix} 0 & -x_3 & x_2 & -x_4 \\ x_3 & 0 & -x_1 & -x_5 \\ -x_2 & x_2 & 0 & -x_6 \\ x_4 & x_5 & x_6 & 0 \end{pmatrix}$$

and $\alpha=\text{diag}(\alpha_1,\cdots,\alpha_4)$, $\beta=(\beta_1,\cdots,\beta_4)$.

$$\begin{aligned} [X,\beta]+[\alpha,\Lambda X]&=0 \\ \implies [X,\beta]&=-[\alpha,\Lambda X] \\ [X,\beta]&=[\Lambda X,\alpha] \end{aligned}$$

$$\implies \begin{pmatrix} 0 & x_3\beta_1-x_3\beta_2 & -x_2\beta_1+x_2\beta_3 & x_4\beta_1-x_4\beta_4 \\ x_3\beta_1-x_3\beta_2 & 0 & x_1\beta_2-x_1\beta_3 & x_5\beta_2-x_5\beta_4 \\ -x_2\beta_1+x_2\beta_3 & x_1\beta_2-x_1\beta_3 & 0 & x_6\beta_3-x_6\beta_4 \\ x_4\beta_1-x_4\beta_4 & x_5\beta_2-x_5\beta_4 & x_6\beta_3-x_6\beta_4 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & x_3\alpha_1\lambda_3-x_3\alpha_2\lambda_3 & -x_2\alpha_1\lambda_2+x_2\alpha_3\lambda_2 & x_4\alpha_1\lambda_4-x_4\alpha_4\lambda_4 \\ x_3\alpha_1\lambda_3-x_3\alpha_2\lambda_3 & 0 & x_1\alpha_2\lambda_1-x_1\alpha_3\lambda_1 & x_5\alpha_2\lambda_5-x_5\alpha_4\lambda_5 \\ -x_2\alpha_1\lambda_2+x_2\alpha_3\lambda_2 & x_1\alpha_2\lambda_1-x_1\alpha_3\lambda_1 & 0 & x_6\alpha_3\lambda_6-x_6\alpha_4\lambda_6 \\ x_4\alpha_1\lambda_4-x_4\alpha_4\lambda_4 & x_5\alpha_2\lambda_5-x_5\alpha_4\lambda_5 & x_6\alpha_3\lambda_6-x_6\alpha_4\lambda_6 & 0 \end{pmatrix}$$

Consider $([X,\beta])_{21}$ and $([[\Lambda X,\alpha]])_{21}$ of the above. That is,

$$x_3\beta_1 - x_3\beta_2 = x_3\alpha_1\lambda_3 - x_3\alpha_2\lambda_3 \tag{4.1}$$

$$x_3(\beta_1 - \beta_2) = \lambda_3 x_3(\alpha_1 - \alpha_2) \tag{4.2}$$

$$\implies \lambda_3 = \frac{\beta_1 - \beta_2}{\alpha_1 - \alpha_2} \tag{4.3}$$

We can obtain the remaining λ_i, $i = 1,\cdots 6$, in the same manner.

4.1.1 Lax pair Framework

In this subsection I aim to explain roughly the framework for constructing the Lax Pair which yields the system described earlier. Following [9] and [10], consider

$$\mathbf{u}_t = \mathbf{F}(\mathbf{u}),\ \mathbf{u} = (u^1,\cdots,u^N)$$

The lax pair for the above ODE is the relation of the form

$$L_t = [A,L],\ (*)$$

where $L = L(\mathbf{u},\lambda)$ and $A = A(\mathbf{u},\lambda)$ are matrices, which is equivalent to the ODE. Geometrically, identifying the system with the Lax pair means that the matrix L evolves via "infinitesimal rotations" that are "orthogonal" to L [10]. It is often the case that $A \in g$ where g is some Lie algebra of some Lie group G [10]. The special idea of Lax pairs are that they lead to conserved quantities. Now we will state and prove the following lemma from [9].

Lemma

- If L_1 and L_2 satisfy $(*)$, then the product $\overline{L} = L_1L_2$ also satisfies $(*)$;

- $\overline{L} = L^n$ satisfies $(*)$ for all $n \in N$;

- $tr(L^n)$ is an integral of motion for our system;

- The coefficients of the characteristic polynomial $det(L-\mu\mathbf{I})$ are integrals of motion

Proof.

