Reaction-diffusion systems with a nonlinear rate of growth

Yubing Wan
University of Texas-Pan American

REACTION-DIFFUSION SYSTEMS WITH A NONLINEAR RATE OF GROWTH

A Thesis

by

YUBING WAN

Submitted to the Graduate School of the
University of Texas-Pan American
In partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

August 2009

Major subject: Mathematics

REACTION-DIFFUSION SYSTEMS WITH A

NONLINEAR RATE OF GROWTH

A Thesis

by

YUBING WAN

Approved as to style and content by:

Zhaosheng Feng, ph.D.
Chair of Committee

Andras Balogh, Ph.D.
Committee Member

Paul Bracken, Ph.D.
Committee Member

Jasang Yoon, Ph.D.
Committee Member

August 2009

ABSTRACT

Wan, Yubing, Reaction-diffusion Systems With A Nonlinear Rate of Growth, Master of Science (MS), August 2009, 43 pages, 4 figures, references, 86 titles.

In the literature there are quite a few elegant approaches which have been proposed to find the first integrals of nonlinear differential equations. Recently, the modified Prelle–Singer method for finding the first integrals of second-order nonlinear ordinary differential equations (ODEs) has attracted considerable attention. Many researchers used this method to derive the first integrals to various systems. In this thesis, we are concerned with the first integrals for reaction-diffusion systems with a nonlinear rate of growth. Under certain parametric conditions we express the first integrals explicitly by applying an analytical method as well as the modified Prelle–Singer method.

ACKNOWLEDGEMENTS

Foremost, I would like to express my utmost gratitude to my supervisor Dr. Zhaosheng Feng for his continuous support of my Master study and research, for his patient guidance, enthusiastic motivation and tremendous help in making this work possible. I will always take pride in saying that I studied under, and worked with Dr. Zhaosheng Feng.

Besides my supervisor, I am sincerely grateful to the members of my advisory committee, Drs. Andras Balogh, Paul Bracken and Jasang Yoon, for their kind service, cordial encouragement and constructive comments on this thesis.

I would also like to thank the Department of Mathematics at the University of Texas – Pan American for providing me with a generous teaching assistantship, which greatly supported my study and research work.

I thank my friends Ruifang Li, Xianqi Li and Guangyue Gao for the inspiring discussions, stimulating study environment and all the fun we have had in the last two years.

Last but not the least, I would like to thank my family: my parents Jianchun Wan and Yanfeng Li, my lovely little sister Yufei Wan. They always give me continuous support and love. To them I dedicate this thesis.

TABLE OF CONTENTS

LIST OF FIGURES

Page

CHAPTER I

INTRODUCTION

The discovery and analysis of traveling waves and spreading speeds for nonlinear partial differential equations with spatial structure has a history which is almost one hundred years long. A solution $u(t,x)$ of an evolutionary system is said to be a traveling wave solution if $u(t,\,x) = U(x-ct)$ for some function U. Usually, U is called the wave profile, and c is called the wave speed. If, in addition, two limits $U(\pm\infty)$ exist, this solution is also called a traveling wavefront. Monostable and bistable nonlinearities frequently appear in spatially homogeneous systems.

Nowadays it has been universally acknowledged in the physical, chemical and biological communities that the reaction-diffusion equation plays an important role in dissipative dynamical systems [1–3]. Typical examples are provided by the fact that there are many phenomena in biology where a key element or precursor of a developmental process seems to be the appearance of a traveling wave of chemical concentration (or mechanical deformation). When reaction kinetics and diffusion are coupled, traveling waves of chemical concentration can effect a biochemical change much faster than straight diffusional processes. This usually gives rise to reaction-diffusion equations which in one dimensional space can look like

$$\frac{\partial u}{\partial t} = k_0 \frac{\partial^2 u}{\partial x^2} + f(u), \tag{1}$$

for a chemical concentration u, where k_0 is the diffusion coefficient, and $f(u)$ represents the kinetics.

When $f(u)$ is linear, in many instances equation (1) can be solved by the separation of variables methods. However if, as in many of the applications considered in [3], $f(u)$ is nonlinear, it is not usually possible to obtain general exact analytical traveling wave solutions and one must analyze such problems numerically [4]. Despite this, however, under some particular circumstances, many nonlinear evolutionary equations have traveling wave solutions of special types, which are of fundamental importance to our understanding of biological phenomena modeled evolutionary equations. The classic and simplest case of the nonlinear reaction-diffusion equation is when $f(u) = k_3u(1-u)$, which is the so-called Fisher equation. It was suggested by Fisher as a deterministic version of a stochastic model for the spatial spread of a favored gene in a population [5]. The discovery and analysis of traveling waves in chemical reactions was first presented by Luther at a conference [6-8]. In the 20th century, the Fisher equation has became the basis for a variety of models for spatial spread [9-11]. The qualitative analysis in the phase plane and traveling wave solutions have been widely investigated. The seminal and now classical references are that by Kolmogorov, Petrovsky and Piscunov [12], Albowitz and Zeppetella [13], Fife [14] and Britten [15]. The first explicit analytic form of a cline solution for the Fisher equation was obtained by Albowitz and Zeppetella making use of the Painlevé analysis [13]. A full discussion of this equation and an extensive bibliography can be seen in [16–20] and references therein.

Since these pioneering works, considerable research has been carried out in an attempt to extend the original results as well as some innovative methods to more complicated cases which arise in several fields. One typical example is that in ecology the first systematic treatment of dispersion models of biological populations assumed random movement [21], thus the classical models of population dispersion were derived based on this assumption in which the diffusion part appears as constant. However, much evidence shows that some species engage in non-random movement. In a general way, for the convenience of the study, we separate this phenomena into two types.

One case is spatial characteristics: some insects move in response to olfactory or visual stimuli. Obviously here the probability is not symmetric. To model this type of movement, McMurtrie [22] considered the case where attractive and repulsive forces are the cause of movement of one species. He assumed that both forces could be measured by a function which depends on position. Letting $\alpha(x)$ and $\beta(x)$ be the concentration at the point x of the attractive and repulsive substances respectively, the probability takes the form

$$p(x_1, x_2) - p(\alpha(x_1), \beta(x_2)).$$

In this case the population density, $u(x,t)$, satisfies the following diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[\Psi(x)\frac{\partial u}{\partial x} + \left\{\frac{\partial \Psi}{\partial \beta}\frac{d\beta}{dx} - \frac{\partial \Psi}{\partial \alpha}\frac{d\alpha}{dx}\right\} u\right],$$

where Ψ is the variance of the motion. (For full details see McMurtrie, [22]). If we include both space-dependent diffusion and non-linear rate of growth the model takes the general form

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[D(x)\frac{\partial u}{\partial x}\right] + g(u).$$

Shiguesada derived and studied a "logistic" model for a dispersing population in a heterogeneous environment in which they also included space variation in the intrinsic rate of growth. Their equation is

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[D(x)\frac{\partial u}{\partial x}\right] + u[\varepsilon(x) - \mu u],$$

where $\mu > 0$, and D and ε are periodic functions in space with period l.

The other case in our study is density-dependent characteristics: some species migrate from densely populated areas into sparsely populated areas to avoid crowding. Thus overcrowding increases population dispersion. Other species have social behavior such that the population only moves from one place to another until its density attains a certain value. Myers and Krebs [23] studied density-dependent dispersion as a regulatory mechanism of the cyclic changes in the density of some small rodents. In these cases, the probability that an animal moves from the point x_1 to x_2 depends on the density at x_1. Here the population density satisfies the following equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[D(u)\frac{\partial u}{\partial x}\right]. \tag{2}$$

The explanations of the derivation of (6) was described by McMurtrie in [22].

When both terms of density-dependent diffusion and non-linear rate of growth are present, one can have the general one-dimensional model for a species

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[D(u)\frac{\partial u}{\partial x}\right] + g(u). \tag{3}$$

It seems that the first model of this type was derived by Gurney and Nisbet [24] in an ecological context, who introduced a probabilistic method to construct the following three-dimensional model for a Malthusian rate of growth

$$\frac{\partial u}{\partial t} = D\nabla \cdot (u\nabla u) + ru, \tag{4}$$

where D and r are positive constants. Gurtin and MacCamy developed a continuum method and investigated a model

$$\frac{\partial u}{\partial t} = \nabla^2[\varphi(u)] + g(u), \tag{5}$$

where $\varphi'(0) = 0$, $\varphi'(u) > 0$ for $u > 0$, to describe the growth of one species, and their analysis was applied for a particular equation

$$\frac{\partial u}{\partial t} = \nabla^2(u^\alpha) + \mu u, \tag{6}$$

with $\alpha \geq 0$. In order to prove the existence of weak solutions, they transformed (6) into

$$\frac{\partial \omega}{\partial \tau} = \nabla^2(\omega^\alpha), \tag{7}$$

where $u = \omega e^{\mu t}$ and $\tau = \frac{e^{\mu(\alpha-1)t}-1}{\mu(\alpha-1)}$.

In equation (3), when the diffusion term $D(u)$ is a constant and the reactive part $g(u)$ is a polynomial of the variable u, this equation was usually regarded as a generalization of the Fisher equation, which is used as a density-dependent diffusion model, in the one-dimensional situation, for studying insect and animal dispersal with growth dynamics [3], and as a generic model arising from the classical theory of population genetics and combustion [19, 20]. Traveling wave solutions and exact solutions in this case have been investigated extensively. For example, when $g(u)$ is a quadratic or cubic polynomial, exact solutions were obtained by Clarkson and Mansfield [25, 26] using the nonclassical reductions method. When both $g(u)$ is a cubic polynomial with three distinct roots, exact solutions have also been presented by Chen and Guo using a truncated Painlevé expansion [27], by Chowdhury, Estévez and Gordoa [28, 29] using a complete Painlevé test, and by Feng and Chen using an analytical method [30]. Herrera and his co-workers [31] investigated a more general case for $g(u)$ and found travelling wave solutions in the phase space, yielding the value of speed. When the wave form can not be written in closed form, the stability of some solutions was studied numerically.

When the nonlinear diffusion term $D(u)$ is strictly positive in the interval $[0, 1]$, Haldeler [32, 33] gave sufficient conditions on the speed v for the existence of solution of front type satisfying $g(0) = 0$ and $g(1) = 0$. Sanchez-Garduno and Maini [34, 35] studied the existence of traveling wave solution when both the diffusion term $D(u)$ and the reactive part $g(u)$ are nonlinear and satisfy the given conditions. It was shown by Engler [36] how travelling wave solutions of $v_t = v_{xx} + f(v)$

and $u_t = (D(u)u_x)_x + g(u)$ are related to each other by a certain transformation. The study of properties of traveling waves and their applications were undertook by de Pablo and Sanchez [37]. When $D \geq 0$ is a constant, spectral analysis of traveling waves for a more general case of equation (2) was studied by Chen and Bates [38]. Under certain conditions on the nonlinear summand in the equation, there exists a travelling wave solution $\phi(x - ct)$ to equation (2) such that $\phi(\infty) = 1$. Moreover, when g is a function of u and t, and g is T-periodic with respect to t, more profound results such as periodic traveling wave solutions of the bistable reaction diffusion were established by Alikakos, Bates and Chen using the Schauder's fixed point theorem and the singular perturbation method [39]. For the more qualitative and numerical results for equation (3) with or without various certain conditions (including boundary conditions), we refer the readers to the literature [40–50].

