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Partial Differential Equations in Curved Spacetimes

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PARTIAL DIFFERENTIAL EQUATIONS IN CURVED SPACETIMES

A Thesis

by

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ABSTRACT

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It is the ambition of this thesis to analyze in a concise and coherent manner the idiosyncratic nature of partial differential equations and their mathematical structure in distinct curved spacetimes. In our work special interest is taken in quantum fields dwelling within the de-Sitter geometry. In Chapters I, II, III, and IV, a meticulous study of general relativity is undertaken with one of its solutions derived, an introduction of quantum mechanics is posed, the relativistic quantum theory of fermions is defined, and a “merging” of the former chapters and results are considered, respectively. With what has been derived we seek to examine the paradigm of the Dirac equation in the de-Sitter spacetime. We determine the vierbein, its dual, and the spin connection of the metric under consideration in order to define the Dirac equation in the de Sitter geometry. Once achieved we seek to redefine the former in Klein-Gordon form such that we may derive an analytic solution to the novel equation. Inevitably, the physical ramifications of said solution will detail the behavior of a relativistic fermion traversing through the de Sitter geometry.

DEDICATION

I dedicate this thesis to my family, specifically my mother, sister, father and brother, whom witnessed my absence from many gatherings. To my grandparents, god-mother and friends, who saw within me the success when at times I couldn't see it within myself. Blessed are the hearts of my family whose infinite patience, love and support made this thesis possible.

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CHAPTER I

INTRODUCTION

The enigma of quantum gravity has perplexed the greatest mind's in mathematics and physics for the past millennia. It is one of mathematical physics greatest ambitions to reconcile two of its most prominent and seminal theories, the theory of quantum mechanics and gravity, into a great *unified theory*, with an auspicious forerunner being "M -Theory" - a true maverick.

1.1 Motivation, features of mathematical physics and historical background

Following suit to the founder of the theory developed in the early twentieth century, we analyze one of the solutions of the field equations, taking special consideration to the properties which it yields. Propelled into the zeitgeist of its era and redefining the dogma of its time, Einstein's theory of general relativity catapulted itself into the physics and mathematics commune by deliberating and deciphering problems older theories succumbed to, such as the the precession of the perihelion of Mercury, which Newtonian gravity failed to solve. The solution to this set of non-linear partial differential equations (or tensor equations) is considered under apt assumptions of the energy and momentum of matter, its independence to time, and symmetry of the coordinate system considered; inevitably yielding the de Sitter-Schwarzschild solution of the Einstein field equations. The obtained solution is derived in a coherent and comprehensive way in which the tensor equations mathematical structure delineates the field of gravitation outside of a spherical mass. From the preliminary solution, we may then extrapolate to a similar scenario, which results in the de Sitter solution by means of an appropriate change of coordinates. Yet this solution to the field equations commenced with a simple assumption and observation: upon looking at the night sky, we assume that most of the universe's matter is clustered together into stars which have the tendency to aggregate into the nebulae of the same character as the galaxy we preside on. Upon astronomic observation, one may posit that most of these nebulae seem to be uniformly distributed, leading to the assumption that the properties of the universe may be described by treating the matter as a perfect homogeneous fluid. From this assumption and, once again, gazing at the heavens and theorizing that the universe is static in nature, we may introduce a system of coordinates $x^i = (r, \theta, \phi, ct)$, where our line element takes a static and spherically symmetric form,

$$ds^2 = a(r)dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) - b(r)c^2dt^2$$

where functions a and b are solely functions of the radial coordinate. We find that the functions $a(r)$ and $b(r)$ are connected to the mass density, $\mathring{\mu}^0$, and the proper pressure, $\mathring{p}$, in the universe by the field equation for a perfect fluid

$$\frac{d\mathring{p}}{dr} + \left(\mathring{\mu}^0 c^2 + \mathring{p}\right)\frac{b'}{2b} = 0$$

From this we arrive at the model of the de Sitter universe by noting that there exists only two manners by which the above equation may be satisfied, with special consideration being made to the following alternative,

$$\mathring{\mu}^0 c^2 + \mathring{p} = 0$$

which details the de Sitter solution of the Einstein equations. Upon solving the resulting differential equation, we find that the space-time interval for the required scenario takes the following form,

$$ds^2 = \frac{dr^2}{1 - r^2/R^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) - \left(1 - \frac{r^2}{R^2}\right)c^2dt^2$$

Extrapolating the attained solution for the field equations, we may transcribe the following spacetime interval, explicitly in matrix form as

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{1-K\left(y^2+z^2\right)}{1-K\left(x^2+y^2+z^2\right)} & \frac{Kxy}{1-K\left(x^2+y^2+z^2\right)} & \frac{Kxz}{1-K\left(x^2+y^2+z^2\right)} \\ 0 & \frac{Kxy}{1-K\left(x^2+y^2+z^2\right)} & \frac{1-K\left(x^2+z^2\right)}{1-K\left(x^2+y^2+z^2\right)} & \frac{Kyz}{1-K\left(x^2+y^2+z^2\right)} \\ 0 & \frac{Kxz}{1-K\left(x^2+y^2+z^2\right)} & \frac{Kyz}{1-K\left(x^2+y^2+z^2\right)} & \frac{1-K\left(x^2+y^2\right)}{1-K\left(x^2+y^2+z^2\right)} \end{pmatrix}$$

where the constant factor K takes values

$$K \in \{-1,0,1\}$$

detailing the curvature of the spacetime.

The resulting "metric tensor", $g_{\mu\nu}$, illustrates the geometry of the spacetime, resulting in the aptly named "de Sitter geometry."

We shift our focus to the realm of the minuscule, pondering upon the theory of quantum fields and particles. Presenting an unprecedented quantum mechanical equation of motion, which ciphered the intricacies unfulfilled by preceding theories, Dirac posited the first truly relativistic equation of motion, which adroitly untangled the dilemma of Lorentz covariance, as well as the issue regarding the duplexity of states. Taking heed of previous attempts, we start with an ansatz

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\Psi = \left[\frac{\hbar c}{\mathrm{i}}\alpha^k\partial_k + \beta m_e c^2\right]\Psi \equiv H^D\Psi$$

where the parameters α and β on the right hand side of the equation, also known as the "Dirac Hamiltonian", are introduced to cope with the deficiencies of the Klein-Gordon equation, as well as retaining an analogous form in order to reproduce the plane wave solutions of its predecessor, with energy eigenvalues $E = \pm\sqrt{p^2c^2 + m_e^2c^4}$. Foreshadowing the properties of the parameters α and β, we note that the explicit form of said objects resembles that of matrices, with eigenvalues of either ± 1, with trace zero, and of dimension $2n \; \forall \; n \in \mathbb{N}$. The significance of these "Dirac" matrices is their embedding of the spin property and internal motions of the fermion into the infrasturcture of the equation, without a need for later, independent introduction. Furthermore, the requisite demanded by the laws of nature of all fundamental physical equations is the covet of invariance under Lorentz transformations. We study said property by analyzing the Dirac equation in its spatial and temporal components, respectively,

$$-\mathrm{i}\hbar\beta\partial_0\Psi - \mathrm{i}\hbar\beta\partial_i\alpha^i\Psi + m_e c\Psi = 0$$

further, define the Dirac "gamma" matrices $\gamma^\mu = \left(\gamma^0, \gamma^i\right)$, as follows,

$$\gamma^0 \equiv \beta = \begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & -\mathbf{1}_2 \end{pmatrix} \quad \text{and} \quad \gamma^i \equiv \beta\alpha^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

with σ_i describing the Pauli spin matrices. After algebraic manipulation of the above equation, we rewrite as follows,

$$\left[-\mathrm{i}\gamma^\mu\partial_\mu + \frac{m_e c}{\hbar}\right]\Psi = 0$$

which is the "covariant" form of the Dirac equation. Naturally, we scrutinize the transformation properties of the "function", in this peculiar scenario the Dirac spinor Ψ, of the equation in order to fathom the possibility of Lorentz covariance of the Dirac system.

Consider two inertial frames of reference IS and IS$'$ related linearly through a Lorentz transformation

$$x' = \Lambda x + a \quad \text{and} \quad x = \Lambda^{-1}\left(x' - a\right)$$

of the space-time 4-vectors x^μ and x'^μ, where Λ is a Lorentz transformation matrix and a is a constant space-time shift. Due to our relation between reference frames being linear, in compliance with the dogma implemented by the special theory of relativity, the corresponding spinors Ψ and Ψ' must also abide by said principles and be related by some linear transformation, say f_Λ. Utilizing the previously defined linear transformation, we note that the spinor in the IS′ frame may be expressed by the components of its counter-part spinor, Ψ, in the IS reference frame as

$$\Psi'\left(x'\right) = f_\Lambda[\Psi(x)] = f_\Lambda\left[\Psi\left(\Lambda^{-1}\left(x'-a\right)\right)\right]$$

and conversely,

$$\Psi(x) = f_\Lambda^{-1}\Psi'\left(x'\right)$$

The gravity that the linear transformation f_Λ has on the transformation is demonstrated by relating the coordinates of the reference frames IS and IS′ by mixing the components of the Dirac 4-spinor Ψ_i in Ψ, in such a manner that for each component of the quantum mechanical state Ψ' we write,

$$\Psi'_\mu = \sum_{\nu=1}^{4} f_{\Lambda,\mu\nu}\Psi_\nu$$

Upon demonstrating Lorentz covariance of the covariant form of the Dirac equation we arrive at the equation

$$\left[-\mathrm{i}\gamma^\mu\Lambda_\mu^\nu\partial'_\nu + \frac{m_e c}{\hbar}\right] f_\Lambda^{-1}\Psi' = 0$$

such that demanding that

$$\Lambda_\mu^\nu\gamma^\mu = f_\Lambda^{-1}\gamma^\nu f_\Lambda$$

which imposes a constraint on f_Λ; thence, once again foreshadowing the results of a calculation later expounded upon in copiousness, we write the Dirac equation in IS′

$$\left[-\mathrm{i}\gamma^\nu\partial'_\nu + \frac{m_e c}{\hbar}\right]\Psi' = 0$$

demonstrating that the Dirac equation satisfies the tenets of special relativity and is Lorentz covariant in form.

Extrapolating the concepts of non-Euclidean space discoursed in the preamble of this introduction, we implement said notions to the principles of relativistic quantum mechanics. Not abandoning the notion of a Minkowskian space-time, i.e. flat-spacetime, but rather using the abstractions and theories founded upon this framework and translating them to curved spacetimes may we arrive at and analyze the behavaior of quantum particles in non-flat spacetime.

We have only considered the natural choice of coordinate basis on the spacetime manifold, with the basis vector being described by $\hat{e}_{(\mu)} = \partial_\mu$ and its dual (or covectors) given by $\hat{\theta}^\mu = dx^\mu$. Since all objects pondered are independent of any specific system of coordinates, we seek a distinct approach for establishing a basis. Using the *Principle of Equivalence* as a guide to cultivate our relativistic fermion theory in curved spacetime, we delineate the fact that this is a precedent for describing space-time as a Lorentzian manifold; the act that the spacetime is locally Minkowskian, we discern that the rules and principles established thus far for a particle moving at relativistic speeds must hold for small enough regions of *curved* spacetime. Thus, we commence by establishing a basis for each point in the curved spacetime, such that locally we may describe it by Minkowskian spacetime, leading to the realization that metric tensor shall be locally inertial when written in terms of the basis. We will define these basis vectors as $\hat{e}_{(a)}$ with corresponding dual vectors $\hat{\theta}^{(b)}$, with a Latin index. Hence, in the neighborhood of each point of spacetime we have,

$$g_{ab} = g_{ab}\hat{\theta}^{(a)}\hat{\theta}^{(b)} = \eta_{ab}$$

This constitutes an orthonormal set with respect to the Minkowskian metric, η. Such an orthonormal set is known as a *tetrad* or *vierbein*, with the practice of employing said technique upon the manifold is referred to as *the tetrad formalism*.

Briefly, we shall refer to a vector written in local indices, or Latin indices whose components are defined with respect to a local inertial basis, as a *local vector*, while vectors written with respect to the coordinate basis shall be referred to as *global vectors*. Moreover, we require that

$$\hat{\theta}^{(a)}\left(\hat{e}_{(b)}\right) = \delta_b{}^a$$

where ${\delta_b}^a$ is the Kronecker delta. Now, we may transform between the "old" coordinate basis and the local inertial basis by

$$\hat{e}_{(\mu)} = {e_\mu}^a \hat{e}_{(a)}$$

similarly for the dual basis

$$\hat{\theta}^{(\mu)} = {e^\mu}_b \hat{\theta}^{(b)}$$

Here ${e_\mu}^a$ and ${e^\mu}_b$ are transformation matrices, and we shall denote them as the *vierbein* and *inverse vierbein*, respectively. We point attention to the fact of the difference between the *vierbein* and the *inverse vierbein* where one may be distinguished from the other by the difference in the positioning of the indices. We have that

$${e_\mu}^a {e^\mu}_b = \delta^a_b \quad {e_\mu}^a {e^\nu}_a = {\delta_\mu}^\nu$$

such that they are inverses of each other. Consider the metric tensor in a global basis and transform it to a local basis. Said transformation will transform the metric to local coordinates, i.e.

$$g_{ab} = g_{\mu\nu} {e^\mu}_a \hat{\theta}^{(a)} {e^\nu}_b \hat{\theta}^{(b)} = \eta_{ab}$$

Suppressing the dual vectors we obtain

$$\eta_{ab} = {e^\mu}_a {e^\nu}_b g_{\mu\nu} \quad \text{and} \quad g_{\mu\nu} = {e_\mu}^a {e_\nu}^b \eta_{ab}$$

Generalizing this notion of rewriting a tensor to local coordinates to any arbitrary tensor, we obtain

$$V^a = {e_\mu}^a V^\mu$$

for the global vector V^μ, in a local basis. Generally, for a tensor we write

$$T^{\mu_1\cdots a\cdots\mu_k}{}_{\nu_1\cdots b\cdots\nu_l} = e_\mu{}^a e^\nu{}_b T^{\mu_1\cdots\mu\cdots\mu_k}{}_{\nu_1\cdots\nu\cdots\nu_l}$$

where we alternate between local and global indices by means of applying the transformation matrices. Now, by means of applying the tetrad formalsim, we may now deal with physical quantities that may only be described in a Minkowskian form.

We have now fulfilled the requirement we had previously imposed, i.e. establishing a basis that is locally Minkowskian at each point in the manifold. We take special interest in studying the manner in which a vector in the manifold parallel transports from one point to another in the same manifold, changing the manner in which its components transform form the local basis to the new basis. The answer to said dilemma comes in the form of a connection, namely *the spin connection*, which will inevitable allow us to study the “covariant derivative” of the spinor.

The equation for parallel transport of local vectors coincides with the form of global vectors, albeit embedded with the spin connection;

$$V^a(x \to x+dx) = V^a(x) - \omega_\mu{}^a{}_b(x) V^b(x) dx^\mu$$

where $\omega_\mu{}^a{}_b(x)$ denotes the spin connection, Within it encloses the information required to parallel transport the vector itself using the Christoffel symbols, as well as adjusting of local coordinates at the starting point to the end point. Inevitably, we arrive at the spin connection

$$\omega_\mu{}^a{}_b = e_\nu{}^a e^\sigma{}_b \Gamma^\nu{}_{\sigma\mu} + e_\nu{}^a \partial_\mu e^\nu{}_b$$

displaying an antisymmetry property,

$$\omega_{\mu ab} = -\omega_{\mu ba}$$

Analogous to the manner in which we examine the transformation properties of the Dirac spinor, Ψ, in flat spacetime, we consider a similar scheme for deriving a covariant derivative for the spinor in curved spacetime. Exploiting the fact that much of the foundation has been established for a Minkowskian template, we employ the tetrad formalism in order to connect distinct neighborhoods in a general curved spacetime using the spin connection. After which we may develop the covariant form of the Dirac equation. Nonetheless, we anticipate the form of the connection coefficient of the spinor field $\Omega_\mu(x)$,

$$\Omega_\mu = -\frac{1}{4}\mathrm{i}\omega_{\mu bc}\sigma^{bc} = \frac{1}{8}\omega_{\mu bc}\left[\gamma^b, \gamma^c\right]$$

Before arriving at the Dirac equation in a curved spacetime, we must first consider the gamma matrices which were defined in a Minkowskian form above. We require that these gamma matrices be written in terms of the global coordinates, hence, in order to mend the situation we contract indices using the inverse vierbein,

$$\gamma^\mu = e^\mu{}_a\gamma^a$$

Now, using the previously formulated formalisms and objects we may write the Dirac equation in a curved spacetime,

$$\left[\mathrm{i}e^\mu{}_a\gamma^a\left(\partial_\mu + \Omega_\mu\right) - m\right]\Psi = 0$$

Ultimately, we have arrived at the Dirac system in a curved spacetime.

It is the intent of this thesis to study such a relativistic fermion theory embedded within the de Sitter geometry; in doing so we seek to derive the explicit form of the spin connection for said spacetime, which will yield the explicit form of the Dirac equation. As demonstrated by (12) for the

FRW metric, we analyze the structure of the spin connection for the aforementioned metric

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{1-K\left(y^2+z^2\right)}{1-K\left(x^2+y^2+z^2\right)} & \frac{Kxy}{1-K\left(x^2+y^2+z^2\right)} & \frac{Kxz}{1-K\left(x^2+y^2+z^2\right)} \\ 0 & \frac{Kxy}{1-K\left(x^2+y^2+z^2\right)} & \frac{1-K\left(x^2+z^2\right)}{1-K\left(x^2+y^2+z^2\right)} & \frac{Kyz}{1-K\left(x^2+y^2+z^2\right)} \\ 0 & \frac{Kxz}{1-K\left(x^2+y^2+z^2\right)} & \frac{Kyz}{1-K\left(x^2+y^2+z^2\right)} & \frac{1-K\left(x^2+y^2\right)}{1-K\left(x^2+y^2+z^2\right)} \end{pmatrix}$$

which delineates the de Sitter geometry. After which we seek solutions to the resulting system of differential equations, which will detail the behavior of electrons within this considered spacetime.

The subsequent sections are laid out and expound upon as a foundation for the ultimate derivation of the Dirac system in a curved spacetime background. As a preliminary step, we consider fields of gravitation, basic concepts, and theories which inevitably lead to a derivation of *Einstein's field equations* and its solutions, among such the de Sitter or de Sitter-Schwarzschild solution. Properties and consequences of the ideologies are deliberated in depth. Next, we shift attention to fundamental principles of quantum mechanics and the foundations established in order to consider the theory in a concise and consistent manner. We consider preliminary theories, their shortcomings and deficiencies, with the imminent relativistic theory of the electron being the final product. Ramifications and features of the equation are analyzed under scrutiny. Finally, our work regarding the spin connection of the Dirac system in the aforementioned spacetime is considered.

CHAPTER II

GENERAL RELATIVITY

2.1 Gravitational fields in nonrelativistic mechanics

Gravitational fields, or fields of gravity have the basic property that all bodies move in them in the same manner, independent of mass, provided the initial conditions are equivalent. For example, the laws of free fall in the gravity field of the earth are the same for all bodies; regardless of their mass, all acquire one and the same acceleration. Such a property of gravitational fields allows the possibility of establishing an analogy between the motion of a body in a gravitational field and the motion of a body not located in any external field, but which is considered from the point of view of a noninertial reference system. Particularly, in an arbitrary reference system, the free motion of all bodies is uniform and rectilinear, and if the velocities coincide, they will be the same for all times. Hence, the properties of motion in a noninertial system are the same as those in an inertial system in the presence of a gravitational field. Namely, a noninertial reference system is equivalent to a certain gravitational field. This is called the *principle of equivalence.* We note that a uniformly accelerated system of reference is equivalent to a constant uniform external field.

We briefly expand on the intrinsic characteristics of fields of gravity and noninertial systems of reference. To this end, consider a body of arbitrary mass moving freely within a uniformly accelerated reference system. It follows, that the body has a constant acceleration, relative to this system, which is equal and opposite to the acceleration of the system itself, appropriately by Newton's Third Law of motion. We find that the same applies to motion within a uniform constant gravitational field, e.g. the gravitational field of Earth over small regions where we may consider the field uniform. Hence, as previously mentioned their exists a correlation between a uniformly accelerated reference system and a constant uniform external field. Analogously, for non-uniformly accelerated linear motion of the system of reference their is an equivalence to a uniform gravitational field.

Nonetheless, the fields to which noninertial systems of reference are equivalent to are not completely identical to “real” fields of gravity which occur in inertial reference frames. As their discrepancy arises when considering the behavior of each of the systems at infinity. At infinite distance from the bodies producing the field, “actual” fields of gravity tend to zero. On the contrary, fields to which noninertial reference systems correspond to increase without limit at infinity, or remain finite in value.

Fields corresponding to noninetial reference systems vanish upon transformation to an inertial reference system. Howbeit, “real” gravitational fields cannot be eliminated by any choice of reference system. All which can be done is by selection of a suitable choice of coordinates we may eliminate the gravitational field in a particular point of space, such that we may consider the field to be uniform over said point. This may be done by considering a system in accelerated motion, such that the acceleration of which is equal to that which would be acquired by the particle if placed in the region of the field considered.

We find that the motion of a particle in a gravitational field is defined, in nonrelativistic mechanics, by a Lagrangian of the form

$$L = \frac{mv^2}{2} - m\phi \tag{2.1}$$

where ϕ is regarded as the *gravitational potential*, which is a function of the coordinates and time which characterizes the field. Consequently, the equation of motion of the particle is

$$\dot{\mathbf{v}} = -\operatorname{grad}\phi \tag{2.2}$$

We note that the above does not contain the mass or any constant characterizing the intrinsic properties of the particle.

2.2 The gravitational field in relativistic mechanics

As described in the preceding section, the rudimentary property of gravitational fields that all bodies moves in them in the same manner remains pertinent in relativistic mechanics.

In an inertial reference system, in Cartesian Coordinates, the interval ds (the line element) is defined by the relation:

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 \tag{2.3}$$

Upon transformation to any other inertial reference system by manner of Lorentz transformation, the space-time interval retains its form, and thus is invariant upon Lorentz transformation. Nonetheless, if we transform to a noninertial reference system , ds^2 will no longer retains its form as the sum of squares of four coordinate differentials.

Consider the transformation to a uniformly rotating system of coordinates,

$$x = x'\cos\Omega t - y'\sin\Omega t, \quad y = x'\sin\Omega t + y'\cos\Omega t, \quad z = z' \quad t = t' \tag{2.4}$$

(Ω is the angular velocity of the rotation, directed along the Z axis), where we define the differential as

$$dx = \frac{\partial x}{\partial x'}dx' + \frac{\partial x}{\partial y'}dy' + \frac{\partial x}{\partial t'}dt' \tag{2.5}$$

Then,

$$dx = \cos\left(\Omega t'\right) dx' - \sin\left(\Omega t'\right) dy' - \Omega x' \sin\left(\Omega t'\right) dt' - \Omega y' \cos\left(\Omega t'\right) dt' \tag{2.6}$$

Similarly, we have

$$dy = \sin\left(\Omega t'\right) dx' + \cos\left(\Omega t'\right) dy' + \Omega x' \cos\left(\Omega t'\right) dt' - \Omega y' \sin\left(\Omega t'\right) dt' \tag{2.7}$$

We note that $dz = dz'$. Then by using Eq.(2.3) we square Eqs.(2.6) & (2.7) such that

$$dx^2 = \cos^2(\Omega t)\, dx'^2 - \sin(\Omega t) dy' \cos(\Omega t) dx' - \Omega x' \sin(\Omega t)\, dt \cos(\Omega t) dx' - \Omega y' \cos(\Omega t)\, dt \cos(\Omega t) dx'$$
$$- \sin(\Omega t) dy' \cos(\Omega t)\, dx' + \sin^2(\Omega t) dy'^2 + \Omega x' \sin(\Omega t) dt \sin(\Omega t) dy' + \Omega y' \cos(\Omega t) dt \sin(\Omega t) dy'$$
$$-\Omega x' \sin(\Omega t) dt \cos(\Omega t) dx' + \Omega x' \sin(\Omega t) dt \sin(\Omega t) dy' + \Omega^2 x'^2 \sin^2(\Omega t) dt^2 + \Omega x' \sin(\Omega t) dt \Omega y' \cos(\Omega t) dt$$
$$-\Omega y' \cos(\Omega t) dt \cos(\Omega t) dx' + \Omega y' \cos(\Omega t) dt \sin(\Omega t) dy' + \Omega y' \cos(\Omega t) dt \Omega x' \sin(\Omega t) dt + \Omega^2 y'^2 \cos^2(\Omega t) dt$$

and

$$dy^2 = \sin^2(\Omega t) dx'^2 + \sin(\Omega t) dx' \cos(\Omega t) dy' + \sin(\Omega t) dx' \Omega x' \cos(\Omega t) dt - \sin(\Omega t) dx' \Omega y' \sin(\Omega t) dt$$
$$+ \cos(\Omega t) dy' \sin(\Omega t) dx' + \cos^2(\Omega t) dy'^2 + \Omega x' \cos(\Omega t) dt \cos(\Omega t) dy' - \Omega y' \sin(\Omega t) \cos(\Omega t) dy'$$
$$+\Omega x' \cos(\Omega t) dt \sin(\Omega t) dx' + \Omega x' \cos(\Omega t) dt \cos(\Omega t) dy' + \Omega^2 x'^2 \cos^2(\Omega t) dt^2 -$$
$$\Omega x' \cos(\Omega t) dt \Omega y' \sin(\Omega t) dt$$
$$-\Omega y' \sin(\Omega t) dt \sin(\Omega t) dx' - \Omega y' \sin(\Omega t) dt \cos(\Omega t) dy' - \Omega y' \sin(\Omega t) dt \Omega x' \cos(\Omega t) dt +$$
$$\Omega^2 y' \sin^2(\Omega t) dt^2$$

After algebraic manipulations we're left with

$$-\left[\cos^2(\Omega t)dx'^2+\sin(\Omega t)dx'^2\right]-\left[\sin^2(\Omega t)dy'+\cos^2(\Omega t)dy'^2\right]-$$
$$\left[\Omega^2x'^2\sin^2(\Omega t)dt^2+\Omega^2x'^2\cos^2(\Omega t)dt^2\right]-\left[\Omega^2y'^2\cos^2(\Omega t)dt+\Omega^2y'^2\sin^2(\Omega t)dt^2\right]$$
$$\implies -dx'^2-dy'^2-\Omega^2x'^2dt^2-\Omega^2y'^2dt^2$$
$$=-dx'^2-dy'^2-\Omega^2\left[x'^2-y'^2\right]dt^2$$

as well as

$$-\Omega y'\cos(\Omega t)dt\cos(\Omega t)dx'-\sin(\Omega t)dx'\Omega y'\sin(\Omega t)dt$$
$$+\Omega x'\sin(\Omega t)dt\sin(\Omega t)dy'+\Omega x'\cos(\Omega t)dt\cos(\Omega t)dy'$$
$$+\Omega x'\sin(\Omega t)dt\sin(\Omega t)dy'+\Omega x'\cos(\Omega t)dt\cos(\Omega t)dy'$$
$$-\Omega y'\cos(\Omega t)\,dt\cos(\Omega t)dx'-\Omega y'\sin(\Omega t)dt\sin(\Omega t)dx'$$

such that after collecting like terms,

$$-2\Omega y'\cos(\Omega t)dt\cos(\Omega t)dx'--2\Omega y'\sin(\Omega t)dx'\sin(\Omega t)dt$$
$$=-2\Omega y'\cos^2(\Omega t)dx'dt--2\Omega y'\sin^2(\Omega t)dx'dt$$
$$=-2\Omega y'\left(\cos^2(\Omega t)\right)+\sin^2(\Omega t))dx'dt$$
$$=-2\Omega y'dx'dt \implies 2\Omega y'dx'dt$$

Similarly, for $2\Omega x'\sin(\Omega t)dt\sin(\Omega t)dy'+2\Omega x'\cos(\Omega t)dt\cos(\Omega t)dy'$ we obtain

$$2\Omega x'dy'dt \implies -2\Omega x'dy'dt$$

Finally, we obtain the result

$$ds^2 = \left[c^2 - \Omega^2\left(x'^2 + y'^2\right)\right]dt^2 - dx'^2 - dy'^2 - dz'^2 + 2\Omega y' dx' dt - 2\Omega x' dy' dt \tag{2.8}$$

There-holds, that in a noninertial reference system the square of the line interval takes a quadratic form of general type in the coordinate differentials, i.e.

$$ds^2 = g_{ik} dx^i dx^k \tag{2.9}$$

where the g_{ik} are functions of the space coordinates x^1, x^2, x^3 and the time coordinate x^0. Hence, we find that when considering a noninertial system, the four-dimensional coordinate system x^0, x^1, x^2, x^3 is curvilinear. The quantities of g_{ik} determine all geometric properties in each of the curvilinear system of coordinates and is called the $space-time\ metric$.

The quantities g_{ik} may always be considered symmetric in the indices i and k ($g_{ik} = g_{ki}$), since they are determined from the symmetric form (2.9), in which g_{ik} and g_{ki} enter as factors of one and the same product $dx^i dx^k$. In general, there are ten distinct quantities g_{ik} - four with equal and six with different indices. When considering an inertial reference system, i.e. when using Cartesian space coordinates $(x^{1,2,3}) = (x, y, z)$ and time $x^0 = ct$, the quantities of the space-time metric are given by

$$g_{00} = 1, \quad g_{11} = g_{22} = g_{33} = -1, \quad g_{ik} = 0 \quad \text{for} \quad i \neq k \tag{2.10}$$

in which we denote any four-dimensional system of coordinates with these values as its metric as $galilean$.

It has been previously shown that a noninertial reference system is equivalent to a given field of force. We now see that in relativistic mechanics, these fields are determined by g_{ik}, with a similar argument made for *actual* fields of gravitation. Any gravitational field is just a change in the space-time metric, which as previously mentioned is determined by the quantities of g_{ik}. Thus, we note that the geometrical properties of space-time are determined by physical phenomena, and are not fixed properties of time and space.

There exists a fundamental difference between *actual* gravitational fields and the fields to which noninertial reference systems are equivalent to. It is possible, by means of a transformation to a noninertial reference system, that the quadratic form (2.9), i.e. the quantities of g_{ik} which are obtained from their galilean values (2.10) by a simple transformation of coordinates, may be reduced over all of space to their galilean values by the inverse coordinate transformation. On the other hand, an "actual" gravitational field cannot be eliminated by any coordinate transformation. Namely, in the presence of a gravitational field the quantities g_{ik} determining its metric cannot, by any transformation of coordinates, be brought to their gaillean values over all space. Such a space-time is said to be *curved*, as opposed to $flat$ space-time, in which such a reduction is permissible. Alternatively, it is possible, by an appropriate choice of coordinates, to bring the quantities of g_{ik} to galilean form at any individual point of the curved spac-time; which amounts to the reduction to diagonal form of a quadratic form with constant coefficients.Such a system is said to be *galilean for the given point.*

Note that after the reduction to diagonal form at a given point, the matrix g_{ik} has one positive and three negative principal values. From this it follows that the determinant g formed from the quantities of g_{ik} is always negative for a real space-time:

$$g < 0 \tag{2.11}$$

There-holds that a change in the space-time metric refers to a change in the in the purely spatial metric. To the galilean space-time metric g_{ik} in flat space-time, there corresponds a euclidean geometry. However, in a gravitational field, the geometry of space in non-euclidean in nature; applying both to "actual" gravitational fields, where space-time is *curved*, as well as to fields resulting from the fact that the system under consideration of non-inertial, which leaves the space-time flat.

For an arbitrary, varying gravitational field, the metric of space not only non-euclidean,but also varies with time, meaning that the relation between different geometrical distances changes with time; hence, the relative position of "test bodies: introduced into the field cannot remain unchanged in any coordinate system, i.e. changing as a result of the dependence to the time coordinate. Thus in the general theory of relativity it is impossible in general to have a system of bodies which are fixed relative to one another.

In connection with the arbitrariness of the choice of a reference system, the laws of nature must be described in the general theory of relativity in a manner which is appropriate to any four-dimensional system of coordinate, i.e. "covariant" in form.

2.3 Curvilinear coordinates

Through the necessity of studying gravitational fields in an arbitrary reference frame, it is essential that we construct four-dimensional geometry in arbitrary curvilinear coordinates.

To this end, let us consider the transformation from one coordinate system, x^0, x^1, x^2, x^3, to another, x'^0, x'^1, x'^2, x'^3, related through:

$$x^i = f^i\left(x'^0, x'^1, x'^2, x^3\right) \tag{2.12}$$

in which f^i are functions to be determined. When we transform said coordinates, the respective differentials transform according to

$$dx^i = \frac{\partial x^i}{\partial x'^k} dx'^k \tag{2.13}$$

Every agglomerate of four quantities $A^i (i = 0, 1, 2, 3)$, which mimics the manner of transformation of the coordinate differentials, we denote as a *contravariant* four-vector:

$$A^i = \frac{\partial x^i}{\partial x'^k} A'^k \tag{2.14}$$

Consider the scalar ϕ. Under coordinate transformation, the four quantities $\partial\phi/\partial x^i$ transform according to the formula

$$\frac{\partial \phi}{\partial x^i} = \frac{\partial \phi}{\partial x'^k}\frac{\partial x'^k}{\partial x^i} \tag{2.15}$$

where chain-rule was employed and we note that this formula differs in form to formula Eq.(2.14). We define every agglomerate of four quantities A_i which transforms according to Eq.(2.15) under a coordinate transformation as a *covariant* four-vector:

$$A_i = \frac{\partial x'^k}{\partial x^i} A'_k \tag{2.16}$$

Next, due to the fact that there are two types of vectors in curvilinear coordinates, there are three types of 2^{nd} rank tensors. We denote a *contravariant tensor* of second rank A^{ik}, as an aggregate of sixteen quantities which transforms like the product of the components of two contravariant vectors:

$$A^{ik} = \frac{\partial x^i}{\partial x'^l}\frac{\partial x^k}{\partial x'^m} A'^{lm} \tag{2.17}$$

Similarly, we define a second rank *covariant tensor* as an aggregate of sixteen quantities which transforms according to the formula

$$A_{ik} = \frac{\partial x'^l}{\partial x^i}\frac{\partial x'^m}{\partial x^k} A'_{lm} \tag{2.18}$$

and *mixed tensor* transforms according to:

$$A^i_k = \frac{\partial x^i}{\partial x'^l}\frac{\partial x'^m}{\partial x^k} A'^l_m \tag{2.19}$$

The rules for creating four-tensors through multiplication or contraction of products of other four-tensors remains the same in curvilinear coordinates as they were in galilean coordinates. For example, by using the transformation laws of the contravarinat and covariant tensors Eqs.(2.14) and (2.16), we see that the scalar product of two four-vectors A^iB_i is invariant under such transformation:

$$A^iB_i = \frac{\partial x^i \partial x'^m}{\partial x'^l \partial x^i} A'^l B'_m = \frac{\partial x'^m}{\partial x'^l} A'^l B'_m = A'^l B'_l \tag{2.20}$$

We define the unit four-tensor δ^i_k with components $\delta^i_k = 0$ for $i \neq k$ and are equal to 1 if $i = k$. If A^k is a four-vector, then multiplication with δ^i_k we obtain:

$$A^k \delta^i_k = A^i \tag{2.21}$$

which is another four-vector; this proves that δ^i_k is a tensor.

The square of the line element in curvilinear coordinates takes a quadratic form in the differentials dx^i:

$$ds^2 = g_{ik} dx^i dx^k \tag{2.22}$$

where g_{ik} are functions of the coordinates; g_{ik} is symmetric in the indices i and k such that

$$g_{ik} = g_{ki} \tag{2.23}$$

The contracted product of g_{ik} and the contravariant tensors $dx^i dx^k$ is a scalar, then g_{ik} forms a covariant tensor which is called the *metric tensor*. We say that two tensors A_{ik} and B^{ik} are said to be *reciprocal* to each other if

$$A_{ik} B^{kl} = \delta^l_i \tag{2.24}$$

The contravariant metric tensor is the tensor g^{ik} which is the reciprocal to the tensor g_{ik}, such that

$$g_{ik} g^{kl} = \delta^l_i \tag{2.25}$$

The components of the metric tensor can then determine the connection between the contra- or co-variant components of the same physical phenomena, which is given by

$$A^i = g^{ik} A_k, \quad A_i = g_{ik} A^k \tag{2.26}$$

The components of the metric tensor in a galilean coordinate system take the values of

$$g_{ik}^{(0)} = g^{(0)ik} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{2.27}$$

Then using formula (2.26) we have the contravariant/covariant relation $A^0 = A_0, A^{1,2,3} = -A_{1,2,3}$, etc. Analogously, the same definitions apply to tensors, in which the transition between tenors of the same physical quantity is fulfilled using the metric tensor according to:

$$A_k^i = g^{il} A_{lk}, \quad A^{ik} = g^{il} g^{km} A_{lm} \tag{2.28}$$

In passing, we define the completely antisymmetric unit pseudotensor e^{iklm}, in which we obtain

$$\varepsilon_{ijk} = \begin{cases} 0, \text{ if any two labels are the same} \\ 1, \text{ if } i, j, k \text{ is an even permutation of } 1,2,3 \\ -1, \text{ if } i, j, k \text{ is an odd permutation of } 1,2,3 \end{cases} \tag{2.29}$$

Let us transform it to an arbitrary system of coordinates and denote it by E^{iklm}. Let x'^i be galilean and x^i be arbitrary curvilinear coordinates. According to the general rules for transformations of tensors, we have:

$$E^{iklm} = \frac{\partial x^i}{\partial x'^p} \frac{\partial x^k}{\partial x'^r} \frac{\partial x^l}{\partial x'^s} \frac{\partial x^m}{\partial x'^t} e^{prst} \tag{2.30}$$

or

$$E^{iklm} = Je^{iklm} \tag{2.31}$$

in which J is the Jacobian of transformation:

$$J = \frac{\partial\left(x^0, x^1, x^2, x^3\right)}{\partial\left(x'^0, x'^1, x'^2, x'^3\right)} \tag{2.32}$$

We can then express the Jacobian in terms of the determinant of the metric tensor g_{ik}; to do this we write the formula for the transformation of the metric tensor:

$$g^{ik} = \frac{\partial x^i}{\partial x'^l}\frac{\partial x^k}{\partial x'^m} g^{(0)lm} \tag{2.33}$$

and equate the determinants of the two sides of the equation. We denote the determinant of the reciprocal tensor $\left|g^{ik}\right| = 1/g$ and the determinant of $|g^{(0)lm}| = -1$. Hence we have $1/g = -J^2$ and so $J = 1/\sqrt{-g}$. Hence, we find that in curvilinear coordinates the antisymmetric unit tensor of rank four must be defined as follows

$$E^{iklm} = \frac{1}{\sqrt{-g}} e^{iklm} \tag{2.34}$$

The indices of this tensor are lowered by using the formula

$$e^{prst} g_{ip} g_{kr} g_{ls} g_{mt} = -g e_{iklm} \tag{2.35}$$

such that its covariant components are

$$E_{iklm} = \sqrt{-g} e_{iklm} \tag{2.36}$$

Now when considering the integral of a scalar in a galilean coordinate system x'^i with respect to $d\Omega' = dx'^0 dx' dx'^2 dx'^3$ is also a scalar. Upon transformation to curvilinear coordinates x^i, the elements of integration $d\Omega'$ goes over into

$$d\Omega' \rightarrow \frac{1}{J} d\Omega = \sqrt{-g} d\Omega \tag{2.37}$$

Hence, in curvilinear coordinates, when integrating over a four-volume the quantity $\sqrt{-g} d\Omega$ behaves as an invariant. It is of upmost importance that we delineate the transformation of an integral over a hypersurface into an integral over a four-volume, i.e. Gauss' theorem, which is fulfilled by the substitution

$$dS_i \rightarrow d\Omega \frac{\partial}{\partial x^i} \tag{2.38}$$

2.4 Distances and time intervals

Next we pose the question of how to define actual distance and time intervals, knowing that the choice of coordinates in the general theory of relativity is not limited in any manner. Let us first consider the relation of "proper time", denoted by τ, to the time coordinate x^0. To this extent, consider two infinitesimally separated events occurring within the same point in space. Then, we find that the space-time interval ds between the corresponding events is defined only by $cd\tau$, in which $d\tau$ is the proper time interval between these events. Since, the events happen in one and the same point in space we may set $dx^1 = dx^2 = dx^3 = 0$ in the expression $ds^2 = g_{ik} dx^i dx^k$, we thus find

$$ds^2 = c^2 d\tau^2 = g_{00} \left(dx^0\right)^2 \tag{2.39}$$

which follows immediately from the definition of the interval ds, since setting $dx^1 = dx^2 = dx^3 = 0$ leaves only the time-like coordinate, which consequently is determined by g_{00} and setting $dx^i = dx^k = dx^0$, we obtain $(dx^0)^2$. Then we may write,

$$d\tau = \frac{1}{c} \sqrt{g_{00}} dx^0 \tag{2.40}$$

Indeed, we have

$$
\begin{aligned}
ds^2 = c^2 d\tau^2 &= g_{00}\left(dx^0\right)^2 \\
\implies d\tau^2 &= \frac{1}{c^2} g_{00}\left(dx^0\right)^2 \\
\implies d\tau &= \frac{1}{c}\sqrt{g_{00}}\left(dx^0\right)
\end{aligned}
$$

The relation determines "actual time interval" (or *proper time* for the given point of space) for a change of the time-like coordinate x^0. We note in passing that the quantity g_{00} is positive:

$$g_{00} > 0 \tag{2.41}$$

We emphasize the distinction between (2.41) and the meaning of the signature of the space-time metric (or the principal values of g_{ik}). A tensor g_{ik} which does not satisfy the signature or principal value of the metric tensor cannot correspond to any real gravitational field; one the other hand non-fulfillment of (2.41) leads to the fact that corresponding system of reference cannot be realized by any real bodies; on the contrary, if the condition of the principal values is satisfied, then a suitable transformation of coordinates can transform g_{00} a positive value.

We proceed by determining the spatial distance element dl. We note that contrary to theory of special relativity, in which the interval dl is defined in such a way as to consider two infinitesimally separated events occurring at one and the same time. Nonetheless, in the general theory of relativity, it is impossible to determine dl by setting $dx^0 = 0$ in ds. This is due to the fact that with in a gravitational field the proper time at different points in space has a distinct dependence on the x^0 coordinate.

To determine dl we proceed as follows. Assume a light signal is shone from some point B in space (with coordinate s$x^\alpha + dx^\alpha$) to an infinitely near point A (with coordinates x^α) and then back over the same path. Writing the space-time interval and separating space and time coordinates:

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta + 2g_{0\alpha}dx^0 dx^\alpha + g_{00}\left(dx^0\right)^2 \tag{2.42}$$

where we note that the events have the same time coordinate for both points A and B, thus leading to two dx^0 and where we employ the Einstein Summation Convention over repeated Greek indices from 1 to 3. The interval between the events corresponding to the departure and arrival of light from one point to the other is zero. Hence, solving the equation $ds^2 = 0$ with respect to dx^0, we find two roots:

$$dx^{0(1)} = \frac{1}{g_{00}}\left\{-g_{0\alpha}dx^\alpha - \sqrt{\left(g_{0\alpha}g_{0\beta} - g_{\alpha\beta}g_{00}\right)dx^\alpha dx^\beta}\right\}$$

$$dx^{0(2)} = \frac{1}{g_{00}}\left\{-g_{0\alpha}dx^\alpha + \sqrt{\left(g_{0\alpha}g_{0\beta} - g_{\alpha\beta}g_{00}\right)dx^\alpha dx^\beta}\right\} \tag{2.43}$$

Indeed, we need only consider Eq.(2.43) in order to verify that the previous two formulas are truly the roots of the equation $ds^2 = 0$. Thus, we proceed as follows:

$$dx^{0(2)} = \frac{1}{g_{00}}\left\{-g_{0\alpha}dx^\alpha + \sqrt{\left(g_{0\alpha}g_{0\beta} - g_{\alpha\beta}g_{00}\right)dx^\alpha dx^\beta}\right\}$$

$$\Longrightarrow\ g_{00}dx^{0(2)} = \left\{-g_{0\alpha}dx^\alpha + \sqrt{\left(g_{0\alpha}g_{0\beta} - g_{\alpha\beta}g_{00}\right)dx^\alpha dx^\beta}\right\}$$

$$\Longrightarrow\ g_{00}dx^{0(2)} + g_{0\alpha}dx^\alpha = \left\{\sqrt{\left(g_{0\alpha}g_{0\beta} - g_{\alpha\beta}g_{00}\right)dx^\alpha dx^\beta}\right\}$$

$$\Longrightarrow\ \left(g_{00}dx^{0(2)} + g_{0\alpha}dx^\alpha\right)^2 = \left\{\sqrt{\left(g_{0\alpha}g_{0\beta} - g_{\alpha\beta}g_{00}\right)dx^\alpha dx^\beta}\right\}^2$$

$$\Longrightarrow\ g_{00}^2 dx^{0(2)^2} + 2g_{00}g_{0\alpha}dx^0 dx^\alpha + g_{0\alpha}^2 dx^{\alpha^2} = \left(g_{0\alpha}g_{0\beta} - g_{\alpha\beta}g_{00}\right)dx^\alpha dx^\beta$$

$$\Longrightarrow\ g_{00}^2 dx^{0(2)^2} + 2g_{00}g_{0\alpha}dx^0 dx^\alpha + g_{0\alpha}^2 dx^{\alpha^2} = g_{0\alpha}g_{0\beta}dx^\alpha dx^\beta - g_{\alpha\beta}g_{00}dx^\alpha dx^\beta$$

$$\Longrightarrow\ g_{00}^2 dx^{0(2)^2} + 2g_{00}g_{0\alpha}dx^0 dx^\alpha + g_{0\alpha}^2 dx^{\alpha^2} - g_{0\alpha}g_{0\beta}dx^\alpha dx^\beta + g_{\alpha\beta}g_{00}dx^\alpha dx^\beta$$

$$\Longrightarrow\ g_{00}^2 dx^{0(2)^2} + 2g_{00}g_{0\alpha}dx^0 dx^\alpha + g_{0\alpha}^2 dx^{\alpha^2} - g_{0\alpha}^2 dx^{\alpha^2} + g_{\alpha\beta}g_{00}dx^\alpha dx^\beta$$

$$\Longrightarrow\ g_{00}^2 dx^{0(2)^2} + 2g_{00}g_{0\alpha}dx^0 dx^\alpha + g_{\alpha\beta}g_{00}dx^\alpha dx^\beta$$

$$\Longrightarrow\ g_{\alpha\beta}dx^\alpha dx^\beta + 2g_{0\alpha}dx^0 dx^\alpha + g_{00}\left(dx^0\right)^2$$

which is exactly (2.42). These two roots corresponding to the propagation of the signal in the two directions between A and B. The in x^0 is the time of arrival of the signal at A, the times when it left B and returns to B are, respectively, $x^0 + dx_0^1$ and $x^0 + dx_0^2$. We may see that the total interval of "time" between the departure and return of the light signal to its original point is given by

$$dx^{0(2)} - dx^{0(1)} = \frac{2}{g_{00}}\sqrt{\left(9_{0a}g_{0\beta} - g_{\alpha\beta}g_{00}\right)dx^{\alpha}dx^{\beta}} \tag{2.44}$$

Then the corresponding interval of proper time is obtained by multiplying by $\sqrt{g_{00}}/c$, while the distance element dl between two points is determined by multiplying once more by $c/2$. Thus, we have

$$dl^2 = \left(-g_{\alpha\beta} + \frac{g_{0\alpha}g_{0\beta}}{g_{00}}\right)dx^{\alpha}dx^{\beta} \tag{2.45}$$

Here we find the required expression, defining the distance in terms of the space coordinate elements. We may rewrite said equation in the form

$$dl^2 = \gamma_{\alpha\beta}dx^{\alpha}dx^{\beta} \tag{2.46}$$

in which

$$\gamma_{\alpha\beta} = \left(-g_{\alpha\beta} + \frac{g_{0a}g_{0\beta}}{g_{00}}\right) \tag{2.47}$$

is the $3-D$ metric tensor, determining the metric, i.e. the geometric properties of space. The relations (2.47) denotes the link between the metric of real space (three dimensional) and the metric of the four-dimensional space-time.

Nonetheless, we note the dependence of the space-time metric on the time-like coordinate, such that the space metric (2.46) changes with time. For this reason, integrating dl is futile; since such an integral would depend on the world line chosen between two given space points. Hence, in the theory of general relativity the concept of a definite distance between bodies loses its meaning, remaining valid only for infinitesimal distances. The only other case in which the distance can be defined over a finite domain is in which g_{ik} does not depend on time, such that the integral $\int dl$ taken along a space curve has a definite meaning.

We note that the tensor $-\gamma_{\alpha\beta}$ is the reciprocal o the contravariant three-dimensional tensor $g^{\alpha\beta}$. In fact, recall the identity $g^{ik}g_{kl} = \delta^i_l$, where we obtain

$$g^{\alpha\beta}g_{\beta\gamma} + g^{\alpha 0}g_{0\gamma} = \delta^{\alpha}_{\gamma,}, \quad g^{\alpha\beta}g_{\beta 0} + g^{\alpha 0}g_{00} = 0, \quad g^{0\beta}g_{\beta 0} + g^{00}g_{00} = 1 \qquad (2.48)$$

The quantities $-g^{\alpha\beta}$ form the contravariant $3-D$ metric tensor corresponding to the metric (2.46):

$$\gamma^{\alpha\beta} = -g^{\alpha\beta} \qquad (2.49)$$

Then we can write the relation

$$-g^{\alpha\beta}\gamma_{\beta y} = \delta^{\alpha}_{\gamma} \qquad (2.50)$$

The determinants of the space-time metric and space metric are related to one another by

$$-g = g_{00}\gamma \qquad (2.51)$$

Define the three dimensional vector **g** with covariant components

$$g_{\alpha} = -\frac{g_{0\alpha}}{g_{00}} \qquad (2.52)$$

We then proceed to define its contravariant components as $g^{\alpha} = \gamma^{a}g_{\beta}$. Utilizing the relation between the space metric and the space-time metric (2.49) and the second equations of (2.48), we see that

$$g^{\alpha} = \gamma^{\alpha\beta}g_{\beta} = -g^{0\alpha} \qquad (2.53)$$

and by using the third equation of (2.48) we obtain

$$g^{00} = \frac{1}{g_{00}} - g_\alpha g^\alpha \tag{2.54}$$

We now explore the notion of simultaneity in the theory of general relativity, i.e. we discuss the question of the possibility of synchronizing clocks at different points in space, hence, crafting a correspondence between the readings of the clocks.

Such an avenue is explored by means of an exchange of light signals between two points. Analogously, we consider the process of light signal propagation between two infinitesimally separated points A and B. We regard as simultaneous with the moment x^0 at point A the reading of the clock at point B which is half-way between the moments of departure and return of the signal to that point, i.e. the moment

$$x^0 + \Delta x^0 = x^0 + \frac{1}{2}\left(dx^{0(2)} + dx^{0(1)}\right)$$

Substitution of Eq.(2.43), we determine that the difference in the values of the "time" x^0 for two simultaneous events happening at infinitely near points is given by

$$\Delta x^0 = -\frac{g_{0\alpha} dx^\alpha}{g_{00}} \equiv g_\alpha dx^\alpha \tag{2.55}$$

which then enables us to synchronize clocks at any infinitesimal region of space. Repeating the procedure described above for point A, we find that we can synchronize clocks, i.e. we can define simultaneity of events, along any open curve.

Nonetheless, we find that synchronization of clocks along a closed contour turns out to be impossible in general; commencing at one point in the close contour and returning to the initial point, we find that the value of Δx^0 is different from zero. Hence, it is, *a fortiori*, impossible to synchronize clocks over all of space, exceptions including those reference systems in which all the components of $g_{0\alpha}$ are equal to zero.

We delineate the fact that the impossibility of the synchronization of clocks is embedded withing the arbitrariness of the reference system and not space-time itself. In any gravitational field, it is possible to select the reference system in such a way that the quantities of $g_{0\alpha}$ become zero, hence, making a complete synchronization of clocks possible. In the general theory of relativity, proper time elapses distinctly even at different points of space in the same reference system, i.e. the interval of proper time between two events in one point in space and the interval of time between two distinct events simultaneous with the previous two at another point in space, are in general different from one another.

2.5 Covariant differentiation

In Galilean coordinates we find that the differentials of a vector A_i form a new vector and the derivatives of said vector $\frac{\partial A_i}{\partial x^k}$ with respect to the coordinates form a tensor. For curvilinear coordinates we find that such is not the case. The differential of the vector A_i does not form a vector and the corresponding derivative is not a tensor. This is due to the fact that the differential dA_i is the difference of vectors located at distinct points of space; at different points we have found that space vectors transform differently, since the coefficients of (2.14) and (2.16) are functions of the coordinates.

Notice that we must know define the formulas for the differentials dA_i in curvilinear coordinates. To this end, we start with the transformation formula of a covariant vector

$$A_i = \frac{\partial x'^k}{\partial x^i} A'_k$$

therefore

$$dA_i = \frac{\partial x'^k}{\partial x^i} dA'_k + A'_k d\frac{\partial x'^k}{\partial x^i} = \frac{\partial x'^k}{\partial x^i} dA'_k + A'_k \frac{\partial^2 x'^k}{\partial x^i \partial x^l} dx^l$$

where we employed the product rule for derivatives applied to differentials.

It is apparent that dA_i does not transform like a vector (the same applies to the differential of a contravariant vector). Then if the second derivative $\frac{\partial^2 x'^k}{\partial x^i \partial x^l} dx^l = 0$, do the transformation formula have the form

$$\frac{\partial x'^k}{\partial x^i} dA'_k$$

hence, the differential transforms like a vector.

We now define the definition of a tensor, which in curvilinear coordinates plays he same role as $\frac{\partial A_i}{\partial x^k}$ in galilean coordinates. Roughly speaking, we must translate $\frac{\partial A_i}{\partial x^k}$ from galilean to curvilinear coordinates.

In curvilinear coordinates, in order to obtain the differential of any vector which behaves like a vector, it is imperative that the two vector to be subtracted by located at the same point in space, i.e. we must "translate" one of the vectors, located at an infinitesimal distance from the second vector, to the point where the second vector is positioned. This translation operation must be defined in such a manner that in galilean coordinates the difference reduces to an ordinary differential dA_i. Since dA_i is defined as the difference between the components of the two separated vectors, this means that upon using galilean coordinates the components of the vector should remain the same after translation; this translation is the translation of a vector paralleled to itself. Nonetheless, upon implementation of such a *parallel translation* in curvilinear coordinates we find that the components of the vector do not remain invariant upon undertaking said translation, since after translation of the vector the difference between the components of the two vectors will not concur with the difference before such a movement.

Hence, to compare two infinitesimally separated vectors we must subject one to parallel translation to the position in which the second is found. Consider an arbitrary contravariant vector with value at x^i of A^i, then the neighboring point, found at an infinitesimal distance, is given by $x^i + dx^i$ with value $A^i + dA^i$. We subject vector A^i to an infinitesimal parallel displacement $x^i + dx^i$; where the change in the vector is denoted by δA^i. The difference DA^i between the two vectors positioned at the same points in space is

$$DA^i = dA^i - \delta A^i \tag{2.56}$$

The change δA^i in the components of the vector under the infinitesimal parallel displacement depends on the values of the components themselves, which is a liner dependence. This follows from the fact that the sum of two vectors must transform according to the same law as each of the constituents. Thus the change takes the form

$$\delta A^i = -\Gamma^i_{kl} A^k dx^l \tag{2.57}$$

where Γ^i_{kl} are functions of the coordinates, where their form depends on the coordinate system; for a galilean coordinates we note $\Gamma^i_{kl} = 0$. From this fact it follows that the quantities Γ^i_{kl} do not form a tensor, since if a tensor is equal to zero in one system of coordinates it must also be equal to zero in every other coordinate system. In curvilinear space we find that it is in fact impossible to make all the Γ^i_{kl} over all of space, but we can, however, choose a coordinate system in which these coefficients vanish over a given infinitesimal region. The coefficients Γ^i_{kl} are called *Christoffel symbols*. Aside from these quantities we shall also use the coefficients $\Gamma_{i,kl}$ defined by:

$$\Gamma_{i,kl} = g_{im} \Gamma^m_{kl} \tag{2.58}$$

Conversely,

$$\Gamma^i_{kl} = g^{im} \Gamma_{m,kl} \tag{2.59}$$

Next, we determine a relation for the variance in the components of a covariant vector under parallel displacement to the Christoffel symbols. To do this we note that under a parallel displacement, a scalar is invariant. In particular, the scalar/dot product of two vectors under parallel displacements is unchanged.

To this end, consider two vectors, one covariant and one contravariant, A_i and B^i. Then from the $\delta(A_iB^i) = 0$ (the change in the scalar product is unchanged), we have,

$$B^i\delta A_i = -A_i\delta B^i = \Gamma^i_{kl}B^kA_idx^l$$

or, changing indices,

$$B^i\delta A_i = \Gamma^k_{il}A_kB^idx^l \tag{2.60}$$

(which translates to the scalar product of a contravariant vector with a covariant vector which has been subjected to parallel displacement). From this, in light of the arbitrary nature of the contravariant B^i,

$$\delta A_i = \Gamma^k_{il}A_kdx^l \tag{2.61}$$

which determines the change in a covariant vector under a parallel displacement. Substituting (2.457) and $dA^i = \left(\partial A^i/\partial x^l\right)dx^l$ in (2.56), we obtain

$$DA^i = \left(\frac{\partial A^i}{\partial x^l} + \Gamma^i_{kl}A^k\right)dx^l \tag{2.62}$$

Indeed,

$$\begin{aligned} DA^i &= dA^i - \delta A^i \\ \Longrightarrow &\left(\frac{\partial A^i}{\partial x^l}\right)dx^l + \Gamma^i_{kl}A^kdx^l \\ \Longrightarrow &\left(\frac{\partial A^i}{\partial x^l} + \Gamma^i_{kl}A^k\right)dx^l \end{aligned}$$

for the covariant vector we find,

$$DA_i = \left(\frac{\partial A_i}{\partial x^l} - \Gamma^k_{il}A_k\right)dx^l \tag{2.63}$$

Formulas (2.62) and (2.63) are tensors, since multiplication with the vector dx^l yields a vector.

Hence, we see that the tensors found are the desired generalization of the concept of a derivative to curvilinear coordinates, we denote these tensors as the *covariant derivatives* of the vectors A_i and A^i, respectively. We denote them by $A^i_{;k}$ and $A_{i;k}$. Therefore,

$$DA^i = A^i_{;l} dx^l; \qquad DA_i = A_{i;l} dx^l \tag{2.64}$$

where the covariant derivatives are given by

$$A^i_{;l} = \frac{\partial A^i}{\partial x^l} + \Gamma^i_{kl} A^k \tag{2.65}$$

$$A_{i;l} = \frac{\partial A_i}{\partial x^l} - \Gamma^k_{il} A_k \tag{2.66}$$

In galilean coordinates, the Christoffel symbols vanish, i.e. $\Gamma^i_{kl} = 0$, and the covariant derivatives reduce to ordinary derivatives.

We now extend this new concept of differentiation to tensors. In order to accomplish such a task we must determine the change in the tensor under an infinitesimal parallel displacement (similar to that of vectors, since at different points in space we have a distinct dependence in the space-time metric). Consider an arbitrary contravariannt tensor expressed as a product of two contravariant vectors $A^i B^k$ and consider its parallel displacement,

$$\delta\left(A^i B^k\right) = A^i \delta B^k + B^k \delta A^i = -A^i \Gamma^k_{lm} B^l dx^m - B^k \Gamma^i_{lm} A^l dx^m \tag{2.67}$$

By virtue of the linearity of this transformation we must also have, for an arbitrary tensor A^{ik},

$$\delta A^{ik} = -\left(A^{im} \Gamma^k_{ml} + A^{mk} \Gamma^i_{ml}\right) dx^l \tag{2.68}$$

Then substituting this into (2.56) we find

$$DA^{ik} = dA^{ik} - \delta A^{ik} \equiv A^{ik}_{;l} dx^l \tag{2.69}$$

in which we get the covariant derivative of the tensor A^{ik} of the form

$$A^{ik}_{;l} = \frac{\partial A^{ik}}{\partial x^l} + \Gamma^i_{ml} A^{mk} + \Gamma^k_{ml} A^{im} \tag{2.70}$$

which follows from

$$\begin{aligned} DA^{ik} &= dA^{ik} - \delta A^{ik} \\ \Longrightarrow & \left(\frac{\partial A^{ik}}{\partial x^l}\right) dx^l + \left(\Gamma^i_{ml} + A^{mk} + \Gamma^k_{ml} + A^{im}\right) dx^l \\ \Longrightarrow & \left(\frac{\partial A^{ik}}{\partial x^l} + \Gamma^i_{ml} + A^{mk} + \Gamma^k_{ml} + A^{im}\right) dx^l \end{aligned}$$

Analogously we may define the covariant derivative of the corresponding mixed and covariant tensors, A^i_k and A_{ik}, respectively,

$$A^i_{k;l} = \frac{\partial A^i_k}{\partial x^l} - \Gamma^m_{kl} A^i_m + \Gamma^i_{ml} A^m_k \tag{2.71}$$

$$A_{ik;l} = \frac{\partial A_{ik}}{\partial x^l} - \Gamma^m_{il} A_{mk} - \Gamma^m_{kl} A_{im} \tag{2.72}$$

Using the above formulation, we find that it is possible to determine the covariant derivative of any arbitrary ranked tensor. Hence, for the tensor $A^{\cdots}_{\cdots}$ its covariant derivative with respect to x^l is found by adding to the ordinary derivative $\frac{\partial A^{\cdots}_{\cdots}}{\partial x^l}$ for each covariant index $iA^{\cdots}_{\cdot i \cdot}$ a term $-\Gamma^k_{il} A^{\cdots}_{\cdot k \cdot}$ and for each contravariant index $iA^{\cdot i \cdot}_{\cdots}$ a term $-\Gamma^i_{kl} A^{\cdot k \cdot}_{\cdots}$.

One can then extend the concept of product rule from ordinary differentiation to the idea of covariant differentiation by the same rule. To do this consider the covariant derivative of a scalar ϕ as an ordinary derivative, i.e. the covariant vector $\frac{\partial \phi}{\partial x^k}$, conforming to the fact that $\delta(\phi) = 0$, hence, $D\phi = d\phi$. For example, the covariant derivative of the product of two covariant tensors A_i and B_k is given by

$$(A_i B_k)_{;l} = A_{i;l} B_k + A_i B_{k;l}$$

Then by raising the index of the covariant derivative, which signifies the differentiation, we obtain the so-called *contravariant derivative*,

$$A_i^{;k} = g^{kl} A_{i;l}, \quad A^{i;k} = g^{kl} A^i_{;l}$$

We prove that the Christoffel symbols are symmetric in the subscripts. Since the covariant derivative of a vector $A_{i;k}$ is a tensor, then it follows that the difference, $A_{i;k} - A_{k;l}$, must also be a tensor.

Consider the scalar ϕ, and let the vector A_i be the gradient of said scalar, i.e. $A_i = \frac{\partial \phi}{\partial x^i}$. Now, since $\partial A_i / \partial x^k = \partial^2 \phi / \partial x^k \partial x^i = \partial A_k / \partial x^i$, then using (2.66) we obtain,

$$A_{k;l} - A_{i;k} = \left(\Gamma^l_{ik} - \Gamma^l_{ki} \right) \frac{\partial \phi}{\partial x^l}$$

We find that,

$$\begin{aligned} A_{k;l} - A_{i;k} &= \frac{\partial A_k}{\partial x^i} - \Gamma^l_{ki} A_l - \frac{\partial A_i}{\partial x^k} - \Gamma^l_{ik} A_l \\ &\implies \left(\Gamma^l_{ik} - \Gamma^l_{ki} \right) A_l \\ &\implies \left(\Gamma^l_{ik} - \Gamma^l_{ki} \right) \frac{\partial \phi}{\partial x^l} \end{aligned}$$

Considering a galilean coordinate system the covariant derivative reduces to an ordinary derivative and the left side of the equation becomes zero. Nonetheless, since the derivative of a vector is a tensor and the difference of two tensors is again a tensor, i.e. $A_{i;k} - A_{k;l}$, we have that being zero in one system then it must also be zero in any coordinate system. Hence, we find,

$$\Gamma^i_{kl} = \Gamma^i_{lk} \tag{2.73}$$

Subsequently,

$$\Gamma_{i,kl} = \Gamma_{i,lk} \tag{2.74}$$

Generally speaking, there exists forty different quantities for the Christoffel symbol; for each of the four values of the i index there are ten distinct pairs of values of the indices k and l.

Next, we analyze the manner in which the Christoffel symbols transform from one coordinate system to another. We obtain these formulas by comparing the laws of transformation of the two sides of the equation defining the covariant derivatives and demanding that they coincide on both sides. We find that

$$\Gamma^i_{kl} = \Gamma'^m_{np} \frac{\partial x^i}{\partial x'^m} \frac{\partial x'^n}{\partial x^k} \frac{\partial x'^p}{\partial x^l} + \frac{\partial^2 x'^m}{\partial x^k \partial x^l} \frac{\partial x^i}{\partial x'^m} \tag{2.75}$$

The above formula depicts that the quantity Γ^i_{kl} behaves like a tensor only under linear transformations (in which the second term vanishes). Equation (2.75) allows us to confirm the aforementioned assertion in which we described the fact that it is always possible to choose a coordinate system in which all the quantities defined by Γ^i_{kl} vanish at a certain point, which we denote as a *locally-geodesic system*.

2.6 Relation of Christoffel symbols to the space-time metric

We commence by showing that the covariant derivative of the metric tensor g_{ik} is zero. To do this we note that

$$DA_i = g_{ik} DA^k$$

which is valid for the vector DA_i, or any other vector. Nonetheless, $A_i = g_{ik}A^k$, such that

$$DA_i = D\left(g_{ik}A^k\right) = g_{ik}DA^k + A^k Dg_{ik}$$

Comparison of the above formula with the aforementioned relation $DA_i = g_{ik} DA^k$, we have remembering that the vector A^k is arbitrary,

$$Dg_{ik} = 0$$

Thus the covariant derivative for the metric takes the form

$$g_{ik:l} = 0 \tag{2.76}$$

Hence, g_{ik} may be considered as a constant during covariant differentiation. The equation $g_{ik;l} = 0$ can be used to relate the Christoffel symbols Γ^i_{kl} to the metric tensor g_{ik}. We proceed by writing the covariant derivative of the metric tensor in accordance to the general definition (2.72):

$$g_{ik;l} = \frac{\partial g_{ik}}{\partial x^l} - g_{mk}\Gamma^m_{il} - g_{im}\Gamma^m_{kl} = \frac{\partial g_{ik}}{\partial x^l} - \Gamma_{k,il} - \Gamma_{i,kl} = 0$$

Thus we obtain that the derivatives of the metric are expressed in terms of the Christoffel symbols. We write the values of the derivatives of g_{ik}, permuting the indices i, k, l:

$$\begin{aligned} \frac{\partial g_{ik}}{\partial x^l} &= \Gamma_{k,il} + \Gamma_{i,kl} \\ \frac{\partial g_{li}}{\partial x^k} &= \Gamma_{i,kl} + \Gamma_{l,ik} \\ -\frac{\partial g_{kl}}{\partial x^i} &= -\Gamma_{l,ki} - \Gamma_{k,li} \end{aligned} \tag{2.77}$$

Then taking a sum of the above equations, dividing by 2 and remembering (2.74),

$$\Gamma_{i,kl} = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^l} + \frac{\partial g_{il}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^i}\right) \tag{2.78}$$

Compute,

$$\begin{aligned} \frac{\partial g_{ik}}{\partial x^l} &= \Gamma_{k,il} + \Gamma_{i,kl} \\ \frac{\partial g_{li}}{\partial x^k} &= \Gamma_{i,kl} + \Gamma_{l,ik} \\ -\frac{\partial g_{kl}}{\partial x^i} &= -\Gamma_{l,ki} - \Gamma_{k,li} \end{aligned} \tag{2.79}$$

Remembering that the Christoffel symbols are symmetric in the lower indices, i.e. $\Gamma_{i,kl} = \Gamma_{i,lk}$ we have that when taking a sum

$$\begin{aligned}
\frac{\partial g_{ik}}{\partial x^l} &= \Gamma_{k,il} + \Gamma_{i,kl} \\
+\frac{\partial g_{li}}{\partial x^k} &= \Gamma_{i,kl} + \Gamma_{l,ik} \\
-\frac{\partial g_{kl}}{\partial x^i} &= -\Gamma_{l,ki} - \Gamma_{k,li} \\
\implies \frac{\partial g_{ik}}{\partial x^l} + \frac{\partial g_{li}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^i} &= \Gamma_{k,il} + \Gamma_{i,kl} + \Gamma_{i,kl} + \Gamma_{l,ik} - \Gamma_{l,ki} - \Gamma_{k,li} \\
\implies \frac{\partial g_{ik}}{\partial x^l} + \frac{\partial g_{li}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^i} &= 2\Gamma_{i,kl} \\
\implies \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^l} + \frac{\partial g_{li}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^i}\right) &= \Gamma_{i,kl}
\end{aligned}$$

From this we raise the indices of the Christoffel symbols using the inverse metric tensor, i.e. $\Gamma^i_{kl} = g^{im}\Gamma_{m,kl}$

$$\Gamma^i_{kl} = \frac{1}{2}g^{im}\left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m}\right) \tag{2.80}$$

We continue by formulating an expression for the contracted Christoffel symbol Γ^i_{ki}. To do this we calculate the differential dg of the determinant g made from the components of the tensor(as previously discussed); dg may be obtained by taking the differential of each component of the tensor g_{ik} and multiplying it by its coefficient in the determinant, i.e. by the corresponding minor. Nonetheless, the components of the reciprocal metric tensor g^{ik} are equal to the minors of the determinant of the g_{ik} divided by the determinant.

Accordingly, the minors of the determinant g are equivalent to gg^{ik}. Hence,

$$dg = gg^{ik}dg_{ik} = -gg_{ik}dg^{ik} \tag{2.81}$$

(since we find that $g_{ik}g^{ik} = \delta^i_i = 4, g^{ik}dg_{ik} = -g_{ik}dg^{ik}$). From (2.80), we write

$$\Gamma^i_{kl} = \frac{1}{2}g^{im}\left(\frac{\partial g_{mk}}{\partial x^i} + \frac{\partial g_{mi}}{\partial x^k} - \frac{\partial g_{ki}}{\partial x^m}\right) \tag{2.82}$$

Interchanging the position of the m and i indices in the third and first terms in the parentheses, observe that the two terms cancel each other such that,

$$\Gamma^i_{ki} = \frac{1}{2} g^{im} \frac{\partial g_{im}}{\partial x^k} \tag{2.83}$$

Indeed,

$$\begin{aligned}
\Gamma^i_{ki} &= \frac{1}{2} g^{im} \left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m} \right) \\
&\implies \frac{1}{2} g^{im} \left(\frac{\partial g_{ik}}{\partial x^m} + \frac{\partial g_{mi}}{\partial x^k} - \frac{\partial g_{ki}}{\partial x^m} \right) \\
&\implies \Gamma^i_{ki} = \frac{1}{2} g^{im} \frac{\partial g_{im}}{\partial x^k}
\end{aligned}$$

Then using (2.81)

$$\Gamma^i_{ki} = \frac{1}{2g} \frac{\partial g}{\partial x^k} = \frac{\partial \ln \sqrt{-g}}{\partial x^k} \tag{2.84}$$

Then multiplying by the reciprocal metric we find the following expression for the mentioned quantity $g^{kl}\Gamma^i_{kl}$, which yields

$$g^{kl}\Gamma^i_{kl} = \frac{1}{2} g^{kl} g^{im} \left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{lm}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m} \right) = g^{kl} g^{im} \left(\frac{\partial g_{mk}}{\partial x^l} - \frac{1}{2} \frac{\partial g_{kl}}{\partial x^m} \right)$$

where we set the k and l indices equivalent to one another, such that ($k = l$),

$$\begin{aligned}
&\frac{1}{2} g^{kl} g^{im} \left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{lm}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m} \right) \\
&\implies g^{kl} g^{im} \left(\frac{1}{2} \frac{\partial g_{mk}}{\partial x^l} + \frac{1}{2} \frac{\partial g_{lm}}{\partial x^k} - \frac{1}{2} \frac{\partial g_{kl}}{\partial x^m} \right) \\
&\implies g^{kl} g^{im} \left(\frac{1}{2} \frac{\partial g_{mk}}{\partial x^l} + \frac{1}{2} \frac{\partial g_{km}}{\partial x^l} - \frac{1}{2} \frac{\partial g_{kl}}{\partial x^m} \right) \\
&\implies g^{kl} g^{im} \left(\frac{\partial g_{mk}}{\partial x^l} - \frac{1}{2} \frac{\partial g_{kl}}{\partial x^m} \right)
\end{aligned}$$

By (2.81) we can thus transform the above quantity to

$$g^{kl}\Gamma^{i}_{kl} = -\frac{1}{\sqrt{-g}}\frac{\partial\left(\sqrt{-g}g^{ik}\right)}{\partial x^{k}} \tag{2.85}$$

We note in passing that the derivatives of the contravariant tensor g^{ik} are related to the derivatives of its counterpart g_{ik} by the relations

$$g_{il}\frac{\partial g^{lk}}{\partial x^{m}} = -g^{lk}\frac{\partial g_{il}}{\partial x^{im}} \tag{2.86}$$

which are obtained by differentiating the equality $g_{il}g^{lk} = \delta_{i}^{k}$, in which δ_{i}^{k} is the Kronecker delta symbol, thus

$$\begin{aligned} \frac{\partial}{\partial x^{m}}\left(g_{il}g^{lk}\right) &= \frac{\partial}{\partial x^{m}}\delta_{i}^{k} \\ \Longrightarrow\ \frac{\partial}{\partial x^{m}}\left(g_{il}\right)g^{lk} + \frac{\partial}{\partial x^{m}}\left(g^{lk}\right)g_{il} &= 0 \\ \Longrightarrow\ -\frac{\partial}{\partial x^{m}}\left(g_{il}\right)g^{lk} &= \frac{\partial}{\partial x^{m}}\left(g^{lk}\right)g_{il} \end{aligned}$$

Finally we point to the fact that the derivatives of g^{ik} can also be expressed in term of the Christoffel symbols Γ^{i}_{kl}. Accordingly, utilizing the identity $g^{ik}_{;l} = 0$ it follows immediately that

$$\frac{\partial g^{ik}}{\partial x^{l}} = -\Gamma^{i}_{ml}g^{mk} - \Gamma^{k}_{ml}g^{im} \tag{2.87}$$

$$\begin{aligned} g^{ik;l} = \frac{\partial g^{ik}}{\partial x^{l}} + g^{mk}\Gamma^{m}_{il} + g^{im}\Gamma^{m}_{kl} &= \frac{\partial g^{ik}}{\partial x^{l}} + \Gamma_{k,il} + \Gamma_{i,kl} = 0 \\ \Longrightarrow\ g^{ik;l} = \frac{\partial g^{ik}}{\partial x^{l}} + g^{mk}\Gamma^{m}_{il} + g^{im}\Gamma^{m}_{kl} &= 0 \\ \Longrightarrow\ \frac{\partial g^{ik}}{\partial x^{l}} &= -g^{mk}\Gamma^{m}_{il} - g^{im}\Gamma^{m}_{kl} \end{aligned}$$

Then using the above formulas we may define the generalized divergence of a vector in curvilinear coordinates $A^i_{;i}$, into a convenient form. Using (2.84), we obtain,

$$A^i_{;i} = \frac{\partial A^i}{\partial x^i} + \Gamma^i_{li} A^l = \frac{\partial A^i}{\partial x^i} + A^l \frac{\partial \ln \sqrt{-g}}{\partial x^l}$$

similarly,

$$A^i_{;i} = \frac{1}{\sqrt{-g}} \frac{\partial \left(\sqrt{-g} A^i\right)}{\partial x^i} \tag{2.88}$$

in which we employed the use of (2.85) as an analogy. Analogously, we may derive an expression for the divergence of the arbitrary antisymmetric tensor A^{ik}, so that by exploiting (2.70) we find

$$A^{ik}_{;k} = \frac{\partial A^{ik}}{\partial x^k} + \Gamma^i_{mk} A^{mk} + \Gamma^k_{mk} A^{im} \tag{2.89}$$

Then by the nature of the antisymmetric tensor, $A^{mk} = -A^{km}$, we have

$$\Gamma^i_{mk} A^{mk} = -\Gamma^i_{km} A^{km} = 0$$

Then using (2.84), we may substitute the Christoffel symbols such that,

$$A^{ik}_{;k} = \frac{1}{\sqrt{-g}} \frac{\partial \left(\sqrt{-g} A^{ik}\right)}{\partial x^k} \tag{2.90}$$

Indeed, working backwards we find,

$$\begin{aligned} A^{ik}_{;k} &= \frac{1}{\sqrt{-g}} \frac{\partial \left(\sqrt{-g} A^{ik}\right)}{\partial x^k} \\ \implies \frac{1}{\sqrt{-g}} &\left[\frac{\partial \sqrt{-g}}{\partial x^k} A^{ik} + \sqrt{-g} \frac{\partial A^{ik}}{x^k} \right] \\ \implies &\frac{1}{\sqrt{-g}} \sqrt{-g} \frac{\partial A^{ik}}{x^k} \\ \implies &\frac{\partial A^{ik}}{\partial x^k} \end{aligned}$$

Next consider the symmetric tensor A_{ik}; we calculate the expression $A^k_{i;k}$ for its mixed components.

We find

$$A^k_{i;k} = \frac{\partial A^k_i}{\partial x^k} + \Gamma^k_{lk} A^l_i - \Gamma^l_{ik} A^k_l = \frac{1}{\sqrt{-g}} \frac{\partial \left(A^k_i \sqrt{-g}\right)}{\partial x^k} - \Gamma^l_{ki} A^k_l \tag{2.91}$$

In which the last term is equal to

$$-\frac{1}{2}\left(\frac{\partial g_{il}}{\partial x^k} + \frac{\partial g_{kl}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^l}\right) A^{kl} \tag{2.92}$$

Due to the symmetry of the tensor A^{kl}, i.e. $A^{kl} = A^{lk}$, two terms in the parentheses cancel each other and we're left with

$$A^k_{i;k} = \frac{1}{\sqrt{-g}} \frac{\partial \left(\sqrt{-g} A^k_i\right)}{\partial x^k} - \frac{1}{2} \frac{\partial g_{kl}}{\partial x^i} A^{kl} \tag{2.93}$$

In a cartesian coordinate system of reference $\partial A_i/\partial x^k - \partial A_k/\partial x^i$ is an antisymmetric tensor; in curvilinear coordinates this tensor is represented by $A_{i;k} - A_{k;i}$. Nonetheless, using the expression for $A_{i;k}$ (2.66) and since the Christoffel symbols are symmetric in the lower indices,$\Gamma^i_{kl} = \Gamma^i_{lk}$, we have

$$A_{i;k} - A_{k;i} = \frac{\partial A_i}{\partial x^k} - \frac{\partial A_k}{\partial x^i} \tag{2.94}$$

Compute,

$$\begin{aligned} A_{i;l} &= \frac{\partial A_i}{\partial x^l} - \Gamma^k_{il} A_k \\ \implies A_{i;k} - A_{k;i} &= \frac{\partial A_i}{\partial x^k} - \Gamma^l_{ik} A_l - \frac{\partial A_k}{\partial x^i} - \Gamma^l_{ki} A_l \\ \implies & \frac{\partial A_i}{\partial x^k} - \frac{\partial A_k}{\partial x^i} \end{aligned}$$

Finally, we transform to curvilinear coordinates the sum $\partial^2\phi/\partial x_i \partial x^i$ of the second derivatives of a scalar ϕ. In curvilinear coordinates the sum goes over into $\phi^{;i}_{;i}$. But, $\phi_{;i} = \partial\phi/\partial x^i$, since the covariant differentiation of a scalar reduces to ordinary differentiation. Then, raising the index by means of the reciprocal metric we have,

$$\phi^{;i} = g^{ik} \frac{\partial \phi}{\partial x^k}$$

and using expression (2.88), we find

$$\phi_{;i}^{;i} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^i} \left(\sqrt{-g} g^{ik} \frac{\partial \phi}{\partial x^k} \right) \tag{2.95}$$

We note that Gauss' theorem (2.38) for the transformation of the integral of a vector over a hypersurface into an integral over a four-volume can, in view of (2.88), be written as

$$\oint A^i \sqrt{-g} dS_i = \int A^i_{;i} \sqrt{-g} d\Omega \tag{2.96}$$

2.7 Motion of a particle in a field of gravity

We determine the motion of a free particle by using the principle of least action, as delineated in the special theory of relativity,

$$\delta S = -mc\delta \int ds = 0 \tag{2.97}$$

conforming to fact that the motion of said particle is given by the world line as an extremal between a given pair of world points.

Analogously, the motion of particle in the presence of a field of gravity is determined by the aforementioned principle of least action given in the form (2.97), since as previously noted, the gravitational field is nothing but a change in the space-time metric, manifesting in a modification of the space-time interval, ds, determined by the dx^i terms. Hence, within the presence of a gravitational field the particle moves along an extremal, or *geodesic line* in the four-space; yet, since in a true gravitational field space-time is not galilean, the path taken is not a "*straight line*" (in a three-dimensional perspective), and the actual spatial motion is nether uniform nor rectilinear.

Logically it ensues that we derive the equations of motion of a particle from the basic principle of least action, nonetheless, we instead consider an eloquent generalization of the differential equations for the free motion of a particle in the theory of special relativity, i.e. in a galilean four-dimensional system. These equations are $du^i/ds = 0$ or $du^i = 0$, where $u^i = dx^i/ds$ is the four velocity. Clearly, in curvilinear coordinates this equation is generalized to the equation

$$Du^i = 0 \tag{2.98}$$

By the expression (2.62), we may write the covariant differential of u^i as follows,

$$du^i + \Gamma^i_{kl} u^k dx^l = 0$$

Dividing the resulting equation by ds, we find

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{kl} \frac{dx^k}{ds} \frac{dx^l}{ds} = 0 \tag{2.99}$$

We briefly note that this is the Geodesic equation, which describes motion of a particle within the presence of a gravitational field. Notice that the motion of a particle within a gravitational field is determined by the Christoffel symbols Γ^i_{kl}. The derivative d^2x^i/ds^2 is the four-acceleration of the particle. Thus, we may call the quantity $-m\Gamma^i_{kl}u^k u^l$ the "four-force" acting on the particle. It is easily seen, if we consider the classical Newtonian force, i.e. $F = ma$, then, we can write

$$\begin{aligned} \frac{d^2x^i}{ds^2} + \Gamma^i_{kl} \frac{dx^k}{ds} \frac{dx^l}{ds} &= 0 \\ \implies \frac{d^2x^i}{ds^2} &= -\Gamma^i_{kl} \frac{dx^k}{ds} \frac{dx^l}{ds} \\ \implies m\frac{d^2x^i}{ds^2} &= -m\Gamma^i_{kl} \frac{dx^k}{ds} \frac{dx^l}{ds} \\ \implies F &= -\Gamma^i_{kl} \frac{dx^k}{ds} \frac{dx^l}{ds} \end{aligned}$$

In this case the metric tensor g_{ik} acts as the "potential" of the gravitational field - its derivatives determine the "intensity" of the field.

As shown previously, we may always select a suitable choice of system of coordinates in which the Christoffel symbols Γ^i_{kl} are zero at an arbitrary point of space-time. The possibility of making such a selection follows from the principle of equivalence, in which the choice of such a locally-inertial system of reference tends to the elimination of the gravitational field in the given infinitesimal element of space-time.

Define the four-momentum of a particle in a field of gravity as

$$p^i = mcu^i \tag{2.100}$$

In which the scalar product with itself is

$$p_i p^i = m^2 c^2 \tag{2.101}$$

Substitution of $-\partial S/\partial x^i$ for the four-momentum p_i, we determine the Hamilton-Jacobi equation for a particle in a gravitational field:

$$g^{ik} \frac{\partial S}{\partial x^i} \frac{\partial S}{\partial x^k} - m^2 c^2 = 0 \tag{2.102}$$

The equation of a geodesic described in (2.99) does not apply to the propagation of a light signal, since alongside the world line of the light ray the interval ds, is zero, thus all terms in (2.99) become infinite. Thus in order to circumvent this dilemma we instead use the fact that the direction of the propagation of a light ray in geometrical optics may be determined by the wave vector tangent to the ray. Hence, we can proceed to writing the four-dimensional wave vector in the form $k^i = dx^i/d\lambda$, where λ us some parameter along the ray. In the theory of special relativity, when studying the propagation of light in vacuum the wave vector does not vary along the path, i.e. $dk^i = 0$. Translating this simple fact into the theory of general relativity this mean that the covariant derivative vanishes, i.e. $Dk^i = 0$ or

$$\frac{dk^i}{d\lambda} + \Gamma^i_{kl} k^k k^l = 0 \tag{2.103}$$

The absolute square of the wave vector is zero, that is,

$$k_i k^i = 0 \tag{2.104}$$

If we define the wave vector as

$$k^i = \left(\frac{\omega}{c}, \mathbf{k}\right)$$

and

$$\mathbf{k} = \frac{\omega}{c}\mathbf{n} \tag{2.105}$$

where the covariant wave vector is given by

$$k_i = \left(\frac{\omega}{c}, -\mathbf{k}\right)$$

Then,

$$\begin{aligned} k^i k_i &= \langle \left(\frac{\omega}{c}, \vec{k}\right), \left(\frac{\omega}{c}, -\vec{k}\right) \rangle \\ &\Longrightarrow \langle \left(\frac{\omega}{c}, \frac{\omega}{c}\mathbf{n}\right), \left(\frac{\omega}{c}, -\frac{\omega}{c}\mathbf{n}\right) \rangle \\ &\Longrightarrow \left(\frac{\omega}{c}\right)^2 - \left(\frac{\omega}{c}\right)^2 \hat{n}^2 \\ &\Longrightarrow \left(\frac{\omega}{c}\right)^2 - \left(\frac{\omega}{c}\right)^2 (1) \\ &\Longrightarrow = 0 \end{aligned}$$

Substitution of $\partial\psi/\partial x^i$ in place of k_i (in which ψ is the eikonal), we find the eikonal equation in a gravitational field

$$g^{ik}\frac{\partial\psi}{\partial x^i}\frac{\partial\psi}{\partial x^k} = 0 \tag{2.106}$$

In the limiting case of small velocities, we impose that the relativistic equations of motion in a gravitational field tend to the corresponding non-relativistic equations. Thus, we remember that at small velocities we require that the gravitational field itself be weak; violating such an assumption would lead to a particle acquiring a high velocity within the gravitational field.

We examine how, in the case of small velocities, the metric tensor g_{ik} determining the field is related to the nonrelativistic potential ϕ of the gravitational field. As previously mentioned, the motion of a particle in a gravitational field in nonrelativistc mechanics is adequately described by the Lagrangian (2.1), which we now write as

$$L = -mc^2 + \frac{mv^2}{2} - m\phi \tag{2.107}$$

adding the $-mc^2$ constant. This is done so that the nonrelativistic Lagrangian in the absence of the field, i.e. $\phi = 0 \implies L = -mc^2 + mv^2/2$, shall correspond to the relativistic function $L = -mc^2\sqrt{1 - v^2/c^2}$ reduces to in the limit as $v/c \to 0$. Ergo, the nonrelativistic action function S for a particle in a gravitational field takes the form

$$S = \int L dt = -mc \int \left(c - \frac{v^2}{2c} + \frac{\phi}{c} \right) dt$$

Comparison of this expression with $S = -mc \int ds$, we find that in the limiting case which we consider

$$ds = \left(c - \frac{v^2}{2c} + \frac{\phi}{c} \right) dt$$

Indeed,

$$\begin{aligned} S = \int L dt = -mc \int \left(c - \frac{v^2}{2c} + \frac{\phi}{c} \right) dt = -mc \int ds \\ \implies \int \left(c - \frac{v^2}{2c} + \frac{\phi}{c} \right) dt = \int ds \\ \implies ds = \left(c - \frac{v^2}{2c} + \frac{\phi}{c} \right) dt \end{aligned}$$

Then squaring and dropping terms which vanish as $c \to \infty$, we find

$$ds^2 = \left(c^2 + 2\phi\right) dt^2 - d\mathbf{r}^2 \tag{2.108}$$

where we computed

$$\begin{aligned}
ds &= \left(c - \frac{v^2}{2c} + \frac{\phi}{c}\right) dt \\
\Longrightarrow ds^2 &= \left(\left(c - \frac{v^2}{2c} + \frac{\phi}{c}\right) dt\right)^2 \\
\Longrightarrow ds^2 &= \left(c - \frac{v^2}{2c} + \frac{\phi}{c}\right)\left(c - \frac{v^2}{2c} + \frac{\phi}{c}\right) dt^2 \\
\Longrightarrow ds^2 &= \left(c^2 - \frac{v^2}{2} + \phi - \frac{v^2}{2} + \left(\frac{v^2}{2c}\right)^2 - \frac{v^2\phi}{2c^2} + \phi - \frac{v\phi}{2c^2} + \left(\frac{\phi}{c}\right)^2\right) dt^2 \\
\Longrightarrow ds^2 &= (c^2 + 2\phi - v^2) dt^2 \\
\Longrightarrow ds^2 &= (c^2 + 2\phi) dt^2 - v^2 dt^2 \\
\Longrightarrow ds^2 &= (c^2 + 2\phi) dt^2 - d\mathbf{r}^2
\end{aligned}$$

Thus in the limiting case the component g_{00} of the metric tensor is

$$g_{00} = 1 + \frac{2\phi}{c^2} \tag{2.109}$$

For the other components of the metric tensor (2.108), it would follows that $g_{\alpha\beta} = \delta_{\alpha\beta} \quad g_{0\alpha} = 0$

Where we note that the metric tensor has the form

$$g_{ik} = \begin{pmatrix} 1 + \frac{2\phi}{c^2} & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{2.110}$$

Actually, the corrections to them are, generally speaking of the same order of magnitude as the corrections to g_{00}. The improbable nature of determining these corrections by methods previously denoted above is related to the fact that the correction to the $g_{\alpha\beta}$, though of the same order of magnitude as the correction to g_{00}, would give rise to terms in the Lagrangian of a higher order of minuteness.

2.8 Rotation

As a special case of a stationary gravitational field, we consider a uniformly rotating system of reference. In order to derive the ds interval we carry out a transformation from an inertial system to a uniformly rotating one. In the coordinate r', ϕ', z', t the inertial system's interval takes the form

$$ds^2 = c^2 dt^2 - dr'^2 - r'^2 d\phi'^2 - dz'^2 \tag{2.111}$$

Let the cylindrical coordinates in the rotating system be r, ϕ, z. Then, if the axis of rotation coincides with the axes Z and Z', then we have the relation between the coordinates of the respective systems $r' = r, z' = z, \phi' = \phi + \Omega t$, where Ω is the angular velocity of rotation. Then substituting the previous relation into (2.111), we find the squared interval expression ds^2 for the rotating reference system:

$$ds^2 = \left(c^2 - \Omega^2 r^2\right) dt^2 - 2\Omega r^2 d\phi dt - dz^2 - r^2 d\phi^2 - dr^2 \tag{2.112}$$

Indeed, introducing said substitution we may calculate

$$\begin{gathered}
ds^2 = c^2 dt^2 - dr'^2 - r'^2 d\phi'^2 - dz'^2 \\
\Longrightarrow = c^2 dt^2 - dr^2 - r^2 d(\phi + \Omega t)^2 - dz^2 \\
\Longrightarrow = c^2 dt^2 - dr^2 - r^2 d(\phi^2 + 2\Omega\phi t + \Omega^2 t^2) - dz^2 \\
\Longrightarrow = c^2 dt^2 - dr^2 - r^2 d\phi^2 - 2\Omega r^2 d\phi dt - \Omega^2 r^2 dt^2 - dz^2 \\
\Longrightarrow = \left(c^2 - \Omega^2 r^2\right) dt^2 - 2\Omega r^2 d\phi dt - dz^2 - r^2 d\phi^2 - dr^2
\end{gathered}$$

It may be pertinent to suggest that the rotating system of reference under consideration can only be used out to distances equal to c/Ω; from (2.112) we see that if $r > c/\Omega$, g_{00} becomes negative, which is not permissible for true gravitational field, as the metric tensor loses its signature and the system cannot be created from real bodies. Physically, the extraneous nature of the rotating reference system at large distances is due to the fact that there the velocity would become greater than the velocity of light, which violates the special theory of relativity. Similar to other stationary fields , the clocks on the rotating body cannot be uniquely synchronized at all points. Attempting to such synchronization along any closed contour, we find, that upon returning to the starting point, a time distinct from the initial value by a factor of

$$\Delta t = -\frac{1}{c}\oint \frac{g_{0\alpha}}{g_{00}} dx^{\alpha} = \frac{1}{c^2}\oint \frac{\Omega r^2 d\phi}{1-\frac{\Omega^2 r^2}{c^2}} \tag{2.113}$$

or, if we assume $\Omega r \ll 1$ (the velocity of rotation is small compared with the velocity of light), Calculate

$$\begin{aligned} \Delta t &= \frac{1}{c^2}\oint \frac{\Omega r^2 d\phi}{1-\frac{\Omega^2 r^2}{c^2}} \\ \Longrightarrow &= \frac{1}{c^2}\oint \Omega r^2 d\phi \\ \Longrightarrow &= \frac{\Omega}{c^2}\int r^2 d\phi \\ \Longrightarrow &= \pm\frac{2\Omega}{c^2} S \end{aligned}$$

in which we estimated the integral by $\pm 2S$, depending on the direction traversed along the contour.

$$\Delta t = \frac{\Omega}{c^2}\int r^2 d\phi = \pm\frac{2\Omega}{c^2} S \tag{2.114}$$

where S is the projected area of the contour on a perpendicular plane to the axis of rotation (in which the sign $+$ or $-$ holds accordingly as we traverse the contour in, or opposite to, the direction of propagation). Next, we presume that a ray of light propagates along a certain closed contour and we calculate the time t that elapses between the starting out of the light ray and its return to the initial point in terms of order v/c. By definition the velocity of light is c if the times are synchronized along the closed curve and if at each point we use the proper time. Since the difference between the world and proper time is of order v^2/c^2, then in calculating the required time interval t the v/c terms of all orders may be neglected. Thus we obtain,

$$t = \frac{L}{c} \pm \frac{2\Omega}{c^2} S$$

where L is he length of the contour. Corresponding to this, the velocity of light, measured as the ratio L/t, appears equal to

$$c \pm 2\Omega \frac{S}{L} \tag{2.115}$$

2.9 The equations of Electrodynamics in the presence of a gravitational field

It is possible to translate the electromagnetic field equations described in the theory of special relativity into the domain of the theory of general relativity, such that they be applicable in an arbitrary four-dimensional curvilinear coordinate system, i.e. within a gravitational filed. The electromagnetic field tensor is defined as

$$F_{ik} = A_{k,i} - A_{i;k} = \frac{\partial A_k}{\partial x^i} - \frac{\partial A_i}{\partial x^k}$$

We may re-express this tensor using covariant differentiation and write $F_{ik} = A_{k;i} - A_{i;k}$. But due to (2.94) we find

$$F_{ik} = A_{k;i} - A_{i;k} = \frac{\partial A_k}{\partial x^i} - \frac{\partial A_i}{\partial x^k} \tag{2.116}$$

thus, the relation of the electromagnetic tensor F_{ik} to the potential A_i does not change. Consequently, the first pair of the Maxwell equations (the homogeneous Maxwell equation) also do not change in form

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0 \tag{2.117}$$

which we refer to as the Bianchi identity. We provide a simple example for the above formulation, as derivation of a proof of the above is not the aim of this paper. We note first that the electromagnetic field tensor is antisymmetric, i.e.

$$F_{ik} = -F_{ki} \tag{2.118}$$

Then we define its components for $\vec{E}$ and $\vec{B}$ as follows

$$F_{0i} = E_i, \qquad F_{i0} = -E_i, \qquad F_{ij} = \varepsilon_{ijk} B_k \tag{2.119}$$

where ε_{ijk} is the totally-antisymmetric tensor, previously defined. Then we, find that

$$\begin{aligned} F_{23} &= -B_x, \quad F_{31} = -B_y, \quad F_{12} = -B_z \\ F_{32} &= B_x, \qquad F_{13} = B_y, \quad F_{21} = B_z \end{aligned} \tag{2.120}$$

which follows from

$$F_{ik} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \tag{2.121}$$

$$F^{ik} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{2.122}$$

in which rows are labeled by i and columns are labeled by k and $1,2,3$ correspond to x,y,z-

components of each of the respective fields. Then consider the indices $(\mu,\nu,\rho)=(0,i,j)$

$$\partial_0 F_{ij}+\partial_i F_{j0}+\partial_j F_{0i}=0 \tag{2.123}$$

which by using (2.119) we write as

$$\varepsilon_{ijk}\frac{\partial B_k}{\partial t}+\partial_i E_j-\partial_j E_i=0 \tag{2.124}$$

Since this is antisymmetric in ij there is no loss in generality in contracting with $\varepsilon_{ij\ell}$, which gives

$$2\frac{\partial B_\ell}{\partial t}+2\varepsilon_{ij\ell}\partial_i E_j=0 \tag{2.125}$$

Which is simply the statement that

$$\vec{\nabla}\times\vec{E}+\frac{\partial \vec{B}}{\partial t}=0 \tag{2.126}$$

which is one of the homogeneous Maxwell Equations. Considering an even simpler case in which we allow $(\mu,\nu,\rho)=(0,2,3)$ such that,

$$\partial_0 F_{23}+\partial_2 F_{30}+\partial_3 F_{02}=0$$
$$\Longrightarrow\ -\partial_t B_x-\partial_y E_z+\partial z E_y=0$$

which is a component of the Maxwell Equation (2.126). Subsequent permutation of indices will lead to the remaining components of the two homogeneous Maxwell equations.

In order to write the second pair of Maxwell equations (inhomogeneous Maxwell equations), we must determine the current four-vector in curvilinear coordinates. We denote the spatial volume element, which we construct form the space coordinate elements dx^1, dx^2, and dx^3, is $\sqrt{\gamma}dV$, where γ is the determinant of the spatial metric tensor (2.47) and $dV=dx^1dx^2dx^3$. We then introduce the charge density ρ according to the definition $de=\rho\sqrt{\gamma}dV$, where de is the charge located within the volume element $\sqrt{\gamma}dV$. The if we multiply this equation on both sides by dx^i, we have:

$$\begin{aligned} de &= \rho\sqrt{\gamma}dV \\ \Longrightarrow dedx^i &= \rho\sqrt{\gamma}dV dx^i \\ \Longrightarrow dedx^i &= \rho\sqrt{\gamma}dx^1 dx^2 dx^3 dx^i \\ \Longrightarrow dedx^i &= \rho\sqrt{\gamma}dx^1 dx^2 dx^3 dx^i \frac{dx^0}{dx^0} \\ \Longrightarrow dedx^i &= \rho\sqrt{\gamma}d\Omega\frac{dx^i}{dx^0} \\ \Longrightarrow dedx^i &= \frac{\rho}{\sqrt{g_{00}}}\sqrt{-g}d\Omega\frac{dx^i}{dx^0} \end{aligned}$$

in which $d\Omega = dx^1 dx^2 dx^3 dx^0$ and we have used the formula $-g = \gamma g_{00}$. The product $\sqrt{-g}d\Omega$ is the invariant element of four-volume, so that the current four-vector is defined by the formula

$$j^i = \frac{\rho c}{\sqrt{g_{00}}}\frac{dx^i}{dx^0} \tag{2.127}$$

we note that the quantities dx^i/dx^0 are the rates of change of the coordinates with 'time" x^0, and *do not* constitute a four vector. The component j^0 of the current four vector, multiplied by $\sqrt{g_{00}}/c$ is the spatial density of charge.

We note that for point charges the density ρ can be expressed as a sum of δ-functions, i.e. $\rho = \sum_a e_a \delta(\mathbf{r} - \mathbf{r}_a)$, nonetheless, we must derive this equation for the case of curvilinear coordinates. By $\delta(\mathbf{r})$ we mean the product between delta functions $\delta\left(x^1\right)\delta\left(x^2\right)\delta\left(x^3\right)$ regardless of the coordinates x^1, x^2, x^3; the integral over dV is unity $\int \delta(\mathbf{r})dV = 1$. With the same definition of the δ-functions, the charge density is

$$\rho = \sum_a \frac{e_a}{\sqrt{\gamma}}\delta(\mathbf{r} - \mathbf{r}_a)$$

and the current four vector is

$$j^i = \sum_a \frac{e_a c}{\sqrt{-g}} \delta\left(\mathbf{r} - r_a\right) \frac{dx^i}{dx^0} \tag{2.128}$$

Namely, the total charge present in all of space may be appropriately described by taking an integral over all space of the charge within a given volume, i.e. $\int \rho dV$. We may then rewrite such an integral in four-dimensions such that:

$$\int \rho dV = \frac{1}{c}\int j^0 dV = \frac{1}{c}\int j^i dS_i \tag{2.129}$$

in which integration is taken over the the entire four-dimensional hyperplane perpendicular to the x^0 axis.