- Take the derivative of $\overline{L}$. That is, $(\overline{L})_t = (L_1)_t L_2 + L_1(L_2)_2 = [A,L_1]L_2 + L_1[A,L_2] = A\overline{L} - \overline{L}A$. Done.

- $\overline{L} = L^n$ follows directly from the proof that $\overline{L} = L_1 L_2$ satisfies $(*)$.

-

$$\begin{aligned}\left(tr(L^k)_t\right) &= k \cdot tr(L)^{k-1}(L)_t \\ &= k \cdot tr(L)^{k-1}[A,L]\end{aligned}$$

and since $tr(A[B,c]) = tr(B[C,A]) = tr(C[A,B])$ and the fact that L^{k-1} commutes with L, we obtain $\left(tr(L^k)\right)_t = 0$.

There are not infinitely many conserved quantities generated by the trace of powers of the L operator as one may think [10] as there are only finitely many independent conserved quantities. When L is a rank one operator this leads to algebraically structured solutions such as solitaary waves (solitons) [10]. The general of reconstructing the solution given the spectral information on L (and "scattering data") is known as the *inverse scattering* method, but this deep subject is not discussed here. Now we will introduce the Lax pair for the *Euler Top* following [9].

Let $U(t)$ be a matrix of dimension $m \times m$

$$L = a\lambda + U,\ A = \frac{U^2}{\lambda}$$

where $a = \text{diag}(a_1, \cdots, a_m)$. Going through the bracket we have the following:

$$\begin{aligned} L_t &= [A, L] \\ &= [\frac{U^2}{\lambda}, a\lambda + U] \\ &= [\frac{U^2}{\lambda}, a\lambda] + \underbrace{[\frac{U^2}{\lambda}, U]}_{0} \\ &= \frac{1}{\lambda}[U^2, a]\lambda \\ &= [U^2, a] \end{aligned}$$

Consider

$$U = \begin{pmatrix} 0 & -u_3 & -u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix}, \; a = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}$$

where the a_1, a_2, a_3 are arbitrary parameters with physical consequence. The resulting system of ODEs is known as the Euler top in three dimensions. Explicitly, this is:

$$(u_1)_t = (a_2 - a_3)u_2u_3, \; (u_2)_t = (a_3 - a_1)u_1u_3, \; (u_3)_t = (a_1 - a_2)u_1u_2$$

By the lemma earlier we may examine the characteristic polynomial and obtain the conserved quantities. The characteristic polynomial is given by the formula:

$$(a_1\lambda - \mu)(a_2\lambda - \mu)(a_3\lambda - \mu) - (u_1^2 + u_2^2 + u_3^2)\mu + (a_1u_1^2 + a_2u_2^2 + a_3u_3^2)\lambda$$

The coefficients of each monomial term in λ and μ are indeed the first integrals of the system. From the characteristic equation $det(L - \mu\mathbf{I}) = 0$, which is elliptic [9], we may write

$$L\Psi = \mu\Psi$$

where $\Psi = \Psi(\lambda, \mu, t)$. The dependence on t is defined via the equation

$$\Psi_t = A\Psi$$

Then we may reconstruct U.

$$\begin{aligned} L_t\Psi + L\Psi_t &= \lambda_t\Psi + \lambda\Psi_t \\ \implies L\Psi_t + LA\Psi &= \lambda_t\Psi + \lambda A\Psi \\ &= \lambda_t\Psi + AL\Psi \end{aligned}$$

Which yields $(L_t + LA - AL)\Psi = \lambda_t\Psi$. Hence, to obtain non-trivial eigenfunctions *Psi*, we must examine

$$L_t + [L,A] = 0 \implies L_t = [A,L]$$

which is true if and only if $\lambda_t = 0$ and therefore the eigenvalues are conserved.

Now we follow material directly from [9] and this is included to provide motivation for why the L, A operators were constructed as they were. To understand this motivation may lead to understanding how to construct the conserved quantities for HK-discretized systems. The following theorem was introduced in [9].