In the past decades, several exact solutions of particular cases of equation (3) have been derived in the literature. For instance, Ablowitz and Zeppetella [13] obtained an exact traveling wave solution of Fisher's equation [5]

$$u_t = u_{xx} + u(1 - u), \tag{8}$$

by finding the special wave speed for which the resulting ODE is of the Painlevé type. Recently Guo and Chen [16] have used the Painlevé expansion method [51–53] to obtain some heteroclinic and homoclinic solutions of (8). Kaliappan [54] and Herrera, Minzoni and Ondarza [55] have derived traveling wave solutions of the generalized Fisher's equations

$$u_t = u_{xx} + u - u^k, \tag{9}$$

$$u_t = u_{xx} + u^p - u^{2p-1}, \tag{10}$$

respectively, Cariello and Tabor [56, 57] found an exact solution of the real Newell-Whitehead equation [58]:

$$u_t = u_{xx} + u(1 - u^2), \tag{11}$$

using a truncated Painlevé expansion and verified that it derives from a nonclassical symmetry reduction [55]. Several authors have studied exact solutions of the Fitzhugh-Nagumo equation

$$u_t = u_{xx} + u(1 - u)(u - a), \tag{12}$$

where a is an arbitrary parameter, which arises in population genetics [19, 20] and models the transmission of nerve impulses [59]. Traveling wave solutions of the Fitzhugh-Nagumo equation (12) have been studied by several authors [19, 20, 60, 61, 62]. Exact solutions of (12) have been obtained using various techniques including Vorob'ev [63] (who calls the associated symmetry a "partial symmetry of the first type"), by Kawahara and Tanaka [64], using Hirota's bilinear method [65]. Exact solutions of the Huxley equation,

$$u_t = u_{xx} + u^p(1-u), \tag{13}$$

have been obtained by Chen and Guo [66], using a truncated Painlevé expansion, and by Clarkson and Mansfield [67], using the nonclassical method. Exact solutions of

$$u_t = u_{xx} - u^3, \tag{14}$$

have been obtained by Clarkson and Mansfield [67], using the nonclassical method.

CHAPTER II

PRELLE-SINGER PROCEDURE

2.1 Prelle-Singer Procedure for First-Order ODEs

There are many methods in solving nonlinear differential equations, but most of them only works for a limited class. Despite its effectiveness in solving FOODEs, the Prelle-Singer (PS) procedure is not very well known outside mathematical circles. This is probably due to its non-standard approach, coupled with the fact that a computer is almost essential to realize its full efficiency. Hence we present a brief overview of the main ideas of the Prelle-Singer procedure [68–77].

Consider the autonomous system of ODEs [69]:

$$\dot{x} = Q(x,y), \qquad \dot{y} = P(x,y), \qquad P,\, Q \in \mathbb{C}[x,y],$$

where an overdot represents a derivative with respect to the independent variable t. This system is equivalent to the class of FOODEs which can be written as

$$y' = \frac{dy}{dx} = \frac{P(x,y)}{Q(x,y)}, \tag{15}$$

in other words those FOODEs which can be isolated in y', leaving a rational function of x and y on the right-hand side.

Prelle and Singer [68] showed that, if an elementary first integral of (15) exists, there exists an integrating factor R with $R^n \in \mathbb{C}[x,y]$ for some integer n, such that

$$\frac{\partial RQ}{\partial x} + \frac{\partial RP}{\partial y} = 0. \tag{16}$$

The key to the success of the PS procedure is that, given the particular form of the FOODE, we know the most general form that the integrating factor can take. We can then realize a computer-assisted exhaustive search for the correct integrating factor. With the integrating factor determined, the ODE can be solved by quadrature. From (16) we see that

$$Q\frac{\partial R}{\partial x} + R\frac{\partial Q}{\partial x} + P\frac{\partial R}{\partial y} + R\frac{\partial P}{\partial y} = 0. \tag{17}$$

Thus, defining the differential operator

$$D \equiv Q\frac{\partial}{\partial x} + P\frac{\partial}{\partial y}, \tag{18}$$

we have that

$$\frac{D[R]}{R} = -\left(\frac{\partial Q}{\partial x} + \frac{\partial P}{\partial y}\right). \tag{19}$$

Now let $R = \prod_i f_i^{n_i}$ where f_i are monic irreducible polynomials and n_i are non-zero rational numbers [69].

From (18) we have

$$\begin{aligned}\frac{D[R]}{R} &= \frac{D[\prod_i f_i^{n_i}]}{\prod_k f_k^{n_k}} = \frac{\sum_i f_i^{n_i-1} n_i D[f_i] \prod_{j\neq i} f_j^{n_j}}{\prod_k f_k^{n_k}}, \\ &= \sum_i \frac{f_i^{n_i-1} n_i D[f_i]}{f_i^{n_i}} = \sum_i \frac{n_i D[f_i]}{f_i}.\end{aligned} \tag{20}$$

From (16), plus the fact that P and Q are polynomials, we conclude that $D[R]/R$ is a polynomial. Therefore, from (20), we see that $f_i | D[f_i]$. Written in the form

$$D[f_i] = f_i g_i, \tag{21}$$

for some polynomial g_i, we see that the equation for the f_i has aspects similar to an eigenvalue equation, and for that reason f_i are sometimes called eigenpolynomials. However current usage seems to prefer the term Darboux polynomials, and we shall refer to the f_i as such in this paper. Given an upper bound, B, on the degree of the Darboux polynomials, f_i, we thus have a criterion for finding them. We can, for example, construct all possible polynomials of degree up to B with monic leading term and arbitrary complex coefficients, construct equation (21) and see if there are non-trivial solutions for the arbitrary coefficients. With this in mind the PS procedure works as follows [69]:

(1) Set the current degree bound, $N = 1$.

(2) Find all Darboux polynomials f_i such that deg $f_i \leq N$ and $f_i | D[f_i]$.

(3) Let $D[f_i] = f_i g_i$. If there exist constants n_i, not all zero, such that

$$\sum_{i=1}^{m} n_i g_i = 0, \tag{22}$$

then from (16) $D[R]/R = 0$ and the ODE is exact. The solution is $w = c$, where c is an arbitrary constant and $\prod_{i=1}^{m} f_i^{n_i}$.

If (22) has no solution then

(4) if there exist constants n_i, not all zero, such that

$$\sum_{i=1}^{m} n_i g_i = -\left(\frac{\partial Q}{\partial x} + \frac{\partial P}{\partial y}\right), \tag{23}$$

then return the solution $w = c$, where c is an arbitrary constant and w is either of

$$\int RPdx - \int \left(RQ + \frac{d}{dy}\int RPdx\right) dy,$$

or

$$-\int RQdy + \int \left(RP + \frac{d}{dx}\int RQdy\right) dx.$$

(5) Set $N = N + 1$. If $N > B$ then exit with no result. Else go to 2.

2.2 Prelle-Singer Procedure for Second-Order ODEs

In this section, we follow [68-77] to modify the techniques developed by Prelle and Singer and apply them to second-order ODEs (SOODEs) with the following rational form. This modified technique was also developed by Chandrasekar et al. [70-72].

Consider the second-order ODE:

$$y'' = \frac{P(x, y, y')}{Q(x, y, y')}, \qquad P, Q \in \mathbb{C}[x, y, y']. \tag{24}$$

We restrict ourselves for the time being to SOODEs which have elementary solutions, i.e. which can be written in the form

$$f(x, y) = 0,$$

where f is an arbitrary combination of exponentials, logarithms and polynomials in its arguments. Since we are working over a complex field, this includes standard trigonometric functions. Our goal is to find elementary first integrals of (24) when such elementary first integrals exist. We believe, given the conditions above, that these first integrals have a very particular form, described later, which permits us to construct a semi-decision procedure analogous to the PS method to find them. Once such a first integral is found, if y' can be isolated, then the PS method (or any other solution method for FOODEs) can then be applied to obtain the full solution.

In this section, in order to present our results in a straightforward way, we start our study by briefly reviewing the Prelle–Singer procedure for solving second-order ODEs developed by Duarte *et al.* [69] and Chandrasekar *et al.* [70-72].

Consider the second-order ODE of the rational form

$$\frac{d^2y}{dx^2} = \phi(x,\, y,\, y') = \frac{P(x,\, y,\, y')}{Q(x,\, y,\, y')}, \qquad P,\, Q \in \mathbb{C}[x,\, y,\, y'], \tag{25}$$

where y' denotes differentiation with respect to x, P and Q are polynomials in x, y and y' with coefficients in the complex field. Suppose that equation (25) admits a first integral $I(x,\, y,\, y') = C$, with C constant on the solutions, so we have the total differential

$$dI = I_x dx + I_y dy + I_{y'} dy' = 0, \tag{26}$$

where the subscript denotes partial differentiation with respect to the corresponding variable. On the solution, since $y'dx = dy$ and equation (25) is equivalent to $\frac{P}{Q}dx = dy'$, adding a null term $S(x,\, y,\, y')y'dx - S(x,\, y,\, y')dy$ to both sides yields

$$\left(\frac{P}{Q} + Sy'\right) dx - Sdy - dy' = 0. \tag{27}$$

Comparing (26) and (27), one can see that on the solutions, the corresponding coefficients of (26) and (27) should be proportional. There exists an integrating factor $R(x,\, y,\, y')$ for (27), such that on the solutions

$$dI = R(\phi + Sy')dx - SRdy - Rdy' = 0. \tag{28}$$

From (26) and (28), we have

$$I_x = R(\phi + Sy'), \quad I_y = -SR, \quad I_{y'} = -R, \tag{29}$$

and the compatibility conditions $I_{xy} = I_{yx}$, $I_{xy'} = I_{y'x}$ and $I_{yy'} = I_{y'y}$, which are equivalent to

$$D[S] = -\phi_y + S\phi_{y'} + S^2, \;\; D[R] = -R(S + \phi_{y'}), \;\; R_y = R_{y'}S + S_{y'}R, \tag{30}$$

where D is an differential operator

$$D = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + \phi\frac{\partial}{\partial y'}.$$

For the given expression of ϕ, one can solve the first equation of (30) for S. Substituting S into the second equation of (30) one can get an explicit form for R by solving it. Once a compatible solution R and S satisfying the extra constraint (the third equation of (30)) is derived, integrating (29), from (26) one may obtain a first integral of motion as follows [69-72]:

$$\begin{aligned} I(x, y, y') &= \int R(\phi + Sy')dx - \int \left[RS + \frac{\partial}{\partial y}\int R(\phi + Sy')dx\right] dy \\ &- \int \left\{R + \frac{\partial}{\partial y'}\left(\int R(\phi + Sy')dx - \int \left[RS + \frac{\partial}{\partial y}\int R(\phi + Sy')dx\right] dy\right)\right\} dy'. \end{aligned}$$

2.3 Prelle-Singer Procedure for n^{th}-Order ODEs

In [72], Chandrasekar et al. generalized the Prelle-Singer Procedure to a class of n^{th}-Order ODEs of the form:

$$y^{(n)} = \frac{P}{Q}, \quad P, Q \in C[x, y, y^{(1)}, y^{(2)}, ..., y^{(n-1)}], \tag{31}$$

where$y^{(1)} = \frac{dy}{dx}$, $y^{(2)} = \frac{d^2y}{dx^2}$ and $y^{(n)} = \frac{d^ny}{dx^n}$ and P and Q are polynomials in $x, y, y^{(1)}, y^{(2)}, ..., y^{(n-2)}$ and $y^{(n-1)}$ with coefficient in field of complex numbers. Let us assume that the ODE (31) admits a first integral $I(x, y, y^{(1)}, y^{(2)}, ..., y^{(n-1)}) = C$, with C constant on the solutions, so that the total differential gives

$$dI = I_x dx + I_y dy + I_{y^{(1)}} dy^{(1)} + ... + I_{y^{(n-1)}} dy^{(n-1)} = 0, \tag{32}$$

where subscript denotes partial differentiation with respect to that variable. Rewriting equation (31) of the form $\frac{P}{Q}dx - dy^{(n-1)} = 0$ and adding null terms

$$S_1(x, y, y^{(1)}, y^{(2)}, ..., y^{(n-1)})y^{(1)}dx - S_1(x, y, y^{(1)}, y^{(2)}, ..., y^{(n-1)})dy$$

and

$$S_i(x, y, y^{(1)}, y^{(2)}, ..., y^{(n-1)})y^{(i)}dx - S_i(x, y, y^{(1)}, y^{(2)}, ..., y^{(n-1)})dy^{(i-1)}, \quad i = 2, 3, ...n - 1,$$

to it we obtain that on the solutions the 1-form

$$\left(\frac{P}{Q} + \sum_{i=1}^{n-1} S_i y^{(i)}\right) dx - S_1 dy - \sum_{i=2}^{n-1} S_i dy^{(i-1)} - dy^{(n-1)} = 0. \tag{33}$$