The conservation of charge is expressed by the equation of continuity, which has a minute discrepancy in that in the theory of special relativity we express such an equation using ordinary derivative, replacing them here by covariant derivatives. Define the charge-current continuity equation as follows:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0 \tag{2.130}$$

If we define the $j^0 = c\rho$ and $\mathbf{j} = j_x, j_y, j_z$, these components form the current-four vector

$$j^i = (c\rho, \mathbf{j}) \tag{2.131}$$

such that we end up with the four divergence equation displayed above, i.e.

$$\frac{\partial j^i}{\partial x^i} = 0 \tag{2.132}$$

Then as aforementioned we may rewrite using covariant differentiation and obtain

$$j^i_{;i} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^i}\left(\sqrt{-g} j^i\right) = 0 \tag{2.133}$$

where we have implemented the use of equation (2.88). Then, we may write the second pair of the Maxwell equations (inhomogeneous Maxwell equations), by replacing the ordinary derivatives by covariant derivatives, when differentiating the electromagnetic field, where we obtain:

$$F^{ik}_{;k} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^k} \left(\sqrt{-g} F^{ik} \right) = -\frac{4\pi}{c} j^i \tag{2.134}$$

where we used formula (2.90). Calculate,

$$\begin{aligned} A^{ik}_{;k} &= \frac{1}{\sqrt{-g}} \frac{\partial \left(\sqrt{-g} A^{ik} \right)}{\partial x^k} \\ \Longrightarrow F^{ik}_{;k} &= \frac{1}{\sqrt{-g}} \frac{\partial \left(\sqrt{-g} F^{ik} \right)}{\partial x^k} \\ \Longrightarrow & \frac{1}{\sqrt{-g}} \left[\frac{\partial \sqrt{-g}}{\partial x^k} F^{ik} + \sqrt{-g} \frac{\partial F^{ik}}{x^k} \right] \\ \Longrightarrow & \frac{1}{\sqrt{-g}} \sqrt{-g} \frac{\partial F^{ik}}{x^k} \\ \Longrightarrow & \frac{\partial F^{ik}}{\partial x^k} \end{aligned}$$

Analogously, we offer a rather simple demonstration of the proof of the above equation, since a full derivation is not the aim of this paper. We rewrite the derivative of the energy-momentum tensor as follows

$$\partial_k F^{ik} = -4\pi j^i \tag{2.135}$$

We first reduce the above equation to a three-dimensional equation, since it is a vector-valued equation; to this end, consider two cases $i = 0$ and $i = n$. For $i = 0$ we have

$$\partial_j F^{0j} = -4\pi j^0 \tag{2.136}$$

which corresponds to

$$-\partial_j E_j = -4\pi\rho, \qquad \text{i.e.} \quad \vec{\nabla} \cdot \vec{E} = 4\pi\rho \tag{2.137}$$

using equation (2.119). Namely, we may substitute $k = 1$ such that,

$$\partial_1 F^{01} = -4\pi j^0$$
$$\implies -\partial_x E_x = -4\pi j^0$$

similarly, substituting in $k = 2,3$ we find the subsequent components of the divergence of the $\vec{E}$ (electric field)

$$-\partial_y E_y = -4\pi j^0 \quad and \quad -\partial_z E_z = -4\pi j^0 \tag{2.138}$$

taking a sum we obtain

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho \tag{2.139}$$

which is the first of the inhomogeneous Maxwell equations. Then for $i = n$,we find that

$$\partial_0 F^{n0} + \partial_j F^{nj} = -4\pi J^j \tag{2.140}$$

which gives

$$\partial_0 E_j + \varepsilon_{ijk}\partial_i B_k = -4\pi J^j \tag{2.141}$$

and this is just

$$-\frac{\partial \vec{E}}{\partial t} + \vec{\nabla} \times \vec{B} = 4\pi\vec{J} \tag{2.142}$$

Thus, equation (2.134) gives the last pair of the Maxwell equations. Finally the equations of motion for a charged particle within the presence of an electromagnetic and gravitational field may be obtained from a simple substitution into the equations of motion of a particle in an electromagnetic field

$$mc\frac{du^i}{ds} = \frac{e}{c}F^{ik}u_k \tag{2.143}$$

in which we replace the four-acceleration du^i/ds, by its contravariant derivative counterpart we have previously defined, i.e. Du^i/ds:

$$mc\frac{Du^i}{ds} = mc\left(\frac{du^i}{ds} + \Gamma^l_{kl}u^k u^l\right) = \frac{e}{c}F^{ik}u_k \tag{2.144}$$

where we substituted in the value of Du^i/ds directly from (2.99) and e is the charge of the particle under study.

We briefly remark upon the statement regarding the total charge present in all of space, as this will prove pertinent in the following sections. As previously introduced, we commented on the fact that we may rewrite the integral of the charge density $\rho = j^0$ over all space as an integral in four-dimensional form taken perpendicular to the $x^0 = const$ axis. Within each moment in time, the total charge is properly given by such an integral over a different hyperplane perpendicular to the x^0 axis. The difference between the integrals $\int j^i dS_i$ taken over two such hyperplanes may be rewritten as a surface integral $\oint j^i dS_i$, in which the integral is taken over the whole closed hypersuface surrounding the four-volume between the two hyperplanes under consideration. Then by Gauss' Theorem we may transform this into an integral over the four-volume between the two hyperplanes and verify that

$$\oint j^i dS_i = \int \frac{\partial j^i}{\partial x^i} d\Omega = 0 \tag{2.145}$$

which follows from (2.132), such that

$$\oint j^i dS_i = \int \frac{\partial j^i}{\partial x^i} d\Omega = \int (0)\, d\Omega = 0$$

2.10 The Riemann curvature tensor

Let us revisit the concept of parallel displacement previously considered. As previously mentioned in the case of a curved four-space, the infinitesimal parallel displacement of a vector is defined as the displacement in which the components of the vector remain invariant under a system of coordinates which is galilean in the given infinitesimal volume element.

Consider $x^i = x^i(s)$ as the parametric equation of an arbitrary curve, in which s is the arc length at a given point of the curve, then the vector $u^i = dx^i/ds$ is the unit tangent vector of the curve. Now, if we consider a geodesic curve, then along it $Du^i = 0$, see (2.99). Thus, if we take the covariant derivative of a the tangent vector along itself then it is equal to zero. In other words, not only is u^i kept parallel along the curve, but the curve continues to point along the same direction along the path - locally.

Contrarily, if we consider the parallel displacement of two vector, the "angle" between them is impervious under such transition. Thus, during the parallel displacement of a vector, its components along the geodesic curve must be the same at all points along the path.

Now we point attention to the fact that when taking the parallel displacement of a vector in a curved space from one point to another gives different results if the displacement is taken along different paths. Notably, it follows that if we displace the vector along a closed contour over a given path, them upon returning to the starting point, its new value will not coincide with its original value.

We provide an illustration in order to make the above notion as unambiguous as possible. To this end, we consider a curved two-dimensional space, or any curved surface. Figure (2.1) shows a portion of such a surface, bounded by three geodesic curves. Let us subject vector 1 to a parallel displacement along this closed contour. Starting at A and moving along line AB, vector 1, always retains its angle along the curve, going over to vector 2. Similarly, following vector 2 through the curve BC and retaining its angle along the path we subsequently arrive at vector 3. Finally, traversing across path CA, maintaining a constant angle, vector 3 goes over to the new vector $1'$. Notice that $1 \neq 1'$, thus, illustrating the above postulated.

To this end, we proceed by deriving a general formula for the change in a vector after parallel displacement around any infinitesimal closed contour. The change of the vector A_k, namely, ΔA_k, is written in the form $\oint \delta A_k$, where the integral is taken over the given contour. Replacing of δA_k in

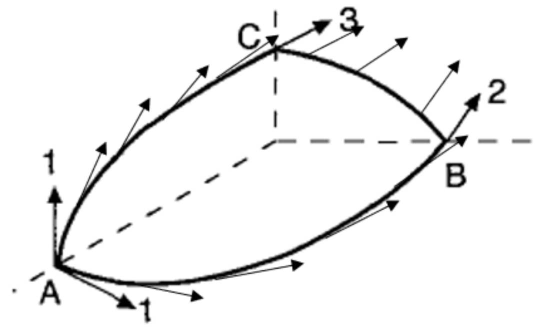


Figure 2.1: Parallel Transport of a vector

the expression (2.61), we obtain

$$\Delta A_k = \oint \Gamma^i_{kl} A_i dx^l \tag{2.146}$$

We note briefly that the reason that the equation for the change of a vector after parallel displacement takes the above form, is due to the fact that δA_k is defined as the change in the components of a vector under an infinitesimal displacement (2.457), which is exactly what was done, while the line integral is considered because we are interested in the path taken by the vector, as taking distinct paths in a curved surface yield contrasting values.

The values of the vector A_i at points inside the contour are not unique, since they yield particular values depending on the path taken to approach the point in question. We note that this non-uniqueness is related to terms of second order. Therefore, with first-order accuracy we may proceed to regard the components of the vector A_i at points inside the infinitesimal contour as being uniquely determined by the value on the contour by the formulas $\delta A_i = \Gamma^n_{ll} A_n dx^i$, i.e. by the derivatives

$$\frac{\partial A_i}{\partial x^l} = \Gamma^n_{il} A_n \tag{2.147}$$

Before we continue we define the general case of Stokes' theorem in four-dimensional space and describe the types of integration in four-space. In four-dimensional space there exists four types of integration; we solely mention those which pertain to the matter under consideration.

Integration over a (two-dimensional) surface in four-space. In four-space the infinitesimal element of surface is given by the antisymmetric tensor of second rank $df^{ik} = dx^i dx'^k - dx^k dx'^i$, in which its components are the projections of the element of area on the coordinate planes.

Integration over a four-dimensional volume, where the element of integration is the scalar

$$d\Omega = dx^0 dx^1 dx^2 dx^3 = c dt dV$$

The element is a scalar, thus, the volume of a portion of four-space remains invariant under rotation.

Analogous to Gauss' and Stokes' theorems in three-dimensional analysis, there exists theorems in four-dimensional space that also allow us to transform four-dimensional integrals. To this end, the integral over a closed hypersurface can be transformed into an integral over the four-volume contained within it by replacing the element of integration dS_i by the operator

$$dS_i \to d\Omega \frac{\partial}{\partial x^i}$$

For instance, for an arbitrary vector A_i the integral over a hypersurface may be written as

$$\oint A^i dS_i = \int \frac{\partial A^i}{\partial x^i} d\Omega$$

which is a generalization of Gauss' theorem, which we have previously seen. We note that in similar fashion as in its three-dimensional counterpart, we may reduce the integral over a hypersurface to an integral over a given volume by adding a derivative to the expression.

The integral over a four-dimensional closed curve is transformed into an integral over the surface spanning it by the substitution

$$dx^i \to df^{ki} \frac{\partial}{\partial x^k}$$

Hence, for the integral of the vector A_i, we obtain:

$$\oint A_i dx^i = \int df^{ki}\frac{\partial A_i}{\partial x^k} = \frac{1}{2}\int df^{ik}\left(\frac{\partial A_k}{\partial x^i} - \frac{\partial A_i}{\partial x^k}\right) \tag{2.148}$$

which is the generalization of Stokes' theorem.

Now applying Stokes' theorem (2.148) to the integral (2.146), and considering that the area enclosed by the contour has the infinitesimal value Δf^{lm}, we have:

$$\Delta A_k = \oint \Gamma^i_{kl} A_i dx^l$$
$$\implies \Delta A_k = \frac{1}{2}\left(\frac{\partial\left(\Gamma^i_{km}A_i\right)}{\partial x^l} - \frac{\partial\left(\Gamma^i_{kl}A_i\right)}{\partial x^m}\right)\Delta f^{lm}$$
$$\implies \Delta A_k = \frac{1}{2}\left(\frac{\partial \Gamma^i_{km}}{\partial x^l}A_i - \frac{\partial \Gamma^i_{kl}}{\partial x^m}A_i + \Gamma^i_{km}\frac{\partial A_i}{\partial x^l} - \Gamma^i_{kl}\frac{\partial A_i}{\partial x^m}\right)\Delta f^{lm}$$

Then substituting in the values obtained from the derivatives of (2.147), we obtain

$$\implies \Delta A_k = \frac{1}{2}\left(\frac{\partial \Gamma^i_{km}}{\partial x^l}A_i - \frac{\partial \Gamma^i_{kl}}{\partial x^m}A_i \Gamma^i_{nl}\Gamma^n_{km} - \Gamma^i_{nm}\Gamma^n_{kl}\right)\Delta f^{lm}$$

$$\Delta A_k = \frac{1}{2}R^i_{klm}A_i\Delta f^{lm} \tag{2.149}$$

where R^i_{klm} is the fourth rank tensor:

$$R^i_{klm} = \frac{\partial \Gamma^i_{km}}{\partial x^l} - \frac{\partial \Gamma^i_{kl}}{\partial x^m} + \Gamma^i_{nl}\Gamma^n_{km} - \Gamma^i_{nm}\Gamma^n_{kl} \tag{2.150}$$

We denote the tensor R^i_{klm} as the *curvature tensor* or the *Riemann tensor*. We claim that the quantity above is indeed a tensor since in (2.149) the left side of the expression is a vector, i.e. ΔA_k.

Next, determine a similar formula for the contravariant vector A^k. To do this, we note that under parallel displacement a scalar does not change in value, i.e. $\Delta\left(A^k B_k\right) = 0$, where B_k is any contravariant vector. Then using (2.149), we have that

$$\Delta\left(A^kB_k\right) = A^k\Delta B_k + B_k\Delta A^k = \frac{1}{2}A^kB_iR_i^{klm}\Delta f^{lm} + B_k\Delta A^k = \\ = B_k\left(\Delta A^k + \frac{1}{2}A^iR^k_{ilm}\Delta f^{lm}\right) = 0 \tag{2.151}$$

in view of the arbitrary nature of the covariant vector B_k, we obtain

$$\Delta A^k = -\frac{1}{2}R^k_{ilm}A^i\Delta f^{lm} \tag{2.152}$$

If we differentiate the vector A_i twice convariantly with respect to x^k and x^l, then the result will generally depend on the order of differentiation, unlike in ordinary differentiation. We find that the difference $A_{i,k;l} - A_{i;l;k}$ is given by the Riemman curvature tensor introduced above. Particularly, we find the expression

$$A_{i;k;l} - A_{i;l,k} = A_mR^m_{ikl} \tag{2.153}$$

which we verify considering a locally-geodesic coordinate system. To this end, consider such a system. Then using (2.66) calculate

$$A_{i;k;l} - A_{i;l;k} = \frac{\partial A_{i;k}}{\partial x^l} - \Gamma^m_{nl}A_m - \frac{\partial A_{i;l}}{\partial x^k} - \Gamma^m_{nl}A_m$$
$$\implies A_{i;k;l} - A_{i;l;k} = \frac{\partial A_{i;k}}{\partial x^l} - \frac{\partial A_{i;l}}{\partial x^k}$$
$$\implies A_{i;k;l} - A_{i;l;k} = \frac{\partial}{\partial x^l}\left(\frac{\partial A_i}{\partial x^k} - \Gamma^m_{ik}A_m\right) - \frac{\partial}{\partial x^k}\left(\frac{\partial A_i}{\partial x^l} - \Gamma^m_{il}A_m\right)$$
$$\implies A_{i;k;l} - A_{i;l;k} = \frac{\partial^2 A_i}{\partial x^k x^l} - \frac{\partial\left(\Gamma^m_{ik}A_m\right)}{\partial x^l} - \frac{\partial^2 A_i}{\partial x^l x^k} + \frac{\partial\left(\Gamma^m_{il}A_m\right)}{\partial x^k}$$
$$\implies A_{i;k;l} - A_{i;l;k} = \frac{\partial\left(\Gamma^m_{il}A_m\right)}{\partial x^k} - \frac{\partial\left(\Gamma^m_{ik}A_m\right)}{\partial x^l}$$
$$\implies A_{i;k;l} - A_{i;l;k} = \left(\frac{\partial \Gamma^m_{il}}{\partial x^k}A_m - \frac{\partial \Gamma^m_{ik}}{\partial x^l}A_m + \Gamma^m_{il}\frac{\partial A_m}{\partial x^k} - \Gamma^m_{ik}\frac{\partial A_m}{\partial x^l}\right)$$

Similarly, substituting in the values obtained from (2.147), factoring A_m, and using the calculation for (2.149) as an analogy we obtain

$$A_{i;k;l} - A_{i;l,k} = A_m R^m_{ikl} \tag{2.154}$$

On the other hand, for the covariant vector we obtain

$$A^i_{;k,l} - A^i_{;l;k} = -A^m R^i_{mkl} \tag{2.155}$$

in which we employed the technique of raising the index i and using the symmetry properties of the curvature tensor.

Equivalently, in a straight forward manner we derive formulas for the second derivatives of tensors (which is achieved by considering a tensor of the form $A_i B_k$, and applying formula (2.153) and (2.155)); then by linearity, the expressions found must hold for an arbitrary tensor A_{ik}. Hence,

$$A_{ik,l;m} - A_{ik,m;l} = A_{in} R^n_{klm} + A_{nk} R^n_{im} \tag{2.156}$$

Considering the intrinsic properties of the curvature tensor, we find that in flat space the curvature tensor is equivalently zero, for, in flat space, we can choose the coordinates in such a way that the Christoffel symbols defining the Riemann tensor are equal to zero over all space, i.e. $\Gamma^i_{kl} = 0$ implying that $R^i_{klm} = 0$. Then by the nature of the Riemann tensor, if it is equal to zero in a given coordinate system it must be zero in every other coordinate system; which is related to the fact that in flat space the parallel displacement of a vector is a single-valued operation such that creating a circuit of an enclosed contour keeps the vector invariant.

Conversely, if the Riemann tensor is equivalently zero, then the space must be flat. Peculiarly, within any space we may choose a coordinate system which is galilean over a given infinitesimal region. If $R^i_{klm} = 0$, then by extending the parallel displacement of the galilean system from the given infinitesimal region to the whole space, we may prove that the space is indeed Euclidean.

We may consider a neighborhood over a given point which is galilean and then extend such a neighborhood further, through the use of a union of a neighborhoods which also feature this galilean requisite, covering all space and demonstrating that the space under consideration is Euclidean.

To this end, we can viably determine the geometry of space, in regards to being curved or flat, by employing the criterion imposed upon the curvature tensor, in which the vanishing or nonvanishing of the tensor determines the circumstances of space.

Finally, we note that we may choose a point in which the coordinate system is locally-geodesic within curved space, yet the curvature tensor need not vanish at this given point, since the derivatives of the Christoffel symbols need not tend to zero when Γ^i_{kl} is zero.

2.11 Properties of the Riemann curvature tensor

We now consider the symmetry properties displayed by the curvature tensor. We start with changing from the mixed components tensor R^i_{klm} to the covariant tensor R_{iklm}, through the use of the space-time metric:

$$R_{iklm} = g_{in} R^n_{klm} \tag{2.157}$$

By means of a transformation we obtain the expression:

$$R_{iklm} = \frac{1}{2}\left(\frac{\partial^2 g_{im}}{\partial x^k \partial x^l} + \frac{\partial^2 g_{kl}}{\partial x^i \partial x^m} - \frac{\partial^2 g_{il}}{\partial x^k \partial x^m} - \frac{\partial^2 g_{km}}{\partial x^i \partial x^l}\right) + g_{np}\left(\Gamma^n_{kl}\Gamma^p_{im} - \Gamma^n_{km}\Gamma^p_{il}\right) \tag{2.158}$$

We compute the not so apparent expression displayed above. Consider the Riemann tensor with Greek indices for the following computation, note that the indices selected is irrelevant to the final outcome, as the yielded product differs not if considering Greek or Latin indices. To this end, rewrite the curvature tensor as:

$$R^\delta_{\mu\gamma\beta} = \frac{\partial \Gamma^\delta_{\mu\beta}}{\partial x^\gamma} - \frac{\partial \Gamma^\delta_{\mu\gamma}}{\partial x^\beta} + \Gamma^\nu_{\mu\beta}\Gamma^\delta_{\nu\gamma} - \Gamma^\nu_{\mu\gamma}\Gamma^\delta_{\nu\beta}$$

Then using (2.157) we write

$$
\begin{aligned}
R_{\alpha\mu\gamma\beta} &= g_{\alpha\delta} R^{\delta}_{\mu\gamma\beta} \\
\Longrightarrow &= g_{\alpha\delta}\left(\frac{\partial \Gamma^{\delta}_{\mu\beta}}{\partial x^{\gamma}} - \frac{\partial \Gamma^{\delta}_{\mu\gamma}}{\partial x^{\beta}} + \Gamma^{\nu}_{\mu\beta}\Gamma^{\delta}_{\nu\gamma} - \Gamma^{\nu}_{\mu\gamma}\Gamma^{\delta}_{\nu\beta}\right) \\
\Longrightarrow &= \frac{\partial \Gamma_{\alpha\mu\beta}}{\partial x^{\gamma}} - \frac{\partial \Gamma_{\alpha\mu\gamma}}{\partial x^{\beta}} + g_{\alpha\delta}\left(\Gamma^{\nu}_{\mu\beta}\Gamma^{\delta}_{\nu\gamma} - \Gamma^{\nu}_{\mu\gamma}\Gamma^{\delta}_{\nu\beta}\right) \\
\Longrightarrow &= \partial_{\gamma}\Gamma_{\alpha\mu\beta} - \partial_{\beta}\Gamma_{\alpha\mu\gamma} + g_{\alpha\delta}\left(\Gamma^{\delta}_{\nu\gamma}\Gamma^{\nu}_{\mu\beta} - \Gamma^{\delta}_{\nu\beta}\Gamma^{\nu}_{\mu\gamma}\right)
\end{aligned}
$$

Then using (2.458) we have

$$
\begin{aligned}
R_{\alpha\mu\gamma\beta} &= \partial_{\gamma}\frac{1}{2}\left(\partial_{\mu}g_{\alpha\beta} + \partial_{\beta}g_{\mu\alpha} - \partial_{\alpha}g_{\mu\beta}\right) - \partial_{\beta}\frac{1}{2}\left(\partial_{\mu}g_{\alpha\gamma} + \partial_{\gamma}g_{\mu\alpha} - \partial_{\alpha}g_{\mu\gamma}\right) + g_{\alpha\delta}\left(\Gamma^{\delta}_{\nu\gamma}\Gamma^{\nu}_{\mu\beta} - \Gamma^{\delta}_{\nu\beta}\Gamma^{\nu}_{\mu\gamma}\right) \\
\Longrightarrow &= \frac{1}{2}\left(\partial_{\gamma}\partial_{\mu}g_{\alpha\beta} + \partial_{\gamma}\partial_{\beta}g_{\mu\alpha} - \partial_{\gamma}\partial_{\alpha}g_{\mu\beta}\right) - \frac{1}{2}\left(\partial_{\beta}\partial_{\mu}g_{\alpha\gamma} + \partial_{\beta}\partial_{\gamma}g_{\mu\alpha} - \partial_{\beta}\partial_{\alpha}g_{\mu\gamma}\right) + g_{\alpha\delta}\left(\Gamma^{\delta}_{\nu\gamma}\Gamma^{\nu}_{\mu\beta} - \Gamma^{\delta}_{\nu\beta}\Gamma^{\nu}_{\mu\gamma}\right) \\
\Longrightarrow &= \frac{1}{2}\left(\partial_{\gamma}\partial_{\mu}g_{\alpha\beta} - \partial_{\gamma}\partial_{\alpha}g_{\mu\beta} - \partial_{\beta}\partial_{\mu}g_{\alpha\gamma} + \partial_{\beta}\partial_{\alpha}g_{\mu\gamma}\right) + g_{\alpha\delta}\left(\Gamma^{\delta}_{\nu\gamma}\Gamma^{\nu}_{\mu\beta} - \Gamma^{\delta}_{\nu\beta}\Gamma^{\nu}_{\mu\gamma}\right)
\end{aligned}
$$

which is equivalent to (2.366). Then from (2.366) we obtain the following relations

$$R_{iklm} = -R_{kilm} = -R_{ikml} \tag{2.159}$$

$$R_{iklm} = R_{lmik} \tag{2.160}$$

where we find that the tensor is antisymmetric in the first and second pair of indices $i,k$and l,m, and symmetric upon interchanging the two pairs with each other. We verify the prior formulations by considering a point in which the coordinate system is locally-geodesic within curved space, i.e. $\Gamma^{i}_{kl} = 0$, thus the second term of the curvature tensor vanishes and we are left with:

$$R_{iklm} = \frac{1}{2}\left(\partial_{k}\partial_{l}g_{im} + \partial_{i}\partial_{m}g_{kl} - \partial_{k}\partial_{m}g_{il} - \partial_{i}\partial_{l}g_{km}\right) \tag{2.161}$$

Then permuting the indices as in (2.402) we find

$$R_{kilm} = \frac{1}{2}\left(\partial_i\partial_l g_{km} + \partial_k\partial_m g_{il} - \partial_i\partial_m g_{kl} - \partial_k\partial_l g_{im}\right)$$
$$\Longrightarrow = \frac{1}{2}\left(-\partial_k\partial_l g_{im} - \partial_i\partial_m g_{kl} + \partial_k\partial_m g_{il} + \partial_i\partial_l g_{km}\right)$$
$$\Longrightarrow = -R_{iklm}$$

Similarly, we have

$$R_{ikml} = \frac{1}{2}\left(\partial_k\partial_m g_{il} + \partial_i\partial_l g_{km} - \partial_k\partial_l g_{im} - \partial_i\partial_m g_{kl}\right)$$
$$\Longrightarrow = \frac{1}{2}\left(-\partial_k\partial_l g_{im} - \partial_i\partial_m g_{kl} + \partial_k\partial_m g_{il} + \partial_i\partial_l g_{km}\right)$$
$$\Longrightarrow = -R_{iklm}$$

Then for (2.160) we find

$$R_{lmik} = \frac{1}{2}\left(\partial_m\partial_i g_{lk} + \partial_l\partial_k g_{mi} - \partial_m\partial_k g_{li} - \partial_l\partial_i g_{mk}\right)$$
$$\Longrightarrow = R_{iklm}$$

where we note that the order of the derivatives is not important, as the partial derivative commute, and that the metric tensor is symmetric, i.e. $g_{ik} = g_{ki}$. Then, using (2.402) and (2.160) we obtain the cyclic sum of the components of R_{iklm} which is derived by permutation of any three indices such that

$$R_{iklm} + R_{imkl} + R_{ilmk} = 0 \tag{2.162}$$

Indeed, we compute directly, again considering a locally-geodesic system, such that

$$\begin{aligned}
R_{iklm} &= \frac{1}{2}\left(\partial_k\partial_l g_{im} + \partial_i\partial_m g_{kl} - \partial_k\partial_m g_{il} - \partial_i\partial_l g_{km}\right) \\
+R_{iklm} &= \frac{1}{2}\left(\partial_m\partial_k g_{il} + \partial_i\partial_l g_{mk} - \partial_m\partial_l g_{ik} - \partial_i\partial_k g_{ml}\right) \\
R_{ilmk} &= \frac{1}{2}\left(\partial_l\partial_m g_{ik} + \partial_i\partial_k g_{lm} - \partial_l\partial_k g_{im} - \partial_i\partial_m g_{lk}\right) \\
&\Longrightarrow = 0
\end{aligned}$$

Subsequently, we prove the Bianchi identity:

$$R^n_{ikl;m} + R^n_{imk;l} + R^n_{ilm;k} = 0 \tag{2.163}$$

Once again we use the strategy of choosing a point which leads to a locally-geodesic coordinate system. Due to its tensor character, the expression described in (2.163) will be valid in all other coordinate systems. Then differentiation (2.150) and noting that $\Gamma^i_{kl} = 0$, we find that for the point under consideration we may write

$$R^n_{ikl;m} = \frac{\partial R^n_{ikl}}{\partial x^m} = \frac{\partial^2 \Gamma^n_{il}}{\partial x^m \partial x^k} - \frac{\partial^2 \Gamma^n_{ik}}{\partial x^m \partial x^l} \tag{2.164}$$

Indeed, we proceed as follows,

$$\begin{aligned}
R^n_{ikl} &= \frac{\partial \Gamma^n_{il}}{\partial x^k} - \frac{\partial \Gamma^n_{ik}}{\partial x^l} + \Gamma^n_{pk}\Gamma^p_{il} - \Gamma^n_{pl}\Gamma^p_{ik} \\
R^n_{imk} &= \frac{\partial \Gamma^n_{ik}}{\partial x^k} - \frac{\partial \Gamma^n_{im}}{\partial x^m} + \Gamma^n_{pm}\Gamma^p_{ik} - \Gamma^n_{pk}\Gamma^p_{im} \\
R^n_{ilm} &= \frac{\partial \Gamma^n_{im}}{\partial x^l} - \frac{\partial \Gamma^n_{il}}{\partial x^m} + \Gamma^n_{pl}\Gamma^p_{im} - \Gamma^n_{pm}\Gamma^p_{il} \\
&\Longrightarrow R^n_{ikl} = \frac{\partial \Gamma^n_{il}}{\partial x^k} - \frac{\partial \Gamma^n_{ik}}{\partial x^l} \\
&\Longrightarrow R^n_{imk} = \frac{\partial \Gamma^n_{ik}}{\partial x^k} - \frac{\partial \Gamma^n_{im}}{\partial x^m} \\
&\Longrightarrow R^n_{ilm} = \frac{\partial \Gamma^n_{im}}{\partial x^l} - \frac{\partial \Gamma^n_{il}}{\partial x^m}
\end{aligned}$$

Since, again we're considering a locally-geodesic coordinate system. Then differentiating each term according to (2.163) and adding we obtain

$$\begin{aligned}
R^n_{ikl;m} &= \frac{\partial R^n_{ikl}}{\partial x^m} = \frac{\partial^2 \Gamma^n_{il}}{\partial x^m \partial x^k} - \frac{\partial^2 \Gamma^n_{ik}}{\partial x^m \partial x^l} \\
+R^n_{imk;l} &= \frac{\partial R^n_{imk}}{\partial x^l} = \frac{\partial^2 \Gamma^n_{ik}}{\partial x^m \partial x^l} - \frac{\partial^2 \Gamma^n_{im}}{\partial x^k \partial x^l} \\
R^n_{ilm;k} &= \frac{\partial R^n_{ilm}}{\partial x^k} = \frac{\partial^2 \Gamma^n_{im}}{\partial x^l \partial x^k} - \frac{\partial^2 \Gamma^n_{il}}{\partial x^m \partial x^k} \\
&\implies = 0
\end{aligned}$$

we note that if we were to consider the covariant derivative for the above Riemann tensors, due to taking into consideration a locally-geodesic system of coordinates, then from (2.72) (and for any arbitrary ranked tensor) we have that the terms formed from the Christoffel symbols vanish and we are left with an ordinary derivative.

By contraction of the Riemann curvature tensor we can construct a second tank tensor. Now, as described by (2.402) it is in the nature of the curvature tensor to be asymmetric when permuting the two adjacent pair of indices i,k and l,m, thus, contraction on the tensor R_{iklm} on the aforementioned pair of indices would lead to a value of zero. Thus, we may contract using any other pair, i.e. i and l for example, such that we obtain the same result, except for sign. Hence, we define the *Ricci tensor* as follows

$$R_{ik} = g^{lm} R_{limk} = R^l_{ilk} \tag{2.165}$$

According to (2.150), we have:

$$R_{ik} = \frac{\partial \Gamma^l_{ik}}{\partial x^l} - \frac{\partial \Gamma^l_{il}}{\partial x^k} + \Gamma^l_{ik}\Gamma^m_{lm} - \Gamma^m_{il}\Gamma^l_{km} \tag{2.166}$$

The tensor is symmetric:

$$R_{ik} = R_{ki} \tag{2.167}$$

We start from the cyclic property of the Riemann tensor and proceed to contract the tensor in order to prove the symmetry of the Ricci tensor. Indeed, consider

$$R_{lmik} + R_{likm} + R_{lkmi} = 0$$

$$\Longrightarrow g^{ln}\left(R_{lmik} + R_{likm} + R_{lkmi}\right) = 0$$

$$\Longrightarrow R^n_{mik} + R^n_{ikm} + R^n_{kmi} = 0$$

$$\Longrightarrow R^n_{mnk} + R^n_{nkm} + R^n_{kmn} = 0$$

$$\Longrightarrow R^n_{mnk} + 0 + R^n_{kmn} = 0$$

$$\Longrightarrow R^n_{mnk} + 0 + R^n_{kmn} = 0$$

$$\Longrightarrow R_{mk} - R_{km} = 0$$

$$\Longrightarrow R_{mk} = R_{km}$$

where we used the symmetry properties of the curvature tensor (2.366) and (2.402), and contraction in the n and i index, noticing that as long as contraction did not occur between pair of indices directly adjacent to each other, the result would remain the same, as well as the fact that interchanging adjacent pair of indices changed the sign of the tensor.

Finally, contraction of the Ricci tensor will lead to the invariant scalar

$$R = g^{ik} R_{ik} = g^{il} g^{km} R_{iklm} \tag{2.168}$$

which is known as the *scalar curvature* of the space.

The components of the Ricci tensor R_{ik} satisfy a differential identity we obtain from contracting the Bianchi identity (2.163) on the pair of indices ik and ln:

$$R^l_{m;l} = \frac{1}{2}\frac{\partial R}{\partial x^m} \tag{2.169}$$

Due to (2.402) - (2.162) not all components of the curvature tensor are independent.

Indeed, starting with the Bianchi identity (2.163) and by contracting twice we prove the formulation above:

$$\begin{aligned}
R_{abmn;l} + R_{ablm;n} + R_{abnl;m} &= 0 \\
\implies g^{bn} g^{am} \left(R_{abmn;l} + R_{ablm;n} + R_{abnl;m}\right) &= 0 \\
\implies g^{bn} \left(R^{m}_{bmn;l} - R^{m}_{bml,n} + R^{m}_{bnl;m}\right) &= 0 \\
\implies g^{bn} \left(R_{bn;l} - R_{bl;n} - R^{m}_{bnl;m}\right) &= 0 \\
\implies R^{n}_{n;l} - R^{n}_{l;n} - R^{nm}_{nl;m} &= 0 \\
\implies R_{;l} - R^{n}_{l;n} - R^{m}_{l;m} &= 0
\end{aligned}$$

we note that the first term contracted to form the Ricci scalar, while the third term contracted to yield a mixed Ricci tensor. Nonetheless replacing indices n to m (or vice versa), we can combine the last two terms and move the result to the right such that:

$$\begin{aligned}
R_{;l} &= 2R^{m}_{l;m} \\
\implies \frac{1}{2}R_{;l} &= R^{m}_{l;m}
\end{aligned}$$

which is equivalent to (2.169). We mention in passing that contraction on the third and fourth indices yields an equivalent tensor, differing only by the sign of the tensor, this is due to the symmetry properties of the curvature tensor displayed by (2.402) and (2.160).

Let us determine the number of independent components. We proceed by taking into account the symmetry and antisymmetry properties of the Riemann curvature tensor, with which we determine the number of independent components in n-dimensions. As previously shown, we begin with the fact that the curvature tensor is antisymmetric in the first and last two pair indices, yet symmetric upon interchanging of the two pairs of indices, as per (2.402) and (2.160). Hence, we may think of the Riemann tensor as a symmetric matrix $R_{[lm][ik]}$, in which we treat the pair of indices $[lm]$ and $[ik]$ as individual indices. Note that an mxm symmetric matrix consists of $m(m+1)/2$ independent components, while an nxn antisymmetric matrix has $n(n-1)/2$ independent components. Thus, we have,

$$\frac{1}{2}\left[\frac{1}{2}n(n-1)\right]\left[\frac{1}{2}n(n-1)+1\right]=\frac{1}{8}\left(n^4-2n^3+3n^2-2n\right)$$

independent components. Nonetheless, by the fact that the Riemann tensor displays a cyclic property (2.162), this leads to the fact that the totally antisymmetric part of the Riemann tensor vanishes

$$R_{[lmik]}=0$$

implying that there exists $n(n-1)(n-2)(n-3)/4!$ further constraints on the independent components leading to,

$$\frac{1}{8}\left(n^4-2n^3+3n^2-2n\right)-\frac{1}{24}n(n-1)(n-2)(n-3)=\frac{1}{12}n^2\left(n^2-1\right) \tag{2.170}$$

independent components of the curvature tensor.

To this end, consider a two-dimensional space, i.e. an ordinary surface. Denote the curvature tensor by P_{abcd} for this two dimensional case and the corresponding metric tensor as γ_{ab}, in which the a, b, indices run through the values $1,2$. Then, computing the number of independent components for the scenario under consideration, we find that there exists only 1 independent component, let it be denoted by P_{1212}. Subsequently, we find that due to the nature of the single independent component there must exist an unambiguous relation between the sole component of the ordinary surface and the resulting scalar curvature. We thus find that the scalar curvature is determined by the relation

$$P=\frac{2P_{1212}}{\gamma}, \quad \gamma\equiv|\gamma_{ab}|=\gamma_{11}\gamma_{22}-(\gamma_{12})^2 \tag{2.171}$$

Indeed, starting with the Ricci tensor we have,

$$R_{ik} = R^{l}_{ilk} = R^{1}_{i1k} + R^{2}_{i2k}$$

then, since the Riemann tensor is antisymmetric in the last two pair of indices we write

$$R_{11} = R^{1}_{111} + R^{2}_{121} = 0 + R^{2}_{121}$$
$$R_{12} = R^{1}_{112} + R^{2}_{122} = R^{1}_{112} + 0$$
$$R_{21} = R^{1}_{211} + R^{2}_{221} = 0 + R^{2}_{221}$$
$$R_{22} = R^{1}_{212} + R^{2}_{222} = R^{1}_{212} + 0$$

Then, using the scalar curvature

$$R = g^{ik} R_{ik} = g^{11} R^{2}_{121} + g^{12} R^{1}_{112} + g^{21} R^{2}_{221} + g^{22} R^{1}_{212} \tag{2.172}$$

and noting that in $2-D$ space the inverse metric takes the explicit form

$$g^{ik} = \frac{1}{g_{11}g_{22} - g_{12}g_{21}} \begin{pmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11} \end{pmatrix} \tag{2.173}$$

Then, we have

$$R = g_{22} R^{2}_{121} - g_{12} R^{1}_{112} - g_{21} R^{2}_{221} + g_{11} R^{1}_{212}$$
$$\implies R_{2121} - g_{21} R^{1}_{112} - g_{12} R^{2}_{221} + R_{1212}$$
$$\implies R_{2121} - R_{2112} - R_{1221} + R_{1212}$$
$$\implies R_{2121} - R_{2112} + R_{2121} + R_{1212}$$
$$\implies R_{1212} - R_{2112}$$

where we used the symmetry properties of the metric tensor, as well as the antisymmetry properties of the Riemann tensor in the first two indices. We also have that

$$\begin{aligned} R_{1212} &= g_{1l}R^{l}_{212} = g_{11}R^{1}_{212} + g_{12}R^{2}_{212} \\ R_{2112} &= g_{2l}R^{l}_{112} = g_{21}R^{1}_{112} + g_{22}R^{2}_{112} \end{aligned} \tag{2.174}$$

Thus, we have that

$$\begin{aligned} R &= \frac{1}{g_{11}g_{22} - g_{12}g_{21}} \left(R_{1212} - R_{2112}\right) \\ \implies &= \frac{1}{g_{11}g_{22} - g_{12}g_{21}} \left(R_{1212} + R_{1212}\right) \\ \implies &= \frac{2}{g_{11}g_{22} - g_{12}g_{21}} R_{1212} \end{aligned}$$

which coincides with (2.171).

Then, we note that the quantity $P/2$ coincides with the *Gaussian curvature* K of the surface:

$$\frac{P}{2} = K = \frac{1}{\rho_1 \rho_2} \tag{2.175}$$

where the ρ_1, ρ_2 are the principal radii of curvature of the surface at a particular point.

We briefly elaborate upon the Gaussian curvature. As previously stated, we consider first a two dimensional case. To this end, we shall define the curvature in a slightly different manner, all be it a geometrical manner. Indeed, as previously mentioned in (2.175) we define the Gaussian curvature in the same manner, nonetheless, we let ρ_1, ρ_2 be the principal radii of curvature for perpendicular geodesics passing through the particular point, p. Then, if we consider the 2-sphere, S^2, we notice that upon constructing a triangle in such a space, in which each side is a geodesic, the interior angles do not, in general, sum up to π or $180°$, rather their sum is larger than they would be if we considered a flat, Euclidean space. Then for the 2-sphere we have that $\rho_1 = \rho_2$, regardless of orientation within the surface, hence, we may write $K = \frac{1}{\rho_1 \rho_2}$. Then, we may deduce that the

Gaussian curvature may take the same value for all points on the surface, and we may then denote this as the space of constant positive curvature. Contrary to the preceding space proposed, we may then find an analogous space of constant negative curvature, in which the value of K is negative, H^2. Here, we note that at all points $p \in H^2$ the radii of curvature of two perpendicular curves that pass through point p are the same but lie on *opposite* sides of the surface, thus we have that $K = -\frac{1}{\rho_1 \rho_2}$. Upon transcription of a triangle in the center of such a space, the sum of the interior angles will yield a value less than π or 180°. We may then illustrate such a space by a drawing a saddle. Below we present simple illustrations of the proposed definitions.

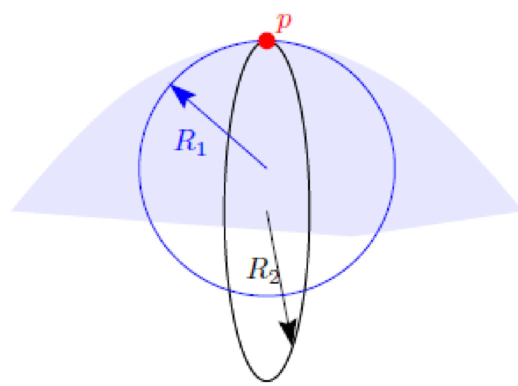


Figure 2.2: Positive Curvature

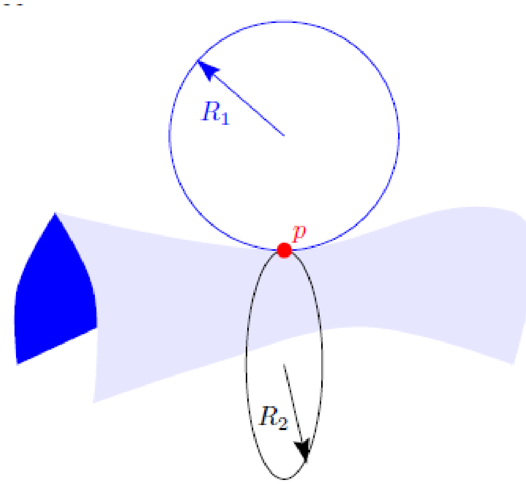


Figure 2.3: Negative Curvature

Hence, we can define the curvature for 2-D surfaces, by taking the sum of the angles θ, as follows,

$$K > 0 : \sum_{i=1}^{3} \theta_i > \pi$$

$$K = 0 : \sum_{i=1}^{3} \theta_i = \pi$$

$$K < 0 : \sum_{i=1}^{3} \theta_i < \pi$$

where if $K > 0$ the we have a positive constant curvature, if $K = 0$ we have a flat, Euclidean space with no curvature, and if $K < 0$ then we have a negative constant curvature.

Next, considering the curvature tensor in three dimensions, we find that $P_{\alpha\beta\gamma\delta}$ (the Riemann tensor for the space under consideration) and the metric tensor $\gamma_{\alpha\beta}$, in which the indices α, β take values from $1,2,3$, have upon using (2.170) we have that the curvature tensor has 6 independent components. The symmetric tensor $P_{\alpha\beta}$ also contains the same number of independent components. Hence, from the linear relation of the Ricci tensor to the Riemann tensor, i.e. $P_{\alpha\beta} = g^{\gamma\delta} P_{\gamma\alpha\delta\beta}$ all the components of the curvature tensor may be expressed in terms of the Ricci tensor and the metric tensor $\gamma_{\alpha\beta}$. Selection of a system of coordinates which takes a cartesian configuration at a given point, and then by a suitable rotation we can bring the tensor $P_{\alpha\beta}$ to principal axes. Hence, the curvature tensor of a three-dimensional space at a give point is determined by three quantities.

Finally, we define the independent components of the curvature tensor in four-dimensions. Again, solving (2.170) for the number of independent components, we find that in 4-D space, the Riemann tensor consists of 20 independent components. Here, the pair of indices ik and lm take six distinct values: $01, 02, 03, 23, 31, 12$. Hence, there exists six components of R_{iklm} which are identical, and $6 \cdot 5/2$ with distinct pair of indices. Nonetheless, the latter of which are still not independent of one another; the three components for which all four indices are different are related by the identity:

$$R_{0123} + R_{0312} + R_{0231} = 0 \tag{2.176}$$

By selecting a coordinate system which is galilean at a particular point and considering transformations which rotate the system (such that g_{ik} remains invariant upon such transformation), we can achieve the vanishing of six of the components of the curvature, leaving 14 quantities which, in general, determine the curvature tensor in 4 space.

We briefly discuss the case in which the curvature tensor can be defined by the value of the Ricci tensor. Hence, if $R_{ik} = 0$, then the curvature tensor has a total of 10 independent components in an arbitrary coordinate system. In this case, by choosing a suitable transformation we can bring the tensor R_{iklm}, at a given point of four-space, to a "canonical" form, in which its components are expressed in terms of four quantities.

However, considering the case in which the Ricci tensor is not equal to zero, $R_{ik} \neq 0$, then we may once again use the same classification for the curvature tensor after one has subtracted from it a particular part that is expressible in terms of the components R_{ik}. Namely, we may write the tensor

$$C_{iklm} = R_{iklm} - \frac{1}{2}R_{il}g_{km} + \frac{1}{2}R_{im}g_{kl} + \frac{1}{2}R_{kl}g_{im} - \frac{1}{2}R_{km}g_{il} + \frac{1}{6}R\left(g_{il}g_{km} - g_{im}g_{kl}\right) \tag{2.177}$$

we note that the above tensor has symmetry properties of the tensor R_{iklm}, but vanishes upon contraction on a pair of indices.

2.12 The action of the gravitational field

In order to derive the equations determining the gravitational field, we must first determine the action of the both the field under consideration alongside the material particles present within the field, after which one must take a sum of the two.

To this end, we proceed much in the same manner as proposed by (2.97), in which we consider the action of the field, rather than that of a particle. We note that the action S_g must be expressed in terms of a scalar integral $\int G\sqrt{-g}d\Omega$ taken over all space and the time coordinate x^0 between two values. We commence by placing a restriction upon the scalar of the action function, in which the equations of the field of gravity must contain the derivatives of the "potentials" no higher than the second order, hence, it is required that the integrand G of the action function contain derivatives of the metric g_{ik} no higher than first order (thus it will be defined by g_{ik} and the quantities Γ^i_{kl}).

Yet, we find that it is general impossible to determine an invariant scalar solely by using the quantities g_{ik} and Γ^i_{kl}, since as we have previously seen, we may always find a coordinate system in which the Christoffel symbols vanish at a particular point, $\Gamma^i_{kl} = 0$. This lingering set-back begs a distinct approach to the situation at hand. Thus, we point attention to the scalar curvature R, which even though it contains derivatives of the metric to the second order $\partial^2 g_{ik}$, alongside the required derivatives of first order and the metric alone, linear in its second order derivatives. Due to this, we may transform the invariant integral $\int R\sqrt{-g}d\Omega$ using Gauss' Theorem to the required integral, in which the second derivatives vanish. Namely, we note that the integral $\int R\sqrt{-g}d\Omega$ can be represented in the form

$$\int R\sqrt{-g}d\Omega = \int G\sqrt{-g}d\Omega + \int \frac{\partial\left(\sqrt{-g}w^i\right)}{\partial x^i}d\Omega$$

in which G contains only the required tensor g_{ik} and its first derivative, while the remaining second order derivatives of the scalar curvature are then compressed into the integrand of the second integral, which is the divergence of a certain quantity w^i. Indeed, we find that this is possible due to the linearity of the scalar curvature in its second derivatives, since we are able to separate the desired quantities of the metric and its first derivative from the vexatious second derivatives. As aforementioned, we may transform the resulting integral over the four-volume into an integral over a closed hypersurface, according to Gauss' Theorem, i.e.

$$\int \frac{\partial\left(\sqrt{-g}w^i\right)}{\partial x^i}d\Omega \to \oint \left(\sqrt{-g}w^i\right)dS_i$$

Then, when varying the action, the variation of the second term (previously presented) vanishes, since by the principle of least action, the variation of the fields at the limits of the region (or the surface) of integration are zero. Consequently, we write

$$\delta \int R\sqrt{-g}d\Omega = \delta \int G\sqrt{-g}d\Omega$$

We note that the left side of the equation is a scalar; hence, the expression on the right must also be a scalar, yet the quantity G is not a scalar.

Hence, we find that the quantity G fulfills the restriction proposed, since it contains only the g_{ik} and its first derivatives. Thus we write,

$$\delta S_g = -\frac{c^3}{16\pi k}\delta \int G\sqrt{-g}d\Omega = -\frac{c^3}{16\pi k}\delta \int R\sqrt{-g}d\Omega \qquad (2.178)$$

in which k is a new universal constant. The constant k is called the *gravitational constant*. The dimensions of k are found by using (2.178), in which we have that the action has dimensions $gm - cm^2 - sec^{-1}$, all coordinates have dimensions cm, the metric g_{ik} is dimensionless, and the scalar curvature has dimensions cm^{-2}. Thus, we find that the required dimensions for the gravitational constant are $cm^3 - gm^{-1} - sec^{-2}$, with value of

$$k = 6.67 \times 10^{-8}\mathrm{cm}^3 - \mathrm{gm}^{-1} - \mathrm{sec}^{-2} \qquad (2.179)$$

Finally, we proceed by determining the quantity G, by using the formulated equation (2.178). Then, substituting in the value of the Ricci tensor R_{ik} given by (2.169), we have

$$\sqrt{-g}R = \sqrt{-g}g^{ik}R_{ik} = \sqrt{-g}\left\{g^{ik}\frac{\partial \Gamma^l_{ik}}{\partial x^l} - g^{ik}\frac{\partial \Gamma^l_{il}}{\partial x^k} + g^{ik}\Gamma^l_{ik}\Gamma^m_{lm} - g^{ik}\Gamma^m_{il}\Gamma^l_{km}\right\} \qquad (2.180)$$

In the first two terms on the right, we have after distributing the quantity $\sqrt{-g}$,

$$\sqrt{-g}g^{ik}\frac{\partial\Gamma^l_{ik}}{\partial x^l} = \frac{\partial}{\partial x^l}\left(\sqrt{-g}g^{ik}\Gamma^l_{ik}\right) - \Gamma^l_{ik}\frac{\partial}{\partial x^l}\left(\sqrt{-g}g^{ik}\right)$$
$$\sqrt{-g}g^{ik}\frac{\partial\Gamma^l_{il}}{\partial x^k} = \frac{\partial}{\partial x^k}\left(\sqrt{-g}g^{ik}\Gamma^l_{il}\right) - \Gamma^l_{il}\frac{\partial}{\partial x^k}\left(\sqrt{-g}g^{ik}\right)$$

We prove the above proposed formulation. It suffices to consider only one case, as the second case follows analogously. Hence,

$$\sqrt{-g}g^{ik}\frac{\partial\Gamma^l_{ik}}{\partial x^l} = \frac{\partial}{\partial x^l}\left(\sqrt{-g}g^{ik}\Gamma^l_{ik}\right) - \Gamma^l_{ik}\frac{\partial}{\partial x^l}\left(\sqrt{-g}g^{ik}\right)$$
$$\Longrightarrow = \left(\partial_k\sqrt{-g}\right)g^{ik}\Gamma^l_{ik} + \sqrt{-g}\left(\partial g^{ik}\right)\Gamma^l_{ik} + \sqrt{-g}g^{ik}\left(\partial_k\Gamma^l_{ik}\right) - \Gamma^l_{ik}\left(\partial_k\sqrt{-g}\right)g^{ik} - \Gamma^l_{ik}\sqrt{-g}\left(\partial_k g^{ik}\right)$$
$$\Longrightarrow = \sqrt{-g}g^{ik}\frac{\partial\Gamma^l_{ik}}{\partial x^l}$$

which follows after canceling like terms. Then dropping the total derivative, i.e. the first two terms of the above equation on the right, we obtain,

$$\sqrt{-g}G = \Gamma^m_{im}\frac{\partial}{\partial x^k}\left(\sqrt{-g}g^{ik}\right) - \Gamma^l_{ik}\frac{\partial}{\partial x^l}\left(\sqrt{-g}g^{ik}\right) - \left(\Gamma^m_{il}\Gamma^l_{km} - \Gamma^l_{ik}\Gamma^m_{lm}\right)g^{ik}\sqrt{-g} \tag{2.181}$$

Indeed, since by (2.178), we may set the integrands equal to each other, i.e.

$$\delta S_g = -\frac{c^3}{16\pi k}\delta\int G\sqrt{-g}d\Omega = -\frac{c^3}{16\pi k}\delta\int R\sqrt{-g}d\Omega$$
$$\Longrightarrow\ G\sqrt{-g} = R\sqrt{-g}$$

it then follows that (2.181) holds. Next, using formulas (2.84)-(2.87), we find that the first two terms on the right are equivalent to $\sqrt{-g}$ multiplied by

$$2\Gamma^l_{ik}\Gamma^i_{lm}g^{mk} - \Gamma^m_{im}\Gamma^i_{kl}g^{kl} - \Gamma^l_{ik}\Gamma^m_{lm}g^{ik} = g^{ik}\left(2\Gamma^l_{mk}\Gamma^m_{li} - \Gamma^m_{lm}\Gamma^l_{ik} - \Gamma^l_{ik}\Gamma^m_{lm}\right)$$
$$= 2g^{ik}\left(\Gamma^m_{il}\Gamma^l_{km} - \Gamma^l_{ik}\Gamma^m_{lm}\right)$$

Starting with (2.85), we have

$$g^{kl}\Gamma^i_{kl} = -\frac{1}{\sqrt{-g}}\frac{\partial\left(\sqrt{-g}g^{ik}\right)}{\partial x^k}$$

$$\implies \sqrt{-g}g^{kl}\Gamma^i_{kl} = -\sqrt{-g}\frac{1}{\sqrt{-g}}\frac{\partial\left(\sqrt{-g}g^{ik}\right)}{\partial x^k}g^{kl} = -\frac{\partial\left(\sqrt{-g}g^{ik}\right)}{\partial x^k}$$

Then we write,

$$\begin{aligned}
\sqrt{-g}G &= \Gamma^m_{im}\left(\sqrt{-g}g^{kl}\Gamma^i_{kl}\right) - \Gamma^l_{ik}\left(\sqrt{-g}g^{km}\Gamma^i_{lm}\right) - \left(\Gamma^m_{il}\Gamma^l_{km} - \Gamma^l_{ik}\Gamma^m_{lm}\right)g^{ik}\sqrt{-g} \\
&= \Gamma^m_{im}\left(\sqrt{-g}g^{kl}\Gamma^i_{kl}\right) - \Gamma^l_{ik}\left(\sqrt{-g}g^{km}\Gamma^i_{lm}\right) - \Gamma^m_{il}\Gamma^l_{km}\sqrt{-g}g^{ik} + \Gamma^l_{ik}\Gamma^m_{lm}\sqrt{-g}g^{ik} \\
&= \Gamma^m_{im}g^{kl}\Gamma^i_{kl} - \Gamma^l_{ik}\Gamma^i_{lm}g^{km} - \Gamma^i_{lm}\Gamma^l_{ik}g^{km} + \Gamma^l_{ik}\Gamma^m_{lm}g^{ik} \\
&= \Gamma^m_{im}g^{kl}\Gamma^i_{kl} - 2\Gamma^m_{il}\Gamma^l_{km}g^{mk} + \Gamma^l_{ik}\Gamma^m_{lm}g^{ik} \\
&= 2\Gamma^m_{il}\Gamma^l_{km}g^{mk} - \Gamma^m_{im}g^{kl}\Gamma^i_{kl} - \Gamma^l_{ik}\Gamma^m_{lm}g^{ik} \\
&= 2\Gamma^l_{mk}\Gamma^m_{li}g^{ik} - \Gamma^m_{lm}g^{ki}\Gamma^l_{ki} - \Gamma^l_{ik}\Gamma^m_{lm}g^{ik} \\
&= 2\Gamma^l_{mk}\Gamma^m_{li}g^{ik} - 2\Gamma^m_{lm}g^{ki}\Gamma^l_{ki} \\
&= 2g^{ik}\left(\Gamma^l_{mk}\Gamma^m_{li} - \Gamma^m_{lm}\Gamma^l_{ki}\right)
\end{aligned}$$

Finally, we obtain

$$G = g^{ik}\left(\Gamma^l_{mk}\Gamma^m_{li} - \Gamma^m_{lm}\Gamma^l_{ki}\right) \tag{2.182}$$

we point attention to the fact that the Christoffel symbols are symmetric in the lower indices, as per (2.73).

As previously mentioned, it is the components of g_{ik} which determine the gravitational field, thus, it follows that the components of g_{ik} be the quantities subjected to variation, when considering Hamilton's principle for the gravitational field. Nonetheless, it is pertinent to create the following fundamental reservation. Specifically, we cannot claim that in an actual field of gravitation the action integral has a minimum in *all* possible variations of g_{ik}, since not every change in g_{ik}

constitutes a change in the space-time metric to a degree in which it will affect the gravitational field. The components g_{ik} also change under a simple transformation of coordinates connected merely through a shift from one system to another within the same space-time, where each such coordinate transformation is a conglomerate of four independent transformations. To this end, we impose four auxiliary conditions and require the fulfillment of said conditions under variation, in order to omit changes in g_{ik} unrelated to a change in the metric. Hence, when applying the principle of least action to the gravitational field under consideration, we necessitate the fulfillment of said restrictions in order to constitute a minimum in the action with respect to variations of the g_{ik}.

Heeding to the assertions made, we show that the gravitational constant must be positive. Pondering upon the four auxiliary conditions mentioned, we utilize the vanishing of the three components $g_{0\alpha}$ and the invariance of the determinant $|g_{\alpha\beta}|$ made up from $g_{\alpha\beta}$:

$$g_{0\alpha} = 0, \quad |g_{\alpha\beta}| = \text{const}$$

from the latter condition we have,

$$g^{\alpha\beta}\frac{\partial g_{\alpha\beta}}{\partial x^0} = \frac{\partial}{\partial x^0}|g_{\alpha\beta}| = 0 \tag{2.183}$$

which follows from the fact that the scalar product of the contravariant and covariant components of $g_{\alpha\beta}$ is consistent with the definition of the determinant of the matrix $g_{\alpha\beta}$, and since the spatial components of the metric, g_{ik} are time-independent, it follows that their derivative with the respect to the time-component x^0 will vanish. Here we consider only the terms in the integrand of the expression for the action which contain derivatives of g_{ik} with respect to x^0. We find with that the terms of interest in G are

$$-\frac{1}{4}g^{\alpha\beta}g^{\gamma\delta}g^{00}\frac{\partial g_{\alpha\gamma}}{\partial x^0}\frac{\partial g_{\beta\delta}}{\partial x^0}$$

which takes a negative value. Namely, if we consider a spatial coordinate system which is Cartesian at a particular point at a given moment of time, such that $g_{\alpha\beta} = g^{\alpha\beta} = -\delta_{\alpha\beta}$, we obtain:

$$-\frac{1}{4}g^{00}\left(\frac{\partial g_{\alpha\beta}}{\partial x^0}\right)^2$$

Certainly,

$$\begin{aligned}
&-\frac{1}{4}g^{\alpha\beta}g^{\gamma\delta}g^{00}\frac{\partial g_{\alpha\gamma}}{\partial x^0}\frac{\partial g_{\beta\delta}}{\partial x^0}\\
\implies &-\frac{1}{4}\left(-\delta_{\alpha\beta}\right)\left(-\delta_{\gamma\delta}\right)g^{00}\frac{g_{\alpha\gamma}}{\partial x^0}\frac{\partial g_{\beta\delta}}{\partial x^0}\\
\implies &-\frac{1}{4}\left(\delta^{\alpha}_{\beta}\right)\left(\delta^{\delta}_{\gamma}\right)g^{00}\frac{g_{\alpha\gamma}}{\partial x^0}\frac{\partial g_{\beta\delta}}{\partial x^0}\\
\implies &-\frac{1}{4}g^{00}\frac{g_{\beta\gamma}}{\partial x^0}\frac{\partial g_{\beta\gamma}}{\partial x^0}\\
\implies &-\frac{1}{4}g^{00}\left(\frac{\partial g_{\beta\gamma}}{\partial x^0}\right)^2
\end{aligned}$$

which is analogous to the above formulated expression. Notice here that we used the properties of the Kronecker delta, namely $\delta_{\alpha\beta} = \delta^{\alpha}_{\beta}$. Then, since we have that $g^{00} = 1/g_{00} > 0$, we have that the sign of the quantity is positive.

We note that through expeditious alterations of the components of the space components of the metric, i.e. $g_{\alpha\beta}$ with the time x^0 (within the time interval of the limits of integration of x^0) the quantity G can be made as large as one wants. If the gravitational constant k were negative, the action would infinitely decrease, without limit, and there could be no minimum, hence, Hamilton's principle would not hold.

2.13 The energy-momentum tensor

It may be pertinent to introduce the general rule for calculating the energy-momentum tensor of any physical system whose action may be described in the form of the following integral:

$$S = \int \Lambda\left(q, \frac{\partial q}{\partial x^i}\right) dV dt = \frac{1}{c}\int \Lambda d\Omega \tag{2.184}$$

in which Λ is some function of the generalized coordinates q, describing the state of the system, and of their first derivatives with respect to coordinates and time, i.e. q' and $\dot{q}$, respectively. Note that the spatial integral $\int \Lambda dV$ is the Lagrangian of the system, so that the quantity Λ may be regarded as the "*Lagrangian density*" of the system.

The Euler-Lagrange equations ("equations of motion" / field equations if considering fields) may be obtained by varying the action S, in accordance to Hamilton's principle. To this end, we have (noting that $q_{,i} \equiv \partial q/\partial x^i$)

$$\delta S = \frac{1}{c}\int\left(\frac{\partial\Lambda}{\partial q}\delta q + \frac{\partial\Lambda}{\partial q_{,i}}\delta q_{,i}\right)d\Omega = \frac{1}{c}\int\left[\frac{\partial\Lambda}{\partial q}\delta q + \frac{\partial}{\partial x^i}\left(\frac{\partial\Lambda}{\partial q_{,i}}\delta q\right) - \delta q\frac{\partial}{\partial x^i}\frac{\partial\Lambda}{\partial q_{,i}}\right]d\Omega = 0 \tag{2.185}$$

Firstly, we verify that the second formula is equivalent to the first expression. Consider

$$\begin{aligned}
\delta S = \frac{1}{c}\int\left[\frac{\partial\Lambda}{\partial q}\delta q + \frac{\partial}{\partial x^i}\left(\frac{\partial\Lambda}{\partial q_{,i}}\delta q\right) - \delta q\frac{\partial}{\partial x^i}\frac{\partial\Lambda}{\partial q_{,i}}\right]d\Omega = 0 \\
= \frac{1}{c}\int\left[\frac{\partial\Lambda}{\partial q}\delta q + \frac{\partial}{\partial x^i}\left(\frac{\partial\Lambda}{\partial q_{,i}}\right)\delta_q + \frac{\partial}{\partial x^i}\left(\delta_q\right)\frac{\partial\Lambda}{\partial q_{,i}} - \delta q\frac{\partial}{\partial x^i}\left(\frac{\partial\Lambda}{\partial q_{,i}}\right)\right]d\Omega \\
= \frac{1}{c}\int\left[\frac{\partial\Lambda}{\partial q}\delta q + \frac{\partial}{\partial x^i}\left(\delta_q\right)\frac{\partial\Lambda}{\partial q_{,i}}\right]d\Omega \\
= \frac{1}{c}\int\left(\frac{\partial\Lambda}{\partial q}\delta q + \frac{\partial\Lambda}{\partial q_{,i}}\delta q_{,i}\right)d\Omega = 0
\end{aligned}$$

Hence, the second expression is equivalent to the first. Then, the second term in the integrand, after transformation by Gauss' theorem, vanishes upon integration over all of space, after which we find the required "Euler-Lagrange equations":

$$\frac{\partial}{\partial x^i}\frac{\partial\Lambda}{\partial q_{,i}} - \frac{\partial\Lambda}{\partial q} = 0 \tag{2.186}$$

Indeed, we rewrite the variation of the action integral:

$$\begin{aligned}\delta S = \frac{1}{c}\int\left[\frac{\partial\Lambda}{\partial q}\delta q + \frac{\partial}{\partial x^i}\left(\frac{\partial\Lambda}{\partial q_{,i}}\delta q\right) - \delta q\frac{\partial}{\partial x^i}\frac{\partial\Lambda}{\partial q_{,i}}\right]d\Omega = 0\\
= \frac{1}{c}\int\left(\frac{\partial\Lambda}{\partial q}\delta q - \delta q\frac{\partial}{\partial x^i}\frac{\partial\Lambda}{\partial q_{,i}}\right)d\Omega + \frac{1}{2}\int\left(\frac{\partial}{\partial x^i}\left(\frac{\partial\Lambda}{\partial q_{,i}}\delta q\right)\right)d\Omega = 0\\
= \frac{1}{c}\int\left(\frac{\partial\Lambda}{\partial q}\delta q - \delta q\frac{\partial}{\partial x^i}\frac{\partial\Lambda}{\partial q_{,i}}\right)d\Omega + \frac{1}{2}\oint\left(\frac{\partial}{\partial x^i}\left(\frac{\partial\Lambda}{\partial q_{,i}}\delta q\right)\right)dS_i = 0\\
= \frac{1}{c}\int\left(\frac{\partial\Lambda}{\partial q}\delta q - \delta q\frac{\partial}{\partial x^i}\frac{\partial\Lambda}{\partial q_{,i}}\right)d\Omega + 0 = 0\\
= \frac{1}{c}\int\left(\frac{\partial\Lambda}{\partial q}\delta q - \delta q\frac{\partial}{\partial x^i}\frac{\partial\Lambda}{\partial q_{,i}}\right)d\Omega = 0\end{aligned}$$

in which we used Gauss' Theorem to reduce the integral in the second term of the integrand from an integral over the given volume to an integral over the hypersurface, then by Hamilton's principle the variation of the fields at the boundary of the region (limits of the region of integration) are zero, hence, we obtain the required integral. The resulting integrand being the "equations of motion". We define the first term of the Euler-Lagrange equations as the canonical or generalized momenta of the system

$$p_i = \frac{\partial\Lambda}{\partial q_{,i}} \tag{2.187}$$

Next, by applying chain rule to the derivative of the function Λ and noting that it is defined by the quantities q and $q_{,i}$, we rewrite it as follows,

$$\frac{\partial\Lambda}{\partial x^i} = \frac{\partial\Lambda}{\partial q}\frac{\partial q}{\partial x^i} + \frac{\partial\Lambda}{\partial q_{,k}}\frac{\partial q_{,k}}{\partial x^i}$$

Then substituting into (2.186) and noting that $q_{,k,i} = q_{,i,k}$

$$\frac{\partial\Lambda}{\partial x^i} = \frac{\partial}{\partial x^k}\left(\frac{\partial\Lambda}{\partial q_{,k}}\right)q_{,i} + \frac{\partial\Lambda}{\partial q_{,k}}\frac{\partial q_{,i}}{\partial x^k} = \frac{\partial}{\partial x^k}\left(q_{,i}\frac{\partial\Lambda}{\partial q_{,k}}\right)$$

Indeed, rewriting (2.186) as:

$$\frac{\partial}{\partial x^i}\frac{\partial\Lambda}{\partial q_{,i}} - \frac{\partial\Lambda}{\partial q} = 0$$
$$\frac{\partial\Lambda}{\partial q} = \frac{\partial}{\partial x^i}\frac{\partial\Lambda}{\partial q_{,i}}$$

substituting into $\partial\Lambda/\partial x^i$ and rewriting the middle term by product rule, we obtain the above expression, i.e.

$$\frac{\partial\Lambda}{\partial x^i} = \frac{\partial\Lambda}{\partial q}\frac{\partial q}{\partial x^i} + \frac{\partial\Lambda}{\partial q_{,k}}\frac{\partial q_{,k}}{\partial x^i}$$
$$\frac{\partial\Lambda}{\partial x^i} = \frac{\partial}{\partial x^k}\left(\frac{\partial\Lambda}{\partial q_{,k}}\right)q_{,i} + \frac{\partial\Lambda}{\partial q_{,k}}\frac{\partial q_{,i}}{\partial x^k} = \frac{\partial}{\partial x^k}\left(q_{,i}\frac{\partial\Lambda}{\partial q_{,k}}\right)$$

By properties of the Kronecker delta, we note that we can always write the derivative $\partial\Lambda/\partial x^i$ as

$$\frac{\partial\Lambda}{\partial x^i} = \delta_i^k\frac{\partial\Lambda}{\partial x^k}$$

such that we introduce the notation

$$T_i^k = q_{,i}\frac{\partial\Lambda}{\partial q_{,k}} - \delta_i^k\Lambda \tag{2.188}$$

we can then express the relation in the form

$$\frac{\partial T_i^k}{\partial x^k} = 0 \tag{2.189}$$

Indeed,

$$
\begin{aligned}
&\frac{\partial T_i^k}{\partial x^k} = 0 \\
\implies &\frac{\partial}{\partial x^k}\left[q_{,i}\frac{\partial \Lambda}{\partial q_{,k}} - \delta_i^k \Lambda\right] \\
\implies &\frac{\partial}{\partial x^k}\left(q_{,i}\frac{\partial \Lambda}{\partial q_{,k}}\right) - \frac{\partial}{\partial x^k}\delta_i^k \Lambda \\
\implies &\frac{\partial \Lambda}{\partial x^i} - \frac{\partial}{\partial x^k}\delta_i^k \Lambda \\
\implies &\frac{\partial \Lambda}{\partial x^i} - \frac{\partial \Lambda}{\partial x^i} = 0
\end{aligned}
$$

We note in passing that if there exits several quantities $q^{(l)}$ describing the recently introduce tensor, we rewrite (2.188) as a sum of all quantities describing the tensor, i.e.

$$
T_i^k = \sum_l q_{,i}^{(l)} \frac{\partial \Lambda}{\partial q_{,k}^{(l)}} - \delta_i^k \Lambda \tag{2.190}
$$

We denote this tensor as the energy-momentum tensor of the system.

We briefly expand on properties of the energy-momentum tensor and subsequent derivations from this formulation. As seen in §2.9, notice that the divergence of a four-vector, i.e. $\partial A^k / \partial x^k = 0$, being equivalently zero may be rewritten as the assertion that the integral $\int A^k dS_k$ of the vector over a constant hypersurface is conserved. At the same time, we may make an analogy of such a statement regarding the divergence of a tensor; i.e. equation (2.189) proclaims that the vector $P^i =$ const $\int T^{ik} dS_k$ is conserved.

This vector must be identified with the four-vector of the momentum of the system. Next, we choose the constant in front of the integral, so that the value of the vector may coincide with the literature definition of the four-momentum, i.e.

$$P^i = \left(\frac{E}{c}, \vec{P}\right) \tag{2.191}$$

in which E is the energy of the system and $\vec{P}$ corresponds to the classical definition of the momentum in three dimensions.; where the time component of the four vector may be rewritten using the integral of the energy-momentum tensor,

$$P^0 = \text{ const} \int T^{0k} dS_k = \text{const} \int T^{00} dV \tag{2.192}$$

where P^0 denotes the energy of the system multiplied by $1/c$ as per (2.191), if the integration is taken over the hyperplane $x^0 = const$. Furthermore, according to (2.188),

$$\begin{aligned} T_i^k &= q_{,i} \frac{\partial \Lambda}{\partial q_{,k}} - \delta_i^k \Lambda \\ \Longrightarrow \ T_0^0 &= q_{,0} \frac{\partial \Lambda}{\partial q_{,0}} - \delta_0^0 \Lambda \\ \Longrightarrow \ T^{00} &= \dot{q} \frac{\partial \Lambda}{\partial \dot{q}} - \Lambda. \quad \left(\dot{q} \equiv \frac{\partial q}{\partial t}\right) \end{aligned}$$

which we obtain from (2.192) and made use of the fact that we extend the integration over the hyperplane $x^0 = const$, which reduced the integral over a hyperplane surrounding the whole volume to an integral over the volume. We may then rewrite the energy-momentum tensor as being a constituent of the four momentum, i.e.

$$P^i = \frac{1}{c} \int T^{ik} dS_k \tag{2.193}$$

Comparison of the above with the formulas relating the energy and the Lagrangian, we see that this quantity must be considered as the energy density of the system, thus, $\int T^{00} dV$ is the total energy of the system. Observe that we may proof such a statement by making use of the Lagrangian formalism, which allows for a general discussion of conservation laws. For this section of the discussion we briefly revert back to 3-D vector coordinates rather than the 4-D formalism we have been employing thus far. Indeed, we define the Lagrangian Λ as follows,

$$\Lambda(q,\dot{q},t) = T(q,\dot{q},t) - U(q,\dot{q},t) \tag{2.194}$$

where $T(q,\dot{q},t)$ is the kinetic energy of the system and $U(q,\dot{q},t)$ is the potential energy of the system. We define the kinetic energy in its usual classical manner, i.e. $T = \frac{1}{2}m\dot{q}_k\dot{q}_l$, where $\dot{q}_k\dot{q}_l$ both denote the velocity of the material in different directions, respectively. Next, we may rewrite the Euler-Lagrange equations as follows,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\Lambda}{\partial\dot{q}_i}\dot{q}_i - \Lambda\right) = -\frac{\partial\Lambda}{\partial t} \tag{2.195}$$

We proof its equivalence to the formulation given in (2.186), yet we use 3-D vector coordinates.

$$\begin{gathered}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\Lambda}{\partial\dot{q}_i}\dot{q}_i - \Lambda\right) = -\frac{\partial\Lambda}{\partial t} \\
\Longrightarrow \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\Lambda}{\partial\dot{q}_i}\dot{q}_i + \frac{\mathrm{d}}{\mathrm{d}t}\dot{q}_i\frac{\partial\Lambda}{\partial\dot{q}_i} \\
-\left[\frac{\partial\Lambda}{\partial q_i}\dot{q}_i + \frac{\partial\Lambda}{\partial\dot{q}_i}\frac{\partial\dot{q}_i}{\partial t} + \frac{\partial\Lambda}{\partial t}\right] = -\frac{\partial\Lambda}{\partial t} \\
\Longrightarrow \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\Lambda}{\partial\dot{q}_i}\dot{q}_i - \frac{\partial\Lambda}{\partial q_i}\dot{q}_i = 0 \\
\Longrightarrow \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\Lambda}{\partial\dot{q}_i} - \frac{\partial\Lambda}{\partial q_i} = 0
\end{gathered}$$

which is equivalent to (2.186), in a three-dimensional formalism. Hence, we have, due to the Euler-Lagrange equations,

$$\frac{\partial\Lambda}{\partial t} = 0 \quad \Longrightarrow \quad \frac{\partial\Lambda}{\partial\dot{q}_i}\dot{q}_i - \Lambda = \text{const.} \tag{2.196}$$

There-holds, that the Lagrangian is independent of time. Furthermore, if we have that the potential energy is independent of the generalized velocities $\dot{q}_i$ we then obtain from direct calculation,

$$\frac{\partial \Lambda}{\partial \dot{q}_i}\dot{q}_i = \frac{\partial T}{\partial \dot{q}_i}\dot{q}_i = m\dot{q}_k\dot{q}_l = 2T \tag{2.197}$$

Then, we find that:

$$\frac{\partial \Lambda}{\partial \dot{q}_i}\dot{q}_i - \Lambda = 2T - T + U = T + U = E \tag{2.198}$$

Finally, using the obtained result, considering once again a four-dimensional coordinate formalism, and reinterpreting the integral $\int T^{00} dV$, we find it must indeed be the total energy of the system. Subsequently, we define the angular momentum of the system in terms of the four-momentum by

$$M^{ik} = \int \left(x^i dP^k - x^k dP^i\right) = \frac{1}{c}\int \left(x^i T^{kl} - x^k T^{il}\right) dS_l \tag{2.199}$$

that is its "density" is expressed in terms of the "density" of momentum by the usual formula.

As aforementioned, if we integrate the formula given in (2.193) over the hyperplane $x^0 = const$, then P^i takes the form

$$P^i = \frac{1}{c}\int T^{i0} dV \tag{2.200}$$

where the integration extends over three-dimensional space. The space components of the P^i constitute the three-dimensional vector of the system, while the time component T^0 is its energy multiplied by $1/c$, i.e. $P^i = \left(\frac{E}{c}, \vec{P}\right) = \left(\frac{E}{c}, P_x, P_y, P_z\right)$. Hence, the vector with components

$$\frac{1}{c}T^{10}, \quad \frac{1}{c}T^{20}, \quad \frac{1}{c}T^{30} \tag{2.201}$$

is denoted as the "momentum density", and the value given by

$$W = T^{00} \tag{2.202}$$

is "energy density".

Ensuingly, we consider the remaining terms of the energy-momentum tensor, by separating the conservation equation (2.189) as follows:

$$\frac{1}{c}\frac{\partial T^{00}}{\partial t}+\frac{\partial T^{0\alpha}}{\partial x^{\alpha}}=0, \quad \frac{1}{c}\frac{\partial T^{\alpha 0}}{\partial t}+\frac{\partial T^{\alpha\beta}}{\partial x^{\beta}}=0 \tag{2.203}$$

Integrating the first equation over a volume V, from which we have

$$\frac{1}{c}\frac{\partial}{\partial t}\int T^{00}dV+\int\frac{\partial T^{0\alpha}}{\partial x^{\alpha}}=0$$

then, transforming the second integral by Gauss' Theorem, we obtain

$$\begin{aligned}\frac{1}{c}\frac{\partial}{\partial t}\int T^{00}dV+\int\frac{\partial T^{0\alpha}}{\partial x^{\alpha}}=0\\ \Longrightarrow \frac{1}{c}\frac{\partial}{\partial t}\int T^{00}dV+\oint T^{0\alpha}df_{\alpha}=0\\ \Longrightarrow \frac{1}{c}\frac{\partial}{\partial t}\int T^{00}dV=-\oint T^{0\alpha}df_{\alpha}\end{aligned}$$

namely,

$$\frac{\partial}{\partial t}\int T^{00}dV=-c\oint T^{0\alpha}df_{\alpha} \tag{2.204}$$

where the integral on the right is taken over the surface surrounding the volume ($df_x.df_y,df_z$ are the components of the vector of the surface element $d\mathbf{f}$). The expression then reads as follows: the left denotes the rate of change of the energy contained within some volume V, from this we deduce that the right must be the amount of energy which "flows" across the boundary of the volume V. From this we may set the vector $\mathbf{S}$ with components

$$cT^{01},cT^{02},cT^{03}$$

as the flux density - or the amount of energy passing through a unit surface in unit time.

Integrating the second equation over some volume V, we find similarly:

$$\frac{\partial}{\partial t}\int \frac{1}{c}T^{\alpha 0}dV = -\oint T^{\alpha\beta}df_{\beta} \tag{2.205}$$

In the left side of the equation we have the change of the momentum of the system in a given volume V per unit time, hence, the right line integral is the momentum emanating from the given volume V per unit time. Thus, the components $T^{\alpha\beta}$ of the energy-momentum tensor constitute the 3-D tensor of the momentum flux density; we denote this "flow" by $-\sigma_{\alpha\beta}$ as the stress tensor. Now, the energy flux density is a vector; the density of flux of momentum, which is itself a vector, must obviously be a tensor (the component $T_{\alpha\beta}$ of this tensor is the amount of the α-component of the momentum passing through a unit surface perpendicular to x^{β} axis per unit time.

We may then write the energy-momentum tensor as follows:

$$T^{ik} = \begin{bmatrix} W & S_x/c & S_y/c & S_z/c \\ S_x/c & -\sigma_{xx} & -\sigma_{xy} & -\sigma_{xz} \\ S_y/c & -\sigma_{yx} & -\sigma_{yy} & -\sigma_{yz} \\ S_z/c & -\sigma_{zx} & -\sigma_{zy} & -\sigma_{zz} \end{bmatrix} \tag{2.206}$$

Now, we consider the case for computing the energy-momentum tensor for a system in curvilinear coordinates. We note the slight discrepancy from (2.184), when considering a system which deviates from flat space-time into non-Euclidean space-time. The action integral for a system in curvilinear coordinates takes the form

$$S = \frac{1}{c}\int \Lambda\sqrt{-g}d\Omega \tag{2.207}$$

this generalization reduces to the accustomed action integral if we consider galilean coordinates, in which $g=-1$, and S returns to the form delineated in (2.184).

In order to calculate the energy-momentum tensor, we proceed by carrying out a slight transformation of coordinates in (2.207) from the coordinates x^i to the coordinates $x'^i = x^i + \eta^i$, where the η^i are small quantities. Under this transformation the inverse metric is transformed according to the formulas

$$g'^{ik}\left(x'^l\right) = g^{lm}\left(x^l\right)\frac{\partial x'^i \partial x'^k}{\partial x^l \partial x^m} = g^{lm}\left(\delta^i_l + \frac{\partial \eta^i}{\partial x^l}\right)\left(\delta^k_m + \frac{\partial \eta^k}{\partial x^m}\right)$$
$$\approx g^{ik}\left(x^l\right) + g^{im}\frac{\partial \eta^k}{\partial x^m} + g^{kl}\frac{\partial \eta^i}{\partial x^l}$$

Remembering the manner in which covariant tensors of 2^{nd} rank transform, i.e.

$$A^{ik} = \frac{\partial x^i}{\partial x'^l}\frac{\partial x^k}{\partial x'^m}A'^{lm} \tag{2.208}$$

and conversely,

$$A'^{lm} = \frac{\partial x'^l}{\partial x^i}\frac{\partial x'^m}{\partial x^k}A^{ik} \tag{2.209}$$

it then follows analogously

$$g'^{ik}\left(x'^l\right) = g^{lm}\left(x^l\right)\frac{\partial x'^i \partial x'^k}{\partial x^l \partial x^m} \tag{2.210}$$

Then by substituting in $x'^i = x^i + \eta^i$, where the η^i , we have

$$g'^{ik}\left(x'^l\right) = g^{lm}\left(x^l\right)\frac{\partial x'^i \partial x'^k}{\partial x^l \partial x^m}$$
$$= g^{lm}\left(x^l\right)\frac{\partial\left(x^i+\eta^i\right)}{\partial x^l}\frac{\partial\left(x^k+\eta^k\right)}{\partial x^m}$$
$$= g^{lm}\left(\frac{\partial x^i}{\partial x^l} + \frac{\partial \eta^i}{\partial x^l}\right)\left(\frac{\partial x^k}{\partial x^m} + \frac{\partial \eta^k}{\partial x^m}\right)$$
$$= g^{lm}\left(\delta^i_l + \frac{\partial \eta^i}{\partial x^l}\right)\left(\delta^k_m + \frac{\partial \eta^k}{\partial x^m}\right)$$

where we briefly note that $\partial x^i/\partial x^l = 1$ if $i = l$ and 0 if $i \neq l \implies \delta^i_l$.

$$
\begin{aligned}
&g^{lm}\left(\delta^i_l\delta^k_m + \delta^l_i\frac{\partial\eta^k}{\partial x^m} + \delta^k_m\frac{\partial\eta^i}{\partial x^l} + \frac{\partial\eta^i}{\partial x^l}\frac{\partial\eta^k}{\partial x^m}\right)\\
&= g^{lm}\delta^i_l\delta^k_m + g^{lm}\delta^l_i\frac{\partial\eta^k}{\partial x^m} + g^{lm}\delta^k_m\frac{\partial\eta^i}{\partial x^l} + g^{lm}\frac{\partial\eta^i}{\partial x^l}\frac{\partial\eta^k}{\partial x^m}\\
&= g^{ik} + g^{im}\frac{\partial\eta^k}{\partial x^m} + g^{lk}\frac{\partial\eta^i}{\partial x^l} + g^{lm}\frac{\partial\eta^i}{\partial x^l}\frac{\partial\eta^k}{\partial x^m}\\
&\approx g^{ik} + g^{im}\frac{\partial\eta^k}{\partial x^m} + g^{lk}\frac{\partial\eta^i}{\partial x^l}
\end{aligned}
$$

which is exactly the expression determined above. In order to represent all terms as functions of the same variables, we expand $g'^{ik}\left(x^l+\eta^l\right)$ in powers of η^l. Moreover, neglecting terms of higher order in η^l, we can in all terms containing η^l, replace, g'^{ik} by g^{ik}. Hence, we obtain

$$
g'^{ik}\left(x^l\right) = g^{ik}\left(x^l\right) - \eta^l\frac{\partial g^{ik}}{\partial x^l} + g^{il}\frac{\partial\eta^k}{\partial x^l} + g^{kl}\frac{\partial\eta^i}{\partial x^l}
$$

We expand $g'^{ik}\left(x^l+\eta^l\right)$ in powers of η^l, by using Taylor series, such that

$$
\begin{aligned}
g'^{ik} &= g^{ik}\left(x^l+\eta^l\right)\\
&= g^{ik}(a) + \frac{\partial g(a)}{\partial x^l}(x-a) + \dots\\
&= g^{ik}(x^l) + \frac{\partial g^{ik}(x^l)}{\partial x^l}(x^l+\eta^l-x^l) + \dots\\
&\approx g^{ik}(x^l) + \frac{\partial g^{ik}(x^l)}{\partial x^l}\eta^l
\end{aligned}
$$

substituting back into the above formulation, we obtain the desired result. We note that the last three terms may be written as the sum of contravariant derivatives of η^i, i.e. $\eta^{i;k}+\eta^{k;i}$. Hence, we obtain the transformation of the g^{ik} in the from

$$
g^{ik} = g^{ik} + \delta g^{ik}, \quad \delta g^{ik} = \eta^{i;k} + \eta^{k,i} \tag{2.211}
$$

in which we write the covariant counterpart of the above formulation as follows

$$g'_{ik} = g_{ik} + \delta g_{ik}, \quad \delta g_{ik} = -\eta_{i;k} - \eta_{k:i} \tag{2.212}$$

Since the action S is a scalar, it remains invariant upon coordinate transformation, while this may be true, the change δS in the action may be written in analogous fashion as was depicted in (2.185). As was communicated at the beginning of the section, we once again consider the quantities q as defining the physical system to which the action applies. It follows, that under coordinate transformation the quantities q fluctuate by δq. In calculating the variation in the action we need not concern ourselves with said terms, since by citing the Euler-Lagrange equations of the system such terms must cancel each other, since the "equations of motion" are obtained by equating the variation of the action with respect to quantities q to zero. Thus, we need only consider the terms associated with changes in the metric g_{ik}. Then, simply using (2.185) as an analog, in which we substitute $\Lambda \to \sqrt{-g}\Lambda$, $q \to g^{ik}$, and $q_{,i} \to \frac{\partial g^{ik}}{\partial x^l}$, we write the variation of the action δS as

$$\delta S = \frac{1}{c}\int \left\{ \frac{\partial \sqrt{-g}\Lambda}{\partial g^{ik}} \delta g^{ik} + \frac{\partial \sqrt{-g}\Lambda}{\partial \frac{\partial g^{ik}}{\partial x^l}} \delta \frac{\partial g^{ik}}{\partial x^l} \right\} d\Omega \tag{2.213}$$

Granted, by duplicating the procedure implemented at the beginning of the section, i.e. setting $\delta g^{ik} = 0$ at the integration limits (boundary) and applying Gauss' Theorem, we obtain an action integral akin to that of the integral of the "equations of motion":

$$= \frac{1}{c}\int \left\{ \frac{\partial \sqrt{-g}\Lambda}{\partial g^{ik}} - \frac{\partial}{\partial x^l} \frac{\partial \sqrt{-g}\Lambda}{\partial \frac{\partial g^{ik}}{\partial x^l}} \right\} \delta g^{ik} d\Omega \tag{2.214}$$

Here we introduce the notation

$$\frac{1}{2}\sqrt{-g}T_{ik} = \frac{\partial \sqrt{-g}\Lambda}{\partial g^{ik}} - \frac{\partial}{\partial x^l} \frac{\partial \sqrt{-g}\Lambda}{\partial \frac{\partial g^{ik}}{\partial x^l}} \tag{2.215}$$

It follows that δS takes the form

$$\delta S = \frac{1}{2c}\int T_{ik}\delta g^{ik}\sqrt{-g}d\Omega = -\frac{1}{2c}\int T^{ik}\delta g_{ik}\sqrt{-g}d\Omega \tag{2.216}$$

where we point notice to the fact that $g^{ik}\delta g_{lk} = -g_{lk}\delta g^{ik}$, thus it follows that $T^{ik}\delta g_{ik} = -T_{ik}\delta g^{ik}$. Certainly, consider

$$\begin{aligned}\delta S &= \frac{1}{c}\int\left\{\frac{\partial\sqrt{-g}\Lambda}{\partial g^{ik}} - \frac{\partial}{\partial x^l}\frac{\partial\sqrt{-g}\Lambda}{\partial\frac{\partial g^{ik}}{\partial x^l}}\right\}\delta g^{ik}d\Omega \\ &= \frac{1}{c}\int\left(\frac{1}{2}\sqrt{-g}T_{ik}\right)\delta g^{ik}d\Omega \\ &= \frac{1}{2c}\int\sqrt{-g}T_{ik}\delta g^{ik}d\Omega\end{aligned}$$

Next, replacing δg^{ik} above with the expression formulated in (2.404), we obtain, using the symmetry properties of the tensor T_{ik},

$$\delta S = \frac{1}{2c}\int T_{ik}\left(\eta^{i;k} + \eta^{k;i}\right)\sqrt{-g}d\Omega = \frac{1}{c}\int T_{ik}\eta^{i;k}\sqrt{-g}d\Omega$$

in which we use the fact that the quantities $\eta^{i;k}$ must be symmetric, i.e. $\eta^{i;k} = \eta^{k;i}$, such that $\eta^{i;k} + \eta^{k;i} = 2\eta^{i;k}$.

Moreover, we re-illustrate the above expression for δS as:

$$\delta S = \frac{1}{c}\int\left(T_i^k\eta^i\right)_{;k}\sqrt{-g}d\Omega - \frac{1}{c}\int T_{i;k}^k\eta^i\sqrt{-g}d\Omega \tag{2.217}$$

Indeed, we verify the assertion above

$$
\begin{aligned}
\delta S &= \frac{1}{c}\int\left(T_i^k\eta^i\right)_{;k}\sqrt{-g}d\Omega-\frac{1}{c}\int T^k_{i;k}\eta^i\sqrt{-g}d\Omega \\
&= \frac{1}{c}\int\left(T^k_{i;k}\eta^i+\eta^{i;k}T_i^k\right)\sqrt{-g}d\Omega-\frac{1}{c}\int T^k_{i;k}\eta^i\sqrt{-g}d\Omega \\
&= \frac{1}{c}\int T^k_{i;k}\eta^i\sqrt{-g}d\Omega+\frac{1}{c}\int\eta^{i;k}T_i^k\sqrt{-g}d\Omega-\frac{1}{c}\int T^k_{i;k}\eta^i\sqrt{-g}d\Omega \\
&= \frac{1}{c}\int T_{ik}\eta^{i;k}\sqrt{-g}d\Omega
\end{aligned}
$$

Implementing (2.90), the first integral may be written as:

$$
\frac{1}{c}\int\frac{\partial}{\partial x^k}\left(\sqrt{-g}T_i^k\eta^i\right)d\Omega \tag{2.218}
$$

then by applying Gauss' Theorem we may transform said integral into an integral over the hypersurface, after which noting the η^i vanishes at the boundary, the integral vanishes. Certainly, we have

$$
\begin{aligned}
\delta S &= \frac{1}{c}\int\left(T_i^k\eta^i\right)_{;k}\sqrt{-g}d\Omega-\frac{1}{c}\int T^k_{i;k}\eta^i\sqrt{-g}d\Omega \\
&= \frac{1}{c}\int\frac{1}{\sqrt{-g}}\frac{\partial\left(\sqrt{-g}T_i^k\eta^i\right)}{\partial x^k}\sqrt{-g}d\Omega-\frac{1}{c}\int T^k_{i;k}\eta^i\sqrt{-g}d\Omega \\
&= \frac{1}{c}\int\frac{\partial\left(\sqrt{-g}T_i^k\eta^i\right)}{\partial x^k}d\Omega-\frac{1}{c}\int T^k_{i;k}\eta^i\sqrt{-g}d\Omega \\
&= \frac{1}{c}\oint\left(\sqrt{-g}T_i^k\eta^i\right)dS_i-\frac{1}{c}\int T^k_{i;k}\eta^i\sqrt{-g}d\Omega \\
&= 0-\frac{1}{c}\int T^k_{i;k}\eta^i\sqrt{-g}d\Omega \\
&= -\frac{1}{c}\int T^k_{i;k}\eta^i\sqrt{-g}d\Omega
\end{aligned}
$$

Therefore, equating δS to zero, we identify the integral

$$
\delta S=-\frac{1}{c}\int T^k_{i;k}\eta^i\sqrt{-g}d\Omega=0 \tag{2.219}
$$

Then, due to the arbitrariness of η^i it follows that

$$T^k_{i:k} = 0 \tag{2.220}$$

Comparison of the product to (2.189), i.e. $\partial T_{ik}/\partial x^k = 0$, which is valid in galilean coordinates, we encounter the fact that the tensor T_{ik}, established by (2.215), must coincide with the energy-momentum tensor - within a constant factor.

Hence, the formula (2.215) enables us to calculate the energy-momentum tensor by differentiating the quantity Λ with respect to the metric tensor g^{ik} and its respective derivatives. We note that in general the energy-momentum tensor is not symmetric, nonetheless, by the means in which it was found, in this particular case it is symmetric, i.e. $T^{ik} = T^{ki}$. Derivation of the energy momentum tensor in the manner described above proves auspicious, since we can use (2.215) to calculate T^{ik} in the presence and absence of a gravitational field; the latter yields to the fact that the metric is has no independent significance and the transition to curvilinear coordinates occurs as an intermediate step in calculating the energy-momentum tensor.

For macroscopic bodies the energy-momentum tensor is

$$T_{ik} = (p + \varepsilon) u_i u_k - p g_{ik} \tag{2.221}$$

We note that the quantity T_{00} (the energy of the system) is always positive

$$T_{00} \geq 0 \tag{2.222}$$

nonetheless, we cannot make such a statement for the mixed component tensor T^0_0.

2.14 The Einstein Field Equations

Finally, we have all that is necessary in order to deduce the equations of the gravitational field. We proceed by invoking the principle of least action, in which we vary the action of the gravitational field and matter, respectively, such that $\delta\left(S_m+S_g\right)=0$, where S_m is the action of the matter and S_g is the action of the field. To this end, as we have previously shown, we shall subject the quantities responsible for the gravitational field to variation, i.e. is the metric tensor g_{ik}.

Now, computing the variation of the action of the gravitational field, i.e. δS_g, we obtain,

$$\delta\int R\sqrt{-g}d\Omega=\delta\int g^{ik}R_{ik}\sqrt{-g}d\Omega$$
$$=\int\left\{R_{ik}\sqrt{-g}\delta g^{ik}+R_{ik}g^{ik}\delta\sqrt{-g}+g^{ik}\sqrt{-g}\delta R_{ik}\right\}d\Omega$$

where we note that we must var every component in the integrand, Then, from formula (2.81), we obtain

$$\delta\sqrt{-g}=-\frac{1}{2\sqrt{-g}}\delta g=-\frac{1}{2}\sqrt{-g}g_{ik}\delta g^{ik} \qquad (2.223)$$

substitution into the expression above yields

$$\delta\int R\sqrt{-g}d\Omega=\int\left(R_{ik}-\frac{1}{2}g_{ik}R\right)\delta g^{ik}\sqrt{-g}d\Omega+\int g^{ik}\delta R_{ik}\sqrt{-g}d\Omega \qquad (2.224)$$

We confirm the above assertion:

$$
\begin{aligned}
&\int \left\{ R_{ik}\sqrt{-g}\delta g^{ik} + R_{ik}g^{ik}\delta\sqrt{-g} + g^{ik}\sqrt{-g}\delta R_{ik} \right\} d\Omega \\
\implies &\int \left\{ R_{ik}\sqrt{-g}\delta g^{ik} + R_{ik}g^{ik}\left(-\frac{1}{2}\sqrt{-g}g_{ik}\delta g^{ik}\right) + g^{ik}\sqrt{-g}\delta R_{ik} \right\} d\Omega \\
\implies &\int \left\{ R_{ik}\sqrt{-g}\delta g^{ik} - \frac{1}{2}g^{ik}R_{ik}\sqrt{-g}g_{ik}\delta g^{ik} + g^{ik}\sqrt{-g}\delta R_{ik} \right\} d\Omega \\
\implies &\int \left\{ R_{ik}\sqrt{-g}\delta g^{ik} - \frac{1}{2}R\sqrt{-g}g_{ik}\delta g^{ik} + g^{ik}\sqrt{-g}\delta R_{ik} \right\} d\Omega \\
\implies &\int \left(R_{ik} - \frac{1}{2}g_{ik}R \right) \delta g^{ik}\sqrt{-g}d\Omega + \int g^{ik}\delta R_{ik}\sqrt{-g}d\Omega
\end{aligned}
$$

in which we used the fact that $g^{ik}R_{ik} = R$, where R is the scalar curvature and substituted the quantity determined in (2.235).

As aforementioned in previous sections (see §2.5 and §2.6) we note that the Christoffel symbols do not constitute a tensor, nonetheless, their variation $\delta\Gamma^i_{kl}$ do embody a tensor, since as we have previously determined $\Gamma^k_{il}A_k dx^l$ is the change in a vector under parallel transport from some point P to an infinitesimally distant point P'. Hence, the quantity $\Gamma^k_{il}A_k dx^l$ is the difference between two vectors, which we obtain from two parallel displacements from point P to one and the same point P'. Since the difference between two vectors at one and the same point is again a vector, it follows that $\Gamma^k_{il}A_k dx^l$ is a tensor.