Theorem 1. *Let* $L = D + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\lambda + \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}$. *There is a unique series*

$$T = \mathbf{I} + \begin{pmatrix} 0 & \alpha_1 \\ \beta_1 & 0 \end{pmatrix}\frac{1}{\lambda} + \begin{pmatrix} 0 & \alpha_2 \\ \beta_2 & 0 \end{pmatrix}\frac{1}{\lambda^2} + \cdots$$

such that

$$T^{-1}LT = L_0$$

where

$$L_0 = D + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lambda + \begin{pmatrix} \rho_0 & 0 \\ 0 & -\rho_0 \end{pmatrix} + \begin{pmatrix} \rho_1 & 0 \\ 0 & -\rho_1 \end{pmatrix} \frac{1}{\lambda} + \begin{pmatrix} \rho_2 & 0 \\ 0 & -\rho_2 \end{pmatrix} \frac{1}{\lambda^2} + \cdots$$

Proof. Multiply on both sides by T to obtain the equation

$$LT = TL_0$$

Now consider only the constant terms. This yields

$$\begin{pmatrix} 0 & 2\alpha_1 \\ -2\beta_1 & 0 \end{pmatrix} - \begin{pmatrix} \rho_0 & 0 \\ 0 & -\rho_0 \end{pmatrix} = - \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}$$

Note that the above is equivalent to

$$\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & \alpha_1 \\ \beta_1 & 0 \end{pmatrix} \right] - \begin{pmatrix} \rho_0 & 0 \\ 0 & -\rho_0 \end{pmatrix} = - \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}$$

Hence,

$$\alpha_1 = -\frac{1}{2}u,\ \beta_1 = \frac{1}{2}v,\ \rho_0 = 0$$

Similarly, we can solve for $\alpha_k, \beta_k, \rho_k$. □

Given that our goal is to understand the Manakov top we wish to generalize this theorem for the $n \times n$ case. We will see this is the same generalization given by [9]. Let

$$L = D + a\lambda + \underbrace{Q}_{[a, P_{n\times n}]},$$

where $Im(ad_a),\ a = \text{diag}(a_1, a_2, \cdot, a_n)$ and G is some Lie algebra and λ is again the spectral param-

eter. Suppose also that a satisfies the Lie algebra $\mathscr{G}$ decomposition

$$G = Ker(ad_a) \oplus Im(ad_a)$$

Theorem 2. *There is a unique series*

$$T = \mathbf{I} + [a, M_1]\frac{1}{\lambda} + [a, M_2]\frac{1}{\lambda^2} + \cdots$$

such that

$$T^{-1}LT = L_0$$

where

$$L_0 = D + a\lambda + \Lambda_0 + +\Lambda_1\frac{1}{\lambda} + \Lambda_2\frac{1}{\lambda^2} + \cdots$$

Proof. The proof is generally the same. We apply induction. For each equation per power of λ we want to check if $(LT)_k - (TL_0)_k = 0$. Let us first consider the coefficients with a λ^1 term. That is,

$$\lambda^1 : \begin{cases} (LT)_{\lambda^1} = a\lambda \\ (TL_0)_{\lambda^1} = a\lambda \end{cases}$$

Indeed, $a\lambda - a\lambda = 0$. Similarly,

$$\lambda^0 : \begin{cases} (LT)_{\lambda^0} = D + [a, P] + a[a, M_1] \\ (TL_0)_{\lambda^0} = D + [a, M_1]a + \Lambda_0 \end{cases}$$

The difference is

$$[a, [a, M_1]] - \Lambda_0 + [a, P] = 0$$

Λ_0 is determined as it is the diagonal terms of $[a, P]$, we know $[a, P]$, so therefore we can solve for the desired terms in $[a, [a, M_1]]$. Now consider the coefficients of λ^{-k}.

$$\lambda^{-k}\begin{cases}(LT)_{-k} = a[a,M_{k+1}] + D[a,M_k] + Q[a,M_k] \\ (TL_0)_{-k} = \Lambda_k + [a,M_k]D + [a,M_{k+1}]a + \sum_{i=1}^{k}[a,M_i]\Lambda_{k-i}\end{cases}$$

The difference is

$$[a,[a,M_{k+1}]] + \underbrace{[D,[a,M_k]]}_{\text{known}} + \underbrace{Q[a,M_k]}_{\text{known}} - \Lambda_k - \underbrace{\sum_{i=1}^{k}[a,M_i]\Lambda_{k-i}}$$

known = 0

By induction $[a,M_k]$, Λ_k are uniquely determined.