So the basic idea is similar to the case of the second-order ODE that, on the solutions, the 1-forms (32) and (33) must be proportional. Multiplying (33) by the factor $R(x, y, y^{(1)}, ..., y^{(n-1)})$ which acts as the integrating factor for (33), we have on the solutions that

$$dI = R\left[\left(\phi + \sum_{i=1}^{n-1} S_i y^{(i)}\right) dx - S_1 dy - \sum_{i=2}^{n-1} S_i dy^{(i-1)} - dy^{(n-1)}\right] = 0, \tag{34}$$

where $\phi \equiv \frac{P}{Q}$. Comparing equations (32) with (34) we have, on the solutions, the relations

$$\begin{aligned} I_x &= R\left(\phi + \sum_{i=1}^{n-1} S_i y^{(i)}\right), \\ I_y &= -RS_1, \\ I_{y^{(i)}} &= -RS_{i+1}, \quad i = 1, 2, ..., n - 2, \\ I_{y^{(n-1)}} &= -R. \end{aligned} \tag{35}$$

Making use of the compatibility conditions, $I_{xy} = I_{yx}$, $I_{xy^{(n-1)}} = I_{y^{(n-1)}x}$, $I_{yy^{(n-1)}} = I_{y^{(n-1)}y}$, $I_{xy^{(i)}} = I_{y^{(i)}x}$, $I_{yy^{(i)}} = I_{y^{(i)}y}$, $I_{y^{(i)}y^{(n-1)}} = I_{y^{(n-1)}y^{(i)}}$, $i = 2, 3, ..., n - 2$, between the equations (35), gives us the following system

$$D[S_1] = -\phi_y + S_1\phi_{y^{(n-1)}} + S_1 S_{n-1}, \tag{36}$$

$$D[S_i] = -\phi_{y^{(i-1)}} + S_i\phi_{y^{(n-1)}} + S_i S_{i+1} - S_{i-1}, \; i = 2, 3, ...n - 2, \tag{37}$$

$$D[S_{n-1}] = -\phi_{y^{(n-2)}} + S_{n-1}\phi_{y^{(n-1)}} + S_{n-1}^2 - S_{n-2}, \tag{38}$$

$$D[R] = -R(S_{n-1} + \phi_{y^{(n-1)}}), \tag{39}$$

$$R_{y^{(i)}}S_1 = -RS_{1y^i} + R_y S_{i+1} + RS_{i+1y}, \ i = 1, 2, ..., n-2, \tag{40}$$

$$R_{y^{(i)}}S_{j+1} = -RS_{j+1y^{(i)}} + R_{y^j}S_{i+1} + RS_{i+1y^j}, \ i, j = 1, 2, ..., n-2, \tag{41}$$

$$R_{y^{(i)}} = R_{y^{(n-1)}}S_{i+1} + RS_{i+1y^{(n-1)}}, \ i = 1, 2, ..., n-2, \tag{42}$$

$$R_y = R_{y^{(n-1)}}S_1 + RS_{1y^{(n-1)}}, \tag{43}$$

where the total differential operator D is defined by $D = \frac{\partial}{\partial x} + y^{(1)}\frac{\partial}{\partial y} + \sum_{i=2}^{n} y^{(i)}\frac{\partial}{\partial y^{(i-1)}}$.

It is easy to see that (36)-(43) form an over determined system for the unknowns, S_i, $i = 1, 2, ..., n-1$ and R. For example, one can check that for a second order ODE, $n = 2$, one gets three equations for two unknowns, say, S and R, whereas for a third order ODE one gets six equations for three unknowns, say, S_1, S_2 and R. Thus in this procedure, for a given n^{th} order ODE, one gets $\frac{n(n+1)}{2}$ number of equations, for n unknowns, out of which $\frac{n(n-1)}{2}$ equations are just extra constraints.

As stated by Chandrasekar et al. in [72], the crux of the problem lies in solving the determining equations and identifying sufficient number of integrating factors and null forms. But the point is that any particular solution will suffice for the purpose. We solve equations (36)-(43) in the following way. Substituting the expression for $\phi = \frac{P}{Q}$, obtained from equation (31) into (36)-(38) we get a system of differential equations for the unknowns S_i, $i = 1, 2, ..., n-1$. Solving them we can obtain expressions for the null forms S_i's. Once S_i's are known then equation (39) becomes the determining equation for the function R. Solving the latter we can get an explicit form for R. Now the functions R and S_i, $i = 1, 2, ..., n-1$, have to satisfy the extra constraints (40)-(43). However, once a compatible solution satisfying all the equations have been found then the functions R and S_i, $i = 1, 2, ..., n-1$, fix the first integral $I[x, y, y^{(1)}, ..., y^{(n-1)}]$ by the relation [72]

$$I[x, y, y^{(1)}, ..., y^{(n-1)}] = \sum_{i=1}^{n} r_i - \int \left[R + \frac{d}{dy^{(n-1)}}\left(\sum_{i=1}^{n} r_i\right)\right] dy^{(n-1)}. \tag{44}$$

where

$$
\begin{aligned}
r_1 &= -\int R\left(\phi + \sum_{i=1}^{n-1} S_i y^i\right) dx,\\
r_2 &= \int \left(RS_1 + \frac{d}{dy} r_1\right) dy,\\
r_j &= \int \left[RS_{j-1} + \frac{d}{dy^{(j-i)}}\left(\sum_{k=1}^{j-1} r_k\right)\right] dy^{(j-1)},\ j = 3, ..., n.
\end{aligned}
\tag{45}
$$

Equation (44) can be derived straightforwardly by integrating the equations (35). Now substituting the expressions for ϕ, R and S_i, $i = 1, 2, ..., n$, into (44) and evaluating the integrals one can get the associated integrals of motion. However, we have to point out that we have not examined the question of existence of consistent solutions to equations (36)-(43) at present.

CHAPTER III

EXAMPLES

3.1 Example 1 (An Exact Solution in General Relativity) [69]

A rich source of nonlinear ODEs in physics are the highly nonlinear equations of general relativity. Einstein's equations are, of course, in general, partial ODEs, but there exist classes of space-times where the symmetry imposed reduces these equations to ODEs in one independent variable. One such class is that of static, spherically symmetric space-times, which depend only on the radial variable, r. The metric for a general static, spherically symmetric space-time has two free functions $\lambda(r)$ and $\mu(r)$ say. On imposing the condition that the matter in the space-time is a perfect fluid, Einstein's equations reduce to two coupled ODEs for $\lambda(r)$ and $\mu(r)$. Specifying one of these functions reduces the problem to solving an ODE (of first or second order) for the other.

Following this procedure, Buchdahl obtained an exact solution for a relativistic fluid sphere by considering the so-called isotropic metric [69]

$$ds^2 = (1-f)^2(1+f)^{-2}dt^2 - (1+f)^4[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)],$$

with $f = f(r)$. The field equations for $f(r)$ reduce to

$$ff'' - 3f'^2 - r^{-1}ff' = 0.$$

Changing to the notation of this paper, with $y(x) = f(r)$, equation (30) assume the form

$$\begin{aligned}
&S_x + y'S_y + \frac{y'(3xy'+y)}{xy}S_{y'} = \frac{3y'^2}{y^2} + S\frac{6xy'+y}{xy} + S^2,\\
&R_x + y'R_y + \frac{y'(3xy'+y)}{xy}R_{y'} = -RS - R\frac{6xy'+y}{xy},\\
&R_y - R_{y'}S - S_{y'}R = 0.
\end{aligned} \tag{46}$$

Testing for rational solutions of (30) with $|N| = 1$ we find $S = -3y'/y$, which leads to $R =$

$1/xy^3$. Substituting into (29) we obtain the reduced FOODE [69]

$$C = y'/xy^3.$$

3.2 Example 2 (A Static Gaseous General-relativistic Fluid Sphere) [69]

In a later paper, Buchdahl approaches the problem of the general-relativistic fluid sphere using a different coordinate system from the previous example. For ease in comparison of the originals, we substitute Buchdahl's $\xi(r)$ by $y(x)$. Einstein's field equations then lead to

$$y'' = \frac{x^2y'^2 + y^2 - 1}{x^2y}. \tag{47}$$

For this SOODE (30) become

$$\begin{aligned} &S_x + y'S_y + \frac{x^2y'^2 + y^2 - 1}{x^2y}S_{y'} = \frac{x^2y'^2 - y^2 - 1}{x^2y} + 2\frac{y'}{y}S + S^2, \\ &R_x + y'R_y + \frac{x^2y'^2 + y^2 - 1}{x^2y}R_{y'} = -RS - 2R\frac{y'}{y}, \\ &R_y - R_{y'}S - S_{y'}R = 0. \end{aligned} \tag{48}$$

This time a solution is found with $N = 4$, namely

$$S = \frac{-x^2y'^2 - xyy' + 1}{xy^2 + x^2yy'}, \quad R = \frac{y + xy'}{xy^2}. \tag{49}$$

Substituting in (29), we get the reduced FOODE [69]

$$C = \frac{2xyy' + y^2 + x^2y'^2 - 1}{2x^2y^2}. \tag{50}$$

It is perhaps worth nothing that our method manages to find this first integral while more established ODE solvers which also look for first integrals (for example the solver in Maple V.5 which is the most recent to which have access) do not.

Assume that S or R is of the particular rational form, for a class of problems, solving the first two equations of (30) usually gives two independent solution sets, (S_i, R_i), $i = 1, 2$. If the set (S_1, R_1) satisfies the third equation of (30) and the other set (S_2, R_2) does not satisfy the third equation of (30), one can use the first set and the formula I to find a first integral for equation (25).

For some examples, one may make the set (S_2, R_2) compatible by seeking a nonzero function $F(x, y, y')$, such that $\bar{R}_2 = F(x, y, y')R_2$ satisfies the second equation of (30) and the new set $(S_2, \bar{R}_2)$ satisfies the third equation of (30). Thus, this new set $(S_2, \bar{R}_2)$ may be used to construct the second integral of equation (25).