Now, let us consider locally-geodesic coordinate system, such that $\Gamma^i_{kl} = 0$ at said point under consideration. Using (2.169), we obtain

$$
g^{ik}\delta R_{ik} = g^{ik}\left\{ \frac{\partial}{\partial x^l}\delta\Gamma^l_{ik} - \frac{\partial}{\partial x^k}\delta\Gamma^l_{il} \right\} = g^{ik}\frac{\partial}{\partial x^l}\delta\Gamma^l_{ik} - g^{il}\frac{\partial}{\partial x^l}\delta\Gamma^k_{ik} = \frac{\partial w^l}{\partial x^l}
$$

Indeed, starting with the second term of (2.224), we write

$$\int g^{ik}\delta R_{ik}\sqrt{-g}d\Omega$$

$$\Longrightarrow g^{ik}\delta R_{ik}$$

$$= g^{ik}\delta\left[\frac{\partial\Gamma^l_{ik}}{\partial x^l} - \frac{\partial\Gamma^l_{il}}{\partial x^k} + \Gamma^l_{ik}\Gamma^m_{lm} - \Gamma^m_{il}\Gamma^l_{km}\right]$$

$$= g^{ik}\delta\left[\frac{\partial\Gamma^l_{ik}}{\partial x^l} - \frac{\partial\Gamma^l_{il}}{\partial x^k} + 0 - 0\right]$$

where we note that the first derivative of the metric vanishes, i.e. $\partial/\partial x^k g^{ik} = 0$, thus we have that the last two terms which are defined solely by the derivatives of the metric vanish. Then,

$$= g^{ik}\delta\left[\frac{\partial\Gamma^l_{ik}}{\partial x^l} - \frac{\partial\Gamma^l_{il}}{\partial x^k}\right]$$

$$= g^{ik}\left[\frac{\partial}{\partial x^l}\delta\Gamma^l_{ik} - \frac{\partial}{\partial x^k}\delta\Gamma^l_{il}\right]$$

$$= g^{ik}\frac{\partial}{\partial x^l}\delta\Gamma^l_{ik} - g^{ik}\frac{\partial}{\partial x^k}\delta\Gamma^l_{il}$$

$$= g^{ik}\frac{\partial}{\partial x^l}\delta\Gamma^l_{ik} - g^{il}\frac{\partial}{\partial x^l}\delta\Gamma^k_{ik}$$

$$= \frac{\partial}{\partial x^l}g^{ik}\delta\Gamma^l_{ik} - \frac{\partial}{\partial x^l}g^{il}\delta\Gamma^k_{ik}$$

$$= \frac{\partial}{\partial x^l}\left[g^{ik}\delta\Gamma^l_{ik} - g^{il}\delta\Gamma^k_{ik}\right]$$

$$= \frac{\partial}{\partial x^l}w^l$$

where we let

$$w^l = \left[g^{ik}\delta\Gamma^l_{ik} - g^{il}\delta\Gamma^k_{ik}\right]$$

Now, since we have that $\delta\Gamma^i_{kl}$ is a tensor, since it is the difference between two vectors, i.e. it is vector, and g^{ik} is obviously a tensor, the difference between the scalar product of the quantity

$g^{ik}\delta\Gamma^l_{ik}$ is a vector, we can write the above relation as follows

$$g^{ik}\delta R_{ik} = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^l}\left(\sqrt{-g}w^l\right)$$

in which we applied (2.88), and rewrote $\frac{\partial}{\partial x^l}w^l$ as $w^l_{;l}$ and made the simple substitution $A^i_{;i} \to w^l_{;l}$. Namely,

$$\begin{aligned}
\frac{\partial}{\partial x^l}w^l &= w^l_{;l} \\
\Longrightarrow &= \frac{1}{\sqrt{-g}}\left[\frac{\partial\sqrt{-g}}{\partial x^l}w^l + \frac{\partial w^l}{\partial x^l}\sqrt{-g}\right] \\
\Longrightarrow &= \frac{1}{\sqrt{-g}}\frac{\partial w^l}{\partial x^l}\sqrt{-g} \\
\Longrightarrow &= \frac{\partial w^l}{\partial x^l}
\end{aligned}$$

Subsequently, we may rewrite the second integral of (2.224) as

$$\begin{aligned}
\int g^{ik}\delta R_{ik}\sqrt{-g}d\Omega &= \int \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^l}\left(\sqrt{-g}w^l\right)\sqrt{-g}d\Omega \\
&= \int \frac{\partial}{\partial x^l}\left(\sqrt{-g}w^l\right)d\Omega
\end{aligned}$$

Once again, applying Gauss' Theorem we may transform the integral over the given volume into an integral over the hypersurface surrounding said four-volume and by Hamilton's principle the variation of the field at the boundary are equivalently zero, thus the second integral vanishes, namely,

$$\begin{aligned}\delta\int R\sqrt{-g}d\Omega &= \int\left(R_{ik}-\frac{1}{2}g_{ik}R\right)\delta g^{ik}\sqrt{-g}d\Omega+\int g^{ik}\delta R_{ik}\sqrt{-g}d\Omega\\ &= \int\left(R_{ik}-\frac{1}{2}g_{ik}R\right)\delta g^{ik}\sqrt{-g}d\Omega+\int\frac{\partial}{\partial x^l}\left(\sqrt{-g}w^l\right)d\Omega\\ &= \int\left(R_{ik}-\frac{1}{2}g_{ik}R\right)\delta g^{ik}\sqrt{-g}d\Omega+\oint\frac{\partial}{\partial x^l}\left(\sqrt{-g}w^l\right)dS_l\\ &= \int\left(R_{ik}-\frac{1}{2}g_{ik}R\right)\delta g^{ik}\sqrt{-g}d\Omega+0\\ &= \int\left(R_{ik}-\frac{1}{2}g_{ik}R\right)\delta g^{ik}\sqrt{-g}d\Omega\end{aligned}$$

where we follow what was delineated upon in §2.10, regarding Gauss' Theorem.

Therefore, the variation of the gravitational field, δS_g, takes the form

$$\delta S_g=-\frac{c^3}{16\pi k}\int\left(R_{ik}-\frac{1}{2}g_{ik}R\right)\delta g^{ik}\sqrt{-g}d\Omega \tag{2.225}$$

Conversely, considering the expression given by (2.178) where

$$\delta S_g=-\frac{c^3}{16\pi k}\delta\int G\sqrt{-g}d\Omega=-\frac{c^3}{16\pi k}\delta\int R\sqrt{-g}d\Omega$$

and selecting the first integral as our incipience and proceeding by varying the action of the field, such that

$$\delta S_g=-\frac{c^3}{16\pi k}\int\left\{\frac{\partial(G\sqrt{-g})}{\partial g^{ik}}-\frac{\partial}{\partial x^l}\frac{\partial(G\sqrt{-g})}{\partial\frac{\partial g^{ik}}{\partial x^l}}\right\}\delta g^{ik}d\Omega \tag{2.226}$$

which is completely analogous to the process used in (2.185) and (2.213).

Since we have the relation given by (2.178), it follows that

$$-\frac{c^3}{16\pi k}\delta\int G\sqrt{-g}d\Omega = -\frac{c^3}{16\pi k}\delta\int R\sqrt{-g}d\Omega$$
$$\Longrightarrow \delta\int G\sqrt{-g}d\Omega = \delta\int R\sqrt{-g}d\Omega$$
$$\Longrightarrow -\frac{c^3}{16\pi k}\int\left\{\frac{\partial(G\sqrt{-g})}{\partial g^{ik}} - \frac{\partial}{\partial x^l}\frac{\partial(G\sqrt{-g})}{\partial\frac{\partial g^{ik}}{\partial x^l}}\right\}\delta g^{ik}d\Omega = -\frac{c^3}{16\pi k}\int\left(R_{ik} - \frac{1}{2}g_{ik}R\right)\delta g^{ik}\sqrt{-g}d\Omega$$
$$\Longrightarrow \left(R_{ik} - \frac{1}{2}g_{ik}R\right)\sqrt{-g} = \left\{\frac{\partial(G\sqrt{-g})}{\partial g^{ik}} - \frac{\partial}{\partial x^l}\frac{\partial(G\sqrt{-g})}{\partial\frac{\partial g^{ik}}{\partial x^l}}\right\}$$
$$\Longrightarrow \left(R_{ik} - \frac{1}{2}g_{ik}R\right) = \frac{1}{\sqrt{-g}}\left\{\frac{\partial(G\sqrt{-g})}{\partial g^{ik}} - \frac{\partial}{\partial x^l}\frac{\partial(G\sqrt{-g})}{\partial\frac{\partial g^{ik}}{\partial x^l}}\right\}$$

Thus, we have the relation:

$$R_{ik} - \frac{1}{2}g_{ik}R = \frac{1}{\sqrt{-g}}\left\{\frac{\partial(G\sqrt{-g})}{\partial g^{ik}} - \frac{\partial}{\partial x^l}\frac{\partial(G\sqrt{-g})}{\partial\frac{\partial g^{ik}}{\partial x^l}}\right\} \tag{2.227}$$

Then, by (2.216), we may write for the variation of the action of the matter as

$$\delta S_m = \frac{1}{2c}\int T_{ik}\delta g^{ik}\sqrt{-g}d\Omega \tag{2.228}$$

where T_{ik} is the energy-momentum tensor of the matter.

Gravitational interaction plays a role only for objects with sufficiently large mass, due partly to the minute magnitude of the gravitational constant, thus in studying the gravitational field we must usually consider macroscopic bodies. Consequently, we must usually write the energy-momentum tensor, T_{ik}, as expressed in (2.221).

Hence, by considering the variation of each action integral in tandem, i.e. $\delta S_m + \delta S_g = 0$, we find:

$$-\frac{c^3}{16\pi k}\int\left(R_{ik}-\frac{1}{2}g_{ik}R-\frac{8\pi k}{c^4}T_{ik}\right)\delta g^{ik}\sqrt{-g}d\Omega=0$$

Indeed, consider

$$\begin{aligned}
&\delta S_g+\delta S_m=0\\
&\Longrightarrow=-\frac{c^3}{16\pi k}\int\left(R_{ik}-\frac{1}{2}g_{ik}R\right)\delta g^{ik}\sqrt{-g}d\Omega+\frac{1}{2c}\int T_{ik}\delta g^{ik}\sqrt{-g}d\Omega=0\\
&\Longrightarrow=\left[-\frac{c^3}{16\pi k}\int\left(R_{ik}-\frac{1}{2}g_{ik}R\right)+\frac{1}{2c}\int T_{ik}\right]\delta g^{ik}\sqrt{-g}d\Omega=0\\
&\Longrightarrow=-\frac{c^3}{16\pi k}\int\left[R_{ik}-\frac{1}{2}g_{ik}R-\frac{16\pi k}{c^3}\frac{1}{2c}T_{ik}\right]\delta g^{ik}\sqrt{-g}d\Omega=0\\
&\Longrightarrow=-\frac{c^3}{16\pi k}\int\left[R_{ik}-\frac{1}{2}g_{ik}R-\frac{8\pi k}{c^4}T_{ik}\right]\delta g^{ik}\sqrt{-g}d\Omega=0
\end{aligned}$$

then, by virtue of the arbitrariness of δg^{ik}, we obtain:

$$R_{ik}-\frac{1}{2}g_{ik}R=\frac{8\pi k}{c^4}T_{ik} \tag{2.229}$$

or in mixed components,

$$R_i^k-\frac{1}{2}\delta_i^k R=\frac{8\pi k}{c^4}T_i^k \tag{2.230}$$

Finally, we obtain the elegant, ingenious equations of the gravitational field, an unprecedented formulation presented by Einstein in 1915. Thus, these equations are aptly deemed the *Einstein equations*.

Contraction of (2.230) on the indices i and k, we have

$$R=-\frac{8\pi k}{c^4}T \tag{2.231}$$

$(T = T_i^i)$. Namely,

$$\begin{aligned}
& R_{ik} - \frac{1}{2} g_{ik} R = \frac{8\pi k}{c^4} T_{ik} \\
\implies & g^{ik} \left[R_{ik} - \frac{1}{2} g_{ik} R \right] = g^{ik} \frac{8\pi k}{c^4} T_{ik} \\
\implies & g^{ik} R_{ik} - \frac{1}{2} g^{ik} g_{ik} R = \frac{8\pi k}{c^4} g^{ik} T_{ik} \\
\implies & R - \frac{1}{2} (4) R = \frac{8\pi k}{c^4} T \\
\implies & R - 2R = \frac{8\pi k}{c^4} T \\
\implies & -R = \frac{8\pi k}{c^4} T \\
\implies & R = -\frac{8\pi k}{c^4} T
\end{aligned}$$

Therefore, the field equations may also be written in the following form

$$R_{ik} = \frac{8\pi k}{c^4} \left(T_{ik} - \frac{1}{2} g_{ik} T \right) \tag{2.232}$$

Certainly,

$$\begin{aligned}
& R_{ik} - \frac{1}{2} g_{ik} R = \frac{8\pi k}{c^4} T_{ik} \\
\implies & R_{ik} - \frac{1}{2} g_{ik} \left(-\frac{8\pi k}{c^4} T \right) = \frac{8\pi k}{c^4} T_{ik} \\
\implies & R_{ik} + \frac{8\pi k}{c^4} \left(\frac{1}{2} g_{ik} T \right) = \frac{8\pi k}{c^4} T_{ik} \\
\implies & R_{ik} = \frac{8\pi k}{c^4} T_{ik} - \frac{8\pi k}{c^4} \left(\frac{1}{2} g_{ik} T \right) \\
\implies & R_{ik} = \frac{8\pi k}{c^4} \left(T_{ik} - \frac{1}{2} g_{ik} T \right)
\end{aligned}$$

We note that the Einstein equations are nonlinear. Thus, for fields of gravity the basic principle of superposition does not hold, rather this holds in the presence of a weak field, which we will see is known as the Newtonian limit where we are able to linearize the field equations, as they take the form of the equations of motion in Newtonian mechanics, which are linear in nature.

Now, if we were to consider a matter-free gravitational field, i.e. empty space $T_{ik} = 0$, we have that the symmetric tensor R_{ik} should vanish, such that the field equations reduce to

$$R_{ik} = 0 \tag{2.233}$$

We briefly touch on the fact that the Ricci tensor vanishing does not mean that in vacuum, the space-time under consideration is flat; as this would require a stronger condition, namely $R^i_{klm} = 0$

the energy-momentum tensor of the electromagnetic field has the property that $T^i_i = 0$, i.e. that the sum of its diagonal terms is zeros. Then, from (2.231), it follows that in the presence of an electromagnetic field without any mass the scalar curvature R is zero.

As previously mentioned, the divergence of the energy-momentum tensor vanishes, (see (2.220):

$$T^k_{i;k} = 0 \tag{2.234}$$

thus, it follows that the divergence of the left side of the field equations (2.230) must be equivalently zero; due to the fact that the identity (2.176) holds. Essentially, we have that

$$\begin{aligned}
R^k_i - \frac{1}{2}\delta^k_i R &= \frac{8\pi k}{c^4} T^k_i \\
\implies \frac{\partial}{\partial x^k}\left(R^k_i - \frac{1}{2}\delta^k_i R\right) &= \frac{\partial}{\partial x^k}\frac{8\pi k}{c^4} T^k_i \\
\implies \frac{\partial}{\partial x^k}\left(R^k_i - \frac{1}{2}\delta^k_i R\right) &= \frac{8\pi k}{c^4}\frac{\partial}{\partial x^k} T^k_i \\
\implies \frac{\partial}{\partial x^k}\left(R^k_i - \frac{1}{2}\delta^k_i R\right) &= 0
\end{aligned}$$

The fact that the above assertion holds, as aforementioned, is due to the identity (2.176), as well as (2.163); nonetheless, we elaborate upon such an observation. Specifically, starting with (2.163), in covariant form, and referencing the proof given for (2.169), we immediately write:

$$
\begin{gathered}
R_{abmn;l} + R_{ablm;n} + R_{abnl;m} = 0 \\
\implies R_{;l} - R^{n}_{l;n} - R^{m}_{l;m} = 0 \\
\implies -2R^{n}_{l;n} + R_{;l} = 0 \\
\implies R^{n}_{l;n} - \frac{1}{2}R_{;l} = 0 \\
\implies \left(R^{n}_{l} - \frac{1}{2}\delta^{n}_{l}R\right)_{;n} = 0
\end{gathered}
$$

Hence, we have the fact that the divergence of the left hand side of equation (2.230) must be zero is true. Therefore, (2.234) is essentially contained within the equations of the gravitational field (2.230), there-holds that equation (2.234) expresses the conservation of energy and momentum, and within this conservation law there exists the second pair of the Maxwell equations, which in turn describes the equations of motion of the matter or the motion of the physical system which the peculiar energy-momentum tensor is considering. Hence, the field equations contain within them the equations for the matter producing the field. Hence, the distribution and motion of the matter which produces such a gravitational field may not be chosen arbitrarily, rather they must be determined through the solving of equations (2.230) under certain initial conditions as we find the field produced by the matter.

We delineate upon the number of field equations in (2.229). Starting with the fact that x^i may be assigned four quantities, due to the fact that we may subject it to arbitrary transformations, we can then assign six of the sixteen quantities of the metric, remembering that g_{ik} is a symmetric tensor; subsequently, the four components of the four-velocity u^i, which appear in the energy-momentum tensor of matter, related to each other by the scalar product $u^i u_i = 1$, give 3 independent components of the total 4, and a final component from the density of the matterr ε/c^2, such that we are left with ten field equations for (2.229) for ten unknowns.

In the case in which we consider the gravitational field in vacuum, there remains six unknown quantities with the number of independent field equations being reduced; we note that the ten equations for the gravitational field in vacuum $R_{ik} = 0$ are connected by the four identities (2.169).

We briefly elucidate upon some idiosyncrasies within the structure of the field equations. As previously affirmed, these equations are non-linear, second-order partial differential equations. We note that the equations do not contain the time derivatives of all ten components of the metric. Rather, from (2.366) we notice that second derivatives with respect to the time are contained solely within the components $R_{0\alpha 0\beta}$ of the curvature tensor, in which they enter in the form of the term $-\frac{1}{2}\ddot{g}_{\alpha\beta}$ (where the dot refers to differentiation with respect to the x^0 component); the second derivatives of the components $g_{0\alpha}$ and g_{00} appear at no time. From this it follows that the tensor R_{ik} and (2.229) contain second derivatives with respect to time for the six spatial components of the metric $g_{\alpha\beta}$ only.

We see that these derivatives are present only within the $^{\beta}_{\alpha}$ -equation if (2.230):

$$R^{\beta}_{\alpha} - \frac{1}{2}\delta^{\beta}_{\alpha} R = \frac{8\pi k}{c^4} T^{\beta}_{\alpha} \tag{2.235}$$

while the equations containing only first order time derivatives are

$$R^0_0 - \frac{1}{2}R = \frac{8\pi k}{c^4} T^0_0, \quad R^0_{\alpha} = \frac{8\pi k}{c^4} T^0_{\alpha} \tag{2.236}$$

We may easily verify such a claim through the use of the identity (2.169) and the proof previously given, such that

$$\begin{aligned}
R^l_{m;l} &= \frac{1}{2}\frac{\partial R}{\partial x^m} \\
\implies \left(R^n_l - \frac{1}{2}\delta^n_l R\right)_{;n} &= 0 \\
\implies \left(R^0_l - \frac{1}{2}\delta^0_l R\right)_{;0} + \left(R^\alpha_l - \frac{1}{2}\delta^\alpha_l R\right)_{;\alpha} &= 0 \\
\implies \left(R^0_l - \frac{1}{2}\delta^0_l R\right)_{;0} &= -\left(R^\alpha_l - \frac{1}{2}\delta^\alpha_l R\right)_{;\alpha}
\end{aligned}$$

in which we simply separated the time and spatial components of (2.236).

$$\left(R^0_i - \frac{1}{2}\delta^0_i R\right)_{;0} = -\left(R^\alpha_i - \frac{1}{2}\delta^\alpha_i R\right)_{;\alpha} \tag{2.237}$$

$(i = 0, 1, 2, 3)$. The highest time derivatives appearing on the right side of the above equation are second derivatives, appearing in the quantities R^α_i , R. Then, since (2.237) is an identity, consequently, its left hand side must not contain any time derivative higher than second order. Nonetheless, one time differentiation appears explicitly within the equation (as we have a derivative with respect to x^0 on the LHS), hence, the formula $R^0_i - \frac{1}{2}\delta^0_i R$ themselves must not contain time derivatives higher than the first order.

In addition, we point to the fact that the left sides of equations (2.236) do not contain first derivatives $\dot{g}_{0\alpha}$ and $\dot{g}_{00}$ (only derivatives $\dot{g}_{\alpha\beta}$). We find that of all the Christoffel symbols which have form $\Gamma_{i,kl}$, only $\Gamma_{\alpha,00}$ and $\Gamma_{0.00}$ contain the condign quantities, but the latter appear only in the components of the curvature tensor of the form $R_{0\alpha0\beta}$, which we note drop out when we devise the LHS of equations (2.236).

If we sought the solution of the field equations for given initial conditions, we must first consider the number of quantities for which the initial spatial distribution may be assigned arbitrarily.

Evidently, the initial conditions required to solve the set of equations of second order must contain both the quantities to be differentiated as well as their respective first derivatives. Yet in the case under consideration the equations contain second derivatives of only the six $g_{\alpha\beta}$ not all the g_{ik} and $\dot{g}_{ik}$ may be arbitrarily assigned. Hence, we may assign the initial values of the functions $g\alpha\beta$ and $\dot{g}_{\alpha\beta}$,after which the equations of (2.236) may devise the admissible parameters of $g_{0\alpha}$ and g_{00}; in (2.235) the initial values of $\dot{g}_{0\alpha}$ still remain arbitrary.

2.15 The Landau-Lifshitz pseudotensor of the gravitational field

As aforementioned, in the absence of a gravitational field, the law of conservation of energy and momentum may be described by the divergence of the energy-momentum tensor of the system vanishing, i.e. $\partial T^{ik}/\partial x^k = 0$. By (2.220) we find that in the presence of a gravitational field, the previous equation generalizes to

$$T^k_{i;k} = \frac{1}{\sqrt{-g}} \frac{\partial \left(T^k_i \sqrt{-g}\right)}{\partial x^k} - \frac{1}{2} \frac{\partial g_{kl}}{\partial x^i} T^{kl} = 0 \tag{2.238}$$

where we used the fact that the energy-momentum tensor is symmetric (i.e. $T^{ik} = T^{ki}$) and (2.93). Yet, we come to find that (2.245) expresses no conservation law whatsoever, as the expression displayed above references only the momentum of matter and not the momentum of the field. Hence, we find that in order to transcribe a conservation law of a gravitational field, we must consider both the four-momentum of the matter producing the field, as well as the field itself. Apropos to this, we proceed as follows. Select a coordinate system in which at a particular point in space-time the first derivative of the metric g_{ik} vanishes, consequently, the second term in (2.238) vanishes accordingly, and in the first term we can remove $\sqrt{-g}$ from under the derivative sign, such that we have

$$\frac{\partial}{\partial x^k} T^k_i = 0$$

Indeed, consider such a coordinate system as the one previously described in which the first derivative of g_{ik} vanishes:

$$
\begin{aligned}
T^k_{i;k} &= \frac{1}{\sqrt{-g}}\frac{\partial\left(T^k_i\sqrt{-g}\right)}{\partial x^k} - \frac{1}{2}\frac{\partial g_{kl}}{\partial x^i}T^{kl} = 0 \\
\Longrightarrow &= \frac{1}{\sqrt{-g}}\frac{\partial\left(T^k_i\sqrt{-g}\right)}{\partial x^k} - 0 \\
\Longrightarrow &= \frac{1}{\sqrt{-g}}\left[\frac{\partial\left(T^k_i\right)}{\partial x^k}\sqrt{-g} + \frac{\partial\left(\sqrt{-g}\right)}{\partial x^k}T^k_i\right] = 0 \\
\Longrightarrow &= \frac{1}{\sqrt{-g}}\left[\frac{\partial\left(T^k_i\right)}{\partial x^k}\sqrt{-g}\right] = 0 \\
\Longrightarrow &\ \frac{\partial}{\partial x^k}T^k_i = 0
\end{aligned}
$$

similarly, in contravariant components we write,

$$
\frac{\partial}{\partial x^k}T^{ik} = 0
$$

Values of the energy-momentum tensor, satisfying the above equation, may be written in the following from,

$$
T^{ik} = \frac{\partial}{\partial x^l}\xi^{ikl}
$$

where the ξ^{ikl} quantities are antisymmetric in the indices k and l:

$$
\xi^{ikl} = -\xi^{ilk}
$$

In order to bring the energy-momentum tensor to this form we start first with the field equations

$$
T^{ik} = \frac{c^4}{8\pi k}\left(R^{ik} - \frac{1}{2}g^{ik}R\right)
$$

and using (2.366) we write

$$
R^{ik} = \frac{1}{2}g^{im}g^{kp}g^{ln}\left\{\frac{\partial^2 g_{lp}}{\partial x^m\partial x^n} + \frac{\partial^2 g_{mn}}{\partial x^l\partial x^p} - \frac{\partial^2 g_{ln}}{\partial x^m\partial x^p} - \frac{\partial^2 g_{mp}}{\partial x^l\partial x^n}\right\}
$$

where we have, according to our selection of the coordinate system, all $\Gamma^i_{kl} = 0$. After a rigorous and protracted calculation we write the tensor T^{ik} as

$$T^{ik} = \frac{\partial}{\partial x^l}\left\{\frac{c^4}{16\pi k}\frac{1}{(-g)}\frac{\partial}{\partial x^m}\left[(-g)\left(g^{ik}g^{lm} - g^{il}g^{km}\right)\right]\right\}$$

First, consider the field equation provided above:

$$\begin{aligned} T^{ik} &= \frac{c^4}{8\pi k}\left(R^{ik} - \frac{1}{2}g^{ik}R\right) \\ \implies \frac{16\pi k}{c^4}T^{ik} &= \frac{16\pi k}{c^4}\left[\frac{c^4}{8\pi k}\left(R^{ik} - \frac{1}{2}g^{ik}R\right)\right] \\ \implies \frac{16\pi k}{c^4}T^{ik} &= 2\left(R^{ik} - \frac{1}{2}g^{ik}R\right) \end{aligned}$$

Then, using (2.165) and (2.168), we rewrite the Ricci tensor and scalar curvature terms as follows,

$$R^{ik} = g^{im}g^{kp}g^{ln}R_{lmnp}$$

$$g^{ik}R = g^{ik}g^{mp}g^{ln}R_{lmnp}$$

Substitution into the field equation provided above, and considering only the RHS of the expression, we obtain:

$$\begin{aligned} &= 2\left(g^{im}g^{kp}g^{ln}R_{lmnp} - \frac{1}{2}g^{ik}g^{mp}g^{ln}R_{lmnp}\right) \\ &= 2\left(g^{im}g^{kp}g^{ln} - \frac{1}{2}g^{ik}g^{mp}g^{ln}\right)R_{lmnp} \end{aligned}$$

Then, since we chose a coordinate system in which all $\Gamma^i_{kl} = 0$, thus, we negate all terms with Christoffel symbols, i.e. the second term, and the Riemann tensor reduces in such a way that we write

$$= 2\left(g^{im}g^{kp}g^{ln} - \frac{1}{2}g^{ik}g^{mp}g^{ln}\right)\left[\frac{1}{2}\left(g_{lp,m,n} + g_{mn,l,p} - g_{ln,m,p} - g_{mp,l,n}\right)\right]$$
$$= \left(g^{im}g^{kp}g^{ln} - \frac{1}{2}g^{ik}g^{mp}g^{ln}\right)\left(g_{lp,m,n} + g_{mn,l,p} - g_{ln,m,p} - g_{mp,l,n}\right)$$
$$= g^{im}g^{kp}g^{ln}g_{lp,m,n} + g^{im}g^{kp}g^{ln}g_{mn,l,p} - g^{im}g^{kp}g^{ln}g_{ln,m,p} - g^{im}g^{kp}g^{ln}g_{mp,l,n}$$
$$-\frac{1}{2}g^{ik}g^{mp}g^{ln}g_{lp,m,n} - \frac{1}{2}g^{ik}g^{mp}g^{ln}g_{mn,l,p} + \frac{1}{2}g^{ik}g^{mp}g^{ln}g_{ln,m,p} + \frac{1}{2}g^{ik}g^{mp}g^{ln}g_{mp,l,n}$$

Next, we exchange dummy indices in such a way that we may write all second order derivatives of the metric in the same way, $g_{np,m,l}$

$$= g^{im}g^{kp}g^{ln}g_{np,m,l} + g^{ip}g^{km}g^{ln}g_{np,m,l} - g^{im}g^{kl}g^{pn}g_{np,m,l} - g^{in}g^{kp}g^{lm}g_{np,m,l}$$
$$-\frac{1}{2}g^{ik}g^{mp}g^{ln}g_{np,m,l} - \frac{1}{2}g^{ik}g^{pm}g^{ln}g_{np,l,m} + \frac{1}{2}g^{ik}g^{np}g^{ml}g_{np,m,l} + \frac{1}{2}g^{ik}g^{lm}g^{np}g_{np,m,l}$$
$$= [g^{im}g^{kp}g^{ln} + g^{ip}g^{km}g^{ln} - g^{im}g^{kl}g^{pn} - g^{in}g^{kp}g^{lm}$$
$$-\frac{1}{2}g^{ik}g^{mp}g^{ln} - \frac{1}{2}g^{ik}g^{pm}g^{ln} + \frac{1}{2}g^{ik}g^{np}g^{ml} + \frac{1}{2}g^{ik}g^{lm}g^{np}]g_{np,m,l}$$
$$= \left[g^{im}g^{kp}g^{ln} + g^{ip}g^{km}g^{ln} - g^{im}g^{kl}g^{pn} - g^{in}g^{kp}g^{lm} - g^{ik}g^{mp}g^{ln} + g^{ik}g^{np}g^{ml}\right]g_{np,m,l}$$

We note in passing the differential identity

$$\left(u_{,m,l}v = (u_{,m}v)_{,l} - u_{,m}v_{,l}\right)$$
$$\Longrightarrow = \left[u_{,m,l}v + u_{,m}v_{,l}\right] - u_{,m}v_{,l}$$
$$\Longrightarrow = u_{,m,l}v$$

Hence,

$$u_{,m,l}v = (u_{,m}v)_{,l} - u_{,m}v_{,l}$$

To this end, if we let $u = g_{np}$ and $v = g^{im}g^{kp}g^{ln} + g^{ip}g^{km}g^{ln} - g^{im}g^{kl}g^{pn} - g^{in}g^{kp}g^{lm} - g^{ik}g^{mp}g^{ln} +$

$g^{ik}g^{np}g^{ml}$ we may write

$$= \left[g_{np,m}\left(g^{im}g^{kp}g^{ln} + g^{ip}g^{km}g^{ln} - g^{im}g^{kl}g^{np} - g^{in}g^{kp}g^{lm} - g^{ik}g^{ln}g^{mp} + g^{ik}g^{np}g^{ml}\right)\right]_{,l}$$
$$-g_{np,m}\left(g^{im}g^{kp}g^{ln} + g^{ip}g^{km}g^{ln} - g^{im}g^{kl}g^{np} - g^{in}g^{kp}g^{lm} - g^{ik}g^{ln}g^{mp} + g^{ik}g^{np}g^{ml}\right)_{,l}$$

We then drop the second term, since it contains only first order derivatives, and we require second order derivatives for the energy-momentum tensor T^{ik}

$$= g_{np,m}\left(g^{im}g^{kp}g^{ln} + g^{ip}g^{km}g^{ln} - g^{im}g^{kl}g^{np} - g^{in}g^{kp}g^{lm} - g^{ik}g^{ln}g^{mp} + g^{ik}g^{np}g^{ml}\right)_{,l}$$
$$= \left(g^{im}g^{kp}g^{ln}g_{np,m} + g^{ip}g^{km}g^{ln}g_{np,m} - g^{im}g^{kl}g^{np}g_{np,m} - g^{in}g^{kp}g^{lm}g_{np,m} - g^{ik}g^{ln}g^{mp}g_{np,m} + g^{ik}g^{np}g^{ml}g_{np,m}\right)_{,l}$$

Then, using (2.86) we may write

$$g_{il}g^{lk}_{,m} = -g^{lk}g_{il,m}$$
$$\implies -g_{np}g^{ln}_{,m} = g^{ln}g_{np,m}$$

Subsequently, using (2.81) in which we have the same upper and lower indices, we obtain

$$dg = g^{ik}dg_{ik} = -gg_{ik}dg^{ik}$$

in which we write

$$g_{,m} = gg^{np}g_{np,m} \implies \frac{g_{,m}}{g} = g^{np}g_{np,m}$$

Therefore,

$$
\begin{aligned}
&= \left[-g^{im}g^{kp}g_{np}g^{ln}_{,m} - g^{ip}g^{km}g_{np}g^{ln}_{,m} - g^{im}g^{kl}\left(g_{,m}/g\right) + g^{in}g^{lm}g_{np}g^{kp}_{,m} + g^{ik}g^{ln}g_{np}g^{mp}_{,m} + g^{ik}g^{ml}\left(g_{,m}/g\right)\right]_{,l} \\
&= \left[-\delta^k_n g^{im}g^{ln}_{,m} - \delta^i_n g^{km}g^{ln}_{,m} - g^{im}g^{kl}\left(g_{,m}/g\right) + \delta^i_p g^{lm}g^{kp}_{,m} + \delta^l_p g^{ik}g^{mp}_{,m} + g^{ik}g^{ml}\left(g_{,m}/g\right)\right]_{,l} \\
&= \left[-g^{im}g^{kl}_{,m} - g^{km}g^{il}_{,m} - g^{im}g^{kl}\left(g_{,m}/g\right) + g^{lm}g^{ik}_{,m} + g^{ik}g^{lm}_{,m} + g^{ik}g^{ml}\left(g_{,m}/g\right)\right]_{,l}
\end{aligned}
$$

Using the symmetry of the differential operator we may exchange the indices m and l in some terms such that

$$
\begin{aligned}
&= \left[-g^{il}g^{km}_{,m} - g^{km}g^{il}_{,m} - g^{il}g^{km}\left(g_{,m}/g\right) + g^{lm}g^{ik}_{,m} + g^{ik}g^{lm}_{,m} + g^{ik}g^{lm}\left(g_{,m}/g\right)\right]_{,l} \\
&= \left[\left(g^{lm}g^{ik}_{,m} + g^{ik}g^{lm}_{,m}\right) - \left(g^{il}g^{km}_{,m} + g^{km}g^{il}_{,m}\right) + \left(g_{,m}/g\right)\left(g^{ik}g^{lm} - g^{il}g^{km}\right)\right]_{,l}
\end{aligned}
$$

Then by product rule and the associative property we may combine terms such that

$$
\begin{aligned}
&= \left[\left(g^{ik}g^{lm}\right)_{,m} - \left(g^{il}g^{km}\right)_{,m} + \left(g_{,m}/g\right)\left(g^{ik}g^{lm} - g^{il}g^{km}\right)\right]_{,l} \\
&= \left[\left(g^{ik}g^{lm} - g^{il}g^{km}\right)_{,m} + \left(g_{,m}/g\right)\left(g^{ik}g^{lm} - g^{il}g^{km}\right)\right]_{,l}
\end{aligned}
$$

Then, multiplying the first term by $\frac{-g}{-g}$ and the second term by $\frac{-1}{-1}$, we have:

$$
\begin{aligned}
&= \left[\left(\frac{-g}{-g}\right)\left(g^{ik}g^{lm} - g^{il}g^{km}\right)_{,m} + \left(\frac{-1}{-1}\right)\left(g_{,m}/g\right)\left(g^{ik}g^{lm} - g^{il}g^{km}\right)\right]_{,l} \\
&= \left[\left(\frac{-g}{-g}\right)\left(g^{ik}g^{lm} - g^{il}g^{km}\right)_{,m} + \left(\frac{-g_{,m}}{-g}\right)\left(g^{ik}g^{lm} - g^{il}g^{km}\right)\right]_{,l}
\end{aligned}
$$

Factoring out $\frac{1}{-g}$ from both terms and applying product rule, we have

$$= \left\{ \left(\frac{1}{-g} \right) \left[(-g) \left(g^{ik} g^{lm} - g^{il} g^{km} \right)_{,m} + (-g)_{,m} \left(g^{ik} g^{lm} - g^{il} g^{km} \right) \right] \right\}_{,l}$$
$$\left\{ \left(\frac{1}{-g} \right) \left[(-g) \left(g^{ik} g^{lm} - g^{il} g^{km} \right) \right]_{,m} \right\}_{,l}$$
$$= \left(\frac{1}{-g} \right) \left[(-g) \left(g^{ik} g^{lm} - g^{il} g^{km} \right) \right]_{,m,l}$$

Finally, adding the LHS of the equation we obtain the required expression for the energy-momentum tensor

$$\frac{16\pi k}{c^4} T^{ik} = \left(\frac{1}{-g} \right) \left[(-g) \left(g^{ik} g^{lm} - g^{il} g^{km} \right) \right]_{,m,l}$$
$$\implies T^{ik} = \frac{c^4}{16\pi k} \left(\frac{1}{-g} \right) \left[(-g) \left(g^{ik} g^{lm} - g^{il} g^{km} \right) \right]_{,m,l}$$

which is equivalent to the expression mentioned above.

We note in passing that the expression derived above is antisymmetric in the indices k and l, and is equivalent to the quantity designated as ξ^{ikl}. Since, we chose a coordinates system in which the first derivatives of $g_{ik} = 0$ at the particular point, we can factor the term $\frac{1}{-g}$ from under the derivative $\partial/\partial x^l$.

We proceed by introducing the notation

$$h^{ikl} = \frac{\partial}{\partial x^m} \lambda^{iklm} \tag{2.239}$$

$$\lambda^{ikkm} = \frac{c^4}{16\pi k} (-g) \left(g^{ik} g^{bm} - g^{il} g^{km} \right) \tag{2.240}$$

The quantities h^{ikl} are antisymmetric in k and l:

$$h^{ikl} = -h^{ilk}$$

Hence, we can write

$$\frac{\partial h^{ikl}}{\partial x^l} = (-g)T^{ik}$$

We note that the above formulation is deduced on the account of allowing the first derivatives of the metric tensor to vanish $\partial g_{ik}/\partial x^l = 0$, which we find is no longer valid when we consider an arbitrary system of coordinates. We find that in general, when we move the term on the right of the above expression (2.241) we have a nonzero difference, i.e. $\partial h^{ikl}/\partial x^l - (-g)T^{ik} \neq 0$. We denote this nonzero difference as $(-g)t^{ik}$, such that we may make the following claim

$$(-g)\left(T^{ik} + t^{ik}\right) = \frac{\partial h^{ikl}}{\partial x^l} \tag{2.241}$$

The reason we have a nonzero difference arises from the fact that when we make a transition from the local geometry, where we assumed that the derivatives of the metric tensor were zero, to global geometry we find that in general $\partial h^{ikl}/\partial x^l - (-g)T^{ik} \neq 0$, hence, the term t^{ik} accounts for such a mishap. We point attention to the fact that in the derivation aforementioned we disregarded first derivatives of the metric in several steps. Furthermore, first derivatives usually combine to form connections (i.e. Christoffel symbols) which account for the curvature of space-time. Thereholds, that outside of our flat neighborhood first derivatives reappear and play a prominent role for curvature, in much the same manner in which they account for such a phenomena in the definition of covariant derivatives.

The quantities of $t^{ik}$are symmetric in i and k:

$$t^{ik} = t^{ki} \tag{2.242}$$

which follows from their definition, since the energy-momentum tensor T^{ik} as well the derivatives $\frac{\partial h^{ikl}}{\partial x^l}$ are symmetric quantities. The if we rewrite the symmetric tensor T^{ik} using the field equations in terms of R^{ik}, we obtain the following relation

$$(-g)\left\{\frac{c^4}{8\pi k}\left(R^{ik}-\frac{1}{2}g^{ik}R\right)+t^{ik}\right\}=\frac{\partial h^{ikl}}{\partial x^l} \tag{2.243}$$

Indeed, we have utilizing (2.229)

$$\begin{aligned}
(-g)\left(T^{ik}+t^{ik}\right)&=\frac{\partial h^{ikl}}{\partial x^l}\\
\Longrightarrow (-g)\left(\left[\frac{c^4}{8\pi k}R^{ik}-\frac{1}{2}g^{ik}R\right]+t^{ik}\right)&=\frac{\partial h^{ikl}}{\partial x^l}\\
\Longrightarrow (-g)\left(\frac{c^4}{8\pi k}\left[R^{ik}-\frac{1}{2}g^{ik}R\right]+t^{ik}\right)&=\frac{\partial h^{ikl}}{\partial x^l}
\end{aligned}$$

using such an expression we may then determine the value of the quantities t^{ik} to be:

$$\begin{aligned}
t^{ik}=\frac{c^4}{16\pi k}[&\left(2\Gamma^n_{lm}\Gamma^p_{np}-\Gamma^n_{lp}\Gamma^p_{mn}-\Gamma^n_{ln}\Gamma^p_{mp}\right)\left(g^{il}g^{km}-g^{ik}g^{lm}\right)+\\
&g^{il}g^{mn}\left(\Gamma^k_{lp}\Gamma^p_{mn}+\Gamma^k_{mn}\Gamma^p_{lp}-\Gamma^k_{np}\Gamma^p_{lm}-\Gamma^k_{lm}\Gamma^p_{np}\right)+\\
&g^{kl}g^{mn}\left(\Gamma^k_{lp}\Gamma^p_{mn}+\Gamma^i_{mn}\Gamma^p_{lp}-\Gamma^i_{np}\Gamma^p_{lm}-\Gamma^i_{lm}\Gamma^p_{np}\right)+\\
&g^{lm}g^{np}\left(\Gamma^i_{ln}\Gamma^k_{mp}-\Gamma^i_{lm}\Gamma^k_{np}\right)]
\end{aligned}$$

a proof of such an expression is excluded as the calculation proves tedious and protracted, and derivation of said formula is not the aim of this paper. We note that we may also write the quantity t^{ik} by using the derivatives of the components of the metric tensor in order to define the tensor $(-g)\,t^{ik}$

$$\begin{aligned}
(-g)t^{ik}=\frac{c^4}{16\pi k}[&g^{ik}{}_{,l}\,g^{lm}_{,m}-g^{il}_{,l}g^{km}_{,m}+\frac{1}{2}g^{ik}g_{lm}g^{ln}{}_{,p}\,g^{pm}_{n}-\\
&\left(g^{il}g_{mn}g^{kn}_{,p}g^{mp}_{,l}+g^{kl}g_{mn}g^{in}_{,p}g^{mp}_{,l}\right)+g_{lm}g^{np}g^{il}_{,n}g^{km}_{,p}+\\
&\frac{1}{8}\left(2g^{il}g^{km}-g^{ik}g^{lm}\right)\left(2g_{np}g_{qr}-g_{pq}g_{nr}\right)g^{nr}_{,l}\,g^{pq}_{,m}]
\end{aligned}$$

in which we have that $g^{ik}=\sqrt{-g}g^{ik}$, and the index i refers to differentiation with respect to x^i.

An imperative property of the quantities $t^{ik}$is that they do not exhibit a tensor; this follows immediately from the fact that in $\partial h^{ikl}/\partial x^l$ we have an ordinary derivative rather than a covariant derivative. Nonetheless, as per (2.15) we see that t^{ik} is expressed in terms of Christoffel symbols Γ^i_{kl}, with the latter quantity behaving like a tensor with respect to linear transformations of coordinates, see (2.75), hence, we have the quantities t^{ik} behaving analogously within the same conditions as the Christoffel symbols.

Then by using the definition previously found, i.e. (2.241), it follows that for the sum $T^{ik}+t^{ik}$ the equation

$$\frac{\partial}{\partial x^k}(-g)\left(T^{ik}+t^{ik}\right)=0 \tag{2.244}$$

is identically fulfilled. Computing the above formulation using the fact that we may consider a system with galilean coordinates such that $\partial T^{ik}/\partial x^k=0$, due to the conservation of energy and momentum, and noting that since T^{ik} is a tensor and if it is zero in one coordinate system it must vanish in all other systems, then we may deduce that $\partial h^{ikl}/\partial x^l=0$ such that

$$\begin{aligned}
(-g)\left(T^{ik}+t^{ik}\right)&=\frac{\partial h^{ikl}}{\partial x^l}\\
\implies \frac{\partial}{\partial x^k}\left((-g)\left(T^{ik}+t^{ik}\right)\right)&=\frac{\partial}{\partial x^k}\left(\frac{\partial h^{ikl}}{\partial x^l}\right)\\
\implies \frac{\partial}{\partial x^k}\left((-g)\left(T^{ik}+t^{ik}\right)\right)&=0
\end{aligned}$$

we remark that due to our selection of coordinate system there lacks a presence of a gravitational field, thus we have that the quantities $t^{ik}=0$. To this end, as aforementioned, we may deduce that (2.244) constitutes a conservation law which takes the form

$$P^i=\frac{1}{c}\int(-g)\left(T^{ik}+t^{ik}\right)dS_k \tag{2.245}$$

Again, if we consider galilean coordinates $t^{ik}=0$, and the integral we have goes over into $(1/c)\int T^{ik}dS_k$, i.e. the four-momentum of the material. We may then deduce that the identity

(2.245) must correspond to the total four-momentum of matter and the gravitational field. To this end, we denote the values t^{ik} as the *energy-momentum pseudotensor* of the gravitational field.

The integration defined in (2.245) may be considered over any infinite hypersurface, including all of three dimensional space. Furthermore, if we consider a hypersurface in which $x^0 = const$, we may reduce the integral over the hypersurface to integral over three dimensional volume, namely,

$$P^i = \frac{1}{c}\int (-g)\left(T^{i0} + t^{i0}\right) dV \tag{2.246}$$

This expression relates the fact that there exists a conservation law for the angular momentum, defined as

$$M^{ik} = \int \left(x^i dP^k - x^k dP^i\right) = \frac{1}{c}\int \left\{x^i \left(T^{kl} + t^{kl}\right) - x^k \left(T^{il} + t^{il}\right)\right\}(-g)dS_l \tag{2.247}$$

which we extracted from the fundamental law of conservation of angular momentum outside the presence of a gravitational field, which is denoted by (2.199), and effecting a simple substitution of $dP^k = T^{kl} + t^{kl}$, which we derived from taking a derivative of (2.245), i.e.

$$\begin{aligned} P^i &= \frac{1}{c}\int (-g)\left(T^{ik} + t^{ik}\right) dS_k \\ \Longrightarrow\ \frac{\partial}{\partial x^k}P^i &= \frac{\partial}{\partial x^k}\frac{1}{c}\int (-g)\left(T^{ik} + t^{ik}\right) dS_k \\ \Longrightarrow\ \frac{\partial}{\partial x^k}P^i &= \frac{1}{c}(-g)\frac{\partial}{\partial x^k}\int \left(T^{ik} + t^{ik}\right) dS_k \\ \Longrightarrow\ \frac{\partial}{\partial x^k}P^i &= \frac{1}{c}(-g)\left(T^{ik} + t^{ik}\right) \end{aligned}$$

in which we applied the Fundamental Theorem of Calculus.

Hence, even in the theory of general relativity, for a closed system of gravitating bodies the total angular momentum is conserved, moreover, it is possible to define a center of inertia which carries out uniform motion. The latter statement is related to the conservation of the components

$M^{0\alpha}$ which is expressed by the equation

$$x^0 \int \left(T^{\alpha 0} + t^{\alpha 0}\right)(-g)dV - \int x^\alpha \left(T^{00} + t^{00}\right)(-g)dV = \text{const} \tag{2.248}$$

where the components of the energy-momentum tensor take their usual values, defined by (2.203) and (2.204). We note that we derive the conservation of the components $M^{0\alpha}$ delineated in (2.248) from the classical mechanics definition of the conservation of the angular momentum components and craft an analogy to the case in the presence of a field of gravity. Nonetheless, we define the classical definition as

$$M^{0\alpha} = \Sigma \left(t\vec{p} - \frac{\mathbf{E}\vec{r}}{c^2}\right) = const \tag{2.249}$$

so that the coordinates of the center of inertia are given by the formula

$$X^\alpha = \frac{\int x^\alpha \left(T^{00} + t^{00}\right)(-g)dV}{\int \left(T^{00} + t^{00}\right)(-g)dV} \tag{2.250}$$

we may use the equation for the center of inertia in a non-gravitational sense as an analogy, by which starting with (2.249), we have

$$M^{0\alpha} = \Sigma \left(t\vec{p} - \frac{\mathbf{E}\vec{r}}{c^2}\right) = const \tag{2.251}$$

due to the conservation of M^{ik} for a closed system. Furthermore, the total energy is also conserved, such that we may rewrite the equality as

$$\frac{\Sigma\mathscr{E}\mathbf{r}}{\Sigma\mathscr{E}} - \frac{c^2\Sigma\mathbf{p}}{\Sigma\mathscr{E}}t = const \tag{2.252}$$

from this we define the radius vector of the angular momentum as

$$\mathbf{R} = \frac{\sum\mathscr{E}\mathbf{r}}{\sum\mathscr{E}} \tag{2.253}$$

which gives the relativistic definition of the coordinates of the *center of inertia* of the system.

Let us select a coordinate system which is inertial in a given volume element, such that all t^{ik} quantities vanish within any point in this space-time, since all derivatives of the metric tend to zero, it follows that all Γ^i_{kl} vanish and with it the energy momentum pseudotensor. Contrarily, we may find values of $t^{ik} \neq 0$ if we only consider curvilinear coordinates within flat space (i.e. outside of a gravitational field) rather than Cartesian coordinates. Consequently, there is no sense in attempting to describe a definite localization of energy of the gravitational field in space. If the energy-momentum tensor is zero at some arbitrary point in space-time, it follows that due to the nature of the tensor it must vanish accordingly in any reference system, so that we may remark that at this specific point there exists no matter or electromagnetic field. On the contrary, the vanishing of the pseudo-tensor at some particular point within a reference system does not constitute the need for it to identically vanish for another reference system, thus it proves futile to discuss whether there exists gravitational energy within the given world point. This comes directly from the fact that by Einstein's equivalence principle we may always adopt suitable coordinates, such that we may "annihilate" the gravitational field in a given volume element, i.e. we may reduce the space-time metric to it's Minkowski counterpart, in which case the pseudotensor t^{ik} tends to zero within this volume element. As oppose to what has been previously mentioned, the values of the four-momentum P^i have definite values and are completely independent of the choice of reference system.

Consider briefly a region of space sufficiently large which encapsulates the masses we shall study, and further suppose that outside of said region there exists no gravitational field. With the passage of time, the region cuts out a "channel" in 4-D space-time. Outside the confines of this channel there is no field, such that the space-time takes on a flat configuration. Describing this in a more mathematical formalism, we may reiterate the aforementioned remark as considering a neighborhood around the point, say y, to which we attach the concept of a "region" which we provided above, after which considering the time coordinate ("the course of time") this neighborhood “cuts” out a *channel*, denote it as Ω, outside of which we have that there is no gravitational field. We thus connect the latter statement to the concept of support of some function, in this case being the gravity, i.e. G for the peculiar case, such that $supp\ G = 0$ outside of the channel crafted, i.e. $y \notin \Omega$. Because of this, we must choose a reference system, such that when we calculate the energy and momentum of the field the reference system goes over into a galilean system and the t^{ik} vanish.

Due to this requirement, the reference system is not uniquely determined and may be chosen arbitrarily within the confines of the channel. Yet, as previously mentioned the P^i values are completely independent of the choice of coordinate system within the interior of the channel. To this end, consider two different coordinate system which are distinct from one another inside of the channel, yet coincide in the exterior channel in both being the same galilean system, and compare the values of the four-momentum of the two distinct systems P^i and P'^i at definite “time” x^0 and x'^0. We introduce a third coordinate system, which coincides with the first system at time x^0 in the interior of the channel, and at the moment x'^0 coincides with the second system, outside of the channel in which it becomes galilean. By invoking the law of conservation of momentum and energy the quantities P^i are constant, i.e. $\partial/\partial x^0 P^i = 0$. It follows that this is the case for the third system, as well as the first two system. There-holds, that the value of $P''^i = const$ in the third system, nonetheless, since the third and first system coincide inside the channel, we have that $P^i = P''^i$, similarly, since the second and third coincide in the exterior of the channel we obtain, $P'^i = P''^i$, hence, we obtain $P^i = P'^i$.

We previously mentioned that the values t^{ik} behave like a tensor with respect to linear transformations of the coordinates. Accordingly, the values P^i constitute a four-vector with respect to these linear transformations, in particular Lorentz transformations which, at infinity, take one galilean reference frame into another. The four-momentum P^i may be expressed as an integral over a distant three-dimensional surface surrounding “all space”. Namely, substitution of (2.241) into

(2.245), we obtain

$$P^i = \frac{1}{c}\int (-g)\left(T^{ik} + t^{ik}\right) dS_k$$
$$\Longrightarrow\; P^i = \frac{1}{c}\int \frac{\partial h^{ikl}}{\partial x^l} dS_k$$

Next, we note that we may rewrite the integral of an antisymmetric tensor A^{ik} as

$$\frac{1}{2}\int A^{ik} df^*_{ik} = \frac{1}{2}\int \left(dS_i \frac{\partial A^{ik}}{\partial x^k} - dS_k \frac{\partial A^{ik}}{\partial x^i}\right) = \int dS_i \frac{\partial A^{ik}}{\partial x^k} \tag{2.254}$$

since we may transform an integral over a two-dimensional surface to an integral over the hypersurface "spanning" it by replacing the element of integration by the df^*_{ik} by the operator,

$$df^*_{ik} \to dS_i \frac{\partial}{\partial x^k} - dS_k \frac{\partial}{\partial x^i} \tag{2.255}$$

Thus, we have that

$$P^i = \frac{1}{c}\int \frac{\partial h^{ikl}}{\partial x^l} dS_k = \frac{1}{2c}\oint h^{ikl} df^*_{kl} \tag{2.256}$$

in which the second rank tensor df^*_{kl} is the "normal" to the surface element. If we select the hypersurface $x^0 = const$ as the surface of integration for (2.245), then in (2.256) the surface of integration turns out to be a surface in ordinary space.

$$P^i = \frac{1}{c}\oint h^{i0\alpha} df_\alpha \tag{2.257}$$

in which the components of df_α are the components of the three-dimensional element of ordinary surface.

Then in order to define the angular momentum, we proceed by substituting (2.241) into

(2.247) and we rewrite the h^{ikl} in the form (2.406). Integration by parts yields,

$$M^{ik} = \int \left(x^i dP^k - x^k dP^i \right) = \frac{1}{c} \int \left[x^i \left(T^{kl} + t^{kl} \right) - x^k \left(T^{il} + t^{il} \right) \right] (-g) dS_l$$

$$\Longrightarrow M^{ik} = \frac{1}{c} \int \left[x^i \left(\frac{\partial h^{kln}}{\partial x^m} \right) - x^k \left(\frac{\partial h^{iln}}{\partial x^m} \right) \right] (-g) dS_l$$

$$\Longrightarrow M^{ik} = \frac{1}{c} \int \left[x^i \left(\frac{\partial^2 \lambda^{klmn}}{\partial x^m \partial x^n} \right) - x^k \left(\frac{\partial^2 \lambda^{ilnm}}{\partial x^m \partial x^n} \right) \right] (-g) dS_l$$

next using integration by parts and letting $u = x^i$, $du = \frac{\partial}{\partial x^m} x^i = \delta^i_m$, $dv = \left(\frac{\partial^2 \lambda^{klmn}}{\partial x^m \partial x^n} \right)$ and $v = \left(\frac{\partial \lambda^{klmn}}{\partial x^n} \right)$, we obtain

$$= \frac{1}{c} \left[x^i \frac{\partial \lambda^{klmn}}{\partial x^n} - x^k \frac{\partial \lambda^{ilmn}}{\partial x^n} \right]_{S_l} - \frac{1}{c} \int \left[\delta^i_m \frac{\partial \lambda^{klmn}}{\partial x^n} - \delta^k_m \frac{\partial \lambda^{ilmn}}{\partial x^n} \right] dS_l$$

after which we may rewrite the first integral by using (2.254) and rewriting $\frac{\partial \lambda^{klmn}}{\partial x^n} = h^{klm}$, we obtain

$$= \frac{1}{2c} \int \left(x^i \frac{\partial \lambda^{klmn}}{\partial x^n} - x^k \frac{\partial \lambda^{ilmn}}{\partial x^n} \right) df^*_{lm} - \frac{1}{c} \int \left(\delta^i_m \frac{\partial \lambda^{klmn}}{\partial x^n} - \delta^k_m \frac{\partial \lambda^{ilmn}}{\partial x^n} \right) dS_l$$

$$= \frac{1}{2c} \int \left(x^i h^{klm} - x^k h^{ilm} \right) df^*_{lm} - \frac{1}{c} \int \frac{\partial}{\partial x^n} \left(\lambda^{klin} - \lambda^{ilkn} \right) dS_l$$

By definition of λ^{iklm} we see that the quantities are antisymmetric upon exchanging the middle two indices as well as a symmetry property from the difference of two quantities λ^{iklm} with slightly varying indices

$$\lambda^{ilkn} - \lambda^{klin} = \lambda^{ilnk}, \quad \lambda^{inlk} = -\lambda^{ilnk}$$

After which the remaining integral over dS_l is equivalent to

$$\frac{1}{c} \int \frac{\partial}{\partial x^n} \left(\lambda^{klin} - \lambda^{ilkn} \right) dS_l$$

$$\Longrightarrow = \frac{1}{c} \int \frac{\partial}{\partial x^n} \left(\lambda^{ilnk} \right) dS_l$$

$$\Longrightarrow = \frac{1}{2c} \int \lambda^{ilnk} df^*_{ln}$$

Evidently, selecting a purely spatial surface of integration, we find:

$$M^{ik} = \frac{1}{c}\int \left(x^i h^{ko\alpha} - x^k h^{io\alpha} + \lambda^{io\alpha k}\right) df_\alpha \tag{2.258}$$

2.16 Newton's Law

We analyze the transition from the Einstein field equations to their reduction to classical Newtonian mechanics. As previously denoted in §2.7, when considering small velocities of particles we require that the gravitational field be weak.

As we determined in §2.7, the formulation for the required component of the metric tensor g_{00} for this limiting case takes the form

$$g_{00} = 1 + \frac{2\phi}{c^2} \tag{2.259}$$

in which ϕ is regarded as the gravitational potential of the field. Then, we may consider the components of the energy-momentum tensor for macroscopic bodies as $T_i^k = \mu c^2 u_i u^k$, in which μ is the mass density of the body, i.e. the sum of rest masses of the particles in a unit volume. For the four-velocity, u^i, since the velocity of the macroscopic object is slow, we may neglect all spatial components and preserve only the time-component of the four-velocity ,i.e. $u^\alpha = 0, u^0 = u_0 = 1$. Hence, we are left solely with the components of the energy-momentum tensor

$$T_0^0 = \mu c^2 \tag{2.260}$$

The value of the scalar $T = T_i^i$ will be the same as T_0^0. Now, we proceed by rewriting the field equations in the form of (2.232):

$$R_i^k = \frac{8\pi k}{c^4}\left(T_i^k - \frac{1}{2}\delta_i^k T\right) \tag{2.261}$$

then for $i = k = 0$, we have

$$R_0^0 = \frac{8\pi k}{c^4}\left(T_0^0 - \frac{1}{2}\delta_0^0 T\right) = \frac{8\pi k}{c^4}\left(\mu c^2 - \frac{1}{2}\mu c^2\right) = \frac{4\pi k}{c^2}\mu \tag{2.262}$$

Calculating the Ricci tensor from the formula given in (2.166), we point to the fact that terms containing products of the Christoffel symbols Γ^i_{kl} are quantities of the second order, while terms containing derivatives with respect to the time-like component $x^0 = ct$ are small as compared to derivatives with respect to the space components x^α and are neglected. Appropriately, there remains $R_{00} = R_0^0 = \partial\Gamma^\alpha_{00}/\partial x^\alpha$. Substitution of

$$\Gamma^\alpha_{00} \simeq -\frac{1}{2}g^{\alpha\beta}\frac{\partial g_{00}}{\partial x^\beta} = \frac{1}{c^2}\frac{\partial \phi}{\partial x^\alpha}$$

into the Ricci tensor yields,

$$R_0^0 = \frac{1}{c^2}\frac{\partial^2\phi}{\partial x^{\alpha 2}} = \frac{1}{c^2}\Delta\phi \tag{2.263}$$

Observe that since the time component of the metric in this limiting case takes the value $g_{00} = 1 + \frac{2\phi}{c^2}$, substitution into the spatial derivative of the Christoffel symbols gives:

$$\begin{aligned}
\Gamma^\alpha_{00} &\simeq -\frac{1}{2}g^{\alpha\beta}\frac{\partial g_{00}}{\partial x^\beta} \\
&= -\frac{1}{2}g^{\alpha\beta}\frac{\partial\left(1 + \frac{2\phi}{c^2}\right)}{\partial x^\beta} \\
&= -\frac{1}{2}g^{\alpha\beta}\left(\frac{2}{c^2}\right)\left(\frac{\partial\phi}{\partial x^\beta}\right) \\
&= -\frac{1}{c^2}g^{\alpha\beta}\left(\frac{\partial\phi}{\partial x^\beta}\right) = \frac{1}{c^2}\frac{\partial\phi}{\partial x^\alpha}
\end{aligned}$$

Subsequently, introducing the found value into the Ricci tensor R_0^0 we find,

$$R_0^0 = \frac{\partial \Gamma_{00}^{\alpha}}{\partial x^{\alpha}} =$$
$$\frac{\partial}{\partial x^{\alpha}} \left(\frac{1}{c^2} \frac{\partial \phi}{\partial x^{\alpha}} \right)$$
$$= \frac{1}{c^2} \frac{\partial^2 \phi}{\partial x^{\alpha 2}}$$
$$= \frac{1}{c^2} \Delta \phi$$

where Δ is the Laplacian differential operator. Thus, substituting in the values acquired into the general expression for the field equations (2.232) we find

$$R_i^k = \frac{8\pi k}{c^4} \left(T_i^k - \frac{1}{2} \delta_i^k T \right)$$
$$\implies R_0^0 = \frac{8\pi k}{c^4} \left(T_0^0 - \frac{1}{2} \delta_0^0 T \right)$$
$$\implies \frac{4\pi k}{c^2} \mu = \frac{1}{c^2} \Delta \phi$$

$$\implies \Delta \phi = 4\pi k \mu \tag{2.264}$$

This is the equation of the field of gravity in nonrelatvistic mechanics. The equation obtained is completely analogous to the Poisson equation, i.e.

$$\Delta \phi = -4\pi \rho \tag{2.265}$$

for the electric potential, where in place of the charge density of matter we instead have the mass density multiplied by $-k$. We know that the potential of an electric field which satisfies the *Poisson equation* is given by

$$\phi = \frac{e}{R} \tag{2.266}$$

where e is the charge of the particle under consideration and R is the distance of the electric field from the charge. Nonetheless, if we have an aggregate of charges, then the field produced by the system is equal, according to the principle of superposition, to the sum of the individual fields of each of the particles. Respectively, we have that the potential of such a system is given by

$$\phi = \sum_a \frac{e_a}{R_a}$$

where R_a is the distance from the charge e_a to the point where we are determining the potential. Furthermore, if we introduce the charge density ρ, the potential takes the form

$$\phi = \int \frac{\rho}{R} dV \tag{2.267}$$

in which $R$is the distance from the volume element dV to the particular point of the field. We can then immediately derive the general solution of (2.411) by using (2.267) as an analogy, such that

$$\phi = -k \int \frac{\mu dV}{R} \tag{2.268}$$

We note that this formula describes the potential of the gravitational field for an arbitrary mass distribution within the nonrelativistic approximation. Notably, we have for the potential of the field of a single particle of mass m

$$\phi = -\frac{km}{R} \tag{2.269}$$

which is analogous to (2.266) for the potential of a single charged particle. Ergo, we find that the force, $F = -m'(\partial\phi/\partial R)$, acting in this field on another particle of mass m' is

$$\frac{\partial \phi}{\partial R} = \frac{\partial}{\partial R}\left(\frac{-km}{R}\right) = -km\left(\frac{\partial}{\partial R}\frac{1}{R}\right) = \frac{km}{R^2}$$

after which,

$$F = -m'(\partial\phi/\partial R) = \frac{-kmm'}{R^2} \tag{2.270}$$

which is Newton's Law of Attraction. The potential energy of a particle within a gravitational field is equal to its mass multiplied by the potential of the field, in analogy to how the potential energy in an electric field is given by the product of the charge density ρ and the potential of the field ϕ_e, i.e.

$$U = \frac{1}{2}\int \rho \phi_e dV \tag{2.271}$$

in much the same way, we may derive the potential energy of an arbitrary mass distribution by

$$U = \frac{1}{2}\int \mu \phi_g dV \tag{2.272}$$

where μ is the mass density and ϕ_g is the potential of the field of gravity.

For the Newtonian potential of a constant gravitational field at arbitrary prolonged distances, producing it, we cam give an expansion for the potential. To this end we select the coordinate origin at the inertial center of the masses. Subsequently, the integral $\int \mu \mathbf{r} dV$, which is analogous to the dipole moment of a system of charges, vanishes identically. Hence, we may always eliminate the "dipole terms" from the expansion. Properly, the expansion of the potential has the form

$$\phi = -k\left\{\frac{M}{R_0} + \frac{1}{6}D_{\alpha\beta}\frac{\partial^2}{\partial X_\alpha \partial X_\beta}\frac{1}{R_0} + \dots\right\} \tag{2.273}$$

in which $M = \int \mu dV$ is the total mass of the system, and the quantity

$$D_{\alpha\beta} = \int \mu \left(3x_\alpha x_\beta - r^2\delta_{\alpha\beta}\right) dV \tag{2.274}$$

we denote as the *mass quadrupole moment tensor*. This tensor is related to the *moment of inertia tensor*

$$J_{\alpha\beta} = \int \mu \left(r^2\delta_{\alpha\beta} - x_\alpha x_\beta\right) dV$$

by the relation

$$D_{\alpha\beta} = J_{\gamma\gamma}\delta_{\alpha\beta} - 2J_{\alpha\beta} \tag{2.275}$$

The assertion of the Newtonian potential from a given mass distribution is within its own right subject to one of the branches of mathematical physics, with entire books being dedicated to the area. Nonetheless, we note that the derivation of such a potential is not the subject of the present paper, and thus we conclude our discussion of said topic.

2.17 Centrally symmetric gravitational field

We now consider a field of gravity with central symmetry. This field is produced by an arbitrary centrally symmetric distribution of matter, in which the motion of the matter is also centrally symmetric, i.e. the velocity at each point must be directed along the radius.

The notion of a centrally symmetric field is as follow: we require that the space-time metric be the same for all points equidistant from the center of the matter. If we consider Euclidean space, we find that this distance is equivalent to the radius vector, nonetheless, in a non-Euclidean space, i.e. in the presence of a gravitational field, such is not the case, as there exists no quantity which has the same properties of the Euclidean radius vector, namely, being equivalent to both the distance from the center as well as the length of the circumference divided by 2π, i.e. $C = 2\pi r \implies r = \frac{C}{2\pi}$. Hence, we leave the choice of the "radius vector" arbitrary.

Utilizing "spherical" space coordinates r, θ, ϕ, then the most general centrally symmetric expression for the space-time metric is

$$ds^2 = h(r,t)dr^2 + k(r,t)\left(\sin^2\theta d\phi^2 + d\theta^2\right) + l(r,t)dt^2 + a(r,t)drdt \qquad (2.276)$$

in which a, h, k, l are all functions of the "radius vector " r and "time" t. Nonetheless, due to the arbitrariness of the choice of system of reference allocated by the general theory of relativity, we can always subject the coordinates to a transformation which does not destroy the central symmetry of ds^2, i.e. the space-time metric, or expression for the interval ds, remains the same for all points at the same distance from the center. To this end, we may transform the coordinates r and t such that they become the functions

$$r = f_1\left(r', t'\right), \quad t = f_2\left(r', t'\right)$$

where f_1, f_2 are functions of the new coordinates r', t'.

Making use of the fact that we have redefined the coordinates r and t by means of the transformation proposed above, we proceed by making the interval ds^2 reduce to the form where we have the coefficient $a(r,t)$ of $dr\,dt$ vanish and the coefficient $k(r,t)$ become equivalent to $-r^2$. The latter requirement implies that the radius vector r proposed is determined in such a way that the circumference of a circle is defined in the same manner as in Euclidean space, i.e. $C = 2\pi r$. For later convenience we allow the coefficients h and l to take the the exponential form, $-e^{\lambda}$ and $c^2 e^{v}$, respectively, where the parameters λ and v are functions of r and t. Appropriately, we obtain the following formula for the space-time interval

$$ds^2 = e^{v} c^2 dt^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) - e^{\lambda} dr^2 \tag{2.277}$$

Denoting the x^0, x^1, x^2, x^3, respectively, using the coordinates ct, r, θ, ϕ, we find for the nonzero components of the metric tensor the expressions

$$g_{00} = e^{v}, \quad g_{11} = -e^{\lambda}, \quad g_{22} = -r^2, \quad g_{33} = -r^2 \sin^2\theta$$

for the contravariant metric we find its components to be

$$g^{00} = e^{-v}, \quad g^{11} = -e^{-\lambda}, \quad g^{22} = -r^{-2}, \quad g^{33} = -r^{-2}\sin^{-2}\theta$$

Then, using the obtained quantities for the components of the metric tensor we proceed to calculating the Christoffel symbols, Γ^{i}_{kl} using formula (2.80). After a protracted calculation we find that the

Christoffel symbols take the following values

$$\begin{array}{lll} \Gamma^1_{11} = \frac{\lambda'}{2} & \Gamma^0_{10} = \frac{\nu'}{2} & \\ \Gamma^0_{11} = \frac{\dot{\lambda}}{2} e^{\lambda-\nu} & \Gamma^1_{22} = -re^{-\lambda} & \Gamma^2_{33} = -\sin\theta\cos\theta \\ \Gamma^2_{12} = \Gamma^3_{13} = \frac{1}{r} & \Gamma^3_{23} = \cot\theta & \Gamma^1_{00} = \frac{\nu'}{2} e^{\nu-\lambda} \\ \Gamma^1_{10} = \frac{\dot{\lambda}}{2} & \Gamma^1_{33} = -r\sin^2\theta e^{-\lambda} & \Gamma^0_{00} = \frac{\dot{\nu}}{2} \end{array} \tag{2.278}$$

where prime denotes differentiation with respect to r and dot denoted differentiation with respect to ct.

We proceed by verifying a few of the proposed values for Γ^i_{kl}, for a complete derivation of all values will prove rather prolonged and redundant. To this end, we note certain assumptions that will reduce the amount of calculations required. Firstly, as previously annotated, we consider a "spherically symmetric" system, namely, the functions of the interval ds^2 are independent of θ and ϕ. Secondly, we consider a vacuum solution to Einstein's field equations, hence, the energy-momentum tensor, appropriately vanishes. Finally, we consider a system which is not static, thus, we may take derivatives with respect to the time-like components ct.

Thence, using (2.80) we calculate the values of the Christoffel symbols. Nonetheless, once again, we simplify the numerous calculations by establishing certain assertions.

1. All non-diagonal terms vanish. Namely, $g_{\mu\nu} = 0$ if $\mu \neq \nu$
2. The Christoffel symbols are symmetric in the lower indices, i.e. $\Gamma^i_{kl} = \Gamma^i_{lk}$
3. The first term in the expression of (2.80) must be on the diagonal, i.e. $g^{ik} = 0$ if $i \neq k$
4. Note that only derivatives of e^{λ} and e^{ν} with respect to r and t are non-vanishing
5. Differentiation with respect to the x^3 coordinate, namely the ϕ variable, yields a vanishing term

We shall proceed term by term, in order of the upper index and note that for this particular calculation we have that Greek indices run over all four indices and Latin indices will run over $1,2,3$.

Hence, consider (2.80) in the following form,

$$\Gamma^{\mu}_{\nu\sigma} = \frac{1}{2} g^{\mu\lambda} \left[g_{\lambda\nu,\sigma} + g_{\lambda\sigma,\nu} - g_{\nu\sigma,\lambda} \right]$$

Consider $\Gamma^{0}_{\mu\nu}$, such that

$$\begin{aligned} \Gamma^{\mu}_{\nu\sigma} &= \frac{1}{2} g^{\mu\lambda} \left[g_{\lambda\nu,\sigma} + g_{\lambda\sigma,\nu} - g_{\nu\sigma,\lambda} \right] \\ \Gamma^{0}_{\nu\sigma} &= \frac{1}{2} g^{0\lambda} \left[g_{\lambda\nu,\sigma} + g_{\lambda\sigma,\nu} - g_{\nu\sigma,\lambda} \right] \end{aligned}$$

Then, as per the proposed deduction made above, the first term must be in the diagonal, hence, we must have $\lambda = 0$, such that,

$$\Gamma^{0}_{\nu\sigma} = \frac{1}{2} g^{00} \left[g_{0\nu,\sigma} + g_{0\sigma,\nu} - g_{\nu\sigma,0} \right]$$

Next, we consider the case in which $\nu = \sigma = 0$ such that

$$\begin{aligned} \Gamma^{0}_{00} &= \frac{1}{2} g^{00} \left[g_{00,0} + g_{00,0} - g_{00,0} \right] \\ \implies \Gamma^{0}_{00} &= \frac{1}{2} g^{00} \left[g_{00,0} \right] = \frac{1}{2} e^{-\nu} \left[\frac{\partial}{\partial t} e^{\nu} \right] \\ \implies &= \frac{1}{2} e^{-\nu} \left[\frac{\partial}{\partial t} e^{\nu} \left(\frac{\partial}{\partial t} \nu(r,t) \right) \right] \\ \implies &= \frac{1}{2} e^{-\nu} \left(e^{\nu} \dot{\nu} \right) = \frac{\dot{\nu}}{2} \end{aligned}$$

Now, consider $\nu = 0$ $\sigma = i$, such that

$$\Gamma^{0}_{0i} = \frac{1}{2} g^{00} \left[g_{00,i} + g_{0i,0} - g_{0i,0} \right]$$

we need only take into account $i=1$, since $i=2,3$ makes the term vanish

$$\Gamma^0_{01} = \frac{1}{2}g^{00}\left[g_{00,1} + g_{01,0} - g_{01,0}\right]$$
$$\Gamma^0_{01} = \frac{1}{2}g^{00}\left[g_{00,1} + 0 - 0\right]$$
$$\implies \frac{1}{2}e^{-v}\left[\frac{\partial}{\partial r}e^{v}\right]$$
$$\implies = \frac{1}{2}e^{-v}\left[\frac{\partial}{\partial r}e^{v}\left(\frac{\partial}{\partial r}v(r,t)\right)\right]$$
$$\implies = \frac{1}{2}e^{-v}\left(e^{v}v'\right) = \frac{v'}{2}$$

Finally, consider $\nu = i$, $\sigma = j$,

$$\Gamma^0_{ij} = \frac{1}{2}g^{00}\left[g_{0i,j} + g_{0j,i} - g_{ij,0}\right]$$

note that no matter what value we chose for the Latin indices i and j, we have terms that are outside of the diagonal, thus, the Christoffel symbols vanish,

$$\Gamma^0_{ij} = \frac{1}{2}g^{00}\left[g_{0i,j} + g_{0j,i} - g_{ij,0}\right] = 0$$

Subsequently, $\Gamma^1_{\mu\nu}$

$$\Gamma^1_{\nu\sigma} = \frac{1}{2}g^{1\lambda}\left[g_{\lambda\nu,\sigma} + g_{\lambda\sigma,\nu} - g_{\nu\sigma,\lambda}\right]$$

The first term must be on the diagonal, consequently, we must have that $\lambda = 1$,

$$\Gamma^1_{\nu\sigma} = \frac{1}{2}g^{1\lambda}\left[g_{\lambda\nu,\sigma} + g_{\lambda\sigma,\nu} - g_{\nu\sigma,\lambda}\right] \implies \Gamma^1_{\nu\sigma} = \frac{1}{2}g^{11}\left[g_{1\nu,\sigma} + g_{1\sigma,\nu} - g_{\nu\sigma,1}\right]$$

Consider the case in which $\nu = \sigma = 0$

$$
\begin{aligned}
\Gamma^1_{00} &= \frac{1}{2}g^{11}\left[g_{10,0} + g_{10,0} - g_{00,1}\right] \\
&\implies \frac{1}{2}g^{11}\left[0 + 0 - g_{00,1}\right] \\
&\implies \frac{1}{2}g^{11}\left[-g_{00,1}\right] = \frac{1}{2}g^{11}\left[\frac{\partial}{\partial r}e^{\nu}\left(\frac{\partial}{\partial r}\nu(r,t)\right)\right] \\
&\implies \frac{1}{2} - e^{-\lambda}\left(-e^{\nu}\nu'\right) = \frac{\nu'}{2}e^{\nu-\lambda}
\end{aligned}
$$

Then, for $\nu = 0$ $\sigma = i$

$$
\Gamma^1_{0i} = \frac{1}{2}g^{11}\left[g_{10,i} + g_{1i,0} - g_{0i,1}\right] \tag{2.279}
$$

Then, we consider only $i = 1$ as this will give the second term in the parentheses to be on the diagonal, such that

$$
\begin{aligned}
\Gamma^1_{01} &= \frac{1}{2}g^{11}\left[g_{10,1} + g_{11,0} - g_{01,1}\right] \\
&\implies \frac{1}{2}g^{11}\left[g_{11,0}\right] \\
&\implies \frac{1}{2} - e^{-\lambda}\left[\frac{\partial}{\partial t}e^{\lambda}\right] \\
&\implies \frac{1}{2} - e^{-\lambda}\left[\frac{\partial}{\partial t}e^{\lambda}\left(\frac{\partial}{\partial t}\lambda(r,t)\right)\right] \\
&\implies \frac{1}{2}e^{-\lambda+\lambda}\left(\dot{\lambda}\right) = \frac{\dot{\lambda}}{2}
\end{aligned}
$$

Next, for $\nu = i$ $\sigma = j \neq i$, we have

$$
\frac{1}{2}g^{11}\left[g_{1i,j} + g_{1j,i} - g_{ij,1}\right] = 0 \tag{2.280}
$$

where we note that no matter the choice of i and j, since $i \neq j$, we have terms which are outside the diagonal and derivatives which vanish due the diagonal entry being independent to the variable being differentiating upon.

Finally, we consider $\nu = \sigma = i$, such that for $\nu = \sigma = 1$

$$
\begin{aligned}
\Gamma^1_{11} &= \frac{1}{2} g^{11} \left[g_{11,1} + g_{11,1} - g_{11,1} \right] \\
\implies \Gamma^1_{11} &= \frac{1}{2} g^{11} \left[g_{11,1} \right] \implies \frac{1}{2} - e^{-\lambda} \left[\frac{\partial}{\partial r} - e^{\lambda} \right] \\
&\implies \frac{1}{2} e^{-\lambda} \left[\frac{\partial}{\partial r} e^{\lambda} \left(\frac{\partial}{\partial r} \lambda(r,t) \right) \right] \\
&\implies \frac{1}{2} e^{-\lambda+\lambda} \left(\lambda' \right) = \frac{\lambda'}{2}
\end{aligned}
$$

For, $\nu = \sigma = 2$

$$
\begin{aligned}
\Gamma^1_{22} &= \frac{1}{2} g^{11} \left[g_{12,2} + g_{12,2} - g_{22,1} \right] \\
&\implies \Gamma^1_{22} = \frac{1}{2} g^{11} \left[-g_{22,1} \right] \\
&\implies = \frac{1}{2} - e^{-\lambda} \left[- \left(\frac{\partial}{\partial r} - r^2 \right) \right] \\
&\implies \frac{1}{2} e^{-\lambda} \left[-2r \right] = -r e^{-\lambda}
\end{aligned}
$$

For $\nu = \sigma = 3$

$$
\begin{aligned}
\Gamma^1_{33} &= \frac{1}{2} g^{11} \left[g_{13,3} + g_{13,3} - g_{33,1} \right] \\
&\implies \Gamma^1_{33} = \frac{1}{2} g^{11} \left[-g_{33,1} \right] \\
&\implies = \frac{1}{2} - e^{-\lambda} \left[- \left(\frac{\partial}{\partial r} - r^2 \sin^2\theta \right) \right] \\
&\implies \frac{1}{2} e^{-\lambda} \left[-2r \sin^2\theta \right] = -r \sin^2\theta \, e^{-\lambda}
\end{aligned}
$$

The remaining terms denoted in (2.278) follow in a similar manner, while all other components (except for those which differ from the ones we have written by a transposition of k and l) are equivalently zero. Nonetheless, we briefly proof the equivalence of $\Gamma^2_{12} = \Gamma^3_{13}$.

Then, $\Gamma^2_{\mu\nu}$ is equivalent to

$$\Gamma^2_{\nu\sigma} = \frac{1}{2} g^{2\lambda} \left[g_{\lambda\nu,\sigma} + g_{\lambda\sigma,\nu} - g_{\nu\sigma,\lambda} \right]$$

The first term must be on the diagonal, consequently, we must have that $\lambda = 2$,

$$\Gamma^2_{\nu\sigma} = \frac{1}{2} g^{2\lambda} \left[g_{\lambda\nu,\sigma} + g_{\lambda\sigma,\nu} - g_{\nu\sigma,\lambda} \right]$$
$$\implies \Gamma^2_{\nu\sigma} = \frac{1}{2} g^{22} \left[g_{2\nu,\sigma} + g_{2\sigma,\nu} - g_{\nu\sigma,2} \right]$$

Then, as per the term given above, consider $\nu = i$ $\sigma = j \neq i$, such that

$$\Gamma^2_{ij} = \frac{1}{2} g^{22} \left[g_{2i,j} + g_{2j,i} - g_{ij,2} \right]$$

It follows that the only viable choice for indices i and j is $i = 1,\ j = 2$, such that

$$\Gamma^2_{ij} = \frac{1}{2} g^{22} \left[g_{2i,j} + g_{2j,i} - g_{ij,2} \right]$$
$$\implies \Gamma^2_{12} = \frac{1}{2} g^{22} \left[g_{21,2} + g_{22,1} - g_{12,2} \right]$$
$$\implies \Gamma^2_{12} = \frac{1}{2} g^{22} \left[g_{22,1} \right]$$
$$\implies \Gamma^2_{12} = \frac{1}{2} \left(-r^{-2} \right) \left[\frac{\partial}{\partial r} - r^2 \right]$$
$$\implies \Gamma^2_{12} = \frac{1}{2} \left(r^{-2} \right) \left[2r \right]$$
$$\implies \Gamma^2_{12} = \frac{1}{r}$$

Similarly, for Γ^3_{13}, we have for indices $i = 1,\ j = 3$

$$\Gamma^3_{ij} = \frac{1}{2} g^{33} \left[g_{3i,j} + g_{3j,i} - g_{ij,3} \right]$$
$$\implies \Gamma^3_{13} = \frac{1}{2} g^{33} \left[g_{31,3} + g_{33,1} - g_{13,3} \right]$$
$$\implies \Gamma^3_{13} = \frac{1}{2} g^{33} \left[g_{33,1} \right]$$
$$\implies \Gamma^3_{13} = \frac{1}{2} \left(-r^{-2} \sin^{-2} \theta \right) \left[\frac{\partial}{\partial r} - r^2 \sin^2 \theta \right]$$
$$\implies \Gamma^3_{13} = \frac{1}{2} \left(r^{-2} \sin^{-2} \theta \right) \left[2r \sin^2 \theta \right]$$
$$\implies \Gamma^3_{13} = \frac{1}{r}$$

Now that we have determined the values for the Christoffel symbols, we can start to assemble the Ricci tensor and scalar. Then by the definition of the Ricci tensor we can proceed by writing the tensor as follows,

$$R_{\mu\nu} = R^{\beta}_{\mu\nu\beta} = \Gamma^{\beta}_{\mu\beta,\nu} - \Gamma^{\beta}_{\mu\nu,\beta} + \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\alpha\nu} - \Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta}$$

where we sum over the β index. Namely,

$$\begin{aligned} R_{\mu\nu} = &\Gamma^0_{\mu0,\nu} - \Gamma^0_{\mu\nu,0} + \Gamma^{\alpha}_{\mu0}\Gamma^0_{\alpha\nu} - \Gamma^{\alpha}_{\mu\nu}\Gamma^0_{\alpha0} \\ &+ \Gamma^1_{\mu1,\nu} - \Gamma^1_{\mu\nu,1} + \Gamma^{\alpha}_{\mu1}\Gamma^1_{\alpha\nu} - \Gamma^{\alpha}_{\mu\nu}\Gamma^1_{\alpha1} \\ &+ \Gamma^2_{\mu2,\nu} - \Gamma^2_{\mu\nu,2} + \Gamma^{\alpha}_{\mu2}\Gamma^2_{\alpha\nu} - \Gamma^{\alpha}_{\mu\nu}\Gamma^2_{\alpha2} \\ &+ \Gamma^3_{\mu3,\nu} - \Gamma^3_{\mu\nu,3} + \Gamma^{\alpha}_{\mu3}\Gamma^3_{\alpha\nu} - \Gamma^{\alpha}_{\mu\nu}\Gamma^3_{\alpha3} \end{aligned} \tag{2.281}$$

Predominantly, we establish the non-diagonal terms of the metric tensor. Remembering that the Ricci tensor is symmetric, i.e. $R_{\mu\nu} = R_{\nu\mu}$, we need only consider one case. Thence, consider first

$R_{\mu\nu}, \mu \neq \nu$, such that $\mu = 0$, $\nu = i$

$$\begin{aligned}
R_{0i} =& \Gamma^0_{00,i} - \Gamma^0_{0i,0} + \Gamma^\alpha_{00}\Gamma^0_{\alpha i} - \Gamma^\alpha_{0i}\Gamma^0_{\alpha 0} \\
&+ \Gamma^1_{01,i} - \Gamma^1_{0i,1} + \Gamma^\alpha_{01}\Gamma^1_{\alpha i} - \Gamma^\alpha_{0i}\Gamma^1_{\alpha 1} \\
&+ \Gamma^2_{02,i} - \Gamma^2_{0i,2} + \Gamma^\alpha_{02}\Gamma^2_{\alpha i} - \Gamma^\alpha_{0i}\Gamma^2_{\alpha 2} \\
&+ \Gamma^3_{03,i} - \Gamma^3_{0i,3} + \Gamma^\alpha_{03}\Gamma^3_{\alpha i} - \Gamma^\alpha_{0i}\Gamma^3_{\alpha 3}
\end{aligned}$$

Next, remembering that Latin indices run from $1-3$, we have considering $i = 1$,

$$\begin{aligned}
R_{01} =& \Gamma^0_{00,1} - \Gamma^0_{01,0} + \Gamma^\alpha_{00}\Gamma^0_{\alpha 1} - \Gamma^\alpha_{01}\Gamma^0_{\alpha 0} \\
&+ \Gamma^1_{01,1} - \Gamma^1_{01,1} + \Gamma^\alpha_{01}\Gamma^1_{\alpha 1} - \Gamma^\alpha_{01}\Gamma^1_{\alpha 1} \\
&+ \Gamma^2_{02,1} - \Gamma^2_{01,2} + \Gamma^\alpha_{02}\Gamma^2_{\alpha 1} - \Gamma^\alpha_{01}\Gamma^2_{\alpha 2} \\
&+ \Gamma^3_{03,1} - \Gamma^3_{01,3} + \Gamma^\alpha_{03}\Gamma^3_{\alpha 1} - \Gamma^\alpha_{01}\Gamma^3_{\alpha 3}
\end{aligned}$$

$$\begin{aligned}
R_{01} =& \Gamma^0_{00,1} - \Gamma^0_{01,0} + \Gamma^\alpha_{00}\Gamma^0_{\alpha 1} - \Gamma^\alpha_{01}\Gamma^0_{\alpha 0} \\
&+ 0 + \Gamma^\alpha_{01}\Gamma^1_{\alpha 1} - \Gamma^\alpha_{01}\Gamma^1_{\alpha 1} \\
&+ 0 + \Gamma^\alpha_{02}\Gamma^2_{\alpha 1} - \Gamma^\alpha_{01}\Gamma^2_{\alpha 2} \\
&+ 0 + \Gamma^\alpha_{03}\Gamma^3_{\alpha 1} - \Gamma^\alpha_{01}\Gamma^3_{\alpha 3}
\end{aligned}$$

Which we obtained after cancellation of certain identical terms. Subsequently, since Greek indices take the values from $0-3$, we start by considering $\alpha = 0$,

$$\begin{aligned}
R_{01} =& \Gamma^0_{00,1} - \Gamma^0_{01,0} + \Gamma^0_{00}\Gamma^0_{01} - \Gamma^0_{01}\Gamma^0_{00} \\
&+ 0 + \Gamma^0_{01}\Gamma^1_{01} - \Gamma^0_{01}\Gamma^1_{01} \\
&+ 0 + \Gamma^0_{02}\Gamma^2_{01} - \Gamma^0_{01}\Gamma^2_{02} \\
&+ 0 + \Gamma^0_{03}\Gamma^3_{01} - \Gamma^0_{01}\Gamma^3_{03}
\end{aligned}$$

$$R_{01} = \Gamma^0_{00,1} - \Gamma^0_{01,0} + \Gamma^0_{00}\Gamma^0_{01} - \Gamma^0_{01}$$

$$\implies R_{01} = \Gamma^0_{00,1} - \Gamma^0_{01,0} + 0$$

Consequently, for $\alpha = 1$ we have

$$\begin{aligned} R_{01} =& \Gamma^0_{00,1} - \Gamma^0_{01,0} + \Gamma^1_{00}\Gamma^0_{11} - \Gamma^1_{01}\Gamma^0_{10} \\ &+ 0 + \Gamma^1_{01}\Gamma^1_{11} - \Gamma^1_{01}\Gamma^1_{11} \\ &+ 0 + \Gamma^1_{02}\Gamma^2_{11} - \Gamma^1_{01}\Gamma^2_{12} \\ &+ 0 + \Gamma^1_{03}\Gamma^3_{11} - \Gamma^1_{01}\Gamma^3_{13} \end{aligned}$$

after which the only surviving terms are

$$R_{01} = \Gamma^1_{00}\Gamma^0_{11} - \Gamma^1_{01}\Gamma^0_{10} - \Gamma^1_{01}\Gamma^2_{12} - \Gamma^1_{01}\Gamma^3_{13}$$

For $\alpha = 2$, we obtain

$$\begin{aligned} R_{01} =& \Gamma^0_{00,1} - \Gamma^0_{01,0} + \Gamma^2_{00}\Gamma^0_{21} - \Gamma^2_{01}\Gamma^0_{20} \\ &+ 0 + \Gamma^2_{01}\Gamma^1_{21} - \Gamma^2_{01}\Gamma^1_{21} \\ &+ 0 + \Gamma^2_{02}\Gamma^2_{21} - \Gamma^2_{01}\Gamma^2_{22} \\ &+ 0 + \Gamma^2_{03}\Gamma^3_{21} - \Gamma^2_{01}\Gamma^3_{23} \end{aligned}$$

For which we find

$$R_{01} = 0 \tag{2.282}$$

for $\alpha = 2$. Finally, for $\alpha = 3$, we find

$$\begin{aligned} R_{01} =& \Gamma^0_{00,1} - \Gamma^0_{01,0} + \Gamma^3_{00}\Gamma^0_{31} - \Gamma^3_{01}\Gamma^0_{30} \\ &+ 0 + \Gamma^3_{01}\Gamma^1_{31} - \Gamma^3_{01}\Gamma^1_{31} \\ &+ 0 + \Gamma^3_{02}\Gamma^2_{31} - \Gamma^3_{01}\Gamma^2_{32} \\ &+ 0 + \Gamma^3_{03}\Gamma^3_{31} - \Gamma^3_{01}\Gamma^3_{33} \end{aligned}$$

analogously, we find that

$$R_{01} = 0 \tag{2.283}$$

Finally, taking a sum over all indices of α, we obtain,

$$R_{01} = \Gamma^0_{00,1} - \Gamma^0_{01,0} + \Gamma^1_{00}\Gamma^0_{11} - \Gamma^1_{01}\Gamma^0_{10} - \Gamma^1_{01}\Gamma^2_{12} - \Gamma^1_{01}\Gamma^3_{13}$$

We proceed to solving this equation. Then, substituting in the values obtained from (2.278) we find,

$$\begin{aligned}
R_{01} &= \Gamma^0_{00,1} - \Gamma^0_{01,0} + \Gamma^1_{00}\Gamma^0_{11} - \Gamma^1_{01}\Gamma^0_{10} - \Gamma^1_{01}\Gamma^2_{12} - \Gamma^1_{01}\Gamma^3_{13} \\
\implies &= \frac{\partial}{\partial r}\left(\frac{\dot{v}}{2}\right) - \frac{\partial}{\partial t}\left(\frac{v'}{2}\right) + \left(\frac{v'}{2}e^{v-\lambda}\right)\left(\frac{\dot{\lambda}}{2}e^{\lambda-v}\right) - \left(\frac{\dot{\lambda}}{2}\right)\left(\frac{v'}{2}\right) - \left(\frac{\dot{\lambda}}{2}\right)\left(\frac{1}{r}\right) - \left(\frac{\dot{\lambda}}{2}\right)\left(\frac{1}{r}\right) \\
\implies &= 0 + \left(\frac{v'}{2}\right)\left(\frac{\dot{\lambda}}{2}\right)\left(e^{v-\lambda+\lambda-v}\right) - \left(\frac{v'}{2}\right)\left(\frac{\dot{\lambda}}{2}\right) - 2\left(\frac{\dot{\lambda}}{2}\right)\left(\frac{1}{r}\right) \\
\implies & \left(\frac{v'}{2}\right)\left(\frac{\dot{\lambda}}{2}\right)(e^0) - \left(\frac{v'}{2}\right)\left(\frac{\dot{\lambda}}{2}\right) - 2\left(\frac{\dot{\lambda}}{2}\right)\left(\frac{1}{r}\right) \\
\implies & R_{01} = -\frac{\dot{\lambda}}{r}
\end{aligned}$$

Thus, we have found one of the values for the Ricci tensor. Then, for $\mu = i$, $\nu = j \neq i$ we find

$$\begin{aligned}
R_{i,j\neq i} =& \Gamma^0_{i0,j} - \Gamma^0_{ij,0} + \Gamma^\alpha_{i0}\Gamma^0_{\alpha j} - \Gamma^\alpha_{ij}\Gamma^0_{\alpha 0} \\
& + \Gamma^1_{i1,j} - \Gamma^1_{ij,1} + \Gamma^\alpha_{i1}\Gamma^1_{\alpha j} - \Gamma^\alpha_{ij}\Gamma^1_{\alpha 1} \\
& + \Gamma^2_{i2,j} - \Gamma^2_{ij,2} + \Gamma^\alpha_{i2}\Gamma^2_{\alpha j} - \Gamma^\alpha_{ij}\Gamma^2_{\alpha 2} \\
& + \Gamma^3_{i3,j} - \Gamma^3_{ij,3} + \Gamma^\alpha_{i3}\Gamma^3_{\alpha j} - \Gamma^\alpha_{ij}\Gamma^3_{\alpha 3}
\end{aligned}$$

$$\begin{aligned}R_{i,j\neq i} =& 0-0+\Gamma^{\alpha}_{i0}\Gamma^{0}_{\alpha j}-\Gamma^{\alpha}_{ij}\Gamma^{0}_{\alpha 0}\\ & 0-0+\Gamma^{\alpha}_{i1}\Gamma^{1}_{\alpha j}-0\\ & 0-0+\Gamma^{\alpha}_{i2}\Gamma^{2}_{\alpha j}-0\\ & +\Gamma^{3}_{i3,j}-0+\Gamma^{\alpha}_{i3}\Gamma^{3}_{\alpha j}-\Gamma^{\alpha}_{ij}\Gamma^{3}_{\alpha 3}\end{aligned}$$

The terms equal to zero follow from the 5 reasons stated above. Next, we examine the remaining terms in detail.

- $\Gamma^{\alpha}_{i0}\Gamma^{0}_{\alpha j}$. Here we find that if α or $j=2,3$ the second symbol vanishes. Then for $\alpha=0$ we require that $i=1$ so that the first symbol gives a non-vanishing term, but then j must be 2 or 3 which leads to the whole term being equal to zero. Then, if $\alpha=1$ then we find that $i=1$ so that the first symbol is non-vanishing, yet $j=2,3$ which leads to the term vanishing.
- $-\Gamma^{\alpha}_{ij}\Gamma^{0}_{\alpha 0}$ Here we have that if $\alpha=0$, then since $i\neq j$ the first symbol is zero and the whole term vanishes. Next, if we consider $\alpha=1$, then the first term must vanish since $i\neq j$.
- $\Gamma^{\alpha}_{i1}\Gamma^{1}_{\alpha j}$ In order to have a non-vanishing term we see that if $\alpha=0$ we must have $i=j=1$, which as has been previously stated cannot be. Then, observe that $\alpha=j$ in order to have a second term that is not zero. Thus we are left with: $\Gamma^{1}_{i1}\Gamma^{1}_{11}+\Gamma^{2}_{i1}\Gamma^{1}_{22}+\Gamma^{3}_{i1}\Gamma^{1}_{33}$. The first term vanishes because $i\neq j$. The second term vanishes, since in order for the first symbol to not vanish we need $i=2$, yet $j=2$, thus the first symbol of the second term is equal to zero. The third term vanishes because i must be equal to 3 in order to be nonzero, yet this cannot happen as $j=3$. Hence, the third vanishes.
- $\Gamma^{\alpha}_{i2}\Gamma^{2}_{\alpha j}$ First we point attention to the fact $\alpha\neq 0$ or else the first symbol is zero. For $\alpha=1$, we require that both i and j be equal to 2 in order for the term to be nonzero, yet this cannot be as $i\neq j$. For $\alpha=2$, then $i=1$ and $j=1$ but this is not possible $(i\neq j)$. Lastly, for $\alpha=3$, we require that $i=j=3$ in order to have a non-vanishing term, but this cannot happen.
- $\Gamma^{3}_{i3,j}$ Here the only possible choice is $i=1,2$, and $j=2,3$ and $j=1,3$ for each choice of i,

respectively, nonetheless, the symbol $\Gamma^3_{23,j}$ is independent of both variables r and ϕ and a derivative with respect to either leads to a vanishing term, analogously the symbol $\Gamma^3_{13,j}$ is independent of θ and ϕ, yielding a vanishing term.