□

The following theorem by [9] is equivalent to the above theorem.

Theorem 3. *There are unique series*

$$u = u_{-1}\lambda^{-1} + \cdots,\ u_i \in Im(ad_a)$$

$$h = h_0 + h_{-1}\lambda^{-1} + \cdots, h_i \in Ker(ad_a)$$

such that

$$e^{ad_a}(L) = L + [u,L] + \frac{1}{2}[u,[u,L]] + \cdots = Dx + a\lambda + h$$

Note that $e^{ad_A}B = e^A B e^{-A}$ where A,B are matrices. Identify $u = -\log(T)$ and $h_{-i} = \Lambda_i$. Recall that $log(T)$ is well-defined if and only if T is invertible. Since T has constant term I then it is indeed invertible. Define $B_{b,n} = e^{-ad_u}(b\lambda^n)$, then according to [9], the corresponding A-operator

$$A_{b,n} = b\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0$$

is determined by the formula

$$A_{b,n} = (B_{b,n})_+$$

When $G = gl_m$, $a = \text{diag}(a_1, \cdots, a_m)$, and $b = \text{diag}(b_1, \cdots, b_m)$, where $a_i \neq a_m$ for $i \neq j$ then the system of equations corresponding to the operator $A_{b,1}$ have the form

$$Q_t = P_x + [Q, P]$$

where Q, P are $m \times m$ matrices and the entries of P and Q are related via

$$p_{ij} = \frac{b_i - b_j}{a_j - a_i} q_{ij}$$

Solutions of the above, independent of x, describe the dynamics of the m-dimensional rigid body [9]. This closely resembles the theorem by [1].

4.1.2 Poisson bracket framework

Definition. A general Hamiltonian system is given by the triple $\mathscr{M}$, the manifold, $\{\cdot,\cdot\}$, the Poisson structure (Poisson bracket), and $H(x)$ a real function on $\mathscr{M}$. The Poisson structure gives the coordinate functions $J_{ij} = \{x_i, x_j\}$ which are called *structure functions*. They satisfy the Leibniz rule and the have the following properties:

- Skew-symmetry

$$J_{ij}(x) = -J_{ji}(x),\ i,j = 1,\cdots,m$$

- Jacobi identity

$$\sum_{i=1} \left[J_{il}\partial_l J_{jk} + J_{kl}\partial_l J_{ij} + J_{jl}\partial_l J_{ki}\right] = 0$$

  for all $x \in \mathscr{M}$ with $\partial_l = \frac{\partial}{\partial x_l}$

The structure functions can be assembled into a skew-symmetric structure matrix $J(x) \in R^{m\times m}$. In this case, the Poisson bracket for the functions F and G take the form:

$$\{F,G\} = (\nabla F(x))^T J(x) (\nabla G(x))$$

and our system can be rewritten as

$$\dot{x} = J(x)\nabla H(x)$$

The Manakov top in $so(4)$ is associated with the 6-dimensional Lie algebra $so(4)$. We may simplify the above Poisson bracket definition to:

$$\{F,G\} = \sum_{j,k=1}^{m} c_{ij}^k x_k \frac{\partial F}{\partial x_i}\frac{\partial G}{\partial x_j}$$

where the c_{ij}^k are the structure constants of the Lie algebra and $J_{ij}(x) = \sum_{k=1}^m c_{ij}^k x_k$.

To find a Poisson bracket that we desire we must find the appropriate structure constants

and function H (our Hamiltonian). Consider the supposed Hamiltonian

$$H = \frac{1}{2}\lambda_1 x_1^2 \frac{1}{2}\lambda_2 x_2^2 + \cdots + \frac{1}{2}\lambda_6 x_6^2$$

Obtaining the structure constants may be done, to my understanding, in two ways:

- Picking an appropriate basis by guesswork/observation/luck
- Solving for the structure functions explicitly from $\dot{\mathbf{x}} = \{\mathbf{x}, H\}$ assuming an appropriate Hamiltonian H has been chosen.