However, this procedure only works for a limited class of equations. We have applied this procedure to tackle some second-order nonlinear ODEs, such as the Liénard-type equations and van der Pol-type nonlinear oscillators, and find that the assumption of the rational form for S or R does not work well and one can not avoid complicated calculations. For instance, Chandrasekar *et al.* applied this procedure to the Helmholtz oscillator with friction

$$u''(t) - ru'(t) - au^2(t) - bu(t) = 0, \tag{51}$$

where r, a and b are arbitrary parameters, which is a simple nonlinear oscillator with quadratic nonlinearity. They assumed that S is of the rational form $S = \frac{A(t,u)+B(t,u)u'}{C(t,u)+D(t,u)u'}$, where A, B, C and D are functions of t and u to be determined. Substituting S into the first equation of (30) and multiplying both sides by the least common denominator, one can obtain a polynomial of third degree in $[\dot{u}]$ which is zero if and only if the corresponding coefficient of each monomial is equal to zero:

$$\begin{aligned}
[\dot{u}^0] &: \ A_tC - AC_t + bBCu - bADu + aBCu^2 - aADu^2 \\
&\quad = A^2 - bC^2 + rAC - 2aC^2u, \\
[\dot{u}^1] &: \ A_tD - AD_t + B_tC - BC_t + A_uC - AC_u + rBC - rAD \\
&\quad = 2AB - 2bCD - 4aCDu + rAD + rBC, \\
[\dot{u}^2] &: \ B_tD - BD_t + A_uD + B_uC - AD_u - BC_u = -bD^2 + B^2 - 2aD^2u + rBD, \\
[\dot{u}^3] &: \ B_uD - BD_u = 0.
\end{aligned}$$

Solving this system, Chandrasekar *et al.* [70] claimed that only under the parametric condition $b = -\frac{6r^2}{25}$, equation (51) has a first integral of the explicit form

$$I_1 = e^{(-6rt/5)}\left[(u')^2 - \frac{4ruu'}{5} + \frac{4r^2u^2}{25} - \frac{2au^3}{3}\right]. \tag{52}$$

The same result was also obtained by Almendral and Sanjuán when they studied the invariance and integrability properties of the Helmholtz oscillator (51).

We note that the developed technique can be used to solve a class of nonlinear second-order ODEs effectively. Here we wish to share some comments on the first integrals of the Helmholtz oscillator and the Duffing–van der Pol oscillator presented in the literature:

1. For the Helmholtz oscillator (51), the solution set (S_2, R_2) presented in ([70]pp. 2464) is incorrect. It is not difficult to check that it does not satisfy equation (4.46) in [70]. As a consequence, the formula (4.53) provides an implicit first integral for the Helmholtz oscillator (51).

2. We note that in [70], pp. 2464) authors stated that "non-trivial solutions only exist for equations (4.45) and (4.46) for the parametric restriction "$c_2 = \frac{6c_1^2}{25}$". With the aid of Mathematica, we find that under the parametric constraint $c_2 = -\frac{6c_1^2}{25}$ there is another nontrivial solution (S_1', R_1') as

$$S_1' = \frac{\frac{2c_1\dot{u}}{5} - \beta u^2 - \frac{8c_1^2 u}{25} - \frac{12c_1^4}{625\beta}}{\dot{u} + \frac{2c_1 u}{5} + \frac{12c_1^3}{125\beta}}, \quad R_1' = -\left(\dot{u} + \frac{2c_1 u}{5} + \frac{12c_1^3}{125\beta}\right) e^{\frac{6c_1 t}{5}},$$

which leads to a new first integral of the Helmholtz oscillator as follows

$$I_2 = \left[\dot{u}^2 + \frac{4c_1}{5}u\dot{u} + \frac{24c_1^3}{125\beta}\dot{u} - \frac{2\beta}{3}u^3 - \frac{8c_1^2}{25}u^2 - \frac{24c_1^4}{625\beta}u\right] e^{\frac{6c_1 t}{5}}. \tag{53}$$

As far as our knowledge goes, this first integral (53) was not presented in the literature [70-72].

3.4 Example 4 (Force-free Duffing-van der Pol Oscillator) [70, 78, 79]

One of the well-studied but still challenging equations in nonlinear dynamics is the Duffing-van der Pol oscillator equation. Its autonomous version (force-free) is

$$\ddot{x} + (\alpha + \beta x^2)\dot{x} - \gamma x + x^3 = 0, \tag{54}$$

where an over-dot denotes differentiation with respect to time and α, β and γ are arbitrary parameters. Equation (54) arises in a model describing the propagation of voltage pulses along a neuronal axon and has recently received much attention from many authors. A vast amount of literature exists on this equation; for details see, for example, Lakshmanan, Rajasekar (2003) and references there in. In this case we have

$$
\begin{aligned}
&S_t + \dot{x}S_x - ((\alpha + \beta x^2)\dot{x} - \gamma x + x^3)S_{\dot{x}} = (2\beta x\dot{x} - \gamma + 3x^2) - (\alpha + \beta x^2)S + S^2,\\
&R_t + \dot{x}R_x + ((\alpha + \beta x^2)\dot{x} - \gamma x + x^3)R_{\dot{x}} = (\alpha + \beta x^2 - S)R,\\
&R_x = SR_{\dot{x}} + RS_{\dot{x}}.
\end{aligned}
\tag{55}
$$

To solve equations (55) we seek an ansatz for S and R of the form

$$
S = \frac{a(t,x) + b(t,x)\dot{x}}{c(t,x) + d(t,x)\dot{x}}, \quad R = A(t,x) + B(t,x)\dot{x}. \tag{56}
$$

From the first equation of (55) we get an equation system

$$
\begin{aligned}
[\dot{x}^1]:\ & b_t c + a_t d - bc_t + a_x c - ac_x - \alpha bc + ad\alpha - \beta bcx^2 + ad\beta x^2\\
&= 2\beta c_2 x - 2\gamma cd + 6cdx_2 - \alpha bc - ad\alpha\\
&-b\beta cx^2 - a\beta dx_2 + 2ab,\\
[\dot{x}^2]:\ & b_t d - bd_t + a_x d + b_x c - ad_x - bc_x\\
&= 4\beta cdx - d^2\gamma + 3d^2x^2 - \alpha bd - bd\beta x^2 + b^2,\\
[\dot{x}^3]:\ & 2\beta d^2 x = 0,\\
[\dot{x}^0]:\ & bc - ad = 0.
\end{aligned}
$$

From the second equation of (55) we get an equation system

$$
\begin{aligned}
[\dot{x}^0]:\ & A_c - \gamma Bcx + \dot{B}cx^3 = Ac\alpha + Ac\beta x^2 - Aa,\\
[\dot{x}^1]:\ & B_t c - A_x c + \alpha Bc + \beta x^2 Bc + A_t d + Bd - \gamma Bdx + Bx^3\\
&= \alpha Bc + Bc\beta x^2 + Ad\alpha + A\beta dx^2 + B\beta x^2 d - Ab - aB,\\
[\dot{x}^2]:\ & B_x c + B_t d + A_x d + B_d + \alpha Bd + \beta x^2 Bd = \alpha Bd - Bb.
\end{aligned}
$$

We find that a nontrivial solution exists under the parametric condition $\alpha = \frac{3}{\beta} - \frac{\gamma\beta}{3}$, and the corresponding forms of S and R reads

$$S = -\frac{\gamma\beta}{3} + \beta x^2, \quad R = e^{\frac{3t}{\beta}}. \tag{57}$$

So using (57) we obtain the first integral of the Duffing-van der Pol equation (54) as

$$\dot{x} + (\alpha - \frac{3}{\beta})x + \frac{\beta}{3}x^3 = Ie^{-\frac{3t}{\beta}}. \tag{58}$$

In works [78] and [79], SenthilVelan and Chandrasekar et al. studied the first integral for equation (54) by using the Lie symmetry method and a nonlinear transformation. In [70], They reproduced the same result by using the Prelle–Singer procedure. In [79, pp.1936], [78, pp.4528], and [70. pp. 2467], they claimed that the non-trivial first integral of equation (54) exists only for the parametric choice:

$$\alpha = \frac{4}{\beta}, \quad \gamma = -\frac{3}{\beta^2}. \tag{59}$$

However, it is remarkable that the new parametric constraint $\alpha = \frac{3}{\beta} - \frac{\gamma\beta}{3}$ is weaker than (59). It is easy to see that the first integral of equation (54) presented in the literature [70, 78, 79] is just a particular case of (58) where $\alpha = 4/\beta$.

CHAPTER IV

LINEARIZATION OF SECOND-ORDER NONLINEAR ODEs

4.1 Generalized Linearizing Transformation (GLT)

Nowadays, linearizing nonlinear differential equations and finding exact solutions is an important subject in the qualitative theory of differential equations [80–81]. For a given nonlinear ODE, it is very natural to ask whether it is linearizable or not, so far to the best of our knowledge, no definitive answer can be given in the general case. So study should be made based on case-by-case.

In the past decades, in a series of research works [80–86], a technique of nonlinear transformations for SNODES has been introduced. In this chapter we follow their technical ideas of making an attempt to unify the linearizing transformations known for the case of SNODES and extending their scope, and apply this technique to a more general nonlinear diffusion-reaction case. In the literature there are two powerful transformations which enable us to linearize the SNODES: one is the so called Invertible Point Transformation (IPT), and the other is the so called Non-point Transformation (NPT).

To make the paper sufficiently self-contained and state our main result in a straightforward way, let us recall the above two transformations described in [80–86] with several examples which help us better understand them. The more description of the IPT can be seen in the literature [82-84] that a class of SNODES that can be linearized through the following transformation,

$$X = F(t,x), \quad T = G(t,x), \tag{60}$$

usually takes the form

$$\ddot{x} = D(t,x)\dot{x}^3 + C(t,x)\dot{x}^2 + B(t,x)\dot{x} + A(t,x), \tag{61}$$

where over dot denotes differentiation with respect to t. When the functions $A(t,x)$, $B(t,x)$,

$C(t,x)$ and $D(t,x)$ satisfy the following system:

$$\begin{aligned} &3D_{tt} + 3BD_t - 3AD_x + 3DB_t + B_{xx} - 6DA_x + CB_x - 2CC_t - 2C_{tx} = 0, \\ &C_{tt} + 6AD_t - 3AC_x + 3DA_t - 2B_{tx} - 3CA_x + 3A_{xx} + 2BB_x - BC_t = 0, \end{aligned} \tag{62}$$

the transformation (60) transforms equation (61) into the second-order linear equation,

$$\frac{d^2X}{dT^2} = 0. \tag{63}$$

For another class of SNODES, one may consider NPT which takes the form

$$X = \hat{F}(t,x), \quad dT = \hat{G}(t,x)dt. \tag{64}$$

The detailed description can be seen in [85, 86]. We may assume that the SNODES which can be linearized through the transformation (64) has the following form

$$\ddot{x} + A_2(t,x)\dot{x}^2 + A_1(t,x)\dot{x} + A_0(t,x) = 0. \tag{65}$$

Here the functions $A_2(t,x)$, $A_1(t,x)$, $A_0(t,x)$ and the transformation (64) should satisfy

$$\begin{cases} A_2 = (\hat{G}\hat{F}_{xx} - \hat{F}_x\hat{G}_x)/K, \\ A_1 = (2\hat{G}\hat{F}_{xt} - \hat{F}_x\hat{G}_t - \hat{F}_t\hat{G}_x)/K, \\ A_0 = (\hat{G}\hat{F}_{tt} - \hat{F}_t\hat{G}_t)/K, \end{cases} \tag{66}$$

where $K = \hat{F}_x\hat{G} \neq 0$. Using the NPT one can also convert equation (65) to the second-order linear equation (63), and obtain the following relations directly:

(i)

$$\begin{cases} S_1(t,x) = A_{1x} - 2A_{2t} = 0, \\ S_2(t,x) = 2A_{0xx} - 2A_{1tx} + 2A_0A_{2x} - A_{1x}A_1 + 2A_{0x}A_2 + 2A_{2tt} = 0. \end{cases} \tag{67}$$

(ii) When $S_1(t,x) \neq 0$ and $S_2(t,x) \neq 0$, then we have

$$\begin{cases} S_1^2 + 2S_{1t}S_2 - 2S_1^2 A_{1t} + 4S_1^2 A_{0x} + 4S_1^2 A_0 A_2 - 2S_1 S_{2t} - S_1^2 A_1^2 = 0, \\ S_{1x}S_2 + S_1^2 A_{1x} - 2S_1^2 A_{2t} - S_1 S_{2x} = 0. \end{cases} \tag{68}$$

As stated in [84], even though both the IPT and NPT transform the second-order nonlinear ODEs to the linear equation (63), the NPT has some disadvantages over the former. For example, in the case of IPT one can unambiguously invert the free particle solution and deduce the solution of the associated nonlinear equation, whereas in the case of NPT it is not so straightforward due to the non-local nature of the independent variable.