- $\Gamma^{\alpha}_{i3}\Gamma^3_{\alpha j}$ First we note that $\alpha \neq 0$ as this leads to a vanishing first term and the term being equal to zero. For $\alpha = 1$ and $\alpha = 2$, we require that both $i = j = 3$ which as we have extensively delineated cannot happen. Lastly, $\alpha = 3$ in order to have a non-vanishing term we must have i be equal to 1 or 2 and j must be 2 or 1, respectively. Hence, this term does not vanish and we have $\Gamma^3_{13}\Gamma^3_{32} + \Gamma^3_{23}\Gamma^3_{31}$

- $-\Gamma^{\alpha}_{ij}\Gamma^3_{\alpha 3}$ For the first term we require that α be equal to 1 or 2. For $\alpha = 1$ we find the $i = j$ in order to have a non-vanishing first term, yet as we know this is impossible. Lastly, for$\alpha = 2$, we find that we have the terms $-\Gamma^2_{12}\Gamma^3_{23} - \Gamma^2_{21}\Gamma^3_{23}$

Thus, we find that we have four non-vanishing terms. We proceed to writing them explicitly.

$$\begin{gathered}
\Gamma^3_{13}\Gamma^3_{32} + \Gamma^3_{23}\Gamma^3_{31} - \Gamma^2_{12}\Gamma^3_{23} - \Gamma^2_{21}\Gamma^3_{23} \\
\implies \left(\frac{1}{r}\right)(\cot\theta) + (\cot\theta)\left(\frac{1}{r}\right) - \left(\frac{1}{r}\right)(\cot\theta) - \left(\frac{1}{r}\right)(\cot\theta) \\
= 0
\end{gathered}$$

Hence, we find that after the protracted calculation we find that $R_{i,j\neq i} = 0$

We proceed by showing only the values for $\mu = \nu = 0$ and $\mu = \nu = 1$ for $R_{\mu\nu=\mu}$, since the remaining terms follow analogously.

$$\begin{aligned}
R_{00} =& \Gamma^0_{00,0} - \Gamma^0_{00,0} + \Gamma^{\alpha}_{00}\Gamma^0_{\alpha 0} - \Gamma^{\alpha}_{00}\Gamma^0_{\alpha 0} \\
& + \Gamma^1_{01,0} - \Gamma^1_{00,1} + \Gamma^{\alpha}_{01}\Gamma^1_{\alpha 0} - \Gamma^{\alpha}_{00}\Gamma^1_{\alpha 1} \\
& + \Gamma^2_{02,0} - \Gamma^2_{00,2} + \Gamma^{\alpha}_{02}\Gamma^2_{\alpha 0} - \Gamma^{\alpha}_{00}\Gamma^2_{\alpha 2} \\
& + \Gamma^3_{03,0} - \Gamma^3_{00,3} + \Gamma^{\alpha}_{03}\Gamma^3_{\alpha 0} - \Gamma^{\alpha}_{00}\Gamma^3_{\alpha 3}
\end{aligned}$$

then,

$$
\begin{aligned}
\implies R_{00} =& 0+0 \\
&+\Gamma^1_{01,0}-\Gamma^1_{00,1}+\Gamma^\alpha_{01}\Gamma^1_{\alpha 0}-\Gamma^\alpha_{00}\Gamma^1_{\alpha 1} \\
&+\Gamma^2_{02,0}-\Gamma^2_{00,2}+\Gamma^\alpha_{02}\Gamma^2_{\alpha 0}-\Gamma^\alpha_{00}\Gamma^2_{\alpha 2} \\
&+\Gamma^3_{03,0}-\Gamma^3_{00,3}+\Gamma^\alpha_{03}\Gamma^3_{\alpha 0}-\Gamma^\alpha_{00}\Gamma^3_{\alpha 3}
\end{aligned}
$$

for $\alpha = 0$, we have

$$\Gamma^1_{01,0}-\Gamma^1_{00,1}+\Gamma^0_{01}\Gamma^1_{00}-\Gamma^0_{00}\Gamma^1_{01}$$

for $\alpha = 1$, we find

$$\Gamma^1_{01}\Gamma^1_{10}-\Gamma^1_{00}\Gamma^1_{11}-\Gamma^1_{00}\Gamma^2_{12}-\Gamma^1_{00}\Gamma^3_{13}$$

For $\alpha = 2$ and $\alpha = 3$ we have vanishing terms for the Ricci tensor. Hence, the Ricci tensor takes the form

$$R_{00} = \Gamma^1_{01,0}-\Gamma^1_{00,1}+\Gamma^0_{01}\Gamma^1_{00}-\Gamma^0_{00}\Gamma^1_{01}+\Gamma^1_{01}\Gamma^1_{10}-\Gamma^1_{00}\Gamma^1_{11}-\Gamma^1_{00}\Gamma^2_{12}-\Gamma^1_{00}\Gamma^3_{13}$$

Substituting the values obtained for the Christoffel symbols, we have

$$
\begin{aligned}
R_{00} &= \Gamma^1_{01,0}-\Gamma^1_{00,1}+\Gamma^0_{01}\Gamma^1_{00}-\Gamma^0_{00}\Gamma^1_{01}+\Gamma^1_{01}\Gamma^1_{10}-\Gamma^1_{00}\Gamma^1_{11}-\Gamma^1_{00}\Gamma^2_{12}-\Gamma^1_{00}\Gamma^3_{13} \\
\implies &= \frac{\partial}{\partial t}\left(\frac{\dot{\lambda}}{2}\right)-\frac{\partial}{\partial r}\left(\frac{\nu'}{2}e^{\nu-\lambda}\right)+\left(\frac{\nu'}{2}\right)\left(\frac{\nu'}{2}e^{\nu-\lambda}\right)-\left(\frac{\dot{\nu}}{2}\right)\left(\frac{\dot{\lambda}}{2}\right) \\
&+\left(\frac{\dot{\lambda}}{2}\right)\left(\frac{\dot{\lambda}}{2}\right)-\left(\frac{\nu'}{2}e^{\nu-\lambda}\right)\left(\frac{\lambda'}{2}\right)-\left(\frac{\nu'}{2}e^{\nu-\lambda}\right)\left(\frac{1}{r}\right)-\left(\frac{\nu'}{2}e^{\nu-\lambda}\right)\left(\frac{1}{r}\right)
\end{aligned}
$$

Similarly, we now consider Ricci tensor for $\mu = \nu = 1$.

$$\begin{aligned} R_{11} =& \Gamma^0_{10,1} - \Gamma^0_{11,0} + \Gamma^\alpha_{10}\Gamma^0_{\alpha 1} - \Gamma^\alpha_{11}\Gamma^0_{\alpha 0} \\ & + \Gamma^1_{11,1} - \Gamma^1_{11,1} + \Gamma^\alpha_{11}\Gamma^1_{\alpha 1} - \Gamma^\alpha_{11}\Gamma^1_{\alpha 1} \\ & + \Gamma^2_{12,1} - \Gamma^2_{11,2} + \Gamma^\alpha_{12}\Gamma^2_{\alpha 1} - \Gamma^\alpha_{11}\Gamma^2_{\alpha 2} \\ & + \Gamma^3_{13,1} - \Gamma^3_{11,3} + \Gamma^\alpha_{13}\Gamma^3_{\alpha 1} - \Gamma^\alpha_{11}\Gamma^3_{\alpha 3} \end{aligned} \tag{2.284}$$

then,

$$\begin{aligned} R_{11} =& \Gamma^0_{10,1} - \Gamma^0_{11,0} + \Gamma^\alpha_{10}\Gamma^0_{\alpha 1} - \Gamma^\alpha_{11}\Gamma^0_{\alpha 0} \\ & + 0 + 0 \\ & + \Gamma^2_{12,1} - \Gamma^2_{11,2} + \Gamma^\alpha_{12}\Gamma^2_{\alpha 1} - \Gamma^\alpha_{11}\Gamma^2_{\alpha 2} \\ & + \Gamma^3_{13,1} - \Gamma^3_{11,3} + \Gamma^\alpha_{13}\Gamma^3_{\alpha 1} - \Gamma^\alpha_{11}\Gamma^3_{\alpha 3} \end{aligned} \tag{2.285}$$

where we find that the Ricci tensor takes the form

$$\begin{aligned} R_{11} = \Gamma^0_{10,1} - \Gamma^0_{11,0} + \Gamma^0_{10}\Gamma^0_{01} + \Gamma^1_{10}\Gamma^0_{11} - \Gamma^1_{11}\Gamma^0_{10} - \Gamma^0_{11}\Gamma^0_{00} + \Gamma^2_{12,1} + \Gamma^2_{12}\Gamma^2_{21} - \Gamma^1_{11}\Gamma^2_{12} \\ + \Gamma^3_{13,1} + \Gamma^3_{13}\Gamma^3_{31} - \Gamma^1_{11}\Gamma^3_{13} \end{aligned}$$

Substituting in the values for the Christoffel symbols we find

$$\begin{aligned} R_{11} = \frac{\partial}{\partial r}\left(\frac{\nu'}{2}\right) - \frac{\partial}{\partial t}\left(\frac{\dot\lambda}{2}e^{\lambda-\nu}\right) + \left(\frac{\nu'}{2}\right)\left(\frac{\nu'}{2}\right) + \left(\frac{\dot\lambda}{2}\right)\left(\frac{\dot\lambda}{2}e^{\lambda-\nu}\right) - \left(\frac{\lambda'}{2}\right)\left(\frac{\nu'}{2}\right) - \left(\frac{\dot\lambda}{2}e^{\lambda-\nu}\right)\left(\frac{\dot\nu}{2}\right) \\ + \frac{\partial}{\partial r}\left(\frac{1}{r}\right) + \left(\frac{1}{r}\right)\left(\frac{1}{r}\right) - \left(\frac{\lambda'}{2}\right)\left(\frac{1}{r}\right) + \frac{\partial}{\partial r}\left(\frac{1}{r}\right) + \left(\frac{1}{r}\right)\left(\frac{1}{r}\right) - \left(\frac{\lambda'}{2}\right)\left(\frac{1}{r}\right) \end{aligned}$$

Similarly, we find for the indices $\mu = \nu = 2$ and $\mu = \nu = 3$, the corresponding Ricci tensors take the form,

$$R_{22} = -\Gamma^1_{22,1} - \Gamma^3_{23,2} - \Gamma^1_{22}\Gamma^0_{10} - \Gamma^1_{22}\Gamma^1_{11} + \Gamma^2_{21}\Gamma^1_{22} - \Gamma^3_{23}\Gamma^3_{32} - \Gamma^1_{22}\Gamma^3_{13}$$

$$\implies R_{22} = -\frac{\partial}{\partial r}\left(-re^{-\lambda}\right) + \frac{\partial}{\partial \theta}\cot\theta - \left(-re^{-\lambda}\right)\left(\frac{\nu'}{2}\right) - \left(-re^{-\lambda}\right)\left(\frac{\lambda'}{2}\right) + \left(\frac{1}{r}\right)\left(-re^{-\lambda}\right) + \cot^2\theta - \left(-re^{-\lambda}\right)\left(\frac{1}{r}\right)$$

and,

$$R_{33} = -\Gamma^1_{33,1} - \Gamma^2_{33,2} - \Gamma^1_{33}\Gamma^0_{10} - \Gamma^1_{33}\Gamma^1_{11} - \Gamma^1_{33}\Gamma^2_{12} - \Gamma^3_{31}\Gamma^1_{33} + \Gamma^3_{32}\Gamma^2_{33}$$

$$\implies R_{33} = -\frac{\partial}{\partial r}\left(-r\sin^2\theta e^{-\lambda}\right) + \frac{\partial}{\partial \theta}\left(-\sin\theta\cos\theta\right) - \left(-r\sin^2\theta e^{-\lambda}\right)\left(\frac{\nu'}{2}\right) - \left(-r\sin^2\theta e^{-\lambda}\right)\left(\frac{\lambda'}{2}\right) - \left(\frac{1}{r}\right)\left(-r\sin^2\theta e^{-\lambda}\right) + \left(\frac{1}{r}\right)\left(-r\sin^2\theta e^{-\lambda}\right) + \cot\theta\left(-\sin\theta\cos\theta\right)$$

After computation of the above derivatives and reduction for each of the Ricci tensors, respectively, we find

$$R_{00} = \frac{\ddot{\lambda}}{2} - \frac{\nu''}{2}e^{\nu-\lambda} - \frac{\nu'}{2}e^{\nu-\lambda}\left(\nu'\right) + \frac{\nu'}{2}e^{\nu-\lambda}\left(\lambda'\right) + \left(\frac{\nu'}{2}\right)^2 e^{\nu-\lambda} - \left(\frac{\dot{\nu}}{2}\right)\left(\frac{\dot{\lambda}}{2}\right) + \left(\frac{\dot{\lambda}}{2}\right)^2 - \left(\frac{\lambda'}{2}\right)\left(\frac{\nu'}{2}\right)e^{\nu-\lambda} - \frac{1}{r}\left(\nu' e^{\nu-\lambda}\right)$$

$$R_{11} = \frac{\nu''}{2} - \frac{\ddot{\lambda}}{2}e^{\lambda-\nu} - \left(\frac{\dot{\lambda}}{2}e^{\lambda-\nu}\right)\left(\dot{\lambda}\right) + \left(\frac{\dot{\lambda}}{2}e^{\lambda-\nu}\right)\left(\dot{\nu}\right) + \left(\frac{\nu'}{2}\right)^2 - \left(\frac{\dot{\lambda}}{2}e^{\lambda-\nu}\right)\left(\frac{\dot{\nu}}{2}\right) + \left(\frac{\dot{\lambda}}{2}\right)^2 e^{\lambda-\nu} - \left(\frac{\lambda'}{2}\right)\left(\frac{\nu'}{2}\right) - \frac{\lambda'}{r}$$

$$R_{22} = -re^{-\lambda}\left(\frac{\lambda'}{2}\right) + re^{-\lambda}\left(\frac{\nu'}{2}\right) + e^{-\lambda} - 1$$

$$R_{33} = \sin^2\theta\left[-re^{-\lambda}\left(\frac{\lambda'}{2}\right) + re^{-\lambda}\left(\frac{\nu'}{2}\right) + e^{-\lambda} - 1\right] = \sin^2\theta R_{22}$$

We find that the Ricci tensor R_{33} is dependent on R_{22}, detailed by the relation $R_{33} = \sin^2\theta R_{22}$. Thus, we obtain only three independent equations when considering coinciding indices $\mu = \nu$. Next, we proceed by computing the Ricci scalar using (2.168),

$$
\begin{aligned}
R = R^{\mu}_{\mu} = g^{\mu\nu}R_{\mu\nu} = g^{00}R_{00} + g^{11}R_{11} + g^{22}R_{22} + g^{33}R_{33} \\
\implies R = e^{-\nu}R_{00} - e^{-\lambda}R_{11} - \frac{1}{r^2}R_{22} - \frac{1}{r^2\sin^2\theta}R_{33} \\
\implies = e^{-\nu}\left[\frac{\ddot{\lambda}}{2} - \frac{\nu''}{2}e^{\nu-\lambda} - \frac{\nu'}{2}e^{\nu-\lambda}\left(\nu'\right) + \frac{\nu'}{2}e^{\nu-\lambda}\left(\lambda'\right) + \left(\frac{\nu'}{2}\right)^2 e^{\nu-\lambda}\right. \\
\left. - \left(\frac{\dot{\nu}}{2}\right)\left(\frac{\dot{\lambda}}{2}\right) + \left(\frac{\dot{\lambda}}{2}\right)^2 - \left(\frac{\lambda'}{2}\right)\left(\frac{\nu'}{2}\right)e^{\nu-\lambda} - \frac{1}{r}\left(\nu' e^{\nu-\lambda}\right)\right] \\
-e^{-\lambda}\left[\frac{\nu''}{2} - \frac{\ddot{\lambda}}{2}e^{\lambda-\nu} - \left(\frac{\dot{\lambda}}{2}e^{\lambda-\nu}\right)\left(\dot{\lambda}\right) + \left(\frac{\dot{\lambda}}{2}e^{\lambda-\nu}\right)\left(\dot{\nu}\right) + \left(\frac{\nu'}{2}\right)^2\right. \\
\left. - \left(\frac{\dot{\lambda}}{2}e^{\lambda-\nu}\right)\left(\frac{\dot{\nu}}{2}\right) + \left(\frac{\dot{\lambda}}{2}\right)^2 e^{\lambda-\nu} - \left(\frac{\lambda'}{2}\right)\left(\frac{\nu'}{2}\right) - \frac{\lambda'}{r}\right] \\
-\frac{1}{r^2}\left[-re^{-\lambda}\left(\frac{\lambda'}{2}\right) + re^{-\lambda}\left(\frac{\nu'}{2}\right) + e^{-\lambda} - 1\right] \\
-\frac{1}{r^2\sin^2\theta}\left(\sin^2\theta\left[-re^{-\lambda}\left(\frac{\lambda'}{2}\right) + re^{-\lambda}\left(\frac{\nu'}{2}\right) + e^{-\lambda} - 1\right]\right)
\end{aligned}
$$

such that we obtain,

$$
\begin{aligned}
R = e^{-\nu}\ddot{\lambda} - e^{-\lambda}\nu'' - \frac{\nu'}{2}e^{-\lambda}\left(\nu'\right) + \frac{\nu'}{2}e^{-\lambda}\left(\lambda'\right) - 2\left(\frac{\nu'}{r}\right)e^{-\lambda} \\
+\frac{\dot{\lambda}}{2}e^{-\nu}\left(\dot{\lambda}\right) - \frac{\dot{\lambda}}{2}e^{-\nu}\left(\dot{\nu}\right) + 2\left(\frac{\lambda'}{r}\right)e^{-\lambda} + \frac{2}{r^2} - e^{-\lambda}\left(\frac{2}{r^2}\right)
\end{aligned}
$$

Subsequently, using the attained values we may solve for the field equations. Firstly, we consider $\mu = \nu = 0$ such that

$$
R_{00} - \frac{1}{2}g_{00}R
$$

For which we have,

$$
\begin{aligned}
&\implies -\frac{1}{2}e^{\nu}\left(e^{-\nu}\ddot{\lambda} - e^{-\lambda}\nu'' - \frac{\nu'}{2}e^{-\lambda}\left(\nu'\right) + \frac{\nu'}{2}e^{-\lambda}\left(\lambda'\right) - 2\left(\frac{\nu'}{r}\right)e^{-\lambda}\right.\\
&\qquad \left. + \frac{\dot{\lambda}}{2}e^{-\nu}\left(\dot{\lambda}\right) - \frac{\dot{\lambda}}{2}e^{-\nu}\left(\dot{\nu}\right) + 2\left(\frac{\lambda'}{r}\right)e^{-\lambda} + \frac{2}{r^2} - e^{-\lambda}\left(\frac{2}{r^2}\right)\right)\\
&\implies -\frac{\ddot{\lambda}}{2} + \frac{\nu''}{2}\left(e^{\nu-\lambda}\right) + \left(\frac{\nu'}{2}\right)^2 e^{-\lambda} - \left(\frac{\nu'}{2}\right)\left(\frac{\lambda'}{2}\right)e^{-\lambda} + \left(\frac{\nu'}{r}\right)e^{\nu-\lambda}\\
&\qquad - \left(\frac{\ddot{\lambda}}{2}\right)^2 + \left(\frac{\dot{\lambda}}{2}\right)\left(\frac{\dot{\nu}}{2}\right) - \left(\frac{\lambda'}{r}\right)\left(e^{\nu-\lambda}\right) - \frac{1}{r^2}e^{\nu} + e^{\nu-\lambda}\left(\frac{1}{r^2}\right)
\end{aligned}
$$

Consequently, we have

$$
\begin{aligned}
R_{00} - \frac{1}{2}g_{00}R = &\frac{\ddot{\lambda}}{2} - \left(\frac{\nu''}{2}\right)e^{\nu-\lambda} - \left(\frac{\nu'}{2}\right)e^{\nu-\lambda}\left(\nu'\right) + \left(\frac{\nu'}{2}\right)e^{\nu-\lambda}\left(\lambda'\right) + \left(\frac{\nu'}{2}\right)^2 e^{\nu-\lambda}\\
&- \left(\frac{\dot{\nu}}{2}\right)\left(\frac{\dot{\lambda}}{2}\right) + \left(\frac{\dot{\lambda}}{2}\right)^2 - \left(\frac{\lambda'}{2}\right)\left(\frac{\nu'}{2}\right)e^{\nu-\lambda} - \frac{1}{r}\left(\nu' e^{\nu-\lambda}\right)\\
&-\frac{\ddot{\lambda}}{2} + \frac{\nu''}{2}\left(e^{\nu-\lambda}\right) + \left(\frac{\nu'}{2}\right)^2 e^{-\lambda} - \left(\frac{\nu'}{2}\right)\left(\frac{\lambda'}{2}\right)e^{-\lambda} + \left(\frac{\nu'}{r}\right)e^{\nu-\lambda}\\
&- \left(\frac{\ddot{\lambda}}{2}\right)^2 + \left(\frac{\dot{\lambda}}{2}\right)\left(\frac{\dot{\nu}}{2}\right) - \left(\frac{\lambda'}{r}\right)\left(e^{\nu-\lambda}\right) - \frac{1}{r^2}e^{\nu} + e^{\nu-\lambda}\left(\frac{1}{r^2}\right)\\
\implies = &-\left(\frac{\nu'}{2}\right)e^{\nu-\lambda}\left(\nu'\right) + \left(\frac{\nu'}{2}\right)e^{\nu-\lambda}\left(\lambda'\right) + 2\left(\frac{\nu'}{2}\right)^2 e^{\nu-\lambda} - 2\left(\frac{\lambda'}{2}\right)\left(\frac{\nu'}{2}\right)e^{\nu-\lambda}\\
&- \left(\frac{\lambda'}{r}\right)\left(e^{\nu-\lambda}\right) - \frac{1}{r^2}e^{\nu} + e^{\nu-\lambda}\left(\frac{1}{r^2}\right)\\
\implies = &-\left(\frac{\lambda'}{r}\right)\left(e^{\nu-\lambda}\right) - \frac{1}{r^2}e^{\nu} + e^{\nu-\lambda}\left(\frac{1}{r^2}\right)
\end{aligned}
$$

Next, we proceed by multiplying both sides by $e^{-\nu}$ and substituting the attained values into (2.229), or (2.230) for mixed components:

$$
\begin{aligned}
e^{-\nu}\frac{8\pi k}{c^4}T_{00} &= e^{-\nu}\left[-\left(\frac{\lambda'}{r}\right)\left(e^{\nu-\lambda}\right)-\frac{1}{r^2}e^{\nu}+e^{\nu-\lambda}\left(\frac{1}{r^2}\right)\right] \\
\Longrightarrow &= -\left(\frac{\lambda'}{r}\right)\left(e^{-\lambda}\right)-\frac{1}{r^2}e^{0}+e^{-\lambda}\left(\frac{1}{r^2}\right) \\
\Longrightarrow &= e^{-\lambda}\left(\frac{1}{r^2}-\frac{\lambda'}{r}\right)-\frac{1}{r^2} \\
\Longrightarrow &= -e^{-\lambda}\left(\frac{1}{r^2}-\frac{\lambda'}{r}\right)+\frac{1}{r^2}
\end{aligned}
$$

substituting into the above field equation we have,

$$
\begin{aligned}
-e^{-\nu}\frac{8\pi k}{c^4}T_{00} &= -e^{-\lambda}\left(\frac{1}{r^2}-\frac{\lambda'}{r}\right)+\frac{1}{r^2} \\
\Longrightarrow \; \frac{8\pi k}{c^4}T_{00} &= -e^{-\lambda}\left(\frac{1}{r^2}-\frac{\lambda'}{r}\right)+\frac{1}{r^2}
\end{aligned}
$$

in which we neglect the $-e^{-\nu}$ term on the left-side for reasons we shall elaborate upon shortly. Furthermore, for mixed components we attain

$$
\frac{8\pi k}{c^4}T_0^0 = -e^{-\lambda}\left(\frac{1}{r^2}-\frac{\lambda'}{r}\right)+\frac{1}{r^2} \tag{2.286}
$$

Similarly, we may compute the corresponding field equation for $\mu = \nu = 1$, such that

$$
R_{11}-\frac{1}{2}g_{11}R
$$

For which we obtain

$$\Longrightarrow \frac{1}{2}\left(e^{\lambda}\right)\left(e^{-\nu}\ddot{\lambda}-e^{-\lambda}\nu''-\frac{\nu'}{2}e^{-\lambda}\left(\nu'\right)+\frac{\nu'}{2}e^{-\lambda}\left(\lambda'\right)-2\left(\frac{\nu'}{r}\right)e^{-\lambda}\right.$$
$$\left.+\frac{\dot{\lambda}}{2}e^{-\nu}\left(\dot{\lambda}\right)-\frac{\dot{\lambda}}{2}e^{-\nu}\left(\dot{\nu}\right)+2\left(\frac{\lambda'}{r}\right)e^{-\lambda}+\frac{2}{r^2}-e^{-\lambda}\left(\frac{2}{r^2}\right)\right)$$
$$\Longrightarrow \frac{\ddot{\lambda}}{2}\left(e^{\lambda-\nu}\right)-\frac{\nu''}{2}-\left(\frac{\nu'}{2}\right)^2+\left(\frac{\lambda'}{2}\right)\left(\frac{\nu'}{2}\right)-2\left(\frac{\nu'}{r}\right)\left(\frac{1}{2}\right)+\left(\frac{\dot{\lambda}}{2}\right)^2e^{\lambda-\nu}$$
$$-\left(\frac{\dot{\lambda}}{2}\right)\left(\frac{\dot{\nu}}{2}\right)e^{\lambda-\nu}+\frac{1}{2}\left(2\frac{\lambda'}{r}\right)+2\frac{1}{r^2}e^{\lambda}\left(\frac{1}{2}\right)-\frac{1}{r^2}$$
$$\Longrightarrow \frac{\ddot{\lambda}}{2}\left(e^{\lambda-\nu}\right)-\frac{\nu''}{2}-\left(\frac{\nu'}{2}\right)^2+\left(\frac{\lambda'}{2}\right)\left(\frac{\nu'}{2}\right)-\frac{\nu'}{r}+\left(\frac{\dot{\lambda}}{2}\right)^2e^{\lambda-\nu}$$
$$-\left(\frac{\dot{\lambda}}{2}\right)\left(\frac{\dot{\nu}}{2}\right)e^{\lambda-\nu}+\frac{\lambda'}{r}+\frac{1}{r^2}e^{\lambda}-\frac{1}{r^2}$$

Then,

$$R_{11}-\frac{1}{2}g_{11}R=\frac{\nu''}{2}-\frac{\ddot{\lambda}}{2}e^{\lambda-\nu}-\left(\frac{\dot{\lambda}}{2}e^{\lambda-\nu}\right)\left(\dot{\lambda}\right)+\left(\frac{\dot{\lambda}}{2}e^{\lambda-\nu}\right)\left(\dot{\nu}\right)+\left(\frac{\nu'}{2}\right)^2-\left(\frac{\dot{\lambda}}{2}e^{\lambda-\nu}\right)\left(\frac{\dot{\nu}}{2}\right)$$
$$+\left(\frac{\dot{\lambda}}{2}\right)^2e^{\lambda-\nu}-\left(\frac{\lambda'}{2}\right)\left(\frac{\nu'}{2}\right)-\frac{\lambda'}{r}+\frac{\ddot{\lambda}}{2}\left(e^{\lambda-\nu}\right)-\frac{\nu''}{2}-\left(\frac{\nu'}{2}\right)^2+\left(\frac{\lambda'}{2}\right)\left(\frac{\nu'}{2}\right)$$
$$-\frac{\nu'}{r}+\left(\frac{\dot{\lambda}}{2}\right)^2e^{\lambda-\nu}-\left(\frac{\dot{\lambda}}{2}\right)\left(\frac{\dot{\nu}}{2}\right)e^{\lambda-\nu}+\frac{\lambda'}{r}+\frac{1}{r^2}e^{\lambda}-\frac{1}{r^2}$$
$$\Longrightarrow=-\frac{\dot{\lambda}}{2}e^{\lambda-\nu}\left(\dot{\lambda}\right)+\frac{\dot{\lambda}}{2}e^{\lambda-\nu}\left(\dot{\nu}\right)-2\left(\frac{\dot{\lambda}}{2}\right)\left(\frac{\dot{\nu}}{2}\right)e^{\lambda-\nu}+2\left(\frac{\dot{\lambda}}{2}\right)^2e^{\lambda-\nu}-\frac{\nu'}{r}+\left(\frac{1}{r^2}\right)e^{\lambda}-\frac{1}{r^2}$$
$$\Longrightarrow=-\frac{\nu'}{r}+\left(\frac{1}{r^2}\right)e^{\lambda}-\frac{1}{r^2}$$

Once again, multiplying both sides by $e^{-\lambda}$ and substituting into the field equation (2.229), we have

$$\begin{aligned} e^{-\lambda}\frac{8\pi k}{c^4}T_{11} &= e^{-\lambda}\left[-\frac{\nu'}{r}+\left(\frac{1}{r^2}\right)e^{\lambda}-\frac{1}{r^2}\right] \\ \Longrightarrow &= -e^{-\lambda}\left(\frac{\nu'}{r}\right)+\frac{1}{r^2}\left(e^0\right)-\frac{1}{r^2}e^{-\lambda} \\ \Longrightarrow &= -e^{-\lambda}\left[\frac{\nu'}{r}+\frac{1}{r^2}\right]+\frac{1}{r^2} \end{aligned}$$

then, substituting into the field equation above we obtain,

$$\begin{aligned} e^{-\lambda}\frac{8\pi k}{c^4}T_{11} &= -e^{-\lambda}\left[\frac{\nu'}{r}+\frac{1}{r^2}\right]+\frac{1}{r^2} \\ \Longrightarrow \quad \frac{8\pi k}{c^4}T_{11} &= -e^{-\lambda}\left[\frac{\nu'}{r}+\frac{1}{r^2}\right]+\frac{1}{r^2} \end{aligned}$$

where we neglect the $e^{-\lambda}$ term from the left hand side of the equation above. Subsequently, in mixed components form we obtain,

$$\frac{8\pi k}{c^4}T_1^1 = -e^{-\lambda}\left[\frac{\nu'}{r}+\frac{1}{r^2}\right]+\frac{1}{r^2} \tag{2.287}$$

For the sake of brevity we simply state the field equations for $\mu=\nu=2$ and $\mu=\nu=3$, such that we find

$$\frac{8\pi k}{c^4}T_2^2 = \frac{8\pi k}{c^4}T_3^3 = -\frac{1}{2}e^{-\lambda}\left(\nu''+\frac{\nu'^2}{2}+\frac{\nu'-\lambda'}{r}-\frac{\nu'\lambda'}{2}\right)+\frac{1}{2}e^{-\nu}\left(\ddot{\lambda}+\frac{\dot{\lambda}^2}{2}-\frac{\dot{\lambda}\dot{\nu}}{2}\right) \tag{2.288}$$

We point attention to the fact that we have one last field equation, namely for the Ricci tensor R_{01} computed above. We note that for this peculiar case when computing $R_{01}-\frac{1}{2}g_{01}R$, the second term vanishes since it is outside of the diagonal in the metric tensor; consequently, we obtain for the

Einstein equations,

$$\frac{8\pi k}{c^4} T_0^1 = R_0^1$$
$$\implies e^{-\lambda} \frac{8\pi k}{c^4} T_0^1 = -e^{-\lambda} \frac{\dot{\lambda}}{r}$$

namely,

$$\frac{8\pi k}{c^4} T_0^1 = -e^{-\lambda} \frac{\dot{\lambda}}{r} \tag{2.289}$$

The remaining components of (2.230) vanish identically.

The equations (2.287) - (2.289) may be integrated exactly in the case of a centrally symmetric field in vacuum, that is, outside of the masses producing the field. Namely, since we consider a system in vacuum, we may proceed to setting the energy-momentum tensor equivalent to zero, such that we obtain the following equations:

$$e^{-\lambda} \left(\frac{\nu'}{r} + \frac{1}{r^2} \right) - \frac{1}{r^2} = 0 \tag{2.290}$$

$$e^{-\lambda} \left(\frac{\lambda'}{r} - \frac{1}{r^2} \right) + \frac{1}{r^2} = 0 \tag{2.291}$$

$$\dot{\lambda} = 0 \tag{2.292}$$

in which (2.292) follows from the fact that

$$\frac{8\pi k}{c^4} T_0^1 = -e^{-\lambda} \frac{\dot{\lambda}}{r}$$
$$\implies -e^{-\lambda} \frac{\dot{\lambda}}{r} = 0$$
$$\implies \frac{\dot{\lambda}}{r} = 0$$
$$\implies \dot{\lambda} = 0$$

where we multiplied both side by the quantity $-r/e^{-\lambda}$. We briefly note that we do not need to write

(2.288), since it follows from the previous trio of equations.

Using (2.292) we see immediately that the quantity λ is independent of time, since the derivative with respect to time of λ is is equivalently zero, this must mean that λ is equal to some constant, namely a function which is independent of time, i.e. $g(r)$. Moreover, adding equation (2.290) and (2.291), we find

$$\begin{aligned} e^{-\lambda}\left(\frac{v'}{r}+\frac{1}{r^2}\right)-\frac{1}{r^2}&=0\\ +e^{-\lambda}\left(\frac{\lambda'}{r}-\frac{1}{r^2}\right)+\frac{1}{r^2}&=0\\ \Longrightarrow e^{-\lambda}\left(\frac{\lambda'}{r}-\frac{1}{r^2}+\frac{v'}{r}+\frac{1}{r^2}\right)&=0\\ \Longrightarrow e^{-\lambda}\left(\frac{\lambda'}{r}+\frac{v'}{r}\right)&=0\\ \Longrightarrow \left(\frac{\lambda'}{r}+\frac{v'}{r}\right)&=0\\ \Longrightarrow \lambda'+v'&=0 \end{aligned}$$

$$\lambda+v=f(t) \tag{2.293}$$

where $f(t)$ is a function of time. But when choosing the interval ds^2 in the form of (2.448), by the arbitrariness of the theory of general relativity, the possibility of an arbitrary transformation of the time of the form $t=f(t')$ remains, as per the remarks aforementioned. Upon consideration of said transformation, we may add to v and arbitrary function of time, say $h(t)$, such that we may always make $f(t)$ in (2.293) vanish, namely,

$$\begin{aligned} \lambda+(v+h(t))&=f(t)\\ \Longrightarrow \frac{\partial}{\partial r}[\lambda+(v+h(t))]&=\frac{\partial}{\partial r}f(t)\\ \Longrightarrow \lambda'+v'&=0 \end{aligned}$$

Therefore, without loss of generality, we have

$$\lambda + (v + h(t)) = f(t)$$

$$\lambda + v = f(t) - h(t)$$

$$\Longrightarrow \lambda + v = 0$$

Note that the centrally symmetric gravitational field in vacuum is automatically static, i.e. it is independent of time.

The equation (2.291) may be integrated accordingly and yields,

$$e^{-\lambda} = e^{v} = 1 + \frac{\text{const}}{r} \tag{2.294}$$

We verify such a claim. Starting with the equation given by (2.291) we write,

$$
\begin{aligned}
e^{-\lambda}\left(\frac{\lambda'}{r}-\frac{1}{r^2}\right)+\frac{1}{r^2}=0 \\
\implies \frac{\lambda'}{r}-\frac{1}{r^2}+e^{\lambda}\frac{1}{r^2}=0 \\
\implies \lambda'-\frac{1}{r}+e^{\lambda}\frac{1}{r}=0 \\
\implies \lambda'=\frac{1}{r}-e^{\lambda}\frac{1}{r} \\
\implies \lambda'=\frac{1}{r}\left(1-e^{\lambda}\right) \\
\implies \frac{\lambda'}{\left(1-e^{\lambda}\right)}=\frac{1}{r} \\
\implies \int\frac{\lambda'}{\left(1-e^{\lambda}\right)}=\int\frac{1}{r} \\
\implies -\ln\left(e^{-\lambda}-1\right)=\ln(r)+c \\
\implies \ln\left(e^{-\lambda}-1\right)=-\ln(r)-c \\
\implies e^{\ln\left(e^{-\lambda}-1\right)}=e^{-\ln(r)}-e^{c} \\
\implies e^{-\lambda}-1=\frac{c}{r} \\
\implies e^{-\lambda}=1+\frac{c}{r}
\end{aligned}
$$

which is equivalent to (2.294).

We note that at infinity $(r\rightarrow\infty)$, (2.294) turns to $e^{-\lambda}=e^{\nu}=1$, that is at an infinite, or rather large distance from the gravitating bodies the metric becomes galilean. We then require that at large distances Newton's law hold, leading to the possibility of expressing the constant in terms of the mass of the body, due to the fact that at large distances the field is weak. Hence, we may use the potential of the field for a single particle within a weak field, such that $g_{00}=1+\left(2\phi/c^2\right)$, where the potential ϕ is given its Newtonian value (2.269), $\phi=-(km/r)$ (m is the total mass of the bodies producing the field). From this we find that the constant takes the value $const=-\left(2km/c^2\right)$

which has dimensions of length and we call it the *gravitational radius* r_g of the body:

$$r_{\rm g} = \frac{2km}{c^2} \tag{2.295}$$

By comparison of the equations (2.269) and (2.294), we have starting with (2.109) and substituting in the value for the gravitational potential given by (2.269), we have

$$\begin{aligned} \phi &= -\frac{km}{r} \\ \implies g_{00} &= 1 + \frac{2\left(-\frac{km}{r}\right)}{c^2} \\ \implies &= 1 - \frac{2km}{rc^2} \end{aligned}$$

Then,

$$\begin{aligned} 1 + \frac{\text{const}}{r} &= 1 - \frac{2km}{rc^2} \\ \implies \text{const} &= -\frac{2km}{c^2} \end{aligned}$$

It follows that for this limiting case, we find

$$\begin{aligned} e^{-\lambda} &= e^{\nu} = 1 + \frac{\text{const}}{r} \\ \implies e^{-\lambda} &= e^{\nu} = 1 - \frac{r_g}{r} \end{aligned}$$

such that we obtain the following space-time metric

$$ds^2 = \left(1 - \frac{r_g}{r}\right) c^2 dt^2 - r^2\left(\sin^2\theta d\phi^2 + d\theta^2\right) - \frac{dr^2}{1 - \frac{r_g}{r}} \tag{2.296}$$

This solution of the field equations was first determined by Karl Schwarzschild (1916). It is the exact solution to the Einstein equations for the gravitational field in vacuum produced by any centrally-symmetric distribution of masses. This solution holds not only for static masses, but also for a moving object, so long as the motion of the object preserves the central symmetry of the system. The metric given by (2.296) depends only on the total mass of the gravitating body.

The corresponding spatial metric is defined by the expression of the spatial distance of (2.296)

$$dl^2 = \frac{dr^2}{1-\frac{r_g}{r}} + r^2\left(\sin^2\theta d\phi^2 + d\theta^2\right) \tag{2.297}$$

The meaning of the coordinate r is determined by the fact that in the metric (2.297) the circumference of a circle with its center at the center of the field is $2\pi r$. We briefly expand upon the above geometric meaning of the coordinate r. Without loss of generality, we consider the two dimensional surface with $r = constant$ and $t = constant$, such that the line element (2.297) takes the form

$$dl^2 = r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

for which the metric coefficients are

$$g_{ij} = \begin{pmatrix} r^2 & 0 \\ 0 & r^2\sin^2\theta \end{pmatrix}; \quad g = \det\left(g_{ij}\right)$$

in which the invariant volume element is given by $\sqrt{g}$, i.e

$$g = \det\left(g_{ij}\right)$$
$$\implies = g = \det\left(\left(r^2\right)\left(r^2\sin^2\theta\right) - 0\right)$$
$$\implies g = r^4\sin^2\theta$$
$$\implies \sqrt{g} = \sqrt{r^4\sin^2\theta}$$
$$\implies \sqrt{g} = r^2\sin\theta$$

Then, finding the proper area of this 2-sphere (or simply sphere) yields:

$$\mathscr{A} = \int_0^{\pi}\int_0^{2\pi}\sqrt{g}d\theta d\phi = r^2\int_0^{\pi}\int_0^{2\pi}\sin\theta d\theta d\phi = 4\pi r^2 \tag{2.298}$$

which is exactly the formula for the area of a sphere. It follows that θ and ϕ are the standard angular coordinates on a sphere. Hence, we may then obtain the radial coordinate r as

$$r = \left(\frac{\mathscr{A}}{4\pi}\right)^{1/2} \tag{2.299}$$

As a consequence, the coordinates (r,θ,ϕ) retain their natural Euclidean interpretation. Next, consider $\theta = const$ and $\phi = const$, such that we may write (2.297) as follows,

$$dl^2 = \frac{dr^2}{1-\frac{r_g}{r}}$$

Integration of both sides as well as taking the square root reveals

$$\int dl^2 = \int \frac{dr^2}{1-\frac{r_g}{r}}$$
$$\implies (l)_{\theta,\phi=\text{ constant}} = \int_{r_1}^{r_2}\frac{dr}{\sqrt{1-\frac{r_g}{r}}}$$

which differs immensely from the usual distance between two points r_1 and r_2 in the Euclidean sense, i.e. $(r_1 - r_2)$. This follows from the fact that vectors within curved space-time transform differently at different locations in space-time, (see §10), which follows from the concept of parallel displacement aforementioned. Hence, the distance between two arbitrary points r_1 and r_2 within the presence of a gravitational field does not only depend on the spatial interval between them, but also the geometry of the space-time. Thus, we may provide the following relation

$$\int_{r_1}^{r_2}\frac{dr}{\sqrt{1-\frac{r_g}{r}}} > r_2 - r_1 \tag{2.300}$$

Further, we see that $g_{00} \leq 1$, which follows from the proposed formula given for (2.109), i.e.

$$g_{00} = 1 - \frac{2km}{rc^2}$$

where we find that for an arbitrary large mass we have that $g_{00} \to 0$ as $m \to \frac{c^2 r}{2k}$. Nonetheless, since we cannot feasibly attain speed equivalent to the speed of light, we note that g_{00} must be bounded above by 1, hence, we have that $g_{00} \leq 1$. Then, combining with the formula (2.40) $d\tau = \sqrt{g_{00}}dt$, which defines the proper time, it follows that

$$d\tau = \sqrt{g_{00}}dt$$
$$\implies d\tau \leq (1)\, dt$$

Thus,

$$d\tau \leq dt \tag{2.301}$$

Here the equal sign holds only at infinity, where t coincides with the proper time. Thus, at finite distances from the masses there is a "slowing down" of the time compared with the time at infinity.

2.17.1 Singularities, pseudosingularities, and Black holes

Now, we rewrite the obtained Schwarzschild metric in (2.296) as follows

$$ds^2 = \left(1 - \frac{2GM}{r}\right) dt^2 - \frac{dr^2}{1 - 2GM/r} - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \tag{2.302}$$

which we will explicitly derive in Section 2.18, but we present here without proof. We note in passing that G is the gravitational constant and is equivalent to the previous annotation used for said constant, i.e. $k = G$. Notice that in the space-time interval presented above a "singularity" as $r \to 2GM$. At this radius,

$$g_{00} = 1 - \frac{2GM}{r} \to 0 \tag{2.303}$$

and

$$g_{11} = -\frac{1}{1-2GM/r} \to -\infty \tag{2.304}$$

As aforementioned, the critical radius

$$r_g = \frac{2km}{c^2} \tag{2.305}$$

is known as the *gravitational radius* or *Schwarzschild radius* of the mass m. For a mass equal to that of the Sun, $m = m_\odot = 2 \times 10^{33}$g, the radius is $r_g = 3.0$km

If a body's radius is larger than r_g, then the singularities (2.303) and (2.304) are disregarded, as the exterior Schwarzschild solution (2.302) applies only in the exterior of the body. Further, as we will see, a body that has collapsed to a radius smaller than that of its gravitational radius is unable to come to equilibrium and will evidently collapse onto itself - the gravitational forces become so strong to the point that nothing can resist them. The critical density at which a body has a smaller size than that of its Schwarzschild radius r_g decreases with mass. For instance, consider that the nucleus of a galaxy consists of 10^{11} stars, each with roughly a mass of $m_\odot$, uniformly distributed over a spherical volume. In this given case the critical density is 10^{-6} g/cm^3; at this density, the typical distance between neighboring stars in the galactic nucleus is about roughly the distance of the Earth to the Sun.

From this point onward, we assume that the body has collapsed completely, such that the mass density is zero at all points (except at the point of singularity $r = 0$). This leads to the possibility of treating the Schwarzschild solution as an exact vacuum solution of the field equations, as aforementioned. The surface $r = r_g$ is a surface of infinite redshift. A clock placed at rest near $r = r_g$ shows a proper time

$$d\tau = \sqrt{1-2GM/r}dt \tag{2.306}$$

which approaches zero as $r \to r_g$, so the clock runs infinitely slow compared with a clock at a larger distance. Take for example, an astronaut at rest near the "singularity" at $r = r_g$ sending out light impulses with a time interval of 1 s (as shown by his clock) between impulses. Then an observer at an arbitrary large distance will receive the pulses within a time interval much larger than 1 s (as show by their clock) between successive pulses. (2.306) details an infinite time dilation for a clock at rest at $r = r_g$. Actually, it follows that a clock cannot remain at rest on this surface.The vanishing of the proper time differential $d\tau$ is a characteristic that follows from the worldline of a light signal, and hence only a light signal, aimed at the radial direction, can remain at rest at r_g. This infinite redshift surface is also known as the static limit, since no material particle can remain at rest on the surface.

If r is in the range $r_g > r > 0$ then the metric solution is free of singularities. Nonetheless, within this region

$$g_{00} = 1 - r_g/r < 0 \quad \text{and} \quad g_{11} = -1/\left(1 - r_g/r\right) > 0 \tag{2.307}$$

Hence, the signs of the time-like and space-like components of the metric are of opposite sign than that of the normal signature. Subsequently, we have that in this region, we have a change in the time-like and space-like coordinates, namely, t becomes a space-like coordinate and r is a time like coordinate. Because the metric is a function of r, and now r corresponds to a time-like component, it follows that when $r < r_g$, the metric tensor is actually time dependent. Namely, in the region $r > r_g$, the static character is a consequence of the particular choice of coordinates, with time-dependence following from consideration of a new coordinate system. Yet, with in the interior region of the spherically symmetric body, $r < r_g$, the time dependence of the metric is not coordinate independent and unavoidable. This follows from the fact that no matter how we define a new set of coordinates, with regard to the old ones, an advance of time $\left(ds^2 > 0\right)$ is impossible unless $dr^2 > 0$; hence the advance of time necessarily entails a change in r, and with it a change in the metric tensor.

We note that the Schwarzschild "singularity" at $r = r_g$ is not a physical singularity, rather it is an artificial singularity or pesudosingularity which arises from our chose of coordinates and may be eliminated by a proper choice of coordinates, as we shall later see. Thus, an astronaut in free fall towards and crossing the surface $r = r_g$ will not experience any unusual phenomena, the physics around him will show no peculiarities. Of course there exists the presence of tidal waves which will increment in strength as he falls deeper into the center, yet the remain finite at $r = r_g$. It isn't until $r = 0$ do the tidal forces become infinite and yielding the presence of a real physical singularity.

We may observe, explicitly, the absence of a genuine singularity at $r = r_g$ by considering the Riemann curvature tensor. It follows that even though the Schwarzschild coordinates g_{00} and g_{11} are a bit tedious to handle, the components of the curvature at $r = r_g$ are finite when we consider local geodesic coordinates. For instance, consider the component $R^0 101$,

$$R^0_{101} = \frac{r_g}{r^3} \frac{1}{1 - r_g/r}$$

At $r = r_g$, this function is singular, yet the singularity is spurious, analogous to the one in (2.304). In order to make this clear, let $x_\mu = (t_0, r_0, \pi/2, \phi_0)$ be a given point in spacetime in the equatorial plane, and further introduce the geodesic coordinates x'^μ at this point by means of the transformation

$$\begin{aligned} x'^0 &= (t - t_0)\sqrt{1 - r_g/r_0} + \cdots, \quad x'^1 = \frac{r - r_0}{\sqrt{1 - r_g/r_0}} + \cdots \\ x'^2 &= r_0(\theta - \pi/2) + \cdots, \quad x'^3 = r_0(\phi - \phi_0) + \ldots \end{aligned} \tag{2.308}$$

where the dots represent quadratic terms; we disregard these terms as we are only interested in the transformation law of the curvature tensor, and this law does not depend on the quadratic terms. The transformation coefficients at x'^μ are

$$\begin{aligned} &\frac{\partial x^0}{\partial x'^0} = \frac{1}{\sqrt{1 - r_g/r_0}}, \quad \frac{\partial x^1}{\partial x'^1} = \sqrt{1 - r_g/r_0} \\ &\frac{\partial x^2}{\partial x'^2} = \frac{1}{r_0}, \quad \frac{\partial x^3}{\partial x'^3} = \frac{1}{r_0} \end{aligned} \tag{2.309}$$

with all other coefficients vanishing.The geodesic coordinates x'^μ can be given the physical interpretation of coordinates of a frame of reference in free fall, and must therefore be regarded as a good choice of coordinates. For the transformed Riemann tensor we find

$$\begin{aligned} R'^{0}_{101} &= \frac{\partial x'^0}{\partial x^\alpha}\frac{\partial x^\beta}{\partial x'^1}\frac{\partial x^\mu}{\partial x'^0}\frac{\partial x^\nu}{\partial x'^1}R^\alpha_{\beta\mu\nu} = \frac{\partial x'^0}{\partial x^0}\frac{\partial x^1}{\partial x'^1}\frac{\partial x^0}{\partial x'^0}\frac{\partial x^1}{\partial x'^1}R^0_{101} \\ &= \left(1-\frac{r_g}{r_0}\right)\frac{r_g}{r_0^3}\frac{1}{1-r_g/r_0} = \frac{r_g}{r_0^3} \end{aligned} \tag{2.310}$$

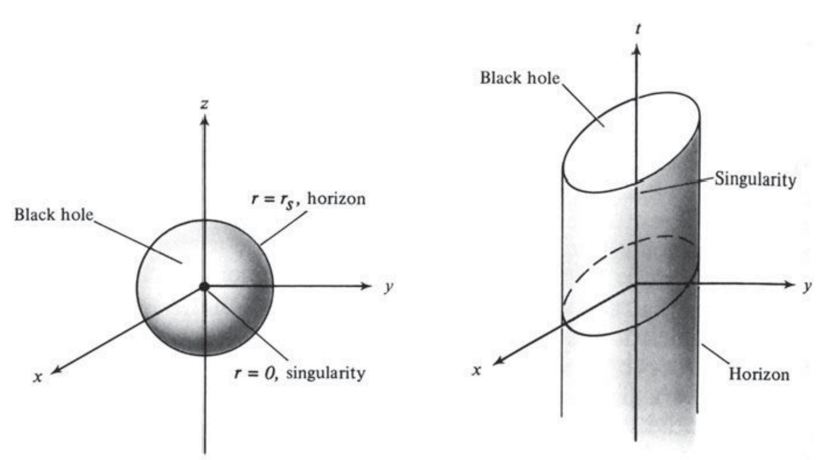
If we consider $r \to r_g$, the curvature tensor component remains finite, hence, in the given x'^μ coordinates the $R'^0 101$ of the curvature tensor is free of singularities. It follows that rest of the components of the Riemann tensor are also singularity free. The tidal force remains finite at $r = r_g$.

Alternatively, we may also compute the scalar curvature in local geodesic coordinates in order to show that all the components of the Riemann tensor remain finite. Using the explicit values of the Riemann tensor in Schwarzschild coordinates, we find that the scalar quantity is finite, $R^\alpha_{\beta\mu\nu}R^{\beta\mu\nu}_\alpha = 48(GM)^2/r_0^6$. Since in local geodesic coordinates, the scalar curvature is the sum of the square of all the components of $R'^\alpha_{\beta\mu\nu}$, it follows that all components must be finite.

Note that at $r = r_g = 0$, the components of $R'^\alpha_{\beta\mu\nu}$ diverge as seen in (2.310). Hence, at the "center", the tidal forces diverge and we obtain a true singularity.Near this point, we have an interchanging of the space-like and time-like coordinates, i.e. t becomes space like and r becomes time-like, so this singularity happens at a given instant of time ($r = 0$) in all of space (all values of t). Hence, the time-dependent, dynamical geometry in the interior of the spherically symmetric body evolves into a singularity and comes to an end. But the geometry on the outside remains static forever.

Although the region of $r < r_g$ has no peculiar properties of a local kind (except at $r = 0$), it does have unusual properties of a global kind. Upon further examination of the spacetime geometry, the region $r < r_g$ is a black hole. By this we mean, that no signal can emerge from the region $r < r_g$ (inside of the body) and reach the region $r > r_g$ (outside of the body). The surface $r = r_g$ acts as a boundary between the region of spacetime that is observable by outside observers, and the region that is unreachable by these observers. This boundary of the black hole is called the *event horizon*.

It is the region of the body beyond which we cannot see, reminiscent to the horizon of a large lake or ocean. We may regard the surface of $r = r_g$ as a “one-way membrane”, through which signals may be sent in, but not out. This is a nonlocal property, since in order to test it, we must analyze the propagation of light signals and their behavior in the long run.



(a) (b)

Figure 2.4: (a) The Schwarzschild black hole and its event horizon in x, y, z coordinates. (b) The Schwarzschild black hole and its event horizon displayed in x, y, t coordinates

It follows that the Schwarzschild event horizon must be thought of an attribute of the Schwarzschild geometry, since it is perceived by all observers. This means that all observers agree on the existence and location of the surface of the event horizon. Event horizons refer to global rather than local properties of spaceime, as well as an intrinsic feature of spacetime that is independent on the state of motion of the observers (9).

In order to understand how signals are interrupted at the Schwarzschild horizon $r = r_g$, consider a light pulse propagating in the radial direction. The velocity of the light signal, may be calculated by (2.302) by taking $ds^2 = 0$:

$$\frac{dr}{dt} = \pm\left(1 - \frac{r_g}{r}\right) \tag{2.311}$$

we see immediately that as the signal approaches $r = r_g$, this coordinate velocity vanishes.

Figure 2.5 (9) shows the forward light cones obtained as a consequence of (2.311).

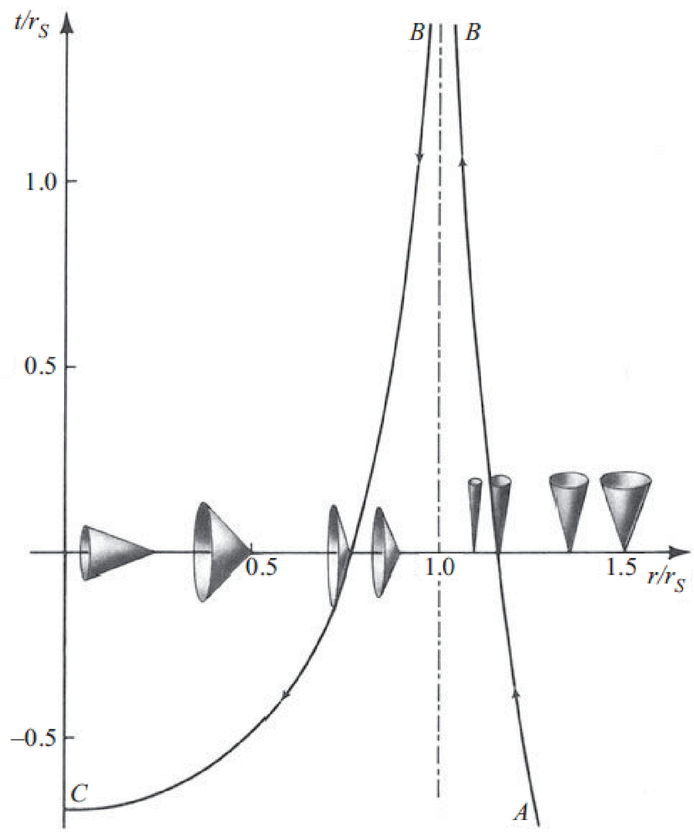


Figure 2.5: Light cone

- For $r > r_g$, the axis of the light cones is parallel to the time t-axis
- For $r < r_g$, the axis of the light cones is parallel to the radial r-axis

The bizarre orientation of the light cones in the interior of the body is a consequence of the reversal of the coordinates, aforementioned: r is a timelike coordinate and t is a spacelike coordinate. Within the interior of the body, the quantity dt/dr gives what would normally be the "velocity". The existence of an event horizon at $r = r_g$ is apparent from observation of the light cones. Any signal must necessarily travel in a spacetime direction that lies within a light cone. Since the light cones in the black hole are oriented towards $r = 0$, any signal which passes the event horizon is pulled toward decreasing values of r and becomes entrapped within the black hole.

The light cones displayed in Figure 5 are tangent to the surface $r = r_g$, this means, that viewed in spacetime, the horizon is a null surface. This is a general property of event horizons, since a light signal that starts exactly on a horizon aimed in the outward direction is compressed between signals just outside of the horizon and the signals just inside of the horizon; this leaves the light signal propagating neither in nor out of the even horizon, leading to a propagation in place indefinitely, indicating that the surface over which it hovers is a null surface. As previously seen, signals may not leave the inside of the black hole, but can enter readily. The curve delineated in Figure 5 is the worldline of a light signal that travels inwards towards a black hole. The signal follows the worldline AB to $t = \infty$ and then BC to $r = 0$. We obtain said curve by integrating (2.311) and notice that signal velocity vanishes as $r \to r_g$ with said signal taking an infinite time t to reach $r = r_g$. From the point of view of an observer at infinity, whose clock is indicated in t-time, as oppose to τ time, the signal never reaches the horizon.

For instance, suppose that a black hole is surrounded by dust-like particles that scatter a small part of the light signal, such that the position of the signal becomes visible to an observer at infinity. The observer will see the lighted spot approach Schwarzschild radius asymptotically without ever reaching it. This is in part due to the the gravitational time-dilation effect. From the point of view of the free falling astronaut near $r = r_g$, no slowing down occurs, since the signal always has the speed of light relative to him. If the observer at infinity remains in constant communication with the astronaut, he will find that the motion and metabolic rate of the astronaut are slowed in much the same way as the motion and oscillation of the light signal. The last syllable muttered by the astronaut before he crosses the event horizon is drawn out to an infinite length when received by the outside observer; all subsequent words remain inside of the black hole. Immediately, we deduce that the astronaut himself is trapped within the body. His worldline intersects the singularity at $r = 0$ and when he is in the vicinity of the singularity the tidal forces rip him apart. We find that for a typical worldline the proper time between r_g and $r = 0$ is of the order r_g/c. This means that after the astronaut falls past the event horizon with approximate mass $m = m_\odot$, he has only $\approx 10^{-5}$ s to live. Since all worldlines within the astronaut's future light cone terminate on $r = 0$, the collision with the singularity is inevitable.

Incidentally, we deduce that if a spherical star is compressed by some astrophysical process to a radius smaller than its Schwarzschild radius, them the gravitational collapse necessarily ensues. If we consider a particle on a surface of the star, and remember that, by Birkhoff's theorem, the exterior geometry must Schwarzschild, then the particle on the surface of such a celestial body has the same equation of motion as that of a spacecraft in an empty Schwarzschild spacetime. The surface of the star must therefore, necessarily fall toward the singularity at $r = 0$, just as the spacecraft does. Gravitational collapse is inevitable.

Consider the geodesic equations for the Schwarzschild metric

$$\frac{\mathscr{E}^2}{1-2GM/r} - \frac{\dot{r}^2}{1-2GM/r} - \frac{\ell^2}{r^2} = 1 \tag{2.312}$$

where $\mathscr{E}$ is the energy and ℓ is the angular momentum. (2.312) gives us $dr/d\tau$ for an astronaut in free fall. For purely radial motion, $\ell = 0$, and thus (2.312) reduces to

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{r_g}{r} - \left(1-\mathscr{E}^2\right)$$

After integration we find

$$\tau = \text{constant} + \frac{r\left(r_g/r - 1 + \mathscr{E}^2\right)^{1/2}}{1-\mathscr{E}^2} + \frac{r_g}{(1-\mathscr{E}^2)^{3/2}} \tan^{-1}\left(\frac{r_g/r - 1 + \mathscr{E}^2}{1-\mathscr{E}^2}\right)^{1/2} \tag{2.313}$$

For a finite change in r, the corresponding change in proper time is finite. For example, suppose that astronaut is initially at rest at the radius $r_0 = r_g/\left(1-\mathscr{E}^2\right)$; this radius is greater than r_g thus the astronaut is outside of the black hole. The proper time to free fall all the way to $r = 0$ is then

$$\Delta\tau = r_g \frac{\pi/2}{(1-\mathscr{E}^2)^{3/2}} = r_g \frac{\pi}{2}\left(\frac{r_0}{r_g}\right)^{3/2} \tag{2.314}$$

If the initial value r_0 is near r_g (and $\mathscr{E}$ is zero), then $r_0/r_g \cong 1$ and $\Delta\tau \cong \pi r_g/2$.

The region within the vicinity of the horizon, not only affects the properties of the speed of light, light cones, and lapse of proper time, but also has a significant effect on the electromagnetic field of particles or currents which may be within its vicinity.

For example, consider Figures (6) and (7), respectively, these show the electric field lines of electrically charged point-like particles fixed at the distances $r = 2r_g$ and $r = 1.1r_g$, respectively. The field lines were attained by solving the Maxwell equations in the curved space-time under consideration, where the electric field is assumed to be weak, such that the Schwarzschild geometry remains unaltered.

Note the asymptotic flow of the field lines in Figure (7), as if the black hole had acquired the electric charge and the field lines originated from within the black hole. Thus, an observer at an arbitrary large distance would perceive the charge distribution as originating from the black hole. This is an example of the so-called "membrane paradigm" for black-hole physics. The black hole then acts as if it were encapsulated by an electrically charged membrane, with the point charge imperceptible to the observer. If the point charge creating such a distribution is moved toward or away of the black hole, the perceived membrane charge distribution rearranges itself, i.e. the currents seem to flow on the membrane, and the resulting magnetic fields can be calculated from these apparent currents. Thus, the electric field lines obviously do not originate from the black-hole horizon, as it may seem from Figures (2.6) and (2.7) (9), but rather they are everywhere continuous in the empty Schwarzschild geometry, except at physical point charges, as imposed by Maxwell's equations.

Finally, we proceed by presenting an approximate expression for the space-time interval at large distances from the origin of the coordinates:

$$ds^2 = ds_0^2 - \frac{2km}{c^2 r}\left(dr^2 + c^2 dt^2\right) \tag{2.315}$$

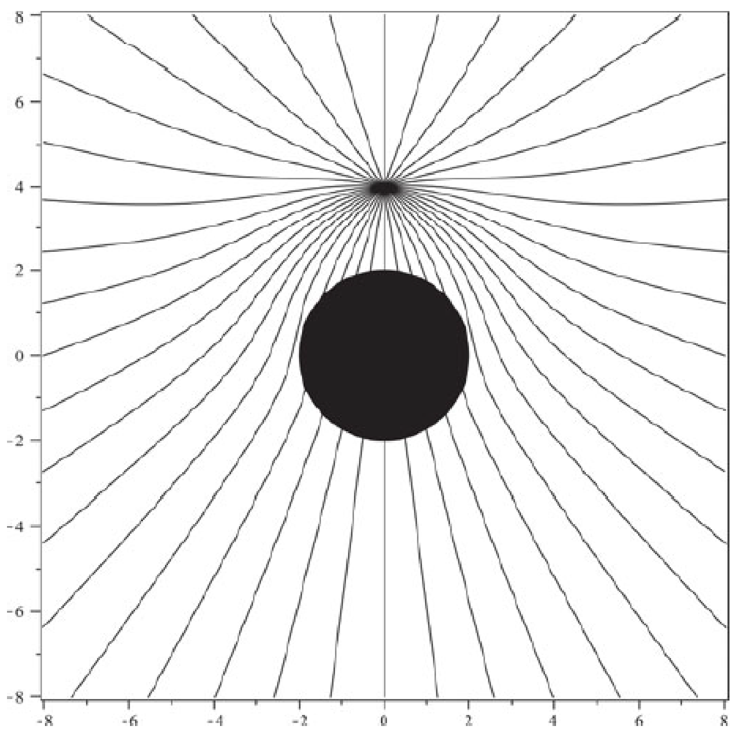


Figure 2.6: (a) Electric filed lines of a point-like charge at $r = 2r_g$

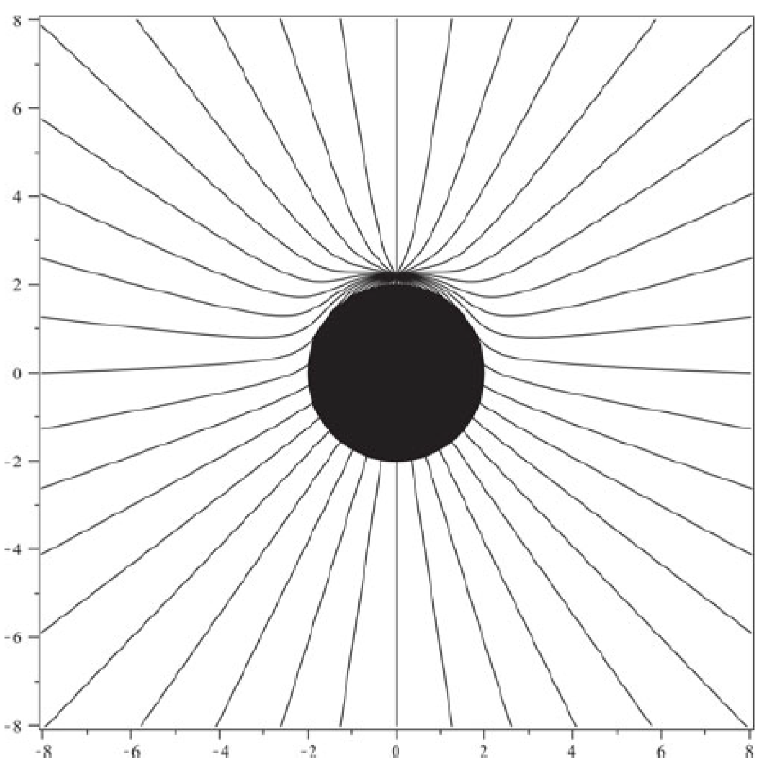


Figure 2.7: (b) Electric filed lines of a point-like charge located at $r = 1.1r_g$

The second term represents a small correction to the galilean metric ds_0^2. At large distances from the masses producing the field, every field appear centrally symmetric, since as $r \to \infty$ the second term vanishes and we are left with the attained space-time metric. Hence, (2.315) determines the metric at large distances from any system of bodies.

We may craft certain general considerations considering the behavior of a centrally symmetric field of gravitation within the interior of gravitating masses. We see that from (2.286) as $r \to 0$, λ must vanish at least like r^2, for if this were not the case then the right side of the equation would become infinite as $r \to 0$, i.e. T_0^0 would have a singularity at $r = 0$. Integration of (2.286) with the condition $\lambda|_{r=0} = 0$ yields

$$\lambda = -\ln\left\{1 - \frac{8\pi k}{c^4 r}\int_0^r T_0^0 r^2 dr\right\} \tag{2.316}$$

Indeed, first we point to the equivalence of the right hand side of (2.286) and the left hand side of (2.291), simple distribution confirms said assertion,

$$-e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right) + \frac{1}{r^2} = e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} \tag{2.317}$$

To this end, consider the equation

$$e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} = \frac{8\pi k}{c^4} T_0^0 \tag{2.318}$$

Consider the following substitutions to the above equation, let $u = e^{-\lambda}$, $u' = -\lambda' u$, and $\lambda' = -\frac{u'}{u}$, such that

$$
\begin{aligned}
e^{-\lambda}\left(\frac{\lambda'}{r}-\frac{1}{r^2}\right)+\frac{1}{r^2}=\frac{8\pi k}{c^4}T_0^0 \\
\implies u\left(\frac{-\frac{u'}{u}}{r}-\frac{1}{r^2}\right)+\frac{1}{r^2}=\frac{8\pi k}{c^4}T_0^0 \\
\implies u'+\frac{u}{r}-\frac{1}{r}=-\frac{8\pi k}{c^4}T_0^0 r \\
\implies u=1-\frac{8\pi k}{3c^4}T_0^0 r^2+\frac{c}{r} \\
\implies e^{-\lambda}=1-\frac{8\pi k}{3c^4}T_0^0 r^2+\frac{c}{r} \\
\implies \ln\left(e^{-\lambda}\right)=\ln\left[1-\frac{8\pi k}{3c^4}T_0^0 r^2+\frac{c}{r}\right] \\
\implies \lambda=-\ln\left[1-\frac{8\pi k}{3c^4}T_0^0 r^2+\frac{c}{r}\right]
\end{aligned}
$$

after which we may set the second and third terms of the RHS equal to the following integral

$$
\begin{aligned}
-\frac{8\pi k}{3c^4}T_0^0 r^2+\frac{c}{r}=-\frac{8\pi k}{c^4 r}\int_0^r T_0^0 r^2 dr \\
\implies \frac{8\pi k}{3c^4}T_0^0 r^2+\frac{c}{r}=\frac{8\pi k}{c^4 r}\int_0^r T_0^0 r^2 dr
\end{aligned}
$$

From (2.292), $T_0^0=e^{-\nu}T_{00}\geq 0$, it follows that $\lambda \geq 0$, since $\dot{\lambda}=0 \to \lambda = const$, that is,

$$
e^{\lambda}\geq 1 \tag{2.319}
$$

Then, subtraction of (2.286) from (2.287), we have;

$$
\begin{aligned}
\frac{8\pi k}{c^4}T_0^0 &= -e^{-\lambda}\left(\frac{1}{r^2}-\frac{\lambda'}{r}\right)+\frac{1}{r^2} \\
-\frac{8\pi k}{c^4}T_1^1 &= -e^{-\lambda}\left(\frac{v'}{r}+\frac{1}{r^2}\right)+\frac{1}{r^2} \\
\Longrightarrow \frac{8\pi k}{c^4}\left(T_0^0-T_1^1\right) &= e^{-\lambda}\left(-\frac{1}{r^2}+\frac{\lambda'}{r}\right)+e^{-\lambda}\left(\frac{v'}{r}+\frac{1}{r^2}\right) \\
\Longrightarrow \frac{8\pi k}{c^4}\left(T_0^0-T_1^1\right) &= \left(\frac{\lambda'}{r}+\frac{v'}{r}\right)e^{-\lambda} \\
\Longrightarrow \frac{8\pi k}{c^4}\left(T_0^0-T_1^1\right) &= \frac{e^{-\lambda}}{r}\left(v'+\lambda'\right)
\end{aligned}
$$

after which setting the difference of the energy-momentum tensors equal to the their values for macroscopic bodies using (2.221), respectively, we write

$$
\frac{e^{-\lambda}}{r}\left(v'+\lambda'\right)=\frac{8\pi k}{c^4}\left(T_0^0-T_1^1\right)=\frac{(\varepsilon+p)\left(1+\frac{v^2}{c^2}\right)}{1-\frac{v^2}{c^2}}\geq 0
$$

hence, we may conclude that $v'+\lambda'\geq 0$. Nonetheless, at an arbitrarily large distance from the masses producing the field, $r\to\infty$, the metric takes its galilean form, i.e. $v\to 0$, $\lambda\to 0$. Thus, from the above computation where we found $v'+\lambda'\geq 0$ it follows that over all of space

$$
v+\lambda\leq 0 \tag{2.320}
$$

Yet, since we found that $\lambda\geq 0$, it follows that $v\leq 0$, i.e.

$$
e^{v}\leq 1 \tag{2.321}
$$

From the inequalities obtained, we surmise that the properties delineated by (2.300) and (2.301) of the spatial metric and the behavior of clocks in a centrally symmetric field in vacuum apply equally to the field in within the interior of the gravitating masses.

We point attention to the circumstance in which the field is produced by a spherical body of "radius" a, such that for $r > a$, we have that $T_0^0 = 0$. Then for point with $r > a$, equation (2.316) can be rewritten as

$$\lambda = -\ln\left\{1 - \frac{8\pi k}{c^4 r}\int_0^a T_0^0 r^2 dr\right\}$$

Howbeit, we can apply the expression (2.296) referring to vacuum, where we consider the gravitational radius of the body and write,

$$\lambda = -\ln\left(1 - \frac{2km}{c^2 r}\right)$$

Equating both expression, we attain the equation

$$m = \frac{4\pi}{c^2}\int_0^a T_0^0 r^2 dr \tag{2.322}$$

namely,

$$\begin{aligned} \frac{2km}{c^2 r} &= \frac{8\pi k}{c^4 r}\int_0^a T_0^0 r^2 dr \\ \Longrightarrow m &= \left(\frac{c^2 r}{2k}\right)\frac{8\pi k}{c^4 r}\int_0^a T_0^0 r^2 dr \\ \Longrightarrow m &= \frac{4\pi}{c^2}\int_0^a T_0^0 r^2 dr \end{aligned}$$

which expresses the mass of the body in terms of its energy-momentum tensor.

Peculiarly, for a static distribution of matter we have that $T_0^0 = \varepsilon$, so that

$$m = \frac{4\pi}{c^2}\int_0^a \varepsilon r^2 dr \tag{2.323}$$

We call attention to the fact that the integration is taken with respect to $4\pi r^2 dr$, whereas the element of spatial volume for the metric (2.448) is $dV = 4\pi r^2 e^{\lambda/2}$, in which according to (2.319), $e^{\lambda/2} > 1$. This discrepancy indicates the gravitational mass defect of the body.

2.18 Schwarzchild's exterior solution

After deriving the solution for the field equations for a centrally symmetric reference system in vacuo, we proceed by considering a similar case. Next, we consider a system which is "static" and "spherically symmetrical", such that we redefine the functions provided in (2.448), where we describe our space-time metric, such that the coefficients of dr^2 and dt^2 are functions independent of time; hence,

$$ds^2 = adr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) - bc^2dt^2 \tag{2.324}$$

in which

$$a = a(r), \quad b = b(r) = 1 + \frac{2\chi}{c^2} \tag{2.325}$$

Subsequently, we obtain the components of the metric, with $x^i = (r, \theta, \phi, ct)$, as follows

$$g_{11} = a(r), \quad g_{22} = r^2, \quad g_{33} = r^2\sin^2\theta, \quad g_{44} = -b \tag{2.326}$$

Note that here the time-like component is denoted by the metric-component g_{44}, as oppose to (2.448) where it was denoted by g_{00}. Analogously, we have that only the diagonal components of the metric remain and point attention to the fact that the signature of the metric has slightly changed. The coordinate system is orthogonal; the non-vanishing components of the contravariant component of the metric tensor g^{ik} are as follows

$$\left.\begin{array}{c} g^{11} = \frac{1}{g_{11}} = \frac{1}{a}, \quad g^{22} = \frac{1}{g_{22}} = \frac{1}{r^2} \\ g^{33} = \frac{1}{g_{33}} = \frac{1}{r^2\sin^2\theta}, \quad g^{44} = \frac{1}{g_{44}} = -\frac{1}{b} \end{array}\right\} \tag{2.327}$$

After which using (2.80) we may calculate the corresponding Christoffel symbols in an equivalent manner as was demonstrated in section 2.17 following the guidelines provided, albeit, making a small alteration to the guidelines. Replacing condition 4 in the procedure provided in section 2.17 with the condition

- Any derivative with respect to time, i.e. x^4 vanishes. That is, any term $g_{\mu\nu,4} = 0$

we obtain the following Christoffel symbols,

$$\left.\begin{array}{l} \Gamma^1_{11} = \frac{a'}{2a}, \quad \Gamma^1_{22} = -\frac{r}{a}, \quad \Gamma^1_{33} = -\frac{r}{a}\sin^2\theta, \quad \Gamma^1_{44} = \frac{b'}{2a} \\ \Gamma^2_{12} = \frac{1}{r}, \quad \Gamma^2_{33} = -\sin\theta\cos\theta, \quad \Gamma^3_{13} = \frac{1}{r}, \quad \Gamma^3_{23} = \cot\theta, \quad \Gamma^4_{14} = \frac{b'}{2b} \end{array}\right\} \tag{2.328}$$

which is completely analogous to the Christoffel symbols derived in (2.278), with only the *time-dependent* terms not present in this rendering of the symbols. Similarly, we may then compute the corresponding Ricci tensors and scalars for the given situation and find that, as mentioned in section 2.17 we can derive four distinct equations, of which only three are independent. We redefine the left hand side of the equation as $R^k_i - \frac{1}{2}\delta^k_i R = G^k_i$, where we denote G^k_i as the Einstein tensor. Essentially, we determine that the left hand side of (2.230) for the diagonal elements $\mu = \nu = 1,2,3,4$

$$G^1_1 = -\frac{b'}{abr} + \frac{1}{r^2}\left(1 - \frac{1}{a}\right) - \lambda \tag{2.329}$$

$$G^4_4 = \frac{a'}{a^2 r} + \frac{1}{r^2}\left(1 - \frac{1}{a}\right) - \lambda \tag{2.330}$$

$$G^2_2 = G^3_3 = -\frac{1}{2a}\left[\left(\frac{b'}{b}\right)' - \frac{1}{2}\frac{a'}{a}\frac{b'}{b} + \frac{1}{2}\left(\frac{b'}{b}\right)^2 + \frac{b'}{br} - \frac{a'}{ar}\right] - \lambda \tag{2.331}$$

which are completely analogous to the derived equations in section 2.17, with the notable difference that the diagonal elements are functions of r only, as well as the substitution of $g_{00} = e^\nu \to g_{44} = -b$ and $g_{11} = -e^\lambda \to g_{11} = a(r)$, where the change in sign value is taken into account by the change in signature of the metric.

We proceed to deriving the solution of field equations in vacuo. In the empty space surrounding a material particle of mass M we have that $T^k_i = 0$ and the Einstein equations (2.230) reduce simply to

$$-\frac{b'}{abr}+\frac{1}{r^2}\left(1-\frac{1}{a}\right)-\lambda=0 \tag{2.332}$$

$$\frac{a'}{a^2r}+\frac{1}{r^2}\left(1-\frac{1}{a}\right)-\lambda=0 \tag{2.333}$$

$$-\frac{1}{2a}\left[\left(\frac{b'}{b}\right)'-\frac{1}{2}\frac{a'}{a}\frac{b'}{b}+\frac{1}{2}\left(\frac{b'}{b}\right)^2+\frac{b'}{br}-\frac{a'}{ar}\right]-\lambda=0 \tag{2.334}$$

Subtraction of (2.332) and (2.333), we obtain

$$\begin{aligned}
\frac{a'}{a^2r}+\frac{1}{r^2}\left(1-\frac{1}{a}\right)-\lambda-\left(-\frac{b'}{abr}+\frac{1}{r^2}\left(1-\frac{1}{a}\right)-\lambda\right)=0\\
\implies \frac{a'}{a^2r}+\frac{b'}{abr}=0\\
\implies \frac{a'}{a^2r}\frac{(b)}{(b)}+\frac{b'}{abr}\frac{(a)}{(a)}=0\\
\implies \frac{a'b}{a^2br}+\frac{b'a}{a^2br}\\
\implies \frac{a'b+ab'}{a^2br}=0\\
\implies \frac{(ab)'}{a^2br}=0
\end{aligned}$$

$$(ab)'=0 \tag{2.335}$$

rather,

$$\implies ab= \text{ constant} \tag{2.336}$$

Let

$$y=\frac{1}{a} \tag{2.337}$$

such that we may rewrite (2.333) as follows,

$$\frac{a'}{a^2r}+\frac{1}{r^2}\left(1-\frac{1}{a}\right)-\lambda=0$$

noting that

$$y = \frac{1}{a(r)} \to y' = -\frac{1}{a^2(r)} \left(\frac{d}{dr} a(r) \right) = -\frac{a'}{a^2(r)} \tag{2.338}$$

which follows from chain rule. We then write

$$-\frac{y'}{r} + \frac{1}{r^2}(1-y) - \lambda = 0$$

$$y'r - 1 + y + r^2\lambda = 0 \tag{2.339}$$

in which we multiplied by $-r^2$ through out the equation and used the fact that $y' = -\frac{a'}{a^2(r)} \to -y' = \frac{a'}{a^2(r)}$. Then,

$$y'r - 1 + y + r^2\lambda = 0$$
$$\implies y'r + y = 1 - r^2\lambda$$

$$(yr)' = 1 - r^2\lambda \tag{2.340}$$

where we used the fact that $r' = \frac{d}{dr} r = 1$, namely,

$$(yr)' = 1 - r^2\lambda$$
$$\implies y'r + yr' = 1 - r^2\lambda$$
$$\implies y'r + y(1) = 1 - r^2\lambda$$

which is equivalent to (2.339). Next, integration of both sides of (2.340) leads to

$$\int (yr)' = \int \left[1 - r^2\lambda\right]$$
$$\implies yr = r - \frac{\lambda r^3}{3} + c$$

after which setting our integration constant equal to $-2m$ we rewrite the above equation as

$$yr = r - \frac{\lambda r^3}{3} - 2m \tag{2.341}$$

or

$$y = 1 - \frac{2m}{r} - \frac{\lambda r^2}{3} \tag{2.342}$$

After which substitution by (2.337) we have,

$$\begin{aligned} y &= \frac{1}{a} \\ \Longrightarrow \frac{1}{a} &= 1 - \frac{2m}{r} - \frac{\lambda r^2}{3} \\ a &= \frac{1}{1 - \frac{2m}{r} - \frac{\lambda r^2}{3}} \end{aligned} \tag{2.343}$$

Then, substituting in the value attained into (2.324) we find for the spatial element of the metric

$$d\sigma^2 = \frac{dr^2}{1 - 2m/r - \lambda r^2/3} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{2.344}$$

Neglecting the λ-term, the spatial element will, as $r \to \infty$ reduce to the usual line element in Euclidean space, without the need of explicit boundary conditions at infinity. Rather, the result also follows due to the arbitrary nature of the general theory of relativity, which allows us to select the variable r such that the geometry on a surface $r =$constant coincides with the geometry of a sphere of radius r in Euclidean space. In actual space, (2.344) will not simply be a radial distance, rather the distance between two arbitrary points (r_1, θ, ϕ) and (r_2, θ, ϕ) measured with standard measuring-rods results in the relation

$$l = \int_{r_1}^{r_2} \left(1 - 2m/r - \lambda r^2/3\right)^{-\frac{1}{2}} dr \tag{2.345}$$

we do not elaborate on the rationale for the given relation between the two arbitrary points in space, since a detailed explanation has been provided in section 2.17.

Then, by (2.336), we obtain

$$b = \frac{\text{constant}}{a} = \text{constant} \times \left(1 - \frac{2m}{r} - \frac{\lambda r^2}{3}\right) \tag{2.346}$$

By a trivial change of the time-scale the constant may be made equal to 1, and thus we obtain the Schwarzchild exterior solution

$$ds^2 = \frac{dr^2}{1 - 2m/r - \lambda r^2/3} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) - \left(1 - \frac{2m}{r} - \frac{\lambda r^2}{3}\right)c^2dt^2 \tag{2.347}$$

then if we disregard the λ-term, which only has true significance for extensive values of r, we obtain,

$$ds^2 = \frac{dr^2}{1 - 2m/r} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) - \left(1 - \frac{2m}{r}\right)c^2dt^2 \tag{2.348}$$

which is completely analogous to the derived equation in section 2.17, i.e. (2.296). The expression must be valid outside a spherical distribution of matter, and since we have a singularity at a distance $r = r_0$, determined by

$$1 - \frac{2m}{r_0} = 0 \quad \rightarrow \quad r_0 = 2m \tag{2.349}$$

we deduce that the "radius" of the mass must be greater than the value determined by (2.349). Indeed, observe that if

$$\begin{gathered}
r_0 = 2m \implies 1 - \frac{2m}{r_0} = 0 \\
r_0 < 2m \implies 1 - \frac{2m}{r_0} = (-) \\
r_0 > 2m \implies 1 - \frac{2m}{r_0} = (+)
\end{gathered}$$

Note that the singularity determined at $r = r_0$ cannot be completely removed by the use of "static" coordinates, rather through the use "isotropic" coordinates r', θ, ϕ, t, defined by the transformations

$$r = r'\left(1 + \frac{m}{2r'}\right)^2, \quad r' = \frac{1}{2}\left\{\left(r^2 - 2mr\right)^{\frac{1}{2}} + r - m\right\} \tag{2.350}$$

may we modify the line element, such that it takes the form

$$ds^2 = \left(1 + \frac{m}{2r'}\right)^4 \left(dr'^2 + r'^2 d\theta^2 + r'^2 \sin^2\theta d\phi^2\right) - \frac{(1 - m/2r')^2}{(1 + m/2r')^2} c^2 dt^2 \tag{2.351}$$

Indeed, we consider only the differentials for dr and dt. To this, end we have by definition of a differential

$$dr = \frac{\partial r}{\partial r'} dr' + \frac{\partial r}{\partial t'} dt'$$

and

$$dt = \frac{\partial t}{\partial r'} dr' + \frac{\partial t}{\partial t'} dt'$$

Using the transformations provided by (2.350), we have,

$$\begin{aligned}
&\frac{\partial}{\partial r}\left[r'\left(1 + \frac{m}{2r'}\right)^2\right] dr' \\
\Longrightarrow\ &\left(\frac{\partial}{\partial r'} r'\right)\left(1 + \frac{m}{2r'}\right)^2 + r'\left(\frac{\partial}{\partial r'}\left(1 + \frac{m}{2r'}\right)^2\right) \\
\Longrightarrow\ &\left(1 + \frac{m}{2r'}\right)^2 + r' 2\left(1 + \frac{m}{2r'}\right)\left(\frac{\partial}{\partial r'}\left(1 + \frac{m}{2r'}\right)\right) \\
\Longrightarrow\ &\left(1 + \frac{m}{2r'}\right)^2 - \frac{m}{r'}\left(1 + \frac{m}{2r'}\right) \\
\Longrightarrow\ &1 + \frac{m}{r'} + \frac{m^2}{4r'^2} - \frac{m}{r'} - \frac{m^2}{2r'^2} \\
\Longrightarrow\ &1 - \frac{m^2}{4r'^2}
\end{aligned}$$

Then, using the relation $a^2 - b^2 = (a+b)(a-b)$, we have

$$
\begin{aligned}
1 - \frac{m^2}{4r'^2} &= \left(1 + \frac{m}{2r'}\right)\left(1 - \frac{m}{2r'}\right) \\
\implies dr &= \left(1 + \frac{m}{2r'}\right)\left(1 - \frac{m}{2r'}\right) dr' \\
\implies dr^2 &= \left(1 + \frac{m}{2r'}\right)^2 \left(1 - \frac{m}{2r'}\right)^2 dr'^2
\end{aligned}
$$

Consider the factor $\frac{1}{\left(1 - \frac{2m}{r}\right)}$, such that

$$
\begin{aligned}
1 - \frac{2m}{r} &= 1 - \frac{2m}{r'\left(1 + \frac{m}{2r'}\right)^2} \\
&\implies \frac{r'\left(1 + \frac{m}{2r'}\right)^2 - 2m}{r'\left(1 + \frac{m}{2r'}\right)^2} \\
&\implies \frac{r'\left(1 + \frac{m}{r'} + \frac{m^2}{4r'^2}\right) - 2m}{r'\left(1 + \frac{m}{2r'}\right)^2} \\
&\implies \frac{r' + m + \frac{m^2}{4r'} - 2m}{r'\left(1 + \frac{m}{2r'}\right)^2} \\
&\implies \frac{r' - m + \frac{m^2}{4r'}}{r'\left(1 + \frac{m}{2r'}\right)^2} \\
&\implies \frac{1 - \frac{m}{r'} + \frac{m^2}{4r'^2}}{\left(1 + \frac{m}{2r'}\right)^2} \\
&\implies \frac{\left(1 - \frac{m}{2r'}\right)^2}{\left(1 + \frac{m}{2r'}\right)^2}
\end{aligned}
$$

Hence, is we consider only the first term of (2.348), we write

$$
\begin{aligned}
ds^2 &= \frac{dr^2}{\left(1 - \frac{2m}{r}\right)} + \dots \\
\implies &= \frac{\left(1 + \frac{m}{2r'}\right)^2 \left(1 - \frac{m}{2r'}\right)^2 dr'^2}{\frac{\left(1 - \frac{m}{2r'}\right)^2}{\left(1 + \frac{m}{2r'}\right)^2}} \\
&\implies \left(1 + \frac{m}{2r'}\right)^4 dr'^2
\end{aligned}
$$

Then, for the time-like components of the line element we have,

$$ds^2 = \ldots - \left(1 - \frac{2m}{r}\right) c^2 dt^2$$

Yet, we have previously derived the value $\left(1 - \frac{2m}{r}\right)$, such that $\left(1 - \frac{2m}{r}\right) = \frac{\left(1 - \frac{m}{2r'}\right)^2}{\left(1 + \frac{m}{2r'}\right)^2}$; hence

$$ds^2 = \ldots - \frac{\left(1 - \frac{m}{2r'}\right)^2}{\left(1 + \frac{m}{2r'}\right)^2} c^2 dt^2 \tag{2.352}$$

Lastly, we need only consider the remaining terms of the space-time interval

$$ds^2 = \ldots + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) - \ldots \tag{2.353}$$

Hence, we have

$$\begin{aligned} r &= r' \left(1 + \frac{m}{2r'}\right)^2 \\ \Longrightarrow r^2 &= \left[r' \left(1 + \frac{m}{2r'}\right)^2\right]^2 \\ \Longrightarrow &= r'^2 \left(1 + \frac{m}{2r'}\right)^4 \end{aligned}$$

Thus, using the attained values we rewrite the line element as follows

$$\begin{aligned} ds^2 &= \left(1 + \frac{m}{2r'}\right)^4 dr'^2 + r'^2 \left(1 + \frac{m}{2r'}\right)^4 \left(d\theta^2 + \sin^2\theta d\phi^2\right) - \frac{\left(1 - \frac{m}{2r'}\right)^2}{\left(1 + \frac{m}{2r'}\right)^2} c^2 dt^2 \\ \Longrightarrow ds^2 &= \left(1 + \frac{m}{2r'}\right)^4 \left(dr'^2 + r'^2 d\theta^2 + r'^2 \sin^2\theta d\phi^2\right) - \frac{(1 - m/2r')^2}{(1 + m/2r')^2} c^2 dt^2 \end{aligned}$$

The singularity at $r_0 = 2m$ is redefined using the radial coordinate $r'_0 = \frac{m}{2}$, and we note that the singularity vanishes from the space-like components of the line element above, (2.351), yet it reappears in the time-like components of the line element, which vanishes at the same place. Upon further investigation we deduce that there exists no non-singular solutions of the Einstein equations for empty space which are stationary and have the form $g_{44} = -1 + \frac{\text{const}}{r}$ at infinity (as shown by Pauli, Einstein, and Serim). We shall then see that this from of the component g_{44} indicates that the field is produced by a mass distribution about the origin. The scalar potential which is invariant under the transformation (2.350) has the following forms in the two corresponding system of coordinates of (2.348) and (2.351)

$$\left.\begin{aligned} \phi &= (-g_{44} - 1)\tfrac{c^2}{2} = mc^2/r \\ \phi' &= (-g'_{44} - 1)\tfrac{c^2}{2} = -\tfrac{mc^2}{r'\left(1+\frac{m}{2r'}\right)^2} \end{aligned}\right\} \tag{2.354}$$

It follows that at arbitrarily large distances where we may consider the field to be weak, both expressions reduce to the Newtonian form $-mc^2/r = -mc^2/r'$, detailing that the constant m must then be connected with the mass of the particle creating the field by the expression

$$m = \frac{k \cdot M}{c^2} = \frac{\kappa c^2 M}{8\pi} \tag{2.355}$$

At protracted distances from the mass distribution creating the field, where we may consider the field weak, we have that the line element of "isotropic" coordinates reduces to the following form

$$ds^2 = \left(1 + \frac{2m}{r'}\right)\left(dx^2 + dy^2 + dz^2\right) - \left(1 - \frac{2m}{r'}\right)c^2dt^2 \tag{2.356}$$

where we have implemented the use of usual spherical coordinate transformation

$$x = r'\sin\theta\cos\phi, \quad y = r'\sin\theta\sin\phi, \quad z = r'\cos\theta \tag{2.357}$$

We may however eliminate the singularity delineated within the line element if we consider a *non-static* coordinate system r', θ, ϕ, t', as first remarked by Lemaître, by means of the transformations

$$r = (9m/2)^{\frac{1}{3}}\left(r' - ct'\right)^{\frac{2}{3}}, \quad dt' = dt - \frac{(2m/r)^{\frac{1}{2}}}{1 - 2m/r}dr \tag{2.358}$$

By which the line element (2.348) takes the following form

$$ds^2 = \frac{2m}{r} dr'^2 + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2 \right) - c^2 dt'^2 \tag{2.359}$$

in which r depends on r' and t' by (2.358).

We proceed as follows to prove the above equation. We commence with (2.359) and using the transformations provided by (2.358) we work in reverse order to obtain (2.348). Thus, we have

$$\begin{cases} \partial_t t' = 1 \\ \partial_r t' = \dfrac{\sqrt{2mr}}{2m-r} \\ \\ \\ \partial_t r' = 1 \\ \partial_r r' = \dfrac{r^{3/2}}{\sqrt{2m}(2m-r)} \end{cases}$$

Then

$$\begin{aligned}
ds^2 &= (dt')^2 - \frac{2m}{r}(dr')^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \\
&= \left(dt + \frac{\sqrt{2mr}}{2m-r} dr \right)^2 - \frac{2m}{r} \left(dt + \frac{r^{3/2}}{\sqrt{2m}(2m-r)} dr \right)^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \\
&= \left(dt + \frac{\sqrt{2mr}}{2m-r} dr \right)^2 - \left(\sqrt{\frac{2m}{r}} dt + \sqrt{\frac{2m}{r}} \frac{r^{3/2}}{\sqrt{2m}(2m-r)} dr \right)^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \\
&= \left(dt + \frac{\sqrt{2mr}}{2m-r} dr \right)^2 - \left(\sqrt{\frac{2m}{r}} dt + \frac{r}{2m-r} dr \right)^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \\
&= \left(1 - \frac{2m}{r} \right) dt^2 - \left\{ -\frac{2mr}{(2m-r)^2} + \frac{r^2}{(2m-r)^2} \right\} dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \\
&= \left(1 - \frac{2m}{r} \right) dt^2 - \left\{ -\frac{r}{(2m-r)} \right\} dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \\
&= \left(1 - \frac{2m}{r} \right) dt^2 - \left(1 - \frac{2m}{r} \right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2)
\end{aligned}$$

which is exactly (2.348).

Here, the new coordinate system does away with the dynamical action of the gravitational field, such that the dynamical potentials (γ_ι, Φ) vanish. The motion of a planet is determined in these coordinates, by the covariant components of the momentum vector of the particle at some time $t' + dt'$ obtained from the corresponding vector at time t' by parallel displacement with line element

$$d\sigma^2 = \frac{2m}{r} dr'^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

Due to a dependence in time of r, we find that the reference system is not rigid, causing the description of planetary motion in the above coordinates to be rather complex.

2.19 Schwarzschild's solution for the interior of a perfect fluid

The energy-momentum tensor of a perfect fluid is given by

$$T_i^k = \left(\mathring{\mu}^0 + \frac{\mathring{p}}{c^2}\right) U_\iota U^k + \mathring{p}\delta_i^k \tag{2.360}$$

in which we denote the contravariant components of the four-velocity by

$$U^\iota \equiv \frac{dx^\iota}{d\tau} = (\Gamma u^\iota, c\Gamma)$$

where $x^\iota = x^\iota(\tau)$ is the equation of the time track of the motion of a particle, $u^\iota = dx^\iota/dt$ are the contravariant components of the usual spatial velocity, τ denotes proper time and

$$\frac{1}{\Gamma} = \left[\left\{\left(1 + \frac{2\phi}{c^2}\right)^{\frac{1}{2}} - \frac{\gamma_\kappa u^\kappa}{c}\right\}^2 - \frac{u^2}{c^2}\right]^{\frac{1}{2}}$$

is the analog of the Lorentz factor in a gravitational field with dynamical potentials (Φ, γ_ι). Note that within a local system of inertia the above definition for the four velocity reduces to

$$U^\iota = \left(\frac{\mathbf{u}}{(1 - u^2/c^2)}, \frac{\iota c}{(1 - u^2/c^2)} \right) \tag{2.361}$$

which is the definition of four-velocity in a local system inertia, as delineated by the special theory of relativity, where

$$\mathbf{u} = \frac{d\mathbf{x}}{dt}$$

We briefly expand upon the basis of the derivation of (2.360). Consider the flow of momentum (or "flux") through a given volume V. We find that the flux of the momentum through the element d**f** of the surface of the volume is equivalent to the force acting on the surface of the element. Thence, $-\sigma_{\alpha\beta} df_\beta$ is the α-component of the force acting on the element. Selecting a system of reference system in which dV is at rest, yields the fact that Pascal's law holds, namely, the pressure p applied to dV is transmitted equally in all directions and is perpendicular to the surface on which it acts at all points. Thus, we can write $\sigma_{\alpha\beta} df_\beta = -p df_\alpha$, such that the stress tensor is $\sigma_{\alpha\beta} = -p\delta_{\alpha\beta}$. Indeed, consider the equation for pressure, i.e.

$$\vec{P} = \frac{\vec{F}}{A}$$

using the information allotted above we have that the force acting on the body under consideration is given by $-\sigma_{\alpha\beta} df_\beta$, while using the fact that Pascal's law holds for the case under consideration, given p as the pressure we may rewrite the above equation as follows:

$$\vec{P} = \frac{\vec{F}}{A}$$

$$\implies \vec{P}A = \vec{F}$$

$$\implies pA = -\sigma_{\alpha\beta} df_\beta$$

$$\implies -p df_\alpha = \sigma_{\alpha\beta} df_\beta$$

where df_α is the surface on which the force is acting upon. As for the components $T^{\alpha 0}$, which represent the momentum density, they vanish for the given volume element in the system of reference in consideration. From (2.204), the T^{00} component is the energy density, W, which for argument's sake we denote ε, then, if we multiply this quantity by a factor of $1/c^2$, we find that ε/c^2 is then the mass density of the body, i.e. the mass per unit volume. Again, we consider the unit "proper" volume, i.e. the volume in the reference system in which the given portion of body considered is at rest.

Hence, in the system of reference under question, the energy-momentum tensor takes the form

$$T^{ik} = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \tag{2.362}$$

In order to generalize the expression for the energy-momentum tensor in an arbitrary reference system we proceed as follows. We introduce the four-velocity u^ι for the macroscopic motion of an element of volume of the body. In the frame of reference in which an element is at rest we have $u^\iota = (1,0)$. The equation for T^{ik} must be chosen such that in the considered reference system it takes the form of (2.362). Thus, we find that the expression satisfying said conditions is

$$T^{ik} = (p+\varepsilon)u^\iota u^k - pg^{ik} \tag{2.363}$$

for mixed components

$$T_i^k = (p+\varepsilon)u_i u^k - p\delta_i^k \tag{2.364}$$

We verify such a claim. First, consider the indices $i = k = 0$, such that expression (2.363) takes the form.

$$T^{ik} = (p+\varepsilon)u^i u^k - pg^{ik}$$
$$\implies T^{00} = (p+\varepsilon)u^0 u^0 - pg^{00}$$
$$\implies T^{00} = (p+\varepsilon)(1)(1) - p(1)$$
$$\implies T^{00} = p+\varepsilon - p$$
$$\implies T^{00} = \varepsilon$$

where we used values of the Minkowskian metric $g^{ik} = (1,-1,-1,-1)$ for corresponding indices $i = k$ from 0-3. Next, for corresponding indices $i = k = 1$, we have

$$T^{ik} = (p+\varepsilon)u^i u^k - pg^{ik}$$
$$\implies T^{11} = (p+\varepsilon)u^1 u^1 - pg^{11}$$
$$\implies T^{11} = (p+\varepsilon)(0)(0) - p(-1)$$
$$\implies T^{11} = -p(-1)$$
$$\implies T^{11} = p$$

With remaining indices following analogously. We need not consider off-diagonal elements, since (2.362) was constructed in such a way that only diagonal elements are non-vanishing. Thus, (2.363) gives the expression for the energy-momentum tensor for a macroscopic body.

Then, dividing the (2.364) by a factor $1/c^2$ and setting the mass density $\varepsilon/c^2 = \mathring{\mu}^0$ and instead of the four-velocity u^i detailed by the special theory of relativity, we instead consider the corresponding four-velocity derived using the contravariant derivative (as per the general theory of relativity) we arrive at (2.360).

Then using the equations of gravitation (2.230),

$$R_i^k - \frac{1}{2}\delta_i^k R = \frac{8\pi k}{c^4} T_i^k$$

and recalling that we had previously defined the Einstein tensor as $G_i^k = R_i^k - \frac{1}{2}\delta_i^k R$, while setting the constant values of the right hand side equivalent to κ, i.e. $\frac{8\pi k}{c^4} = -\kappa$, so that we may arrive at

$$G_i^k = -\kappa T_i^k \tag{2.365}$$

we may use (2.329), (2.330), and (2.331) to deduce that a static spherically symmetric field denoted by (2.324) is possible only if we allow the velocity u^ι of the matter to be equivalently zero and if we set the proper mass density $\mathring{\mu}^0$ and the proper pressure $\mathring{p}$ to be function of r only.

Thus, we find

$$\begin{aligned} U^\iota &= \left(0, 0, 0, \frac{c}{\sqrt{b}}\right) \\ U_\iota &= g_{ik} U^k = (0, 0, 0, -c\sqrt{b}) \\ T_i^k &= -\left(\mathring{\mu}^0 + \frac{\mathring{p}}{c^2}\right) c^2 \delta_{i4}\delta_{k4} + \mathring{p}\delta_i^k \\ \mathring{\mu}^0 &= \mathring{\mu}^0(r), \quad \mathring{p} = \mathring{p}(r) \end{aligned} \tag{2.366}$$

Using the attained information above alongside the conservation law of energy and momentum, i.e.

$$T_{i,k}^k = \frac{1}{\sqrt{|g|}} \frac{\partial\left(\sqrt{|g|} T_i^k\right)}{\partial x^k} - \Gamma_{is}^r T_r^s = 0 \tag{2.367}$$

where we have simply employed the definition of the contravariant derivative to the energy-momentum tensor, i.e. (2.91), and noted that we may always replace the invariant volume element $\sqrt{-g}$ by $\sqrt{g}$, since the sign does not change the value when integrating accordingly. Then, for $i = 1$, we have

$$\frac{1}{\sqrt{|g|}} \frac{\partial}{\partial r}(\sqrt{|g|}\mathring{p}) + \Gamma_{14}^4 \left(\mathring{\mu}^0 c^2 + \mathring{p}^0\right) - \Gamma_{1r}^r \mathring{p} = 0 \tag{2.368}$$

Surely, we have setting $i = 1$ the following expression

$$T^k_{1,k} = \frac{1}{\sqrt{|g|}} \frac{\partial \left(\sqrt{|g|} T^k_1\right)}{\partial x^k} - \Gamma^r_{1s} T^s_r = 0$$

Now, since the energy-momentum tensor of a perfect fluid is diagonal, this means that the only non-vanishing term is given by $k = 1$, i.e.

$$\begin{aligned} T^1_{1,1} &= \frac{1}{\sqrt{|g|}} \frac{\partial \left(\sqrt{|g|} T^1_1\right)}{\partial x^1} - \Gamma^r_{1s} T^s_r = 0 \\ \implies T^1_{1,1} &= \frac{1}{\sqrt{|g|}} \frac{\partial \left(\sqrt{|g|} T^1_1\right)}{\partial r} - \Gamma^r_{1s} T^s_r = 0 \\ \implies T^1_{1,1} &= \frac{1}{\sqrt{|g|}} \frac{\partial \left(\sqrt{|g|} \mathring{p}\right)}{\partial r} - \Gamma^r_{1s} T^s_r = 0 \end{aligned}$$

Again, using the fact that the energy-momentum tensor is diagonal the only non-vanishing terms for the second term occur with coinciding indices $r = s$ such that

$$\implies T^1_{1,1} = \frac{1}{\sqrt{|g|}} \frac{\partial \left(\sqrt{|g|} \mathring{p}\right)}{\partial r} - \left(\Gamma^1_{11} T^1_1 + \Gamma^2_{12} T^2_2 + \Gamma^3_{13} T^3_3 + \Gamma^4_{14} T^4_4\right) = 0$$

Then, using the information allotted by (2.366), we write for the energy-momentum tensor T^4_4, making use of the properties of the Kronecker delta $\delta_{ik}=1$ if $i=k$ and 0 if $i\neq k$

$$
\begin{aligned}
T_i^k &= -\left(\mathring{\mu}^0+\frac{\mathring{p}}{c^2}\right)c^2\delta_{i4}\delta_{k4}+\mathring{p}\delta_i^k \\
\implies T_4^4 &= -\left(\mathring{\mu}^0+\frac{\mathring{p}}{c^2}\right)c^2\delta_{44}\delta_{44}+\mathring{p}\delta_4^4 \\
\implies T_4^4 &= -\left(\mathring{\mu}^0+\frac{\mathring{p}}{c^2}\right)c^2(1)(1)+\mathring{p}(1) \\
\implies T_4^4 &= -\mathring{\mu}^0c^2-\mathring{p}+\mathring{p} \\
\implies T_4^4 &= -\left(\mathring{\mu}^0c^2+\mathring{p}^0\right)
\end{aligned}
$$

Then inserting the attained value into the previous expression we find

$$
\begin{aligned}
\implies T_{1,1}^1 &= \frac{1}{\sqrt{|g|}}\frac{\partial\left(\sqrt{|g|}\mathring{p}\right)}{\partial r}-\left(\Gamma_{11}^1T_1^1+\Gamma_{12}^2T_2^2+\Gamma_{13}^3T_3^3-\Gamma_{14}^4\left(\mathring{\mu}^0c^2+\mathring{p}^0\right)\right)=0 \\
\implies T_{1,1}^1 &= \frac{1}{\sqrt{|g|}}\frac{\partial\left(\sqrt{|g|}\mathring{p}\right)}{\partial r}+\Gamma_{14}^4\left(\mathring{\mu}^0c^2+\mathring{p}^0\right)-\left(\Gamma_{11}^1T_1^1+\Gamma_{12}^2T_2^2+\Gamma_{13}^3T_3^3\right)=0 \\
\implies T_{1,1}^1 &= \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial r}(\sqrt{|g|}\mathring{p})+\Gamma_{14}^4\left(\mathring{\mu}^0c^2+\mathring{p}^0\right)-\Gamma_{1r}^r\mathring{p}=0
\end{aligned}
$$

where we have set the remaining terms for the Christoffel symbols equal to Γ^r_{1r}, i.e. $\Gamma^r_{1r}=\Gamma^1_{11}+\Gamma^2_{12}+\Gamma^3_{13}$ and have made use of the fact that the remaining terms of the energy-momentum tensor are all equivalent to the pressure of the fluid, i.e. $T_1^1=T_2^2=T_3^3=\mathring{p}$. Next, by means of (2.328) and using the definition,

$$
\Gamma_{ik}^k=\frac{1}{2g}\frac{\partial g}{\partial x^i}=\frac{\partial}{\partial x^i}\ln\sqrt{|g|} \tag{2.369}
$$

we have,

$$
\frac{d\mathring{p}}{dr}+\left(\mathring{\mu}^0c^2+\mathring{p}\right)\frac{b'}{2b}=\frac{d\mathring{p}}{dr}+\frac{\mathring{\mu}^0+\mathring{p}/c^2}{1+2\phi/c^2}\frac{d\phi}{dr}=0 \tag{2.370}
$$

Indeed, consider the following

$$\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial r}(\sqrt{|g|}\mathring{p})+\Gamma^4_{14}\left(\mathring{\mu}^0c^2+\mathring{p}^0\right)-\Gamma^r_{1r}\mathring{p}=0$$

using the fact that $\sqrt{|g|}$ is independent of r, being a scalar, we may extract it from the parentheses in the first term, following the same course of logic we have that from (2.369) the third term vanishes accordingly, since we may always choose a coordinate system such that $\partial/\partial x^i\sqrt{|g|}=0$, hence,

$$\begin{aligned}\frac{1}{\sqrt{|g|}}\left(\sqrt{|g|}\right)\frac{\partial}{\partial r}(\mathring{p})+\Gamma^4_{14}\left(\mathring{\mu}^0c^2+\mathring{p}^0\right)-0=0\\ \implies \frac{\partial\mathring{p}}{\partial r}+\Gamma^4_{14}\left(\mathring{\mu}^0c^2+\mathring{p}^0\right)=0\\ \implies \frac{d\mathring{p}}{dr}+\left(\mathring{\mu}^0c^2+\mathring{p}\right)\frac{b'}{2b}=0\\ \implies \frac{d\mathring{p}}{dr}+\frac{\left(\mathring{\mu}^0c^2+\mathring{p}\right)}{2\left(1+\frac{2\phi}{c^2}\right)}\frac{2}{c^2}\frac{d\phi}{dr}=0\\ \implies \frac{d\mathring{p}}{dr}+\frac{\left(\mathring{\mu}^0+\frac{\mathring{p}}{c^2}\right)}{\left(1+\frac{2\phi}{c^2}\right)}\frac{d\phi}{dr}=0\end{aligned}$$

The physical significance of the above equation is as follows: (2.370) details the dependence of the pressure on the scalar field potential ϕ in the equilibrium state of a fluid, since we have that the equation is equivalently zero and the energy-momentum tensor denotes a perfect fluid, under the influence of its own gravitational field. We find that the remaining equations of (2.367) to give no novel information, hence, they are disregarded.