I have opted to solve for structure functions explicitly since it is relatively easy to do so.

$$
\begin{aligned}
\dot{x} &= \{\mathbf{x}, H\} \\
&= \{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}, \frac{1}{2}\lambda_1 x_1^2 + \cdots + \frac{1}{2}\lambda_6 x_6^2 \} \\
&= I_{6\times 6} \cdot \begin{pmatrix} J_{11} & J_{12} & J_{13} & J_{14} & J_{15} & J_{16} \\ J_{21} & J_{22} & J_{23} & J_{24} & J_{25} & J_{26} \\ J_{31} & J_{32} & J_{33} & J_{34} & J_{35} & J_{36} \\ J_{41} & J_{42} & J_{43} & J_{44} & J_{45} & J_{46} \\ J_{51} & J_{52} & J_{53} & J_{54} & J_{55} & J_{56} \\ J_{61} & J_{62} & J_{63} & J_{64} & J_{65} & J_{66} \end{pmatrix} \cdot \begin{pmatrix} \lambda_1 x_1 \\ \lambda_2 x_2 \\ \lambda_3 x_3 \\ \lambda_4 x_4 \\ \lambda_5 x_5 \\ \lambda_6 x_6 \end{pmatrix} \\
&= \begin{pmatrix} J_{11}\lambda_1 x_1 + J_{12}\lambda_2 x_2 + \cdots + J_{16}\lambda_6 x_6 \\ J_{21}\lambda_1 x_1 + J_{22}\lambda_2 x_2 + \cdots + J_{26}\lambda_6 x_6 \\ J_{31}\lambda_1 x_1 + J_{32}\lambda_2 x_2 + \cdots + J_{36}\lambda_6 x_6 \\ J_{41}\lambda_1 x_1 + J_{42}\lambda_2 x_2 + \cdots + J_{46}\lambda_6 x_6 \\ J_{51}\lambda_1 x_1 + J_{52}\lambda_2 x_2 + \cdots + J_{56}\lambda_6 x_6 \\ J_{61}\lambda_1 x_1 + J_{62}\lambda_2 x_2 + \cdots + J_{66}\lambda_6 x_6 \end{pmatrix}
\end{aligned}
$$

Now pick the appropriate values for the structures constants. For example, the values for the structure constants for the first equation of the Manakov Top are $J_{12} = -x_3$, $J_{13} = x_2$, $J_{15} = -x_6$,

$J_{16} = x_5$, and $J_{11} = J_{14} = 0$. The corresponding structure matrix (Poisson matrix) is:

$$\underbrace{\begin{pmatrix} 0 & -x_3 & x_2 & 0 & -x_6 & x_5 \\ x_3 & 0 & -x_1 & x_6 & 0 & -x_4 \\ -x_2 & x_1 & 0 & -x_5 & x_4 & 0 \\ 0 & -x_6 & x_5 & 0 & -x_3 & x_2 \\ x_6 & 0 & -x_4 & x_3 & 0 & -x_1 \\ -x_5 & x_4 & 0 & -x_2 & x_1 & 0 \end{pmatrix}}_{J}$$

The above matrix has really nice structure. It is clearly skew-symmetric and notice that it is of the form

$$\begin{pmatrix} P & Q \\ Q & P \end{pmatrix}$$

where both $P, Q \in so(3)$, but P, Q are independent in the sense that P depends only on x_1, x_2, x_3 and Q depends only on x_4, x_5, x_6. Because of this we can verify the Jacobi identity (not shown here). Therefore,

$$\{F, G\} = (\nabla F)^T J(x)(\nabla G)$$

with J defined earlier constitutes a Poisson bracket. The Manakov Top looks like 4 "copies" of the 3D rigid body system. There are 4 invariant submanifolds which are all 2-spheres:

$x_1^2 + x_2^2 + x_3^2$ is constant when $x_4 = x_5 = x_6 = 0$

$x_1^2 + x_5^2 + x_6^2$ is constant when $x_2 = x_3 = x_4 = 0$

$x_3^2 + x_4^2 + x_5^2$ is constant when $x_1 = x_2 = x_6 = 0$

$x_2^2 + x_4^2 + x_6^2$ is constant when $x_1 = x_3 = x_5 = 0$

The earlier calculations have shown that

$$H = \frac{1}{2}\lambda_1 x_1^2 + \frac{1}{2}\lambda_2 x_2^2 + \frac{1}{2}\lambda_3 x_3^2 + \frac{1}{2}\lambda_4 x_4^2 + \frac{1}{2}\lambda_5 x_5^2 + \frac{1}{2}\lambda_6 x_6^2$$

is indeed the Hamiltonian of the Manakov Top and therefore the Manakov Top is Hamiltonian. We have found one conserved quantity of the system. If there are two more (which are independent) that can be found and they are all in involution then the system will be completely integrable.