In [84], Chandrasekar and his co-authors developed a more general transformation which can be utilized to linearize a wider class of SNODEs as follows

$$X = F(t,x), \quad dT = G_1(t,x,\dot{x})dt. \tag{69}$$

This transformation is called the generalized linearizing transformation (GLT). One of advantages of GLT lies in a fact that it sometimes enables us to convert a class of nonlinear equations into the free particle equation, which cannot be linearized by the NPT and IPT. Note that when the function G in (69) is independent of the variable $\dot{x}$ then it becomes an NPT. When we choose a perfect differentiable function as G, then it becomes an IPT, that is $G(t,x,\dot{x}) = \frac{d}{dt}\hat{G}(t,x)$, then $dT = \frac{d\hat{G}}{dt}dt \implies T = \hat{G}(t,x)$. So one can see that (69) is a combined transformation which includes IPT and NPT as its two particular cases, respectively.

In our applications, we only consider the case where G is a polynomial function in $\dot{x}$, particularly where it is linear in $\dot{x}$ with coefficients which are arbitrary functions of t and x. For such a simple case we also can see some interesting results. For example, we consider the case

$$X = F(t,x), \quad dT = (G_1(t,x)\dot{x} + G_2(t,x))dt. \tag{70}$$

Substituting the above transformation (70) into the second-order linear equation (63), one can derive that the most general SODEs that can be linearized through the GLT (70) are in the form

$$\ddot{x} + A_3(t,x)\dot{x}^3 + A_2(t,x)\dot{x}^2 + A_1(t,x)\dot{x} + A_0(t,x) = 0. \tag{71}$$

where over dot denotes differentiation with respect to t and the functions $A_i(t,x)'s$ $(i = 0, 1, 2, 3)$, and the transformation functions F and G satisfy the following relations:

$$\begin{cases} A_3 = (G_1 F_{xx} - F_x G_{1x})/M, \\ A_2 = (G_2 F_{xx} + 2G_1 F_{xx} - F_x G_{2x} - F_t G_{1x} - F_x G_{1t})/M, \\ A_1 = (2G_2 F_{xt} + G_1 F_{tt} - F_x G_{2t} - F_t G_{2x} - F_t G_{1t})/M, \\ A_0 = (G_2 F_{tt} - F_t G_{2t})/M, \end{cases} \tag{72}$$

where $M = F_x G_2 - F_t G_1 \neq 0$.

When the nonlinear equation (71) is given, i.e. one knows explicit forms of the functions $A_i(t,x)'s$ $(i = 0, 1, 2, 3)$, one can solve equation (72) for the linearizing transformation functions F, G_1 and G_2. Once F, G_1 and G_2 are found, one can plug them into (70) and use it to transform (71) to the second-order linear equation (63). Then through solving the latter one can get the associated first integral. In many cases, however, it is still difficult to integrate it further unambiguously to obtain the general solution due to the nonlocal nature of the transformation (70). Fortunately, in some cases, we are able to overcome this problem by performing the following procedure [84]:

Integrating the linear equation (63) once yields

$$\frac{dX}{dT} = I_1 = C(t, x, \dot{x}), \tag{73}$$

where I_1 is the first integral. Rewriting (73) for $\dot{x}$ gives

$$\dot{x} = f(t, T, I_1), \tag{74}$$

where f is a function of the indicated variable. Due to non-local nature of the independent variable, we need to consider only a particular solution for the second-order linear equation (63), that is

$$X(t, x) = I_1 T, \tag{75}$$

from which we get

$$x = g(t, T, I_1). \tag{76}$$

where g is a function of t, T and I_1. By virtue of relations (74) and (76), equation (70) can be re-expressed as

$$dT = h(t, T, I_1)dt, \tag{77}$$

here h is a function of t, T and I_1. Usually it is found that in many cases of linearizable equations one can separate the variables T and t in equation (77), and obtain the general solution by integrating the resultant equation.

The above roughly gives us the general idea for finding linearizing transformation and the general solution for the given equation. In this chapter we will focus on a particular but important case of equation (71), namely, $A_3 = A_2 = 0$ in equation (72). However, the other choices, like $A_3 = A_1 = 0$ and $A_2 \neq 0$, $A_0 \neq 0$, can also lead to many new linearizable equations, which should be handled separately. Solving the first and second equation in equation (72) with this restriction, we obtain

$$G_1 = a(t)F_x, \quad G_2 = a(t)F_t - (a_t x + b(t))F_x, \tag{78}$$

where $a(t)$ and $b(t)$ are arbitrary functions of t. Combining equation (78) and the last two equations of (72) gives

$$A_1 = S_x + \frac{a_t}{(a_t x + b)}S + \frac{a_{tt}x + b_t}{(a_t x + b)}, \tag{79}$$

$$A_0 = S_t + \frac{a_t}{(a_t x + b)}S^2 + \frac{a_{tt}x + b_t}{(a_t x + b)}S, \tag{80}$$

where

$$S(t, x) = \frac{F_t}{F_x}. \tag{81}$$

Solving equation (79) yields

$$S = \frac{(c(t) - b_t x - \frac{1}{2}a_{tt}x^2 + \int A_1(a_t x + b)dx)}{a_t x + b}, \tag{82}$$

where $c(t)$ is an arbitrary function of t. From (80) and (82), we deduce [84]

$$\begin{aligned} A_0 \;=\; & \frac{a_t(c - b_t x - \frac{1}{2}a_{tt}x^2 + \int A_1(a_t x + b)dx)^2}{(a_t x + b)^3} + \\ & \frac{c_t - b_{tt}x - \frac{1}{2}a_{ttt}x^2 + \int A_1(a_{tt}x + b_t) + (a_t x + b)A_{1t}dx}{(a_t x + b)}. \end{aligned} \tag{83}$$

Consequently, one can find the explicit form of F by substituting the expression for S into (81) and solving the associated first-order partial differential equation for F. Once F is known, one can derive G_1 and G_2 from the relation (78) which in turn provides us the GLT through (70). The corresponding linearizable equation assumes the form $\ddot{x} + A_1(t,x)\dot{x} + A_0(t,x) = 0$, where A_0 is given in equation (83) and A_1 is the given function in this analysis.

4.2 Two Examples by GLT

In this section, we will re-illustrate two simple but non-trivial examples from [84-86] to show how effective and feasible GLT is.

Example 1.

Firstly consider the case $A_1 = kx^q$, where k and q are arbitrary parameters, and fix the arbitrary functions a, b and c such as $a(t) = t,\ b(t) = c(t) = 0$, so that the equation (82) gives us:

$$S = \frac{k}{q+2}x^{q+1}. \tag{84}$$

Once S is known, F and A_0 can be fixed through the relations (81) and (83) of the form

$$A_0 = \frac{k^2}{(q+2)^2}x^{2q+1}, \quad F = \frac{k}{q+2}t - \frac{1}{qx^q}. \tag{85}$$

The forms of A_0 and A_1 fix the linearizable equation (71) to the form

$$\ddot{x} + kx^q\dot{x} + \frac{k^2}{(q+2)^2}x^{2q+1} = 0. \tag{86}$$

Since $a(t) = t$ and $b(t) = 0$, from (78) we have

$$G_1 = \frac{t}{x^{q+1}}, \quad G_2 = \frac{kt}{q+2} - \frac{1}{x^q}. \tag{87}$$

As a consequence the linearizing transformation turns out to be

$$X = \frac{kt}{q+2} - \frac{1}{qx^q}, \quad dT = [-t(\frac{k}{q+2} + \frac{\dot{x}}{x^{q+1}}) + \frac{1}{x^q}]dt. \tag{88}$$

It is easy to check that the equation (86) can be linearized to the free particle equation (63) through the transformation (88).

Equation (86) and its sub-cases have been widely discussed in the contemporary literature. In particular, Mahomed and Leach [4] have shown that equation (86) with $q = 1$ is one of the SNODEs that can be linearized to the free particle equation (63) through the IPT $X = \frac{k}{3}t - \frac{1}{x}$ and $T = \frac{t}{x} - \frac{kt^2}{6}$. Consequently, the group invariance and integrability properties of this sub-case, namely, $q = 1$ and the general equation (86) has been studied extensively by different authors, see for example[12-16]. However in the literature, equation (86) has been shown to be linearizable to free particle equation only for the value $q = 1$. For other values of q, the linearization of this equation through IPT or NPT was not known. But in the present work we have proved above that one can linearize the entire class of equation (86) under the one general transformation (88), irrespective of the value of q. One may note that choosing $q = 1$ the GLT (88) coincides exactly with the point transformation for equation (86) with the same parametric restriction. This example further confirms the arguments that IPT is a sub-case of GLT.

In the following, we derive the general solution of (86) using our procedure discussed through equations (73)-(77). Using (88) into equation (73), we obtain the first integral in the form

$$I_1 = \frac{(\frac{k}{q+2}x^{q+1} + \dot{x})}{-t(\frac{k}{q+2}x^{q+1} + \dot{x}) + x}. \tag{89}$$

Rewriting (89) for $\dot{x}$, we get

$$\dot{x} = -\frac{k}{q+2}x^{q+1} + \frac{I_1}{1+I_1 t}x. \tag{90}$$

Making use of the particular solution for the free particle equation given in equation (75) and rewriting this for x in equation (88), we get

$$x = \left(\frac{1}{q(\frac{k}{q+2}t - I_1 T)}\right)^{\frac{1}{q}}. \tag{91}$$

Substituting (90) and (91) in the second equation in (88), we obtain

$$dT = q\left(\frac{1}{1+I_1 t}\right)\left(\frac{k}{q+2}t - I_1 T\right)dt. \tag{92}$$

Rewrite equation (92) in the form

$$\frac{dT}{dt} + \frac{qI_1}{1+I_1 t}T = \frac{kq}{q+2}\left(\frac{t}{1+I_1 t}\right). \tag{93}$$

and integrating the resultant equation (93), we get

$$T = (1 + I_1 t)^{-q}\left(I_2 + \frac{(1 + I_1 t)^q (q I_1 t - 1)}{I_1^2 (2 + 3q + q^2)}\right), \tag{94}$$

where I_2 is the second integration constant. Substituting the resultant expression for T into (91), we obtain the general solution of (86), that is

$$x(t) = \left(\frac{I_1 (q+1)(q+2)(1 + I_1 t)^q}{q(k(1 + I_1 t)^{q+1} - I_1^2 I_2 (2 + 3q + q^2))}\right)^{\frac{1}{q}}, \tag{95}$$

which is the same as the one obtained by Feix et al [84].

Equation (86) is not an isolated example that can be linearized through the GLT. In fact, one can linearize a larger class of equations through this GLT and obtain the general solution. This is mainly due to the presence of arbitrary functions, namely, $a(t)$, $b(t)$ and $c(t)$ in the determining equation for given $A_1(x, t)$.

Example 2.

Now we consider a slightly more general form [84-86]

$$A_1 = k_1 x^q + k_2, \quad A_3 = A_2 = 0, \tag{96}$$

where k_1, k_2 and q are arbitrary constants in equation (70). Let us again fix the arbitrary functions a, b and c of the same form as in the previous example, that is $a(t) = t$, $b(t) = 0$ and $c(t) = 0$, so that we get $S = \frac{k_2}{2}x + \frac{k_1}{q+2}x^{q+1}$. The respective linearizable equation turns out to be

$$\ddot{x} + (k_1 x^q + k_2)\dot{x} + \frac{k_1^2}{(q+2)^2}x^{2q+1} + \frac{k_1 k_2}{(q+2)}x^{q+1} + \frac{k_2^2}{4}x = 0. \tag{97}$$

Proceeding further, we obtain the GLT in the form

$$\begin{cases} X = \frac{2k_1}{q(q+1)k_2} - \left(\frac{1}{qx^q} + \frac{2k_1}{q(q-1)k_2}\right)\exp\left(-\frac{q}{2}k_2 t\right), \\ dT = \left[\frac{1}{x^q}(1 - \frac{k_2}{2}t) - t(\frac{\dot{x}}{x^{q+1}} + \frac{k_1}{(q-1)})\right]\exp\left(-\frac{q}{2}k_2 t\right)dt. \end{cases} \tag{98}$$

One may note that in the limit $k_2 \to 0$ both the linearizing transformations (98), and the linearizable equation (97), reduce to the earlier example (vide equation (87) and (85), respectively).