Furthermore, the field equations given by (2.365) reduce to only two independent equations from which may take the equations

$$G^1_1=-\kappa T^1_1,\quad G^4_4=-\kappa T^4_4 \tag{2.371}$$

the remaining equations of (2.365) being results of (2.371) on account of the conservation expression denoted by (2.370).

Then from equations (2.329), (2.330), (2.366) and (2.371) we obtain

$$\frac{b'}{abr} - \frac{1}{r^2}\left(1 - \frac{1}{a}\right) + \lambda = \kappa \mathring{p} \tag{2.372}$$

$$\frac{a'}{a^2 r} + \frac{1}{r^2}\left(1 - \frac{1}{a}\right) - \lambda = \kappa \mathring{\mu}^0 c^2 \tag{2.373}$$

The equations (2.370), (2.372), and (2.373) together with the equation of the state of the matter, i.e. $\kappa\left(\mathring{p} + \mathring{\mu}^0 c^2\right)$, give the connection between the proper pressure and the proper mass density, $\mathring{p}$ and $\mathring{\mu}^0$, respectively, which determine the interior state and the gravitational field of the fluid.

For the sake of simplicity, we assume that the fluid under consideration is practically incompressible, meaning that the mass density of the fluid remains constant in space and in time. Thus, we treat the proper mass density as a constant and the solution of (2.373) may be obtained from the solution of (2.343) of (2.333) through the substitution $\lambda \to \lambda + \kappa \mu^0 c^2$. We require that our solution be regular as $r \to 0$, i.e. there is no singularity as $r \to 0$, then the constant of integration $2m$ in (2.343) must be set equal to zero. Thus, we obtain

$$a = \frac{1}{1 - \frac{\lambda + \kappa \mathring{\mu}^0 c^2}{3} r^2} = \frac{1}{1 - \frac{r^2}{R^2}} \tag{2.374}$$

where we set

$$R^2 = \frac{3}{\lambda + \kappa \mathring{\mu}^0 c^2} \tag{2.375}$$

Further since we assumed an incompressible fluid and set the proper mass density as a constant, integrating (2.370), yields

$$\left(\mathring{\mu}^0 c^2 + \mathring{p}\right)\sqrt{b} = \text{ const}$$

By taking a sum of equations (2.372) and (2.373) and multiplication by $\sqrt{b}$, we obtain

$$\begin{aligned}
&\frac{b'}{abr}-\frac{1}{r^2}\left(1-\frac{1}{a}\right)+\lambda=\kappa\mathring{p}\\
&+\frac{a'}{a^2r}+\frac{1}{r^2}\left(1-\frac{1}{a}\right)-\lambda=\kappa\mathring{\mu}^0c^2\\
&\Longrightarrow\ \frac{b'}{abr}+\frac{a'}{a^2r}=\kappa\mathring{p}+\kappa\mathring{\mu}^0c^2\\
&\Longrightarrow\ \frac{b'}{abr}+\frac{a'}{a^2r}=\kappa\left(\mathring{\mu}^0c^2+\mathring{p}\right)\\
&\Longrightarrow\ \frac{b'\sqrt{b}}{abr}+\frac{a'\sqrt{b}}{a^2r}=\kappa\left(\mathring{\mu}^0c^2+\mathring{p}\right)\sqrt{b}\\
&\Longrightarrow\ \frac{b'}{a\sqrt{b}r}+\frac{a'\sqrt{b}}{a^2r}=\text{comst}
\end{aligned}$$

Inserting the attained value in the expression for a, given by (2.374) we obtain

$$\sqrt{b}+R^2\frac{1-r^2/R^2}{r}\frac{d\sqrt{b}}{dr}=A \tag{2.376}$$

in which we set A a constant. Letting $y=\sqrt{b}$ and introducing the a new variable $x=\left(1-r^2/R^2\right)^{\frac{1}{2}}$ instead of r, (2.376) may be rewritten as

$$y-x\frac{dy}{dx}=A \tag{2.377}$$

The solution of which is

$$y=A-Bx$$

where B is an integration constant. Indeed, starting with (2.377), we proceed in the reverse direction to prove the validity of the above claim. Thus,

$$\begin{aligned}
&y-x\frac{dy}{dx}=A\\
&\Longrightarrow\ y-xy'=A\\
&\Longrightarrow\ y-A=xy'
\end{aligned}$$

Let $u = y - A$ such that $u + A = y$ and $u' = y'$. Hence,

$$
\begin{aligned}
& u = xu' \\
\implies & \int \frac{1}{x} dx = \int \frac{du}{u} \\
\implies & \ln|x| + C = \ln|u| \\
\implies & e^{\ln|x|+C} = e^{\ln|u|} \\
\implies & |x|e^{C} = |u| \\
\implies & Cx = u \\
\implies & Cx = y - A \\
\implies & y = A + Cx \\
\implies & y = A - Bx
\end{aligned}
$$

where we proceeded to using the method of separation of variables to verify the above claim and noted that C is an arbitrary constant of integration which may assume any appropriate value we choose.

Hence,

$$
b = y^2 = \left(A - B\sqrt{(1 - r^2/R^2)} \right)^2 \tag{2.378}
$$

Ultimately, using the expressions defined by (2.374) and (2.378) with (2.372), we obtain the expression for the pressure $\mathring{p}$ measured in a local system of inertia to equal

$$
\kappa \mathring{p} = \frac{3B\sqrt{(1 - r^2/R^2)} - A}{R^2 \left\{ A - B\sqrt{(1 - r^2/R^2)} \right\}} + \lambda \tag{2.379}
$$

Indeed, consider (2.374), (2.378) and (2.372) such that,

$$
\frac{b'}{abr} - \frac{1}{r^2}\left(1 - \frac{1}{a}\right) + \lambda = \kappa \mathring{p} \tag{2.380}
$$

Then, consider first (2.378) such that

$$
\begin{aligned}
b' &= \frac{d}{dr}\left(A - B\sqrt{(1 - r^2/R^2)}\right)^2 \\
\implies &= 2\left(A - B\sqrt{\left(1 - \frac{r^2}{R^2}\right)}\right)\frac{d}{dr}\left(A - B\sqrt{\left(1 - \frac{r^2}{R^2}\right)}\right) \\
\implies &= 2\left(A - B\sqrt{\left(1 - \frac{r^2}{R^2}\right)}\right)\frac{d}{dr}\left(-B\sqrt{\left(1 - \frac{r^2}{R^2}\right)}\right) \\
\implies &= 2\left(A - B\sqrt{\left(1 - \frac{r^2}{R^2}\right)}\right)\left(-B\left(\frac{1}{2}\right)\left(1 - \frac{r^2}{R^2}\right)^{-1/2}\right)\left(-\frac{2r}{R^2}\right) \\
\implies &= \frac{\left(A - B\sqrt{1 - \frac{r^2}{R^2}}\right)2Br}{R^2\left(1 - \frac{r^2}{R^2}\right)^{1/2}}
\end{aligned}
$$

Then, we have

$$
a = \frac{1}{1 - \frac{r^2}{R^2}} \implies \frac{1}{a} = \left(1 - \frac{r^2}{R^2}\right)
$$

$$
b = \left(A - B\sqrt{(1 - r^2/R^2)}\right)^2 \implies \frac{1}{b} = \left(\frac{1}{\left(A - B\sqrt{(1 - r^2/R^2)}\right)^2}\right)
$$

Thus,

$$
\begin{aligned}
\frac{b'}{abr} &= \frac{\left(A - B\sqrt{1 - \frac{r^2}{R^2}}\right)2Br}{R^2\left(1 - \frac{r^2}{R^2}\right)^{1/2}}\left(\frac{\left(1 - \frac{r^2}{R^2}\right)}{\left(A - B\sqrt{(1 - r^2/R^2)}\right)^2}\right) \\
&= \frac{2B\left(1 - \frac{r^2}{R^2}\right)^{1/2}}{R^2\left(A - B\sqrt{\left(1 - \frac{r^2}{R^2}\right)}\right)}
\end{aligned}
$$

Next, for the second term, we have

$$
\begin{aligned}
-\frac{1}{r^2}\left(1-\frac{1}{a}\right) &= -\frac{1}{r^2}\left(1-\left(1-\frac{r^2}{R^2}\right)\right)\\
&= -\frac{1}{r^2}+\frac{1}{r^2}\left(1-\frac{r^2}{R^2}\right)\\
&= -\frac{1}{r^2}+\frac{1}{r^2}\left(1-\frac{r^2}{r^2R^2}\right)\\
&= -\frac{1}{R^2}
\end{aligned}
$$

Combining the attained expression we find

$$
\begin{aligned}
&\frac{2B\left(1-\frac{r^2}{R^2}\right)^{1/2}}{R^2\left(A-B\sqrt{\left(1-\frac{r^2}{R^2}\right)}\right)}-\frac{1}{R^2}\\
\implies &\frac{2B\left(1-\frac{r^2}{R^2}\right)^{1/2}}{R^2\left(A-B\sqrt{\left(1-\frac{r^2}{R^2}\right)}\right)}-\frac{1}{R^2}\frac{\left(A-B\sqrt{\left(1-\frac{r^2}{R^2}\right)}\right)}{\left(A-B\sqrt{\left(1-\frac{r^2}{R^2}\right)}\right)}\\
\implies &\frac{2B\sqrt{1-\frac{r^2}{R^2}}+B\sqrt{1-\frac{r^2}{R^2}}-A}{R^2(A-B\sqrt{1-\frac{r^2}{R^2}})}\\
\implies &\frac{3B\sqrt{1-\frac{r^2}{R^2}}-A}{R^2(A-B\sqrt{1-\frac{r^2}{R^2}})}
\end{aligned}
$$

which is what is given on the right hand side of (2.379), after which reintroducing the λ term, we have exactly the expression for (2.379)

Further, substitution of the attained values into (2.324) gives the Schwarzchild interior solution

$$
ds^2 = \frac{dr^2}{1-r^2/R^2}+r^2\left(d\theta^2+\sin^2\theta d\phi^2\right)-\left\{A-B\sqrt{(1-r^2/R^2)}\right\}^2 c^2dt^2 \qquad (2.381)
$$

where the spatial line element is given by the terms dependent on space only, which detail the spatial geometry of the system

$$d\sigma^2 = \frac{dr^2}{1-r^2/R^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{2.382}$$

It follows that the geometry on the surface of $r = r_1 = const$ is equivalent to that of the surface of a sphere of radius r_1 in Euclidean space, nonetheless, r_1 is not the distance to the origin $r_0 = 0$ measured by standard rods, but rather the distance is given by

$$l_1 = \int_0^{r_1} \frac{dr}{\sqrt{(1-r^2/R^2)}} = R\sin^{-1}\frac{r_1}{R} = r_1\left(1 + \frac{1}{6}\left(\frac{r_1}{R}\right)^2 + \frac{3}{40}\left(\frac{r_1}{R}\right)^4 + \dots\right) \tag{2.383}$$

Indeed, consider the following integral,

$$l_1 = \int_0^{r_1} \frac{dr}{\sqrt{(1-r^2/R^2)}}$$

substitute $r = R\sin(u)$ and $dr = R\cos(u)du$. Then, we have

$$\sqrt{1-\frac{r^2}{R^2}} = \sqrt{1-\sin^2(u)} = \cos(u)$$

and $u = \sin^{-1}(\frac{r}{R})$. Thus, we have

$$\int_0^{r_1} \frac{dr}{\sqrt{(1-r^2/R^2)}} = R\int_0^{r_1} 1du$$
$$\implies = Ru + C$$

substituting back for $u = \sin^{-1}(\frac{r}{R})$, we obtain,

$$= R\sin^{-1}(\frac{r_1}{R})$$

Then applying a Taylor Series of $\sin^{-1}(\frac{r_1}{R})$, we obtain,

$$
\begin{aligned}
R\sin^{-1}(\frac{r_1}{R}) &= R\left(\left(\frac{r_1}{R}\right)+\frac{1}{6}\left(\frac{r_1}{R}\right)^3+\frac{3}{40}\left(\frac{r_1}{R}\right)^5+\dots\right)\\
\implies &= R\left(\frac{r_1}{R}\right)\left(1+\frac{1}{6}\left(\frac{r_1}{R}\right)^2+\frac{3}{40}\left(\frac{r_1}{R}\right)^4+\dots\right)\\
\implies &= r_1\left(1+\frac{1}{6}\left(\frac{r_1}{R}\right)^2+\frac{3}{40}\left(\frac{r_1}{R}\right)^4+\dots\right)
\end{aligned}
$$

Thus, we obtain (2.383).

Then, for the volume of the sphere we have,

$$V_1 = \int_0^{r_1}\int_0^{\pi}\int_0^{2\pi}\sqrt{\gamma}dr d\theta d\phi = \iiint \frac{r^2\sin\theta}{\sqrt{(1-r^2/R^2)}}dr d\theta d\phi = 4\pi\int_0^{r_1}\frac{r^2 dr}{\sqrt{(1-r^2/R^2)}}$$

$$\text{i.e. } V_1 = \frac{4\pi R^3}{2}\left[\sin^{-1}\frac{r_1}{R}-\frac{r_1}{R}\sqrt{\left(1-\frac{r_1^2}{R_1^2}\right)}\right] = \frac{4\pi r_1^3}{3}\left[1+\frac{3}{10}\left(\frac{r_1}{R}\right)^2+\dots\right] \tag{2.384}$$

Consider a fluid filling a sphere in the space inside $r = r_1$ with constant mass density $\mathring{\mu}^0$. We have that for $r < r_1$ the interior solution (2.381) is satisfied, while for $r > r_1$ the Schwarzchild exterior solution (2.347) proved valid, since if $r < r_1$ then we have that the fluid resides within the sphere, hence, we use solution (2.381) for the "interior", while if $r > r_1$ we then consider the exterior solution, (2.347), as the fluid "leaves" or resides outside of the sphere. Next, we modify the constant A and B in (2.381), such that both solutions coincide for $r = r_1$, furthermore, we note that proper pressure must be set to zero at the surface of the sphere. If we disregard the λ-term, which exhibits a minute effect inside of the solar system and is significant only for large values of r, we have that the conditions imply

$$
\begin{aligned}
1-\tfrac{2m}{r_1} = 1-\tfrac{r_1^2}{R^2} &= \left\{A-B\sqrt{\left(1-r_1^2/R^2\right)}\right\}^2\\
3B\sqrt{\left(1-r_1^2/R^2\right)}-A = 0, &\quad R^2 = \tfrac{3}{\kappa\mathring{\mu}^0 c^2}
\end{aligned}
$$

which have solutions

$$\left.\begin{array}{c} A = \frac{3}{2}\sqrt{\left(1 - r_1^2/R^2\right)}, \quad B = \frac{1}{2}, \quad R^2 = \frac{3}{\kappa \mathring{\mu}^0 c^2} \\ m = \frac{r_1^3}{2R^2} = \frac{\kappa c^2 \mathring{\mu}^0}{6} r_1^3 = \frac{k\mathring{\mu}^0}{c^2}\frac{4\pi}{3} r_1^3 \end{array}\right\} \tag{2.385}$$

Starting with (2.379), we consider

$$\kappa \mathring{p} = \frac{3B\sqrt{(1-r^2/R^2)} - A}{R^2\left\{A - B\sqrt{(1-r^2/R^2)}\right\}} + \lambda$$

Neglecting the λ term and setting $\mathring{p} = 0$ at the surface of the sphere, we may write,

$$\implies 0 = \frac{3B\sqrt{(1-r^2/R^2)} - A}{R^2\left\{A - B\sqrt{(1-r^2/R^2)}\right\}} + 0$$
$$\implies 0 = 3B\sqrt{(1-r^2/R^2)} - A$$

for (2.375), since $\lambda = 0$, we write

$$R^2 = \frac{3}{\lambda + \kappa \mathring{\mu}^0 c^2}$$
$$\implies R^2 = \frac{3}{\kappa \mathring{\mu}^0 c^2}$$

which is equivalent to the value displayed by (2.385). We proceed to finding the solutions,

$$3B\sqrt{(1-r^2/R^2)} - A = 0$$
$$\implies 3B\sqrt{(1-r^2/R^2)} = A$$

Then, using the relations delineated by the aforementioned equations, we have,

$$
\begin{aligned}
1-\frac{r_1^2}{R^2} &= \left\{A-B\sqrt{\left(1-r_1^2/R^2\right)}\right\}^2 \\
\implies \sqrt{1-\frac{r_1^2}{R^2}} &= A-B\sqrt{\left(1-r_1^2/R^2\right)} \\
\implies \sqrt{1-\frac{r_1^2}{R^2}}+B\sqrt{\left(1-r_1^2/R^2\right)} &= A \\
\implies \sqrt{1-\frac{r_1^2}{R^2}}+B\sqrt{\left(1-r_1^2/R^2\right)} &= 3B\sqrt{(1-r^2/R^2)} \\
\implies \sqrt{1-\frac{r_1^2}{R^2}} &= 2B\sqrt{(1-r^2/R^2)} \\
\implies B &= \frac{\sqrt{1-\frac{r_1^2}{R^2}}}{2\sqrt{1-\frac{r_1^2}{R^2}}} \\
\implies B &= \frac{1}{2}
\end{aligned}
$$

Then, plugging in the attained values into A, we have

$$
\begin{aligned}
A &= 3B\sqrt{(1-r^2/R^2)} \\
\implies A &= \frac{3}{2}\sqrt{(1-r^2/R^2)}
\end{aligned}
$$

Finally, for m, we find

$$
\begin{aligned}
1-\frac{2m}{r_1} &= 1-\frac{r_1^2}{R^2} \\
\implies \frac{2m}{r_1} &= \frac{r_1^2}{R^2} \\
\implies m = \frac{r_1^3}{2R^2} &= \frac{r_1^3}{(2)\,(3)}\kappa\mathring{\mu}^0c^2 = \frac{r_1^3}{6}\kappa\mathring{\mu}^0c^2
\end{aligned}
$$

where we multiplied both sides by $\frac{r_1}{2}$ and substituted in the attained value for R^2.

Referring back to (2.355), we note that the formulation provided by (2.385) denotes that the gravitational field of the spherical fluid at large distances corresponds to a mass,

$$M = \frac{4\pi}{3} r_1^3 \mathring{\mu}^0 \tag{2.386}$$

which is equivalent to that of a constant distribution of Newtonian mass over a sphere of radius r_1 in a given Euclidean space.

By (2.384) we find that the volume of a sphere is larger than that of its Euclidean counterpart, i.e. $\frac{4\pi}{3} r_1^3$, while the proper mass density $\mathring{\mu}^0$ which was measured in a local system of inertia differs from the attained value in the selected system of coordinates. We deduce that the deviation of the volume V_1 attained, given by (2.384), and that of Euclidean space varies by a small amount within an astronomical perspective. We introduce here without proof that in the case of the sun, we may set

$$\mathring{\mu}^0 = 1.4 \frac{\text{gm}}{\text{cm}^3}, \qquad r_1 = 6.95\text{X}10^{10}\text{cm} \tag{2.387}$$

Thus, we obtain

$$\begin{gathered} R = 3.5\text{x}10^{13}\text{cm} \\ \tfrac{r_1}{R} \approx 2\text{x}10^{-3} \\ \tfrac{V_1 - 4\pi r_1^3/3}{4\pi r_1^3/3} \approx 10^{-6} \end{gathered} \tag{2.388}$$

It is found that the difference between the distance l_1 defined by (2.383) and the radial coordinate r_1 is undetectable by the astronomical determination of r_1. We also find that the condition $r_1 > r_0 = 2m$ for the applicability of the exterior solution (2.348) is satisfied, since from (2.385) details the following:

$$\frac{2m}{r_1} = \frac{r_1^2}{R^2} \approx 10^{-6} < 1$$

From (2.385) we have

$$
\begin{aligned}
m &= \frac{r_1^3}{2R^2} \\
\implies \left(\frac{2}{r_1}\right) m &= \left(\frac{2}{r_1}\right) \frac{r_1^3}{2R^2} \\
\implies \frac{2m}{r_1} &= \frac{r_1^2}{R^2}
\end{aligned}
$$

and by (2.387) and (2.388), we have,

$$
\begin{aligned}
\frac{r_1}{R} &\approx .0019857 \\
\implies \frac{r_1^2}{R^2} &\approx 3.94\text{x}10^{-6} \to 10^{-6} < 1
\end{aligned}
$$

We proceed by introducing isotropic coordinates, similar to the case of empty space, such that we may define arbitrary function $a(r)$ and $b(r)$ in (2.324) by means of the transformation $r' = r'(r)$ satisfying the differential equation

$$\frac{dr'}{r'} = \sqrt{a(r)}\frac{dr}{r} \tag{2.389}$$

which has general solution

$$r' = C\exp\left(\int^r \frac{\sqrt{a(r)}dr}{r}\right) \tag{2.390}$$

Indeed, starting with (2.389) and proceeding by means of the method of separation of variables, we find

$$
\begin{aligned}
\frac{dr'}{r'} &= \sqrt{a(r)}\frac{dr}{r} \\
\implies \int^{r'} \frac{dr'}{r'} &= \int^{r} \sqrt{a(r)}\frac{dr}{r} \\
\implies \ln|r'| &= \int^{r} \sqrt{a(r)}\frac{dr}{r} + C \\
\implies e^{\ln|r'|} &= \exp\left(\int^{r} \sqrt{a(r)}\frac{dr}{r} + C\right) \\
\implies |r'| &= \exp\left(\int^{r} \sqrt{a(r)}\frac{dr}{r}\right) e^{C} \\
\implies r' &= C\exp\left(\int^{r} \sqrt{a(r)}\frac{dr}{r}\right)
\end{aligned}
$$

where C is an arbitrary constant, further, we assume $e^C > 0$. The line element then transforms such that we obtain

$$
\begin{aligned}
ds^2 &= \tfrac{r^2}{r'^2}\left[dr'^2 + r'^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) - bc^2dt^2\right] \\
&= \tfrac{r^2}{r'^2}\left(dx^2 + dy^2 + dz^2\right) - bc^2dt^2
\end{aligned}
\tag{2.391}
$$

in which (x, y, z) are all given by (2.357)

For $a(r)$ of the form (2.374) we thus obtain,

$$
r' = C\frac{r/R}{1+\sqrt{(1-r^2/R^2)}}, \quad r = r'\frac{2CR}{C^2+r'^2} \tag{2.392}
$$

with line element for an incompressible fluid taking the form

$$
ds^2 = \frac{4C^2R^2}{\left(C^2+r'^2\right)^2}\left(dx^2+dy^2+dz^2\right) - \left[A - B\frac{C^2-r'^2}{C^2+r^2}\right]^2 c^2dt^2 \tag{2.393}
$$

The constant C is to be determined in such a way as to make (2.392) coincide with (2.350) at the boundary of the fluid.

Finally, we delineate that the Schwarzschild solution represents the only exact solution of the field equations in vacuo which has found any true application from an astronomical perspective.

We introduce, without proof, that Weyl and Reissner solved the problem of the gravitational field produced by the electromagnetic energy in the surrounding of a charged particle. The result obtained is a line element which takes the form

$$ds^2 = \frac{dr^2}{1 - 2m/r - \kappa e^2/r^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) - \left(1 - \frac{2m}{r} - \frac{\kappa e^2}{r^2}\right)c^2 dt^2 \tag{2.394}$$

The ratio of the two terms depending on the charge and the mass, respectively, is thus, by (2.355)

$$\frac{\kappa e^2}{2mr} = \frac{4\pi e^2}{rMc^2} \tag{2.395}$$

If we consider an electron, the aforementioned terms above will therefore become of the same magnitude at a distance corresponding to the classical electron radius $a = e^2/Mc^2$. Nonetheless, both terms will have negligible effects on the interaction between electrons as compared with the Coulomb interaction.

2.20 Cosmological Models

We delineate the consequences brought about by the equations presented in section 2.19, in which we considered a perfect fluid under static, homogeneous and isotropic assumptions.

Upon gazing at the heavens, we find that it is true that most of the matter within the observable universe is partly aggregated into stars which have a tendency to cluster into nebulae of the same character as our own galaxy. Yet, in the portion of space which may be astronomically observed, these nebulae seem to be to fairly uniformly distributed, leading to the sensible assumption that at large scales, the properties of the universe may be properly described by treating the matter as a perfect homogeneous fluid. In the models proposed by Einstein and de Sitter the universe is assumed to be static in nature, leading to the fact that we may introduce a system of coordinates

$x^\iota = (r, \theta, \phi, ct)$ in which the line element takes the static and spherically symmetric form

$$ds^2 = a(r)dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) - b(r)c^2dt^2 \tag{2.396}$$

(referenced in (2.324)), where a and b are simply functions of the r coordinate. On the basis of assumption of the homogeneity of the universe any point in space may be taken as the origin $r = 0$ of the spatial system of coordinates.

We see that the functions $a(r)$ and $b(r)$ are connected to the proper mass density $\mathring{\mu}^0$ and the proper pressure $\mathring{p}$ in the universe by the field equations (2.372), (2.373) & (2.370) for a perfect fluid

$$\frac{d\mathring{p}}{dr} + \left(\mathring{\mu}^0c^2 + \mathring{p}\right)\frac{b'}{2b} = 0 \tag{2.397}$$

$$\frac{b'}{abr} - \frac{1}{r^2}\left(1 - \frac{1}{a}\right) + \lambda = \kappa\mathring{p} \tag{2.398}$$

$$\frac{a'}{a^2r} + \frac{1}{r^2}\left(1 - \frac{1}{a}\right) - \lambda = \kappa\mathring{\mu}^0c^2 \tag{2.399}$$

Here, $\mathring{p}$ and $\mathring{\mu}^0$ are constants due to the assumed homogeneity of the model of the universe and we look for possible regular solutions of these equations.

Since, by assumption we have that space-time, i.e. the universe, is homogeneous throughout, we have that the pressure and mass density are equivalent for all points in space-time, thus, the change of the proper pressure with respect to the radial coordinate r, must be equivalently zero, $d\mathring{p}/dr = 0$, thus leading to the reduction of (2.397) as follows ,

$$\left(\mathring{\mu}^0c^2 + \mathring{p}\right)b' = 0$$

which follows from,

$$\frac{d\mathring{p}}{dr} + \left(\mathring{\mu}^0 c^2 + \mathring{p}\right)\frac{b'}{2b} = 0$$
$$0 + \left(\mathring{\mu}^0 c^2 + \mathring{p}\right)\frac{b'}{2b} = 0$$
$$\left(\mathring{\mu}^0 c^2 + \mathring{p}\right)\frac{b'}{2b} = 0$$
$$\left(\mathring{\mu}^0 c^2 + \mathring{p}\right) b' = 0$$

where we multiplied both side by $(2b)$. Now, we have two explicit conditions that satisfy (2.397), either

$$b' = 0 \tag{2.400}$$

or

$$\mathring{\mu}^0 c^2 + \mathring{p} = 0 \tag{2.401}$$

These two alternatives detail distinct solutions to the field equations, the Einstein and de Sitter, respectively, with corresponding cosmological models.

2.21 The de Sitter Universe

We explore the consequences denoted in section 2.20, specifically the conditions given as a result of (2.401), namely,

$$\mathring{\mu}^0 c^2 + \mathring{p}^0 = 0 \tag{2.402}$$

Adding equations (2.398) and (2.399) we obtain in the given case

$$\begin{aligned}
\frac{b'}{abr} - \frac{1}{r^2}\left(1-\frac{1}{a}\right) + \lambda &= \kappa\mathring{p} \\
+\frac{a'}{a^2r} + \frac{1}{r^2}\left(1-\frac{1}{a}\right) - \lambda &= \kappa\mathring{\mu}^0 c^2 \\
\implies \frac{b'}{abr} + \frac{a'}{a^2r} &= \kappa\mathring{p} + \kappa\mathring{\mu}^0 c^2 \\
\implies \frac{b'}{abr} + \frac{a'}{a^2r} &= \kappa\left(\mathring{p} + \mathring{\mu}^0 c^2\right) \\
\implies \frac{b'}{abr}\frac{(a)}{(a)} + \frac{a'}{a^2r}\frac{(b)}{(b)} &= \kappa\left(\mathring{p} + \mathring{\mu}^0 c^2\right) \\
\implies \frac{b'a}{a^2br} + \frac{a'b}{a^2br} &= \kappa\left(\mathring{p} + \mathring{\mu}^0 c^2\right) \\
\implies \frac{(ab)'}{a^2br} &= \kappa\left(\mathring{p} + \mathring{\mu}^0 c^2\right) \\
\implies (ab)' &= \kappa\left(\mathring{p} + \mathring{\mu}^0 c^2\right)\left(a^2br\right) = 0 \\
\implies (ab)' &= 0 \\
\implies ab &= \text{const}
\end{aligned}$$

By a rather elementary change of scale of the time variable (this in due part to the arbitrariness of the time coordinate) the constant can be set equal to 1, by which we may then express a and b as corresponding reciprocals to one another:

$$ab = 1 \tag{2.403}$$

Introduction of the new variable $y = 1/a$, we may rewrite (2.399) as

$$(yr)' \equiv y'r + y = +1 - \left(\lambda + \kappa\mathring{\mu}^0 c^2\right) r^2$$

since we have

$$y'r + yr' = \frac{dy}{dr}r + y\frac{dr}{dr} = y'r + y(1) = \left(\frac{1}{a(r)}\right)'\frac{1}{r} + y = \left(-\frac{a'}{a^2r}\right) + \frac{1}{a}$$

after which starting with (2.399) we see that

$$\begin{aligned}
&\frac{a'}{a^2r} + \frac{1}{r^2}\left(1 - \frac{1}{a}\right) - \lambda = \kappa\mathring{\mu}^0c^2 \\
\implies &\frac{a'}{a^2r} + \frac{1}{r^2}\left(1 - \frac{1}{a}\right) = \left(\lambda + \kappa\mathring{\mu}^0c^2\right) \\
\implies &\frac{a'}{a^2r} + 1 - \frac{1}{a} = \left(\lambda + \kappa\mathring{\mu}^0c^2\right)r^2 \\
\implies &\frac{a'}{a^2r} - \frac{1}{a} = -1 + \left(\lambda + \kappa\mathring{\mu}^0c^2\right)r^2 \\
\implies &\frac{a'}{a^2r} - \frac{1}{a} = -1 + \left(\lambda + \kappa\mathring{\mu}^0c^2\right)r^2 \\
\implies &-\frac{a'}{a^2r} + \frac{1}{a} = 1 - \left(\lambda + \kappa\mathring{\mu}^0c^2\right)r^2 \\
\implies &y'r + y = 1 - \left(\lambda + \kappa\mathring{\mu}^0c^2\right)r^2 \\
\implies &(yr)' = 1 - \left(\lambda + \kappa\mathring{\mu}^0c^2\right)r^2 \\
\implies &\int (yr)'\,dr = \int 1dr - \int \left(\lambda + \kappa\mathring{\mu}^0c^2\right)r^2dr \\
\implies &\int (yr)'\,dr = \int 1dr - \left(\lambda + \kappa\mathring{\mu}^0c^2\right)\int r^2dr
\end{aligned}$$

integration with respect to the radial coordinate gives

$$\implies yr = r - \frac{\left(\lambda + \kappa\mathring{\mu}^0c^2\right)}{3}r^3 + \mathrm{C} \tag{2.404}$$

Since we require that y be regular for $r = 0$, i.e. that there is no singularity at $r = 0$, the constant “C” on the RHS of (2.404) must be equivalently zero, hence,

$$y(r=0)=0$$
$$\implies y(r=0)=0-\frac{\left(\lambda+\kappa\mathring{\mu}^0c^2\right)}{3}(0)^3+\mathrm{C}=0$$
$$\implies \mathrm{C}=0$$

Therefore, we obtain from (2.403) and (2.404)

$$b=\frac{1}{a}=y=1-\frac{r^2}{R^2} \tag{2.405}$$

where we set

$$\frac{1}{R^2}=\frac{\lambda+\kappa\mathring{\mu}^0c^2}{3} \tag{2.406}$$

Furthermore, using (2.405) in (2.396), we find that

$$ds^2=a(r)dr^2+r^2\left(d\theta^2+\sin^2\theta d\phi^2\right)-b(r)c^2dt^2$$

namely, since we set $y=\frac{1}{a} \implies a=\frac{1}{y}$, hence,

$$\frac{1}{a}=y=1-\frac{r^2}{R^2}$$
$$\implies a=\frac{1}{\left(1-\frac{r^2}{R^2}\right)}$$

further, since b is the reciprocal of a, as per (2.403), we have,

$$b=\frac{1}{a}=1-\frac{r^2}{R^2}$$

such that the space-time interval takes the form

$$ds^2 = \frac{dr^2}{1 - r^2/R^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) - \left(1 - \frac{r^2}{R^2}\right)c^2 dt^2 \tag{2.407}$$

which is the aptly named de Sitter - Schwarzschild line element.

The components of the spatial metric tensor are given by

$$\left.\begin{aligned} &\gamma_{11} = \tfrac{1}{1-r^2/R^2}, \quad \gamma_{22} = r^2, \quad \gamma_{33} = r^2\sin^2\theta \\ &\gamma_{\iota\kappa} = 0 \text{ for } \iota \neq \kappa, \qquad\quad \gamma = |\gamma_{\iota\kappa}| = \tfrac{r^4\sin^2\theta}{1-r^2/R^2} \end{aligned}\right\} \tag{2.408}$$

in which the dynamical potentials are given by

$$\gamma_\iota = 0, \quad \Phi = -\frac{r^2c^2}{2R^2} \tag{2.409}$$

Namely, we find that taking the determinant of the spatial metric, i.e $|\gamma_{\iota\kappa}|$, we find

$$|\gamma_{\iota\kappa}| = \begin{pmatrix} \frac{1}{\left(1-\frac{r^2}{R^2}\right)} & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix} = \left(\frac{1}{1-\frac{r^2}{R^2}}\right)\left(r^2\right)\left(r^2\sin^2\theta\right) = \frac{r^4\sin^2\theta}{\left(1-\frac{r^2}{R^2}\right)}$$

The spatial line element takes the form

$$d\sigma^2 = \frac{dr^2}{1 - r^2/R^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{2.410}$$

which is real only for the values $r < R$, which defines the extension of the physical space (in the Einstein universe).

Since, the dynamical properties in the chosen scenario are nonzero, we find that a free particle is acted upon by a gravitational force, by the following relation

$$\begin{aligned}
\mathbf{K} &= -m\,\mathrm{grad}\,\Phi = -m\nabla\Phi \\
&= -m\left(\frac{\partial}{\partial r}\Phi, \frac{\partial}{\partial \theta}\Phi, \frac{\partial}{\partial \phi}\Phi\right) \\
&= -m\left(\frac{\partial}{\partial r}\left(-\frac{r^2c^2}{2R^2}\right), 0, 0\right) \\
&= -m\left(\left(\frac{-2rc^2}{2R^2}\right), 0, 0\right) \\
&= m\left(\frac{rc^2}{R^2}, 0, 0\right)
\end{aligned}$$

hence,

$$\mathbf{K} = -m\,\mathrm{grad}\,\Phi = m\left(\frac{rc^2}{R^2}, 0, 0\right) \tag{2.411}$$

where it is clear that the gravitational force is proportional to the variable r, which means that Newton's law of inertia does not hold over arbitrarily large regions of space in the de Sitter view of the universe.

We note in passing that we may always describe the force $\vec{F}$, rather a conservative vector field or conservative force, as any one of the following if it satisfies any of the following 3 conditions:

1. The curl of the vector field is equivalent to the zero vector, i.e.

$$\boldsymbol{\nabla} \times \vec{F} = \vec{0} \tag{2.412}$$

   similarly, we may define the conservative field by $\vec{F} = \nabla f$, such that

$$\boldsymbol{\nabla} \times (\nabla f) = \vec{0} \tag{2.413}$$

   where f is a scalar function.

2. There is a zero net work W done by the force when moving a particle through a trajectory

that begins and ends in the same place, i.e. the line integral of a conservative vector field is independent of path, such that from the Fundamental Lemma of line integrals, we have,

$$W = \oint_C \vec{F} \cdot d\vec{r} = 0 \tag{2.414}$$

where C is the path taken by the particle, i.e. the curve.

3. The force is equivalent to the negative gradient of a potential, say ϕ

$$\vec{F} = -\nabla\phi \tag{2.415}$$

We briefly prove the 3rd statement above. Assume that statement 2 holds. Then, let C be a simple curve from the origin to a point x and define a function

$$\phi = -\int_C \vec{F} \cdot d\vec{r} \tag{2.416}$$

The fact that this function is well defined (i.e. independent of path C) follows from statement 2. Further, from the Fundamental Theorem of Calculus, it follows that

$$\begin{aligned} \nabla\phi &= -\nabla \int_C \vec{F} \cdot d\vec{r} \\ \implies \nabla\phi &= -\vec{F} \\ \implies \vec{F} &= -\nabla\phi \end{aligned}$$

Thus, statement 2 implies 3. We exclude the remainder of the proof, as providing a proof of such an elementary concept is outside of the scope of this paper.

Only in regions for which

$$\frac{r^2}{R^2} \ll 1$$

will the line element given by (2.407) reduce to the line element of special relativity and the law of inertia hold approximately, namely,

$$ds^2 = \frac{dr^2}{1 - r^2/R^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) - \left(1 - \frac{r^2}{R^2}\right)c^2dt^2 \tag{2.417}$$

for $\frac{r^2}{R^2} \ll 1$, reduces to

$$\Longrightarrow ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) - c^2dt^2$$

We may now derive the equations of motion of a free particle using the expression provided by (2.411). Let

$$x^\iota = x^\iota(\tau) \tag{2.418}$$

be the equation of the time track of the motion of some particle in an arbitrary system of coordinates $\left(x^i\right)$. Then, as provided in section 2.19, we let the contravariant components of the four-velocity be

$$U^\iota \equiv \frac{dx^i}{d\tau} = (\Gamma u^\iota, c\Gamma)$$

where

$$\frac{1}{\Gamma} = \left[\left\{\left(1 + \frac{2\phi}{c^2}\right)^{\frac{1}{2}} - \frac{\gamma_\kappa u^\kappa}{c}\right\}^2 - \frac{u^2}{c^2}\right]^{\frac{1}{2}}$$

is the analog to the Lorentz factor in the presence of a field of gravity with dynamical potentials (Φ, γ_ι). Then, by using the metric to lower indices we find the the covariant components of the four velocity are

$$\left.\begin{aligned} U_i &= g_{\iota\kappa}U^k = g_{ik}u^\kappa\Gamma + g_{i4}c\Gamma \\ U_\iota &= \Gamma\left[u_\iota - \gamma_\iota\left(\gamma_\kappa u^\kappa\right) + c\gamma_\iota\left(1 + \tfrac{2\Phi}{c^2}\right)^{\frac{1}{2}}\right] \\ U_4 &= -\Gamma c\left(1 + \tfrac{2\Phi}{c^2}\right)^{\frac{1}{2}}\left\{\left(1 + \tfrac{2\Phi}{c^2}\right)^{\frac{1}{2}} - \tfrac{\gamma_\kappa u^\kappa}{c}\right\} \end{aligned}\right\} \tag{2.419}$$

where $u_\iota = \gamma_{\iota\kappa}u^\kappa$ are the convariant components of the three-dimensional velocity vector, calculated from the contravariant components by implementation of the spatial metric $\gamma_{\iota\kappa}$.

By purely spatial transformations

$$x'^{\iota} = x'^{\iota}\left(x^{\kappa}\right), \quad x'^{4} = x^{4} \tag{2.420}$$

the spatial parts of the U^{ι} and U_{ι} of the four-velocity transform according to the contravariant and covariant components of a vector, respectively; nonetheless, unless the coordinate system is time-orthogonal, the corresponding contravariant and covariant vectors will represent different space vectors. We have the analog of the inner/dot product of the four-velocity to be equivalently $-c^2$, since

$$U_{\iota}U^{\iota} = (\Gamma u^{\iota}, c\Gamma)^2 = (\Gamma u^{\iota}, c\Gamma)\cdot(\Gamma u^{\iota}, c\Gamma) = \Gamma^2\vec{u}^2 - c^2\Gamma^2 = 0 - c^2\Gamma^2 = -c^2 \tag{2.421}$$

The components of the four-acceleration in curvilinear coordinates takes the form

$$A^i = \frac{DU^i}{d\tau} = \frac{dU^i}{d\tau} + \Gamma^i_{kl}U^kU^l, \quad A_i = \frac{DU_i}{d\tau} = \frac{dU_i}{d\tau} - \Gamma_{l,ik}U^kU^l \tag{2.422}$$

which is obtained by covariant differentiation, see section 2.5. In a local system of inertia, i.e. locally-geodesic, we see that (2.422) reduces to

$$A^i = \frac{DU^i}{d\tau} = \frac{dU^i}{d\tau} \tag{2.423}$$

since $\Gamma^i_{kl} = 0$. Then by (4.21) we have

$$\begin{aligned} U_{\iota}U^{\iota} &= -c^2 \\ \Longrightarrow \frac{d}{d\tau}\left(U_{\iota}U^{\iota}\right) &= \frac{d}{d\tau}\left(-c^2\right) \\ \Longrightarrow \frac{d}{d\tau}\left(U_{\iota}\right)\left(U^{\iota}\right) + \left(U_{\iota}\right)\frac{d}{d\tau}\left(U^{\iota}\right) &= 0 \\ \Longrightarrow -A_iU^i + U_iA^i &= 0 \end{aligned}$$

$$U_i A^i = U^i A_i \tag{2.424}$$

The contravariant and covariant components of the four-momentum vector take the form,

$$P^\iota = \mathring{m}_0 U^\iota, \quad P_\iota = \mathring{m}_0 U_\iota \tag{2.425}$$

where we have that $\mathring{m}_0$ is the proper mass of the particle in question, i.e. the mass measured in a local system of inertia. We see that the four-momentum P_ι is proportional to the four velocity and the factor of proportionality is equal to the proper mass $\mathring{m}_0$ of the particle, i.e.

$$P_\iota = \mathring{m}_0 U_\iota$$

Then, taking the norm of the linear momentum vector, we obtain

$$P_\iota P^\iota = \vec{p}^2 - \frac{E^2}{c^2} = \mathring{m}_0 \sum_\iota U_\iota^2 = -\mathring{m}_0^2 c^2 \tag{2.426}$$

which we find is accordance with the energy-momentum relation, since

$$\begin{aligned}
\mathbf{p}^2 - \frac{E^2}{c^2} &= -\mathring{m}_0^2 c^2 \\
\implies -\mathbf{p}^2 + \frac{E^2}{c^2} &= \mathring{m}_0^2 c^2 \\
\implies \frac{E^2}{c^2} &= \mathring{m}_0^2 c^2 + \mathbf{p}^2 \\
\implies E^2 &= \mathring{m}_0^2 c^4 + \mathbf{p}^2 c^2 \\
\implies E &= \pm\sqrt{\mathring{m}_0^2 c^4 + \mathbf{p}^2 c^2}
\end{aligned}$$

For a particle in free fall, i.e. a particle acted only by gravitational forces, we have in a local system of inertia

$$\frac{d\mathring{P}^\iota}{d\tau} = 0, \quad \frac{d\mathring{P}_\iota}{d\tau} = 0$$

which delineates the law of conservation of energy and momentum.

In a general system of coordinates (x^{ι}) we take a covariant derivative of the linear momentum, such that we find the following form

$$\frac{DP^{\iota}}{d\tau}=0, \quad \frac{DP_{\iota}}{d\tau}=0 \tag{2.427}$$

in which

$$\frac{DP^{\iota}}{d\tau}=\frac{dP^{\iota}}{d\tau}+\Gamma^{\iota}_{kl}U^{k}P^{l} \tag{2.428}$$

$$\frac{DP_{\iota}}{d\tau}=\frac{dP_{\iota}}{d\tau}-\Gamma_{l,\iota k}U^{k}P^{l} \tag{2.429}$$

Equations (2.427) express the fact that the four-momentum vector at some arbitrary time $\tau+d\tau$ may be obtained form the four-momentum vector at time τ by means of a parallel displacement. Then using the following relations, (see section 2.6),

$$\Gamma^{i}_{kl}=\Gamma^{i}_{lk}, \quad \Gamma_{i,kl}=\Gamma_{i,lk}, \quad \frac{\partial g_{ik}}{\partial x^{l}}=\Gamma_{i,kl}+\Gamma_{k,il} \tag{2.430}$$

we may rewrite (2.427) as follows

$$\frac{DP_{\iota}}{d\tau}=\frac{dP_{\iota}}{d\tau}-\frac{\mathring{m}_{0}}{2}\left(\Gamma_{l,ik}+\Gamma_{k,il}\right)U^{k}U^{l}=\frac{dP_{\iota}}{d\tau}-\frac{1}{2}\frac{\partial g_{kl}}{\partial x^{\iota}}U^{k}P^{l}=0 \tag{2.431}$$

Thus, the given equations detail the motion of a particle in a given external gravitational field. We note in passing that the particle itself will create a gravitational field which may be described using the quantities of the metric. Nonetheless, of sake of simplicity, we consider this field of gravity to be weak compared to that of the external gravitational field, such that we may neglect its influence on the metric.

We briefly expand upon the physical interpretation of the quantities displayed in (2.431) by providing a three-dimensional analog to the equations. To this end, define the "purely" spatial linear momentum vector as follows,

$$
\begin{aligned}
p^{\iota} &= mu^{\iota} \\
p_{\iota} &= mu_{\iota} = m\gamma_{\iota\kappa}u^{\kappa} = \gamma_{\iota\kappa}p^{\kappa} \\
m = \mathring{m}_0\Gamma &= \mathring{m}_0\left[\left\{\left(1+\tfrac{2\Phi}{c^2}\right)^{\frac{1}{2}} - \tfrac{\gamma_{\kappa}u^{\kappa}}{c}\right\}^2 - \tfrac{u^2}{c^2}\right]^{-\frac{1}{c^2}}
\end{aligned}
\tag{2.432}
$$

where the mass m, is given by the rest mass of the particle in question and the and the analog of the Lorentz factor in curvilinear coordinates.

Let $p_{\iota} = p_{\iota}(t)$ be a function of time t such that we may construct a new space vector $\frac{d_c p_{\iota}}{dt}$ by

$$
\frac{d_c p_{\iota}}{dt} \equiv \frac{dp_{\iota}}{dt} - \gamma_{\lambda,\iota\kappa}u^{\kappa}p^{\lambda} = \frac{dp_{\iota}}{dt} - \frac{1}{2}\frac{\partial\gamma_{\kappa\lambda}}{\partial x^{\iota}}u^{\kappa}p^{\lambda} \tag{2.433}
$$

which is the analog to the 4-D covariant derivative in three dimensions. Here $\gamma_{\lambda,\iota\kappa}$ are the three dimensional Christoffel symbols, derived from the spatial metric $\gamma_{\iota\kappa}$, such that the vector $d_c p_{\iota}/dt$ is the three-dimensional analog of the four vector derived in (2.429), (2.431). Indeed, recalling the definition of a covariant derivative, see section 2.6, we have

$$
\left.
\begin{aligned}
\frac{Da^{\iota}}{d\lambda} &= \frac{da^{\iota}}{d\lambda} + \Gamma^{\iota}_{kl}U^{k}a^{l} \\
U^{\iota} &= \frac{dx^{\iota}}{d\lambda}
\end{aligned}
\right\}
$$

where we have a vector $a^{\iota}(\lambda)$. Upon simple substitution of $a^{\iota} = p^{\iota}$, $\lambda \to t$, and $\Gamma^{\iota}_{kl} \to \gamma^{\lambda}_{ik}$, we obtain

$$
\frac{d_c p^{\iota}}{dt} = \frac{dp^{\iota}}{dt} + \gamma^{\iota}_{\lambda k}u^{k}p^{\lambda}
$$

but, since the we seek the covariant vector, upon contraction, i.e. lowering of indices, we have

$$
\frac{d_c p_{\iota}}{dt} \equiv \frac{dp_{\iota}}{dt} - \gamma_{\lambda,\iota\kappa}u^{\kappa}p^{\lambda} = \frac{dp_{\iota}}{dt} - \frac{1}{2}\frac{\partial\gamma_{\kappa\lambda}}{\partial x^{\iota}}u^{\kappa}p^{\lambda} \tag{2.434}
$$

Recall that

$$\frac{dt}{d\tau} = \Gamma, \quad \gamma_{\iota\kappa} = g_{\iota\kappa} + \gamma_\iota \gamma_\kappa \tag{2.435}$$

where, as aforementioned, Γ is the Lorentz analog in curvilinear coordinates, which may be derived in a similar manner as what was depicted in section 2.4; we see that (2.431) may be rewritten, for $i = 1, 2, 3$ in the form

$$\frac{d_c p_\iota}{dt} = K_\iota = mG_\iota \tag{2.436}$$

in which G_ι is a space vector depending on the dynamical gravitational potentials (Φ, γ_ι) and their derivatives.

Now, if we interpret $p^\iota = mu^\iota$ as the momentum of the particle, then it immediately follows that K_ι must be interpreted as the gravitational force acting in the particle, in accordance to

$$\begin{gathered} K_\iota = F = ma^\iota \\ \implies F = m\frac{du^\iota}{dt} \\ \implies F = \frac{d}{dt}(mu^\iota) \\ \implies F = \frac{d}{dt}p^\iota \end{gathered}$$

The factor of proportionality m defined by (2.432) appears as the inertial mass of the particle moving with the velocity u^ι in the field of gravity with potentials (Φ, γ_ι). Now, since $K_\iota = mG_\iota$, m also represents the gravitational mass of the particle. If we consider a particle at rest, then mass reduces to

$$\begin{gathered} m = \mathring{m}_0 \Gamma = \mathring{m}_0 \left[\left\{ \left(1 + \frac{2\Phi}{c^2}\right)^{\frac{1}{2}} - \frac{\gamma_\kappa u^\kappa}{c} \right\}^2 - \frac{u^2}{c^2} \right]^{-\frac{1}{c^2}} \\ \implies m = \mathring{m}_0 \Gamma = \mathring{m}_0 \left[\left\{ \left(1 + \frac{2\Phi}{c^2}\right)^{\frac{1}{2}} - \frac{\gamma_\kappa (0)}{c} \right\}^2 - \frac{(0)}{c^2} \right]^{-\frac{1}{c^2}} \end{gathered}$$

$$m_0 = \frac{\mathring{m}_0}{\sqrt{(1+2\Phi/c^2)}} \tag{2.437}$$

which represents the rest mass of a particle in a gravitational field.

For small particle velocities, where we neglect terms of order u/c, the quantity of G_ι reduces to a_ι, but in general the gravitational force K_ι will be an expression of the potentials, their derivatives, the velocity of the particle, and its acceleration. We point attention to one specific case in which the gravitational force K_ι becomes simple, which is the case where the coordinate system is time-orthogonal so that the vector potentials $\gamma_\iota = 0$. In the present case, the equations of (2.431) become identical to (2.436) if we set

$$G_\iota = -\frac{\partial \Phi}{\partial x^\iota}, \quad \mathbf{K} = m\mathbf{G} = -m\,\mathrm{grad}\,\Phi \tag{2.438}$$

thus, the force of gravitation is connected to the scalar potential, Φ, in the same was as in Newton's Theory. Since we may now write

$$\mathbf{K} = m\mathbf{G} = m\left(-\frac{\partial \Phi}{\partial x^\iota}\right) = -m\,\mathrm{grad}\,\Phi \tag{2.439}$$

Recall that in Newton's Theory/Law the metric reduces to the value of the gravitational potential, see §2.7, i.e. $g_{00} = 1 + \frac{2\Phi}{c^2}$. We find this to be true for arbitrarily strong fields and for all velocities. Further, if we set $\gamma_\iota = 0$, the expression for (2.432) for the "relativistic" mass reduces to

$$m = \mathring{m}_0\Gamma = \mathring{m}_0\left[\left\{\left(1+\frac{2\Phi}{c^2}\right)^{\frac{1}{2}} - \frac{\gamma_\kappa u^\kappa}{c}\right\}^2 - \frac{u^2}{c^2}\right]^{-\frac{1}{c^2}}$$

$$\Longrightarrow \; m = \mathring{m}_0\Gamma = \mathring{m}_0\left[\left\{\left(1+\frac{2\Phi}{c^2}\right)^{\frac{1}{2}} - \frac{(0)u^\kappa}{c}\right\}^2 - \frac{u^2}{c^2}\right]^{-\frac{1}{c^2}}$$

$$m = \frac{\mathring{m}_0}{\sqrt{\left(1+\frac{2\Phi}{c^2}-\frac{u^2}{c^2}\right)}} \tag{2.440}$$

Analogously, the spatial covariant derivative of the contravariant components p^ι with respect to t is defined by

$$\frac{d_c p^\iota}{dt} = \frac{dp^\iota}{dt} + \gamma^\iota_{\kappa\lambda} u^\kappa p^\lambda \tag{2.441}$$

Since the spatial metric tensor $\gamma_{\iota\kappa}$ may depend on t, we find that the contravariant and covariant components of the three dimensional analog of the four-momentum will not be equivalent in general. We find that

$$\frac{d_c p_\iota}{dt} = \gamma_{\iota\kappa}\frac{d_c p^\kappa}{dt} + \frac{\partial \gamma_{\iota\kappa}}{\partial t} p^\kappa \tag{2.442}$$

which is easily found using (2.432) and applying product rule to each side of the relation, or the reciprocal relation

$$\frac{d_c p^\iota}{dt} = \gamma^{\iota\kappa}\frac{d_c p_\kappa}{dt} - \gamma^{\iota\kappa}\frac{\partial \gamma_{\kappa\lambda}}{\partial t} p^\lambda = K^\iota - \gamma^{\iota\kappa}\frac{\partial \gamma_{\kappa\lambda}}{\partial t} p^\lambda \tag{2.443}$$

where

$$K^\iota = \gamma^{\iota\kappa} K_\kappa$$

since, by (2.436) we have

$$\begin{aligned} \frac{d_c p^\iota}{dt} &= K_\iota \\ \implies \gamma^{\iota\kappa}\frac{d_c p^\iota}{dt} &= \gamma^{\iota\kappa} K_\iota = K^\kappa \end{aligned}$$

which are the contravariant components of the force of gravitation. The time derivative of the norm of the four-momentum is, by (2.442) and (2.443)

$$\frac{d}{dt}(p_\iota p^\iota) = \frac{dp_\iota}{dt} p^\iota + p_\iota \frac{dp^\iota}{dt} = \frac{d_c p_\iota}{dt} p^\iota + p_\iota \frac{d_c p^\iota}{dt} = 2K_\iota p^\iota - \frac{\partial \gamma_{\iota\kappa}}{\partial t} p^\iota p^\kappa \tag{2.444}$$

we state the RHS of (2.444) without proof for the sake of brevity.

If we consider a system in which the dynamical potentials vanish, it follows that the gravitational force vanishes accordingly (which follows from (2.436)), and the motion of the particle reduces to

$$\frac{d_c p_\iota}{dt} = 0 \tag{2.445}$$

i.e. the covariant components of the four momentum vectors are at distinct times attained by parallel displacement in the three dimensional sense. This does not translate to the magnitude of the momentum being constant in time, since from (2.444) we have

$$\begin{aligned} \frac{d}{dt}\left(p_\iota p^\iota\right) &= 2K_\iota p^\iota - \frac{\partial \gamma_{\iota\kappa}}{\partial t} p^\iota p^\kappa \\ \Longrightarrow &= 0 - \frac{\partial \gamma_{\iota\kappa}}{\partial t} p^\iota p^\kappa \end{aligned}$$

i.e.

$$\frac{dp^2}{dt} = -\frac{\partial \gamma_{\iota\kappa}}{\partial t} p^\iota p^\kappa \tag{2.446}$$

Thus, p is constant if and only if our reference system is rigid, i.e. the spatial metric, $\gamma_{\iota\kappa}$, is time-independent. Further, if the dynamical potentials are zero, $\gamma_{\iota\kappa} = \Phi = 0$, we have that

$$m = \frac{\mathring{m}_0}{\sqrt{(1-u^2/c^2)}}, \quad p^2 = m^2 u^2 = \frac{\mathring{m}_0^2 u^2}{1-u^2/c^2}$$

and, if we further assume that the reference frame is rigid, we have that the velocity as we all the mass become constant, and the equations of motion of (2.445) may be written as

$$\frac{d_c u_\iota}{dt} = 0 \tag{2.447}$$

Indeed,

$$\begin{aligned} \frac{d_c p_\iota}{dt} &= 0 \\ \implies \frac{d_c(mu_\iota)}{dt} &= 0 \\ \implies m\frac{d_c(mu_\iota)}{dt} &= 0 \\ \implies \frac{d_c(mu_\iota)}{dt} &= 0 \end{aligned}$$

where we multiplied both sides by m. Therefore, in the present case the orbit of the particle is given by a geodesic in physical space, i.e. the particle moves with a constant velocity in the "straightest" line compatible with the geometry of space. The motion is analogous to the motion of a free particle bound to motion on a smooth curved two dimensional surface in a system of inertia where the only forces exerted on the particle are the normal reactions of the surface. The only apparent distinction is that we must consider the motion of a particle in curved 3-D space.

To this end, if the spatial metric varies with respect to t, the motion of the particle is analogous to the motion of a particle on a smooth *variable* surface in an inertial system. Hence, if the potentials are zero, the action of the field of gravity has the character of a "normal reaction" from the curved three-dimensional space.

Thus, as aforementioned, the equations of motion of a free particle may be obtained by (2.436) simultaneously with the gravitational force given by (2.411). Nonetheless, a more convenient approach comes from the possibility of elimination of the dynamical potentials by using a suitable transformation of space-time coordinates, which is achieved by defining new variables r', θ', ϕ', t' by the following transformation

$$\left.\begin{aligned} r' = \frac{r}{\sqrt{(1-r^2/R^2)}}e^{-ct/R}, \qquad t' = t + \frac{R}{2c}\ln\left(1-\frac{r^2}{R^2}\right) \\ \theta' = \theta, \qquad \phi' = \phi \end{aligned}\right\} \tag{2.448}$$

As previously shown by Robertson and Lemaître, respectively, the transformation leads to the line element taking the form

$$ds^2 = e^{2ct'/R}\left(dr'^2 + r'^2 d\theta'^2 + r'^2 \sin^2\theta' d\phi'^2\right) - c^2 dt'^2 \tag{2.449}$$

We proceed by proving the above claim, in the reverse order. Recall that the definition of the differential for r' and t', respectively, are as follows,

$$dr' = \frac{\partial r'}{\partial r}dr + \frac{\partial r'}{\partial t}dt \quad \text{and} \quad dt' = \frac{\partial t'}{\partial r}dr + \frac{\partial t'}{\partial t}dt \tag{2.450}$$

Consider first dr' such that

$$dr' = \frac{\partial}{\partial r}\left(\frac{r}{\sqrt{1-\frac{r^2}{R^2}}}e^{-ct/R}\right) + \frac{\partial}{\partial t}\left(\frac{r}{\sqrt{1-\frac{r^2}{R^2}}}e^{-ct/R}\right)$$

Then, by applying the quotient rule to the first term of the differential we have

$$\frac{\partial}{\partial r}\left(\frac{r}{\sqrt{1-\frac{r^2}{R^2}}}\right) = \frac{\left(\frac{d}{dr}r\right)\left(1-\frac{r^2}{R^2}\right)^{1/2} - r\frac{d}{dr}\left(1-\frac{r^2}{R^2}\right)^{1/2}}{\left(1-\frac{r^2}{R^2}\right)}$$

$$\implies = \frac{(1)\left(1-\frac{r^2}{R^2}\right)^{1/2} - r\left(\frac{1}{2}\right)\left(1-\frac{r^2}{R^2}\right)^{-1/2}\frac{d}{dr}\left(1-\frac{r^2}{r^2}\right)}{\left(1-\frac{r^2}{R^2}\right)}$$

$$\implies = \frac{\left(1-\frac{r^2}{R^2}\right)^{1/2} - \frac{r}{2}\left(1-\frac{r^2}{R^2}\right)^{-1/2}\left(-\frac{2r}{R^2}\right)}{\left(1-\frac{r^2}{R^2}\right)}$$

$$\implies = \frac{\left(1-\frac{r^2}{R^2}\right)^{1/2} + \frac{r^2}{R^2}\left(1-\frac{r^2}{R^2}\right)^{-1/2}}{\left(1-\frac{r^2}{R^2}\right)}$$

$$\implies = \frac{\left(1-\frac{r^2}{R^2}\right)^{1/2} + \frac{r^2}{R^2\left(1-\frac{r^2}{R^2}\right)^{1/2}}}{\left(1-\frac{r^2}{R^2}\right)}$$

$$\implies = \frac{\frac{R^2\left(1-\frac{r^2}{R^2}\right)+r^2}{R^2\left(1-\frac{r^2}{R^2}\right)^{1/2}}}{\left(1-\frac{r^2}{R^2}\right)}$$

$$\implies = \frac{R^2\left(1-\frac{r^2}{R^2}\right)+r^2}{R^2\left(1-\frac{r^2}{R^2}\right)^{3/2}}$$

$$\implies = \frac{R^2-r^2+r^2}{R^2\left(1-\frac{r^2}{R^2}\right)^{3/2}}$$

$$\implies = \frac{R^2}{R^2\left(1-\frac{r^2}{R^2}\right)^{3/2}}$$

$$\implies = \frac{1}{\left(1-\frac{r^2}{R^2}\right)^{3/2}}$$

Hence, we obtain,

$$dr' = \frac{1}{\left(1-\frac{r^2}{R^2}\right)^{3/2}} e^{-ct/R} dr + \frac{\partial}{\partial t}\left(\frac{r}{\sqrt{1-\frac{r^2}{R^2}}} e^{-ct/R}\right) dt$$

After which, computing the partial derivative with respect to time of the second term leads to

$$\begin{aligned} &\frac{r}{\sqrt{1-\frac{r^2}{R^2}}} \frac{\partial}{\partial t}\left(e^{-ct/R}\right) \\ \implies &\frac{r}{\sqrt{1-\frac{r^2}{R^2}}} - \frac{c}{R} e^{-ct/R} \\ \implies &-\frac{cr}{R\sqrt{1-\frac{r^2}{R^2}}} e^{-ct/R} \end{aligned}$$

Therefore, we find

$$dr' = \frac{1}{\left(1-\frac{r^2}{R^2}\right)^{3/2}} e^{-ct/R} dr - \frac{cr}{R\sqrt{1-\frac{r^2}{R^2}}} e^{-ct/R} dt$$

Similarly, we have for the differential of t' the following,

$$\begin{aligned} dt' &= \frac{\partial t'}{\partial r} dr + \frac{\partial t'}{\partial t} dt \\ \implies &= \frac{\partial}{\partial r}\left(t + \frac{R}{2c}\ln\left(1-\frac{r^2}{R^2}\right)\right) dr + \frac{\partial}{\partial t}\left(t + \frac{R}{2c}\ln\left(1-\frac{r^2}{R^2}\right)\right) dt \end{aligned}$$

Thence, consider the first derivative

$$\frac{\partial}{\partial r}\left(t+\frac{R}{2c}\ln\left(1-\frac{r^2}{R^2}\right)\right)dr$$
$$\Longrightarrow \frac{R}{2c}\frac{\partial}{\partial r}\ln\left(1-\frac{r^2}{R^2}\right)$$
$$\Longrightarrow \frac{R}{2c}\frac{1}{\left(1-\frac{r^2}{R^2}\right)}\frac{\partial}{\partial r}\left(1-\frac{r^2}{R^2}\right)$$
$$\Longrightarrow \frac{R}{2c}\frac{1}{\left(1-\frac{r^2}{R^2}\right)}\left(-\frac{2r}{R^2}\right)$$
$$\Longrightarrow -\frac{r}{cR\left(1-\frac{r^2}{R^2}\right)}dr$$

next,

$$\frac{\partial}{\partial t}\left(t+\frac{R}{2c}\ln\left(1-\frac{r^2}{R^2}\right)\right)dt$$
$$\Longrightarrow = \frac{\partial}{\partial t}tdt = 1dt$$

Thus, we have for the differential of t' the following

$$dt' = -\frac{r}{cR\left(1-\frac{r^2}{R^2}\right)}dr + 1dt$$

Squaring both dr' and dt', we have

$$dr'^2 = \left(\frac{1}{\left(1-\frac{r^2}{R^2}\right)^{3/2}} e^{-ct/R} dr - \frac{cr}{R\sqrt{1-\frac{r^2}{R^2}}} e^{-ct/R} dt \right)^2$$
$$\implies dr'^2 = \left(\frac{1}{\left(1-\frac{r^2}{R^2}\right)^{3/2}} e^{-ct/R} dr \right)^2 - 2\left(\frac{1}{\left(1-\frac{r^2}{R^2}\right)^{3/2}} e^{-ct/R} dr \right)\left(\frac{cr}{R\sqrt{1-\frac{r^2}{R^2}}} e^{-ct/R} dt \right) + \left(\frac{cr}{R\sqrt{1-\frac{r^2}{R^2}}} e^{-ct/R} dt \right)^2$$
$$\implies dr'^2 = \frac{e^{-2ct/R}}{\left(1-\frac{r^2}{R^2}\right)} \left(\frac{1}{\left(1-\frac{r^2}{R^2}\right)^2} dr^2 - 2\frac{cr}{R\left(1-\frac{r^2}{R^2}\right)} drdt + \frac{c^2r^2}{R^2} dt^2 \right)$$

Similarly,

$$dt'^2 = \left(dt - \frac{r}{cR\left(1-\frac{r^2}{R^2}\right)} dr \right)^2$$
$$\implies dt^2 - 2\left(\frac{r}{cR\left(1-\frac{r^2}{R^2}\right)} \right) drdt + \frac{r^2}{c^2R^2\left(1-\frac{r^2}{R^2}\right)^2} dr^2$$

Then, multiplying dr'^2 by $e^{2ct/R}$ we have,

$$e^{2ct'/R} dr'^2 = e^{2ct'/R} \frac{e^{-2ct/R}}{\left(1-\frac{r^2}{R^2}\right)} \left(\frac{1}{\left(1-\frac{r^2}{R^2}\right)^2} dr^2 - 2\frac{cr}{R\left(1-\frac{r^2}{R^2}\right)} drdt + \frac{c^2r^2}{R^2} dt^2 \right)$$

Recall that

$$t' = t + \frac{R}{2c} \ln\left(1 - \frac{r^2}{R^2}\right)$$

Hence,

$$e^{2ct'/R} = e^{2c\left(t+\frac{R}{2c}\ln\left(1-\frac{r^2}{R^2}\right)\right)/R}$$
$$\implies = e^{2ct/R}e^{\ln\left(1-\frac{r^2}{R^2}\right)}$$
$$\implies = e^{2ct/R}\left(1-\frac{r^2}{R^2}\right)$$

Then,

$$e^{2ct'/R}dr'^2 = \left[e^{2ct/R}\left(1-\frac{r^2}{R^2}\right)\right]\frac{e^{-2ct/R}}{\left(1-\frac{r^2}{R^2}\right)}\left(\frac{1}{\left(1-\frac{r^2}{R^2}\right)^2}dr^2 - 2\frac{cr}{R\left(1-\frac{r^2}{R^2}\right)}drdt + \frac{c^2r^2}{R^2}dt^2\right)$$
$$\implies e^{2ct'/R}dr'^2 = \left(\frac{1}{\left(1-\frac{r^2}{R^2}\right)^2}dr^2 - 2\frac{cr}{R\left(1-\frac{r^2}{R^2}\right)}drdt + \frac{c^2r^2}{R^2}dt^2\right)$$

Then, multiplying dt'^2 by c^2, we find

$$c^2dt'^2 = c^2dt^2 - 2\left(\frac{cr}{R\left(1-\frac{r^2}{R^2}\right)}\right)drdt + \frac{r^2}{R^2\left(1-\frac{r^2}{R^2}\right)^2}dr^2 \tag{2.451}$$

After which, we have, considering only the terms $e^{2ct'/R}dr'^2$ as well as $c^2dt'^2$

$$e^{2ct'/R}dr'^2 \cdots - c^2dt'^2 =$$
$$\left(\frac{1}{\left(1-\frac{r^2}{R^2}\right)^2}dr^2 - 2\frac{cr}{R\left(1-\frac{r^2}{R^2}\right)}drdt + \frac{c^2r^2}{R^2}dt^2\right)\dots$$
$$-c^2dt^2 + 2\left(\frac{cr}{R\left(1-\frac{r^2}{R^2}\right)}\right)drdt - \frac{r^2}{R^2\left(1-\frac{r^2}{R^2}\right)^2}dr^2$$
$$\Longrightarrow = \left(\frac{1}{\left(1-\frac{r^2}{R^2}\right)^2}dr^2 + \frac{c^2r^2}{R^2}dt^2\right)\dots$$
$$-c^2dt^2 + \frac{r^2}{R^2\left(1-\frac{r^2}{R^2}\right)^2}dr^2$$
$$\Longrightarrow = \frac{dr^2}{\left(1-\frac{r^2}{R^2}\right)^2}\left(1-\frac{r^2}{R^2}\right)\cdots + \left(\frac{c^2r^2}{R^2} - c^2\right)dt^2$$
$$\Longrightarrow = \frac{dr^2}{\left(1-\frac{r^2}{R^2}\right)}\cdots + c^2dt^2\left(\frac{r^2}{R^2} - 1\right)$$
$$\Longrightarrow = \frac{dr^2}{\left(1-\frac{r^2}{R^2}\right)}\cdots - c^2dt^2\left(1-\frac{r^2}{R^2}\right)$$

Recall that $\phi = \phi'$ and $\theta = \theta'$, thus, we recover the line element detailed in (2.407), namely,

$$ds^2 = \frac{dr^2}{1 - r^2/R^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) - \left(1 - \frac{r^2}{R^2}\right)c^2dt^2$$

We may then define some new space coordinates x', y', z' connected with r', θ', ϕ' by the equations connecting the Cartesian coordinates and polar coordinates in a Euclidean space, i.e. $x = r\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = r\cos\phi$ and $t = t'$, such that we may rewrite (2.449) as

$$ds^2 = e^{2ct'/R}\left(dx'^2 + dy'^2 + dz'^2\right) - c^2dt'^2 \qquad (2.452)$$

The new coordinates defined x', y', z', t' can take values from $-\infty$ to $+\infty$.

With the attained coordinates we find that the spatial metric for (2.452) is

$$\left.\begin{aligned} \gamma_{11} = \gamma_{22} = \gamma_{33} = \ & \tfrac{1}{\gamma^{11}} = \tfrac{1}{\gamma^{22}} = \tfrac{1}{\gamma^{33}} = e^{2ct'/R} \\ \gamma_{\iota\kappa} = \gamma^{\iota\kappa} = 0 \quad & \text{for } \iota \neq \kappa \\ \gamma_{\iota} = 0, \quad & \Phi = 0 \end{aligned}\right\} \tag{2.453}$$

it is immediately apparent that the spatial Christoffel symbols are all zero.

We find that the time variable t' is the time shown by a standard clock at rest at any reference point. Nonetheless, at any fixed time t' the spatial geometry is Euclidean with the new defined coordinates being Cartesian aside from the common factor $e^{2ct'/R}$. The distance from the origin of our coordinate system $r' = 0$ to a point (r', θ', ϕ'), as measured by standard measuring rods, is

$$l = e^{ct'/R} r' \tag{2.454}$$

The velocity of light is constant and equal to c.

$$w = \frac{d\sigma}{dt'} = c \tag{2.455}$$

where $d\sigma$ is the spatial line element.

By (2.455), we see that the trajectories of light rays are straight lines. Consider a free falling particle in an accelerated system with coordinates $\left(x^i\right)$. If we consider non-permanent gravitational fields, by the equivalence principle, we may transform these fields away by an appropriate pseudo-Cartesian coordinates transformation, for example coordinates (X^ι) of the system of inertia S. In the present system the motion of the particle is uniform, i.e. its time track is a straight line defined by the equation

$$\frac{d^2 X^\iota}{d\lambda^2} = 0 \tag{2.456}$$

where λ is an arbitrary parameter; namely, the time track of the particle is given by a geodesic in 4-space. We find that the geodesics are defined by the variational principle, see section 2.13 for a derivation of Hamilton's principle. Introducing proper time $\tau = s/ic$ as a parameter, such that we may rewrite the variational principle as

$$= \delta \int \left(g_{\iota k} \frac{dx^\iota}{d\lambda} \frac{dx^k}{d\lambda} \right)^{1/2} d\lambda = \delta \int ds = 0$$

Since we assume $\tau = s/ic$ as a parameter we may write

$$\begin{aligned} \tau &= \frac{s}{ic} \\ \implies \tau &= \frac{1}{c} \frac{s}{i} \\ \implies d\tau &= \frac{1}{c} \frac{ds}{i} \end{aligned}$$

If we allow the Lagrangian to assume the following form, i.e. $L = \left(g_{\iota k} \frac{dx^\iota}{d\lambda} \frac{dx^k}{d\lambda} \right)^{1/2} = ds$, after which substituting the parameter $\lambda = \tau$ we may write

$$\implies d\tau = \frac{1}{c} \frac{\sqrt{\left(g_{\iota k} \frac{dx^\iota}{d\lambda} \frac{dx^k}{d\lambda} \right)}}{\sqrt{-1}} = \frac{1}{c} \sqrt{\frac{\left(g_{\iota k} \frac{dx^\iota}{d\lambda} \frac{dx^k}{d\lambda} \right)}{-1}} = \frac{1}{c} \sqrt{-g_{\iota k} \frac{dx^\iota}{d\lambda} \frac{dx^k}{d\lambda}}$$

By Hamilton's principle, the variation of the integral must vanish at the boundary of the surface, i.e. the integral must be zero for all variations δx^ι for $\tau = \tau_1$ and $\tau = \tau_2$. Hence,

$$\delta \int_{\tau_1}^{\tau_2} d\tau = \delta \int_{\tau_1}^{\tau_2} \frac{1}{c} \sqrt{\left(-g_{\iota k} \frac{dx^\iota}{d\tau} \frac{dx^k}{d\tau} \right)} d\tau$$

$$= \frac{1}{c^2} \int \left[\frac{d}{d\tau} \left(g_{\iota k} \frac{dx^k}{d\tau} \right) - \frac{1}{2} \frac{\partial g_{kl}}{\partial x^\iota} \frac{dx^k}{d\tau} \frac{dx^l}{d\tau} \right] \delta x^\iota d\tau \tag{2.457}$$

Evidently, we arrive at the Euler-Lagrange equations, which follows from the Fundamental Lemma of Variational Calculus,

$$\frac{d}{d\tau}\left(g_{\iota k}\frac{dx^k}{d\tau}\right) = \frac{1}{2}\frac{\partial g_{kl}}{\partial x^\iota}\frac{dx^k}{d\tau}\frac{dx^l}{d\tau}, \quad g_{\iota k}\frac{dx^\iota}{d\tau}\frac{dx^k}{d\tau} = -c^2 \tag{2.458}$$

Consider a light ray in empty space. If we consider a system of inertia S, the time track of the ray is given by (2.456), with a caveat, we impose an extra condition on the track of the ray. We require that $ds^2 = G_{\iota k}dX^\iota dX^k = 0$, where $G_{\iota k}$ is the metric for the pseudo-Cartesian coordinates considered. Hence, the track of the light ray is a geodesic of zero length, leading to the conclusion that we cannot use the length as a parameter.

Hence, instead of considering (2.458), we have in an arbitrary system of coordinates (x^ι) the substitution of the equations provided in (2.458) by a more general principle, where we substitute $\tau \to \lambda$, and have the RHS of the second equation vanish, i.e.

$$\frac{d}{d\lambda}\left(g_{\iota k}\frac{dx^k}{d\lambda}\right) = \frac{1}{2}\frac{\partial g_{kl}}{dx^\iota}\frac{dx^l}{d\lambda}\frac{dx^l}{d\lambda}, \quad g_{\iota k}\frac{dx^\iota}{d\lambda}\frac{dx^k}{d\lambda} = 0 \tag{2.459}$$

where

$$x^\iota = x^\iota(\lambda) \tag{2.460}$$

can be regarded as an arbitrary parametric representation of the time track.

Again, since the trajectories of light rays are straight lines we may utilize the second equation of (2.459), namely,

$$g_{\iota k}\frac{dx^\iota}{d\lambda}\frac{dx^k}{d\lambda} = 0$$

Taking a derivative of both sides, we find

$$g_{\iota k}\frac{dx^\iota}{d\lambda}\frac{dx^k}{d\lambda} = 0 \implies \frac{\partial}{\partial x^\iota}g_{kl}\frac{dx^k}{d\lambda}\frac{dx^l}{d\lambda} = 0 \tag{2.461}$$

Hence, by the relation provided in (2.459) we may write

$$\frac{d}{d\lambda}\left(g_{\iota k}\frac{dx^k}{d\lambda}\right) = 0 \tag{2.462}$$

or, similarly, taking $\iota = k = 1,2,3$ in the equation we may substitute the metric for the purely spatial metric, i.e.

$$\begin{aligned} \frac{d}{d\lambda}\left(g_{\iota k}\frac{dx^k}{d\lambda}\right) &= 0 \\ \implies \frac{d}{d\lambda}\left(\gamma_{\iota k}\frac{dx'^{\iota}}{d\lambda}\right) &= 0 \end{aligned}$$

using (2.453) we write $\gamma_{\iota k} = e^{2ct'/R}$

$$\frac{d}{d\lambda}\left(e^{2ct'/R}\frac{dx'^{\iota}}{d\lambda}\right) = 0$$

By integration we have,

$$\begin{aligned} \int \frac{d}{d\lambda}\left(e^{2ct'/R}\frac{dx'^{\iota}}{d\lambda}\right) &= \int 0 \\ \implies \left(e^{2ct'/R}\frac{dx'^{\iota}}{d\lambda}\right) &= a^{\iota} \\ \implies \frac{dx'^{\iota}}{d\lambda} &= a^{\iota} e^{2ct'/R} \end{aligned}$$

Thus,

$$\frac{dx'^{\iota}}{d\lambda} = a^{\iota} e^{2ct'/R} \tag{2.463}$$

where a^{ι} are constants of integration. Immediately, we see that

$$\frac{dx'/dt'}{a^1} = \frac{dy'/dt'}{a^2} = \frac{dz'/dt'}{a^3} \tag{2.464}$$

and the equations of the trajectory of a light ray are linear equations of the following form

$$\frac{x'-x'_0}{a^1}=\frac{y'-y'_0}{a^2}=\frac{z'-z'_0}{a^3} \tag{2.465}$$

We see that this holds true since in (2.463) when dividing by a^ι on both sides we have that the derivative is equivalent to some constant, hence, breaking up the derivative with respect to lambda into its three components, setting $\lambda = t'$, and noting that the constants a^ι correspond to the three different spatial coordinates, we attain three distinct equations which are equivalent through their equivalence to the same constant factor $e^{-2ct'/R}$. Indeed, if we consider the trajectory of the light ray from point x_0 to x_1 in space we deduce that

$$\begin{aligned}\frac{dx'/dt'}{a^1} &= \frac{1}{a^1}\frac{dx'}{dt'}\\ \implies &= \frac{1}{a^1}\int_{x_0}^{x_1}\frac{dx'}{dt'}\\ \implies &= \frac{x'_1-x'_0}{a^1}\end{aligned}$$

By equations (2.465), the light rays are rectilinear in (x', y', z')-space and we may deduce parallax of celestial bodies and in this determine the distance (2.454) by direct measurements. We examine the propagation of a light ray along the x'-axis. Using (2.453) and (2.455) we attain

$$e^{ct'/R}\frac{dx'}{dt'}=\pm c \tag{2.466}$$

or, rather

$$\frac{dx'}{dt'}=\pm ce^{-ct'/R} \tag{2.467}$$

The trajectory of a light ray and its given motion starting off at the point $x' = x'_0 > 0$ at time t'_0 in the negative x' axis is given by

$$x'=x'_0+R\left(e^{-ct'/R}-e^{-ct'_0/R}\right) \tag{2.468}$$

Since,

$$\frac{dx'}{dt'} = \pm c e^{-ct'/R}$$
$$\implies \int_{x_0}^{x} \frac{dx'}{dt'} = \pm c \int_{t_0}^{t} e^{-ct'/R}$$
$$\implies x' - x_0' = \pm c \frac{-R}{c} \left[e^{-ct'/R} \right|_{t_0}^{t}$$
$$\implies x' - x_0' = R \left[e^{-ct'/R} \right|_{t_0}^{t}$$
$$\implies x' = x_0' + R \left[e^{-ct'/R} \right|_{t_0}^{t}$$

immediately we see that, unless

$$x_0' < Re^{-ct_0'/R} \tag{2.469}$$

this signal will never reach the origin $x' = 0$. We briefly elaborate upon the consequences provided by (2.468). Set $x' = 0$, such that

$$x' = x_0' + R \left[e^{-ct'/R} \right|_{t_0}^{t}$$
$$\implies 0 = x_0' + R \left[e^{-ct'/R} \right|_{t_0}^{t}$$
$$\implies x_0' = -R \left[e^{-ct'/R} \right|_{t_0}^{t}$$
$$\implies x_0' = -Re^{-ct'/R} + Re^{-ct_0'/R} > Re^{-ct_0'/R}$$
$$\implies -Re^{-ct'/R} > 0$$

which is not true even for $t' = \infty$.

Even though the space-time interval given in (2.452) may take all values (i.e. $+\infty$ to $-\infty$), an observer standing at the origin will never be able to attain any info. about the regions outside the "horizon" delineated in (2.469). We find that the greatest distance that an observer placed at the origin at some time t'_0 can perceive is dictated by (2.454) and (2.469), such that

$$L = e^{ct'_0/R} R e^{-ct'_0/R} = R \tag{2.470}$$

i.e. it is the same for all times in congruity with the intrinsic static nature of the de Sitter universe. It follows, considering only x'-axis of the radial coordinate r', we find that from (2.454) and (2.469) at time t_0

$$\begin{gathered} L = e^{ct'_0/R} r' \\ \Longrightarrow = e^{ct'_0/R} R e^{-ct'/R} = R \\ \Longrightarrow L = R \end{gathered}$$

We may proceed in a similar fashion for the remaining components of the radial coordinate, i.e. y' and z'. Moreover, since (2.452) is invariant against any displacement from the origin, the above deduction holds for any observer at rest in the reference system considered. Yet, the position of the horizon is dependent on the observer, nonetheless, it is the same for all of the distinct "equivalent" observers.

In the analyzed coordinate system we have that the gravitational force on a free particle vanishes, since we assume that the potentials $(\Phi, \gamma_\iota) = 0$, and the equations of motion of the particle are portrayed by (2.445)

$$\frac{d_c p_\iota}{dt'} = 0$$

in which

$$\left.\begin{aligned} p_\iota &= m u_\iota \\ m &= \frac{m_0}{\sqrt{\left(1 - u^2/c^2\right)}} \\ u_\iota &= \gamma_{\iota\kappa} \frac{dx'^\kappa}{dt'} \end{aligned}\right\} \tag{2.471}$$

As aforementioned, since the spatial Christoffel symbols are zero, we have,

$$\frac{d_c p_\iota}{dt'} = \frac{dp_\iota}{dt'} = 0$$

By (2.433) we write,

$$\frac{d_c p_\iota}{dt} = \frac{dp_\iota}{dt} - \gamma_{\lambda,\iota\kappa} u^\kappa p^\lambda$$
$$\implies \frac{d_c p_\iota}{dt} = \frac{dp_\iota}{dt} - 0 = 0$$
$$\implies \frac{d_c p_\iota}{dt} = \frac{dp_\iota}{dt}$$

where $\gamma_{\lambda,\iota\kappa}$ are the three dimensional Christoffel symbols. This shows that the covariant components of the four-momentum are constant,

$$p_\iota = \text{const} \tag{2.472}$$

We note however that the magnitude of the momentum-vector is in general no constant in time, for from (2.446), $\frac{dp^2}{dt} = -\frac{\partial \gamma_{\iota\kappa}}{\partial t} p^\iota p^\kappa$.

Granting all this, the contravariant components of the four-momentum are not constant, since by lowering of indices using the spatial metric we find

$$p^\iota = \gamma^{\iota\kappa} p_\kappa = e^{-2ct'/R} p_\iota \tag{2.473}$$

in consonance with (2.443). By (2.473) we obtain,

$$p^\iota = mu^\iota$$
$$\implies = m\gamma_{\iota k} \frac{dx'^k}{dt'}$$
$$\implies = m\frac{dx'^k}{dt'} \gamma_{\iota k}$$
$$\implies = p_\iota e^{-2ct'/R}$$

$$\left.\begin{array}{l} mu^{\iota} = m\frac{dx'^{\iota}}{dt'} = p_{\iota}e^{-2ct'/R} \\ \frac{dx'/dt'}{b^1} = \frac{dy'/dt'}{b^2} = \frac{dz'/dt'}{b^3} \end{array}\right\} \tag{2.474}$$

in which b^1, b^2, b^3 are constants. A proof of the (2.474) is completely analogous to the proof provided for (2.465). Evidently, analogous to (2.465), we see that the orbits of the free particles are the canonical equations of a line in space, i.e.

$$\frac{x' - x'_0}{b^1} = \frac{y' - y'_0}{b^2} = \frac{z' - z'_0}{b^3} \tag{2.475}$$

yet the velocity of the particle is not constant in general. We may observe that if the velocity vanishes at some time relative the the reference system it will remain zero; thence, from (2.474),

$$\frac{dx'^{\iota}}{dt'} = 0 \quad \text{for } t' = t'_0$$

needs the four-momentum vector to be zero so that the velocity remains zero for all times. Select an arbitrary reference point $(x', y', z') =$ constant, in the given reference system the chosen point may represent a freely falling particle, i.e. in an inertial reference system, howbeit, if we examine a non-rigid reference system, then the distance from the origin of a point with values that are constant will depend on time, as per (2.454). Thus, assuming that the nebula may be treated as free particles at rest in the considered coordinate system, their distance from the origin will increment with a radial velocity

$$v_r = \frac{dl}{dt'} = \frac{c}{R}e^{ect'R}r' = \frac{c}{R}l \tag{2.476}$$

which is proportional to the distance l.

We proceed by investigating the red shift phenomena expressed by nebula due to their radial velocity. To investigate such a phenomena we consider the light emitted by a particle which is located permanently at the reference point (r', θ', ϕ'). Then by (2.455) and (2.453) the radial velocity of light is

$$\frac{dr'}{dt'} = \pm c e^{-ct'/R} \tag{2.477}$$

which is equivalent to (2.467), since considering the propagation of a light ray along the x'-axis we obtain, from (2.466)

$$\begin{aligned} e^{ct'/R}\frac{dx'}{dt'} &= \pm c \\ \Longrightarrow \frac{dx'}{dt'} &= \pm c e^{-ct'/R} \end{aligned}$$

Then considering every component of the radial coordinate, i,e, x', y', z' and noting their propagation along their respective axes, we may write

$$\frac{dr'}{dt'} = \pm c e^{-ct'/R}$$

Hence, if t'_1 and t'_2 are the corresponding times of emission of radiation and reception at the origin $r' = 0$, respectively, we find by integration

$$\begin{aligned} \frac{dr'}{dt'} &= \pm c e^{-ct'/R} \\ \Longrightarrow dr' &= \pm c e^{-ct'/R} dt' \\ \Longrightarrow \int_{r'}^{0} dr' &= \int_{t'_1}^{t'_2} \pm c e^{-ct'/R} dt' \\ \Longrightarrow \int_{r'}^{0} dr' &= -c \int_{t'_1}^{t'_2} e^{-ct'/R} dt' \\ \Longrightarrow r'\Big|_{r\prime}^{0} &= -c \left(\frac{-R}{c}\right) e^{-ct'/R}\Big|_{t'_1}^{t'_2} \\ \Longrightarrow r'\Big|_{r\prime}^{0} &= R\, e^{-ct'/R}\Big|_{t'_1}^{t'_2} \\ \Longrightarrow -r' &= R\left(e^{-ct'_2/R} - e^{-ct'_1/R}\right) \\ \Longrightarrow r' &= -R\left(e^{-ct'_2/R} - e^{-ct'_1/R}\right) \end{aligned}$$

$$\implies r' = R\left(e^{-ct_1'/R} - e^{-ct_2'/R}\right) \tag{2.478}$$

Upon differentiation of the above expression we find for the time-interval Δt_2 between the arrival of two successive waves crests at the origin and the time interval Δt_1 between their emission the following expression:

$$\Delta t_2' = \Delta t_1' e^{c(t_2-t_1)/R} \tag{2.479}$$

We find that through the study of the difference between the frequencies emission and arrival, we arrive at the phenomena of red shift. Now, since t' is the time shown by a standard clock at rest and since the velocity of light is constant, c, everywhere, we have that for the proper wavelength λ^0 of the emitted light was

$$\lambda^0 = c\Delta t_1'$$

Indeed, since

$$\begin{aligned} \lambda^0 = c\Delta t_1' &\implies \Delta t_1' \\ \implies &= \left(\frac{\Delta x}{\Delta t_1'}\right)\left(\Delta t_1'\right) \\ \implies \Delta x &= \lambda^0 = \text{wavelength} \end{aligned}$$

Similarly, the wavelength measured by an observer at the origin will be

$$\lambda = c\Delta t_2'$$

Thus, from (2.479) we find

$$\begin{aligned} \Delta t_2' &= \Delta t_1' e^{c\left(t_2'-t_1'\right)/R} \\ \implies \frac{\lambda}{c} &= \frac{\lambda^0}{c} e^{c\left(t_2'-t_1'\right)/R} \end{aligned}$$

$$\lambda = \lambda^0 e^{c\left(t_2'-t_1'\right)/R} \tag{2.480}$$

Furthermore, the distance to the particle at the time of observation of the light is, by (2.454) and (2.478)

$$l = r' e^{ct_2'/R} = R\left(e^{c\left(t_2'-t_1'\right)/R} - 1\right) \tag{2.481}$$

Indeed, from (2.454) and (2.478) we find

$$\begin{aligned}
l &= e^{ct_2'/R} r' \\
\Longrightarrow &= e^{ct_2'/R} R\left(e^{-ct_1'/R} - e^{-ct_2'/R}\right) \\
\Longrightarrow &= R\left(e^{-c\left(t_2'-t_1'\right)/R} - e^{-c\left(t_2'-t_2'\right)/R}\right) \\
\Longrightarrow &= R\left(e^{-c\left(t_2'-t_1'\right)/R} - e^{0}\right) \\
\Longrightarrow &= R\left(e^{c\left(t_2'-t_1'\right)/R} - 1\right)
\end{aligned}$$

which is exactly what was derived in (2.481). We find that the frequency between the proper wavelength and perceived wavelength differ by an exponential factor that depends on time, leading to the red shift, which shrinks or elongates the wavelength. Then, by (2.480) we obtain

$$\frac{\Delta\lambda}{\lambda^0} = \frac{\lambda - \lambda^0}{\lambda^0} = \frac{l}{R} \tag{2.482}$$

Certainly, from (2.481), we have

$$\begin{aligned}
l &= R\left(e^{c\left(t_2'-t_1'\right)/R} - 1\right) \\
\Longrightarrow \ \frac{l}{R} &= \left(e^{c\left(t_2'-t_1'\right)/R} - 1\right) \\
\Longrightarrow \ \frac{l}{R} + 1 &= e^{c\left(t_2'-t_1'\right)/R}
\end{aligned}$$

By (2.480) we find

$$\lambda = \lambda^0 e^{c(t_2-t_1)/R}$$
$$\implies \frac{\lambda}{\lambda^0} = e^{c(t_2-t_1)/R}$$

Thence, neglecting the "1" term, we write

$$\frac{\lambda}{\lambda^0} = e^{c(t_2-t_1)/R}$$
$$\implies \frac{\lambda}{\lambda^0} = \frac{l}{R} + 1$$
$$\implies \frac{\lambda}{\lambda^0} = \frac{l}{R}$$
$$\implies \frac{\Delta\lambda}{\lambda^0} = \frac{l}{R}$$

Hence, if we assume that the nebula are at rest in the coordinate system considered, we obtain an explanation for the red shift observed by Huble and Humason. To attain the quantitative agreement the constant R in (2.482) must take the value

$$R \simeq 1.66 \times 10^{27}\text{cm} \simeq 1.75 \times 10^{9}\text{light years.} \tag{2.483}$$

Upon juxtapose of the considered de Sitter universe and the Einstein universe, we notice that the value for the "radius" of the universe is smaller than the Einstein universe. The explanation for such an observation is best explained by Weyl's hypothesis, in which we assumed that the nebula in the mean are at rest relative to the reference system described in the coordinates (x', y', z', t'). This hypothesis is unrefined and natural as it has the alluring feature of considering all nebula on the same footing, such that an observer at any other reference point would observe the same red shift of the light coming from the nebula as the observer at the arbitrarily chosen origin of the coordinate system.

In the original coordinate system the motion of the nebula is obtained from (2.448) by solving the first equation with respect to r. Hence,

$$r^2 = \frac{r'^2}{e^{-2ct/R} + r'^2/R^2} \tag{2.484}$$

where r' is a constant for each nebula. Thence we see that the nebula according to the Weyl hypothesis are freely falling particles which start at the point $r = 0$ at $t = -\infty$ and end at the antipodal point $r = R$ for $t \to +\infty$.

Belatedly, introducing five variables $z_\mu = (z_0, z_1, z_2, z_3, z_4)$ by the equations

$$\left.\begin{aligned} z_1 = x' e^{ct'/R}, \quad z_2 &= y' e^{ct'/R}, \quad z_3 = z' e^{ct'/R} \\ z_0 \quad &= R\left(\cosh\tfrac{ct'}{R} - \tfrac{r'^2}{2R^2} e^{ct'/R}\right) \\ z_4 \quad &= iR\left(\sinh\tfrac{ct'}{R} + \tfrac{r'^2}{2R^2} e^{ct'/R}\right) \end{aligned}\right\} \tag{2.485}$$

we have

$$\sum_{\mu=0}^{4} z_\mu^2 = R^2 \tag{2.486}$$

such that the line element takes the form

$$ds^2 = \sum_{\mu=0}^{4} \left(dz_\mu\right)^2 \tag{2.487}$$

The space-time of the de Sitter universe may thus be pictured as the 4-D space of a sphere of radius R in a 5-D pseudo-Euclidean space (through embedding). Since the equations (2.486) and (2.487) are form-invariant (due to being the space-time interval) under the group of 5-D orthogonal transformations of the variable $\left(z_\mu\right)$, the line element (2.452) will be form invariant under the group L of corresponding transformations of the variables $x^\iota = (x', y', z', ct')$ connected with $(z_0, \ldots, z_4)$ by the equations (2.485). These transformations play the same role in the de Sitter universe as the inhomogeneous Lorentz transformations in the flat space of the special theory of relativity.

Despite the fact that the de Sitter models provides a natural explanation of the observed red shift of spectral lines, this model has within it its own deficiencies. According to the condition (2.402) underlying the model, the density and pressure of the celestial bodies satisfy the equation

$$\mathring{\mu}^0 c^2 + \mathring{p} = 0 \tag{2.488}$$

It follows immediately that $\mathring{\mu}^0 c^2 \gg 0$, then (2.488) leads to the assumption that the proper pressure, $\mathring{p}$, is negative and very large. Permitting the existence of cohesive forces in the ideal fluid filling of our model, a value of $\mathring{p}$ of the order of $-\mathring{\mu}^0 c^2$ would be incompatible with the properties of any known material. Thus, (2.488) may be satisfied only if we take the density to be zero or at least very much smaller than the mean density of the actual celestial matter. Consequently, the the de Sitter universe corresponds to an empty universe containing no appreciable amount of matter and radiation, with stars and nebula being treated as a kind of test bodies which do not contribute essentially to the gravitational field.