Let $H_1 = ||x||^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2$ and now we wish to compute

$$\{H_1, H\} = \nabla_x(H_1) J(x) \nabla_x H \stackrel{?}{=} 0$$

We have the following:

$$\begin{aligned}
\{H_1, H\} &= \begin{pmatrix} 2x_1 & 2x_2 & 2x_3 & 2x_4 & 2x_5 & 2x_6 \end{pmatrix} \cdot J \cdot \begin{pmatrix} \lambda_1 x_1 \\ \lambda_2 x_2 \\ \lambda_3 x_3 \\ \lambda_4 x_4 \\ \lambda_5 x_5 \\ \lambda_6 x_6 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \lambda_1 x_1 \\ \lambda_2 x_2 \\ \lambda_3 x_3 \\ \lambda_4 x_4 \\ \lambda_5 x_5 \\ \lambda_6 x_6 \end{pmatrix} \\
&= 0
\end{aligned}$$

Hence, H_1 is the first conserved quantity not counting the Hamiltonian. This function is a 5-sphere in 6D phase space.

Consider the function $H_2 = x_1x_4 + x_3x_6 + x_2x_5$. The motivation being that the indices are three apart every time. Then upon computation of the products we should obtain nice cancellation.

$$\begin{aligned}
\{H_2, H\} &= \begin{pmatrix} x_4 & x_5 & x_6 & x_1 & x_2 & x_3 \end{pmatrix} \cdot J \cdot \begin{pmatrix} \lambda_1 x_1 \\ \lambda_2 x_2 \\ \lambda_3 x_3 \\ \lambda_4 x_4 \\ \lambda_5 x_5 \\ \lambda_6 x_6 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \lambda_1 x_1 \\ \lambda_2 x_2 \\ \lambda_3 x_3 \\ \lambda_4 x_4 \\ \lambda_5 x_5 \\ \lambda_6 x_6 \end{pmatrix} \\
&= 0
\end{aligned}$$

Therefore $H_2 = x_1x_4 + x_3x_6 + x_2x_5$ is a constant of motion. Similarly the function described by [1] $\sum_{1 \le i < j \le 4} \frac{\gamma_i - \gamma_j}{\alpha_i - \alpha_j} X_{ij}^2$ with $\gamma_i \in C$ is an invariant. Hence, the top is completely integrable. If they are set to a constant, then they geometrically intersect in the affine part of an Abelian variety of genus 2 [1].

Claim. The Manakov Top is bi-hamiltonian

Recall our first Poisson bracket is

$$F,G_1 = (\nabla F)^T \begin{pmatrix} 0 & -x_3 & x_2 & 0 & -x_6 & x_5 \\ x_3 & 0 & -x_1 & x_6 & 0 & -x_4 \\ -x_2 & x_1 & 0 & -x_5 & x_4 & 0 \\ 0 & -x_6 & x_5 & 0 & -x_3 & x_2 \\ x_6 & 0 & -x_4 & x_3 & 0 & -x_1 \\ -x_5 & x_4 & 0 & -x_2 & x_1 & 0 \end{pmatrix} (\nabla G)$$

We obtained the above Poisson bracket by first assuming the Hamiltonian was

$$H = \frac{1}{2}\sum_{i=1}^{6} \lambda_i x_i^2$$

Now suppose the Hamiltonian is

$$H_* = \frac{1}{2}\sum_{i=1}^{6} x_i^2$$

Going through the same process as before to construct the Poisson matrix we obtain

$$\underbrace{\begin{pmatrix} 0 & \lambda_3 x_3 & -\lambda_2 x_2 & 0 & \lambda_6 x_6 & -\lambda_5 x_5 \\ -\lambda_3 x_3 & 0 & \lambda_1 x_1 & -\lambda_6 x_6 & 0 & \lambda_4 x_4 \\ \lambda_2 x_2 & -\lambda_1 x_1 & 0 & \lambda_5 x_5 & -\lambda_4 x_4 & 0 \\ 0 & \lambda_6 x_6 & -\lambda_5 x_5 & 0 & \lambda_3 x_3 & -\lambda_2 x_2 \\ -\lambda_6 x_6 & 0 & \lambda_4 x_4 & -\lambda_3 x_3 & 0 & \lambda_1 x_1 \\ \lambda_5 x_5 & -\lambda_4 x_4 & 0 & \lambda_2 x_2 & -\lambda_1 x_1 & 0 \end{pmatrix}}_{J_*}$$

We have simply shifted over the moments of inertia constants as well as some negative signs. This

yields us the second Poisson bracket:

$$\{F,G\}_2 = (F)^T J_*(\nabla G)$$

Hence, the Manakov Top is bi-hamiltonian.