The associated first integral reads

$$I_1 = \frac{dX}{dT} = \frac{(\frac{k_2}{2}x + \frac{k_1}{q+2}x^{q+1} + \dot{x})}{-t(\frac{k_2}{2}x + \frac{k_1}{q+2}x^{q+1} + \dot{x}) + x}. \tag{99}$$

Repeating the same steps given in the previous example, one can get the general solution for equation (97) in the form

$$x(t) = (I_1 + t)\exp\left(-\frac{k_2}{2}t\right)\left(I_2 + \frac{qk_1}{(q+2)}\int_0^t \exp\left(-\frac{qk_2}{2}t'\right)(I_1 + t')^q dt'\right)^{-\frac{1}{q}}, \tag{100}$$

where I_2 is the second integration constant.

CHAPTER V

NEW RESULTS

5.1 Applications of Generalized Linearizing Transformation (GLT)

When we apply the traveling wave transformation $u(x,t) = u(\xi)$, $\xi = x - vt$ to the reaction-diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[D(u)\frac{\partial u}{\partial x}\right] + R(u)\left(\frac{\partial u}{\partial x}\right)^2 + g(u),$$

it would turn out to be

$$\frac{d}{d\xi}\left[D(u(\xi))\frac{du}{d\xi}\right] + R(u)\left(\frac{du}{d\xi}\right)^2 + v\frac{du}{d\xi} + g(u(\xi)) = 0. \tag{101}$$

Apparently, this is a second-order nonlinear ordinary differential equation (SNODE). For the unification purpose of this thesis, we change the variable ξ to t , and unify the expression of equation (101) into the form

$$\ddot{u} + h(t, u(t))\dot{u} + l(t, u(t)) = 0, \tag{102}$$

where double dots $\ddot{u}$ indicates the second differentiation with respect to the variable t, and the dot $\dot{u}$ indicates the differentiation with respect to the variable t. When we choose proper functions for $D(u)$ and $g(u)$, the form of (102) will become

$$\ddot{x} + (k_1x^q + k_2x^p)\dot{x} + \frac{k_1^2x^{2q+1}}{(q+2)^2} + \frac{2k_1k_2x^{p+q+1}}{(q+2)(p+2)} + \frac{k_2^2x^{2p+1}}{(p+2)^2} = 0. \tag{103}$$

Our goal in this chapter is to show how to linearize this equation by using the generalized linearizing transformation and obtain its first integral.

In chapter IV, we discussed the applications of generalized linearizing transformation in the cases when $A_1 = kx^q$, and when $A_1 = k_1x^q + k_2$. Through this generalized linearizing transformation, we can linearize more general SNODEs, and get their first integrals.

It is natural to consider the more general case when $A_1 = k_1x^q + k_2x^p$, where p, q are arbitrary parameters. In this thesis, we extend the linearization method described in the proceeding chapter to this general case, and establish a generalized formula for the first integrals of this kind SNODEs.

Consider,

$$A_1 = k_1 x^q + k_2 x^p, \quad A_3 = A_2 = 0, \tag{104}$$

where k_1, k_2 and p, q are arbitrary parameters. Following the procedures introduced before, we fix the arbitrary functions a, b and c such as $a(t) = t,\ b(t) = c(t) = 0$. The equation (82) then gives us

$$S = \frac{k_1}{(q+2)} x^{q+1} + \frac{k_2}{(p+2)} x^{p+1}. \tag{105}$$

Once A_1 is fixed, by (83), we can easily derive

$$A_0 = \frac{k_1^2 x^{2q+1}}{(q+2)^2} + \frac{2k_1 k_2 x^{p+q+1}}{(q+2)(p+2)} + \frac{k_2^2 x^{2p+1}}{(p+2)^2}. \tag{106}$$

The forms of A_0 and A_1 fix the linearizable equation (71) into the form

$$\ddot{x} + (k_1 x^q + k_2 x^p)\dot{x} + \frac{k_1^2 x^{2q+1}}{(q+2)^2} + \frac{2k_1 k_2 x^{p+q+1}}{(q+2)(p+2)} + \frac{k_2^2 x^{2p+1}}{(p+2)^2} = 0.$$

This is a new and more general SNODEs that has not been studied before as far as we know.

Following the generalized linearizing transformation procedure, we should substitute (105) into (81), and then the equation assumes the form

$$\frac{k_1}{(q+2)} x^{q+1} + \frac{k_2}{(p+2)} x^{p+1} = \frac{F_x}{F_t}. \tag{107}$$

If we could solve above equation for F, and get the explicit form of F, we just need substitute it into (78), solve for G_1, G_2, and then through the GLT (70), we could finally convert the equation (103) into the free particle form of (63). However, the key problem arises here is that there is apparently no explicit general solution to the equation (107).

Even though we cannot get the explicit expression of function F by (107), we eventually solve this problem by finding out a general formula to get the first integrals with the expression of $S(t, x)$ directly. In this case, once $S(t, x)$ is figured out, we can get the first integral straightforwardly by the generalized formula

$$I_1 = \frac{dX}{dT} = \frac{\dot{x} + S(t,x)}{(\dot{x} + S(t,x))t - x}. \tag{108}$$

Since $S = \frac{k_1}{(q+2)}x^{q+1} + \frac{k_2}{(p+2)}x^{p+1}$, the first integral for the general equation (103) turns exactly to be

$$I_1 = \frac{dX}{dT} = \frac{\dot{x} + \frac{k_1 x^{q+1}}{q+2} + \frac{k_2 x^{p+1}}{p+2}}{t\left(\dot{x} + \frac{k_1 x^{q+1}}{q+2} + \frac{k_2 x^{p+1}}{p+2}\right) - x}. \tag{109}$$

As the proofs of our conclusion, we go back to check the two examples in chapter IV, we can easily find that they are just two particular cases to this general formula. Since the parameters k_1, k_2 and p, q in (104) are arbitrary, if we fix $k_2 = 0$, then we have $A_1 = k_1 x^q$. Then the equations (82) and (83) give us $S = \frac{k_1}{q+2}x^{q+1}$ and $A_0 = \frac{k_1^2}{(q+2)^2}x^{2q+1}$. If we substitute the expression of S into the formula (108), then the first integral for the equation

$$\ddot{x} + k_1 x^q \dot{x} + \frac{k_1^2}{(q+2)^2}x^{2q+1} = 0,$$

turns out directly to be

$$I_1 = \frac{\left(\dot{x} + \frac{k_1}{q+2}x^{q+1}\right)}{t\left(\dot{x} + \frac{k_1}{q+2}x^{q+1}\right) - x},$$

which is actually the same as the result showed in the first example of Section 4.2.

Similarly, if we fix $p = 0$, the equations (82) and (83) gives us

$$S = \frac{k_2}{2}x + \frac{k_1}{q+2}x^{q+1},$$

and

$$A_0 = \frac{k_1^2}{(q+2)^2}x^{2q+1} + \frac{k_1 k_2}{(q+2)}x^{q+1} + \frac{k_2^2}{4}x.$$

By the formula (108), the first integral for equation (97) turns out to be

$$I_1 = \frac{\left(\dot{x} + \frac{k_2}{2}x + \frac{k_1}{q+2}x^{q+1}\right)}{t\left(\dot{x} + \frac{k_2}{2}x + \frac{k_1}{q+2}x^{q+1}\right) - x},$$

which is actually the same as the result showed in the second example of Section 4.2.

5.2 Another Example for Particular Cases of GLT

Now we take another particular example to illustrate the applications of this generalized linearizing transformation and compare it with our formula. If we Let $q = 2$, $p = 1$, and k_1 and k_2 are arbitrary parameters, then $A_1(t,x) = k_1x^2 + k_2x$. If we fix the arbitrary functions a, b and c such as $a(t) = t$, $b(t) = c(t) = 0$, so that the equation (82) gives us

$$S(t,x) = \frac{k_1x^3}{4} + \frac{k_2x^2}{3} = \frac{F_x}{F_t}. \tag{110}$$

For this particular case, we can solve (110) for F and A_0 to be

$$F = \left(3k_1 + \frac{4k_2}{x}\right) e^{\frac{4k_2^2}{9k_1}\left(t - \frac{3}{k_2 x}\right)}, \tag{111}$$

$$A_0 = \frac{k_1^2x^5}{16} + \frac{k_1k_2x^4}{6} + \frac{k_2^2x^3}{9}. \tag{112}$$

The forms of A_0 and A_1 fix the linearizable equation (71) to the form

$$\ddot{x} + (k_1x^2 + k_2x)\dot{x} + \frac{k_1^2x^5}{16} + \frac{k_1k_2x^4}{6} + \frac{k_2^2x^3}{9} = 0. \tag{113}$$

This is a nonlinear second-order differential equation, with $k_1x^2 + k_2x$ as a quadratic coefficient for $\dot{x}$, and $\frac{k_1^2x^5}{16} + \frac{k_1k_2x^4}{6} + \frac{k_2^2x^3}{9}$ as a nonlinear reaction term.

Following the GLT procedure introduced in Chapter IV, we get equations

$$F_x = \frac{4k_2^2}{9k_1x^3} e^{\frac{4k_2^2}{9k_1}\left(t - \frac{3}{k_2 x}\right)}, \tag{114}$$

$$F_t = \left(\frac{k_2^2}{9} + \frac{4k_2^3}{27k_1x}\right) e^{\frac{4k_2^2}{9k_1}\left(t - \frac{3}{k_2 x}\right)}. \tag{115}$$

Then

$$dF = (F_x\dot{x} + F_t)dt = \left(\frac{4k_2^2}{9k_1x^3} + \frac{k_2^2}{9} + \frac{4k_2^3}{27k_1x}\right) e^{\frac{4k_2^2}{9k_1}\left(t - \frac{3}{k_2 x}\right)} dt, \tag{116}$$

$$dT = \left[\left(\frac{4k_2^2}{9k_1x^3}\dot{x} + \frac{k_2^2}{9} + \frac{4k_2^3}{27k_1x}\right)t - \frac{4k_2^2}{9k_1x^2}\right] e^{\frac{4k_2^2}{9k_1}\left(t - \frac{3}{k_2 x}\right)} dt. \tag{117}$$

The first integral turns out to be

$$I_1 = \frac{dX}{dT} = \frac{\dot{x} + \frac{k_1x^3}{4} + \frac{k_2x^2}{3}}{t\left(\dot{x} + \frac{k_1x^3}{4} + \frac{k_2x^2}{3}\right) - x}. \tag{118}$$

Actually, this result can be found easily by substituting $S(t,x) = \frac{k_1x^3}{4} + \frac{k_2x^2}{3}$ into the formula (108) without doing so many calculations above. Here, even though we can get the explicit expression of function of F in this case, this linearizable equation (71) is unsolvable by the generalized linearizing transformation. What's more, we can also show many cases of equation (103) that fail to be linearized by this generalized linearizing transformation while can be found the first integrals by our formula.

5.3 Numerical Simulation Analysis

In most particular cases of the linearizable equation (103), the generalized linearizing transformation is unapplicable for the reason of failing to get the explicit expression of function F. However, once we get the first integral by our formula, we can apply it to many scientific fields directly.