We proceed by deriving the geodesics of a particle in the de Sitter Universe. The motion of a test particle is denoted by the geodesic equation (2.99),

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{kl}\frac{dx^k}{ds}\frac{dx^l}{ds} = 0$$

Using the line element before transformation, i.e. the de Sitter - Schwarzschild line element (2.407), given by

$$ds^2 = \frac{dr^2}{1 - r^2/R^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) - \left(1 - \frac{r^2}{R^2}\right)c^2dt^2$$

Under the assumptions used in order to attain the de Sitter - Schwarzschild line element, i.e. assuming a homogeneous, isotropic, and spherically symmetric solution, we may use the Christoffel symbols denoted in (2.278) for the centrally symmetric solution of the field equations, namely,

$$\begin{array}{lll}
\Gamma^1_{11} = \frac{\lambda'}{2} & \Gamma^0_{10} = \frac{\nu'}{2} & \Gamma^2_{33} = -\sin\theta\cos\theta \\
\Gamma^0_{11} = \frac{\dot{\lambda}}{2}e^{\lambda-\nu} & \Gamma^1_{22} = -re^{-\lambda} & \Gamma^1_{00} = \frac{\nu'}{2}e^{\nu-\lambda} \\
\Gamma^2_{12} = \Gamma^3_{13} = \frac{1}{r} & \Gamma^3_{23} = \cot\theta & \Gamma^0_{00} = \frac{\dot{\nu}}{2} \\
\Gamma^1_{10} = \frac{\dot{\lambda}}{2} & \Gamma^1_{33} = -r\sin^2\theta e^{-\lambda} &
\end{array}$$

Then, substituting the surviving terms of the Christoffel symbols into the geodesic equation we may attain the geodesic of the test particle in the considered space-time. Thence, consider $i = 1$ such that,

$$\frac{d^2x^1}{ds^2} + \Gamma^1_{kl}\frac{dx^k}{ds}\frac{dx^l}{ds} = 0$$

Now, since the only surviving Christoffel symbols are those of the form $l = k$, we have

$$\frac{d^2x^1}{ds^2} + \left[\Gamma^1_{00}\frac{dx^0}{ds}\frac{dx^0}{ds} + \Gamma^1_{11}\frac{dx^1}{ds}\frac{dx^1}{ds} + \Gamma^1_{22}\frac{dx^2}{ds}\frac{dx^2}{ds} + \Gamma^1_{33}\frac{dx^3}{ds}\frac{dx^3}{ds}\right] = 0$$
$$\implies \frac{d^2x^1}{ds^2} + \left[\Gamma^1_{00}\left(\frac{dx^0}{ds}\right)^2 + \Gamma^1_{11}\left(\frac{dx^1}{ds}\right)^2 + \Gamma^1_{22}\left(\frac{dx^2}{ds}\right)^2 + \Gamma^1_{33}\left(\frac{dx^3}{ds}\right)^2\right] = 0$$

$$\frac{d^2r}{ds^2} + \left[\frac{\nu'}{2}e^{\nu-\lambda}\left(\frac{dt}{ds}\right)^2 + \frac{\lambda'}{2}\left(\frac{dr}{ds}\right)^2 - re^{-\lambda}\left(\frac{d\theta}{ds}\right)^2 - re^{-\lambda}\sin^2\theta\left(\frac{d\phi}{ds}\right)^2\right] = 0 \qquad (2.489)$$

Then, for $i = 2$, we have

$$\frac{d^2x^2}{ds^2} + \Gamma^2_{kl}\frac{dx^k}{ds}\frac{dx^l}{ds} = 0$$

The only terms which have an upper index of 2 are those with the following form, $k = 1$, $l = 2$ and

$k = 3 = l$, taking a sum yields

$$\frac{d^2x^2}{ds^2} + \left[\Gamma^1_{12}\frac{dx^1}{ds}\frac{dx^2}{ds} + \Gamma^1_{33}\frac{dx^3}{ds}\frac{dx^3}{ds}\right] = 0$$
$$\implies \frac{d^2x^2}{ds^2} + \left[\Gamma^1_{12}\frac{dx^1}{ds}\frac{dx^2}{ds} + \Gamma^1_{33}\left(\frac{dx^3}{ds}\right)^2\right] = 0$$
$$\implies \frac{d^2\theta}{ds^2} + \left[\frac{1}{r}\frac{dr}{ds}\frac{d\theta}{ds} - \sin\theta\cos\theta\left(\frac{d\phi}{ds}\right)^2\right] = 0$$

$$\frac{d^2\theta}{ds^2} + \left[\frac{2}{r}\frac{dr}{ds}\frac{d\theta}{ds} - \sin\theta\cos\theta\left(\frac{d\phi}{ds}\right)^2\right] = 0 \tag{2.490}$$

where the "2" in the last term comes from the symmetry in the lower indices of the Christoffel symbols, i.e. $\Gamma^2_{12} = \Gamma^2_{21}$. Similarly, for the remaining terms we $i = 3$,

$$\frac{d^2x^3}{ds^2} + \Gamma^3_{kl}\frac{dx^k}{ds}\frac{dx^l}{ds} = 0$$

Only surviving terms with upper index $i = 3$ have the following form $k = 1$, $l = 3$ and $k = 2$, $l = 3$,

$$\implies \frac{d^2x^3}{ds^2} + \left[\Gamma^3_{13}\frac{dx^1}{ds}\frac{dx^3}{ds} + \Gamma^3_{23}\frac{dx^2}{ds}\frac{dx^3}{ds}\right] = 0$$
$$\implies \frac{d^2\phi}{ds^2} + \left[\frac{1}{r}\frac{dr}{ds}\frac{d\phi}{ds} + \cot\theta\frac{d\theta}{ds}\frac{d\phi}{ds}\right] = 0$$

$$\frac{d^2\phi}{ds^2} + \left[\frac{2}{r}\frac{dr}{ds}\frac{d\phi}{ds} + 2\cot\theta\frac{d\theta}{ds}\frac{d\phi}{ds}\right] = 0 \tag{2.491}$$

where, once again, "2's" in the last term come from the symmetry in the lower indices of the Christoffel symbols. Finally, for $i = 0$, we attain

$$\frac{d^2x^0}{ds^2} + \Gamma^0_{kl}\frac{dx^k}{ds}\frac{dx^l}{ds} = 0$$

Only surviving terms with upper index $i = 0$ are of the form $k = 1$, $l = 0$, such that

$$\implies \frac{d^2t}{ds^2} + \Gamma^0_{10}\frac{dx^1}{ds}\frac{dx^0}{ds} = 0$$
$$\implies \frac{d^2t}{ds^2} + \frac{\nu'}{2}\frac{dr}{ds}\frac{dt}{ds} = 0$$

$$\frac{d^2t}{ds^2} + \nu'\frac{dr}{ds}\frac{dt}{ds} = 0 \tag{2.492}$$

Next, assume the motion to be in the plane $\theta = \frac{\pi}{2}$, such that

$$\sin\theta = 1 \quad \cos\theta = 0 = \frac{d\theta}{ds} \tag{2.493}$$

From (2.490), we derive

$$\frac{d^2\theta}{ds^2} + \left[\frac{2}{r}\frac{dr}{ds}(0) - (1)(0)\left(\frac{d\phi}{ds}\right)^2\right] = 0$$
$$\implies \frac{d^2\theta}{ds^2} = 0$$

which implies that the particle will move in the plane $\theta = \frac{\pi}{2}$ continuously. Thence, using (2.493) we may reduce (2.490), (2.491), and (2.492) such that using the result attained above, i.e. $\frac{d^2\theta}{ds^2} = 0$, we may then plug it back into (2.490) to find

$$(0) + \left[\frac{2}{r}\frac{dr}{ds}(0) - \sin\theta\cos\theta\left(\frac{d\phi}{ds}\right)^2\right] = 0$$
$$\implies \left(\frac{d\phi}{ds}\right)^2 = 0$$

Then, (2.489) becomes

$$\frac{d^2r}{ds^2} + \left[\frac{\nu'}{2}e^{\nu-\lambda}\left(\frac{dt}{ds}\right)^2 + \frac{\lambda'}{2}\left(\frac{dr}{ds}\right)^2 - re^{-\lambda}\left(\frac{d\theta}{ds}\right)^2\right] = 0 \tag{2.494}$$

Similarly, (2.491) & (2.492) become

$$\frac{d^2\phi}{ds^2} + \left[\frac{2}{r}\frac{dr}{ds}\frac{d\phi}{ds}\right] = 0 \tag{2.495}$$

$$\frac{d^2t}{ds^2} + v'\frac{dr}{ds}\frac{dt}{ds} = 0 \tag{2.496}$$

From (2.495) we obtain,

$$\frac{d}{ds}\left(r^2\frac{d\phi}{ds}\right) = 0 \tag{2.497}$$

since,

$$\begin{aligned}
\frac{d}{ds}\left(r^2\right)\frac{d\phi}{ds} + r^2\left(\frac{d^2\phi}{ds^2}\right) &= 0 \\
\implies 2r\frac{d\phi}{ds} + r^2\left(\frac{d^2\phi}{ds^2}\right) &= 0 \\
\implies \frac{2}{r}\frac{d\phi}{ds} + \left(\frac{d^2\phi}{ds^2}\right) &= 0
\end{aligned}$$

which is equivalent to (2.495). Integration, yields,

$$\begin{aligned}
\int \frac{d}{ds}\left(r^2\frac{d\phi}{ds}\right) &= \int 0 \\
\implies r^2\frac{d\phi}{ds} &= \text{const} \\
\implies r^2\frac{d\phi}{ds} &= h
\end{aligned}$$

$$\frac{d\phi}{ds} = \frac{h}{r^2} \tag{2.498}$$

By the same logic, from (2.496) we obtain,

$$\frac{d}{ds}\left(e^{v}\frac{dt}{ds}\right) = 0$$

namely,

$$
\begin{aligned}
&\frac{d}{ds}\left(e^{v}\right)\frac{dt}{ds}+e^{v}\left(\frac{d^2t}{ds^2}\right)=0\\
\implies &\frac{d}{ds}\left(e^{v}\right)\frac{dt}{ds}+e^{v}\left(\frac{d^2t}{ds^2}\right)=0\\
\implies &e^{v}\left(\frac{d}{ds}v(s,r)\right)\frac{dt}{ds}+e^{v}\left(\frac{d^2t}{ds^2}\right)=0\\
\implies &v'e^{v}\frac{dr}{ds}\frac{dt}{ds}+e^{v}\left(\frac{d^2t}{ds^2}\right)=0\\
\implies &e^{-v}\left(v'e^{v}\frac{dr}{ds}\frac{dt}{ds}+e^{v}\left(\frac{d^2t}{ds^2}\right)\right)=0\left(e^{-v}\right)\\
\implies &v'\frac{dr}{ds}\frac{dt}{ds}+\left(\frac{d^2t}{ds^2}\right)=0
\end{aligned}
$$

Integration yields,

$$
\begin{aligned}
&\int\frac{d}{ds}\left(e^{v}\frac{dt}{ds}\right)=\int 0\\
\implies &e^{v}\frac{dt}{ds}=k\\
\implies &\frac{dt}{ds}=ke^{-v}
\end{aligned}
$$

Then, substituting the value attained for $e^{-v}=\left(1-\frac{r^2}{R^2}\right)^{-1}$, thus,

$$\frac{dt}{ds}=k\left(1-\frac{r^2}{R^2}\right)^{-1} \tag{2.499}$$

After which, inserting (2.493) into the line element we derive

$$
\begin{aligned}
ds^2 &= e^{\nu} dt^2 - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) - e^{\lambda} dr^2 \\
\implies -ds^2 &= -e^{\nu} dt^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) + e^{\lambda} dr^2 \\
\implies -\frac{ds^2}{ds^2} &= -e^{\nu} \left(\frac{dt^2}{ds^2}\right) + r^2 \left(\frac{d\theta^2}{ds^2} + \sin^2\theta \frac{d\phi^2}{ds^2}\right) + e^{\lambda} \frac{dr^2}{ds^2} \\
\implies -1 &= e^{\lambda} \left(\frac{dr}{ds}\right)^2 + r^2 \left(\frac{d\theta}{ds}\right)^2 + r^2 \sin^2\theta \left(\frac{d\phi}{ds}\right)^2 - e^{\nu} \left(\frac{dt}{ds}\right)^2 \\
\implies -r^2 \left(\frac{d\theta}{ds}\right)^2 &= e^{\lambda} \left(\frac{dr}{ds}\right)^2 + r^2 \sin^2\theta \left(\frac{d\phi}{ds}\right)^2 - e^{\nu} \left(\frac{dt}{ds}\right)^2 + 1
\end{aligned}
$$

evaluating the line element at $\theta = \frac{\pi}{2}$ and substituting the values obtained in (2.498) and (2.499) into the above line element we find,

$$
\begin{aligned}
-r^2 \left(\frac{d\theta}{ds}\right)^2 &= e^{\lambda} \left(\frac{dr}{ds}\right)^2 + r^2 \sin^2\theta \left(\frac{d\phi}{ds}\right)^2 - e^{\nu} \left(\frac{dt}{ds}\right)^2 + 1 \\
\implies 0 &= e^{\lambda} \left(\frac{dr}{ds}\right)^2 + r^2 \left(\frac{d\phi}{ds}\right)^2 - e^{\nu} \left(\frac{dt}{ds}\right)^2 + 1 \\
\implies e^{\lambda} \left(\frac{dr}{ds}\right)^2 &+ r^2 \left(\frac{h}{r^2}\right)^2 - e^{\nu} \left(k\left(1 - \frac{r^2}{R^2}\right)^{-1}\right)^2 + 1 = 0 \\
\implies e^{\lambda} \left(\frac{dr}{ds}\right)^2 &+ \frac{h^2}{r^2} - e^{\nu} k^2 \left(1 - \frac{r^2}{R^2}\right)^{-2} + 1 = 0 \\
\implies \left(\frac{dr}{ds}\right)^2 &+ \frac{h^2}{r^2} e^{-\lambda} - e^{\nu-\lambda} k^2 \left(1 - \frac{r^2}{R^2}\right)^{-2} + e^{-\lambda} = 0
\end{aligned}
$$

Using the attained values for $e^{-\lambda} = e^{\nu} = 1 - \frac{r^2}{R^2}$, we have

$$\implies \left(\frac{dr}{ds}\right)^2 + \frac{h^2}{r^2}\left(1 - \frac{r^2}{R^2}\right) - \left(1 - \frac{r^2}{R^2}\right)^2 k^2 \left(1 - \frac{r^2}{R^2}\right)^{-2} + e^{-\lambda} = 0$$

$$\implies \left(\frac{dr}{ds}\right)^2 + \frac{h^2}{r^2}\left(1 - \frac{r^2}{R^2}\right) - k^2 + e^{-\lambda} = 0$$

$$\implies \left(\frac{dr}{ds}\right)^2 + \frac{h^2}{r^2}\left(1 - \frac{r^2}{R^2}\right) - k^2 + \left(1 - \frac{r^2}{R^2}\right) = 0$$

$$\implies \left(\frac{dr}{ds}\right)^2 = k^2 - \frac{h^2}{r^2}\left(1 - \frac{r^2}{R^2}\right) - \left(1 - \frac{r^2}{R^2}\right)$$

$$\implies \left(\frac{dr}{ds}\right)^2 = k^2 - \frac{h^2}{r^2} + \frac{h^2}{R^2} - 1 + \frac{r^2}{R^2}$$

From which we obtain,

$$\frac{dr}{ds} = \left[= k^2 - \frac{h^2}{r^2} + \frac{h^2}{R^2} - 1 + \frac{r^2}{R^2}\right]^{1/2} \tag{2.500}$$

From (2.500) we find

$$\left(\frac{h}{r^2}\frac{dr}{ds}\right)^2 = k^2 - \frac{h^2}{r^2} + \frac{h^2}{R^2} - 1 + \frac{r^2}{R^2} \tag{2.501}$$

Let $u = \frac{1}{r}$, such that $\frac{du}{d\phi} = -\frac{1}{r^2}\frac{dr}{d\phi}$, and (2.501) becomes

$$\left(\frac{du}{d\phi}\right)^2 = \frac{k^2 - 1}{h^2} + \frac{1}{u^2 h^2 R^2} - u^2 + \frac{1}{R^2} \tag{2.502}$$

Differentiation yields

$$\frac{d^2u}{d\phi^2} + u = -\frac{1}{u^2 h^2 R^2} \tag{2.503}$$

which expresses an orbital equation of the particle in the de Sitter universe.

Alternatively, we now consider the de Sitter line element, namely, (2.452), such that we have

$$ds^2 = e^{2ct'/R}\left(dx'^2 + dy'^2 + dz'^2\right) - c^2 dt'^2$$

with corresponding metric $g_{\mu\nu}$,

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & e^{2t} & 0 & 0 \\ 0 & 0 & e^{2t} & 0 \\ 0 & 0 & 0 & e^{2t} \end{pmatrix} \tag{2.504}$$

with corresponding inverse (contravariant) metric,

$$g^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & e^{-2t} & 0 & 0 \\ 0 & 0 & e^{-2t} & 0 \\ 0 & 0 & 0 & e^{-2t} \end{pmatrix} \tag{2.505}$$

where we have set the constant $c = R = 1$, for the sake of simplicity, as well as set (t', x', y', z') to (t, x, y, z). Thus, computing the corresponding Christoffel symbols for the considered line element, we have that

$$\begin{aligned} \Gamma^0_{11} &= e^{2t} \quad \Gamma^1_{10} = 1 = \Gamma^1_{01} \\ \Gamma^0_{22} &= e^{2t} \quad \Gamma^2_{20} = 1 = \Gamma^2_{02} \\ \Gamma^0_{33} &= e^{2t} \quad \Gamma^3_{30} = 1 = \Gamma^3_{03} \end{aligned}$$

where we point attention to the fact that all spatial Christoffel symbols vanish, as delineated by (2.453).

Indeed, consider the following Christoffel symbols,

$$\Gamma^0_{11} = \frac{1}{2} g^{0\sigma} \left(g_{1\sigma,1} + g_{\sigma 1,1} + g_{11,\sigma} \right)$$

The only viable solution for the Christoffel symbol is obtained by letting $\sigma = 0$ such that

$$\begin{aligned}
\Gamma^0_{11} &= \frac{1}{2} g^{00} \left(g_{10,1} + g_{01,1} - g_{11,0} \right) \\
\implies &= \frac{1}{2} g^{00} \left(0 + 0 - g_{11,0} \right) \\
\implies &= \frac{1}{2} g^{00} \left(0 + 0 - g_{11,0} \right) \\
\implies & \frac{1}{2}(-1)\left(-\frac{\partial}{\partial t} e^{2t} \right) = e^{2t}
\end{aligned}$$

with the rest of the Christoffel symbols of the form Γ^0_{ii} where $i = 1, 2, 3$ following analogously. Next, consider

$$\Gamma^1_{10} = \frac{1}{2} g^{1\sigma} \left(g_{1\sigma,0} + g_{\sigma 0,1} + g_{01,\sigma} \right)$$

The only choice which keeps the Christoffel symbol from vanishing is obtained by setting $\sigma = 1$, thus,

$$\begin{aligned}
\Gamma^1_{10} &= \frac{1}{2} g^{11} \left(g_{11,0} + g_{10,1} + g_{01,1} \right) \\
\implies &= \frac{1}{2} g^{11} \left(g_{11,0} + 0 - 0 \right) \\
\implies &= \frac{1}{2} \left(e^{-2t} \right) \left(\frac{\partial}{\partial t} e^{2t} \right) = \frac{1}{2} (2) e^{2t-2t} = 1
\end{aligned}$$

due to symmetry of the Christoffel symbol in the lower indices we have $\Gamma^1_{10} = \Gamma^1_{01} = 1$. Thus, we find that all symbols of the form $\Gamma^i_{i0} = \Gamma^i_{0i} = 1$, in which $i = 1, 2, 3$, are equivalent.

Then, substituting the surviving terms of the Christoffel symbols into the geodesic equation we find the following geodesics, letting $i = 0$, we have

$$\frac{d^2 x^0}{ds^2} + \Gamma^0_{kl} \frac{dx^k}{ds} \frac{dx^l}{ds} = 0$$

correspondingly, letting $k = l = 1,2,3$, such that

$$\frac{d^2x^0}{ds^2} + \left[\Gamma^0_{11}\frac{dx^1}{ds}\frac{dx^1}{ds} + \Gamma^0_{22}\frac{dx^2}{ds}\frac{dx^2}{ds} + \Gamma^0_{33}\frac{dx^3}{ds}\frac{dx^3}{ds}\right] = 0$$
$$\frac{d^2t}{ds^2} + \left[e^{2t}\left(\frac{dx}{ds}\right)^2 + e^{2t}\left(\frac{dy}{ds}\right)^2 + e^{2t}\left(\frac{dz}{ds}\right)^2\right] = 0$$
$$\frac{d^2t}{ds^2} + e^{2t}\left[\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2\right] = 0$$

Next, consider the square of the magnitude of the four velocity of a particle with mass traveling along a time-like world line such that

$$g_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = u^\alpha u_\beta = -c^2$$

which holds when $\alpha = \beta$, such that

$$g_{\alpha\alpha}\frac{dx^\alpha}{ds}\frac{dx^\alpha}{ds} = u^\alpha u_\alpha = -c^2$$

Nonetheless, since we set $c = 1$, we rewrite the above equation as

$$g_{\alpha\alpha}\frac{dx^\alpha}{ds}\frac{dx^\alpha}{ds} = u^\alpha u_\alpha = -1$$

Thence,

$$g_{00}\frac{dx^0}{ds}\frac{dx^0}{ds} + g_{11}\frac{dx^1}{ds}\frac{dx^1}{ds} + g_{22}\frac{dx^2}{ds}\frac{dx^2}{ds} + g_{33}\frac{dx^3}{ds}\frac{dx^3}{ds} = -1$$
$$\implies g_{00}\left(\frac{dt}{ds}\right)^2 + e^{2t}\left[\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2\right] = -1$$
$$\implies (-1)\left(\frac{dt}{ds}\right)^2 + e^{2t}\left[\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2\right] = -1$$

Thus, consider the above two equations, such that we proceed to solving them as a system of equations, namely,

$$
\begin{aligned}
&\frac{d^2t}{ds^2} + e^{2t}\left[\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2\right] = 0 \\
&-\left[-\left(\frac{dt}{ds}\right)^2 + e^{2t}\left[\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2\right] + 1 = 0\right] \\
&\implies \frac{d^2t}{ds^2} + \left(\frac{dt}{ds}\right)^2 - 1 = 0 \\
&\implies \frac{d^2t}{ds^2} + \left(\frac{dt}{ds}\right)^2 = 1
\end{aligned}
$$

which is a second-order nonlinear ordinary differential equation, which we may solve by means of substitution. Consider the substitution, $\frac{dt}{ds} = u$, such that we may rewrite the above as follows

$$
u'' + u'^2 - 1 = 0 \tag{2.506}
$$

We proceed by solving by means of substitution, letting $\frac{du}{ds} = v(s)$, we have that

$$
\begin{aligned}
&v' = -v^2 + 1 \\
&\implies \frac{v'}{-v^2+1} = 1 \\
&\implies \int \frac{v'}{-v^2+1} = \int 1 ds \\
&\implies \int -\frac{v'}{(v+1)(v-1)} = \int 1 ds \\
&\implies -\frac{1}{2}\ln(-v+1) + \frac{1}{2}\ln(v+1) = s + c_1 \\
&\implies \exp\left[-\frac{1}{2}\ln(-v+1) + \frac{1}{2}\ln(v+1)\right] = e^{2(s+c_1)} \\
&\implies v(s) = \frac{e^{2(s+c_1)} - 1}{e^{2(s+c_1)} + 1}
\end{aligned}
$$

Then, reverting back to $du/ds = v(s)$, we have that

$$\begin{aligned} \frac{du}{ds} &= \frac{e^{2(s+c_1)}-1}{e^{2(s+c_1)}+1} \\ \implies \int \frac{du}{ds} &= \int \frac{e^{2(s+c_1)}-1}{e^{2(s+c_1)}+1} ds \\ \implies u &= \ln\left(e^{2(s+c_1)}+1\right) - s + c_2 \\ \implies u &= \ln\left(c_1 e^{2s}+1\right) - s + c_2 \end{aligned}$$

Letting $c_1 = 1$ and $c_2 = 0$, we have the solution,

$$t = u = \ln\left(e^{2s}+1\right) - s \tag{2.507}$$

Then,

$$\frac{dt}{ds} = \frac{d}{ds}\left[\ln\left(e^{2s}+1\right) - s\right] = \frac{1}{(e^{2s}+1)}\left(\frac{d}{ds}(e^{2s}+1)\right) - 1 = \frac{1}{(e^{2s}+1)}\left(2e^{2s}-1\right)$$
$$\frac{\left(2e^{2s}-1\right)}{(e^{2s}+1)} - \frac{(e^{2s}+1)}{(e^{2s}+1)} = \frac{(e^{2s}-1)}{(e^{2s}+1)} = \tanh(s)$$

Now, substituting the remaining terms of the Christoffel symbols into the geodesic equation we find the following geodesics, letting $i = 1, 2, 3$, respectively, we find the following geodesic equations

$$\frac{d^2x^1}{ds^2} + \Gamma^1_{kl}\frac{dx^k}{ds}\frac{dx^l}{ds} = 0$$

and letting $k = 0$ and $l = 1$, we have

$$\begin{aligned}
&\frac{d^2x^1}{ds^2} + \Gamma^1_{01}\frac{dx^0}{ds}\frac{dx^1}{ds} = 0 \\
\implies &\frac{d^2x}{ds^2} + \Gamma^1_{01}\frac{dt}{ds}\frac{dx}{ds} = 0 \\
\implies &\frac{d^2x}{ds^2} + (1)\frac{dt}{ds}\frac{dx}{ds} = 0 \\
\implies &\frac{d^2x}{ds^2} + (2)\frac{dt}{ds}\frac{dx}{ds} = 0
\end{aligned}$$

where the "2" follows from the symmetry of the Christoffel symbols. Consequently, we have analogous equations for the x^2 and x^3 coordinates, namely,

$$\begin{aligned}
\frac{d^2y}{ds^2} + (2)\frac{dt}{ds}\frac{dy}{ds} = 0 \\
\frac{d^2z}{ds^2} + (2)\frac{dt}{ds}\frac{dz}{ds} = 0
\end{aligned}$$

Note that the resulting equations are of the same form, thus, it suffices to solve only one of the above geodesic equations. To this end, we consider the above equation of the x-coordinate, such that we denote it as a second-order linear differential equation.

$$\begin{aligned}
&\frac{d^2x}{ds^2} + (2)\frac{dt}{ds}\frac{dx}{ds} = 0 \\
\implies &\frac{d^2x}{ds^2} + (2)\tanh(s)\frac{dx}{ds} = 0
\end{aligned}$$

Rewriting the ODE above and considering the substitution $\frac{du}{ds} = v(s)$, we have that

$$
\begin{aligned}
v' + 2\tanh(s)v = 0 \\
\implies \frac{v'}{v} = -2\tanh(s) \\
\implies \int \frac{v'}{v} = -2\int \tanh(s) \\
\implies \ln|v| = -2\ln(\cosh(s)) + c_1 \\
\implies |v| = e^{-2\ln(\cosh(s)) + c_1} \\
\implies |v| = e^{c_1} e^{-2\ln(\cosh(s))} \\
\implies |v| = c_1 e^{\ln(\cosh^{-2}(s))} \\
\implies |v| = c_1 \cosh^{-2}(s) = c_1 \operatorname{sech}^2(s)
\end{aligned}
$$

Then, reverting back to u we find that

$$
\begin{aligned}
\frac{du}{ds} = c_1 \operatorname{sech}^2(s) \\
\implies \int \frac{du}{ds} = \int c_1 \operatorname{sech}^2(s) \\
\implies u = c_1 \tanh(s) + c_2
\end{aligned}
$$

Thus, we have that

$$
x = c_1 \tanh(s) + c_2 \tag{2.508}
$$

Similarly, we write for the remaining coordinates

$$
y = c_3 \tanh(s) + c_4
$$

$$
z = c_5 \tanh(s) + c_6
$$

Next we consider the geodesics for the aptly named Schwarzschild exterior solution. We define the metric as follows

$$ds^2 = \frac{dr^2}{1-2m/r-\lambda r^2/3} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) - \left(1 - \frac{2m}{r} - \frac{\lambda r^2}{3}\right)c^2 dt^2 \tag{2.509}$$

Set $c = 1$ and consider following line-element

$$ds^2 = -\left(1 - \frac{2m}{r} - \frac{\lambda r^2}{3}\right)dt^2 + \frac{dr^2}{1-2m/r-\lambda r^2/3} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{2.510}$$

To this end, we have the following metric and inverse metric,

$$g_{\mu\nu} = \begin{pmatrix} -\left(1-\frac{2m}{r}-\frac{\lambda r^2}{3}\right) & 0 & 0 & 0 \\ 0 & \frac{1}{1-2m/r-\lambda r^2/3} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \tag{2.511}$$

$$g^{\mu\nu} = \begin{pmatrix} -\frac{1}{1-2m/r-\lambda r^2/3} & 0 & 0 & 0 \\ 0 & \left(1-\frac{2m}{r}-\frac{\lambda r^2}{3}\right) & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{pmatrix} \tag{2.512}$$

and denote the following $x^\mu = (x^0, x^1, x^2, x^3) = (t, r, \theta, \phi)$. As found previously, in multiple occasions, we consider the Christoffel symbols of the corresponding metric, such that we arrive at the following symbols

$$\Gamma^0_{01} = -\frac{3m+\lambda r^3}{r\left(3r-6m-\lambda r^3\right)} = \Gamma^0_{10} \qquad \Gamma^1_{22} = -\left(r - 2m - \frac{\lambda r^3}{3}\right)$$

$$\Gamma^1_{00} = \frac{(3m+\lambda r^3)(6m-3r+\lambda r^3)}{9r^3} \qquad \Gamma^1_{33} = -\left(r - 2m - \frac{\lambda r^3}{3}\right)\sin^2\theta \qquad \Gamma^2_{33} = -\sin\theta\cos\theta$$

$$\Gamma^1_{11} = \frac{3m-\lambda r^3}{r\left(\lambda r^3+6m-3r\right)} \qquad \Gamma^3_{23} = \cot\theta \qquad \Gamma^2_{21} = \frac{1}{r} = \Gamma^2_{12}$$

$$\Gamma^3_{13} = \frac{1}{r} = \Gamma^3_{31}$$

To this end, we have the following geodesic equations

$$\frac{d^2t}{ds^2} - \frac{6m+2\lambda r^3}{r(3r-6m-\lambda r^3)}\frac{dt}{ds}\frac{dr}{ds} = 0$$

$$\frac{d^2r}{ds^2} + \frac{(3m+\lambda r^3)(6m-3r+\lambda r^3)}{9r^3}\left(\frac{dt}{ds}\right)^2 + \frac{3m-\lambda r^3}{r(\lambda r^3+6m-3r)}\left(\frac{dr}{ds}\right)^2$$
$$-\left(r-2m-\frac{\lambda r^3}{3}\right)\left(\frac{d\theta}{ds}\right)^2 - \left(r-2m-\frac{\lambda r^3}{3}\right)\sin^2\theta\left(\frac{d\phi}{ds}\right)^2 = 0$$

$$\frac{d^2\theta}{ds^2} - \sin\theta\cos\theta\left(\frac{d\phi}{ds}\right)^2 + \left(\frac{2}{r}\right)\frac{d\theta}{ds}\frac{dr}{ds} = 0$$

$$\frac{d^2\phi}{ds^2} + \left(\frac{2}{r}\right)\frac{dr}{ds}\frac{d\phi}{ds} + 2\cot\theta\frac{d\theta}{ds}\frac{d\phi}{ds} = 0$$

Then, we make the following assumption: $\theta = \frac{\pi}{2}$ such that $d\theta = 0$. We may arrive at such an assumption, since we know that the metric is symmetric, and any geodesic which starts to move within the plane $\theta = \frac{\pi}{2}$ will remain confined in the plane indefinitely. We also point attention to the fact that our assumption satisfies the fourth equation above; this is easily verified by direct substitution onto the above equation. Thus, we can reduce the following equations, namely the fourth equation above, such that

$$\begin{aligned}
&\frac{d^2\phi}{ds^2} + \left(\frac{2}{r}\right)\frac{dr}{ds}\frac{d\phi}{ds} + 2\cot\theta\frac{d\theta}{ds}\frac{d\phi}{ds} = 0 \\
&\implies \frac{d^2\phi}{ds^2} + \left(\frac{2}{r}\right)\frac{dr}{ds}\frac{d\phi}{ds} = 0 \\
&\implies \frac{d^2\phi}{ds^2} = -\left(\frac{2}{r}\right)\frac{dr}{ds}\frac{d\phi}{ds} \\
&\implies -\frac{\phi''}{2\phi'} = \frac{r'}{r} \\
&\implies -\int\frac{\phi''}{2\phi'} = \int\frac{r'}{r} \\
&\implies -\frac{\ln(\phi')}{2} = \ln(r) + C \\
&\implies C = \ln(r) + \frac{\ln(\phi')}{2} \\
&\implies C = 2\ln(r) + \ln\left(\phi'\right) \\
&\implies C = \ln\left(r^2\right) + \ln\left(\phi'\right) \\
&\implies C = \ln\left(r^2\right) + \ln\left(\frac{d\phi}{ds}\right)
\end{aligned}$$

CHAPTER III

FUNDAMENTAL OF QUANTUM MECHANICS

3.1 Basics of Quantum Mechanics

We now shift our attention to the study of matter on a molecular and sub-molecular scale. We provide a brief overview of the tenets governing the processes of subatomic matter, by briefly discussing the axioms and postulates required in order to craft a theory of matter at such a minute scale. We note, however, unlike previous theories which deal with matter in a macroscopic scale, i.e. in Newtonian mechanics, we do not have an intuitive access to quantum mechanics. The postulates presented bear with them minimum to no rationale for their consideration, yet they form the foundation of the theory. The formalism presented allows the computation of observables which in turn conform to experimental data, as such we consider such conventionalism to be the basis of the theory.

3.1.1 The Quantum Mechanical State

In order to study the processes of the particles in a sub molecular state we require that we define the current "circumstance" which the particle/system is experiencing within the moment in time we wish to consider. To this end, it is convenient we define the *state* of a system in quantum mechanics. The *wave function*, Ψ_n, contains all information that can be known about the quantum mechanical system, and is distinguished between states by the subscript n. We note in passing that the term deemed "quantum mechanical system" denotes an elementary particle, i.e. an electron, or an aggregate of elementary particles.

3.1.2 Bracket Notation

We introduce the Dirac *bra-ket notation*, $|\Psi_n\rangle = |n\rangle$, used to represent the n-th quantum mechanical state of Ψ_n. The *ket* $|n\rangle$ refers to the quantum mechanical state Ψ_n, while the *bra* $\langle n|$ refers to its complex conjugate $\Psi_n^\star$ (this is to be replaced by $\Psi_n^\dagger = \Psi_n^{\star T}$ if the state contains more than one component and is a vector quantity). We find that the bra vector $\langle n|$ is not an element of the vector space, but rather of the dual vector space, which is the space of linear mappings

$$\begin{aligned} \langle n| : \quad & V \to \mathbb{R} \\ & |m\rangle \mapsto \langle n|m\rangle \end{aligned} \tag{3.1}$$

here $\langle n|m\rangle$ denotes the scalar product on the vector space V - the space of square-integrable functions. The scalar product is defined over all dynamical variables ω,

$$\langle n|m\rangle = \int_{-\infty}^{+\infty} \mathrm{d}\sigma \Psi_n^\star(\sigma)\Psi_m(\sigma) \tag{3.2}$$

3.1.3 Expansion in a Complete Basis Set

A state can be described by known basis functions $\Phi_k(\sigma)$,

$$\Psi_n(\sigma) = \sum_{k=1}^{K} C_{nk}\Phi_k(\sigma) \tag{3.3}$$

with unknown coefficients C_{nk}. The basis functions may be chosen in such a way that they constitute an orthonormal set, i.e.

$$\int_{-\infty}^{+\infty} \mathrm{d}\sigma \quad \Phi_k^\star(\sigma)\Phi_l(\sigma) = \delta_{kl} \tag{3.4}$$

Thus, we may write the inner product in the basis expansion as

$$\langle n|m\rangle = \sum_{k,l=1}^{K} C_{nk}^{\star} C_{ml} \int_{-\infty}^{+\infty} \mathrm{d}\sigma \Phi_k^{\star}(\sigma)\Phi_l(\sigma) = \sum_{k=1}^{K} C_{nk}^{\star} C_{mk} \equiv C_n^{\dagger} C_m \tag{3.5}$$

where we incorporated the fact that we may always choose the basis functions as an orthonormal set, i.e. (3.4), and we evidently set $l = k$. Here the C_m denotes a vector of all expansion coefficients C_{mk} of the state $|m\rangle$. In general, we have that the number of basis functions K is in infinite if we are to consider an exact representation of the state. Thus, if $K = \infty$ the basis is *complete*. In this case, we obtain,

$$\sum_{k=1}^{\infty} |\Phi_k\rangle \langle\Phi_k| = \mathbf{1} = id \tag{3.6}$$

which is deemed the resolution of the identity. If K is finite, then the basis is said to be *finite*.

3.1.4 Born's Interpretation

The state $|n\rangle$ has no physical meaning, yet the absolute square (or squared modulus) $|\Psi_n|^2 = \Psi_n^{\star}\Psi_n$, may be interpreted as a probability density distribution, i.e the probability of observing the given system in a certain condition in a given moment in time. This is the *Born-interpretation* of quantum mechanics, which implies that for a single particle the wave function must be normalized, i.e. integration over all variables of the system must result in unity,

$$\langle n|n\rangle \stackrel{!}{=} 1 \tag{3.7}$$

i.e. the probability of observing the particle over all of space (if the dynamical variable chosen is one of spatial configuration) must be one at any time. We may extend the concept of the Born interpretation for a system of an arbitrary number of identical particles. Consider an N number of particles with positions $\{r_i\}$, such that they are described by the wave function $\Psi_n(r_1, r_2, \ldots, r_N)$, describing the n-th state. (3.7) dictates that the wave function will be normalized as follows,

$$\langle n|n\rangle = \int_{-\infty}^{+\infty} \mathrm{d}^3 r_1 \cdots \int_{-\infty}^{+\infty} \mathrm{d}^3 r_N \Psi_n^{\star}(r_1, r_2, \ldots, r_N)\, \Psi_n(r_1, r_2, \ldots, r_N) \stackrel{!}{=} 1 \tag{3.8}$$

Subsequently, the integrand of (3.8) details the probability of finding particle 1 in the volume element $\mathrm{d}^3 r_1$, particle 2 in volume element $\mathrm{d}^3 r_2$, etc.

Define the particle density distribution of the state Ψ_n as

$$\rho_n(r) \equiv N \int_{-\infty}^{+\infty} \mathrm{d}^3 r_2 \cdots \int_{-\infty}^{+\infty} \mathrm{d}^3 r_N \Psi_n^{\star}(r, r_2, \ldots, r_N)\, \Psi_n(r, r_2, \ldots, r_N) \tag{3.9}$$

such that $[\rho_n(r)/N]\, \mathrm{d}^3 r$ is the probability of finding *any* given particle in the volume element dr^3. Observe that integration occurs over all but the first particle coordinate, for which the index has been dropped. This possibility arises due to the fact that we consider N identical particles and thus any particle coordinate could be selected to survive integration. Consequently, we assume that an interchanging of particle coordinates within the wave function produces a sign change, i.e. $\Psi(\ldots r_i \ldots r_j \ldots) \rightarrow \pm\Psi(\ldots r_j \ldots r_i \ldots)$ that will cancel when taking the square modulus of the wave function. The particle coordinate with the dropped index is the spatial coordinate considered (whose infinitesimally small surrounding we inquire) when attempting to locate *any* particle of the set of identical particles.

It follows that integration of the density of any state over all of space the number of particles is recovered,

$$\int_{-\infty}^{+\infty} \mathrm{d}^3 r \rho(r) = N \tag{3.10}$$

where we dropped the n index for the sake of brevity. In the case of electronic system, the particle density is referred to as the *electron density*, with the *charge density* connected to the electron density by

$$\rho_c(r) = q_e \rho(r) = -e\rho(r) \tag{3.11}$$

where q_e is the charge of an electron, which is the simply the charge-weighted particle density. Subsequently, $-e\rho(r)\mathrm{d}^3r$ represents charge in the volume of dr^3. For the probability in a finite volume, integration is required.

3.1.5 Hilbert Space

We define a Hilbert space as a vector space $\mathscr{H}$ with inner product $\langle f, g\rangle$, such that the norm

$$|f| = \sqrt{\langle f, f\rangle} \tag{3.12}$$

turns H into a complete metric space, i.e. every Cauchy sequence in $\mathscr{H}$ has a limit in $\mathscr{H}$. Namely, we have that the physical state of a system represented by a vector $|\Psi_n\rangle$ are square integrable complex-valued functions of space and time. Integration is positive definite and finite, i.e.

$$\int_{-\infty}^{+\infty} \mathrm{d}^3r \Psi_n^\star(r)\Psi_m(r) < \infty \tag{3.13}$$

with such an imposition we guarantee normalizability of the vector as delineated by (3.8). To each particle we may assign a given Hilbert space, such that particle 1 resides within $\mathscr{H}_1$ and particle 2 resides within the corresponding Hilbert space $\mathscr{H}_2$. The Hilbert space of the system, $\mathscr{H}_{\mathrm{tot}}$ is comprised by the aggregate of the respective particles Hilbert space's, such that we comprise the system's Hilbert space via direct product,

$$\mathscr{H}_{\mathrm{tot}} = \mathscr{H}_1 \otimes \mathscr{H}_2 \tag{3.14}$$

Formally, any physical observable is denoted by a linear, hermitean operator $\hat{O}$,

$$\hat{O}^\dagger = \hat{O} \tag{3.15}$$

or a self-adjoint operator. As an operator $\hat{O}$ acts on the functions $|n\rangle$ of the Hilbert space,

$$\begin{aligned} \hat{O}: \quad & \mathscr{H} \to \mathscr{H} \\ & |n\rangle \mapsto \hat{O}|n\rangle \end{aligned} \tag{3.16}$$

Thus, when discussing any physical properties of the system we use the formal language delineated on the Hilbert space in order to adequately describe the system.

We note in passing that we shall drop the hat designator when referring to operators in subsequent sections, as it should become apparent what quantity is an operator form the context.

3.1.6 The Schrödinger Equation

The time evolution of a quantum mechanical state may be described by

$$\mathrm{i}\hbar \frac{\partial}{\partial t} \Psi_n(r,t) = \hat{H} \Psi_n(r,t) \tag{3.17}$$

which we note is an equation of motion. We point attention to the following, we see that our wave function has an explicit dependence on time t, as well as the *Hamiltonian operator* $\hat{H}$, which operates on the wave function. We later show that this operator in fact represents the energy observable of the system, due to its relation to the Hamiltonian function of classical physics. As aforementioned, we have no intuitive reasoning for the derivation of the presented equation of motion, as well as the Hamiltonian operator, rather we deduce the existence of such a postulate by making use of the correspondence principle which relates the $\hat{H}$ to the classical Hamiltonian function.

3.1.7 Constraints of the Schrödinger Equation

(3.17) must comply with the principle of special relativity, i.e. it must be Lorentz invariant in distinct reference frames. Thus, we find that the Hamiltonian operator $\hat{H}$ is further restricted in order to comply with the Lorentz invariance, i.e. retaining form invariance under Lorentz transformations. Further, we have that the quantum mechanical state equation must further comply with certain conditions:

- must be homogeneous in order to satisfy (3.7)
- must abide by the *superposition principle*, i.e. a linear combination of solutions is again a solution

The latter requirement is fundamental in describing interference phenomena. Nonetheless, it is equally well justified to consider these requirements as the consequences of the time evolution equation in accordance with experiment if the equation of motion and the form of $\hat{H}$ are postulated.

3.1.8 Probabilistic Character and Time Evolution

Since (3.6) contains a first derivative in time, the wave function at time t is solely determined by the initial state $\Psi(r,0)$ which fixes the only integration constant. However, this delineates the fact the time evolution of a quantum mechanical system is completely deterministic, since the state evolves uniquely according to the linear differential equation of motion at some time t out of an earlier state. Nonetheless, the probabilistic character of quantum mechanics solely arises from the measurement process from which there exists no consistent mathematical description.

3.1.9 Stationary States

If the Hamiltonian operator is time independent, we may then consider the following ansatz, such that we may separate the wave function into functions of space and time,

$$\Psi_n(r,t) = \psi_n(t)\Psi_n(r) \tag{3.18}$$

This time-independent state is called a *stationary state*. Substitution of (3.18) into (3.17) yields

$$\Psi_n(r)\mathrm{i}\hbar\frac{\partial\psi_n(t)}{\partial t} = \psi_n(t)\hat{H}\Psi_n(r) \tag{3.19}$$

Proceeding by the Method of Separation of Variables, we find that we may rearrange (3.19) as follows

$$\frac{1}{\psi_n(t)}\mathrm{i}\hbar\frac{\partial\psi_n(t)}{\partial t} = \frac{1}{\Psi_n(r)}\hat{H}\Psi_n(r) \equiv E_n \tag{3.20}$$

where E_n is the separation constant. Consequently, we arrive at two distinct equations, namely,

$$\mathrm{i}\hbar\frac{\partial\psi_n(t)}{\partial t} = E_n\psi_n(t) \tag{3.21}$$

$$\hat{H}\Psi_n(r) = E_n\Psi_n(r) \tag{3.22}$$

We find that the constant E_n becomes an energy eigenvalue of the system since (3.22) is an eigenvalue equation of the operator $\hat{H}$, which, as aforementioned, is the energy observable. Solving the above ODE, i.e. (3.21), we find that,

$$\begin{aligned}
\mathrm{i}\hbar\frac{\partial\psi_n(t)}{\partial t} &= E_n\psi_n(t) \\
\implies \mathrm{i}\hbar\frac{\partial\psi_n(t)}{\partial t} - E_n\psi_n(t) &= 0 \\
\implies \mathrm{i}\hbar\mu - E_n &= 0 \\
\implies \mu &= \frac{E_n}{\mathrm{i}\hbar}
\end{aligned}$$

Thus, we have the general solution

$$\psi_n(t) = \exp\left(\frac{E_n t}{\mathrm{i}\hbar}\right) = \exp\left(-\frac{\mathrm{i}E_n t}{\hbar}\right) \tag{3.23}$$

we note, that the solution of (3.22) depends explicitly on the representation of the Hamiltonian operator, which itself depends on the system under consideration.

3.1.10 Observables & Expectation Values

As aforementioned, any observable of a physical system is adequately described by an hermitean operator acting on the corresponding Hilbert space. Furthermore, we posit that any experimentally measured value o_k of an observable O must coincide with an eigenvalue of the following eigenvalue equation

$$\hat{O}\Phi_k = o_k\Phi_k \tag{3.24}$$

Due to the fact that eigenstates $\{\Phi_k\}$ form a complete orthonormal set, i.e. $\langle\Phi_k|\Phi_l\rangle = \delta_{kl}$, the state function Φ of the system may always be expressed as a superposition of the eigenstates. Nonetheless, once a measurement is performed the given superposition of the system collapses to a single eigenstate and the value measured is the corresponding eigenvalue. The probability of measuring a given eigenvalue o_k is given by overlying the corresponding eigenstate Φ_k and the actual state Ψ, $\langle\Phi_k|\Psi\rangle$.

Next, we define the mean result of a measured value. Nonetheless,since taking a measurement of an observable collapses the superposition yielding the resulting eigenstate, we cannot calculate the mean value from a sequence of measurements of the same system, but rather we take a mean value of a sequence of identical systems equally described by the vector Ψ. Thence, we define the average of these measurements as

$$o = \langle\hat{O}\rangle_\Psi = \frac{\langle\Psi|\hat{O}|\Psi\rangle}{\langle\Psi|\Psi\rangle} \tag{3.25}$$

which we denote as the *expectation value* of the system. The average taken is considered over all dependent variables of the state function. For a normalized state function $\Psi(\{r_i\})$, the denominator of (3.22) reduces to unity and we obtain

$$o = \langle\Psi|\hat{O}|\Psi\rangle = \int_{-\infty}^{\infty} \mathrm{d}^3r_1 \cdots \int_{-\infty}^{\infty} \mathrm{d}^3r_N \Psi^\dagger(\{r_i\})\,\hat{O}(\{r_i\})\,\Psi(\{r_i\}) \tag{3.26}$$

where Ψ is assumed to be described by N particles with positions $\{r_i\}$, as the only dynamical variables. Further, we generalize the complex conjugate $\Psi^\star$ to $\Psi^\dagger$, which is the complex conjugate and transpose of the state vector, in cases where Ψ takes the form of a vector.

We note that the expectation value delineates the average of a broad amount of measurements of the observable in question. If the vector Ψ is given by the superposition described in (3.3), the expectation value reduces to

$$o = \sum_{kl} C_k^\star C_l \left\langle \Phi_k | \hat{O} | \Phi_l \right\rangle = \sum_k |C_k|^2 o_k \tag{3.27}$$

We point attention to the fact the expectation value does not coincide with the most probable eigenvalue, where the latter is found by determining the largest squared weight $|C_k|^2$ in the expansion. The expectation values corresponds to an eigenvalue o_k if the system was prepared in the eigenstate Φ_k of the operator corresponding to the observable.

3.1.11 Hermitean Operators and Unitary Transformations

Consider a general operator $\hat{B}$, its adjoint operator $\hat{B}^\dagger$ is defined by

$$\left\langle \Psi_n | \hat{B} \Psi_m \right\rangle = \left\langle \hat{B}^\dagger \Psi_n | \Psi_m \right\rangle \tag{3.28}$$

which in the case of an hermitean operator $\hat{O} = \hat{O}^\dagger$ reads

$$\left\langle \Psi_n | \hat{O} \Psi_m \right\rangle = \left\langle \hat{O} \Psi_n | \Psi_m \right\rangle \equiv \left\langle \Psi_n | \hat{O} | \Psi_m \right\rangle \tag{3.29}$$

thus, it makes no difference on which side of the inner product the operator acts. Indeed,

$$\langle \psi | O | \psi \rangle^\star = \left[\int_{-\infty}^{\infty} \psi^\star(x) O \psi(x) dx \right]^\star = \int_{-\infty}^{\infty} \psi(x) (O\psi(x))^\star dx = \langle O\psi | \psi \rangle$$

such that

$$\langle O\psi|\psi\rangle = \langle\psi|O\psi\rangle = \left\langle O^{\dagger}\psi|\psi\right\rangle$$
$$\implies O^{\dagger} = O$$

We note in passing that it matters not which side of the inner product we consider the complex conjugate, as the computation provided above will result in the same relation between the operator and its adjoint operator. This symmetry is reflected in the formulation of the Dirac bracket and is denoted by two vertical dashes used to separate the operator from the states. Since all physical variables must have real expectation values and eigenvalues, ti follows that the hermitean operators feature strictly real eigenvalue spectra, such that $o_n \in \mathbb{R}$. We note in passing, that there exists no procedure which we may use to find the explicit form of the operator $\hat{O}$ to be associated with the corresponding observable O.

It is pertinent to consider *unitary transformations*, in order to uncover the uniqueness of the state function Ψ, we define the unitary operator U on the corresponding Hilbert space. We consider U a unitary transformation if it satisfies the following,

$$UU^{\dagger} = U^{\dagger}U = U^{\star,T}U = 1 \tag{3.30}$$

such that we have that the inverse of the operator is equal to its corresponding adjoint U^{dagger}. Explicitly, we have that (3.30) may be written as satisfying the following properties for the unitary operator U:

1. U is invertible, i.e., U^{-1} exists,

2. U preserves all scalar products, i.e.,

$$\langle U\Psi_n|U\Psi_m\rangle = \langle\Psi_n|\Psi_m\rangle \quad \forall\Psi_n,\Psi_m \in \mathscr{H}$$

The unitary operator preserves the lengths of vectors and the angles between them; thus, we may compare the transformations performed by the unitary operator to be analogous to rotations in 3-D Euclidean space,

Then, if we consider the unitary operator acting on some observable operator $\hat{O}$, we have that the operation $U\hat{O}U^\dagger$ is hermitean, since it follows that

$$\left(U\hat{O}U^\dagger\right)^\dagger = \left(U^\dagger\right)^\dagger \hat{O}^\dagger U^\dagger = U\hat{O}U^\dagger$$

where we have used the hermitean property of the operator $\hat{O}$ as delineated in (3.29). Nonetheless, we have using (3.30),

$$\hat{O}\Psi = a\Psi \implies \hat{O}\left(U^\dagger U\right)\Psi = a\Psi \implies U\hat{O}U^\dagger U\Psi = aU\Psi$$

We have that $\hat{O}$ and $U\hat{O}U^\dagger$ have the same set of eigenvalues, i.e.

$$\hat{O}\Psi = a\Psi \quad \text{and} \quad \left(U\hat{O}U^\dagger\right)(U\Psi) = a(U\Psi) \tag{3.31}$$

To this end, we have that applying the same reasoning to the expectation values over the operator $\hat{O}$, the expectation values remain invariant upon implementation of unitary transformations,

$$\langle\hat{O}\rangle = \langle\Psi|\hat{O}|\Psi\rangle = \left\langle\Psi\middle|U^\dagger U\hat{O}U^\dagger U\Psi\middle|\Psi\right\rangle = \left\langle U\Psi\middle|U\hat{O}U^\dagger\middle|U\Psi\right\rangle \tag{3.32}$$

Thus, any wave function is defined up to an arbitrary transformation: a physical state may be equivalently described by itself, Ψ or by $(U\Psi)$ and the matrix element (integral containing an operator) may be computed as $\langle\Psi|\hat{O}|\Psi\rangle$ or equivalently as $\left\langle U\Psi\middle|\left(U\hat{O}U^\dagger\right)\middle|U\Psi\right\rangle$. We point attention to the fact that this property will prove vital for the calculation of observables as well as their corresponding expectation values.

3.1.12 Heisenberg Equation of Motion

Next, we consider the equation of motion described by time-propagated observables. Note, that the aforementioned description of the equations of motion of particles, given by (3.17), defines the so-called *Schrödinger picture*, in which we have a time-propagated state function while the observables remain static, e.g., $H \to H^{(\mathrm{S})}$, where $'(\mathrm{S})'$ will denote the *Schrödinger* picture. (3.17) accurately describes all "normal" observables, i.e. energy, momentum, etc. yet fails to provide such an explanation for a system interacting with a time-dependent external potential. We shift focus to the so-called *Heisenberg picture* which is concerned with static wave functions and time-propagated observables. Thus, we commence by introducing the unitary transformation

$$U(t) = \exp\left(-\frac{\mathrm{i}}{\hbar}H^{(\mathrm{S})}t\right) \quad \Rightarrow \quad U^{\dagger}(t) = \exp\left(\frac{\mathrm{i}}{\hbar}H^{(\mathrm{S})}t\right) \tag{3.33}$$

where we recall that the Hamiltonian is an hermitean operator, i.e. $\hat{H}^{(\mathrm{S}),\dagger} = \hat{H}^{(\mathrm{S})}$. The particular selection of the unitary allows us to define the following ansatz for the propagation of any state $\Psi(t)$ in time

$$\Psi(t) = U(t)\Psi(t_0) \tag{3.34}$$

where $\Psi(t_0) = \Psi(t = 0)$ is *time-independent*. Fulfillment of (3.17) is easily verified by direct substitution, but is left out for the sake of brevity. Using the facts delineated in the previous section, we may then write the following operator

$$O^{(\mathrm{H})} = U^{\dagger}(t)O^{(\mathrm{S})}U(t) \tag{3.35}$$

where the superscript $'(\mathrm{H})'$ denotes the Heisenberg picture, pointing attention to the fact that the unitary transformation allows us to switch among operators. Observe that the expectation values remain the same for both pictures as the physical assertions remain unchanged.

$$\begin{aligned} o &= \left\langle \Psi(t) \middle| \hat{O}^{(\mathrm{S})} \middle| \Psi(t) \right\rangle = \left\langle U(t)\Psi(t_0) \middle| \hat{O}^{(S)} \middle| U(t)\Psi(t_0) \right\rangle \\ &= \left\langle \Psi(t_0) \middle| U^{\ dagger}(t) O^{(S)} U(t) \middle| \Psi(t_0) \right\rangle = \left\langle \Psi(t_0) \middle| \mathscr{O}^{(H)} \middle| \Psi(t_0) \right\rangle \end{aligned} \tag{3.36}$$

Since the Hamiltonian commutes with the unitary $U(t)$,

$$\hat{H}^{(\mathrm{S})} U(t) = U(t) \hat{H}^{(\mathrm{S})} \tag{3.37}$$

The Hamiltonian is equivalent in both perspectives

$$\hat{H}^{(\mathrm{H})} = U^{\dagger}(t) \hat{H}^{(S)} U(t) = \hat{H}^{(S)} U^{\dagger}(t) U(t) = \hat{H}^{(S)} \equiv \hat{H} \tag{3.38}$$

Differentiating (3.35), we obtain,

$$\begin{aligned} \frac{d}{dt} O^{(\mathrm{H})} &= \frac{d}{dt} \left[U^{\dagger}(t) O^{(\mathrm{S})} U(t) \right] \\ \Longrightarrow &= \frac{d}{dt} \left(U^{\dagger}(t) \right) O^{(\mathrm{S})} U(t) + U^{\dagger} \frac{d}{dt} \left(O^{(\mathrm{S})} \right) U(t) + U^{\dagger}(t) O^{(\mathrm{S})} \frac{d}{dt} (U(t)) \\ \Longrightarrow &= \frac{d}{dt} \left(U^{\dagger}(t) \right) O^{(\mathrm{S})} U(t) + U^{\dagger}(t) O^{(\mathrm{S})} \frac{d}{dt} (U(t)) \end{aligned}$$

where we note that $O^{(\mathrm{S})}$ is time-independent. Then differentiation of the unitary transformation yields,

$$\begin{aligned} \frac{d}{dt} U(t) &= \frac{d}{dt} \exp\left(-\frac{\mathrm{i}}{\hbar} \hat{H} t \right) \\ \Longrightarrow &= -\frac{\mathrm{i}}{\hbar} \hat{H} \exp\left(-\frac{\mathrm{i}}{\hbar} \hat{H} t \right) \end{aligned}$$

such that we have

$$\frac{\mathrm{d}U(t)}{\mathrm{d}t} = -\frac{\mathrm{i}}{\hbar} \hat{H} U(t) \tag{3.39}$$

where we used the fact that in the Schr*ö*dinger picture all observables are time-independent. Immediately, we may deduce the form of its complex-conjugate-transpose counter-part to be

$$\frac{\mathrm{d}U^{\dagger}(t)}{\mathrm{d}t} = \frac{\mathrm{i}}{\hbar}U^{\dagger}(t)\hat{H}^{+} = \frac{\mathrm{i}}{\hbar}U^{\dagger}(t)\hat{H} \tag{3.40}$$

Substitution of the attained values into (3.35), we obtain,

$$\frac{\mathrm{d}O^{(\mathrm{H})}}{\mathrm{d}t} = \frac{\mathrm{i}}{\hbar}U^{\dagger}(t)\hat{H}\hat{O}^{(\mathrm{S})}U(t) - U^{\dagger}(t)O^{(S)}\frac{\mathrm{i}}{\hbar}\hat{H}U(t) \tag{3.41}$$

Now, inserting two different instances of unities to the RHS of the equation above in order to arrive at the Heisenberg picture observable, hence,

$$\begin{aligned}\frac{\mathrm{d}\hat{O}^{(\mathrm{H})}}{\mathrm{d}t} &= \frac{\mathrm{i}}{\hbar}U^{\dagger}(t)\hat{H}\hat{O}^{(\mathrm{S})}U(t) - U^{\dagger}(t)O^{(S)}\frac{\mathrm{i}}{\hbar}\hat{H}U(t) \\ &= \frac{\mathrm{i}}{\hbar}U^{\dagger}(t)\hat{H}\left[U(t)U^{\dagger}(t)\right]\hat{O}^{(\mathrm{S})}U(t) - U^{+}(t)O^{(S)}\left[U(t)U^{\dagger}(t)\right]\frac{\mathrm{i}}{\hbar}\hat{H}U(t) \\ &= \frac{\mathrm{i}}{\hbar}\hat{H}\underbrace{U^{\dagger}(t)\hat{O}^{(S)}U(t)}_{O^{(\hat{H})}} - \underbrace{U^{\dagger}(t)\hat{O}^{(S)}U(t)}_{O^{(\hat{H})}}\frac{\mathrm{i}}{\hbar}\hat{H}\end{aligned} \tag{3.42}$$

where we have used the fact delineated in (3.37). Thus, we arrive at Heisenberg's equations of motion

$$\frac{\mathrm{d}\hat{O}^{(\mathrm{H})}}{\mathrm{d}t} = \frac{\mathrm{i}}{\hbar}\left[\hat{H},\hat{O}^{(\mathrm{H})}\right] = \frac{1}{\mathrm{i}\hbar}\left[\hat{O}^{(\mathrm{H})},\hat{H}\right] \tag{3.43}$$

where the operator $\hat{O}^{(H)}$ is propagated in time as oppose to the state function $\Psi(t_0)$, which remains static, and we use the commutator of two operators for short-hand. Eqn. (3.43) allows us to form the connection to classical physics. We note that the classical equations of motion and the quantum mechanical equations are analogs if we consider the commutator $\left[\hat{O}^{(\mathrm{H})},\hat{H}\right]/(\mathrm{i}\hbar)$ to be the analog of the Poisson bracket $\{O,H\}$ for an equivalent observable $O = O(p,q)$ which does not explicitly depend on time. Observe that if the operator $\hat{O}^{(H)}$ commutes with $\hat{H}$, then (3.43) yields zero, to which corresponds a constant of motion, in analogy to the classical case.

3.1.13 Hamiltonian in Nonrelativistic Quantum Theory

As aforementioned there exists no tenet to follow in order to arrive at the theory of quantum mechanics. Rather, by applying the so-called correspondence principle to the (nonrelativistic) Hamiltonian mechanics, may we devise the Schr*ö*dinger quantum mechanics. We note, however, that this recipe fails at times, notably for observables which have no classical analog - such as spin. This "rule" represents the desire to have some sort of link to the classical realm, which we may think of as a limiting case of quantum mechanics for large quantum numbers or vanishing $\hbar$.

The correspondence principle dictates that we may arrive at the Schr*ö*dinger Hamiltonian by a simple substitution of the momentum p and position r vectors, respectively in the total energy function H, by the operators $\hat{p}$ and $\hat{r}$, such that $\hat{H}$ takes the form

$$H = \frac{p^2}{2m} + V(r) \quad \longrightarrow \quad \hat{H} = \frac{\hat{p}^2}{2m} + \hat{V}(\hat{r}) \tag{3.44}$$

for a *single* particle of mass m in an external scalar potential V without vector potentials.

Note that the potential energy operator $\hat{V}(\hat{r})$ depends solely on the position operator, hence, its explicit form may only be derived once $\hat{r}$ and the interaction under consideration is defined. Analogously, the kinetic energy operator is not yet known

$$\hat{T} = \frac{\hat{p}^2}{2m} \tag{3.45}$$

However, we may generalize the single particle Hamiltonian operator to N particles,

$$\hat{H} = \sum_{i=1}^{N} \frac{\hat{p}_i^2}{2m_i} + V(\hat{r}_1, \hat{r}_1, \ldots, \hat{r}_N) \tag{3.46}$$

to the sum of $\hat{H}$'s of the N particles.

Following Schr*ö*dinger, we select the momentum and position operators to take the following forms,

$$\hat{p} \equiv \mathrm{i}\hbar\nabla = \frac{\hbar}{\mathrm{i}}\nabla \tag{3.47}$$

$$\hat{r} \equiv r \tag{3.48}$$

in order to explicitly derive the form of the Hamiltonian operator. The reasoning behind such a selection stems on the fact that the operators yield energy eigenvalues as solution os (3.22) whose differences coincide with transition energies measured in spectroscopy. Once again, we point attention to the fact that we may not derive these assumptions from any previous underlying principles, rather we must postulate such a principle. Squaring of the momentum operator yields,

$$\hat{p}^2 = \hat{p}\hat{p} = (-\mathrm{i}\hbar\nabla)(-\mathrm{i}\hbar\nabla) = -\hbar^2\nabla^2 = -\hbar^2\Delta \tag{3.49}$$

where Δ si the Laplacian operator. Observe that the explicit Schr$\ddot{o}$dinger equation, for a single particle, contains a first derivative with respect to time and second derivatives with respect to space

$$i\hbar\frac{\partial}{\partial t}\Psi_n(r,t) = \left[-\frac{\hbar^2}{2m}\Delta + \hat{V}(\hat{r})\right]\Psi_n(r,t) \tag{3.50}$$

Thus, we have that due to this reason the Schr$\ddot{o}$dinger equation fails to be Lorentz invariant, i.e. it does not retain its form under a Lorentz coordinate transformation. Since, we require that any physical observable be hermitean, we proceed to showing that the Hamiltonian satisfies this property. Indeed,we prove the hermiticity of the scalar potential

$$\left\langle \Psi_n|\hat{V}\Psi_m \right\rangle = \left\langle \hat{V}\Psi_m|\Psi_n \right\rangle^\star = \left\langle \Psi_m|\hat{V}\Psi_n \right\rangle^\star = \left\langle \hat{V}\Psi_n|\Psi_m \right\rangle \tag{3.51}$$

where we exploit the fact that the operator $\hat{V}$ is real and defined by a multiplicative operator. For the kinetic energy operator $\hat{T}$ it suffices to prove that the momentum operator is hermitean, which we may arrive at by integration by parts and noting that the function $(\Psi_n^\star\Psi_m)$ vanishes at the boundaries

$+\infty$ and $-\infty$ due to (3.7)

$$\langle\Psi_n|\hat{p}\Psi_m\rangle = \langle\Psi_n|-\mathrm{i}\hbar\nabla\Psi_m\rangle \overset{p.I.}{=} \langle-\mathrm{i}\hbar\nabla\Psi_n|\Psi_m\rangle = \langle\hat{p}\Psi_n|\Psi_m\rangle \tag{3.52}$$

explicitly we may write for a single variable, say x, the following

$$\begin{gathered}
\int_{-\infty}^{\infty}\Psi_n^{\star}\left(-i\hbar\frac{\partial^2}{\partial x^2}\right)\Psi_m dx \\
\Longrightarrow -i\hbar\int_{-\infty}^{\infty}\Psi_n^{\star}\frac{\partial^2}{\partial x^2}\Psi_m dx \\
\Longrightarrow -i\hbar\int_{-\infty}^{\infty}\left(\Psi_n^{\star}\Psi_m\right)\frac{\partial^2}{\partial x^2}dx \\
\Longrightarrow -i\hbar\left[\frac{\partial}{\partial x}\left(\Psi_n^{\star}\Psi_m\right)|_{-\infty}^{\infty}+\int_{-\infty}^{\infty}\frac{\partial}{\partial x}\frac{\partial}{\partial x}\left(\Psi_n^{\star}\Psi_m\right)dx\right] \\
\Longrightarrow -i\hbar\left[0+\int_{-\infty}^{\infty}\frac{\partial}{\partial x}\frac{\partial}{\partial x}\left(\Psi_n^{\star}\Psi_m\right)dx\right] \\
\Longrightarrow -i\hbar\left[0+\int_{-\infty}^{\infty}\frac{\partial^2}{\partial x^2}\left(\Psi_n^{\star}\Psi_m\right)dx\right] \\
\Longrightarrow \int_{-\infty}^{\infty}-i\hbar\frac{\partial^2}{\partial x^2}\Psi_n^{\star}\Psi_m dx
\end{gathered}$$

Extending the above derivation to the y and z coordinates, we arrive at

$$\begin{gathered}
\Longrightarrow \int_{-\infty}^{\infty}\hat{p}\left(\Psi_n^{\star}\Psi_m\right)dx \\
\Longrightarrow \int_{-\infty}^{\infty}\hat{p}\Psi_n^{\star}\Psi_m dx \\
\Longrightarrow \langle\hat{p}\Psi_n|\Psi_m\rangle
\end{gathered}$$

3.1.14 Commutation Relations for Momentum and Position operators

Measurement of observables in different directions, i and j, corresponds to a sequential application of operators for these two directions. If their results are independent of the first measurement, then the commutator of the product of both operators is equivalently zero. Thus, for the conjugate position and momentum operators we find that

$$\left[x_i, p_j\right] = 0 \quad \forall i \neq j \text{ and } \left[x_i, x_j\right] = \left[p_i, p_j\right] = 0 \quad \forall i, j \in \{x, y, z\} \tag{3.53}$$

whereas conjugate operators facing in the same direction do not commute

$$[x_i, p_i] = \mathrm{i}\hbar \quad \forall i \in \{x, y, z\} \tag{3.54}$$

Thus, measurement of the position and momentum in the same direction is dependent on the first measurement performed. These relations are required in order to fulfill the Heisenberg uncertainty principle. Perhaps one of the only guiding principles that the founders of the theory abided by was that the results of observations must be reproduce by the theory even if it violates classical concepts.

Eqn. (3.54) imposes a constraint on the choice of momentum and position operators. Nonetheless, one may immediately note that the following ans*ä*tze fulfill the requirements of (3.54), i.e.

$$\hat{p} = -\mathrm{i}\hbar\nabla_{\boldsymbol{r}} + F(\hat{\boldsymbol{r}}) \quad \hat{r} = r \quad \text{or} \qquad \hat{p} = -\mathrm{i}\hbar\nabla_{\boldsymbol{r}} \quad \hat{r} = r + G(\hat{p}) \tag{3.55}$$

in which $\hat{r}$ is the multiplicative operator so that $\hat{p}$ must contain the derivative with respect to the position coordinate, and

$$\begin{aligned} &\hat{p}' = p && \hat{p}' = p + F'(\hat{r}') \\ &\hat{r}' = \mathrm{i}\hbar\nabla_p + G'(\hat{p}') && \hat{r}' = \mathrm{i}\hbar\nabla p \end{aligned} \tag{3.56}$$

where we have that $\hat{p}$ is the multiplicative operator and $\hat{r}$ contains the derivative with respect to this momentum coordinate. The operator ∇_r denotes the standard gradient vector, while ∇_p is described by

$$\nabla_p = \left(\frac{\partial}{\partial p_x}, \frac{\partial}{\partial p_y}, \frac{\partial}{\partial p_z}\right)^T \tag{3.57}$$

Thus, one of the operators must be a multiplicative operator while the second must be a differential operator. The first choice is denoted as the *poistion-space representation*, while the second is called the *momentum-space representation*. Further, we require that all arbitrary functions of position and momentum vanish (upon taking a derivative),

$$F(\hat{r}) = G(\hat{p}) = F'(\hat{r}) = G'(\hat{p}) = 0 \tag{3.58}$$

in order to comply with the correspondence principle.

3.1.15 Ehrenfest and Hellmann-Feynman & The Schrödinger Velocity Operator

We consider Ehrenfest's theorem for the time evolution of expectation vales. Disregarding any dependence on dynamical variables by both the state function and the operator, we thereby make no explicit reference to the picture under consideration. Consider the time derivative of the expectation value of an observable O for a normalized stat Ψ

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{O}\rangle_\Psi &= \int_{-\infty}^{\infty}\mathrm{d}^3r_1\cdots\int_{-\infty}^{\infty}\mathrm{d}^3r_N\left[\frac{\partial\Psi^\dagger}{\partial t}\hat{O}\Psi+\Psi^\dagger\frac{\partial\hat{O}}{\partial t}\Psi+\Psi^\dagger O\frac{\partial\Psi}{\partial t}\right] \\ &= \frac{\mathrm{i}}{\hbar}\langle[\hat{H},\hat{O}]\rangle_\Psi+\left\langle\frac{\partial\hat{O}}{\partial t}\right\rangle_\Psi \end{aligned} \tag{3.59}$$

where the total time derivatives of the right-hand side have been substituted by partial derivatives due to the quantities in the integrand depending on position and time, and positions being time-independent (as there is no trajectories in quantum mechanics). This fact is thus proven by the equation above, as it holds for a system of an arbitrary number of particles N.

It follows that if the operators $\hat{H}$ and $\hat{O}$ commute or if Ψ is an eigenfunction of the Hamiltonian operator, then the time derivative of the expectation value is equivalent to the time derivative of the operator $\hat{O}$,

$$\frac{\mathrm{d}\langle\hat{O}\rangle_\Psi}{\mathrm{d}t} = \left\langle\frac{\partial\hat{O}}{\partial t}\right\rangle_\Psi \tag{3.60}$$

In the case in which the observable in question is the Hamiltonian $\hat{H}$, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{O}\rangle_\Psi &= \frac{\mathrm{i}}{\hbar}\langle[\hat{H},\hat{H}]\rangle_\Psi + \left\langle\frac{\partial\hat{H}}{\partial t}\right\rangle_\Psi \\
\Longrightarrow &= \frac{\mathrm{i}}{\hbar}\langle H\cdot H - H\cdot H\rangle_\Psi + \left\langle\frac{\partial\hat{H}}{\partial t}\right\rangle_\Psi \\
\Longrightarrow &= 0 + \left\langle\frac{\partial\hat{H}}{\partial t}\right\rangle_\Psi
\end{aligned}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{O}\rangle_\Psi = \left\langle\frac{\partial\hat{H}}{\partial t}\right\rangle_\Psi \tag{3.61}$$

which holds true if Ψ statisfies (3.17). We note that this is a special case of the Hellmann-Feynman theorem which can be generalized for first derivatives of any arbitrary parameter λ, on which the state function may depend, such that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\lambda}\langle\hat{H}\rangle_\Psi &= \int_{-\infty}^{\infty}\mathrm{d}^3r_1\cdots\int_{-\infty}^{\infty}\mathrm{d}^3r_N\left[\frac{\mathrm{d}\Psi^\dagger}{\mathrm{d}\lambda}\hat{H}\Psi + \Psi^\dagger\frac{\mathrm{d}\hat{H}}{\mathrm{d}\lambda}\Psi + \Psi^\dagger\hat{H}\frac{\mathrm{d}\Psi}{\mathrm{d}\lambda}\right] \\
&= \left\langle\frac{\mathrm{d}\hat{H}}{\mathrm{d}\lambda}\right\rangle_\Psi
\end{aligned} \tag{3.62}$$

where we imposed that the state function must be an eigenfunction of $\hat{H}$. Note that in this generalized derivation we did not substitute the right hand side of the equation with partial derivatives, this is due to the fact that λ may be considered a general variable whose variation might affect quantities that depend implicitly on it.

We now consider the force law in quantum mechanics, also known as Ehrenfest's Theorem. Firstly, considering Heisenberg's equation of motion with the observable being equivalent to the Schr*ö*dinger Hamiltonian we may write

$$\frac{\mathrm{d}\hat{p}}{\mathrm{d}t} = \frac{1}{\mathrm{i}\hbar}[\hat{p},\hat{H}] = \frac{1}{\mathrm{i}\hbar}[\hat{p},\hat{V}(\hat{r})] = -\nabla\hat{V}(\hat{r}) \tag{3.63}$$

using the following identity for the commutator, $[A, B+C] = [A, B] + [A, C]$, we have

$$\begin{aligned} \frac{\mathrm{d}\hat{p}}{\mathrm{d}t} &= \frac{1}{\mathrm{i}\hbar}[\hat{p}, \hat{H}] \\ \Longrightarrow &= \frac{1}{\mathrm{i}\hbar}[\hat{p}, \frac{\hat{p}^2}{2m} + \hat{V}(\hat{r})] \\ \Longrightarrow &= \frac{1}{\mathrm{i}\hbar}\left([\hat{p}, \frac{\hat{p}^2}{2m}] + [\hat{p}, \hat{V}(\hat{r})]\right) \\ \Longrightarrow &= \frac{1}{\mathrm{i}\hbar}\left([0 + [\hat{p}, \hat{V}(\hat{r})]\right) \\ \Longrightarrow &= \frac{1}{\mathrm{i}\hbar}[\hat{p}, \hat{V}(\hat{r})] \end{aligned}$$

then using the commutation relations delineated by (3.54) we have

$$\Longrightarrow = -\nabla \hat{V}(\hat{r}) \tag{3.64}$$

Analogously, we define the velocity operator $\dot{\hat{r}}$ by using (3.43) such that

$$\hat{v} \equiv \frac{\mathrm{d}\hat{r}}{\mathrm{d}t} = \frac{\mathrm{i}}{\hbar}[\hat{H}, \hat{r}] \tag{3.65}$$

Using the Schr*ö*dinger Hamiltonian, we may derive the Schr*ö*dinger velocity operator

$$\dot{\hat{r}} = \frac{\mathrm{i}}{\hbar}\left[\frac{\hat{p}^2}{2m} + \hat{V}(r), \hat{r}\right] = \frac{\mathrm{i}}{\hbar}\left[\frac{\hat{p}^2}{2m}, \hat{r}\right] = \frac{\mathrm{i}}{2m\hbar}(\hat{p}[\hat{p}, \hat{r}] + [\hat{p}, \hat{r}]\hat{p}) = \frac{\mathrm{i}}{2m\hbar}(2\hat{p}i\hbar) = \frac{\hat{p}}{m} \tag{3.66}$$

which is the canonical momentum operator divided by the mass, analogous to its classical mechanics counterpart. Furthermore, since we have that the position and momentum operators of momentum and position always commute for distinct particles, then (3.66) holds for the velocity operator of any particle in a many-particle system described by $\hat{H}$.

Using the previously derived velocity operator, we proceed by taking a time derivative of the operator such that

$$\ddot{\hat{r}} = \frac{d\dot{\hat{r}}}{dt} = \frac{1}{m}\frac{d\hat{p}}{dt} \tag{3.67}$$

which we can then use to rewrite (3.63) as

$$\begin{aligned}\ddot{\hat{r}} &= \frac{d\dot{\hat{r}}}{dt} = \frac{1}{m}\frac{d\hat{p}}{dt}\\ \implies m\ddot{\hat{r}} &= \frac{d\hat{p}}{dt}\end{aligned}$$

$$m\ddot{\hat{r}} = -\nabla\hat{V}(\hat{r}) \tag{3.68}$$

which is the quantum mechanical analog of Newton's second law. Taking the expectation values of both sides, with respect to a Heisenberg state, which is time-independent, yields

$$m\langle\ddot{\hat{r}}\rangle_{\Psi(t_0)} = m\frac{\mathrm{d}^2}{\mathrm{d}t^2}\langle r\rangle_{\Psi(t_0)} = -\langle\nabla\hat{V}(\hat{r})\rangle_{\Psi(t_0)} \tag{3.69}$$

3.1.16 Current Density and the Continuity Equation

Analogous to the classical definition of current density and the corresponding continuity equation, we derive the quantum mechanical counterpart of said postulates. We deduce the explicit form of the particle density operator $\hat{\rho}_r$ at position r, by relating its expectation value for an N-particle system such that

$$\langle\hat{\rho}_{\boldsymbol{r}}\rangle \equiv \int_{-\infty}^{\infty}\mathrm{d}^3r_1\int_{-\infty}^{\infty}\mathrm{d}^3r_2\cdots\int_{-\infty}^{\infty}\mathrm{d}^3r_N\Psi^{\dagger}(r_1,r_2,\ldots,r_N,t)\,\hat{\rho}_r\Psi(r_1,r_2,\ldots,r_N,t) \tag{3.70}$$

to the Born interpretation $\rho(r,t) = \langle\hat{\rho}_r\rangle$, such that we must satisfy

$$\langle\hat{\rho}_r\rangle = N\int_{-\infty}^{\infty}\mathrm{d}^3r_2\cdots\int_{-\infty}^{\infty}\mathrm{d}^3r_N\Psi^{\dagger}(r,r_2,\ldots,r_N,t)\,\Psi(r,r_2,\ldots,r_N,t) \tag{3.71}$$

for N identical particles. We fulfill the requirement imposed by (3.71) by writing the particle density operator as

$$\hat{\rho}_r = \sum_{i=1}^{N} \delta^{(3)}(r_i - r) \tag{3.72}$$

in whihc r is a general position coordinate and r_i is the coordinate of the ith particle. For the time evolution of the particle density we use (3.59),

$$\frac{\mathrm{d}\rho(r,t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{\rho}r\rangle = \frac{\mathrm{i}}{\hbar}\left\langle\left[\hat{H},\hat{\rho}_r\right]\right\rangle + \left\langle\frac{\partial\hat{\rho}_r}{\partial t}\right\rangle \tag{3.73}$$

where we have the second term on the RHS of (3.73) vanishes due to the fact that the density operator is time-independent. Similarly, we may substitute the total derivative on the LHS with partial derivatives with respect to time, i.e.

$$\frac{\mathrm{d}\rho(r,t)}{\mathrm{d}t} = \frac{\partial\rho(r,t)}{\partial t} \tag{3.74}$$

We select the Schr*ö*dinger Hamiltonian for the RHS of (3.73) for an N-particle system, where the kinetic energy operators of the individual particles are summed. All potential energy operators $V(r_1, r_1, \ldots, r_N)$ commute with $\hat{\rho}_r$ operator and thus vanish. Hence, in a nonrelativistic frame the time evolution of the density is simply

$$\begin{aligned}\frac{\partial}{\partial t}\rho(r,t) &= \frac{\mathrm{i}}{\hbar}\left\langle\left[\sum_{i=1}^{N}\frac{p_i^2}{2m_i},\hat{\rho}_r\right]\right\rangle = \frac{\mathrm{i}}{\hbar}\left\langle\left[\sum_{i=1}^{N}\frac{p_i^2}{2m_i},\sum_{j=1}^{N}\delta^{(3)}\left(r_j - r\right)\right]\right\rangle \\ &= \frac{\mathrm{i}}{\hbar}\left\langle\sum_{i=1}^{N}\left[\frac{p_i^2}{2m_i},\delta^{(3)}(r_i - r)\right]\right\rangle\end{aligned} \tag{3.75}$$

If we assume that the particles are identical we may replace the summation with the N number of particles under consideration and write

$$\frac{\partial\rho(r,t)}{\partial t} = \frac{\mathrm{i}}{\hbar}N\left\langle\left[\frac{p_1^2}{2m_1},\delta^{(3)}(r_1 - r)\right]\right\rangle \tag{3.76}$$

The above equation can be rewritten as to provide the *current density* of a system of N particles with mass $m \equiv m_i$. Thus, setting $p_1 = -i\hbar\nabla_1$ we attain the *continuity equation* for an N-particle system,

$$\begin{aligned}
\frac{\partial\rho(r,t)}{\partial t} &= \frac{\mathrm{i}}{\hbar}\left\langle \sum_{i=1}^{N}\left[\frac{p_1^2}{2m_1}, \delta^{(3)}(r_i - r)\right]\right\rangle \\
\Longrightarrow &= N\frac{\mathrm{i}}{\hbar}\left\langle\left[\frac{-\hbar^2\nabla_1^2}{2m_1}, \delta^{(3)}(r_1 - r)\right]\right\rangle \\
\Longrightarrow &-\left(\frac{\mathrm{i}}{\mathrm{i}}\right)N\frac{\mathrm{i}\hbar}{2m_1}\left\langle\left[\nabla_1^2, \delta^{(3)}(r_1 - r)\right]\right\rangle \\
\Longrightarrow &\; N\frac{\hbar}{2m_1\mathrm{i}}\left\langle \nabla_1^2\delta^{(3)}(r_1 - r) - \delta^{(3)}(r_1 - r)\nabla_1^2\right\rangle \\
\Longrightarrow &\; N\frac{\hbar}{2m_1\mathrm{i}}\left\langle \Psi|\nabla_1^2\delta^{(3)}(r_1 - r) - \delta^{(3)}(r_1 - r)\nabla_1^2|\Psi\right\rangle \\
\Longrightarrow &\; N\frac{\hbar}{2m_1\mathrm{i}}\left\langle \Psi|\nabla_1^2\delta^{(3)}(r_1 - r)|\Psi\right\rangle - \left\langle\Psi|\delta^{(3)}(r_1 - r)\nabla_1^2|\Psi\right\rangle
\end{aligned}$$

Hence,

$$\frac{\partial\rho(r,t)}{\partial t} = N\frac{\hbar}{2m_1\mathrm{i}}\left\langle \nabla_1^2\Psi|\delta^{(3)}(r_1 - r)|\Psi\right\rangle - \left\langle\Psi|\delta^{(3)}(r_1 - r)|\nabla_1^2\Psi\right\rangle \tag{3.77}$$

where we exploited the hermiticity of the Hamiltonian, namely, $\nabla_1^2 = \nabla_1^{2\dagger}$.

In the most elementary case of a single particle, we have the wavefunction $\Psi(r_1, \ldots, r_N, t) \to \Psi(r,t)$, and (3.77) reduces to

$$\begin{aligned}
\frac{\partial}{\partial t}(\Psi^\star\Psi) &= \frac{\hbar}{2m\mathrm{i}}\left\{\left(\nabla^2\Psi\right)^\star\Psi - \Psi^\star\nabla^2\Psi\right\} \\
&= -\nabla\cdot\frac{\hbar}{2m\mathrm{i}}\{\Psi^\star\nabla\Psi - (\nabla\Psi)^\star\Psi\} \\
&= -\operatorname{div} j
\end{aligned} \tag{3.78}$$

where we have used the identity $\nabla^2 f = (\nabla\cdot\nabla)f$ and substituted nonrevlativistic current density j

for the time-dependent state $\Psi(r,t)$,

$$\begin{aligned} j &\equiv \frac{\hbar}{2m\mathrm{i}}\left\{\Psi^{\star}\nabla\Psi-(\nabla\Psi)^{\star}\Psi\right\} \\ &= \frac{\hbar}{m}\operatorname{Im}\left\{\Psi^{\star}\nabla\Psi\right\} \end{aligned} \tag{3.79}$$

We note that since the squared wave function gives the probability, and considering the simplest case of a single particle, where the Dirac delta function turns to 1, we may rewrite the LHS of (3.77) as (3.78), with $N=1$ in the given case.

The connection to the current density displayed in classical mechanics is made apparent by introducing the Schr*ö*dinger velocity operator, (3.66), into the nonrelativistic current density. For a single particle, (3.79) may be rewritten in terms of the momentum operator as oppose to the gradient, such that we have

$$\begin{aligned} j(r) &= \frac{1}{2}\left\{\Psi^{\star}(r)\frac{p}{m}\Psi(r)+\left(\frac{p}{m}\Psi(r)\right)^{\star}\Psi(r)\right\} \\ &= \frac{1}{2}\left\{2\left(\Psi^{\star}(r)\frac{p}{m}\Psi(r)\right)\right\} \\ &= \operatorname{Re}\left\{\Psi^{\star}(r)\frac{p}{m}\Psi(r)\right\} = \operatorname{Re}\left\{\Psi^{\star}(r)\dot{r}\Psi(r)\right\} \end{aligned} \tag{3.80}$$

Then analogous to the density-operator definition of (3.72), the expression delineated for j in terms of the velocity operator $\dot{r}$ motivates the definition of a current-density operator for a particle i,

$$\hat{j}_{r,i} = \dot{r}_i\delta^{(3)}\left(r_i - r\right) \tag{3.81}$$

so that we may find the current density of the ith particle from an expectation value

$$j(r) = \operatorname{Re}\int_{-\infty}^{\infty}\mathrm{d}^3 r_i \Psi^{\star}\left(r_i\right)\hat{j}_{r,j}\Psi\left(r_i\right) \tag{3.82}$$

We may then generalize the current-density operator defined by (3.81) for N particles, where we have

$$\hat{j}_r = \sum_{i=1}^{N} \left(\dot{r}_i \delta^{(3)} (r_i - r) \right) \tag{3.83}$$

returning the the general case. Further, we may then use the current-density operator in order to compute the current density from an expectation value using (3.26)

$$j(r) = \mathrm{Re} \int_{-\infty}^{\infty} \mathrm{d}^3 r_1 \cdots \int_{-\infty}^{\infty} \mathrm{d}^3 r_N \Psi^{\star} (\{r_i\}) \, \hat{j}_r \Psi (\{r_i\}) \tag{3.84}$$

Hence, the continuity equation for an N-particle system, may be rewritten as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot j = 0 \tag{3.85}$$

The above delineates the tendency demonstrated by nature when a change of density at a certain position with time results in a net current from or to the environment (depending on the sign of the divergence) of that position. Notice that the time propagation of the particle's density remains constant if the divergence of the current density vanishes. This details the situation where a current (whether that may be probability or charge) ensure the influx and efflux are of the same magnitude in the volume element under inspection. We may also consider other cases where the divergence vanishes, since we the divergence of the current density is the sum of the changes of the current density in each spatial direction, thus, a change in one direction must be compensated by a change in another in order to ensure a constant density. Since, the probability distribution found from the wave function of a stationary state is independent of time, this means that the divergence of the current density must be equivalently zero. A non-vanishing divergence of the current density yield s a probability density influx and efflux that do not compensate. In a quantum mechanical framework this may be realized by nonstationary states.

3.1.17 Angular Momenta, Rotations and Classical Angular Momentum

It is conventional to describe rotational motion by means of spherical coordinates, yet the transformation to spherical coordinates has an impact on the definition of the angular momentum and the squared angular momentum operators which are used in the explicit description of the kinetic energy expression. Thus, in order to bypass the tedious calculations required in order to transform the coordinates from Cartesian to spherical, we first consider the situation in classical mechanics and subsequently apply the correspondence principle.

Referring to the *classical* definition of the angular momentum first, we write

$$l^2 = (r \times p)^2 = r^2 p^2 - (r \cdot p)^2 \tag{3.86}$$

which can use to solve for the squared momentum as

$$\frac{l^2 + (r \cdot p)^2}{r^2} = p^2$$

where we may redefine the above as follows

$$p^2 = \frac{(r \cdot p)^2}{r^2} + \frac{l^2}{r^2} = p_r^2 + \frac{l^2}{r^2} \tag{3.87}$$

where we set the first term of the RHS equal to the square of the radial momentum p_r, which thus is defined as

$$p_r \equiv \frac{r}{r} \cdot p = \hat{r} \cdot p \tag{3.88}$$

where $\hat{r}$ denotes the *unit* vector in the r-direction and where for the sake of brevity the "transpose" sign has been omitted for the scalar product. With the denoted definitions, the classical KE for a single particle may be written as a sum of the radial and rotational kinetic energies

$$T = \frac{p^2}{2m} = \frac{p_r^2}{2m} + \frac{l^2}{2mr^2} \equiv T_{\text{rad}} + T_{\text{rot}} \tag{3.89}$$

which for a fixed radial coordinate r simplifies to the pure rotational energy expression

$$T_{\text{rot}} = \frac{l^2}{2mr^2} = \frac{l^2}{2I} \tag{3.90}$$

wikth $I \equiv mr^2$ denoting the moment of inertia of the point-like particle.

3.1.18 Orbital Angular Momentum

Using the correspondence principle, we define the quantum mechanical analog of the angular momentum as an operator of the following form

$$\hat{l} \equiv \hat{r} \times \hat{p} = \begin{pmatrix} \hat{y}\hat{p}_z - \hat{z}\hat{p}_y \\ 2\hat{p}_x - \hat{x}\hat{p}_z \\ \hat{x}\hat{p}_y - \hat{y}\hat{p}_x \end{pmatrix} \equiv \begin{pmatrix} \hat{l}_x \\ \hat{l}_y \\ \hat{l}_z \end{pmatrix} \tag{3.91}$$

from which we may directly obtain the squared angular momentum. Note that upon derivation of the squared angular momentum there arises an addition term, as oppose to the equation detailed by (3.86),

$$\hat{l}^2 = (\hat{r} \times \hat{p})^2 = \hat{r}^2\hat{p}^2 - (\hat{r} \cdot \hat{p})^2 + \text{ ih } \hat{r} \cdot \hat{p} \tag{3.92}$$

which may be attributed to the non-commutative nature of the position and momentum operators. The dimension of the calculated squared angular momentum is the dimension of an action squared, thus, is equal to the unit of Planck's constant squared, h^2. Consequently, every given term in (3.92) contains a squared Planck's constant $h^2 = 4\pi^2\hbar^2$, induced from the canonical momentum operator.

We know introduce the transformation from Cartesian coordinates to spherical coordinates

(r, ϑ, φ):

$$
\begin{aligned}
x &= r \sin\theta \cos\varphi \\
y &= r \sin\theta \sin\varphi \\
z &= r \cos\theta
\end{aligned}
\tag{3.93}
$$

conversely,

$$
\begin{aligned}
r &= \sqrt{x^2 + y^2 + z^2} \\
\theta &= \arccos\left(\frac{z}{r}\right) \\
\varphi &= \arctan\left(\frac{y}{x}\right)
\end{aligned}
\tag{3.94}
$$

The projection of the momentum operator onto the position operator is given by

$$
\hat{r} \cdot \hat{p} = -\mathrm{i}\hbar (r \cdot \nabla) = -\mathrm{i}\hbar \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} \right) \tag{3.95}
$$

and can be re-written in spherical coordinates using the chain rule

$$
\frac{\partial}{\partial u} = \frac{\partial r}{\partial u}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial u}\frac{\partial}{\partial \theta} + \frac{\partial \varphi}{\partial u}\frac{\partial}{\partial \varphi}, \quad u \in \{x, y, z\} \tag{3.96}
$$

The explicit evaluation of the scalar product then yields

$$
\hat{r} \cdot \hat{p} = -\mathrm{i}\hbar \frac{\partial}{\partial r} \tag{3.97}
$$

Thus, we may write for (3.92) the following,

$$
\begin{aligned}
\hat{l}^2 = (\hat{r} \times \hat{p})^2 &= \hat{r}^2 \hat{p}^2 - (\hat{r} \cdot \hat{p})^2 + \text{ in } \hat{r} \cdot \hat{p} \\
\Longrightarrow &= \hat{r}^2 \hat{p}^2 - \left(-\mathrm{i}\hbar r\frac{\partial}{\partial r}\right)^2 + \mathrm{i}\hbar\left(-\mathrm{i}\hbar r\frac{\partial}{\partial r}\right) \\
\Longrightarrow &= \hat{r}^2 \hat{p}^2 + \hbar^2 r^2 \left(\frac{\partial}{\partial r}\right)^2 + \hbar^2 r\left(\frac{\partial}{\partial r}\right) \\
\Longrightarrow &= \hat{r}^2 \hat{p}^2 + \hbar^2 \left[\left(r\frac{\partial}{\partial r}\right)^2 + r\frac{\partial}{\partial r}\right]
\end{aligned}
$$

thus,

$$\hat{l}^2 = \hat{r}^2\hat{p}^2 + \hbar^2\left[\left(r\frac{\partial}{\partial r}\right)^2 + r\frac{\partial}{\partial r}\right] \tag{3.98}$$

from which we derive the expression for the squared canonical momentum operator,

$$\hat{p}^2 = \frac{1}{r^2}\hat{l}^2 - \frac{\hbar^2}{r^2}\left[\left(r\frac{\partial}{\partial r}\right)^2 + r\frac{\partial}{\partial r}\right] \tag{3.99}$$

where we simply solved for the squared momentum term from (3.98), by dividing both sides by r^2 and subtracting the operators terms in brackets. We may then rewrite the second term of the RHS of the above equation as

$$\frac{1}{r^2}\left[\left(r\frac{\partial}{\partial r}\right)^2 + r\frac{\partial}{\partial r}\right] = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} = \left(\frac{1}{r}\frac{\partial}{\partial r}r\right)^2 \tag{3.100}$$

Indeed, since

$$\begin{aligned}
&\frac{1}{r^2}\left[\left(r\frac{\partial}{\partial r}\right)^2 + r\frac{\partial}{\partial r}\right] \\
\Longrightarrow\ &\frac{1}{r^2}\left[r^2\left(\frac{\partial}{\partial r}\right)^2 + r\left(\frac{\partial}{\partial r}\right) + r\frac{\partial}{\partial r}\right] \\
\Longrightarrow\ &\frac{1}{r^2}\left[r^2\left(\frac{\partial}{\partial r}\right)^2 + 2r\left(\frac{\partial}{\partial r}\right)\right] \\
\Longrightarrow\ &\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}
\end{aligned}$$

in order to derive the analog of (3.87) for the squared momentum

$$\hat{p}^2 = \frac{\hat{l}^2}{r^2} - \hbar^2\left(\frac{1}{r}\frac{\partial}{\partial r}r\right)^2 \equiv \frac{\hat{l}^2}{r^2} + \hat{p}_r^2 \tag{3.101}$$

Thus, we find for the radial momentum operator

$$\hat{p}_r = -\mathrm{i}\hbar\frac{1}{r}\frac{\partial}{\partial r}r = -\mathrm{i}\hbar\left(\frac{\partial}{\partial r} + \frac{1}{r}\right) \tag{3.102}$$

in which we have manipulated the sign in such a way that the standard commuatiation relation with the conjugated radial position operator yields

$$[\hat{r}, \hat{p}_r] = \mathrm{i}\hbar \tag{3.103}$$

instead of $-\mathrm{i}\hbar$. We note that the form of the radial momentum operator may seem peculiar at first, since we have an extra term which the operator is proportional to, namely $\propto 1/r$. Yet, the additional term $1/r$ commutes with the r term and vanishes form the commutator.

Using the radial momentum operator previously derived, we may now write the kinetic energy operator in spherical coordinates as

$$T = \frac{\hat{p}^2}{2m} = \frac{\hat{l}^2}{2mr^2} + \frac{\hat{p}_r^2}{2m} \tag{3.104}$$

Thus, analogous to the previously derived classical mechanics kinetic energy expression and applying the correspondence principle, the operator for purely rotational energy of a single particle with fixed distance r from the rotational axis yields

$$\hat{T}_{\text{rot}} = \frac{\hat{l}^2}{2mr^2} \tag{3.105}$$

We may analyze the validity of (3.104) by transforming the squared momentum operator term from Cartesian coordinates to spherical coordinates. We find that the components of the angular momentum operator read

$$\begin{aligned} l_x &= \frac{\hbar}{\mathrm{i}}\left(-\sin\varphi\frac{\partial}{\partial\theta} - \cos\varphi\cot\theta\frac{\partial}{\partial\varphi}\right) \\ l_y &= \frac{\hbar}{\mathrm{i}}\left(\cos\varphi\frac{\partial}{\partial\theta} - \sin\varphi\cot\theta\frac{\partial}{\partial\varphi}\right) \\ l_z &= \frac{\hbar}{i}\frac{\partial}{\partial\varphi} \end{aligned} \tag{3.106}$$

whose proof we exclude for the sake of brevity.Using the components of the angular momentum we may then define the squared angular momentum operator in angular coordinates as follows,

$$\hat{l}^2 = l_x^2 + l_y^2 + l_z^2 = -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \right] \tag{3.107}$$

the Legendre operator. The components of the angular momentum operator fulfill the following commutation relations:

$$[l_x, l_y] = \mathrm{i}hl_z, \quad [l_y, l_z] = \mathrm{i}hl_x, \quad [l_z, l_x] = \mathrm{i}hl_y \tag{3.108}$$

and

$$\left[l^2, l_i\right] = 0, \quad \text{for} \quad i \in \{x, y, z\} \tag{3.109}$$

which may be directly proved by direct substitution of the derived components in (3.106). Indeed, it suffices to prove one of the cases denoted by (3.108), namely the second term. To this end, consider the second term demonstrated in (3.108),

$$[l_y, l_z] = i\hbar l_x$$

$$\Longrightarrow = l_y l_z - l_z l_y$$

$$\Longrightarrow = \frac{\hbar}{\mathrm{i}} \left(\cos\varphi \frac{\partial}{\partial\theta} - \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \right) \left(\frac{\hbar}{i} \frac{\partial}{\partial\varphi} \right) - \frac{\hbar}{i} \left(\frac{\partial}{\partial\varphi} \right) \frac{\hbar}{\mathrm{i}} \left(\cos\varphi \frac{\partial}{\partial\theta} - \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \right)$$

$$\Longrightarrow = \frac{\hbar}{\mathrm{i}} \left[\left(\cos\varphi \frac{\partial}{\partial\theta} - \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \right) \frac{\partial}{\partial\varphi} - \frac{\partial}{\partial\varphi} \left(\cos\varphi \frac{\partial}{\partial\theta} - \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \right) \right]$$

$$\Longrightarrow = \frac{\hbar}{\mathrm{i}} \left[\left(\cos\varphi \frac{\partial}{\partial\theta} - \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} \right) \frac{\partial}{\partial\varphi} - \left(-\sin\varphi \frac{\partial}{\partial\theta} + \cos\varphi \frac{\partial}{\partial\varphi} \frac{\partial}{\partial\theta} - \cos\varphi \cot\theta \frac{\partial}{\partial\varphi} - \sin\varphi \cot\theta \frac{\partial^2}{\partial\varphi^2} \right) \right]$$

$$\Longrightarrow = \frac{\hbar}{\mathrm{i}} \left[\left(\cos\varphi \frac{\partial}{\partial\varphi} \frac{\partial}{\partial\theta} - \sin\varphi \cot\theta \frac{\partial^2}{\partial\varphi^2} \right) - \left(-\sin\varphi \frac{\partial}{\partial\theta} + \cos\varphi \frac{\partial}{\partial\varphi} \frac{\partial}{\partial\theta} - \cos\varphi \cot\theta \frac{\partial}{\partial\varphi} - \sin\varphi \cot\theta \frac{\partial^2}{\partial\varphi^2} \right) \right]$$

$$\Longrightarrow = -\frac{\hbar}{\mathrm{i}} \left[-\sin\varphi \frac{\partial}{\partial\theta} - \cos\varphi \cot\theta \frac{\partial}{\partial\varphi} \right]$$

$$\Longrightarrow = -\left(\frac{\mathrm{i}}{\mathrm{i}} \right) \frac{\hbar}{\mathrm{i}} \left[-\sin\varphi \frac{\partial}{\partial\theta} - \cos\varphi \cot\theta \frac{\partial}{\partial\varphi} \right]$$

$$\Longrightarrow = \mathrm{i}\hbar \left(-\sin\varphi \frac{\partial}{\partial\theta} - \cos\varphi \cot\theta \frac{\partial}{\partial\varphi} \right)$$

$$\Longrightarrow = \mathrm{i}\hbar l_x$$

which is equivalent to the second term of (3.108), with the rest of the terms following analogously. We note in passing that the three relations delineated in (3.106) may be conveniently written with the Levi-Civitá symbol as follows

$$[l_i, l_j] = i\hbar \sum_{k=1}^{3} \varepsilon_{ijk} l_k, \quad \text{for} \quad i,j,k \in \{x,y,z\} \tag{3.110}$$

The eigenfunctions of the Legendre operator $\hat{l}^2$,

$$l^2 Y_{lm}(\theta,\varphi) = l(l+1)\hbar^2 Y_{lm}(\theta,\varphi) \tag{3.111}$$

are the spherical harmonics $Y_{lm}(\theta,\varphi)$. Since the operator for the rotational energy is proportional to the (scalar) Legendre operator, they have the same eigenfunctions. Due to the commutation relations of the angular momentum operator, the spherical harmonics are also eigenfunctions for only one of the components of the aforementioned operator, yet due to (3.108) not of the other two components. Letting this component be the l_z operator, we have

$$l_z Y_{lm}(\vartheta,\varphi) = m\hbar Y_{lm}(\vartheta,\varphi) \tag{3.112}$$

This *orbital angular momentum quantum number l* (or *azimuthal* quantum number) can assume the values $0,1,2,3,\ldots$ while the values of the *magnetic quatntum number m* assumes values $\{-l,-l+1,\ldots,l\}$. The general expression for the spherical harmonics is expressed by

$$Y_{lm}(\vartheta,\varphi) = (-1)^m \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} P_{lm}(\cos\vartheta) e^{im\varphi} \tag{3.113}$$

in angular coordinates, where we have $P_{lm}(\xi)$ exhibits the associated Legendre polynomials

$$P_{lm}(\xi) = \left(1-\xi^2\right)^{\frac{m}{2}} \frac{d^m}{d\xi^m} P_l(\xi) \tag{3.114}$$

where the Legendre polynomials are given by

$$P_l(\xi) = \frac{1}{2^l l!} \frac{\mathrm{d}^l}{\mathrm{d}\xi^l} \left(\xi^2 - 1\right)^l \tag{3.115}$$

where in the case under consideration we have that the polynomials are

$$P_l(\cos\theta) = \frac{1}{2^l l!} \frac{\mathrm{d}^l \left(\cos^2\vartheta - 1\right)^l}{\mathrm{d}(\cos\theta)^l} = \frac{(-1)^l}{2^l l!} \frac{\mathrm{d}^l \sin^{2l}\vartheta}{\mathrm{d}(\cos\vartheta)^l} \tag{3.116}$$

where we obtain the associated Legendre polynomials to be

$$P_{lm}(\cos\vartheta) = \frac{(-1)^l}{2^l l!} \sin^m\vartheta \frac{\mathrm{d}^{l+m} \sin^{2l}\vartheta}{\mathrm{d}(\cos\vartheta)^{l+m}} \tag{3.117}$$

The first four Legendre polynomials are explicitly,

$$P_0 = 1, P_1 = \cos\vartheta, P_2 = \frac{1}{2}\left(3\cos^2\vartheta - 1\right), P_3 = \frac{1}{2}\left(5\cos^3\vartheta - 3\cos\vartheta\right) \tag{3.118}$$

which may be derived from (3.115) with a simple substitution from $\xi \to \cos\vartheta$, namely, if we consider the Legendre polynomial P_1 and apply the substitution just mentioned we obtain,

$$\begin{aligned}
P_l(\xi) &= \frac{1}{2^l l!} \frac{\mathrm{d}^l}{\mathrm{d}\xi^l} \left(\xi^2 - 1\right)^l \\
\Longrightarrow P_1(\xi) &= \frac{1}{2^1 1!} \frac{\mathrm{d}^1}{\mathrm{d}\xi^1} \left(\xi^2 - 1\right)^1 \\
\Longrightarrow &= \frac{1}{2} \frac{d}{d\xi} \left(\xi^2 - 1\right) \\
\Longrightarrow &= \frac{1}{2} \frac{d}{d\xi} 2\xi \\
\Longrightarrow &= \xi \to \cos\vartheta
\end{aligned}$$

The representation of the spherical harmonics Y_{lm} are defined with a specific phase factor that varies from presentation to presentation. Here, we follow Edmonds, which the convention by Condon and

Shortley commonly used, in whihc the $(-1)^m$ prefactor of the spherical harmonics multiplied by the $(-1)^l$ prefactor of the Legendre polynomials, yields a total prefactor of $(+1)$ for Y_{lm} in the case $l = m$.