4.2 HK discretization

Recall the discretization scheme:

- The derivative is discretized by a forward difference

$$\frac{dx_j}{dt} \mapsto \frac{\left[x_j(n+1) - x_j(n)\right]}{h}$$

  where h is the lattice parameter;

- a first-order monomial is discretized by taking average:

$$x_j \mapsto \frac{1}{2}\left[x_j(n+1) + x_j(n)\right](1 - \delta_k \delta_i \tilde{x}_j^2)$$

- a quadratic term is discretized by

$$x_j x_k \mapsto \frac{1}{2}[x_j(n+1)x_k(n) + x_j(n)x_k(n+1)]$$

Write $\overline{x} \mapsto x(n+1)$ and $x \mapsto x(n)$. Consider the cycle product $(ijk)(\tilde{i}\tilde{j}\tilde{k})$ where $i, \tilde{i}$, $j, \tilde{j}$, and $k, \tilde{k}$ can be $1, 2, 3$. Let δ_{kj}, $\delta_{\tilde{k}\tilde{j}}$ denote $\frac{A_{kj}}{2}$ and $\frac{A_{k\tilde{j}}}{2}$ respectively for the first three equations. In the second three equations we let $\delta_{k\tilde{j}}$ and $\delta_{\tilde{k}j}$ denote $\frac{A_{k\tilde{j}}}{2}$ and $\frac{A_{\tilde{k}j}}{2}$ respectively. Finally, let $h = 1$. The

HK discretization of the Manakov Top (dMT) is:

$$dMT = \begin{cases} \overline{x}_1 - x_1 = \delta_{32}(\overline{x}_2 x_3 + x_2 \overline{x}_3) + \delta_{65}(\overline{x}_5 x_6 + x_5 \overline{x}_6) \\ \overline{x}_2 - x_2 = \delta_{13}(\overline{x}_1 x_3 + x_1 \overline{x}_3) + \delta_{46}(\overline{x}_4 x_6 + x_4 \overline{x}_6) \\ \overline{x}_3 - x_3 = \delta_{21}(\overline{x}_1 x_2 + x_1 \overline{x}_2) + \delta_{54}(\overline{x}_4 x_5 + x_4 \overline{x}_5) \\ \overline{x}_4 - x_4 = \delta_{35}(\overline{x}_3 x_5 + x_3 \overline{x}_5) + \delta_{62}(\overline{x}_2 x_6 + x_2 \overline{x}_6) \\ \overline{x}_5 - x_5 = \delta_{43}(\overline{x}_3 x_4 + x_3 \overline{x}_4) + \delta_{16}(\overline{x}_1 x_6 + x_1 \overline{x}_6) \\ \overline{x}_6 - x_6 = \delta_{24}(\overline{x}_2 x_4 + x_2 \overline{x}_4) + \delta_{51}(\overline{x}_1 x_5 + x_1 \overline{x}_5) \end{cases}$$

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BIOGRAPHICAL SKETCH

Austin Marstaller was accepted to the Southern Methodist University (SMU) Applied Mathematics PhD program in April 2020 and will begin his coursework there in late August 2020. He has received his Bachelors of Science in pure mathematics from The University of Texas at Dallas and earned a Masters of Science in Applied Mathematics at The University of Texas Rio Grande Valley in August 2020. Since January 2020 he has been working as a Research Assistant at the MAIA lab at UT Southwestern working on improving Deep Learning methods for reducing bias in models. In years past he worked as a faculty member for the AwesomeMath Summer Program teaching elementary number theory. With assistance from SMU, he will continue to play an active role in math education through participating in local Math Circles and helping to organize them. He may be contacted at: amarstaller@mail.smu.edu