In the two examples of Section 4.2, both of the two linearizable equations are solved to have general solutions, now we take particular values for these parameters, and make numerical simulation to analyze their properties.

For the linearizable equation (86)

$$\ddot{x} + kx^q\dot{x} + \frac{k^2}{(q+2)^2}x^{2q+1} = 0,$$

the first integral is

$$I_1 = \frac{\left(\frac{k}{q+2}x^{q+1} + \dot{x}\right)}{-t\left(\frac{k}{q+2}x^{q+1} + \dot{x}\right) + x},$$

and the general solution is

$$x(t) = \left(\frac{I_1(q+1)(q+2)(1+I_1t)^q}{q\left(k(1+I_1t)^{q+1} - I_1^2I_2(2+3q+q^2)\right)}\right)^{\frac{1}{q}}.$$

The graph of the solution is shown in Figure 1 in the case of $q = 1$, $I_1 = 10$, $I_2 = 10$, $k = 10$.

For the linearizable equation (97)

$$\ddot{x} + (k_1x^q + k_2)\dot{x} + \frac{k_1^2}{(q+2)^2}x^{2q+1} + \frac{k_1k_2}{(q+2)}x^{q+1} + \frac{k_2^2}{4}x = 0,$$

the first integral is

$$I_1 = \frac{dX}{dT} = \frac{\left(\frac{k_2}{2}x + \frac{k_1}{q+2}x^{q+1} + \dot{x}\right)}{-t\left(\frac{k_2}{2}x + \frac{k_1}{q+2}x^{q+1} + \dot{x}\right) + x},$$

and the general solution is

$$x(t) = (I_1 + t)\exp(-\frac{k_2}{2}t)\left(I_2 + \frac{qk_1}{(q+2)}\int_0^t \exp\left(-\frac{qk_2}{2}t'\right)(I_1+t')^q dt'\right)^{-\frac{1}{q}}.$$

The graph of the solution is illustrated in Figure 2 in the case of $q = 3$, $I_1 = 1$, $I_2 = 1$, $k_1 = 1$, $k_2 = 1$.

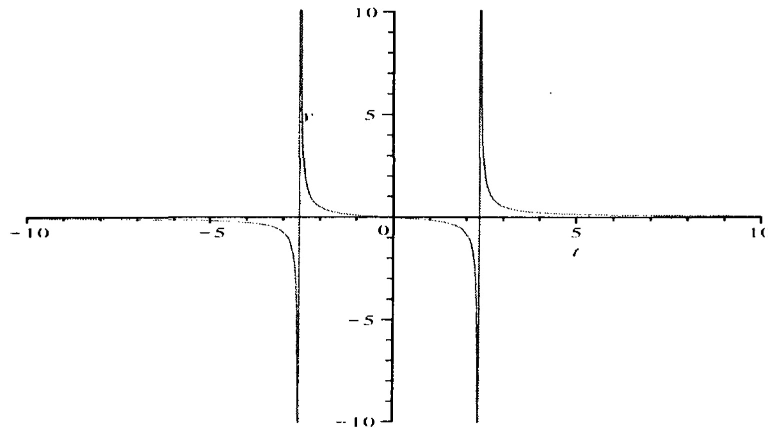

Figure 1: For case $q = 1$, $I_1 = 10$, $I_2 = 10$, $k = 10$.

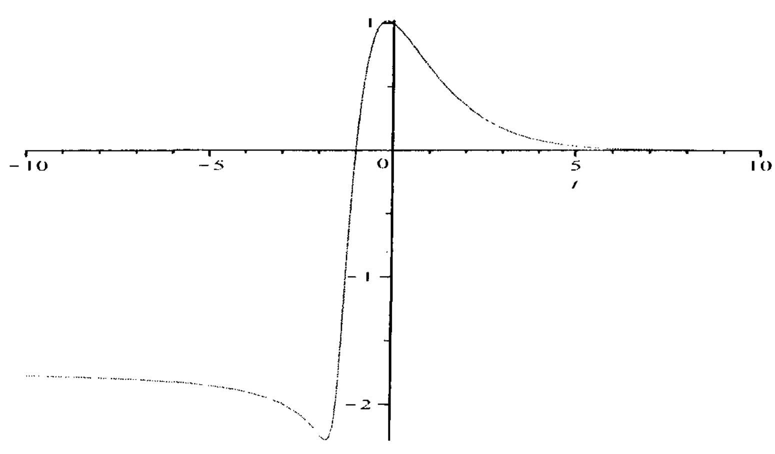

Figure 2: For case $q = 3$, $I_1 = 1$, $I_2 = 1$, $k_1 = 1$, $k_2 = 1$.

Rewriting (118) for $\dot{x}$, we get

$$\dot{x} = -\left(\frac{k_1 x^3}{4} + \frac{k_2 x^2}{3}\right) + \frac{I_1 x}{I_1 t - 1}. \tag{119}$$

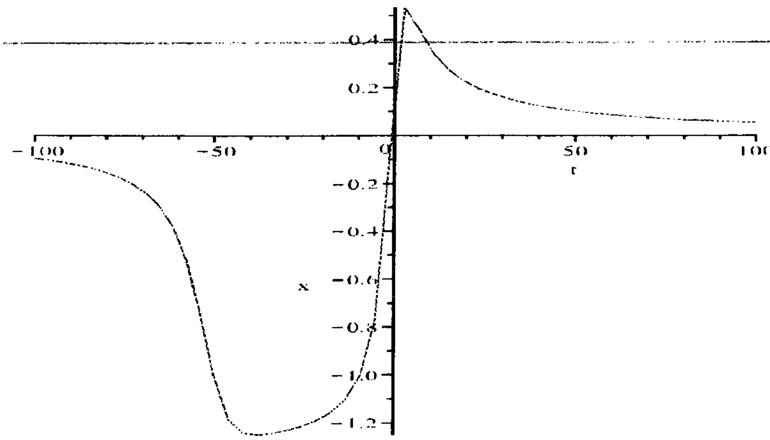

Figure 3: For case $I_1 = 1, k_1 = 1, k_2 = 1$.

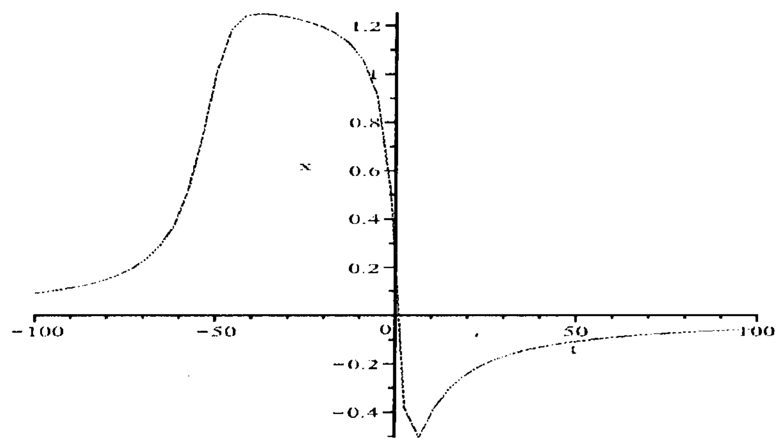

Figure 4: For case $I_1 = 1, k_1 = 1, k_2 = -1$.

By using Maple, we also get two typical numerical graphs of this equation as shown in Figures 3 and 4.

REFERENCES

[1] Whitham, G. B. (1974). Linear and Nonlinear Waves, *Springer-Verlag*, New York.

[2] Kuramoto, Y. (1984). Chemical Oscillations, Waves and Turbulence. *Springer*, Berlin.

[3] Murray, J. D. (1993). Mathematical Biology, *Springer-Verlag*, New York.

[4] Mitchell, A. R. and Griffiths, D. F. (1980). The Finite Difference Method in Partial Differential Equations, *John Wiley and Sons*, New York.

[5] Fisher, F. A. (1937). The wave of advance of advantageous genes, *Ann. Eugenics*, 7, 353–369.

[6] Luther, R. L. (1906). Räumliche Fortpflanzung Chemischer Reaktinen. *Z. für Elektrochemie undangew Physikalische Chemie,* 12, 506–600.

[7] Arnold, R., Showalter, K. and Tyson, J. J. (1987). Propagation of chemical reactions in space, *J. Chem. Educ.*, 64, 740–742.

[8] Showalter, K. and Tyson, J. J. (1987). *Luther's* 1906 discovery and analysis of chemical waves, *J. Chem. Educ.* 64, 742–744.

[9] Aoki, K. Gene-culture waves of advance, *J. Math. Biol.*, 25 (1987), 453–464.

[10] Ammerman, A. J. and Cavalli-Sforza, L. L. (1971). Measuring the rate of spread of early farming. *Man*, 6, 674–688.

[11] Ammerman, A. J. and Cavalli-Sforza, L. L. (1983). The Neolithic Transition and the Genetics of Populations in Europe. *Princeton University Press*, Princeton.

[12] Kolmogorov, A., Petrovsky, I. and Piskunov, N. (1937). Etude de l'equation de la diffusion aveccroissance de la quantite de matiere et son application a un probleme biologique. *Moscow Bull Univ. Math.*, 1, 1–25.

[13] Ablowitz, M. J. and Zeppetella, A. (1979). Explicit solution of Fisher's equation for a special wave speed. *Bull. Math. Biol.*, 41, 835–840.

[14] Fife, P. C. (1979). Mathematical aspects of reacting and diffusing systems, Lecture Notes in Biomathematics. *Springer-Verlag*, Berlin.

[15] Britton, N. F. (1986). Reaction-Diffusion Equations and Their Applications to Biology, Academic Press, New York.

[16] Guo, B. Y. and Chen, Z. Y. (1991). Analytic solutions of the Fisher equation. *J. Phys. A (Math. Gen.)*, 24, 645–650.

[17] Chen, Z. X. and Guo, B. Y. (1989). Analytic solutions of carrier flow equations via the Painlevé analysis approach. *J. Phys. A (Math. Gen.)*, 22, 5187–5194.

[18] Volpert, A. I., Volpert, V. A. and Volpert, V. A. (1994). Traveling Wave Solutions of Parabolic Systems (Translations of Mathematical Monographs), *American Mathematical Society Providence*, Vol. 14, RI.

[19] Aronson, D. G. and Weinberger, H. F. (1978). Multidimensional nonlinear diffusion arising in population genetics. *Adv. Math.*, 30, 33–76.

[20] Aronson, G. G. and Weinberger, H. F. (1975). Nonlinear diffusion in population genetics, combustion, and nerve pulse propagation. *Partial Differential Equations and Related Topics, (ed. Goldstein, J. A.), Lecture Notes in Math.*, Vol. 446, pp. 5–49. Springer, Berlin.

[21] Skellam, J. G. (1951). Random dispersal in theoretical populations. *Biometrika*, 38, 218–236.

[22] McMurtrie, R. (1978). Persistence and stability of single-species and prey-predator systems in spatially heterogeneous environments. *Math. Biosci.*, 39, 11–51.

[23] Myers, J. H. and Krebs, C. J. (1974). Population cycles in rodents. *Sci. Amer.*, 6, 38–46.

[24] Gurney, W. S. and Nisbet, R. M. (1975). The regulation of inhomogeneous population, *J. Theor. Biol*, 52, 441–457.

[25] Clarkson, P. A. and Mansfield, E. L. (1993). Nonclassical symmetry reductions and exact solutions of nonlinear reaction-diffusion equations. *Applications of Analytic and Geometric Methods to Nonlinear Differential Equations, (ed. P.A. Clarkson)*, Vol. 413, pp. 375–389. Kluwer, Dordrecht.