We provide a few of the (normalized) spherical harmonics for the small angular momentum numbers,

$$Y_{00} = \sqrt{\frac{1}{4\pi}}, \quad Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\vartheta, \quad Y_{20} = \sqrt{\frac{5}{16\pi}} \left(3\cos^2\vartheta - 1\right) \tag{3.119}$$

$$Y_{1(\pm 1)} = \mp\sqrt{\frac{3}{8\pi}} \sin\vartheta e^{\pm i\varphi}, \quad Y_{2(\pm 1)} = \mp\sqrt{\frac{15}{8\pi}} \sin\vartheta \cos\vartheta e^{\pm i\varphi} \tag{3.120}$$

$$Y_{2(\pm 2)} = \sqrt{\frac{15}{32\pi}} \sin^2\vartheta e^{\pm 2i\varphi} \tag{3.121}$$

For the sake of brevity we prove the first two formulae. Consider the element Y_{00}, such that

$$\begin{aligned} Y_{00}(\vartheta, \varphi) &= (-1)^0 \sqrt{\frac{2(0)+1}{4\pi} \frac{(0-0)!}{(0+0)!}} P_{00}(\cos\vartheta) e^{i0\varphi} \\ \implies &= (1) \sqrt{\frac{1}{4\pi} (1)} (1) (1) \\ \implies &= \sqrt{\frac{1}{4\pi}} \end{aligned}$$

where we have used the Legendre polynomials shown in (3.118) together with (3.114). Next, consider the element Y_{10},

$$Y_{10}(\vartheta, \varphi) = (-1)^0 \sqrt{\frac{2(1)+1}{4\pi} \frac{(1-0)!}{(1+0)!}} P_{10}(\cos\vartheta) e^{i0\varphi}$$

Consider first the associated Legendre polynomial P_{10}, where we use (3.114) to explicitly derive its solution,

$$P_{10}(\xi) = \left(1 - \xi^2\right)^{\frac{0}{2}} \frac{\mathrm{d}^0}{\mathrm{d}\xi^0} P_1(\xi) \implies = (1) P_1(\xi) \implies = \xi \to \cos\vartheta \tag{3.122}$$

Substitution into the spherical harmonic under consideration we have,

$$
\begin{aligned}
Y_{10}(\vartheta,\varphi) &= (1)\sqrt{\frac{3}{4\pi}}\cos\vartheta(1)\\
\implies &= \sqrt{\frac{3}{4\pi}}\cos\vartheta
\end{aligned}
$$

The spherical harmonics fulfill the completeness condition

$$
\begin{aligned}
\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_{lm}(\vartheta,\varphi)Y_{lm}^{\star}\left(\vartheta',\varphi'\right) &= \delta\left(\cos\vartheta-\cos\vartheta'\right)\delta\left(\varphi-\varphi'\right)\\
&= \frac{1}{\sin\vartheta}\delta\left(\vartheta-\theta'\right)\delta\left(\varphi-\varphi'\right)
\end{aligned}
\tag{3.123}
$$

and are orthogonal, since

$$
\langle Y_{lm}|Y_{l'm'}\rangle = \int_0^{\pi}\sin\vartheta\mathrm{d}\vartheta\int_0^{2\pi}\mathrm{d}\varphi Y_{lm}^{\star}(\vartheta,\varphi)Y_{l'm'}(\vartheta,\varphi) = \delta_{ll'}\delta_{mm'} \tag{3.124}
$$

Furthermore, they satisfy the following theorem,

$$
\sum_{m=-l}^{l} Y_{lm}(\vartheta,\varphi)Y_{lm}^{\star}\left(\vartheta',\varphi'\right) = \frac{2l+1}{4\pi}P_l(\cos\Theta) \tag{3.125}
$$

where Θ is the angle between the directions (ϑ,φ) and (ϑ',φ') with the following symmetry properties

$$
Y_{l(-m)} = (-1)^m Y_{lm}^{\star} \tag{3.126}
$$

Inversion of coordinates $r\rightarrow -r$, may be expressed by the partiy operator P. In spherical coordinates, the inversion of coordinates reduces to the change of angles only, $r\rightarrow r, \varphi\rightarrow(\varphi+\pi)_{\text{ mod }2\pi'}$ and $\vartheta\rightarrow\pi-\vartheta$. For the spherical harmonics under parity transformation, we find

$$
PY_{lm}(\vartheta,\varphi) = Y_{lm}\left(\pi-\vartheta,(\varphi+\pi)_{\text{ mod }2\pi}\right) = (-1)^l Y_{lm}(\vartheta,\varphi) \tag{3.127}
$$

Thus, for even l the spherical harmonics denote an even parity, while for odd l we have a corresponding odd parity, yielding a sign inversion under point reflection at the origin.

3.1.19 Coupling of Angular Momenta

If a system is comprised of N particles, each with a unique angular momentum j_i with $i = 1, \ldots, N$, the total angular momentum J is the sum of the N individual angular momenta j_i. Consider for example a system of two particles each with an angular momenta, j_1 and j_2, respectively. Then, the angular momentum operator J is the sum of the individual angular momentum vector operators,

$$J = j_1 + j_2 \tag{3.128}$$

in analogy to classical physics. For the total angular momentum, there exists a system of eigenstates for which angular-momentum eigenvalue equations hold

$$J^2 |JM_J\rangle = J(J+1)\hbar^2 |JM_J\rangle \tag{3.129}$$

$$J_z |JM_J\rangle = M_J\hbar |JM_J\rangle \tag{3.130}$$

(Dirac's ket symbol denotes the unknown eigenfunctions which are distinguished by their eigenvalues, i.e. by the *total* angular momentum quantum numbers J amd M_J). The properties proposed for a single orbital angular momentum transfer to the case of total orbital angular momentum which can be shown by making all formal equations explicit, which we skip here as this is not the aim of this paper.

Since the uncouples states $\left|j_1 m_{j(1)}\right\rangle$ and $\left|j_2 m_{j(2)}\right\rangle$ each form a complete basis, the direct product basis built from these forms itself a complete basis for the Hilbert space representation of the total eigenstate $|JM_J\rangle$ in accordance to (3.14). Nonetheless, it is pertinent to mention that the product basis created from this makes J^2 non-diagonal. In order to obtain a coupled representation which leaves J^2 diagonal, we must expand the coupled state $|JM_J\rangle$ into the product basis of the uncoupled states

$$|JM_J\rangle = \sum_{m_{j(1)},m_{j(2)}} \left|j_1 m_{j(1)}\right\rangle \left|j_2 m_{j(2)}\right\rangle \left\langle j_1 m_{j(1)} j_2 m_{j(2)} | JM_J\right\rangle \tag{3.131}$$

We note that the projection coefficients $\left\langle j_1 m_{j(1)} j_2 m_{j(2)} | JM_J\right\rangle$ are called the Clebsch-Gordon vector coupling coefficients; they are required in order to yield an eigenfunction from an expansion into the uncoupled product basis. These coefficients are defined for $J = j_1 + j_2$ as

$$\begin{aligned} \left\langle j_1 m_{j(1)} j_2 m_{j(2)} | JM_J\right\rangle &= \sqrt{\frac{(2j_1)!\,(2j_2)!}{(2J)!}} \\ \times\sqrt{\frac{(J+M_J)!\,(J-M_J)!}{\left(j_1+m_{j(1)}\right)!\left(j_1-m_{j(1)}\right)!\left(j_2+m_{j(2)}\right)!\left(j_2-m_{j(2)}\right)!}} \end{aligned} \tag{3.132}$$

We stress that the above equation holds for $J = j_1 + j_2$ only, which implies that $J > j_1$ and $J > j_2$, and a more general expression may be derived.

3.1.20 Spin

Briefly, we expand on the origin of the spin quantum number. In 1922, Otto Stern and Walter Gerlach demonstrated in the aptly deemed Stern-Gerlach experiment that a beam of silver atoms when fired at an inhomogeneous magnetic field split into two parts. What was discovered from said experiment is that a magnetic moment (intrinsic actually) must be associated with the beam of silver atoms. Considering said problem in a quantum mechanical manner we find that this observation may be explained by matrix eigenvalue equations, which are consistent with angular momentum. This points to a magnetic dipole originating from an angular momentum. The result yielded by the aforementioned experiment, led to a component of the angular momentum which is not an integer multiple of the reduced Planck constant, but only half of it, thus, leading to the conclusion that the magnetic moment cannot be a byproduct from an orbital angular momentum.

In spectroscopy, a splitting of the lines in the presence of a static magnetic field was observed. This spectroscopic effect is called the *Zeeman effect*. It is mostly described by the orbital angular momenta, yet not all features in a line spectrum affected by a static magnetic field may be explained by the orbital angular momentum alone. The *duplexity problem* was resolved by the unprecedented introduction of an intrinsic angular momentum called *spin* - first introduced by Goudsmit and Uhlenbeck. This novel concept has no classical analog, and thus we cannot apply the correspondence principle. This spin observable can be portrayed by two eigenvalue equations,

$$s^2 \rho_{sm_s} = s(s+1)\hbar^2 \rho_{sm_s} \tag{3.133}$$

$$s_z \rho_{sm_s} = m_s \hbar \rho_{sm_s} \quad \text{with} \quad m_s \in \{-s, -s+1, \ldots, s\} \tag{3.134}$$

which are identical to the angular-momentum eigenvalue equations. For this reason we find that this new spin property is an *internal* angular momentum of an elementary particle. The quantum number s may adopt positive or half-integer values or can vanish, while the explicit form of the eigenfunction ρ_{sm_s} depends on the system considered. Experimentally, it is determined that the spin quantum number of an electron is $s = 1/2$. The eigenvalue equations may be written in vector form with the electron spin operator

$$s = \frac{\hbar}{2}\sigma \tag{3.135}$$

where σ is a vector consisting of the three Pauli spin matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.136}$$

Consequently, the squared spin operator is

$$s^2 = \frac{\hbar^2}{4} \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \frac{1}{2}\left(\frac{1}{2}+1\right)\hbar^2 \mathbf{1}_2 \tag{3.137}$$

which retains two different eigenvectors

$$\rho_{\frac{1}{2}\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \rho_{\frac{1}{2}(-\frac{1}{2})} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.138}$$

with magnetic spin quantum numbers $m_s = 1/2$ and $m_s = -1/2$, which may be easily verified by direct substitution into (3.133) and (3.134). The z-component of the spin operator for spin-1/2 particles satisfies

$$s_z = \frac{\hbar}{2}\sigma_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.139}$$

As aforementioned regarding quantum mechanical measurements, each individual measurement of the z-components of the spin of a particle yields either of the allowable eigenvalues, $1/2$ or $-1/2$, respectively. The expectation value of very many measurements on equally prepared particles is, nonetheless, zero.

The spin angular momentum may also coupled to another spin angular momentum, analogous to other angular momenta. Two spin operators couple to a total spin operator S via

$$S = s_1 + s_2 \tag{3.140}$$

Coupling of all spins of an aggregate of particles, say electrons, to a total spin angular momentum S, leads to the construction of the total spin eigenfunction ρ_{SM_S}.

3.1.21 Coupling of Orbital and Spin Angular Momenta

Spin may be coupled with the orbital angular momentum to yield a *total angular momentum for a single particle*. The total angular momentum operator j for a single particle is defined as the vectorial sum of all angular momenta, i.e.

$$j = l \otimes \mathbf{1}_2 + \mathbf{1}_1 \otimes s \quad \rightarrow \quad l\mathbf{1}_2 + s \tag{3.141}$$

where l is the orbital angular momentum operator and s is the spin momentum operator, with

$$l\mathbf{1}_2 \equiv l \otimes \mathbf{1}_2 = (l_x\mathbf{1}_2, l_y\mathbf{1}_2, l_z\mathbf{1}_2) \tag{3.142}$$

For the coupling of two general angular momenta, we seek a formula to create the eigenstates of the total angular momentum from those of the individual angular momenta, i.e. from the uncoupled representations of the orbital angular momentum and spin eigenfunctions.

The eigenvalue equation for the squared two-component total angular momentum j^2 is of identical form as its orbital and spin angular momentum counterparts, which explicitly written takes the form

$$j^2\chi_{jm_j} = j(j+1)\hbar^2\chi_{jm_j} \tag{3.143}$$

The corresponding eigenstates also satisfies the equation for the z-component (rather one-component) of j,

$$j_z\chi_{jm_j} = m_j\hbar\chi_{jm_j} \tag{3.144}$$

as the subscript m_j has indicated. The eigenstates χ_{jm_j} are called *spherical spinors* or Pauli spinors which may be constructed from the uncoupled product states $\phi_{lm_l} = Y_{lm_l}\rho_{\frac{1}{2}m_s}$. The total angular quantum number is expressed in terms of the orbital and spin quantum numbers, yielding $j = l - \frac{1}{2}$ or $j = l + \frac{1}{2}$. For given l, the two possible product states can be explicitly written as

$$\phi_{l\left(m_j-\frac{1}{2}\right)} = Y_{l\left(m_j-\frac{1}{2}\right)}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \phi_{l\left(m_j+\frac{1}{2}\right)} = Y_{l\left(m_j+\frac{1}{2}\right)}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.145}$$

Note that the subscripts of the product states indicate that two different m_l components must be combined with the spin eigenfunctions. The reasoning behind this is that in order to yield the coupled states with given m_j, we must fulfill

$$m_j = m_l + m_s \quad \text{originating from} \quad j_z = l_z + s_z \tag{3.146}$$

such that

$$m_l = m_j - m_s = m_j \mp \frac{1}{2} \tag{3.147}$$

resulting in two different spherical harmonics Y_{Im_l} and $Y_{lm_l'}$ to enter the product basis such that $m_l = m_j - 1/2$ and $m_l' = m_j + 1/2$. From the product basis obtained for (3.145)a Clebsch-Gordan summation yields the coupled eigenstates

$$\chi_{jm_j} = \sum_{m_s} Y_{l\left(m_j - m_s\right)} \rho_{\frac{1}{2}m_s} \left\langle l\left(m_j - m_s\right)\frac{1}{2}m_s | jm_j \right\rangle \tag{3.148}$$

$$= \begin{pmatrix} Y_{l\left(m_j - \frac{1}{2}\right)} & \left\langle l\left(m_j - \frac{1}{2}\right)\frac{1}{2}\frac{1}{2}|jm_j\right\rangle \\ Y_{l\left(m_j + \frac{1}{2}\right)} & \left\langle l\left(m_j + \frac{1}{2}\right)\frac{1}{2}\left(-\frac{1}{2}\right)|jm_j\right\rangle \end{pmatrix} \tag{3.149}$$

The vector coupling coefficients $\left\langle l\left(m_j - m_s\right)\frac{1}{2}m_s|jm_j\right\rangle$ are the Clebsch-Gordan coefficients. For total angular quantum number $j = l + \frac{1}{2}$ we point attention to the fact that $j > l$ and $j > \frac{1}{2}$. Thus, we may use (3.132) to explicitly resolve the Clebsch-Gordan coefficients and obtain

$$\begin{aligned} \chi^{(+)}_{jm_j} &= \sqrt{\frac{l + m_j + 1/2}{2l+1}} \phi_{l\left(m_j - \frac{1}{2}\right)} + \sqrt{\frac{l - m_j + 1/2}{2l+1}} \phi_{l\left(m_j + \frac{1}{2}\right)} \\ &= \begin{pmatrix} \sqrt{\frac{l+m_j+1/2}{2l+1}} Y_{l\left(m_j - \frac{1}{2}\right)} \\ \sqrt{\frac{l-m_j+1/2}{2l+1}} Y_{l\left(m_j + \frac{1}{2}\right)} \end{pmatrix} \end{aligned} \tag{3.150}$$

where the superscript $'(+)'$ indicates that $j = l + \frac{1}{2}$.

We may also consider the case with l' larger than j so that $j = l' - \frac{1}{2}$. We find that for $j = l' - \frac{1}{2}$ we must carry on via a distinct approach in which we must rearrange the Clebsch-

Gordan coefficient first so that the constraint for (3.132), $J = j_1 + j_2$, is satisfied. From well-known symmetry relations for Clebsch-Gordan coefficients we find

$$\left\langle l'\left(m_j - m_{\mathrm{s}}\right)\frac{1}{2}m_{\mathrm{s}}|jm_j\right\rangle = (-1)^{3/2-m_{\mathrm{s}}}\sqrt{\frac{2l'}{2l'+1}}\left\langle \frac{1}{2}m_{\mathrm{s}}j\left(-m_j\right)|l' - \left(m_j - m_{\mathrm{s}}\right)\right\rangle \tag{3.151}$$

where the Clebsch-Gordan coefficient on the right-hand side may be evaluated via (3.132). Thus, we find that

$$\chi_{jm_j}^{(-)} = -\sqrt{\frac{l'-m_j+1/2}{2l'+1}}\phi_{l'\left(m_j-\frac{1}{2}\right)} + \sqrt{\frac{l'+m_j+1/2}{2l'+1}}\phi_{l'\left(m_j+\frac{1}{2}\right)} = \begin{pmatrix} -\sqrt{\frac{l'-m_j+1/2}{2l'+1}}Y_{l'\left(m_j-\frac{1}{2}\right)} \\ \sqrt{\frac{l'+m_j+1/2}{2l'+1}}Y_{l'\left(m_j+\frac{1}{2}\right)} \end{pmatrix} \tag{3.152}$$

in which the superscript $'(-)'$ denotes the case $j = l' - \frac{1}{2}$.

Omitting an explicit reference to the orbital angular momentum quantum number l, we write the two sets of spherical spinors as follows,,

$$\chi_{jm_j}^{(\pm)} = \sum_{m_s} Y_{\left(j\mp\frac{1}{2}\right)\left(m_j-m_s\right)}\rho_{\frac{1}{2}m_s}\left\langle \left(j\mp\frac{1}{2}\right)\left(m_j-m_s\right)\frac{1}{2}m_s|jm_j\right\rangle \tag{3.153}$$

$$= \begin{pmatrix} Y_{(j\mp1)\left(m_j-\frac{1}{2}\right)} & \left\langle\left(j\mp\frac{1}{2}\right)\left(m_j-\frac{1}{2}\right)\frac{1}{2}\frac{1}{2}|jm_j\right\rangle \\ Y_{\left(j\mp\frac{1}{2}\right)\left(m_j+\frac{1}{2}\right)} & \left\langle\left(j\mp\frac{1}{2}\right)\left(m_j+\frac{1}{2}\right)\frac{1}{2}\left(-\frac{1}{2}\right)|jm_j\right\rangle \end{pmatrix} \tag{3.154}$$

Now by using the definitions of the various angular momenta, (3.91) and (3.135), we find that follwoing set of eigenvalue equations hold for the coupled spherical spinors of (3.150) and (3.152)

$$j^2\chi_{jm_j}^{(\pm)} = j(j+1)\hbar^2\chi_{jm_j}^{(\pm)}, \quad j = \frac{1}{2}, \frac{3}{2}, \dots \tag{3.155}$$

$$l^2\chi_{jm_j}^{(\pm)} = l(l+1)\hbar^2\chi_{jm_j}^{(\pm)}, \quad l = j\mp\frac{1}{2} \tag{3.156}$$

$$s^2\chi_{jm_j}^{(\pm)} = s(s+1)\hbar^2\chi_{jm_j}^{(\pm)}, \quad s = \frac{1}{2} \tag{3.157}$$

$$j_z \chi^{(\pm)}_{jm_j} = m_j \hbar \chi^{(\pm)}_{jm_j}, \quad m_j = -j, -j+1, \ldots, j \tag{3.158}$$

Thus, the sign in the subscript of the spherical spinor, $\chi^{(\pm)}_{jm_j}$, affects only the eigenvalue of the equations of l^2.

We present the scalar operator $(\sigma \cdot r)/r = (\sigma \cdot \hat{r})$, whose significance is justified by leaving the total angular momentum invariant upon application of said operator. This scalar operator is a $(2 \times 2)-$matrix which may be written explicitly as

$$\sigma \cdot \hat{r} = \left(\frac{\sigma \cdot r}{r} \right) = \frac{1}{r} \left(\sigma_x x + \sigma_y y + \sigma_z z \right) = \frac{1}{r} \begin{pmatrix} z & x - \mathrm{i}y \\ x + \mathrm{i}y & -z \end{pmatrix} \tag{3.159}$$

in which we used the transformation to spherical coordinate we previously defined as well as Euler's formula $\exp(\pm \mathrm{i}\varphi) = \cos\varphi \pm \mathrm{i}\sin\varphi$ in order to rewrite the operator in spherical coordinates

$$\sigma \cdot \hat{r} = \begin{pmatrix} \cos\theta & \sin\theta e^{-i\varphi} \\ \sin\theta e^{+i\varphi} & -\cos\theta \end{pmatrix} \tag{3.160}$$

In order to show that the operator does not change the total angular momentum, it suffices to show that commutator of said operator vanishes,

$$\left[(\sigma \cdot \hat{r}), j^2 \right] = 0 \tag{3.161}$$

Thus, we find that j^2 and $(\sigma \cdot \hat{r})$ possess the same eigenfunctions, namely $\chi^{(\pm)}_{jm_j}$. Hence, if the scalar operator is to operate on the eigenstates it *must not* change the total angular momentum quantum number j but rather is only allowed to switch between the two possible spherical spinors which are available to j,

$$\left(\frac{\sigma \cdot r}{r} \right) \chi^{(\xi)}_{jm_j} = \alpha \chi^{(\zeta)}_{jm_j} \quad \text{with} \quad \xi, \zeta \in \{+, -\} \tag{3.162}$$

We expound on the properties of the scalar operator and whether it keeps the spherical spinor invariant or whether it switches from the $(+)$ to the $(-)$ spinor of the pair (or conversely), by considering the parity properties of the spherical spinors. We use the parity operator P on the Pauli spinors where we find

$$P\chi^{(\pm)}_{jm_j}(r) = \chi^{(\pm)}_{jm_j}(-r) = (-1)^l \chi^{(\pm)}_{jm_j}(r) \tag{3.163}$$

due to the parity of the spherical harmonics, denoted in (3.127). Since, we previously considered $j = l + \frac{1}{2}$ for $(+)$ spherical spinors, we rearrange to obtain $l = l^{(+)} = j - \frac{1}{2}$ for $\chi^{(+)}_{jm_j}$ and $l = l^{(-)} = j + 1/2$ for $\chi^{(-)}_{jm_j}$, such that the parity of a pair of Pauli spinors for given $\left(j, m_j\right)$ are not equivalent. Rather if the $l^{(+)}$ is even, then the corresponding spherical spinors $\chi^{(+)}_{jm_j}$ possess even parity, while $l^{(-)}$ is then odd, with $\chi^{(-)}_{jm_j}$ of odd parity as well. Operating on the scalar operator with the parity operator P, we find that the parity of $(\sigma \cdot \hat{r})$ is odd, since

$$P(\sigma \cdot \hat{r}) = [\sigma \cdot (-\hat{r})] = -(\sigma \cdot \hat{r}) \tag{3.164}$$

which then leads to a change in the parity of the spherical spinor such that operating on the spherical spinor, such as in (3.162), results in

$$\left(\frac{\sigma \cdot r}{r}\right) \chi^{(\pm)}_{jm_j} = \alpha \chi^{(\mp)}_{jm_j} \tag{3.165}$$

Next, we identify the unknown prefactor α of the eigenvalue equation of (3.165). To this end, we consider the squared operator $(\sigma \cdot \hat{r})^2$. Thus, evaluating such a squared operator using Dirac's relation, we obtain

$$(\sigma \cdot A)(\sigma \cdot B) = A \cdot B 1_2 + i\sigma \cdot (A \times B) \tag{3.166}$$

for arbitrary vector operators A and B. Then since $A = B = r$, we obtain,

$$
\begin{aligned}
(\sigma \cdot r)(\sigma \cdot r) &= r \cdot r 1_2 + i\sigma \cdot (r \times r) \\
\Longrightarrow &= |r|^2 1_2 + i\sigma \cdot (r \times r) \\
\Longrightarrow &= |r|^2 1_2 + 0
\end{aligned}
$$

since the cross-product of r with itself is zero, $r \times r = 0$. Thus, we have,

$$
(\sigma \cdot r)(\sigma \cdot r) = |r|^2 \mathbf{1}_2 = r^2 \mathbf{1}_2 \Rightarrow \left(\frac{\sigma \cdot r}{r}\right)\left(\frac{\sigma \cdot r}{r}\right) = \mathbf{1}_2 \tag{3.167}
$$

Thus, the corresponding eigenvalue equation

$$
\left(\frac{\sigma \cdot r}{r}\right)^2 \chi_{jm_j}^{(\pm)} = \alpha^2 \chi_{jm_j}^{(\pm)} \overset{(3.167)}{=} \chi_{jm_j}^{(\pm)} \tag{3.168}
$$

where $\alpha^2 = 1$, such that $\alpha = \pm 1$. We now determine the value of the α factor, by considering a special set of coordinates which will reduce our calculations. To this end, we select the set such that $\theta = 0$, so that the Cartesian coordinates become $x = y = 0$ and $z = r$, and (3.159) and (3.160) become

$$
\left(\frac{\sigma \cdot r}{r}\right) \overset{\theta=0}{=} \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.169}
$$

Indeed, considering (3.159) we find that

$$
\begin{aligned}
\sigma \cdot \hat{r} &= \frac{1}{r}\begin{pmatrix} r & 0 \\ 0 & -r \end{pmatrix} \\
\Longrightarrow &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
\Longrightarrow &= \sigma_z
\end{aligned}
$$

similarly, by (3.160) we have

$$\sigma\cdot\hat{r} = \begin{pmatrix} \cos(0) & \sin(0)e^{-\mathrm{i}\varphi} \\ \sin(0)e^{+\mathrm{i}\varphi} & -\cos(0) \end{pmatrix}$$
$$\Longrightarrow = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$\Longrightarrow = \sigma_z$$

For $\theta = 0$, (3.168) may be simplified since the only spherical harmonics which survive are those whose magnetic quantum number is zero, with all other spherical harmonics vanishing due to a sine function which tends to zero. Non-vanishing harmonics occur when $m_j = 1/2$ or $m_j = -1/2$. For the case in which $m_j = 1/2$, we surmise from (3.168) the following

$$\begin{aligned} \left(\tfrac{\sigma\cdot r}{r}\right)\chi^{(+)}_{j(1/2)} &\overset{\theta=0}{=} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \sqrt{\frac{l^{(+)}+1}{2l^{(+)}+1}}Y_{l^{(+)}0}(\varphi,0) \\ 0 \end{pmatrix} \\ = -&\begin{pmatrix} -\sqrt{\frac{l^{(-)}}{2l^{(+)}+1}}\sqrt{\frac{2l^{(+)}+1}{2l^{(-)}+1}}Y_{l^{(-)}}(\varphi,0) \\ 0 \end{pmatrix} = -\chi^{(-)}_{j(-1/2)}(\varphi,0) \end{aligned} \tag{3.170}$$

for the remaining selection of $m_j = -1/2$, (3.168) gives

$$\begin{aligned} \left(\tfrac{\sigma\cdot r}{r}\right)\chi^{(+)}_{j(-1/2)} &\overset{\theta=0}{=} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ \sqrt{\frac{l^{(+)}+1}{2l^{(+)}+1}}Y_{l^{(+)}0}(\varphi,0) \end{pmatrix} \\ = -&\begin{pmatrix} 0 \\ \sqrt{\frac{l^{(-)}}{2l^{(+)}+1}}\sqrt{\frac{2l^{(+)}+1}{2l^{(-)}+1}}Y_{l^{(-)}0}(\varphi,0) \end{pmatrix} = -\chi^{(-)}_{j(-1/2)}(\varphi,0) \end{aligned} \tag{3.171}$$

We analyze the above equations in order to attain a better understanding. We commence by noting once more that the l quantum number is different for a given j; in particular, we have $l^{(+)} = l^{(-)} - 1$. Thus, the spherical harmonics with opposite sign, $(+)$ or $(-)$, are different for any given (j, m_j).

Nonetheless, for $\theta = 0$ all spherical harmonics with $m = 0$ take the elementary form

$$Y_{l0}(\varphi, 0) = \sqrt{\frac{1}{4\pi}}\sqrt{2l+1} \tag{3.172}$$

such that the relation between two functions is given by

$$Y_{l0}(\varphi, 0) = \sqrt{\frac{2l+1}{2l'+1}}Y_{l'0}(\varphi, 0) \tag{3.173}$$

Finally, we obtain from (3.170) and (3.171) that α assumes a *negative* sign such that we finally obtain,

$$(\sigma \cdot \hat{r})\chi^{(\pm)}_{jm_j} = -\chi^{(\mp)}_{jm_j} \tag{3.174}$$

We note that the action of the scalar operator on a Pauli spinor may also be evaluated explicitly using the representation of the operator of (3.160).

3.2 Relativistic Theory of the Electron

3.2.1 Classical Energy Expression and Correspondence Principle

Now, we shift our focus to what is perhaps the first step toward a relativistic electronic structure theory. We consider as a preliminary step the option of setting up a quantum mechanical equation of motion which abides by the correspondence principle. Applying the correspondence to the classical *nonrelativistic* kinetic energy expression $E = p^2/(2m)$, we attain the time-dependent Schrödinger equation, with first derivatives with respect to time yet we arrive at second derivatives with respect to space. Since we have disparity between the form of the derivatives of time and space, we arrive at the conclusion that the Schrödinger equation does not fulfill the requirement of covariance under Lorentz transformations. To this end, we consider a different approach. We consider the classical expression for the *relativistic* energy of the freely moving particle

$E = \sqrt{p^2c^2 + m_e^2c^4}$ and using the correspondence principle apply the following substitutions

$$E \longrightarrow \mathrm{i}\hbar\frac{\partial}{\partial t} \tag{3.175}$$

$$p \longrightarrow -\mathrm{i}\hbar\nabla \tag{3.176}$$

obtaining the following result as

$$E = \sqrt{p^2c^2 + m_e^2c^4} \tag{3.177}$$

$$\mathrm{i}\hbar\frac{\partial}{\partial t} \quad \Psi = \sqrt{-\hbar^2c^2\nabla^2 + m_e^2c^4}\Psi \tag{3.178}$$

where the resulting energy operator of (3.178) is known as the $square - root\ operator$. We point attention to the fact that evaluation of the square root of the spatial differentiation, ∇^2, would be tedious and vexing, while an expansion of said square root would lead to infinitely high derivatives with respect to the spatial coordinates, leading to the above equation not being covariant under Lorentz transformations, again.

In order to bypass said arduous calculation, we proceed by using the squared energy expression to derive the quantum mechanical equation of motion

$$E^2 \quad = \quad p^2c^2 + m_e^2c^4 \tag{3.179}$$

$$-\hbar^2\frac{\partial^2}{\partial t^2} \quad \Psi = \left(-\hbar^2c^2\nabla^2 + m_e^2c^4\right)\Psi \tag{3.180}$$

The resulting (3.180) is a second-order partial differential equation for a freely moving particle, in which the spatial and time coordinates are considered in equal footing. Historically, (3.180) is deemed the Klein-Gordon equation and was derived in respective occasions by W. Gordon in 1926 and O. Klein in 1927, among others.

The Klein-Gordon equation may be written in a compact, explicitly covariant form

$$\left[\partial_\mu \partial^\mu + \left(\frac{m_e c}{\hbar}\right)^2\right]\Psi = \left[\Box + \left(\frac{m_e c}{\hbar}\right)^2\right]\Psi = 0 \tag{3.181}$$

where we consider the covariant vector $\partial_\mu = \partial/\partial x^\mu$ with $x^\mu = (ct, x) = (ct, x_1, x_2, x_3)$. We briefly note that the d'Alembertian operator $\Box = \partial_\mu \partial^\mu$ is Lorentz covariant, leading to the fact that the covariance of the rest of the equation holds under transformations.

We note that the physical interpretation of the Klein-Gordon, is only the energy-momentum relation. It does not make reference to the quantum mechanical spin of a particle, such as the Schrödinger equation, and therefore cannot explain its experimentally observed occurrence.

3.2.2 Solutions of the Klein-Gordon Equation

In order to attain a profound understanding of the Klein-Gordon equation we take special interest in its solutions. The Klein-Gordon eigenfunction is given by the plane wave

$$\Psi(r,t) = \exp\left[\frac{\mathrm{i}}{\hbar}(Et - p \cdot r)\right] \tag{3.182}$$

which we may verify by direct substitution into the left-hand side of (3.180)

$$-\hbar^2 \frac{\partial^2}{\partial t^2} \exp\left[\frac{\mathbf{i}}{\hbar}(Et - p \cdot r)\right] = E^2 \exp\left[\frac{\mathbf{i}}{\hbar}(Et - p \cdot r)\right] \tag{3.183}$$

and into the right hand side,

$$\left(-\hbar^2 c^2 \nabla^2 + m_e^2 c^4\right) \exp\left[\frac{\mathrm{i}}{\hbar}(Et - p \cdot r)\right] = \left(c^2 p^2 + m_e^2 c^4\right) \exp\left[\frac{\mathrm{i}}{\hbar}(Et - p \cdot r)\right] \tag{3.184}$$

respectively. Comparison of coefficients in (3.183) and (3.184) gives the eigenvalue

$$E^2 = c^2 p^2 + m_e^2 c^4 \tag{3.185}$$

Which leads to the following energy values of the freely moving particle

$$E = \pm\sqrt{c^2p^2 + m_e^2c^4} \tag{3.186}$$

which denotes the expected positive energies, yet also yields negative-energy values a bit perplexing to our current understanding. Thus, the quadratic form in (3.180) leads to negative energy eigenvalue for freely moving particles. Returning to the expected finite positive kinetic energy displayed by particles, it is arduous to interpret the negative-energy state yielded, yet we hint at their occurrence in the Dirac equation.

We anticipate the fact that the Klein-Gordon equation is unfit to properly describe the properties of an electron, yet under a properly quantized field-theoretical form, it describes neutral mesons of spin 0. Mesons are strongly interacting bosons, i.e. they are integer spin particles subject to the strong force.

3.2.3 Klein-Gordon Density Distribution

Our derivation of the Klein-Gordon equation has been attributed to the *ad hoc* hypothesis that the correspondence principle is aptly applied in the relativistic scheme. Even though the relativistic equation is Lorentz covariant in nature there still remains the peculiarity of the unexplained negative-energy solutions. It is pertinent to further study properties of the Klein-Gordon equation, such that we may obtain a better understanding of the properties of said equation. To this end, we choose to analyze the density distribution, which we may derive from a continuity equation. In order to attain said continuity equation from the Klein-Gordon equation we proceed to multiplying (3.181) by $\Psi^\star$,

$$\Psi^\star\left[\partial_\mu\partial^\mu + \left(\frac{m_ec}{\hbar}\right)^2\right]\Psi = 0 \tag{3.187}$$

and similarly consider its complex conjugate

$$\Psi\left[\partial_\mu\partial^\mu+\left(\frac{m_e c}{\hbar}\right)^2\right]\Psi^\star=0 \tag{3.188}$$

Subtracting of (3.188) from (3.187) gives

$$\Psi^\star\partial_\mu\partial^\mu\Psi-\Psi\partial_\mu\partial^\mu\Psi^\star=0 \tag{3.189}$$

since

$$\begin{gathered}
\Psi^\star\left[\partial_\mu\partial^\mu+\left(\frac{m_e c}{\hbar}\right)^2\right]\Psi=0\\
-\Psi\left[\partial_\mu\partial^\mu+\left(\frac{m_e c}{\hbar}\right)^2\right]\Psi^\star=0\\
\Longrightarrow\ \Psi^\star\partial_\mu\partial^\mu\Psi+\Psi^\star\left(\frac{m_e c}{\hbar}\right)^2\Psi-\Psi\partial_\mu\partial^\mu\Psi^\star-\Psi\left(\frac{m_e c}{\hbar}\right)^2\Psi^\star\\
\Longrightarrow\ \Psi^\star\partial_\mu\partial^\mu\Psi-\Psi\partial_\mu\partial^\mu\Psi^\star=0
\end{gathered}$$

since, the $\left(\frac{m_e c}{\hbar}\right)^2$ term is a scalar which commutes with $\Psi^\star$ and Ψ, respectively. (3.189) may be rewritten as

$$\partial_\mu\left[\Psi^\star\partial^\mu\Psi-\Psi\partial^\mu\Psi^\star\right]=0 \tag{3.190}$$

due to the product rule yielding term by term

$$\partial_\mu\Psi^\star\partial^\mu\Psi=\frac{\partial}{\partial x^\mu}\left(\Psi^\star\frac{\partial\Psi}{\partial x_\mu}\right)=\Psi^\star\frac{\partial^2\Psi}{\partial x^\mu\partial x_\mu}+\left(\frac{\partial\Psi^\star}{\partial x^\mu}\right)\left(\frac{\partial\Psi}{\partial x_\mu}\right) \tag{3.191}$$

$$\partial_\mu\Psi\partial^\mu\Psi^\star=\frac{\partial}{\partial x^\mu}\left(\Psi\frac{\partial\Psi^\star}{\partial x_\mu}\right)=\Psi\frac{\partial^2\Psi^\star}{\partial x^\mu\partial x_\mu}+\left(\frac{\partial\Psi}{\partial x^\mu}\right)\left(\frac{\partial\Psi^\star}{\partial x_\mu}\right) \tag{3.192}$$

such that the last terms on the RHS of (3.191) and (3.192) cancel upon subtraction.

Recall the definition of the nonrelativistic current density derived in (3.79)

$$j = \frac{\hbar}{2m_e \mathrm{i}} \left[\Psi^\star \nabla \Psi - \Psi \nabla \Psi^\star\right] \tag{3.193}$$

and write (3.190) after multiplication with the term $i/(2m_e\mathrm{i})$ as

$$\frac{\partial}{\partial t} \frac{\mathrm{i}\hbar}{2m_e c^2} \left[\Psi^\star \frac{\partial}{\partial t}\Psi - \Psi \frac{\partial}{\partial t}\Psi^\star\right] + \operatorname{div} j = 0 \tag{3.194}$$

such that we may rewrite the above continuity equation in covariant form as

$$\partial_\mu j^\mu = \dot{\rho} + \operatorname{div} j = 0 \tag{3.195}$$

with the zeroth component of the 4-current $j^\mu = (c\rho, j)$ as

$$\rho \equiv \frac{\mathrm{i}\hbar}{2m_e c^2} \left[\Psi^\star \frac{\partial}{\partial t}\Psi - \Psi \frac{\partial}{\partial t}\Psi^\star\right] \tag{3.196}$$

such that it defines the Klein-Gordon density distribution. Note that we have a significant problem when dealing with the Klein-Gordon equation. Since this equation is a second-order differential equation in t, we arrive at two integration constants which allows one to choose the initial values of the differential equation, namely $\partial\Psi/\partial t$ and Ψ independently from one another, such that ρ need not be a positive definite quantity. Consequently, ρ does not represent the probability density distribution, as this quantity must be strictly positive for any coordinates t and x_n. Due to this, we may reject the Klein-Gordon as a fundamental quantum mechanical equation, with apt use for describing spinless particles.

Thus rejection of the Klein-Gordon as a fundamental quantum mechanical equation, comes from the fact that intrinsic angular momentum, i.e. spin, does not emerge naturally and in much the same way as in the nonrelativistic regime must be included *a posteriori*. For this reason, the Klein-Gordon cannot properly describe the motion for a freely moving electron. Yet, we ponder upon what results may be of importance in our search for a fundamental relativistic quantum mechanical equation of motion. Certainly, we would like to recover the plane wave solutions of (3.182) for a freely moving particle, yet in order to attain a single integration constant for a positive definite density distribution we shift our focus to *first-order differential equation in time*, with corresponding *first-order differential equation in space*, in order to fulfill the Lorentz covariance requirement.

3.2.4 Derivation of the Relativistic Quantum Mechanical Equation of Motion for an Electron

In 1928, Paul Dirac posited an unprecedented quantum mechanical equation for the electron, which resolved the problems of Lorentz covariance and the duplexity of atomic states, which was accounted for by the introduction of spin. In order to derive this fundamental quantum mechanical equation for the electron, whose result is relativistic covariance, we start with an ansatz for this equation based on the results derived in the preceding section

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\Psi = \left[\frac{\hbar c}{\mathrm{i}}\alpha^k\partial_k + \beta m_e c^2\right]\Psi \equiv H^D\Psi \tag{3.197}$$

where we introduced four parameters in order to cope with the defects of the Klein-Gordon equation. Note that the components of the vector α,

$$\alpha = \left(\alpha^i\right) = \left(\alpha^1, \alpha^2, \alpha^3\right) \tag{3.198}$$

imply that Einstein's summation convention is to be employed.

3.2.5 Relation to the Klein-Gordon Equation

The Klein-Gordon equation implements the relativistic energy-momentum relation $E^2 = p^2c^2 + m_e^2c^4$ for a freely moving particle. Appropriately, the solution of the field-free Klein-Gordon equation are plane waves with energy eigenvalue $E = \pm\sqrt{p^2c^2 + m_e^2c^4}$, which must be reproduced by the solutions of the Dirac equation. Thus, in order to create a relationship between the Klein-Gordon and Dirac equations we apply the operator identity of (3.197), $\mathrm{i}\hbar\partial/\partial t = \hbar c/\mathrm{i}\alpha^k\partial_k + \beta m_e c^2$, to the left and right hand sides, respectively, of the Dirac (3.197)

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\cdot\mathrm{i}\hbar\frac{\partial}{\partial t}\Psi = \left[\frac{\hbar c}{\mathrm{i}}\alpha^k\partial_k + \beta m_e c^2\right]\cdot\left[\frac{\hbar c}{\mathrm{i}}\alpha^k\partial_k + \beta m_e c^2\right]\Psi$$
$$\Longrightarrow -\hbar^2\frac{\partial^2}{\partial t^2}\Psi = \left[\frac{\hbar c}{\mathrm{i}}\alpha^k\partial_k + \beta m_e c^2\right]\cdot\left[\frac{\hbar c}{\mathrm{i}}\alpha^k\partial_k + \beta m_e c^2\right]\Psi$$

such that,

$$-\hbar^2\frac{\partial^2}{\partial t^2}\Psi = \left[\frac{\hbar c}{\mathrm{i}}\alpha^k\partial_k + \beta m_e c^2\right]\left[\frac{\hbar c}{\mathrm{i}}\alpha^k\partial_k + \beta m_e c^2\right]\Psi \tag{3.199}$$

$$= -\frac{\hbar^2 c^2}{2}\sum_{i,j=1}^{3}\left(\alpha^i\alpha^j + \alpha^j\alpha^i\right)\partial_i\partial_j\Psi + \frac{\hbar m_e c^3}{\mathrm{i}}\sum_{i=1}^{3}\left(\alpha^i\beta + \beta\alpha^i\right)\partial_i\Psi + \beta^2 m_e^2 c^4\Psi \tag{3.200}$$

where we obtain (3.200) through the algebraic FOIL process, we also introduced the anticommutator $\alpha^i\alpha^j + \alpha^j\alpha^i$ instead of the product $\alpha^i\alpha^j$ and dealt with the resulting double counting by the prefactor of $1/2$. Comparison with (3.180) details the explicit representation of the unknown parameters - the correspondence principle is implicitly implanted within the Dirac equation. Comparison of coefficients in (3.180) and (3.200) yields

$$\alpha^i\alpha^j + \alpha^j\alpha^i = 2\delta^{ij} \Longrightarrow \left(\alpha^i\right)^2 = 1 \tag{3.201}$$

$$\alpha^i\beta + \beta\alpha^i = 0 \tag{3.202}$$

$$\beta^2 = 1 \quad \Longrightarrow \quad \beta = \beta^{-1} \tag{3.203}$$

Hence, we have that the α^k and β parameters must satisfy the following commutations relations

$$\left\{\alpha^i, \alpha^j\right\} = 2\delta^{ij} \quad \text{and} \quad \left\{\alpha^i, \beta\right\} = 0 \tag{3.204}$$

3.2.6 Explicit Expressions for the Dirac Parameters

From (3.201)-(3.203), we may derive explicitly the parameters $\alpha^1, \alpha^2, \alpha^3$, and β. Using (3.201) and (3.203), we have $\left(\alpha^i\right)^2 = \beta^2 = 1$, where it follows that the parameters possess eigenvalues of either $+1$ or -1, thus, suggesting that the α and β parameters are matrices, then we may rewrite (3.202) as

$$\alpha^i = -\beta^{-1}\alpha^i\beta = -\beta\alpha^i\beta \tag{3.205}$$

after which we can use the cyclic property of the traces, namely $\mathrm{Tr}(ABC) = \mathrm{Tr}(BCA)$, to attain

$$\mathrm{Tr}\left(\alpha^i\right) = -\mathrm{Tr}\left(\beta\alpha^i\beta\right) = -\mathrm{Tr}\left(\alpha^i\beta^2\right) = -\mathrm{Tr}\left(\alpha^i\right) \tag{3.206}$$

Hence, we find that the trace can only be zero, since $\mathrm{Tr}\left(\alpha^i\right) = -\mathrm{Tr}\left(\alpha^i\right)$, and the number of positive and negative eigenvalues of all α^i must be equal. Accordingly, we attain,

$$\beta = -\left(\alpha^i\right)^{-1}\beta\alpha^i = -\alpha^i\beta\alpha^i \tag{3.207}$$

$$\Longrightarrow \mathrm{Tr}(\beta) = -\mathrm{Tr}\left(\alpha^i\beta\alpha^i\right) = -\mathrm{Tr}\left(\beta\left(\alpha^i\right)^2\right) = -\mathrm{Tr}(\beta) \tag{3.208}$$

Thus, this trace also vanishes and β must also have an equal number of positive and negative eigenvalues, suggesting an even dimension of the matrices.

Since we found that the matrices are of even dimension, we have that the smallest possible dimension for the parameters would be two. The only possible (2×2)-matrices which are *linearly independent* and feature the required properties (i.e. one eigenvalue which is $+1$, one which is -1 and a zero trace) are the three Pauli spin matrices

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1, \quad \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} = \sigma_2, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3 \tag{3.209}$$

$\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$. Yet, we have only three parameters, but our ansatz requires four parameters. Hence, the parameters cannot be two dimensional. Hence, the subsequent dimension is four and we find the following set of (4×4)-matrices,

$$\alpha^i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix} \tag{3.210}$$

which satisfy all properties delineated in (3.201)-(3.203).

We note in passing that the peculiar choice of matrices neither unique no is 4 dimensions the only dimension which truly satisfies the requirements delineated in (3.201)-(3.203), with a possibility of higher dimensions. This particular choice of matrices in (3.210) is known as the *standard representation* of the Dirac matrices α^k and β. Another example which equally satisfies the conditions imposed above is the Weyl representation

$$\alpha^i_{\text{Weyl}} = \begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix} \quad \text{and} \quad \beta_{\text{Weyl}} = \begin{pmatrix} 0 & \mathbf{1}_2 \\ \mathbf{1}_2 & 0 \end{pmatrix} \tag{3.211}$$

which leads to the fact that $c(\sigma \cdot p)$ and $m_e c^2$ then change places in the Dirac matrix equation compared to the standard representation.

Since the Pauli matrices define the Dirac matrices we note the connection to the spin operator through (3.135). For the physical interpretation of the $\boldsymbol{\alpha}$ parameters we quote Dirac himself [2] : *The α's are new dynamical variables which it is necessary to introduce in order to satisfy the conditions of the problem. They may be regarded as describing some internal motions of the electron, which for most purposes may be taken to be the spin of the electron postulated in previous theories*

Due to Ψ necessarily being the same dimension as Dirac matrices, we understand that it is, in general, an n-component vector of functions if the dimension of the Dirac matrices is n. In the standard representation, the quantum mechanical state function Ψ is a vector of four functions, denoted as a 4-spinor.

3.2.7 Continuity Equation and Definition of the 4-Current

We proceed by deriving a continuity equation corresponding to (3.197) by multiplying by the complex conjugate of Ψ. This adjoint row vector of Ψ reads $\Psi^\dagger = (\Psi_1^\star, \Psi_2^\star, \dots, \Psi_n^\star)$ with $n = 4$. Then, we have for (3.197)

$$\mathrm{i}\hbar\Psi^\dagger \frac{\partial}{\partial t}\Psi = \frac{\hbar c}{\mathrm{i}}\Psi^\dagger \alpha^k \partial_k \Psi + m_e c^2 \Psi^\dagger \beta \Psi \tag{3.212}$$

whose complex conjugate reads

$$-i\hbar\left(\frac{\partial}{\partial t}\Psi^\dagger\right)\Psi = -\frac{\hbar c}{\mathrm{i}}\left(\partial_k \Psi^\dagger\right)\alpha^{k,\dagger}\Psi + m_e c^2 \Psi^\dagger \beta^\dagger \Psi \tag{3.213}$$

Then, subtracting (3.213) from (3.212), divide by $(\mathrm{i}\hbar)$ to attain

$$\frac{\partial}{\partial t}\left(\Psi^\dagger\Psi\right) = -c\left[\Psi^\dagger \alpha^k \partial_k \Psi + \left(\partial_k \Psi^\dagger\right)\alpha^{k,\dagger}\Psi\right] + \frac{m_e c^2}{\mathrm{i}\hbar}\left[\Psi^\dagger \beta \Psi - \Psi^\dagger \beta^\dagger \Psi\right] \tag{3.214}$$

since

$$\begin{aligned}
\mathrm{i}\hbar\Psi^\dagger \frac{\partial}{\partial t}\Psi &= \frac{\hbar c}{\mathrm{i}}\Psi^\dagger \alpha^k \partial_k \Psi + m_e c^2 \Psi^\dagger \beta \Psi \\
+i\hbar\left(\frac{\partial}{\partial t}\Psi^\dagger\right)\Psi &= \frac{\hbar c}{\mathrm{i}}\left(\partial_k \Psi^\dagger\right)\alpha^{k,\dagger}\Psi - m_e c^2 \Psi^\dagger \beta^\dagger \Psi \\
\mathrm{i}\hbar\frac{\partial}{\partial t}\left(\Psi^\dagger\Psi\right) &= \frac{\hbar c}{\mathrm{i}}\left[\Psi^\dagger \alpha^k \partial_k \Psi + \left(\partial_k \Psi^\dagger\right)\alpha^{k,\dagger}\Psi\right] + m_e c^2\left[\Psi^\dagger \beta \Psi - \Psi^\dagger \beta^\dagger \Psi\right] \\
\implies \frac{\partial}{\partial t}\left(\Psi^\dagger\Psi\right) &= -c\left[\Psi^\dagger \alpha^k \partial_k \Psi + \left(\partial_k \Psi^\dagger\right)\alpha^{k,\dagger}\Psi\right] + \frac{m_e c^2}{\mathrm{i}\hbar}\left[\Psi^\dagger \beta \Psi - \Psi^\dagger \beta^\dagger \Psi\right]
\end{aligned}$$

We note that we may further simplify the above equation by noting that the parameters α^k, β

are hermitean,

$$\alpha^{k,\dagger} = \alpha^k \quad \text{and} \quad \beta^\dagger = \beta \tag{3.215}$$

The physical reason for the hermiticity of the parameters is the necessity of the Dirac Hamiltonian H^D in (3.197) must be hermitean in order to attain real eigenvalues. Hence, the matrices α^k, β must be hermitean. Then, using the properties delineated in (3.215), (3.214) reduces to

$$\frac{\partial}{\partial t}\left(\Psi^\dagger \Psi\right) = -c\left[\Psi^\dagger \alpha^k \partial_k \Psi + \left(\partial_k \Psi^\dagger\right)\alpha^k \Psi\right] \tag{3.216}$$

which then yields the continuity equation

$$\dot{\rho} = -\operatorname{div} j \tag{3.217}$$

Then, defining the Dirac density distribution as

$$\rho \equiv \Psi^\dagger \Psi \tag{3.218}$$

and the Dirac current density as

$$\boldsymbol{j} \equiv c\Psi^\dagger \boldsymbol{\alpha} \Psi, \quad \text{i.e.,} \quad j^k \equiv c\Psi^\dagger \alpha^k \Psi \tag{3.219}$$

Both of which can be combined into the 4-current by

$$j^\mu = \left(j^0, j^k\right), \quad \text{with} \quad j^0 \equiv c\rho \tag{3.220}$$

such that the continuity equation may be written as

$$\partial_\mu j^\mu = 0, \quad \text{or explicitly} \quad \frac{1}{c}\frac{\partial}{\partial t} j^0 + \frac{\partial}{\partial x^k} j^k = 0 \tag{3.221}$$

3.3 Lorentz Covariance of the Field-Free Dirac Equation

3.3.1 Covariant Form

One of the fundamental necessities which all fundamental physical equations must abide by is invariance in form under Lorentz transformations. In order to analyze the behavior of the Dirac equation under Lorentz transformation, we commence by rewriting it as follows

$$-\mathrm{i}\hbar\beta\partial_0\Psi - \mathrm{i}\hbar\beta\partial_i\alpha^i\Psi + m_e c\Psi = 0 \tag{3.222}$$

where we simply divided the Dirac equation into its corresponding space and time like components and equated the equation to 0. Next, we define the set of four new Dirac "gamma" matrices $\gamma^\mu = (\gamma^0, \gamma^i)$, as follows

$$\gamma^0 \equiv \beta = \begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & -\mathbf{1}_2 \end{pmatrix} \quad \text{and} \quad \gamma^i \equiv \beta\alpha^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \tag{3.223}$$

which possess the following properties

$$\left(\gamma^0\right)^2 = 1 \quad \text{while} \quad \left(\gamma^i\right)^2 = -\mathbf{1}_4 \tag{3.224}$$

and

$$\left(\gamma^0\right)^\dagger = \gamma^0 \quad \text{while} \quad \left(\gamma^i\right)^\dagger = -\gamma^i \tag{3.225}$$

fulfilling the following anticommutation relation

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbf{1}_4 = \begin{cases} 0 & \text{if } \mu \neq \nu \\ 2 & \text{if } \mu = \nu = 0 \\ -2 & \text{if } \mu = \nu \in \{1,2,3\} \end{cases} \tag{3.226}$$

After division by $\hbar$, the Dirac equation (3.222) may be expressed in terms of the γ matrices as follows,

$$\left[-\mathrm{i}\gamma^{\mu}\partial_{\mu}+\frac{m_e c}{\hbar}\right]\Psi=0 \tag{3.227}$$

Indeed, we have

$$\begin{aligned}
&-\mathrm{i}\hbar\beta\partial_0\Psi-\mathrm{i}\hbar\beta\partial_i\alpha^i\Psi+m_e c\Psi=0\\
\implies &-\mathrm{i}\hbar\beta\partial_0\Psi-\mathrm{i}\hbar\partial_i\beta\alpha^i\Psi+m_e c\Psi=0\\
\implies &-\mathrm{i}\hbar\gamma^0\partial_0\Psi-\mathrm{i}\hbar\partial_i\gamma^i\Psi+m_e c\Psi=0\\
\implies &\left[-\mathrm{i}\gamma^{\mu}\partial_{\mu}+\frac{m_e c}{\hbar}\right]\Psi=0
\end{aligned}$$

We point attention to the fact that object $\gamma^{\mu}\partial_{\mu}$ is a four-component object consisting of (4×4)-matrices rather than a Lorentz scalar, since γ^{μ} is not a Lorentz 4-vector. Next, we must determine the transformation properties of Ψ such that (3.227) is covariant under Lorentz transformations.

3.3.2 Lorentz Transformations of the Dirac Spinor

Consider two inertial reference frames IS and IS$'$ related through a Lorentz transformation

$$x'=\Lambda x+a \quad \text{and} \quad x=\Lambda^{-1}\left(x'-a\right) \tag{3.228}$$

of the space-time 4-vectors x^{μ} and x'^{μ}, where Λ is a Lorentz transformation matrix and a is a constant space-time shift. Due to the fact that the Dirac equation must comply with the tenets delineated by the special theory of relativity, it is linear in IS and must also be linear in IS$'$, hence, Ψ and Ψ' must be related by some linear transformation f_{Λ} otherwise the transformation would create nonlinear components of the state in the new coordinates system. The quantum mechanical state function Ψ' in IS$'$ can be expressed by the components of Ψ defined in IS,

$$\Psi'\left(x'\right)=f_{\Lambda}[\Psi(x)]=f_{\Lambda}\left[\Psi\left(\Lambda^{-1}\left(x'-a\right)\right)\right] \tag{3.229}$$

where we used the relation denoted in (3.228) for the 4-vector x'^{μ}, and

$$\Psi(x) = f_{\Lambda}^{-1}\Psi'\left(x'\right) \tag{3.230}$$

The (4×4)-matrix operator f_{Λ} acts on the Dirac 4-spinors as we will determine. The operator f_{Λ} of the Lorentz transformation, which relates the coordinates of IS and IS$'$, mixes the components of Ψ_i of Ψ. (3.229) may be written for each component of the new state vector of reference frame IS$'$ as

$$\Psi'_{\mu} = \sum_{\nu=1}^{4} f_{\Lambda,\mu\nu}\Psi_{\nu} \tag{3.231}$$

We demonstrate the Lorentz covariance of (3.227) by introducing the transformation of coordinates step by step with the four component differential operator ∂_{μ},

$$\partial_{\mu} = \frac{\partial}{\partial x^{\mu}} = \frac{\partial x'^{\nu}}{\partial x^{\mu}}\frac{\partial}{\partial x'^{\nu}} = \Lambda^{\nu}_{\mu}\partial'_{\nu} \tag{3.232}$$

where we attained the intermediate step by chain rule, which yields together with (3.229)

$$\left[-\mathrm{i}\gamma^{\mu}\Lambda^{\nu}_{\mu}\partial'_{\nu} + \frac{m_e c}{\hbar}\right] f_{\Lambda}^{-1}\Psi' = 0 \tag{3.233}$$

Multiplying with the linear transformation matrix f_{Λ} from the left we attain

$$\left[-\mathrm{i}f_{\Lambda}\gamma^{\mu}\Lambda^{\nu}_{\mu}\partial'_{\nu}f_{\Lambda}^{-1} + f_{\Lambda}\frac{m_e c}{\hbar}f_{\Lambda}^{-1}\right]\Psi' = 0 \tag{3.234}$$

which yields

$$\left[-\mathrm{i}f_{\Lambda}\gamma^{\mu}\Lambda^{\nu}_{\mu}\partial'_{\nu}f_{\Lambda}^{-1} + f_{\Lambda}\frac{m_e c}{\hbar}f_{\Lambda}^{-1}\right]\Psi' = 0 \tag{3.235}$$

where the transformation matrix and its inverse vanish from the second term since the quantity $\left(\frac{m_e c}{\hbar}\right)$ is a scalar and commutes with both matrices. Now we note that the explicit form of the gamma matrices yields the same results in any frame of reference. Subsequently, the γ^{μ} matrices must remain invariant under Lorentz transformations. If we require that

$$\Lambda^{\nu}_{\mu}\gamma^{\mu} = f_{\Lambda}^{-1}\gamma^{\nu} f_{\Lambda} \tag{3.236}$$

which dictates a constraint on the linear transformation f_{Λ}, we then obtain for the term in parentheses in (3.235)

$$f_{\Lambda}\gamma^{\mu}\Lambda^{\nu}_{\mu} f_{\Lambda}^{-1} = f_{\Lambda}\Lambda^{\nu}_{\mu}\gamma^{\mu} f_{\Lambda}^{-1} = f_{\Lambda}\left(f_{\Lambda}^{-1}\gamma^{\nu} f_{\Lambda}\right) f_{\Lambda}^{-1} = \gamma^{\nu} \tag{3.237}$$

Hence, for the Dirac equation in IS′ ,

$$\left[-\mathrm{i}\gamma^{\nu}\partial'_{\nu} + \frac{m_e c}{\hbar}\right]\Psi' = 0 \tag{3.238}$$

Hence, we have shown that the Dirac equation for a freely moving electron satisfies the principle of relativity and is Lorentz covariant.

3.3.3 Particle at Rest

Naturally, after deriving the true relativistic quantum mechanical equation we seek to understand its solutions. It is pertinent to note that the solution of the field-free Dirac equation may be determined in two ways: (i) we may consider the solution of the electron at rest after which we may Lorentz transform said solution in accordance to (3.229) to a frame of reference which moves with constant velocity $(-v)$ with respect to the reference frame that observes the electron as standing idle or (ii) we may consider a more direct approach from the (full field free) Dirac equation (3.197) for the electron advancing at a constant velocity v.

Due to both approaches being conceptually noteworthy, we proceed by considering the solutions for a particle at rest first, in order to analyze the way in which the Dirac reduces under such a reference frame. Note, that in a reference frame in which the electron is observed at rest, there is no contribution from the kinetic energy operators to the energy expectation value and the spatial derivatives may be disregarded. Consequently, the only surviving term in the Hamiltonian pertains to the rest energy term proportional to $m_e c^2$. Thus, the Dirac equation (3.197) yields

$$i\hbar\frac{\partial}{\partial t}\Psi = \left[\beta m_e c^2\right]\Psi \tag{3.239}$$

Its four solutions are

$$\Psi_1^{(+)} = \exp\left(-i\frac{m_e c^2}{\hbar}t\right)\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad \Psi_2^{(+)} = \exp\left(-i\frac{m_e c^2}{\hbar}t\right)\begin{pmatrix}0\\1\\0\\0\end{pmatrix} \tag{3.240}$$

$$\Psi_1^{(-)} = \exp\left(i\frac{m_e c^2}{\hbar}t\right)\begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \quad \Psi_2^{(-)} = \exp\left(i\frac{m_e c^2}{\hbar}t\right)\begin{pmatrix}0\\0\\0\\1\end{pmatrix} \tag{3.241}$$

where we recall that Ψ is a 4-spinor. We may verify such a claim by explicitly writing (3.239), such that we attain the following four equations

$$\begin{aligned}
i\hbar\frac{\partial}{\partial t}\Psi &= m_e c^2\Psi \\
i\hbar\frac{\partial}{\partial t}\Psi &= m_e c^2\Psi \\
i\hbar\frac{\partial}{\partial t}\Psi &= -m_e c^2\Psi \\
i\hbar\frac{\partial}{\partial t}\Psi &= -m_e c^2\Psi
\end{aligned}$$

It suffices to solve one of the above first order differential equations, as the rest follow analogously.

Thus, consider,

$$
\begin{aligned}
\mathrm{i}\hbar\frac{\partial}{\partial t}\Psi &= m_e c^2\Psi \\
\mathrm{i}\hbar\frac{\partial}{\partial t}\Psi - m_e c^2\Psi &= 0 \\
\implies \mathrm{i}\hbar\mu - m_e c^2 &= 0 \\
\implies \mu &= \frac{m_e c^2}{\mathrm{i}\hbar}
\end{aligned}
$$

such that we have the following solution

$$
\Psi = \exp\left(\frac{m_e c^2}{\mathrm{i}\hbar}t\right) \equiv \exp\left(-\frac{\mathrm{i}m_e c^2}{\hbar}t\right)
$$

where we have used the characterstic polynomial to find the solution of the differential equation. The β matrix entries control the sign on the right-hand side of (3.239) in the case of a negative coefficient in the exponential. The solutions in (3.240) and (3.241) are normalized to one.

If we now consider the eigenvalue equation of the Hamiltonian, we may write the energy expectation value for an electron at rest using the eigensolutions of (3.240),

$$
\begin{aligned}
E^{(+)} &= \left\langle \Psi_{1,2}^{(+)} \left| H^D \right| \Psi_{1,2}^{(+)} \right\rangle \\
&= \exp\left(\mathrm{i}\frac{m_e c^2}{\hbar}t\right)\left[m_e c^2\right]\exp\left(-\mathrm{i}\frac{m_e c^2}{\hbar}t\right) = m_e c^2
\end{aligned} \tag{3.242}
$$

and similarly using the solutions of (3.241) we have,

$$
\begin{aligned}
E^{(-)} &= \left\langle \Psi_{1,2}^{(-)} \left| H^D \right| \Psi_{1,2}^{(-)} \right\rangle \\
&= \exp\left(-\mathrm{i}\frac{m_e c^2}{\hbar}t\right)\left[-m_e c^2\right]\exp\left(\mathrm{i}\frac{m_e c^2}{\hbar}t\right) = -m_e c^2
\end{aligned} \tag{3.243}
$$

We note that the spatial integration has been neglected since it simply reduces to unity by suitable normalization of the plane wave spinors.

It is simple to fathom the thought that the β matrix divides the four solutions of (3.240) and (3.241) into two categories. On one hand we have a class of two eigenvectors which adequately describe a spin $1/2$-particle with positive rest energy $m_e c^2$ as one would expect. Yet, the second set of solutions in (3.241) present a peculiar enigma, as they describe a spin $1/2$-particle with negative energy $-m_e c^2$. While the first set of upper solutions are physically germane, the negative energy solutions in (3.241) present a conundrum, and require further analysis. Nonetheless, recall that we have previously encountered negative energies when handling the Klein-Gordon equation and the Dirac equation was forged in such a manner as to mirror the results of the Klein-Gordon equation for freely moving particles, and consequently for particles at a static position.

3.3.4 Freely Moving Particle

We proceed to derive the Dirac states for a freely moving particle of mass m_e. Here we note that the charge of the particle (fermion) does not factor into the Dirac equation for this particle being at rest or moving at a constant velocity v. The solutions derived in (3.240) and (3.241) may be subjected to a general Lorentz boost into an inertial reference frame moving relatively to the previous one with velocity $(-v)$. Nonetheless, we venture a direct solution of the Dirac equation (3.197) as it is unambiguous and clear-cut. For this approach we select the following ansatz of plane waves,

$$\Psi(x) = u(p) \exp\left[-\mathrm{i}\frac{p \cdot x}{\hbar}\right] \tag{3.244}$$

with the scalar product for the linear momentum and position 4-vectors given by

$$p \cdot x = Et - \boldsymbol{p} \cdot \boldsymbol{r} \tag{3.245}$$

Here we have that the $\boldsymbol{p}$ in this particular situation denotes the momentum eigenvalue and not the momentum operator. Due to the manner in which the standard representation of the 4×4 Dirac matrices $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3) = (\alpha_x, \alpha_y, \alpha_z)$ was determined, allows us to rewrite (3.197) for a freely moving fermion in the following form

$$\left[c\boldsymbol{\alpha}\cdot\hat{p}+\beta m_e c^2\right]\Psi = \mathrm{i}\hbar\frac{\partial}{\partial t}\Psi \tag{3.246}$$

Indeed, from (3.197) we have

$$\begin{aligned}
\mathrm{i}\hbar\frac{\partial}{\partial t}\Psi &= \left[\frac{\hbar c}{\mathrm{i}}\alpha^k\partial_k+\beta m_e c^2\right]\Psi \equiv H^D\Psi \\
\Longrightarrow\ \mathrm{i}\hbar\frac{\partial}{\partial t}\Psi &= \left[\left(\frac{\mathrm{i}}{\mathrm{i}}\right)\frac{\hbar c}{\mathrm{i}}\alpha^k\partial_k+\beta m_e c^2\right]\Psi \equiv H^D\Psi \\
\Longrightarrow\ \mathrm{i}\hbar\frac{\partial}{\partial t}\Psi &= \left[-\mathrm{i}\hbar c\alpha^k\partial_k+\beta m_e c^2\right]\Psi \equiv H^D\Psi \\
\Longrightarrow\ \mathrm{i}\hbar\frac{\partial}{\partial t}\Psi &= \left[c\boldsymbol{\alpha}\cdot\hat{p}+\beta m_e c^2\right]\Psi
\end{aligned}$$

where we made the substitution delineated in (3.176), namely, $\hat{p}\longrightarrow -\mathrm{i}\hbar\nabla$. Recall that we had previously denoted the eigenstate Ψ as a 4-spinor. Now, due to the block structure of the α^i in (3.210), the spinor is often divided into an upper and a lower 2-spinor, Ψ^L and Ψ^S

$$\Psi = \begin{pmatrix}\Psi_1\\ \Psi_2\\ \Psi_3\\ \Psi_4\end{pmatrix} = \begin{pmatrix}\Psi^L\\ \Psi^S\end{pmatrix} \tag{3.247}$$

(in which L denotes "large" and S "small"). Then, the Dirac equation may be drafted in split notation as follows

$$c\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}}\Psi^S+m_e c^2\Psi^L = \mathrm{i}\hbar\frac{\partial}{\partial t}\Psi^L \tag{3.248}$$

$$c\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}}\Psi^L-m_e c^2\Psi^S = \mathrm{i}\hbar\frac{\partial}{\partial t}\Psi^S \tag{3.249}$$

We prove the above claim as follows, starting with (3.246), we write its explicit form as well as that of the Dirac matrices, such that

$$
\begin{aligned}
&\left[c\left[\alpha_x p_x+\alpha_y p_y+\alpha_z p_z\right]+\beta m_e c^2\right]\Psi = \mathrm{i}\hbar\frac{\partial}{\partial t}\Psi \\
\Longrightarrow &\left[c\left[\begin{pmatrix} 0 & \sigma_x p_x \\ \sigma_x p_x & 0\end{pmatrix}+\begin{pmatrix} 0 & \sigma_y p_y \\ \sigma_y p_y & 0\end{pmatrix}+\begin{pmatrix} 0 & \sigma_z p_z \\ \sigma_z p_z & 0\end{pmatrix}\right]+\begin{pmatrix} I_2 & 0 \\ 0 & -I_2\end{pmatrix} m_e c^2\right]\Psi = \mathrm{i}\hbar\frac{\partial}{\partial t}\Psi \\
\Longrightarrow &\left[c\begin{pmatrix} 0 & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & 0\end{pmatrix}+\begin{pmatrix} I_2 & 0 \\ 0 & -I_2\end{pmatrix} m_e c^2\right]\begin{pmatrix}\Psi^L \\ \Psi^S\end{pmatrix} = \mathrm{i}\hbar\frac{\partial}{\partial t}\begin{pmatrix}\Psi^L \\ \Psi^S\end{pmatrix} \\
\Longrightarrow &\begin{pmatrix} I_2 m_e c^2 & c\vec{\sigma}\cdot\vec{p} \\ c\vec{\sigma}\cdot\vec{p} & -I_2 m_e c^2\end{pmatrix}\begin{pmatrix}\Psi^L \\ \Psi^S\end{pmatrix} = \mathrm{i}\hbar\frac{\partial}{\partial t}\begin{pmatrix}\Psi^L \\ \Psi^S\end{pmatrix}
\end{aligned}
$$

such that we ma write the above matrix equation inn the same manner as that of (3.248) and (3.249). For the sake of completeness, we present the explicit form of the Dirac equation as four coupled first-order partial differential equation,

$$c\hat{p}_z\Psi_3+c\left(\hat{p}_x-\mathrm{i}\hat{p}_y\right)\Psi_4+m_e c^2\Psi_1=\mathrm{i}\hbar\frac{\partial}{\partial t}\Psi_1 \tag{3.250}$$

$$c\left(\hat{p}_x+\mathrm{i}\hat{p}_y\right)\Psi_3-c\hat{p}_z\Psi_4+m_e c^2\Psi_2=\mathrm{i}\hbar\frac{\partial}{\partial t}\Psi_2 \tag{3.251}$$

$$c\hat{p}_z\Psi_1+c\left(\hat{p}_x-\mathrm{i}\hat{p}_y\right)\Psi_2-m_e c^2\Psi_3=\mathrm{i}\hbar\frac{\partial}{\partial t}\Psi_3 \tag{3.252}$$

$$c\left(\hat{p}_x+\mathrm{i}\hat{p}_y\right)\Psi_1-c\hat{p}_z\Psi_2-m_e c^2\Psi_4=\mathrm{i}\hbar\frac{\partial}{\partial t}\Psi_4 \tag{3.253}$$

For the explicit solutions of the above equations, we must ascertain $u(p)$ in (3.244). Nonetheless, it may be pertinent to derive the explicit form of (3.250) - (3.253). Now, starting with (3.248), we write

$$c\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}}\Psi^S + m_ec^2\Psi^L = \mathrm{i}\hbar\frac{\partial}{\partial t}\Psi^L$$
$$\Longrightarrow c\left[\sigma_x p_x + \sigma_y p_y + \sigma_z p_z\right]\Psi^S + m_ec^2\Psi^L = \mathrm{i}\hbar\frac{\partial}{\partial t}\Psi^L$$
$$\Longrightarrow c\left[\sigma_x p_x\Psi^S + \sigma_y p_y\Psi^S + \sigma_z p_z\Psi^S\right] + m_ec^2\Psi^L = \mathrm{i}\hbar\frac{\partial}{\partial t}\Psi^L$$

Note that $\hat{\boldsymbol{p}} = \frac{\nabla\hbar}{i} \to p_x = \frac{\frac{\partial}{\partial x}\hbar}{i}, p_y = \frac{\frac{\partial}{\partial y}\hbar}{i}, p_z = \frac{\frac{\partial}{\partial z}\hbar}{i}$, then, using the Pauli matrices we obtain

$$\sigma_x p_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{\partial}{\partial x} \Longrightarrow \begin{pmatrix} 0 & p_x \\ p_x & 0 \end{pmatrix}\begin{pmatrix} \Psi_3 \\ \Psi_4 \end{pmatrix} = \begin{pmatrix} p_x\Psi_4 \\ p_x\Psi_3 \end{pmatrix}$$

then similarly for the remaining terms in the above equation we have,

$$\begin{pmatrix} 0 & -\mathrm{i}p_y \\ \mathrm{i}p_y & 0 \end{pmatrix}\begin{pmatrix} \Psi_3 \\ \Psi_4 \end{pmatrix} = \begin{pmatrix} -\mathrm{i}p_y\Psi_4 \\ \mathrm{i}p_y\Psi_3 \end{pmatrix} \quad \begin{pmatrix} p_z & 0 \\ 0 & -p_z \end{pmatrix}\begin{pmatrix} \Psi_3 \\ \Psi_4 \end{pmatrix} = \begin{pmatrix} p_z\Psi_3 \\ -p_z\Psi_4 \end{pmatrix}$$

hence, we may write the following

$$c\left[\begin{pmatrix} p_x\Psi_4 \\ p_x\Psi_3 \end{pmatrix} + \begin{pmatrix} -\mathrm{i}p_y\Psi_4 \\ \mathrm{i}p_y\Psi_3 \end{pmatrix} + \begin{pmatrix} p_z\Psi_3 \\ -p_z\Psi_4 \end{pmatrix}\right] + \begin{pmatrix} m_ec^2\Psi_1 \\ m_ec^2\Psi_2 \end{pmatrix} = \mathrm{i}\hbar\frac{\partial}{\partial t}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$$

thus, explicitly written we have the equations (3.250) and (3.251) with the remaining equations following analogously.

Upon examination of (3.240) and (3.241) we find that $u(p)$ is a 4-vector, such that $u(p=0)$ is the exact same to either of the four vectors in the above equations. Considering the case of nonvanishing momentum we attain the solution for $u(p)$ by substitution of the ansatz of (3.244) into (3.246), i.e.

$$\left[c\boldsymbol{\alpha}\cdot\hat{\boldsymbol{p}} + \beta m_ec^2\right]u(p)\exp\left[-\mathrm{i}\frac{p\cdot x}{\hbar}\right] = \mathrm{i}\hbar\frac{\partial}{\partial t}u(p)\exp\left[-\mathrm{i}\frac{p\cdot x}{\hbar}\right] \tag{3.254}$$

Then multiplying both sides of the above equation by a factor of $\exp[+\mathrm{i}(p\cdot x)/\hbar]$ from the left we attain

$$\begin{aligned} & c\exp\left[+\mathrm{i}\frac{p\cdot x}{\hbar}\right]\boldsymbol{\alpha}\cdot\hat{p}\exp\left[-\mathrm{i}\frac{p\cdot x}{\hbar}\right]u(p)+\beta m_e c^2 u(p) \\ & \qquad = \mathrm{i}\hbar\exp\left[+\mathrm{i}\frac{p\cdot x}{\hbar}\right]\frac{\partial}{\partial t}u(p)\exp\left[-\mathrm{i}\frac{p\cdot x}{\hbar}\right] \end{aligned} \tag{3.255}$$

where we used the fact that the β matrix commutes with the exponential factors, thus, yielding unity upon multiplication.

Using (3.245) we may write

$$c\exp\left[+\mathrm{i}\frac{p\cdot x}{\hbar}\right]\boldsymbol{\alpha}\cdot\hat{p}\exp\left[-\mathrm{i}\frac{p\cdot x}{\hbar}\right] = c\boldsymbol{\alpha}\cdot p \tag{3.256}$$

we have for every direction i

$$\hat{p}_i\exp\left[-\mathrm{i}\frac{p\cdot x}{\hbar}\right] = -\mathrm{i}\hbar\frac{\partial}{\partial x_i}\exp\left[-\mathrm{i}\frac{p\cdot x}{\hbar}\right] = p_i\exp\left[-\mathrm{i}\frac{p\cdot x}{\hbar}\right] \tag{3.257}$$

Then, we may write for the right-hand side of (3.255) the following,

$$\mathrm{i}\hbar\exp\left[+\mathrm{i}\frac{p\cdot x}{\hbar}\right]\frac{\partial}{\partial t}\exp\left[-\mathrm{i}\frac{p\cdot x}{\hbar}\right] = \mathrm{i}\hbar\left[-\frac{\mathrm{i}}{\hbar}E\right] = E \tag{3.258}$$

such that we obtain

$$\left[c\boldsymbol{\alpha}\cdot p+\beta m_e c^2\right]u(p) = Eu(p) \tag{3.259}$$

in which we have used the fact that the derivatives of $u(p)$ with respect to spatial coordinates are zero, $\hat{p}u(p) = -\mathrm{i}\hbar\nabla u(p) = 0$ We point attention to the fact that p in (3.259) delineates the eigenvalue of the momentum 3-vector of (3.246) - in order to make this difference concrete, the momentum operator in (3.246) and (3.256) is denoted by a hat. Now, we may write (3.259)in split notation as follows

$$\begin{pmatrix} m_ec^2 - E & c\sigma\cdot p \\ c\sigma\cdot p & -m_ec^2 - E \end{pmatrix} \begin{pmatrix} u^L(p) \\ u^S(p) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.260}$$

which gives the relation between the upper and lower components as

$$u^L = \left[\frac{c\sigma\cdot p}{E - m_ec^2}\right] u^S \tag{3.261}$$

$$u^S = \left[\frac{c\sigma\cdot p}{E + m_ec^2}\right] u^L \tag{3.262}$$

Indeed starting form (3.259) and following the proposed steps for the previous derivations of (3.250) - (3.253) we write

$$\begin{aligned}
&\left[c\left[\alpha_x p_x + \alpha_y p_y + \alpha_z p_z\right] + \beta m_ec^2\right] u(p) = Eu(p) \\
\Longrightarrow &\left[c\left[\begin{pmatrix} 0 & \sigma_x p_x \\ \sigma_x p_x & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma_y p_y \\ \sigma_y p_y & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma_z p_z \\ \sigma_z p_z & 0 \end{pmatrix}\right] + \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} m_ec^2\right] u(p) = Eu(p) \\
\Longrightarrow &\left[c\left[\begin{pmatrix} 0 & \sigma_x p_x \\ \sigma_x p_x & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma_y p_y \\ \sigma_y p_y & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma_z p_z \\ \sigma_z p_z & 0 \end{pmatrix}\right] + \begin{pmatrix} m_ec^2 I_2 & 0 \\ 0 & -m_ec^2 I_2 \end{pmatrix}\right] u(p) = Eu(p \\
\Longrightarrow &\begin{pmatrix} I_2 m_ec^2 & c\vec{\sigma}\cdot\vec{p} \\ c\vec{\sigma}\cdot\vec{p} & -I_2 m_ec^2 \end{pmatrix} u(p) = Eu(p) \\
\Longrightarrow &\begin{pmatrix} I_2 m_ec^2 & c\vec{\sigma}\cdot\vec{p} \\ c\vec{\sigma}\cdot\vec{p} & -I_2 m_ec^2 \end{pmatrix} u(p) - I_4 Eu(p) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\Longrightarrow &\begin{pmatrix} m_ec^2 I_2 - E & c\vec{p}\cdot\vec{\sigma} \\ \vec{p}\cdot\vec{\sigma} & -m_ec^2 I_2 - E \end{pmatrix} \begin{pmatrix} u^L(p) \\ u^S(p) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\end{aligned}$$

From which we obtain the following two equations

$$\left(m_e c^2 - E\right) u^L(p) + u^S(p) c\vec{\sigma}\cdot\vec{p} = 0$$
$$c\vec{\sigma}\cdot\vec{p} u^L(p) - \left(m_e c^2 - E\right) u^S(p) = 0$$

After which we may then solve the first equation for $u^L(p)$, such that

$$u^L(p) = -\frac{c\vec{\sigma}\cdot\vec{p}}{m_e c^2 - E}$$
$$\implies = \frac{c\vec{\sigma}\cdot\vec{p}}{E - m_e c^2}$$

with the second equation, i.e. $u^S(p)$, following analogously. Upon insertion of (3.261) into the lower (3.262), i.e.

$$u^S = \left[\frac{c\sigma\cdot p}{E + m_e c^2}\right]\left[\frac{c\sigma\cdot p}{E - m_e c^2}\right] u^S \tag{3.263}$$

we posit that

$$\left[\frac{c\sigma\cdot p}{E + m_e c^2}\right]\left[\frac{c\sigma\cdot p}{E - m_e c^2}\right] = 1 \tag{3.264}$$

in order to maintain consistency. The equation above, allows us to determine the energy E to be,

$$E = \pm\sqrt{c^2 p^2 + m_e^2 c^4} \tag{3.265}$$

which remains consistent with the comments pertaining the the free-particle Klein-Gordon equation. We select the ansätze

$$u_1^L = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad u_2^L = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.266}$$

and substitute the values into (3.262), such that we obtain the explicit form of u^S, where the scalar product of the vector of the momentum eigenvalues and the Pauli spin matrices, is given by

$$\boldsymbol{\sigma}\cdot\boldsymbol{p}=\begin{pmatrix} p_z & p_x-\mathrm{i}p_y \\ p_x+\mathrm{i}p_y & -p_z \end{pmatrix} \tag{3.267}$$

yielding the following 4-spinors $u(p)$ as

$$u_1^{(+)}(p)=\mathscr{N}\begin{pmatrix} 1 \\ 0 \\ \frac{cp_z}{E+m_ec^2} \\ \frac{c\left(p_x+\mathrm{i}p_y\right)}{E+m_ec^2} \end{pmatrix},\quad u_2^{(+)}(p)=\mathscr{N}\begin{pmatrix} 0 \\ 1 \\ \frac{c\left(p_x-\mathrm{i}p_y\right)}{E+m_ec^2} \\ \frac{-cp_z}{E+m_ec^2} \end{pmatrix} \tag{3.268}$$

Before continuing we proof the above claim posit by (3.268). Starting with (3.262) we have

$$u^S=\left[\frac{c\boldsymbol{\sigma}\cdot p}{E+m_ec^2}\right]u^L$$

$$\Longrightarrow=\left[\frac{c\begin{pmatrix} p_z & p_x-\mathrm{i}p_y \\ p_x+\mathrm{i}p_y & -p_z \end{pmatrix}}{E+m_ec^2}\right]u^L$$

$$\Longrightarrow=\begin{pmatrix} \frac{cp_z}{E+m_ec^2} & \frac{c\left(p_x-\mathrm{i}p_y\right)}{E+m_ec^2} \\ \frac{c\left(p_x+\mathrm{i}p_y\right)}{E+m_ec^2} & \frac{-cp_z}{E+m_ec^2} \end{pmatrix}u^L$$

recall that,

$$u(p)=\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}=\begin{pmatrix} u^L \\ u^S \end{pmatrix}$$

Then, choose $u_L^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ in order to attain,

$$u_1^{(+)}(p) = \mathscr{N} \begin{pmatrix} 1 \\ 0 \\ \frac{cp_z}{E+m_ec^2} \\ \frac{c(p_x+\mathrm{i}p_y)}{E+m_ec^2} \end{pmatrix}$$

similarly, choosing $u_L^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ we obtain,

$$u_2^{(+)}(p) = \mathscr{N} \begin{pmatrix} 0 \\ 1 \\ \frac{c(p_x-\mathrm{i}p_y)}{E+m_ec^2} \\ \frac{-cp_z}{E+m_ec^2} \end{pmatrix}$$

Now, choosing u^S to be

$$u_1^S = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad u_2^S = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.269}$$

in much the same way we may attain u^L from (3.261) so that we may write the resulting 4-spinor $u(p)$ as

$$u_1^{(-)}(p) = \mathscr{N} \begin{pmatrix} \frac{cp_z}{E-m_ec^2} \\ \frac{c(p_x+\mathrm{i}p_y)}{E-m_ec^2} \\ 1 \\ 0 \end{pmatrix}, \quad u_2^{(-)}(p) = \mathscr{N} \begin{pmatrix} \frac{c(p_x-\mathrm{i}p_y)}{E-m_ec^2} \\ \frac{-cp_z}{E-m_ec^2} \\ 0 \\ 1 \end{pmatrix} \tag{3.270}$$

which correspond to the negative energy eigenvalues $E = -|E|$. The normalization constant $\mathscr{N}$ is selected as to satisfy the following properties ,

$$u_i^{(\pm),\dagger}(p)\cdot u_j^{(\pm)}(p)=\delta_{ij} \tag{3.271}$$

$$u_i^{(\pm),\dagger}(p)\cdot u_j^{(\mp)}(p)=0 \tag{3.272}$$

$\forall i,j\in\{1,2\}$. For $i=j$, (3.271) yields for any of the free-particle spinors denoted above,

$$u_i^{(\pm),\dagger}\cdot u_i^{(\pm)}=\mathscr{N}^2\left[1+\frac{c^2p^2}{\left(E\pm m_ec^2\right)^2}\right]=1 \tag{3.273}$$

which results for the squared normalization constant

$$\begin{aligned}\mathscr{N}^2&=\frac{1}{1+\frac{c^2p^2}{\left(E\pm m_ec^2\right)^2}}=\frac{\left(E\pm m_ec^2\right)^2}{\left(E\pm m_ec^2\right)^2+c^2p^2}\\&=\frac{\left(E\pm m_ec^2\right)^2}{\left(E\pm m_ec^2\right)^2+E^2-m_e^2c^4}=\frac{\left(E\pm m_ec^2\right)^2}{\left(E\pm m_ec^2\right)^2+\left(E+m_ec^2\right)\left(E-m_ec^2\right)}\\&=\frac{\left(E\pm m_ec^2\right)}{\left(E\pm m_ec^2\right)+\left(E\mp m_ec^2\right)}=\frac{E\pm m_ec^2}{2E}\end{aligned} \tag{3.274}$$

note that the counter and the denominator carry either a global $(+)$ or a global $(-)$ sign for the $u_i^{(+)}$ and $u_i^{(-)}$ solutions, respectively, so that the normalization constant is equivalent for all four spinors $u_i^{(\pm)}(p)$,

$$\mathscr{N}=\sqrt{\frac{|E|+m_ec^2}{2|E|}} \tag{3.275}$$

Each of the four solutions correspond to two different kinds of eigenvalues. The $u_1^{(+)}$ and $u_2^{(+)}$ times the exponential factor yield positive eigenvalues, the solutions $u_1^{(-)}$ and $u_2^{(-)}$ times the exponential possess negative energy eigenvalues. Such a claim may be verified b y direct substitution that these four spinors satisfy the free-particle Dirac equation and are orthogonal to one another.

Due to the positive- and negative-energy solutions for all possible p forming a complete orthonormal basis, we may write the most general free-particle wavve function as a superposition of the basis spinors,

$$\Psi(x) = \Psi(r,t) = \mathcal{N} \left(\sum_p \sum_{\nu=1,2} c_{p,\nu}^{(+)} u_\nu^{(+)}(p) \exp\left[\mathrm{i} \frac{p \cdot r - |E| t}{\hbar} \right] - \sum_p \sum_{\nu=1,2} c_{p,\nu}^{(-)} u_\nu^{(-)}(p) \exp\left[\mathrm{i} \frac{p \cdot r + |E| t}{\hbar} \right] \right) \tag{3.276}$$

with expansion coefficients $c_{p,\nu}^{(\pm)}$ of the wave packet established from a Fourier expansion of Ψ at $t = 0$.

3.3.5 Dirac Equation in External Electromagnetic Potentials

The covariant form of the Dirac equation, (3.227), allows us to incorporate arbitrary external electromagnetic fields. These fields allow us to describe the interaction of electrons with light, albeit classically. The only guiding principle for the derivation of the field-dependent Dirac equation for an electron interacting with an external electromagnetic field is that the resulting Dirac equation is still being Lorentz covariant. Nonetheless, this is easily achieved if we use the covariant form of the Dirac equation as a starting point. This form allows is to introduce various additional fields and fix possible phases solely on the principle of symmetry, i.e. that the form of the equation has to be conserved under Lorentz transformations. We briefly note that the minimal coupling procedure preserves Lorentz covariance, thus we analyze this procedure for the Dirac equation.

For the consideration of the Dirac equation in covariant form with external electromagnetic fields, it may be pertinent to define suitable 4-quantities, in such a way, the space and time coordinates are re-unified for the relativistic space-time framework. The momentum reads

$$p^\mu = \mathrm{i}\hbar \frac{\partial}{\partial x_\mu} = \mathrm{i}\hbar \partial^\mu \tag{3.277}$$

with components

$$p^0 = p_0 = \mathrm{i}\hbar \frac{\partial}{\partial(ct)} \quad \text{and} \quad p^i = -p_i = \mathrm{i}\hbar \frac{\partial}{\partial x_i} \quad \text{with } i \in \{1,2,3\} \tag{3.278}$$

The components of the 4-potential are determined by $A^{\mu} = (\phi, A)$, where ϕ denotes the scalar potential of the EM field and the vector potential $A = \left(A^1, A^2, A^3\right)$ contains the contravariant components of the 4-potential. We must add a Lorentz scalar to the Dirac Hamiltonian in order to preserve Lorentz covariance. We note that the Lorentz scalar shall depend on the 4-potential, hence, the simplest choice is a linear dependence on the 4-potential and by multiplication with γ^{μ}, we obtain the desired Lorentz scalar. The minimal coupling which preserves the Lorentz invariance is attained by the following substitution for the 4-momentum operator

$$\mathrm{i}\hbar\partial_{\mu} \longrightarrow \mathrm{i}\hbar\partial_{\mu} - \frac{q_e}{c}A_{\mu} \tag{3.279}$$

explicitly written we have

$$\mathrm{i}\hbar\frac{\partial}{\partial t} \quad \longrightarrow \quad \mathrm{i}\hbar\frac{\partial}{\partial t} - q_e\phi \tag{3.280}$$

$$\mathrm{i}\hbar\frac{\partial}{\partial x^i} \quad \longrightarrow \quad \mathrm{i}\hbar\frac{\partial}{\partial x^i} - \frac{q_e}{c}A_i \quad \text{with } i \in \{1,2,3\} \tag{3.281}$$

Insertion of the attained minimal coupling substitution into (3.279) in the field-free Dirac equation as displayed in (3.227) yields the covariant form of the Dirac equation with external electromagnetic fields,

$$\left[-\gamma^{\mu}\left(\mathrm{i}\hbar\partial_{\mu} - \frac{q_e}{c}A_{\mu}\right) + m_e c\right]\Psi = 0 \tag{3.282}$$

Separation of (3.282) into its spatial and time coordinates we may write

$$\gamma^0\left(\mathrm{i}\hbar\partial_0 - \frac{q_e}{c}A_0\right)\Psi = \left[-\gamma^i\left(\mathrm{i}\hbar\partial_i - \frac{q_e}{c}A_i\right) + m_e c\right]\Psi \tag{3.283}$$

where all which was done was that we moved the time-like term of the Dirac equation with external electromagnetic fields to the right-hand side of our equation. Multiplying the attained equation by γ^0, we attain,

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\Psi = \left[c\boldsymbol{\alpha}\cdot\left(p - \frac{q_e}{c}A\right) + \beta m_e c^2 + q_e\phi\right]\Psi \tag{3.284}$$

were we note that the vector potential $A = \left(A^1, A^2, A^3\right)$ contains its contravariant components and we note that $A^i = -A_i$, such that the sign in front of the q_e/cA term has to be changed.

Indeed, we prove the claim above. Starting with (3.283), we attain

$$\begin{aligned}
(\gamma^0)\,\gamma^0\left(\mathrm{i}\hbar\partial_0 - \frac{q_e}{c}A_0\right)\Psi &= (\gamma^0)\left[-\gamma^i\left(\mathrm{i}\hbar\partial_i - \frac{q_e}{c}A_i\right) + m_e c\right]\Psi \\
\implies \left(\mathrm{i}\hbar\partial_0 - \frac{q_e}{c}A_0\right)\Psi &= (\gamma^0)\left[-\gamma^i\left(\mathrm{i}\hbar\partial_i - \frac{q_e}{c}A_i\right) + m_e c\right]\Psi \\
\implies \mathrm{i}\hbar\frac{1}{c}\partial_0\Psi &= (\gamma^0)\left[-\gamma^i\left(\mathrm{i}\hbar\partial_i - \frac{q_e}{c}A_i\right) + m_e c\right]\Psi + \frac{q_e\phi}{c}\Psi \\
\implies \mathrm{i}\hbar\frac{1}{c}\partial_0\Psi &= \left[-\gamma^0\gamma^i\left(\mathrm{i}\hbar\partial_i - \frac{q_e}{c}A_i\right) + \gamma^0 m_e c\right]\Psi + \frac{q_e\phi}{c}\Psi
\end{aligned}$$

Note that $\gamma^0 = \beta$ and $\gamma^i = \beta\alpha^i$. Further, $\left(\gamma^0\right)^\dagger = \gamma^0$, i.e. is its own inverse. Also, $\gamma^{0,\dagger} \equiv \beta^\dagger = \beta = \gamma^0$. Thus,

$$\begin{aligned}
\implies \mathrm{i}\hbar\frac{1}{c}\partial_0\Psi &= \left[-\beta^\dagger\beta\alpha^i\left(\mathrm{i}\hbar\partial_i - \frac{q_e}{c}A_i\right) + \beta m_e c\right]\Psi + \frac{q_e\phi}{c}\Psi \\
\implies \mathrm{i}\hbar\frac{1}{c}\partial_0\Psi &= \left[-\mathrm{i}\hbar\alpha^i\partial_i - \frac{q_e}{c}A_i + \beta m_e c\right]\Psi + \frac{q_e\phi}{c}\Psi
\end{aligned}$$

Now, replacing $p \to \frac{\hbar\nabla}{\mathrm{i}} \implies \nabla = \frac{p\mathrm{i}}{\hbar}$, we have

$$\begin{aligned}
\implies \mathrm{i}\hbar\frac{1}{c}\partial_0\Psi &= \left[-\mathrm{i}\hbar\alpha^i\left(\frac{p\mathrm{i}}{\hbar}\right) - \frac{q_e}{c}A_i + \beta m_e c\right]\Psi + \frac{q_e\phi}{c}\Psi \\
\implies \mathrm{i}\hbar\frac{1}{c}\partial_0\Psi &= \left[\alpha\cdot p - \frac{q_e}{c}\boldsymbol{A} + \beta m_e c\right]\Psi + \frac{q_e\phi}{c}\Psi
\end{aligned}$$

Multiplying across by c we obtain.

$$\begin{aligned}
\implies \mathrm{i}\hbar\partial_0\Psi &= \left[c\alpha\cdot p - q_e\boldsymbol{A} + \beta m_e c^2\right]\Psi + q_e\phi\Psi \\
\implies \mathrm{i}\hbar\frac{\partial}{\partial t}\Psi &= \left[c\boldsymbol{\alpha}\cdot\left(p - \frac{q_e}{c}A\right) + \beta m_e c^2 + q_e\phi\right]\Psi
\end{aligned}$$

Hence, it is understood that the structure of the product of gamma matrices, i.e. $\gamma^0\gamma^i = \alpha^i$ determines where the components of the vector potential A^i enter the Dirac Hamiltonian, thus reducing the Dirac equation form γ matrices to α matrices. Yet, the *scalar* potential $A_0 = \phi$ always enters the Dirac Hamiltonian on the diagonal, since $\gamma^0\gamma^0 = 1_4$ by virtue of (3.224) in any representation.

Aside from Lorentz covariance, the invariance under gauge transformation of the 4-potential must be guaranteed in order for the physical observables to remain invariant under said transformations, hence, we require the following substitution

$$A_\mu(x) \quad \longrightarrow \quad A_\mu(x) + \partial_\mu \chi(x) \tag{3.285}$$

After gauge transformation the covariant Dirac (3.282) reads

$$\left[-\gamma^\mu \left\{ \mathrm{i}\hbar\partial_\mu - \frac{q_e}{c}A_\mu - \frac{q_e}{c}\left(\partial_\mu \chi\right)\right\} + m_e c\right]\Psi' = 0 \tag{3.286}$$

Since $|\Psi|^2 = |\Psi'|^2$ must be fulfilled, Ψ and Ψ' can only differ by a phase factor,

$$\Psi'(x) = \exp[\mathrm{i}\omega(x)]\Psi(x) \tag{3.287}$$

With the space-time function $\omega(x)$ to be determined from the condition that if Ψ is a solution of (3.282), then Ψ' must be a solution of (3.286). If we insert the ansatz in (3.287) into (3.286), we find the following terms must be identical

$$\mathrm{i}\hbar\partial_\mu \Psi' - \frac{q_e}{c}\left(\partial_\mu \chi\right)\Psi' = \exp(\mathrm{i}\omega)\mathrm{i}\hbar\left(\partial_\mu \Psi\right) \tag{3.288}$$

if we shall obtain the original (3.282). Thence, after applying the product and chain rules, respectively, for the evaluation of $\partial_\mu \Psi'$, we may eliminate the exponential and the Ψ depending terms in order to attain,

$$-\hbar\left(\partial_\mu \omega\right) = \frac{q_e}{c}\left(\partial_\mu \chi\right) \quad \Longrightarrow \quad \partial_\mu \omega = \partial_\mu\left(-\frac{q_e}{\hbar c}\chi\right) \tag{3.289}$$

so that $\omega = -(q_e/\hbar c)\chi$ is uniquely given up to a constant. The wave function after gauge transformation thence reads

$$\Psi(x) \quad \longrightarrow \quad \Psi'(x) = \exp\left[-\mathrm{i}\frac{q_e}{\hbar c}\chi(x)\right]\Psi(x) \tag{3.290}$$

Thus, it follows that minimal coupling also preserves gauge invariance of the Dirac equation.