[26] Clarkson, P. A. and Mansfield, E. L. (1994). Symmetry reductions and exact solutions of a class of nonlinear heat equations. *Phys. D*, 70, 250–288.

[27] Chen, Z. X. and Guo, B. Y. (1992). Analytic solutions of the Nagumo equation. *J. Appl. Math.*, 48, 107–15.

[28] Chowdhury, S. R. (1991). Painlevé analysis and special solutions of two families of reaction diffusion equations. *Phys Lett.*, A 159, 311–317.

[29] Estévez, P. G. and Gordoa, P. R. (1990). Painlevé analysis of the generalized Burgers-Huxley equation. *J. Phys. A (Math. Gen.)*, 23, 4831–4837.

[30] Feng, Z. and Chen, G. (2005). Traveling wave behavior for a nonlinear reaction-diffusion equation. *Dyn. Contin. Discrete Impuls. Syst.*, (Ser. A) 12, 543–664.

[31] Herrera, J. J., Minzoni, A. and Ondarza, R. (1992). Reaction-diffusion equations in one dimension: particular solutions and relaxation. *Phys. D* 57, 249–266.

[32] Hadeler, K. P. (1981). Diffusion in Fisher's population model. *Rocky Mountain J. Math.*, 11, 39–45.

[33] Hadeler, K. P. (1983). Free boundary problems in biological models. *Free Boundary Problems: Theorey and Applications, (eds. Fasano, A. and Primicerio, M.)*, Vol. II, pp. 665–671, Pitman, London.

[34] Sanchez-Garduno, F. and Maini, P. K. (1995). Travelling wave phenomena in some degenerate reaction-diffusion equations. *J. Diff. Equs.*, 117, 281–319.

[35] Sanchez-Garduno, F. and Maini, P. K. (1997). Travelling wave phenomena in nonlinear diffusion degenerate Nagumo equations. *J. Math. Biol.*, 35, 713–728.

[36] Engler, H. (1985). Relations between travelling wave solutions of quasilinear parabolic equations. *Proc. Amer. Math. Soc.*, 93, 297–302.

[37] de Pablo, A. and Sanchez, A. (1998). Travelling wave behaviour for a porous-Fisher equation. *European J. Appl. Math.*, 9, 285–304.

[38] Bates, P. W. and Chen, F. (2006). Spectral analysis of traveling waves for nonlocal evolution equations. *SIAM J. Math. Anal.*, 38, 116–126.

[39] Alikakos, N. D., Bates, P. W. and Chen, X. F. (1999). Periodic traveling waves and locating oscillating patterns in multidimensional domains. *Trans. Amer. Math. Soc.*, 351, 2777–2805.

[40] Feng, Z. and Chen, G. (2009). Traveling wave solutions in parametric forms for a diffusion model with a nonlinear rate of growth. *Discrete Contin. Dyn. Syst.*, Ser. A, 24, 763–780.

[41] Feng, Z. (2008). Traveling wave behavior for a generalized Fisher equation. *Chaos Solitons Fractals*, 38, 481–488.

[42] Feng, Z. and Li, Y. (2006). Complex traveling wave solutions to the Fisher equation. *Phys.*, A 366, 115–123.

[43] Feng, Z. and Chen, G. (2005). Traveling wave behavior for a nonlinear reaction-diffusion equation. *Dyn. Contin. Discrete Impuls. Syst.*, Ser. A 12, 643–664.

[44] Feng, Z. (2004). Traveling wave solutions to nonlinear evolution equations. *Variational Methods: Open Problems, Recent Progress, And Numerical Algorithms*, pp. 269–285, *Contemp. Math.*, Vol. 357, *Amer. Math. Soc., Providence*, RI.

[45] Chen, L. H. and Mo, J. Q. (2008). The homotopic mapping method for reaction diffusion system in biomathematics. *J. Biomath*, 23, 385–389.

[46] Jin, Y. and Zhao, X. Q. (2009). Spatial dynamics of a nonlocal periodic reaction-diffusion model with stage structure. *SIAM J. Math. Anal.*, 40, 2496–2516.

[47] Lombardo, S., Mulone, G. and Trovato, M. (2008). Nonlinear stability in reaction-diffusion systems via optimal Lyapunov functions. *J. Math. Anal. Appl.*, 342, 461–476.

[48] Xu, D. S. and Zhao, X. Q. (2003). A nonlocal reaction-diffusion population model with stage structure. *Can. Appl. Math. Q.*, 11, 303–319.

[49] Murakawa, H. (2007). Reaction-diffusion system approximation to degenerate parabolic systems, *Nonlinearity*, 20, 2319–2332.

[50] Heinze, S. and Schweizer, B. (2005). Creeping fronts in degenerate reaction-diffusion systems. *Nonlinearity*, 18, 2455–2476.

[51] Newell, A. C., Tabor, M. and Zeng, Y. B. (1987). A unified approach to Painlevé expansions. *Phys., D 29*, 1–68.

[52] Weiss, J. (1984). The sine-Gordon equations: complete and partial Integrability. *J. Math. Phys.*, 25, 2226–2235.

[53] Weiss, J., Tabor, M. and Carnevale, G. (1983). The Painlevé property for partial differential equations. *J. Math. Phys.*, 24, 522–526.

[54] Kaliappan, P. (1984). An exact solution for travelling waves of $u_t = Du_{xx} + u - u^k$. *Physica*, D 11, 368–374.

[55] Herrera, J. J. E., Minzoni, A. & Ondarza, R. (1992). Reaction-diffusion equations in one dimension: particular solutions and relaxation. *Physica,* D 57, 249–266.

[56] Cariello, F. and Tabor, M. (1989). Painlevé expansions for nonintegrable evolution equations. *Phys.*, D 39, 77–94.

[57] Cariello, F. and Tabor, M. (1991). Similarity reductions from extended Painlevé expansions for nonintegrable evolution equations. *Phys.*, D 53, 59–70.

[58] Newell, A. C. and Whitehead, J. A. (1969). Finite bandwidth, finite amplitude convection. *J. Fluid Mech.*, 38, 279–303.

[59] FitzHugh, R. (1961). Impulses and physiological states in theoretical models of nerve membrane. *Biophysical J.*, 1, 445–466.

[60] Fife, P. C. and McLeod, J. B. (1977). The approach of solutions of nonlinear diffusion equations to travelling front solutions. *Arch. Ration. Mech. Anal.*, 65, 335–361.

[61] Hadeler, K. P. and Rothe, F. (1975). Travelling fronts in nonlinear diffusion equations. *J. Math. Biol.*, 2, 251–263.

[62] Veling, E. J. M. (1981). Travelling waves in an initial-boundary value problem. *Proc. Roy. Soc. Edinburgh Sect.*, A 90, 41–61.

[63] Vorobev, E. M. (1991). Reduction and quotient equations for differential equations with symmetries. *Acta. Appl. Math.*, 23, 1–24.

[64] Kawahara, T. and Tanaka, M. (1983). Interactions of traveling fronts: an exact solution of a nonlinear diffusion equation. *Phys. Lett.,* A 97, 311–314.

[65] Hirota, R. (1980). Topics Current Physics. *Solitons* (R.K. Bullough and P.J. Caudrey, eds.), Springer, Berlin, Vol. 17, pp. 157–176.

[66] Chen, Z. X. and Guo, B. Y. (1992). Analytic solutions of the Nagumo equation. *IMA J. Appl. Math.*, 48, 107–115.

[67] Clarkson, P. A. and Kruskal, M. D. (1989). New similarity reductions of the Boussinesq equation. *J. Math. Phys.*, 30, 2201–2213.

[68] Prelle, M., Singer, M. (1983). Elementary first integrals of differential equations. *Trans. Am. Math. Soc.* 279, 215–229.

[69] Duarte, L. G. S., Duarte, S. E. S., da Mota, A. C. & Skea, J. E. F. (2001). Solving the second-order ordinary differential equations by extending the Prelle–Singer method. *J. Phys. A (Math. Gen.),* 34, 3015–3024.

[70] Chandrasekar, V. K., Senthilvelan, M. & Lakshmanan, M. (2005). On the complete integrability and linearization of certain second-order nonlinear ordinary differential equations. *Proc. R. Soc. Lond.*, Ser. A 461, 2451–2476.

[71] Chandrasekar, V. K., Senthilvelan, M. & Lakshmanan, M. (2006). On the complete integrability and linearization of nonlinear ordinary differential equations–(II) Third-order equations. *Proc. R. Soc. Lond. Ser. A* 462, 1831–1852.

[72] Man, Y. K. (1993). Computing closed form solutions of first-order ODEs using the Prelle–Singer procedure. *J. Symb. Comput.* 16, 423–443.

[73] Man, Y. K. (1994). First integrals of autonomous systems of differential equations and the Prelle–Singer procedure. *J. Phys. A (Math. Gen.)* 27, L329–L332.

[74] Man, Y. K. & MacCallum, M. A. H. (1996). A rational approach to the Prelle–Singer algorithm. *J. Symb. Comput.*, 11, 1–11.

[75] Shtokhamer, R. (1988). Solving first-order differential equations using the Prelle–Singer algorithm, *Technical Report*, 88-09 Center for Mathematical Computation, University of Delaware.

[76] Singer, M. (1990). Formal solutions of differential equations. *J. Symb. Comput.* 10, 59–94.

[77] Singer, M. (1992). Liouvillian first integrals of differential equations, *Trans. Am. Math. Soc.* 333, 673–688.

[78] Chandrasekar, V. K., Senthilvelan, M. and Lakshmanan, M. (2004). New aspects of integrability of force-free Duffing-van der Pol oscillator and related nonlinear systems. *J. Phys. A (Math. Gen.)*, 37, 4527–4534.

[79] SenthilVelan, M. and Lakshmanan, M. (1995). Lie symmetries and infinite-dimensional Lie algebras of certain nonlinear dissipative systems. *J. Phys. A (Math. Gen.)*, 28, 1929–1942.

[80] Ibragimov, N. H. (1999). Elementary Lie group analysis and ordinary differential equations. Wiley, New York.

[81] Steeb, W. H. (1993). Invertible point transformations and nonlinear differential equations, World Scientific, London.

[82] Duarte, L. G. S., Duarte, S. E. S. & Moreira, I. C. (1989). N-dimensional equations with the maximum number of symmetry generators. *J. Phys. A (Math. Gen.)* 22, L201–L206.

[83] Duarte, L. G. S., Euler, N., Moreira, I. C. & Steeb, W. H. (1990). Invertible point transformations, Painleve analysis and anharmonic oscillators, *J. Phys. A (Math. Gen.)* 23, 1457–1464.

[84] Chandrasekar, V. K., Senthilvelan, M. and Lakshmanan, M. (2006). A unification in the theory of linearization of second-order nonlinear ordinary differential equations, *J. Phys. A (Math. Gen.)* 39, L69–L76.

[85] Duarte, L. G. S., Moreira, I. C. & Santos, F. C. (1994). Linearization under nonpoint transformations. *J. Phys. A (Math. Gen,)*, 27, L739U" -L744.

[86] Euler, N. & Euler, M. (2004). Sundman symmetries of nonlinear second-order and third-order ordinary differential equations. *J. Nonlinear Math. Phys.* 11, 399–421.

BIOGRAPHICAL SKETCH

Yubing Wan, the son of Jianchun Wan and Yanfeng Li, was born in China in 1983. He received his bachelor degree in Applied Mathematics from China University of Mining and Technology, Xuzhou, Jiangsu, China in July of 2007. In August of 2007, he joined the Mathematical Master's Program at the University of Texas-Pan American, Edinburg, Texas. His main research interests were in Differential Equations, Dynamic Systems and Applied Biomathematics. His permanent mailing address is: 685 Dongma Cun Tuanwang Zheng, Laiyang City, Shandong Province, P.R.China, 221008.