3.3.6 Kinematic Momentum, Non-relativistic Limit and Pauli Equation

The connection of the canonical momentum operator p with the effect of external vector potentials $\boldsymbol{A}$ on a moving electron with charge $q_e = -e$ is written as

$$p \quad \longrightarrow \quad p - \frac{q_e}{c}A = p + \frac{e}{c}A \equiv \pi \tag{3.291}$$

where π is the mechanical-momentum operator, such that we may rewrite the Dirac electron in external electromagnetic fields as follows

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\Psi = \left[c\boldsymbol{\alpha}\cdot\boldsymbol{\pi} + \beta m_e c^2 - e\phi\right]\Psi \tag{3.292}$$

We proceed to deriving the Pauli two-component equation from the Dirac equation in external EM fields. It may be practical to recover the Schrödinger in order to decipher the connection between the relativistic theory and its nonrelativistic counter part. Hence, we rewrite (3.292) in split notation

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\begin{pmatrix}\Psi^L\\ \Psi^S\end{pmatrix} = c\begin{pmatrix}(\sigma\cdot\pi)\Psi^S\\ (\sigma\cdot\pi)\Psi^L\end{pmatrix} + m_e c^2\begin{pmatrix}\Psi^L\\ -\Psi^S\end{pmatrix} + V\begin{pmatrix}\Psi^L\\ \Psi^S\end{pmatrix} \tag{3.293}$$

with $V = -e\phi$ defining the electrostatic energy operator. Due to the lowest possible nonrelativistic energy permissible for a free particle by Schrödinger mechanics being zero instead of the $+m_e c^2$ (rest energy) of Dirac's theory, we require a shift of the origin of the energy scale by $-m_e c^2$, in order to conform to the tenets of nonrelativistic quantum mechanics.

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\Psi^L\\ \Psi^S\end{pmatrix} = c\begin{pmatrix}(\sigma\cdot\pi)\Psi^S\\ (\sigma\cdot\pi)\Psi^L\end{pmatrix} - 2m_e c^2\begin{pmatrix}0\\ \Psi^S\end{pmatrix} + V\begin{pmatrix}\Psi^L\\ \Psi^S\end{pmatrix} \tag{3.294}$$

Since,

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\Psi^L\\ \Psi^S\end{pmatrix} = c\begin{pmatrix}(\sigma\cdot\pi)\Psi^S\\ (\sigma\cdot\pi)\Psi^L\end{pmatrix} + m_e c^2\begin{pmatrix}\Psi^L\\ -\Psi^S\end{pmatrix} + \left(V - m_e c^2\right)\begin{pmatrix}\Psi^L\\ \Psi^S\end{pmatrix}$$

$$\Longrightarrow i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\Psi^L\\ \Psi^S\end{pmatrix} = c\begin{pmatrix}(\sigma\cdot\pi)\Psi^S\\ (\sigma\cdot\pi)\Psi^L\end{pmatrix} + m_e c^2\begin{pmatrix}\Psi^L\\ -\Psi^S\end{pmatrix} + -m_e c^2\begin{pmatrix}\Psi^L\\ \Psi^S\end{pmatrix} + V\begin{pmatrix}\Psi^L\\ \Psi^S\end{pmatrix}$$

$$\Longrightarrow i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\Psi^L\\ \Psi^S\end{pmatrix} = c\begin{pmatrix}(\sigma\cdot\pi)\Psi^S\\ (\sigma\cdot\boldsymbol{\pi})\Psi^L\end{pmatrix} - 2m_e c^2\begin{pmatrix}0\\ \Psi^S\end{pmatrix} + V\begin{pmatrix}\Psi^L\\ \Psi^S\end{pmatrix}$$

Next, we shift our attention to the small component and write only the inferior part of (3.294) as

$$\left(i\hbar\frac{\partial}{\partial t} + 2m_e c^2 - V\right)\Psi^S = c(\sigma\cdot\pi)\Psi^L \tag{3.295}$$

where we shifted all Ψ^S dependent terms to the left of the equality.

Using the Klein-Gordon equation as a reference, we have for the non-relativistic energies, the energy $i\hbar\partial/\partial t \to E$ and the potential V as compared to the rest energy, so that we may approximate the small component of the 4-spinor as

$$\Psi^S \approx \frac{\sigma\cdot\pi}{2m_e c}\Psi^L \tag{3.296}$$

Indeed‘,

$$\left(\mathrm{i}\hbar\frac{\partial}{\partial t} + 2m_ec^2 - V\right)\Psi^S = c(\sigma\cdot\pi)\Psi^L$$
$$\Longrightarrow \Psi^S = \frac{c(\sigma\cdot\pi)\Psi^L}{\left(\mathrm{i}\hbar\frac{\partial}{\partial t} + 2m_ec^2 - V\right)}$$
$$\Longrightarrow \Psi^S \approx \frac{\sigma\cdot\pi}{2m_ec}\Psi^L$$

as per the above remarks made regarding the energy and potential of the system. This relation among the components of the 4-spinor guarantees that any state below $-2m_ec^2$ is omitted (otherwise $\mathrm{i}\hbar\partial/\partial t \to E$ would not be small compared to the rest energy). This shows that the lower component of the Ψ^S spinor is smaller than the Ψ^L spinor by a factor of $1/c$, leading to the reason behind why Ψ^L is known as the *large component* and Ψ^S the *small component*. If we consider a limit $c \to \infty$, the small component tends to zero, $\Psi^S(r,t) = 0$. We arrive at a final solution which depends on operators describing the interaction of an electron with external magnetic fields with the Schrödinger KE operator.

Substitution of the kinetic-balance condition (3.296) into the remaining upper component of (3.294), we have

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\Psi^L = \frac{(\sigma\cdot\pi)(\sigma\cdot\pi)}{2m_e}\Psi^L + V\Psi^L \tag{3.297}$$

where using Dirac’s relation for two arbitrary vector operators $\boldsymbol{A}$ and $\boldsymbol{B}$ yields

$$(\sigma\cdot\boldsymbol{A})(\sigma\cdot\boldsymbol{B}) = \boldsymbol{A}\cdot\boldsymbol{B}\mathbf{1}_2 + \mathrm{i}\sigma\cdot(\boldsymbol{A}\times\boldsymbol{B})$$

such that applying said relation to (3.297) we attain

$$(\sigma\cdot\pi)(\sigma\cdot\pi) = \pi^2 - \frac{q_e\hbar}{c}\sigma\cdot B \tag{3.298}$$

which results in

$$\frac{1}{2m_e}\left(p-\frac{q_e}{c}A\right)^2\Psi^L-\frac{q_e\hbar}{2m_ec}(\sigma\cdot B)\Psi^L+V\Psi^L=\mathrm{i}\hbar\frac{\partial}{\partial t}\Psi^L \tag{3.299}$$

which is the Pauli equation known form nonrelativistic quantum mechanics. Observe,

$$\begin{aligned}\left(p-\frac{q_e}{c}A\right)^2&=p^2-\frac{q_e}{c}(p\cdot A+A\cdot p)+\frac{q_e^2}{c^2}A^2\\&=p^2-2\frac{q_e}{c}p\cdot A+\frac{q_e^2}{c^2}A^2\end{aligned} \tag{3.300}$$

in which we used the fact that $A\cdot p=p\cdot A-(p\cdot A)$ and the fact that the Coulomb gauge makes the divergence of the vector potential vanish, i.e. $(p\cdot A)=-\mathrm{i}\hbar(\nabla\cdot A)=-\mathrm{i}\hbar(\mathrm{div}A)=0$. We note in passing that the linear term in the momentum operator, $(2q_e/c)\,p\cdot A$, is *imaginary*. The mixed term, i.e. since p represents the momentum and A the vector potential, hence, defining the magnetic field, may be expressed by the angular momentum coupled to a constant and a homogeneous magnetic field, $B=\mathrm{curl}A$,

$$2p\cdot A=p\cdot(B\times r)=(r\times p)\cdot B=l\cdot B \tag{3.301}$$

where we now redefine the Pauli equation using the spin operato, (3.135), as

$$\left[\frac{p^2}{2m_e}+\frac{q_e^2A^2}{2m_ec^2}-\underbrace{\frac{q_e}{2m_ec}(l+2s)}_{\equiv\mu}\cdot B+V\right]\Psi^L=\mathrm{i}\hbar\frac{\partial}{\partial t}\Psi^L \tag{3.302}$$

since, we have

$$
\begin{aligned}
&\frac{1}{2m_e}\left(p-\frac{q_e}{c}A\right)^2\Psi^L-\frac{q_e\hbar}{2m_ec}(\sigma\cdot B)\Psi^L+V\Psi^L=\mathrm{i}\hbar\frac{\partial}{\partial t}\Psi^L\\
\implies &\frac{1}{2m_e}\left[p^2-2\frac{q_e}{c}p\cdot A+\frac{q_e^2}{c^2}A^2\right]\Psi^L-\frac{q_e\hbar}{2m_ec}(\sigma\cdot B)\Psi^L+V\Psi^L=\mathrm{i}\hbar\frac{\partial}{\partial t}\Psi^L\\
\implies &\frac{1}{2m_e}\left[p^2-\frac{q_e}{c}(l\cdot B)+\frac{q_e^2}{c^2}A^2\right]\Psi^L-\frac{q_e\hbar}{2m_ec}(\sigma\cdot B)\Psi^L+V\Psi^L=\mathrm{i}\hbar\frac{\partial}{\partial t}\Psi^L\\
\implies &\left[\frac{p^2}{2m_e}+\frac{q_e^2A^2}{2m_ec^2}-\frac{q_e}{2m_ec}(l\cdot B)-\frac{q_e}{2m_ec}(\sigma\cdot B)+V\right]\Psi^L=\mathrm{i}\hbar\frac{\partial}{\partial t}\Psi^L\\
\implies &\left[\frac{p^2}{2m_e}+\frac{q_e^2A^2}{2m_ec^2}-\frac{q_e}{2m_ec}[(l\cdot B)+(\sigma\cdot B)]+V\right]\Psi^L=\mathrm{i}\hbar\frac{\partial}{\partial t}\Psi^L\\
\implies &\left[\frac{p^2}{2m_e}+\frac{q_e^2A^2}{2m_ec^2}-\underbrace{\frac{q_e}{2m_ec}(l+2s)}_{\equiv\mu}\cdot B+V\right]\Psi^L=\mathrm{i}\hbar\frac{\partial}{\partial t}\Psi^L
\end{aligned}
$$

where it is understood that the magnetic moment $\boldsymbol{\mu}$ of the Pauli electron, which interacts with the external magnetic field $\boldsymbol{B}$, is generated by the spin and angular momenta,

$$
\mu=\mu_{\text{orbit}}+\mu_{\text{spin}}=-\frac{\mu_{\mathrm{B}}}{\hbar}l-g\frac{\mu_{\mathrm{B}}}{\hbar}s \tag{3.303}
$$

Here, two quantities were introduced: (i) the *Bohr magneton* $\mu_{\mathrm{B}}=e\hbar/2m_ec$ amd (ii) the *gyromagnetic ratio* of the electron $g=2$ also known as the Landé factor. The value reproduces the experimental result for the gyromagnetic ration very well. The quadratic term in the vector potential, $q_e^2A^2/2m_ec^2$, is known as the *diamagnetic* term while the remaining linear potential operators $\boldsymbol{B}$ are known as the *paramagnetic terms*.

Note that considering the nonrelativistic limit for infinite speed of light $c\to\infty$ yields the Schrödinger equation. One must consider the Pauli equation when analyzing magnetic phenomena, as simple Schrödinger mechanics does not suffice. Furthermore, we note in passing that the Schrödinger may be employed upon the condition in which V is small compared to the rest energy, m_ec^2. If we consider V arbitrarily large, the Schrödinger equation is no longer a valid approximation.

CHAPTER IV

DIRAC EQUATION IN CURVED SPACE-TIME

4.0.1 The tetrad formalism

Up until this point we have considered only the natural choice of the coordinate basis on the spacetime manifold. In the coordinate basis the basis vector are described by $\hat{e}_{(\mu)} = \partial_\mu$ and its dual (or covectors) is given by $\hat{\theta}^\mu = dx^\mu$. All objects analyzed are independent of any specific coordinate system, hence, we seek a different approach for establishing a basis. Using *Einstein's Principle of Equivalence*, referred to in chapter 1, we recognize this as a precedent for describing space-time as a Lorentzian maninfold; the act that the space-time is locally Minkowskian. Since we have ventured and pondered over the steps in which to formulate the relativistic quantum mechanics for spin- $\frac{1}{2}$ particles, i.e. describe their behavior in local flat space-time, we know that such must hold for small enough regions of *curved* space-time. Thence, we use this as our preliminary point for our conception of the Dirac equation in curved space-time.

Primarily, we are concerned with establishing a basis for each point in space-time, a curved space-time, with the basis being locally inertial (Minkowskian). This inevitably points to the fact that the metric tensor shall be Minkowskian in form when written in terms of the basis. We denote this basis vectors as $\hat{e}_{(a)}$ with corresponding dual vectors $\hat{\theta}^{(b)}$, with a Latin index. Thus, in the neighborhood of each point in space-time we have

$$g_{ab} = g_{ab}\hat{\theta}^{(a)}\hat{\theta}^{(b)} = \eta_{ab} \tag{4.1}$$

As per Einstein summation convention, we recognize this as a summation over the repeated indices of each of the respective dual vectors with the metric, namely constituting an inner product of the two dual basis vectors, forming an orthonormal set with respect to the Minkowskain metric. Said orthonormal set is called a *tetrad* or *vierbein*, and the practice of employing such an orthonormal frame at each point on a manifold is called the *the tetrad formalism*.

We will now utilize the convention that when a tensor is described by Latin indices, then its components will be defined with respect to a local inertial basis. From here on out we shall refer to a vector written in local indices as *local vectors*, while vectors written in terms of the coordinate basis shall be referred to as *global vectors*. Further, we require that,

$$\hat{\theta}^{(a)}\left(\hat{e}_{(b)}\right) = \delta_b{}^a \tag{4.2}$$

where δ_b^a is the usual Kronecker delta. We may transform between old coordinate basis and the local inertial basis by the following,

$$\hat{e}_{(\mu)} = e_\mu{}^a \hat{e}_{(a)} \tag{4.3}$$

similarly for the dual basis,

$$\hat{\theta}^{(\mu)} = e^\mu{}_b \hat{\theta}^{(b)} \tag{4.4}$$

Here $e_\mu{}^a$ and $e^\mu{}_b$ are transformation matrices, and we shall denote them as the *vierbein* and *inverse vierbein*, respectively. We point attention to the fact of the difference between the *vierbein* and the *inverse vierbein* where one may be distinguished from the other by the difference in the positioning of the indices. The fact that they are inverses follows from the precedent employed in (4.2). We have that

$$e_\mu{}^a e^\mu{}_b = \delta^a_b \quad e_\mu{}^a e^\nu{}_a = \delta_\mu{}^\nu \tag{4.5}$$

Consider the metric in global basis, and transform the global basis to the local basis. Such a transformation will yield the metric in local coordinates as in equation (4.1),

$$g_{ab} = g_{\mu\nu} e^{\mu}{}_{a} \hat{\theta}^{(a)} e^{\nu}{}_{b} \hat{\theta}^{(b)} = \eta_{ab} \tag{4.6}$$

Now neglecting the dual vectors we attain

$$\eta_{ab} = e^{\mu}{}_{a} e^{\nu}{}_{b} g_{\mu\nu} \quad \text{and} \quad g_{\mu\nu} = e_{\mu}{}^{a} e_{\nu}{}^{b} \eta_{ab} \tag{4.7}$$

We may generalize this notion of rewriting a tensor to local coordinates to any arbitrary tensor. Using the appropriate vierbeins and inverse vierbeins we may consider any tensor with respect to the local basis and transform it between local and global indices. Hence, for the global vector V^{μ}, its components in the local basis are described by

$$V^{a} = e_{\mu}{}^{a} V^{\mu} \tag{4.8}$$

For a general tensor we write

$$T^{\mu_1 \cdots a \cdots \mu_k}{}_{\nu_1 \cdots b \cdots \nu_l} = e_{\mu}{}^{a} e^{\nu}{}_{b} T^{\mu_1 \cdots \mu \cdots \mu_k}{}_{\nu_1 \cdots \nu \cdots \nu_l} \tag{4.9}$$

where one may observe the possibility of alternating from local and global indices by applying said transformation matrices. Thus, using this convention of transformation matrices we have that the local indices get lowered and raised with the Minkowskian metric, while the global indices get raised and lowered with the metric. With these tetrads at hand we have a manner of dealing with physical quantities that are only formulated in a Minkowskian template.

4.0.2 The Spin Connection

With the previously developed formalism, we have created a local inertial coordinate system at each point on the manifold. It is of peculiar interest to study the way in which a vector located in one point of the manifold parallel transports to another point on the same manifold, changing with it the way in which its components transform from the local ("old") basis to the new basis. The answer to this paradigm will provide a connection to this form of parallel transport; it is known as *the spin connection* and it derives from the fact that it allows us to formulate the covariant derivative of the spinor.

The equation for parallel transport of local vectors coincides with the form of global vectors, albeit embedded with the spin connection;

$$V^a(x \to x+dx) = V^a(x) - \omega_\mu{}^a{}_b(x)V^b(x)dx^\mu \tag{4.10}$$

where $\omega_\mu{}^a{}_b(x)$ denotes the spin connection. Within it confines the information required to parallel transport the vector itself using the Christoffel symbols, as well as adjusting the local coordinates at the starting point with the local coordinates at the end point via the vierbeins.

In order to derive the explicit form for the spin connection, we first recall

$$V^\mu(x) = e^\mu{}_a(x)V^a(x) \tag{4.11}$$

Transporting from the point x to the infinitesimal point $x+dx$ yields

$$V^\mu(x \to x+dx) = e^\mu{}_a(x+dx)V^a(x \to x+dx) \tag{4.12}$$

Now, Taylor expanding the parallel transported vierbein to the first order in dx reads

$$V^\mu(x \to x+dx) = e^\mu{}_a(x)V^a(x \to x+dx) + \partial_\nu e^\mu{}_a(x)V^a(x \to x+dx)dx^\nu \tag{4.13}$$

Substituting the expression for $V^a(x \to x+dx)$ in (4.10), we obtain (keeping only first order terms in dx)

$$\begin{aligned} V^{\mu}(x \to x+dx) &= e^{\mu}{}_{a}(x)V^{a}(x) - \left[e^{\mu}{}_{a}(x)\omega_{\lambda}{}^{a}{}_{b}(x) - \partial_{\lambda}e^{\mu}{}_{b}(x)\right]V^{b}(x)dx^{\lambda} \\ &= V^{\mu}(x) - \left[e^{\mu}{}_{a}(x)\omega_{\lambda}{}^{a}{}_{b}(x) - \partial_{\lambda}e^{\mu}{}_{b}(x)\right]e_{\sigma}{}^{b}(x)V^{\sigma}(x)dx^{\lambda} \end{aligned} \tag{4.14}$$

where we briefly note, that relabeling with "dummy" indices is convention, so long as we do not change the value of the tensor. From the equation for the parallel transport of vectors, we recognize the previous as a slight variation of the aforementioned,

$$\Gamma^{\mu}{}_{\sigma\lambda} = \left[e^{\mu}{}_{a}\omega_{\lambda}{}^{a}{}_{b} - \partial_{\lambda}e^{\mu}{}_{b}\right]e_{\sigma}{}^{b} \tag{4.15}$$

Solving for the spin connection, we find it to be

$$\omega_{\mu}{}^{a}{}_{b} = e_{\nu}{}^{a}e^{\sigma}{}_{b}\Gamma^{\nu}{}_{\sigma\mu} + e_{\nu}{}^{a}\partial_{\mu}e^{\nu}{}_{b} \tag{4.16}$$

with lowered indices read as:

$$\omega_{\mu ab} \equiv \eta_{ac}e_{\nu}{}^{c}e^{\sigma}{}_{b}\Gamma^{\nu}{}_{\sigma\mu} + \eta_{ac}e_{\nu}{}^{c}\partial_{\mu}e^{\nu}{}_{b} \tag{4.17}$$

Then, since the $\Gamma^{\nu}{}_{\sigma\mu}$ is the Christoffel symbols, we have conjectured that the connection is metric compatible. In terms of local coordinates (or a "locally-geodesic" coordinate system) this would mean that the covariant derivative of the Minkowskian metric should vanish. Using the spin connection we find that

$$\begin{aligned} \nabla_{\mu}\eta_{ab} &= \partial_{\mu}\eta_{ab} - \omega_{\mu}{}^{c}{}_{a}\eta_{cb} - \omega_{\mu}{}^{c}{}_{b}\eta_{ac} \\ &= -\omega_{\mu ba} - \omega_{\mu ab} = 0 \end{aligned} \tag{4.18}$$

thus the $\omega_{\mu ab}$ exhibits an antisymmetry in the last two indices;

$$\omega_{\mu ab} = -\omega_{\mu ba} \tag{4.19}$$

4.0.3 The spinor covariant derivative

We start by noting the nature of the Dirac field. We note that due to the Dirac system not possessing the characteristics of a tensorial field we may not simply interchange the partial derivative present within the system via the its covariant counterpart. Rather, we seek a different approach. We find it appropriate to exploit the fact that we may derive the Dirac equation in flat spacetime. Then, utilizing the tetrad formalism we may connect distinct neighborhoods in a general curved spacetime using the spin connection. After which, we may derive an unprecedented covariant derivative appropriate to the spinors of the system. Thus, with this at hand we may formulate the covariant version of the Dirac equation.

First we require the covariant derivative of the spinor field,

$$\nabla_\mu \Psi(x) = \left[\partial_\mu + \Omega_\mu(x)\right]\Psi(x) \tag{4.20}$$

where $\Omega_\mu(x)$ is the connection coefficient of the spinor field. With the connection at hand, the spinor must obey the following rule for parallel transport:

$$\Psi(x \to x+dx) = \Psi(x) - \Omega_\mu(x)\Psi(x)dx^\mu \tag{4.21}$$

To determine the spinor connection we analyze certain parallel transport properties of the Dirac bilinears, to implicitly derive the rule for parallel transport of the spinor. To this end, we have the scalar quantity $S(x) = \bar{\Psi}(x)\Psi(x)$ which should remain invariant under parallel transport:

$$\begin{aligned} S(x \to x+dx) &= \overline{\Psi}(x \to x+dx)\Psi(x \to x+dx) \\ &= \left[\Psi^\dagger(x)\gamma^0 - \Psi^\dagger(x)\Omega^\dagger_\mu(x)\gamma^0 dx^\mu\right]\left[\Psi(x) - \Omega_\nu(x)\Psi(x)dx^\nu\right] \\ &= S(x) - \overline{\Psi}(x)\left[\Omega_\mu(x) + \gamma^0\Omega^\dagger_\mu(x)\gamma^0\right]\Psi(x)dx^\mu \end{aligned} \tag{4.22}$$

where we have used (4.21), the definition of the Dirac adjoint, $\overline{\psi} = \psi^\dagger\gamma^0$, and $\left(\gamma^0\right)^2 = 1$. Moreover, we have neglected the term proportional to $dx^\mu dx^\nu$ since it is an infinitesimal difference. We briefly prove the above claim, since

$$
\begin{aligned}
S(x \to x+dx) &= \overline{\Psi}(x \to x+dx)\Psi(x \to x+dx) \\
&= \left[\Psi^\dagger(x)\gamma^0 - \Psi^\dagger(x)\Omega_\mu^\dagger(x)\gamma^0 dx^\mu\right]\left[\Psi(x) - \Omega_\nu(x)\Psi(x)dx^\nu\right] \\
&= \Psi^\dagger(x)\gamma^0\Psi(x) - \Psi^\dagger(x)\Omega_\nu(x)\Psi(x)dx^\nu - \Psi^\dagger(x)\Omega_\mu^\dagger(x)\gamma^0\Psi(x)dx^\mu + \Psi^\dagger\Omega_\mu^\dagger\gamma^0 dx^\mu\Omega_\nu\Psi(x)dx^\nu \\
&\implies \Psi^\dagger(x)\gamma^0\Psi(x) - \Psi^\dagger(x)\gamma^0\Omega_\nu(x)\Psi(x)dx^\nu - \Psi^\dagger\Omega_\mu^\dagger(x)\gamma^0\Psi(x)dx^\mu \\
&\implies \overline{\Psi}(x)\Psi(x) - \overline{\Psi}(x)\Omega_\nu(x)\Psi(x)dx^\nu - \Psi^\dagger\,(1)\,\Omega_\mu^\dagger(x)\gamma^0\Psi(x)dx^\mu \\
&\implies \overline{\Psi}(x)\Psi(x) - \overline{\Psi}(x)\Omega_\nu(x)\Psi(x)dx^\nu - \Psi^\dagger\gamma^0\gamma^0\Omega_\mu^\dagger(x)\gamma^0\Psi(x)dx^\mu \\
&\implies S(x) - \overline{\Psi}(x)\Omega_\nu(x)\Psi(x)dx^\nu - \overline{\Psi}(x)\gamma^0\Omega_\mu^\dagger(x)\gamma^0\Psi(x)dx^\mu \\
&\implies S(x) - \overline{\Psi}(x)\left[\Omega_\mu(x) + \gamma^0\Omega_\mu^\dagger(x)\gamma^0\right]\Psi(x)dx^\mu
\end{aligned}
$$

in order for (4.22) to hold we require that

$$\gamma^0\Omega_\mu^\dagger\gamma^0 = -\Omega_\mu \tag{4.23}$$

which makes sense, since the Dirac scalar, i.e. $S(x) = \bar{\Psi}(x)\Psi(x)$ should remain unchanged under parallel transport. Next, we examine the local vector $j^a(x) = \overline{\Psi}(x)\gamma^a\Psi(x)$ which should transport in an analogous fashion to that of (4.10);

$$
\begin{aligned}
j^a(x \to x+dx) &= \left[\Psi^\dagger(x) - \Psi^\dagger(x)\Omega_\mu^\dagger(x)dx^\mu\right]\gamma^0\gamma^a\left[\Psi(x) - \Omega_\mu(x)\Psi(x)dx^\mu\right] \\
&= \overline{\Psi}(x)\gamma^a\Psi(x) - \overline{\Psi}(x)\left[\gamma^a\Omega_\mu(x) - \Omega_\mu(x)\gamma^a\right]\Psi(x)dx^\mu
\end{aligned} \tag{4.24}
$$

where we have taken into account (4.23). Since, we have that

$$
\begin{aligned}
j^a(x \to x+dx) &= \left[\Psi^\dagger(x) - \Psi^\dagger(x)\Omega_\mu^\dagger(x)dx^\mu\right]\gamma^0\gamma^a\left[\Psi(x) - \Omega_\mu(x)\Psi(x)dx^\mu\right] \\
&= \left[\Psi^\dagger(x) - \Psi^\dagger(x)\Omega_\mu^\dagger(x)dx^\mu\right]\left[\gamma^0\gamma^a\Psi(x) - \gamma^0\gamma^a\Omega_\mu(x)\Psi(x)dx^\mu\right] \\
&= \Psi^\dagger(x)\gamma^0\gamma^a\Psi(x) - \Psi^\dagger(x)\gamma^0\gamma^a\Omega_\mu(x)\Psi(x)dx^\mu - \Psi^\dagger(x)\Omega_\mu^\dagger(x)dx^\mu\gamma^0\gamma^a\Psi(x) + \\
&\Psi^\dagger(x)\Omega_\mu^\dagger(x)dx^\mu\gamma^0\gamma^a\Omega_\mu(x)\Psi(x)dx^\mu \\
&\implies \Psi^\dagger(x)\gamma^0\gamma^a\Psi(x) - \Psi^\dagger(x)\gamma^0\gamma^a\Omega_\mu(x)\Psi(x)dx^\mu - \Psi^\dagger(x)\Omega_\mu^\dagger(x)dx^\mu\gamma^0\gamma^a\Psi(x) \\
&\implies \overline{\Psi}(x)\gamma^a\Psi(x) - \overline{\Psi}(x)\gamma^a\Omega_\mu(x)\Psi(x)dx^\mu - \Psi^\dagger(x)\Omega_\mu^\dagger(x)\gamma^0\gamma^a\Psi(x)dx^\mu \\
&\implies \overline{\Psi}(x)\gamma^a\Psi(x) - \overline{\Psi}(x)\gamma^a\Omega_\mu(x)\Psi(x)dx^\mu + \Psi^\dagger(x)\gamma^0\Omega_\mu\gamma^a\Psi(x)dx^\mu \\
&\implies \overline{\Psi}(x)\gamma^a\Psi(x) - \overline{\Psi}(x)\gamma^a\Omega_\mu(x)\Psi(x)dx^\mu + \overline{\Psi}(x)\Omega_\mu(x)\gamma^a\Psi(x)dx^\mu \\
&\implies \overline{\Psi}(x)\gamma^a\Psi(x) - \overline{\Psi}(x)\left[\gamma^a\Omega_\mu(x) - \Omega_\mu(x)\gamma^a\right]\Psi(x)dx^\mu \\
&\implies \overline{\Psi}(x)\gamma^a\Psi(x) - \overline{\Psi}(x)\left[\gamma^a, \Omega_\mu(x)\right]\Psi(x)dx^\mu
\end{aligned}
$$

Namely, we used the fact that,

$$
\begin{aligned}
\gamma^0\Omega_\mu^\dagger\gamma^0 &= -\Omega_\mu \\
\implies \left(\gamma^0\right)^2\Omega_\mu^\dagger\gamma^0 &= -\gamma^0\Omega_\mu \\
\implies \Omega_\mu^\dagger\gamma^0 &= -\gamma^0\Omega_\mu \\
\implies -\Omega_\mu^\dagger\gamma^0 &= \gamma^0\Omega_\mu
\end{aligned}
$$

By the requirement that this must obey the parallel transport of a vector, (4.10), we attain our second condition on the spinor connection:

$$\left[\gamma^a, \Omega_\mu\right] = \omega_\mu{}^a{}_b\gamma^b \tag{4.25}$$

From the resulting commutator we surmise that Ω_μ should be comprised of some combination of the spin connection, together with a product of gamma matrices satisfying the commutation relation. If we note that that the spin gamma matrices satisfyt the following commutation relation $\left[\gamma^a,\sigma^{bc}\right]=2i\left(\gamma^c\eta^{ba}-\gamma^b\eta^{ca}\right)$, where $\omega^{\mu\nu}\equiv\frac{\mathrm{i}}{2}\left[\gamma^\mu,\gamma^\nu\right]$, and making the indices conform to (4.25), we attain the following ansatz for Ω_μ:

$$\Omega_\mu = C\omega_{\mu bc}\sigma^{bc} \tag{4.26}$$

where C is a complex constant. Substitution into (4.25):

$$\left[\gamma^a,\Omega_\mu\right] = 2\mathrm{i}C\omega_{\mu bc}\left(\gamma^c\eta^{ba}-\gamma^b\eta^{ca}\right) = 4\mathrm{i}C\omega_\mu{}^a{}_b\gamma^b \tag{4.27}$$

Indeed, consider.

$$\begin{aligned}
\left[\gamma^a,\Omega_\mu\right] &= \left[\gamma^a, C\omega_{\mu bc}\sigma^{bc}\right]\\
&= C\omega_{\mu bc}\left[\gamma^a,\sigma^{bc}\right]\\
&= 2\mathrm{i}C\omega_{\mu bc}\left(\gamma^c\eta^{ba}-\gamma^b\eta^{ca}\right)\\
&= 2\mathrm{i}C\left[\omega_{\mu bc}\gamma^c\eta^{ba}-\omega_{\mu bc}\gamma^b\eta^{ca}\right]\\
&= 2\mathrm{i}C\left[\omega_{\mu db}\gamma^b\eta^{da}-\left(-\omega_{\mu cb}\gamma^b\eta^{ca}\right)\right]\\
&= 2\mathrm{i}C\left[\omega_\mu{}^a{}_b\gamma^b-\left(-\omega_\mu{}^a{}_b\gamma^b\right)\right]\\
&= 2\mathrm{i}C\left[\omega_\mu{}^a{}_b\gamma^b+\omega_\mu{}^a{}_b\gamma^b\right]\\
&= 2\mathrm{i}C\left(2\omega_\mu{}^a{}_b\gamma^b\right)\\
&= 4\mathrm{i}C\left(\omega_\mu{}^a{}_b\gamma^b\right)
\end{aligned}$$

Here we used the antisymmetry property of the spin connection in the second equality. Using this, we determine the complex constant C to assume the value $C=\frac{1}{4\mathrm{i}}$. Now, note that $\left(\sigma^{bc}\right)^\dagger=\gamma^0\sigma^{bc}\gamma^0$, we notice that with the constant C the spin connection satisfies (4.23). Finally, we have derived the covariant derivative of the spinor. It has the following connection:

$$\Omega_\mu = -\frac{1}{4}\mathrm{i}\omega_{\mu bc}\sigma^{bc} = \frac{1}{8}\omega_{\mu bc}\left[\gamma^b,\gamma^c\right] \tag{4.28}$$

4.0.4 Dirac equation in curved spacetime

To arrive at the Dirac equation valued in curved spacetime we must also consider the gamma matrices which conform to the Minkowskian form of the Dirac equation, i.e. (3.197) or (3.227). Now, presently these matrices are written in terns of local coordinates, thus, to write then in terms of global coordinates we must contract using the inverse vierbein;

$$\gamma^{\mu} = e^{\mu}{}_{a}\gamma^{a} \tag{4.29}$$

The global gamma matrices satisfy the generalized Clifford algebra:

$$\{\gamma^{\mu}, \gamma^{\nu}\} = 2g^{\mu\nu} \tag{4.30}$$

We are now prepared to write down the generalized Dirac system in a curved spacetime agenda. It is

$$\left[\mathrm{i}e^{\mu}{}_{a}\gamma^{a}\left(\partial_{\mu} + \Omega_{\mu}\right) - m\right]\Psi = 0 \tag{4.31}$$

we may further analyze the connection terms, such that expanded it reads,

$$\mathrm{i}e^{\mu}{}_{c}\gamma^{c}\Omega_{\mu} = \mathrm{i}\gamma^{c}\Omega_{c} = \frac{1}{8}\mathrm{i}\omega_{cab}\left(\gamma^{c}\gamma^{a}\gamma^{b} - \gamma^{c}\gamma^{b}\gamma^{a}\right) \tag{4.32}$$

which involves the product of three gamma matrices. Since, we have,

$$\begin{aligned}
&\mathrm{i}e^{\mu}{}_{c}\gamma^{c}\Omega_{\mu} = \mathrm{i}e^{\mu}{}_{c}\gamma^{c}\left(\frac{1}{8}\omega_{\mu ab}\left[\gamma^{a}, \gamma^{b}\right]\right)\\
&= \frac{\mathrm{i}}{8}e^{\mu}{}_{c}\omega_{\mu ab}\gamma^{c}\left[\gamma^{a}, \gamma^{b}\right]\\
&= \frac{\mathrm{i}}{8}\omega_{cab}\gamma^{c}\left(\gamma^{a}\gamma^{b} - \gamma^{b}\gamma^{a}\right)\\
&= \frac{\mathrm{i}}{8}\omega_{cab}\left(\gamma^{c}\gamma^{a}\gamma^{b} - \gamma^{c}\gamma^{b}\gamma^{a}\right)
\end{aligned}$$

Where we have defined $\omega_{cab} \equiv e^{\mu}{}_{c}\omega_{\mu ab}$. Using the identity

$$\gamma^c\gamma^a\gamma^b = \eta^{ca}\gamma^b + \eta^{ab}\gamma^c - \eta^{cb}\gamma^a - \mathrm{i}\varepsilon^{dcab}\gamma_d\gamma^5 \tag{4.33}$$

results in

$$\gamma^c\gamma^a\gamma^b - \gamma^c\gamma^b\gamma^a = 2\eta^{ca}\gamma^b - 2\eta^{cb}\gamma^a - 2\mathrm{i}\varepsilon^{cabd}\gamma_d\gamma^5 \tag{4.34}$$

Indeed, we have

$$\begin{aligned}
&\gamma^c\gamma^a\gamma^b = \eta^{ca}\gamma^b + \eta^{ab}\gamma^c - \eta^{cb}\gamma^a - \mathrm{i}\varepsilon^{dcab}\gamma_d\gamma^5 \\
&-\gamma^c\gamma^b\gamma^a = \eta^{cb}\gamma^a + \eta^{ba}\gamma^c - \eta^{ca}\gamma^b - \mathrm{i}\varepsilon^{dcba}\gamma_d\gamma^5 \\
&\implies \eta^{ca}\gamma^b - \eta^{cb}\gamma^a - \mathrm{i}\varepsilon^{dcab}\gamma_d\gamma^5 - \eta^{cb}\gamma^a + \eta^{ca}\gamma^b + \mathrm{i}\varepsilon^{dcba}\gamma_d\gamma^5 \\
&\implies 2\eta^{ca}\gamma^b - 2\eta^{cb}\gamma^a - \mathrm{i}\varepsilon^{dcab}\gamma_d\gamma^5 - \mathrm{i}\varepsilon^{dcab}\gamma_d\gamma^5 \\
&\implies 2\eta^{ca}\gamma^b - 2\eta^{cb}\gamma^a - 2\mathrm{i}\varepsilon^{cabd}\gamma_d\gamma^5
\end{aligned}$$

where we have used the properties of the Levi-Civita symbol. Thus, the Dirac equation, (4.31), may be written explicitly as

$$\mathrm{i}e^{\mu}{}_{a}\gamma^a\partial_\mu\Psi + \frac{1}{4}\mathrm{i}\omega_{cab}\left(\eta^{ca}\gamma^b - \eta^{cb}\gamma^a\right)\Psi - \frac{1}{4}\varepsilon^{abcd}\omega_{cab}\gamma_d\gamma^5\Psi - m\Psi = 0 \tag{4.35}$$

where the properties of the Levi-Civita symbol have been implemented. We foreshadow the fact that the term involving this symbol actually vanishes for the metric in question.

4.0.5 The Dirac System in Curved Spacetime

We define the metric given by the following line element in matrix form,

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{1-K\left(y^2+z^2\right)}{1-K\left(x^2+y^2+z^2\right)} & \frac{Kxy}{1-K\left(x^2+y^2+z^2\right)} & \frac{Kxz}{1-K\left(x^2+y^2+z^2\right)} \\ 0 & \frac{Kxy}{1-K\left(x^2+y^2+z^2\right)} & \frac{1-K\left(x^2+z^2\right)}{1-K\left(x^2+y^2+z^2\right)} & \frac{Kyz}{1-K\left(x^2+y^2+z^2\right)} \\ 0 & \frac{Kxz}{1-K\left(x^2+y^2+z^2\right)} & \frac{Kyz}{1-K\left(x^2+y^2+z^2\right)} & \frac{1-K\left(x^2+y^2\right)}{1-K\left(x^2+y^2+z^2\right)} \end{pmatrix} \tag{4.36}$$

with corresponding inverse metric,

$$g^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -Kx^2+1 & -Kxy & -Kxz \\ 0 & -Kxy & -Ky^2+1 & -Kyz \\ 0 & -Kxz & -Kyz & -Kz^2+1 \end{pmatrix} \tag{4.37}$$

Subsequently, we define the Christoffel symbols, Riemann and Ricci tensors, as well as the scalar curvature:

$$\Gamma^{\eta}_{\mu\nu} = \begin{array}{llll} \Gamma^2_{22} = \frac{xK\left(-1+K\left(y^2+z^2\right)\right)}{-1+K\left(x^2+y^2+z^2\right)}, & \Gamma^2_{23} = -\frac{K^2yx^2}{-1+K\left(x^2+y^2+z^2\right)}, & \Gamma^2_{24} = -\frac{K^2zx^2}{-1+K\left(x^2+y^2+z^2\right)}, & \Gamma^2_{33} = \frac{xK\left(-1+K\left(x^2+z^2\right)\right)}{-1+K\left(x^2+y^2+z^2\right)} \\ \Gamma^2_{34} = -\frac{K^2xyz}{-1+K\left(x^2+y^2+z^2\right)}, & \Gamma^2_{44} = \frac{xK\left(-1+K\left(x^2+y^2\right)\right)}{-1+K\left(x^2+y^2+z^2\right)}, & \Gamma^3_{22} = \frac{Ky\left(-1+K\left(y^2+z^2\right)\right)}{-1+K\left(x^2+y^2+z^2\right)}, & \Gamma^3_{23} = -\frac{K^2xy^2}{-1+K\left(x^2+y^2+z^2\right)} \\ \Gamma^3_{24} = -\frac{K^2xyz}{-1+K\left(x^2+y^2+z^2\right)}, & \Gamma^3_{33} = \frac{K\left(-1+K\left(x^2+z^2\right)\right)y}{-1+K\left(x^2+y^2+z^2\right)}, & \Gamma^3_{34} = -\frac{K^2zy^2}{-1+K\left(x^2+y^2+z^2\right)}, & \Gamma^3_{44} = \frac{K\left(-1+K\left(x^2+y^2\right)\right)y}{-1+K\left(x^2+y^2+z^2\right)} \\ \Gamma^4_{22} = \frac{zK\left(-1+K\left(y^2+z^2\right)\right)}{-1+K\left(x^2+y^2+z^2\right)}, & \Gamma^4_{23} = -\frac{K^2xyz}{-1+K\left(x^2+y^2+z^2\right)}, & \Gamma^4_{24} = -\frac{K^2xz^2}{-1+K\left(x^2+y^2+z^2\right)}, & \Gamma^4_{33} = \frac{zK\left(-1+K\left(x^2+z^2\right)\right)}{-1+K\left(x^2+y^2+z^2\right)} \\ \Gamma^4_{34} = -\frac{K^2yz^2}{-1+K\left(x^2+y^2+z^2\right)}, & \Gamma^4_{44} = \frac{zK\left(-1+K\left(x^2+y^2\right)\right)}{-1+K\left(x^2+y^2+z^2\right)} & & \end{array} \tag{4.38}$$

$$R_{\alpha\beta\mu\nu} = \tag{4.39}$$

$$R_{\mu\nu} = \left.\begin{array}{lll} R_{22} = \frac{2K\left(-1+K\left(y^2+z^2\right)\right)}{-1+K\left(x^2+y^2+z^2\right)}, & R_{23} == -\frac{2K^2yx}{-1+K\left(x^2+y^2+z^2\right)}, & R_{24} = -\frac{2K^2zx}{-1+K\left(x^2+y^2+z^2\right)} \\ R_{32} = -\frac{2K^2yx}{-1+K\left(x^2+y^2+z^2\right)}, & R_{33} = \frac{2K\left(-1+K\left(x^2+z^2\right)\right)}{-1+K\left(x^2+y^2+z^2\right)}, & R_{34} = -\frac{2K^2yz}{-1+K\left(x^2+y^2+z^2\right)} \\ R_{42} = -\frac{2K^2zx}{-1+K\left(x^2+y^2+z^2\right)}, & R_{43} = -\frac{2K^2yz}{-1+K\left(x^2+y^2+z^2\right)} & R_{44} = \frac{2K\left(-1+K\left(x^2+y^2\right)\right)}{-1+K\left(x^2+y^2+z^2\right)} \end{array}\right\} \tag{4.40}$$

$$R = 6K \tag{4.41}$$

such that we find the vierbein (tetrad) by employing the following diagonalization

$$A = PDP^{-1} \tag{4.42}$$

where P is given by the eigenvectors of matrix A, in this case being represented by the metric $g_{\mu\nu}$, P^{-1} being its inverse and D the eigenvalues of the metric. Hence,

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{y}{x} & -\frac{z}{x} & \frac{x}{z} \\ 0 & 1 & 0 & \frac{y}{z} \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad P^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{xy}{x^2+y^2+z^2} & \frac{x^2+z^2}{x^2+y^2+z^2} & -\frac{yz}{x^2+y^2+z^2} \\ 0 & -\frac{xz}{x^2+y^2+z^2} & -\frac{yz}{x^2+y^2+z^2} & \frac{x^2+y^2}{x^2+y^2+z^2} \\ 0 & \frac{xz}{x^2+y^2+z^2} & \frac{yz}{x^2+y^2+z^2} & \frac{z^2}{x^2+y^2+z^2} \end{pmatrix} \tag{4.43}$$

such that we find the following,

$$P^{-1}\,gP=\begin{pmatrix}1&0&0&0\\0&-\frac{xy}{x^2+y^2+z^2}&\frac{x^2+z^2}{x^2+y^2+z^2}&-\frac{yz}{x^2+y^2+z^2}\\0&-\frac{xz}{x^2+y^2+z^2}&-\frac{yz}{x^2+y^2+z^2}&\frac{x^2+y^2}{x^2+y^2+z^2}\\0&\frac{xz}{x^2+y^2+z^2}&\frac{yz}{x^2+y^2+z^2}&\frac{z^2}{x^2+y^2+z^2}\end{pmatrix}.$$

$$\begin{pmatrix}-1&0&0&0\\0&\frac{1-K\left(y^2+z^2\right)}{1-K\left(x^2+y^2+z^2\right)}&\frac{Kxy}{1-K\left(x^2+y^2+z^2\right)}&\frac{Kxz}{1-K\left(x^2+y^2+z^2\right)}\\0&\frac{Kxy}{1-K\left(x^2+y^2+z^2\right)}&\frac{1-K\left(x^2+z^2\right)}{1-K\left(x^2+y^2+z^2\right)}&\frac{Kyz}{1-K\left(x^2+y^2+z^2\right)}\\0&\frac{Kxz}{1-K\left(x^2+y^2+z^2\right)}&\frac{Kyz}{1-K\left(x^2+y^2+z^2\right)}&\frac{1-K\left(x^2+y^2\right)}{1-K\left(x^2+y^2+z^2\right)}\end{pmatrix}.\begin{pmatrix}1&0&0&0\\0&-\frac{y}{x}&-\frac{z}{x}&\frac{x}{z}\\0&1&0&\frac{y}{z}\\0&0&1&1\end{pmatrix}$$

$$=\begin{pmatrix}-1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&\frac{1}{1-K\left(x^2+y^2+z^2\right)}\end{pmatrix}$$

Then, consider the following

$$B=PD^{1/2}P^{-1} \tag{4.44}$$

where B denotes the vierbein. Now, consider the following matrix,

$$\sqrt{D}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&\frac{1}{\sqrt{1-K\left(x^2+y^2+z^2\right)}}\end{pmatrix} \tag{4.45}$$

Such that, we construct the vierbein as follows,

$$B=\begin{pmatrix}1&0&0&0\\0&-\frac{y}{x}&-\frac{z}{x}&\frac{x}{z}\\0&1&0&\frac{y}{z}\\0&0&1&1\end{pmatrix}\cdot\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&\frac{1}{\sqrt{1-K\left(x^2+y^2+z^2\right)}}\end{pmatrix}\cdot$$

$$\begin{pmatrix}1&0&0&0\\0&-\frac{xy}{x^2+y^2+z^2}&\frac{x^2+z^2}{x^2+y^2+z^2}&-\frac{yz}{x^2+y^2+z^2}\\0&-\frac{xz}{x^2+y^2+z^2}&-\frac{yz}{x^2+y^2+z^2}&\frac{x^2+y^2}{x^2+y^2+z^2}\\0&\frac{xz}{x^2+y^2+z^2}&\frac{yz}{x^2+y^2+z^2}&\frac{z^2}{x^2+y^2+z^2}\end{pmatrix}$$

$$=\begin{pmatrix}1&0&0&0\\0&\frac{x^2\sqrt{\frac{1}{1-K\left(x^2+y^2+z^2\right)}}+y^2+z^2}{x^2+y^2+z^2}&\frac{xy\left(\sqrt{\frac{1}{1-K\left(x^2+y^2+z^2\right)}}-1\right)}{x^2+y^2+z^2}&\frac{xz\left(\sqrt{\frac{1}{1-K\left(x^2+y^2+z^2\right)}}-1\right)}{x^2+y^2+z^2}\\0&\frac{xy\left(\sqrt{\frac{1}{1-K\left(x^2+y^2+z^2\right)}}-1\right)}{x^2+y^2+z^2}&\frac{y^2\sqrt{\frac{1}{1-K\left(x^2+y^2+z^2\right)}}+x^2+z^2}{x^2+y^2+z^2}&\frac{yz\left(\sqrt{\frac{1}{1-K\left(x^2+y^2+z^2\right)}}-1\right)}{x^2+y^2+z^2}\\0&\frac{xz\left(\sqrt{\frac{1}{1-K\left(x^2+y^2+z^2\right)}}-1\right)}{x^2+y^2+z^2}&\frac{yz\left(\sqrt{\frac{1}{1-K\left(x^2+y^2+z^2\right)}}-1\right)}{x^2+y^2+z^2}&\frac{z^2\sqrt{\frac{1}{1-K\left(x^2+y^2+z^2\right)}}+x^2+y^2}{x^2+y^2+z^2}\end{pmatrix}$$

such that simplified we have,

$$B=\begin{pmatrix}1&0&0&0\\0&\frac{\frac{x^2}{\sqrt{1-K\left(x^2+y^2+z^2\right)}}+y^2+z^2}{x^2+y^2+z^2}&\frac{xy\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)}{\sqrt{1-K\left(x^2+y^2+z^2\right)}\left(x^2+y^2+z^2\right)}&\frac{xz\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)}{\sqrt{1-K\left(x^2+y^2+z^2\right)}\left(x^2+y^2+z^2\right)}\\0&\frac{xy\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)}{\sqrt{1-K\left(x^2+y^2+z^2\right)}\left(x^2+y^2+z^2\right)}&\frac{\frac{y^2}{\sqrt{1-K\left(x^2+y^2+z^2\right)}}+x^2+z^2}{x^2+y^2+z^2}&\frac{yz\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)}{\sqrt{1-K\left(x^2+y^2+z^2\right)}\left(x^2+y^2+z^2\right)}\\0&\frac{xz\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)}{\sqrt{1-K\left(x^2+y^2+z^2\right)}\left(x^2+y^2+z^2\right)}&\frac{yz\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)}{\sqrt{1-K\left(x^2+y^2+z^2\right)}\left(x^2+y^2+z^2\right)}&\frac{\frac{z^2}{\sqrt{1-K\left(x^2+y^2+z^2\right)}}+x^2+y^2}{x^2+y^2+z^2}\end{pmatrix}\qquad(4.46)$$

with corresponding inverse vierbein defined as,

$$B^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\frac{x^2}{\sqrt{\frac{1}{1-K(x^2+y^2+z^2)}}}+y^2+z^2}{x^2+y^2+z^2} & -\frac{xy\left(\sqrt{\frac{1}{1-K(x^2+y^2+z^2)}}-1\right)}{(x^2+y^2+z^2)\sqrt{\frac{1}{1-K(x^2+y^2+z^2)}}} & -\frac{xz\left(\sqrt{\frac{1}{1-K(x^2+y^2+z^2)}}-1\right)}{(x^2+y^2+z^2)\sqrt{\frac{1}{1-K(x^2+y^2+z^2)}}} \\ 0 & -\frac{xy\left(\sqrt{\frac{1}{1-K(x^2+y^2+z^2)}}-1\right)}{(x^2+y^2+z^2)\sqrt{\frac{1}{1-K(x^2+y^2+z^2)}}} & \frac{x^2+z^2+\frac{y^2}{\sqrt{\frac{1}{1-K(x^2+y^2+z^2)}}}}{x^2+y^2+z^2} & -\frac{yz\left(\sqrt{\frac{1}{1-K(x^2+y^2+z^2)}}-1\right)}{(x^2+y^2+z^2)\sqrt{\frac{1}{1-K(x^2+y^2+z^2)}}} \\ 0 & -\frac{xz\left(\sqrt{\frac{1}{1-K(x^2+y^2+z^2)}}-1\right)}{(x^2+y^2+z^2)\sqrt{\frac{1}{1-K(x^2+y^2+z^2)}}} & -\frac{yz\left(\sqrt{\frac{1}{1-K(x^2+y^2+z^2)}}-1\right)}{(x^2+y^2+z^2)\sqrt{\frac{1}{1-K(x^2+y^2+z^2)}}} & \frac{x^2+y^2+\frac{z^2}{\sqrt{\frac{1}{1-K(x^2+y^2+z^2)}}}}{x^2+y^2+z^2} \end{pmatrix} \tag{4.47}$$

such that simplified we have,

$$B^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{x^2\sqrt{1-K(x^2+y^2+z^2)}+y^2+z^2}{(x^2+y^2+z^2)} & -\frac{xy\left(1-\sqrt{1-K(x^2+y^2+z^2)}\right)}{(x^2+y^2+z^2)} & -\frac{xz\left(1-\sqrt{1-K(x^2+y^2+z^2)}\right)}{(x^2+y^2+z^2)} \\ 0 & -\frac{xy\left(1-\sqrt{1-K(x^2+y^2+z^2)}\right)}{(x^2+y^2+z^2)} & \frac{y^2\sqrt{1-K(x^2+y^2+z^2)}+x^2+z^2}{(x^2+y^2+z^2)} & -\frac{yz\left(1-\sqrt{1-K(x^2+y^2+z^2)}\right)}{(x^2+y^2+z^2)} \\ 0 & -\frac{xz\left(1-\sqrt{1-K(x^2+y^2+z^2)}\right)}{(x^2+y^2+z^2)} & -\frac{yz\left(1-\sqrt{1-K(x^2+y^2+z^2)}\right)}{(x^2+y^2+z^2)} & \frac{z^2\sqrt{1-K(x^2+y^2+z^2)}+x^2+y^2}{(x^2+y^2+z^2)} \end{pmatrix} \tag{4.48}$$

We calculate

$$B^{-1}gB^{-1} = \eta$$

where

$$\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Similarly, we compute

$$B\eta B = g$$

where

$$g_{\mu\nu} = g = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{1-K\left(y^2+z^2\right)}{1-K\left(x^2+y^2+z^2\right)} & \frac{Kxy}{1-K\left(x^2+y^2+z^2\right)} & \frac{Kxz}{1-K\left(x^2+y^2+z^2\right)} \\ 0 & \frac{Kxy}{1-K\left(x^2+y^2+z^2\right)} & \frac{1-K\left(x^2+z^2\right)}{1-K\left(x^2+y^2+z^2\right)} & \frac{Kyz}{1-K\left(x^2+y^2+z^2\right)} \\ 0 & \frac{Kxz}{1-K\left(x^2+y^2+z^2\right)} & \frac{Kyz}{1-K\left(x^2+y^2+z^2\right)} & \frac{1-K\left(x^2+y^2\right)}{1-K\left(x^2+y^2+z^2\right)} \end{pmatrix} \tag{4.49}$$

That shows that

$$e^a{}_\mu(x) = (B(x))_{\mu a}, \quad e^b{}_\nu(x) = (B(x))_{b\nu}$$

That is

$$e^0{}_0(x) = 1,$$

$$e^0{}_1(x) = 0,$$

$$e^0{}_2(x) = 0,$$

$$e^0{}_3(x) = 0,$$

$$e^1{}_0(x) = 0,$$

$$e^1{}_1(x) = \frac{\frac{x^2}{\sqrt{1-K\left(x^2+y^2+z^2\right)}} + y^2 + z^2}{x^2+y^2+z^2},$$

$$e^1{}_2(x) = \frac{xy\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)}{\sqrt{1-K\left(x^2+y^2+z^2\right)}\left(x^2+y^2+z^2\right)},$$

$$e^1{}_3(x) = \frac{xz\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)}{\sqrt{1-K\left(x^2+y^2+z^2\right)}\left(x^2+y^2+z^2\right)},$$

$$e^2{}_0(x) = 0,$$

$$e^2{}_1(x) = \frac{xy\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)}{\sqrt{1-K\left(x^2+y^2+z^2\right)}\left(x^2+y^2+z^2\right)},$$

$$e^2{}_2(x) = \frac{\frac{y^2}{\sqrt{1-K\left(x^2+y^2+z^2\right)}} + x^2 + z^2}{x^2+y^2+z^2},$$

$$e^2{}_3(x) = \frac{yz\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)}{\sqrt{1-K\left(x^2+y^2+z^2\right)}\left(x^2+y^2+z^2\right)},$$

$$e^3{}_0(x) = 0,$$

$$e^3{}_1(x) = \frac{xz\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)}{\sqrt{1-K\left(x^2+y^2+z^2\right)}\left(x^2+y^2+z^2\right)},$$

$$e^3{}_2(x) = \frac{yz\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)}{\sqrt{1-K\left(x^2+y^2+z^2\right)}\left(x^2+y^2+z^2\right)},$$

$$e^3{}_3(x) = \frac{\frac{z^2}{\sqrt{1-K\left(x^2+y^2+z^2\right)}} + x^2 + y^2}{x^2+y^2+z^2},$$

We will call $e^a{}_\mu(x)$ the vierbein used in four spacetime dimensions. $e^a{}_\mu(x)$ can be thought of as the components of a set of covariant vector fields $\mathbf{e}^a$, labeled by the orthonormal frame index a. Then

$$g_{\mu\nu}(x) = e^a{}_\mu(x) e^b{}_\nu(x) \eta_{ab}$$

The inverse, or dual, of the vierbein will be denoted by $e_a{}^\mu(x)$ and satisfies

$$e_a{}^\mu(x) e^a{}_\nu(x) = \delta^\mu_\nu$$

$$e_a{}^\mu(x) e^b{}_\mu(x) = \delta^a_b$$

Thus,

$$e_0{}^0(x) = 1,$$

$$e_1{}^0(x) = 0,$$

$$e_2{}^0(x) = 0,$$

$$e_3{}^0(x) = 0,$$

$$e_0{}^1(x) = 0,$$

$$e_0{}^2(x) = 0,$$

$$e_0{}^3(x) = 0,$$

$$e_1{}^1(x) = \frac{x^2\sqrt{1-K\left(x^2+y^2+z^2\right)}+y^2+z^2}{\left(x^2+y^2+z^2\right)},$$

$$e_2{}^1(x) = -\frac{xy\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)}{\left(x^2+y^2+z^2\right)},$$

$$e_3{}^1(x) = -\frac{xz\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)}{\left(x^2+y^2+z^2\right)},$$

$$e_1{}^2(x) = -\frac{xy\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)}{\left(x^2+y^2+z^2\right)},$$

$$e_1{}^3(x) = -\frac{xz\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)}{\left(x^2+y^2+z^2\right)},$$

$$e_2{}^2(x) = \frac{y^2\sqrt{1-K\left(x^2+y^2+z^2\right)}+x^2+z^2}{\left(x^2+y^2+z^2\right)},$$

$$e_2{}^3(x) = -\frac{yz\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)}{\left(x^2+y^2+z^2\right)},$$

$$e_3{}^2(x) = -\frac{yz\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)}{\left(x^2+y^2+z^2\right)},$$

$$e_3{}^3(x) = \frac{z^2\sqrt{1-K\left(x^2+y^2+z^2\right)}+x^2+y^2}{\left(x^2+y^2+z^2\right)},$$

The Dirac equation for a curved spacetime background is

$$\left[ie_a^\mu \gamma^a\left(\partial_\mu+\Omega_\mu\right)-m\right]\Psi = 0 \tag{4.50}$$

The connection term in this equation can be further analyzed. This term when written out,

$$ie_c^\mu \gamma^c \Omega_\mu = i\gamma^c \Omega_c = \frac{1}{8} i\omega_{cab} \left(\gamma^c \gamma^a \gamma^b - \gamma^c \gamma^b \gamma^a\right)$$

involves products of three gamma matrices. Here we have defined $\omega_{cab} \equiv e^\mu{}_c \omega_{\mu ab}$.

We have calculated the spin connection ω_{cab} for the curved metric. Its non-zero components up to antisymmetry are:

Indices $\{c,a,b\}$	**Expression for** ω_{cab}
$\{1,2,1\}$	$\frac{y-y\sqrt{1-K(x^2+y^2+z^2)}}{x^2+y^2+z^2}$
$\{1,3,1\}$	$\frac{z-z\sqrt{1-K(x^2+y^2+z^2)}}{x^2+y^2+z^2}$
$\{2,1,2\}$	$\frac{x-x\sqrt{1-K(x^2+y^2+z^2)}}{x^2+y^2+z^2}$
$\{2,3,2\}$	$\frac{z-z\sqrt{1-K(x^2+y^2+z^2)}}{x^2+y^2+z^2}$
$\{3,1,3\}$	$\frac{x-x\sqrt{1-K(x^2+y^2+z^2)}}{x^2+y^2+z^2}$
$\{3,2,3\}$	$\frac{y-y\sqrt{1-K(x^2+y^2+z^2)}}{x^2+y^2+z^2}$

(4.51)

It may be pertinent to take a moment to mention the high degree of symmetry which this connection exhibits. Observe that the patterns on the table are highly regular. The expressions for equal middle index are equal. We have only index combinations of the form $\{i, j, i\}$, where $i = 2,3,4$ and $j \neq i$. Secondly we notice that there is no components with all different indices. Thus

$$\frac{1}{4}\varepsilon^{abcd}\omega_{cab}\gamma_d\gamma^5 = 0 \tag{4.52}$$

due to properties of the Levi-Civita symbol.

Consider the first spin connection term above, such that we may explicitly write (4.48), as follows:

$$\mathrm{i}e^{\mu}{}_{a}\gamma^{a}\partial_{\mu}\Psi$$
$$\implies \mathrm{i}e^{\mu}{}_{0}\gamma^{0}\partial_{\mu}+\mathrm{i}e^{\mu}{}_{1}\gamma^{1}\partial_{\mu}+\mathrm{i}e^{\mu}{}_{2}\gamma^{2}\partial_{\mu}+\mathrm{i}e^{\mu}{}_{3}\gamma^{3}\partial_{\mu}$$

Now, taking a sum over μ from $0-3$ with 0 being the time component, and $e^{\mu}{}_{b}$ representing the inverse vierbein, we have

$$\implies \mathrm{i}e^{0}{}_{0}\gamma^{0}\partial_{0}+\mathrm{i}e^{0}{}_{1}\gamma^{1}\partial_{0}+\mathrm{i}e^{0}{}_{2}\gamma^{2}\partial_{0}+\mathrm{i}e^{0}{}_{3}\gamma^{3}\partial_{0}+$$
$$\mathrm{i}e^{1}{}_{0}\gamma^{0}\partial_{1}+\mathrm{i}e^{1}{}_{1}\gamma^{1}\partial_{1}+\mathrm{i}e^{1}{}_{2}\gamma^{2}\partial_{1}+\mathrm{i}e^{1}{}_{3}\gamma^{3}\partial_{1}+$$
$$\mathrm{i}e^{2}{}_{0}\gamma^{0}\partial_{2}+\mathrm{i}e^{2}{}_{1}\gamma^{1}\partial_{2}+\mathrm{i}e^{2}{}_{2}\gamma^{2}\partial_{2}+\mathrm{i}e^{2}{}_{3}\gamma^{3}\partial_{2}+$$
$$\mathrm{i}e^{3}{}_{0}\gamma^{0}\partial_{3}+\mathrm{i}e^{3}{}_{1}\gamma^{1}\partial_{3}+\mathrm{i}e^{3}{}_{2}\gamma^{2}\partial_{3}+\mathrm{i}e^{3}{}_{3}\gamma^{3}\partial_{3}$$

$$\implies \mathrm{i}\left[e^{0}{}_{0}\gamma^{0}+e^{0}{}_{1}\gamma^{1}+e^{0}{}_{2}\gamma^{2}+e^{0}{}_{3}\gamma^{3}\right]\partial_{0}+$$
$$\mathrm{i}\left[e^{1}{}_{0}\gamma^{0}+e^{1}{}_{1}\gamma^{1}+e^{1}{}_{2}\gamma^{2}+e^{1}{}_{3}\gamma^{3}\right]\partial_{1}+$$
$$\mathrm{i}\left[e^{2}{}_{0}\gamma^{0}+e^{2}{}_{1}\gamma^{1}+e^{2}{}_{2}\gamma^{2}+e^{2}{}_{3}\gamma^{3}\right]\partial_{2}+$$
$$\mathrm{i}\left[e^{3}{}_{0}\gamma^{0}+e^{3}{}_{1}\gamma^{1}+e^{3}{}_{2}\gamma^{2}+e^{3}{}_{3}\gamma^{3}\right]\partial_{3}$$

$$\implies \mathrm{i}\left[(1)\gamma^{0}+(0)\,\gamma^{1}+(0)\gamma^{2}+(0)\gamma^{3}\right]\partial_{0}+$$
$$\mathrm{i}\left[(0)\gamma^{0}+e^{1}{}_{1}\gamma^{1}+e^{1}{}_{2}\gamma^{2}+e^{1}{}_{3}\gamma^{3}\right]\partial_{1}+$$
$$\mathrm{i}\left[(0)\gamma^{0}+e^{2}{}_{1}\gamma^{1}+e^{2}{}_{2}\gamma^{2}+e^{2}{}_{3}\gamma^{3}\right]\partial_{2}+$$
$$\mathrm{i}\left[(0)\gamma^{0}+e^{3}{}_{1}\gamma^{1}+e^{3}{}_{2}\gamma^{2}+e^{3}{}_{3}\gamma^{3}\right]\partial_{3}$$

$$\implies \mathrm{i}\left[(1)\gamma^0\right]\partial_0 +$$

$$\mathrm{i}\left[\frac{x^2\sqrt{1-K\left(x^2+y^2+z^2\right)}+y^2+z^2}{\left(x^2+y^2+z^2\right)}\gamma^1 - \frac{xy\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)}{\left(x^2+y^2+z^2\right)}\gamma^2\right.$$

$$\left. - \frac{xz\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)}{\left(x^2+y^2+z^2\right)}\gamma^3\right]\partial_1 +$$

$$\mathrm{i}\left[-\frac{xy\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)}{\left(x^2+y^2+z^2\right)}\gamma^1 + \frac{y^2\sqrt{1-K\left(x^2+y^2+z^2\right)}+x^2+z^2}{\left(x^2+y^2+z^2\right)}\gamma^2\right.$$

$$\left. - \frac{yz\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)}{\left(x^2+y^2+z^2\right)}\gamma^3\right]\partial_2 +$$

$$\mathrm{i}\left[-\frac{xz\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)}{\left(x^2+y^2+z^2\right)}\gamma^1 - \frac{yz\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)}{\left(x^2+y^2+z^2\right)}\gamma^2 +\right.$$

$$\left. \frac{z^2\sqrt{1-K\left(x^2+y^2+z^2\right)}+y^2+x^2}{\left(x^2+y^2+z^2\right)}\gamma^3\right]\partial_3$$

such that we may further reduce the expression above as follows,

$$\implies \mathrm{i}\gamma^0\partial_0 +$$

$$\frac{\mathrm{i}}{\left(x^2+y^2+z^2\right)}\left[\left(x^2\sqrt{1-K\left(x^2+y^2+z^2\right)}+y^2+z^2\right)\gamma^1 - xy\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)\gamma^2\right.$$

$$\left. -xz\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)\gamma^3\right]\partial_1 +$$

$$\frac{\mathrm{i}}{\left(x^2+y^2+z^2\right)}\left[-xy\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)\gamma^1 + \left(y^2\sqrt{1-K\left(x^2+y^2+z^2\right)}+x^2+z^2\right)\gamma^2\right.$$

$$\left. -yz\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)\gamma^3\right]\partial_2 +$$

$$\frac{\mathrm{i}}{\left(x^2+y^2+z^2\right)}\left[-xz\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)\gamma^1 - yz\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)\gamma^2 +\right.$$

$$\left.\left(z^2\sqrt{1-K\left(x^2+y^2+z^2\right)}+y^2+x^2\right)\gamma^3\right]\partial_3$$

Now, for the second term we have,

$$\frac{1}{4}\mathrm{i}\omega_{cab}\left(\eta^{ca}\gamma^{b}-\eta^{cb}\gamma^{a}\right)$$

such that we must take a sum over each respective index from $0-3$. To this end, consider first the index c, such that

$$\omega_{0ab}\left(\eta^{0a}\gamma^{b}-\eta^{0b}\gamma^{a}\right)+\omega_{1ab}\left(\eta^{1a}\gamma^{b}-\eta^{1b}\gamma^{a}\right)+\omega_{2ab}\left(\eta^{2a}\gamma^{b}-\eta^{2b}\gamma^{a}\right)+\omega_{3ab}\left(\eta^{3a}\gamma^{b}-\eta^{3b}\gamma^{a}\right)$$

Next, taking a sum of a from $0-3$, we have, for $a=0$:

$$\omega_{00b}\left(\eta^{00}\gamma^{b}-\eta^{0b}\gamma^{0}\right)+\omega_{10b}\left(\eta^{10}\gamma^{b}-\eta^{1b}\gamma^{0}\right)+\omega_{20b}\left(\eta^{20}\gamma^{b}-\eta^{2b}\gamma^{0}\right)+\omega_{30b}\left(\eta^{30}\gamma^{b}-\eta^{3b}\gamma^{0}\right)$$

similarly, for the remaining terms we attain, for $a=1$

$$\omega_{01b}\left(\eta^{01}\gamma^{b}-\eta^{0b}\gamma^{1}\right)+\omega_{11b}\left(\eta^{11}\gamma^{b}-\eta^{1b}\gamma^{1}\right)+\omega_{21b}\left(\eta^{21}\gamma^{b}-\eta^{2b}\gamma^{1}\right)+\omega_{31b}\left(\eta^{31}\gamma^{b}-\eta^{3b}\gamma^{1}\right)$$

for $a=2$,

$$\omega_{02b}\left(\eta^{02}\gamma^{b}-\eta^{0b}\gamma^{2}\right)+\omega_{12b}\left(\eta^{12}\gamma^{b}-\eta^{1b}\gamma^{2}\right)+\omega_{22b}\left(\eta^{22}\gamma^{b}-\eta^{2b}\gamma^{2}\right)+\omega_{32b}\left(\eta^{32}\gamma^{b}-\eta^{3b}\gamma^{2}\right)$$

for $a=3$,

$$\omega_{03b}\left(\eta^{03}\gamma^{b}-\eta^{0b}\gamma^{3}\right)+\omega_{13b}\left(\eta^{13}\gamma^{b}-\eta^{1b}\gamma^{3}\right)+\omega_{23b}\left(\eta^{23}\gamma^{b}-\eta^{2b}\gamma^{3}\right)+\omega_{33b}\left(\eta^{33}\gamma^{b}-\eta^{3b}\gamma^{3}\right)$$

We note that for the term

$$\omega_{00b}\left(\eta^{00}\gamma^{b}-\eta^{0b}\gamma^{0}\right)+\omega_{10b}\left(\eta^{10}\gamma^{b}-\eta^{1b}\gamma^{0}\right)+\omega_{20b}\left(\eta^{20}\gamma^{b}-\eta^{2b}\gamma^{0}\right)+\omega_{30b}\left(\eta^{30}\gamma^{b}-\eta^{3b}\gamma^{0}\right)$$

there is no spin connection with middle index “0”, thus, all terms here vanish. Thus, consider the following term,

$$\omega_{01b}\left(\eta^{01}\gamma^{b}-\eta^{0b}\gamma^{1}\right)+\omega_{11b}\left(\eta^{11}\gamma^{b}-\eta^{1b}\gamma^{1}\right)+\omega_{21b}\left(\eta^{21}\gamma^{b}-\eta^{2b}\gamma^{1}\right)+\omega_{31b}\left(\eta^{31}\gamma^{b}-\eta^{3b}\gamma^{1}\right)$$

Then for $b=0$,

$$\omega_{010}\left(\eta^{01}\gamma^{0}-\eta^{00}\gamma^{1}\right)+\omega_{110}\left(\eta^{11}\gamma^{0}-\eta^{10}\gamma^{1}\right)+\omega_{210}\left(\eta^{21}\gamma^{0}-\eta^{20}\gamma^{1}\right)+\omega_{310}\left(\eta^{31}\gamma^{0}-\eta^{30}\gamma^{1}\right)$$
$$\Longrightarrow=0+0+0+0$$

for $b=1$,

$$\omega_{011}\left(\eta^{01}\gamma^{1}-\eta^{01}\gamma^{1}\right)+\omega_{111}\left(\eta^{11}\gamma^{1}-\eta^{11}\gamma^{1}\right)+\omega_{211}\left(\eta^{21}\gamma^{1}-\eta^{21}\gamma^{1}\right)+\omega_{311}\left(\eta^{31}\gamma^{1}-\eta^{31}\gamma^{1}\right)$$
$$\Longrightarrow=0+0+0+0$$

for $b=2$,

$$\omega_{012}\left(\eta^{01}\gamma^{2}-\eta^{02}\gamma^{1}\right)+\omega_{112}\left(\eta^{11}\gamma^{2}-\eta^{12}\gamma^{1}\right)+\omega_{212}\left(\eta^{21}\gamma^{2}-\eta^{22}\gamma^{1}\right)+\omega_{312}\left(\eta^{31}\gamma^{2}-\eta^{32}\gamma^{1}\right)$$
$$\Longrightarrow=\omega_{112}\eta^{11}\gamma^{2}-\omega_{212}\eta^{22}\gamma^{1}$$

for $b=3$,

$$\omega_{013}\left(\eta^{01}\gamma^{3}-\eta^{03}\gamma^{1}\right)+\omega_{113}\left(\eta^{11}\gamma^{3}-\eta^{13}\gamma^{1}\right)+\omega_{213}\left(\eta^{21}\gamma^{3}-\eta^{23}\gamma^{1}\right)+\omega_{313}\left(\eta^{31}\gamma^{3}-\eta^{33}\gamma^{1}\right)$$
$$\Longrightarrow=\omega_{113}\eta^{11}\gamma^{3}-\omega_{313}\eta^{33}\gamma^{1}$$

For the remaining terms, we attain, for $b = 0$

$$\omega_{020}\left(\eta^{02}\gamma^0 - \eta^{00}\gamma^2\right) + \omega_{120}\left(\eta^{12}\gamma^0 - \eta^{10}\gamma^2\right) + \omega_{220}\left(\eta^{22}\gamma^0 - \eta^{20}\gamma^2\right) + \omega_{320}\left(\eta^{32}\gamma^0 - \eta^{30}\gamma^2\right)$$
$$\Longrightarrow = 0 + 0 + 0 + 0$$

for $b = 1$,

$$\omega_{021}\left(\eta^{02}\gamma^1 - \eta^{01}\gamma^2\right) + \omega_{121}\left(\eta^{12}\gamma^1 - \eta^{11}\gamma^2\right) + \omega_{221}\left(\eta^{22}\gamma^1 - \eta^{21}\gamma^2\right) + \omega_{321}\left(\eta^{32}\gamma^1 - \eta^{31}\gamma^2\right)$$
$$\Longrightarrow = -\omega_{121}\eta^{11}\gamma^2 + \omega_{221}\eta^{22}\gamma^1$$

for $b = 2$,

$$\omega_{022}\left(\eta^{02}\gamma^2 - \eta^{02}\gamma^2\right) + \omega_{122}\left(\eta^{12}\gamma^2 - \eta^{12}\gamma^2\right) + \omega_{222}\left(\eta^{22}\gamma^2 - \eta^{22}\gamma^2\right) + \omega_{322}\left(\eta^{32}\gamma^2 - \eta^{32}\gamma^2\right)$$
$$\Longrightarrow = 0 + 0 + 0 + 0$$

for $b = 3$,

$$\omega_{023}\left(\eta^{02}\gamma^3 - \eta^{0b}\gamma^2\right) + \omega_{123}\left(\eta^{12}\gamma^3 - \eta^{13}\gamma^2\right) + \omega_{223}\left(\eta^{22}\gamma^3 - \eta^{23}\gamma^2\right) + \omega_{323}\left(\eta^{32}\gamma^3 - \eta^{33}\gamma^2\right)$$
$$\Longrightarrow = \omega_{223}\eta^{22}\gamma^3 - \omega_{323}\eta^{33}\gamma^2$$

then we have, for $b = 0$

$$\omega_{030}\left(\eta^{03}\gamma^0 - \eta^{00}\gamma^3\right) + \omega_{130}\left(\eta^{13}\gamma^0 - \eta^{10}\gamma^3\right) + \omega_{230}\left(\eta^{23}\gamma^0 - \eta^{20}\gamma^3\right) + \omega_{330}\left(\eta^{33}\gamma^0 - \eta^{30}\gamma^3\right)$$
$$\Longrightarrow = 0 + 0 + 0 + 0$$

for $b=1$,

$$\omega_{031}\left(\eta^{03}\gamma^{1}-\eta^{01}\gamma^{3}\right)+\omega_{131}\left(\eta^{13}\gamma^{1}-\eta^{11}\gamma^{3}\right)+\omega_{231}\left(\eta^{23}\gamma^{1}-\eta^{21}\gamma^{3}\right)+\omega_{331}\left(\eta^{33}\gamma^{1}-\eta^{31}\gamma^{3}\right)$$
$$\Longrightarrow\ -\omega_{131}\eta^{11}\gamma^{3}+\omega_{331}\eta^{33}\gamma^{1}$$

for $b=2$,

$$\omega_{032}\left(\eta^{03}\gamma^{2}-\eta^{02}\gamma^{3}\right)+\omega_{132}\left(\eta^{13}\gamma^{2}-\eta^{12}\gamma^{3}\right)+\omega_{232}\left(\eta^{23}\gamma^{2}-\eta^{22}\gamma^{3}\right)+\omega_{332}\left(\eta^{33}\gamma^{2}-\eta^{32}\gamma^{3}\right)$$
$$\Longrightarrow=-\omega_{232}\eta^{22}\gamma^{3}+\omega_{332}\eta^{33}\gamma^{2}$$

for $b=3$,

$$\omega_{033}\left(\eta^{03}\gamma^{3}-\eta^{03}\gamma^{3}\right)+\omega_{133}\left(\eta^{13}\gamma^{3}-\eta^{13}\gamma^{3}\right)+\omega_{233}\left(\eta^{23}\gamma^{3}-\eta^{23}\gamma^{3}\right)+\omega_{333}\left(\eta^{33}\gamma^{3}-\eta^{33}\gamma^{3}\right)$$
$$\Longrightarrow=0+0+0+0$$

Then, remembering that the Minkowskian takes the form

$$\eta=\begin{pmatrix}-1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

and recalling that the spin connection has an antisymmetry in the last two indices ,

$$\omega_{\mu ab}=-\omega_{\mu ba}$$

Thus, the surviving terms read

$$\omega_{112}\eta^{11}\gamma^2 - \omega_{212}\eta^{22}\gamma^1 + \omega_{113}\eta^{11}\gamma^3 - \omega_{313}\eta^{33}\gamma^1$$
$$-\omega_{121}\eta^{11}\gamma^2 + \omega_{221}\eta^{22}\gamma^1 + \omega_{223}\eta^{22}\gamma^3 - \omega_{323}\eta^{33}\gamma^2$$
$$-\omega_{131}\eta^{11}\gamma^3 + \omega_{331}\eta^{33}\gamma^1 - \omega_{232}\eta^{22}\gamma^3 + \omega_{332}\eta^{33}\gamma^2$$

and considering the following terms

$$-\omega_{212}\eta^{22}\gamma^1 + \omega_{221}\eta^{22}\gamma^1$$

such that by the antisymmetry of the spin connection, we observe

$$\omega_{\mu ab} = -\omega_{\mu ba}$$
$$\implies \omega_{221} = -\omega_{212}$$
$$\implies \omega_{212} = -\omega_{221}$$

Hence, we obtain

$$-\omega_{212}\eta^{22}\gamma^1 + \omega_{221}\eta^{22}\gamma^1$$
$$\implies -(-\omega_{221})\,\eta^{22}\gamma^1 + \omega_{221}\eta^{22}\gamma^1$$
$$\implies 2\omega_{221}\eta^{22}\gamma^1$$
$$\implies -2\left(\frac{x - x\sqrt{1-K(x^2+y^2+z^2)}}{x^2+y^2+z^2}\right)\gamma^1$$

Thus, we have that all terms follow an analogous pattern such that,

$$2\omega_{112}\eta^{11}\gamma^2 + 2\omega_{113}\eta^{11}\gamma^3 + 2\omega_{221}\eta^{22}\gamma^1 + 2\omega_{223}\eta^{22}\gamma^3 + 2\omega_{331}\eta^{33}\gamma^1 + 2\omega_{332}\eta^{33}\gamma^2$$

such that we further reduce,

$$
\begin{aligned}
&2\omega_{112}\gamma^2+2\omega_{113}\gamma^3+2\omega_{221}\gamma^1+2\omega_{223}\gamma^3+2\omega_{331}\gamma^1+2\omega_{332}\gamma^2\\
\implies &2\omega_{221}\gamma^1+2\omega_{331}\gamma^1+2\omega_{332}\gamma^2+2\omega_{112}\gamma^2+2\omega_{113}\gamma^3+2\omega_{223}\gamma^3\\
\implies &2\left[\omega_{221}+\omega_{331}\right]\gamma^1+2\left[\omega_{332}+\omega_{112}\right]\gamma^2+2\left[\omega_{113}+\omega_{223}\right]\gamma^3
\end{aligned}
$$

Observe that,

$$
\begin{aligned}
&2\left[-\frac{x-x\sqrt{1-K(x^2+y^2+z^2)}}{x^2+y^2+z^2}-\frac{x-x\sqrt{1-K(x^2+y^2+z^4)}}{x^2+y^2+z^2}\right]\gamma^1\\
&+2\left[-\frac{y-y\sqrt{1-K(x^2+y^2+z^2)}}{x^2+y^2+z^2}-\frac{y-y\sqrt{1-K(x^2+y^2+z^4)}}{x^2+y^2+z^2}\right]\gamma^2\\
&+2\left[-\frac{z-z\sqrt{1-K(x^2+y^2+z^2)}}{x^2+y^2+z^2}-\frac{z-z\sqrt{1-K(x^2+y^2+z^4)}}{x^2+y^2+z^2}\right]\gamma^3
\end{aligned}
$$

such that we have,

$$
\begin{aligned}
&-\frac{4}{(x^2+y^2+z^2)}\left[x-x\sqrt{1-K(x^2+y^2+z^2)}\right]\gamma^1\\
&-\frac{4}{(x^2+y^2+z^2)}\left[y-y\sqrt{1-K(x^2+y^2+z^2)}\right]\gamma^2\\
&-\frac{4}{(x^2+y^2+z^2)}\left[z-z\sqrt{1-K(x^2+y^2+z^2)}\right]\gamma^3\\
\implies &-\frac{4}{(x^2+y^2+z^2)}\Big[x-x\sqrt{1-K(x^2+y^2+z^2)}\gamma^1+y-y\sqrt{1-K(x^2+y^2+z^2)}\gamma^2\\
&+z-z\sqrt{1-K(x^2+y^2+z^2)}\gamma^3\Big]
\end{aligned}
$$

Then, plugging in the attained value into the original term,

$$\frac{1}{4}\mathrm{i}\omega_{cab}\left(\eta^{ca}\gamma^{b}-\eta^{cb}\gamma^{a}\right)$$
$$\implies -\frac{\mathrm{i}}{(x^2+y^2+z^2)}\left[x-x\sqrt{1-K(x^2+y^2+z^2)}\gamma^1+y-y\sqrt{1-K(x^2+y^2+z^2)}\gamma^2\right.$$
$$\left.+z-z\sqrt{1-K(x^2+y^2+z^2)}\gamma^3\right]$$

Then, combining both terms attained into the original equation,

$$\mathrm{i}e^{\mu}{}_{a}\gamma^{a}\partial_{\mu}\Psi+\frac{1}{4}\mathrm{i}\omega_{cab}\left(\eta^{ca}\gamma^{b}-\eta^{cb}\gamma^{a}\right)\Psi-\frac{1}{4}\varepsilon^{abcd}\omega_{cab}\gamma_{d}\gamma^{5}\Psi-m\Psi=0$$

we have neglecting the spinor,

$$\implies \mathrm{i}\gamma^0\partial_0+$$
$$\frac{\mathrm{i}}{(x^2+y^2+z^2)}\left[\left(x^2\sqrt{1-K\left(x^2+y^2+z^2\right)}+y^2+z^2\right)\gamma^1-xy\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)\gamma^2\right.$$
$$\left.-xz\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)\gamma^3\right]\partial_1+$$
$$\frac{\mathrm{i}}{(x^2+y^2+z^2)}\left[-xy\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)\gamma^1+\left(y^2\sqrt{1-K\left(x^2+y^2+z^2\right)}+x^2+z^2\right)\gamma^2\right.$$
$$\left.-yz\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)\gamma^3\right]\partial_2+$$
$$\frac{\mathrm{i}}{(x^2+y^2+z^2)}\left[-xz\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)\gamma^1-yz\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)\gamma^2+\right.$$
$$\left.\left(z^2\sqrt{1-K\left(x^2+y^2+z^2\right)}+y^2+x^2\right)\gamma^3\right]\partial_3$$
$$-\frac{\mathrm{i}}{(x^2+y^2+z^2)}\left[x-x\sqrt{1-K(x^2+y^2+z^2)}\gamma^1+y-y\sqrt{1-K(x^2+y^2+z^2)}\gamma^2\right.$$
$$\left.+z-z\sqrt{1-K(x^2+y^2+z^2)}\gamma^3\right]-m=0$$

Now, multiplying both sides by a factor of $\frac{\left(x^2+y^2+z^2\right)}{\mathrm{i}}$, we attain,

$$\implies \left(x^2+y^2+z^2\right)\gamma^0\partial_0 +$$

$$\Bigg[\left(x^2\sqrt{1-K\left(x^2+y^2+z^2\right)}+y^2+z^2\right)\gamma^1 - xy\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)\gamma^2$$

$$-xz\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)\gamma^3\Bigg]\partial_1 +$$

$$\Bigg[-xy\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)\gamma^1 + \left(y^2\sqrt{1-K\left(x^2+y^2+z^2\right)}+x^2+z^2\right)\gamma^2$$

$$-yz\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)\gamma^3\Bigg]\partial_2 +$$

$$\Bigg[-xz\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)\gamma^1 - yz\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)\gamma^2 +$$

$$\left(z^2\sqrt{1-K\left(x^2+y^2+z^2\right)}+y^2+x^2\right)\gamma^3\Bigg]\partial_3$$

$$-\Bigg[x - x\sqrt{1-K(x^2+y^2+z^2)}\gamma^1 + y - y\sqrt{1-K(x^2+y^2+z^2)}\gamma^2$$

$$+z - z\sqrt{1-K(x^2+y^2+z^2)}\gamma^3\Bigg] - \frac{\left(x^2+y^2+z^2\right)}{\mathrm{i}}m = 0$$

We note that we can rewrite the last term as $-\frac{\left(x^2+y^2+z^2\right)}{\mathrm{i}}=\left(x^2+y^2+z^2\right)\mathrm{i}$, thus,

$$\Longrightarrow \left(x^2+y^2+z^2\right)\gamma^0\partial_0+$$

$$\Bigg[\left(x^2\sqrt{1-K\left(x^2+y^2+z^2\right)}+y^2+z^2\right)\gamma^1-xy\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)\gamma^2$$

$$-xz\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)\gamma^3\Bigg]\partial_1+$$

$$\Bigg[-xy\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)\gamma^1+\left(y^2\sqrt{1-K\left(x^2+y^2+z^2\right)}+x^2+z^2\right)\gamma^2$$

$$-yz\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)\gamma^3\Bigg]\partial_2+$$

$$\Bigg[-xz\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)\gamma^1-yz\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)\gamma^2+$$

$$\left(z^2\sqrt{1-K\left(x^2+y^2+z^2\right)}+y^2+x^2\right)\gamma^3\Bigg]\partial_3$$

$$-\Bigg[x-x\sqrt{1-K(x^2+y^2+z^2)}\gamma^1+y-y\sqrt{1-K(x^2+y^2+z^2)}\gamma^2$$

$$+z-z\sqrt{1-K(x^2+y^2+z^2)}\gamma^3\Bigg]+\left(x^2+y^2+z^2\right)m\mathrm{i}=0$$

$$\implies (x^2+y^2+z^2)\,\gamma^0\partial_0\Psi +$$

$$\left(x^2\sqrt{1-K\,(x^2+y^2+z^2)}+y^2+z^2\right)\gamma^1\partial_1\Psi - xy\left(1-\sqrt{1-K\,(x^2+y^2+z^2)}\right)\gamma^2\partial_1\Psi$$

$$-xz\left(1-\sqrt{1-K\,(x^2+y^2+z^2)}\right)\gamma^3\partial_1\Psi$$

$$-xy\left(1-\sqrt{1-K\,(x^2+y^2+z^2)}\right)\gamma^1\partial_2\Psi + \left(y^2\sqrt{1-K\,(x^2+y^2+z^2)}+x^2+z^2\right)\gamma^2\partial_2\Psi$$

$$-yz\left(1-\sqrt{1-K\,(x^2+y^2+z^2)}\right)\gamma^3\partial_2\Psi$$

$$-xz\left(1-\sqrt{1-K\,(x^2+y^2+z^2)}\right)\gamma^1\partial_3\Psi - yz\left(1-\sqrt{1-K\,(x^2+y^2+z^2)}\right)\gamma^2\partial_3 +$$

$$\left(z^2\sqrt{1-K\,(x^2+y^2+z^2)}+y^2+x^2\right)\gamma^3\partial_3\Psi$$

$$-\left(x-x\sqrt{1-K(x^2+y^2+z^2)}\right)\gamma^1\Psi - \left(y-y\sqrt{1-K(x^2+y^2+z^2)}\right)\gamma^2\Psi$$

$$-\left(z-z\sqrt{1-K(x^2+y^2+z^2)}\right)\gamma^3\Psi + (x^2+y^2+z^2)\,m\mathrm{i}\Psi = 0$$

Hence, we write the Dirac operator as follows

$$\mathscr{D} := \frac{1}{(x^2+y^2+z^2)}\Bigg[(x^2+y^2+z^2)\,\gamma^0\partial_0\Psi +$$

$$\left(x^2\sqrt{1-K\,(x^2+y^2+z^2)}+y^2+z^2\right)\gamma^1\partial_1\Psi - xy\left(1-\sqrt{1-K\,(x^2+y^2+z^2)}\right)\gamma^2\partial_1\Psi$$

$$-xz\left(1-\sqrt{1-K\,(x^2+y^2+z^2)}\right)\gamma^3\partial_1\Psi$$

$$-xy\left(1-\sqrt{1-K\,(x^2+y^2+z^2)}\right)\gamma^1\partial_2\Psi + \left(y^2\sqrt{1-K\,(x^2+y^2+z^2)}+x^2+z^2\right)\gamma^2\partial_2\Psi$$

$$-yz\left(1-\sqrt{1-K\,(x^2+y^2+z^2)}\right)\gamma^3\partial_2\Psi$$

$$-xz\left(1-\sqrt{1-K\,(x^2+y^2+z^2)}\right)\gamma^1\partial_3\Psi - yz\left(1-\sqrt{1-K\,(x^2+y^2+z^2)}\right)\gamma^2\partial_3\Psi +$$

$$\left(z^2\sqrt{1-K\,(x^2+y^2+z^2)}+y^2+x^2\right)\gamma^3\partial_3\Psi$$

$$-\left(x-x\sqrt{1-K(x^2+y^2+z^2)}\right)\gamma^1\Psi - \left(y-y\sqrt{1-K(x^2+y^2+z^2)}\right)\gamma^2\Psi$$

$$-\left(z-z\sqrt{1-K(x^2+y^2+z^2)}\right)\gamma^3\Psi\Bigg]$$

The calculations lead to the following expression for the squared Dirac operator

$$\begin{aligned}\mathscr{D}^2 := \frac{1}{x^2+y^2+z^2} K\left(x^2+y^2+z^2\right)\Big(& x^2 f^{(0,2,0,0)}(t,x,y,z)+2\mathrm{xy} f^{(0,1,1,0)}(t,x,y,z)+2\mathrm{xz} f^{(0,1,0,1)}(t,x,y,z)\\ &+y^2 f^{(0,0,2,0)}(t,x,y,z)+2\mathrm{yz} f^{(0,0,1,1)}(t,x,y,z)+z^2 f^{(0,0,0,2)}(t,x,y,z)\\ &-\left(x^2+y^2+z^2\right)\left(f^{(0,0,0,2)}(t,x,y,z)+f^{(0,0,2,0)}(t,x,y,z)+f^{(0,2,0,0)}(t,x,y,z)\right)\end{aligned}$$

for the second order terms of the first component of Ψ, with the subsequent components reading as follows,

$$\begin{aligned}\frac{1}{x^2+y^2+z^2} K\left(x^2+y^2+z^2\right)\Big(& x^2 g^{(0,2,0,0)}(t,x,y,z)+2\mathrm{xy} g^{(0,1,1,0)}(t,x,y,z)+2\mathrm{xz} g^{(0,1,0,1)}(t,x,y,z)\\ &+y^2 g^{(0,0,2,0)}(t,x,y,z)+2\mathrm{yz} g^{(0,0,1,1)}(t,x,y,z)+z^2 g^{(0,0,0,2)}(t,x,y,z)\\ &-\left(x^2+y^2+z^2\right)\left(g^{(0,0,0,2)}(t,x,y,z)+g^{(0,0,2,0)}(t,x,y,z)+g^{(0,2,0,0)}(t,x,y,z)\right)\\ \frac{1}{x^2+y^2+z^2} K\left(x^2+y^2+z^2\right)\Big(& x^2 h^{(0,2,0,0)}(t,x,y,z)+2\mathrm{xy} h^{(0,1,1,0)}(t,x,y,z)+2\mathrm{xz} h^{(0,1,0,1)}(t,x,y,z)\\ &+y^2 h^{(0,0,2,0)}(t,x,y,z)+2\mathrm{yz} h^{(0,0,1,1)}(t,x,y,z)+z^2 h^{(0,0,0,2)}(t,x,y,z)\\ &-\left(x^2+y^2+z^2\right)\left(h^{(0,0,0,2)}(t,x,y,z)+h^{(0,0,2,0)}(t,x,y,z)+h^{(0,2,0,0)}(t,x,y,z)\right)\\ \frac{1}{x^2+y^2+z^2} K\left(x^2+y^2+z^2\right)\Big(& x^2 j^{(0,2,0,0)}(t,x,y,z)+2\mathrm{xy} j^{(0,1,1,0)}(t,x,y,z)+2\mathrm{xz} j^{(0,1,0,1)}(t,x,y,z)\\ &+y^2 j^{(0,0,2,0)}(t,x,y,z)+2\mathrm{yz} j^{(0,0,1,1)}(t,x,y,z)+z^2 j^{(0,0,0,2)}(t,x,y,z)\\ &-\left(x^2+y^2+z^2\right)\left(j^{(0,0,0,2)}(t,x,y,z)+j^{(0,0,2,0)}(t,x,y,z)+j^{(0,2,0,0)}(t,x,y,z)\right)\end{aligned}$$

where we have defined Ψ as follows

$$\Psi = \begin{pmatrix} f(t,x,y,z) \\ g(t,x,y,z) \\ h(t,x,y,z) \\ j(t,x,y,z) \end{pmatrix}$$

correspondingly, the term $f^{(0,2,0,0)}(t,x,y,z)$ denotes a second order derivative with respect to x, i.e. $\frac{\partial^2}{\partial x^2}(f(t,x,y,z)) \equiv f^{(0,2,0,0)}(t,x,y,z)$ and correspondingly for derivatives with respect to t, y and z. Then, for the first order terms of the squared operator we find the following,

$$\begin{aligned}
&\left(\frac{1}{x^2+y^2+z^2}\right) 3K\left(x^2+y^2+z^2\right)\left(zf^{(0,0,0,1)}(t,x,y,z)+yf^{(0,0,1,0)}(t,x,y,z)+\mathrm{xf}^{(0,1,0,0)}(t,x,y,z)\right)\\
&+\mathrm{iy}\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right) f^{(0,1,0,0)}(t,x,y,z)-ix\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right) f^{(0,0,1,0)}(t,x,y,z)\\
&+x\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right) g^{(0,0,0,1)}(t,x,y,z)-iy\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right) g^{(0,0,0,1)}(t,x,y,z)\\
&+iz\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right) g^{(0,0,1,0)}(t,x,y,z)-z\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right) g^{(0,1,0,0)}(t,x,y,z)
\end{aligned}$$

for the remaining components we have,

$$\begin{aligned}
&\left(\frac{1}{x^2+y^2+z^2}\right) 3K\left(x^2+y^2+z^2\right)\left(zg^{(0,0,0,1)}(t,x,y,z)+yg^{(0,0,1,0)}(t,x,y,z)+x\mathrm{g}^{(0,1,0,0)}(t,x,y,z)\right)\\
&-\mathrm{iy}\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right) g^{(0,1,0,0)}(t,x,y,z)+ix\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right) g^{(0,0,1,0)}(t,x,y,z)\\
&-x\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right) f^{(0,0,0,1)}(t,x,y,z)-iy\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right) f^{(0,0,0,1)}(t,x,y,z)\\
&-iz\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right) f^{(0,0,1,0)}(t,x,y,z)+z\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right) f^{(0,1,0,0)}(t,x,y,z)
\end{aligned}$$

and

$$\left(\frac{1}{x^2+y^2+z^2}\right)3K\left(x^2+y^2+z^2\right)\left(zh^{(0,0,0,1)}(t,x,y,z)+yh(0,0,1,0)(t,x,y,z)+h\mathrm{x}^{(0,1,0,0)}(t,x,y,z)\right)$$
$$+\mathrm{iy}\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)h^{(0,1,0,0)}(t,x,y,z)-ix\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)h^{(0,0,1,0)}(t,x,y,z)$$
$$+x\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)j^{(0,0,0,1)}(t,x,y,z)-iy\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)j^{(0,0,0,1)}(t,x,y,z)$$
$$+iz\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)j^{(0,0,1,0)}(t,x,y,z)-z\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)j^{(0,1,0,0)}(t,x,y,z)$$
$$\left(\frac{1}{x^2+y^2+z^2}\right)3K\left(x^2+y^2+z^2\right)\left(zj^{(0,0,0,1)}(t,x,y,z)+yj^{(0,0,1,0)}(t,x,y,z)+x\mathrm{j}^{(0,1,0,0)}(t,x,y,z)\right)$$
$$-\mathrm{iy}\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)j^{(0,1,0,0)}(t,x,y,z)+ix\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)j^{(0,0,1,0)}(t,x,y,z)$$
$$-x\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)h^{(0,0,0,1)}(t,x,y,z)-iy\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)h^{(0,0,0,1)}(t,x,y,z)$$
$$-iz\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)h^{(0,0,1,0)}(t,x,y,z)+z\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)h^{(0,1,0,0)}(t,x,y,z)$$

we point attention to the intertwining of components with in the first order terms of the operator. Thus, separation of said terms is warranted in order to write the operator in terms of single components, rather than the above intertwining. In addition we note that the this intertwining happens in pairs, such that the first two components of Ψ are entwined, as well as the last two components, hinting to the possibility of having two systems of differential equations rather than the possible four - reducing our assignment.

Finally, for the zeroth order terms of the operator we find

$$\left(\frac{1}{x^2+y^2+z^2}\right)\left(K\left(x^2+y^2+z^2\right)f(t,x,y,z)+\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)f(t,x,y,z)\right)$$
$$\left(\frac{1}{x^2+y^2+z^2}\right)\left(K\left(x^2+y^2+z^2\right)g(t,x,y,z)+\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)g(t,x,y,z)\right)$$
$$\left(\frac{1}{x^2+y^2+z^2}\right)\left(K\left(x^2+y^2+z^2\right)h(t,x,y,z)+\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)h(t,x,y,z)\right)$$
$$\left(\frac{1}{x^2+y^2+z^2}\right)\left(K\left(x^2+y^2+z^2\right)j(t,x,y,z)+\left(1-\sqrt{1-K\left(x^2+y^2+z^2\right)}\right)j(t,x,y,z)\right)$$

where all terms equivalent with the slight difference of the corresponding component of Ψ.

Thus, we are posed with a peculiar dilemma. We seek to redefine the aforementioned Dirac equation, in the considered geometry, in Klein-Gordon form in order to attain an analytic solution to the problem. Yet, we are posed with the difficulty of the above presented intertwining of components of Ψ in the first order terms of the operator. Hence, in order to remedy the situation we seek to explicitly write the operator squared in terms of the Dirac gamma matrices, in hope that using the properties of the γ-matrices we may be able to properly separate the operator and proceed to deriving a solution to the equation

$$\left(\mathscr{D}^2 - m^2\right)\Psi = 0$$

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BIOGRAPHICAL SKETCH

Jorge Andres Garcia studied for two years at South Texas College, McAllen, TX, where he attained an Associate's Degree in Biology. He later obtained a Bachelor's in Chemistry from the University of Texas - Rio Grande Valley with a minor in Pure Mathematics. He obtained a MS of Mathematics from the University of Texas - Rio Grande Valley, where he took peculiar interest to partial differential equations and differential geometry, with a concentration in quantum mechanics and quantum field theory, as well as gravity.