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Reduction of the KP Hierarchy

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REDUCTION OF THE KP HIERARCHY

A Thesis

by

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ABSTRACT

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This thesis will delve into the Kadomtsev-Petviashvili equation or KP equation and it's heirarchy. More specifically, the solition theory around it. To do so, we first explore the soliton theory for the Korteweg de-Vries equation or KdV equation by analysing it through the inverse scattering transform method and presenting it's soliton solutions. Second, we will introduce, Hirota's bilinear form, and understand its main idea. Third, introduce Sato Theory, and use it to formulate the KP hierarchy, via using pseudo-differential operators, presenting the lax operator, the dressing operator, Sato equation, and the zero curvature equation (Zakharov-Shabat Equation). Fourth, find the general solution and one-soliton solution to the KP hierarchy and peform a 2-reduction and 3-reduction on the KP heirarchy. Finally, use Hirota's bilinear method (direct method) to find the multiple solition solutions for the KP hierarchy.

DEDICATION

The completion of my master studies would not have been possible without the love and support of my father, Eugenio Torres. Who wholeheartedly inspired, motivated and supported me by all means to accomplish this degree. Thank you for your love, support, and patience.

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CHAPTER I

INTRODUCTION

In this report we will look into the soliton theory around the KP (Kadomtsev-Petviashvili) equation. To do so we will look into the inverse scattering transform of the KdV (Korteweg de-Vries) equation, and brief descriptions of Hirota's bilinear form and Sato's Theory.

1.1 Inverse Scattering Transform of the KdV (Korteweg de-Vries) equation

We will show how to solve the Korteweg de-Vries (KdV) equation via the inverse scattering transformation method. We will present the direct scattering problem, as well as the time evolution of the scattering data. Then, we will show the Gelfand-Levitan-Marchenko (GLM) equation in order to solve the inverse scattering problem of the KdV equation. Finally, we will construct and show explicit one- and two-soliton solutions for the reflectionless case.

The KdV equation is a nonlinear dispersive partial differential equation for a function u of two variables: x (space variable) and t (time variable), which can be written as follows

$$\frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0. \tag{1.1}$$

This lax pair contains two linear operators, L and M. According to the Schrödinger equation [2] we have that

$$L = -\partial_x^2 + u(x), \tag{1.2}$$

and that the compatibility condition is as follows

$$\frac{dL}{dt} + LM - ML = 0. \tag{1.3}$$

We can derive that $M = 4\partial_x^3 - 6u\partial_x - 3u_x$ which produces the KdV equation (1.1). Therefore the lax pair is given as follows

$$L\psi = \lambda\psi \tag{1.4}$$

$$M\psi = \frac{d\psi}{dt} \tag{1.5}$$

One can easily show that $\lambda_t = 0$ by taking the derivative with respect to time for (1.4) and plugging it into (1.5).

In mathematics, scattering means the passing of plane waves $\psi \sim e^{\pm ikx}$ through the field of the potential $u(x)$ from $x = -\infty$ to $x = +\infty$. Let $k \in R$ such that $k^2 = \lambda$, and let $u(x)$ be a smooth real function that does not depend on t, such that $| u(x) | \to 0$ for $| x | \to +\infty$. The solutions satisfy $\psi \sim e^{ikx}$, $\overline{\psi} \sim e^{-ikx}$. By Picard's method [2] it can be shown that for every k the differential equation

$$\psi'' + k^2\psi = u\psi, \tag{1.6}$$

with a prescribed asymptotic behavior that has a pair of linearly independent solutions.

Now for $\phi \sim e^{-ikx}$ and $\overline{\phi} \sim e^{ikx}$ as $x \to -\infty$, the Wronskian of two solutions is defined as $W(\psi, \phi) = \psi_x\phi - \psi\phi_x$ [2]. Also, we have that $W(\psi, \overline{\psi}) = W(\phi, \overline{\phi}) = 2ik$ [2], meaning it is not equal to zero, thus forming the basis for the space of solutions. Now the transition matrix from this basis is formed by ψ and $\overline{\psi}$ to a basis formed by ϕ and $\overline{\phi}$ and is given by following

$$\begin{bmatrix} \phi \\ \overline{\phi} \end{bmatrix} = \begin{bmatrix} a(k) & b(k) \\ \overline{b}(k) & \overline{a}(k) \end{bmatrix} \begin{bmatrix} \overline{\psi} \\ \psi \end{bmatrix}, \tag{1.7}$$

which is the scattering matrix [2].

Since W is an invariant skew symmetric bilinear form, $|a(k)|^2 - |b(k)|^2 = 1$, then $a(k)$ can be analytically extended to $\Im k > 0$ since

$$\begin{aligned} W(\psi,\phi) &= \psi_x\phi - \psi\phi_x && (1.8)\\ &= a(k)W(\overline{\psi},\psi) + b(k)W(\psi,\psi) && (1.9)\\ &= a(k)W(\overline{\psi},\psi) && (1.10)\\ \Rightarrow\ & a(k) = \frac{W(\psi,\phi)}{2ik}. && (1.11)\end{aligned}$$

Similarly,

$$b(k) = \frac{-W(\psi,\overline{\phi})}{2ik}. \qquad (1.12)$$

Also, $a(k) = 0$ if and only if there exists a solution to $L\psi = k^2\psi$ that exponentially decays at infinity. It follows that $a(k)$ has at most a finite number of zeros in the upper half plane. In other words, the zeros of $a(k)$ correspond to eigenvalues of the discrete spectrum: $a(k) = 0$ if and only if $\lambda = k^2$ is a point of the discrete spectrum. We denote the zeros $i\kappa_1, ..., i\kappa_n$ with $\kappa_1 > ... > \kappa_n > 0$. Then $\lambda_i = -\kappa_i^2$ and $\phi_s(x) := \phi(x, i\kappa_s)$ are the eigenvalues and eigenfunctions of the discrete spectrum of L respectively [2].

Which can be seen as follows

$$\phi_s(x) = \begin{cases} e^{\kappa_s x}, & \text{if } x \to -\infty \\ b_s e^{-\kappa_s x}, & \text{if } x \to +\infty \end{cases} \qquad (1.13)$$

where $b_s \in R$[2].

Thus, we obtain the following scattering data:

1. The reflection coefficient $r(k) = \frac{b(k)}{a(k)}$, $k \in R$
2. $\kappa_1, ..., \kappa_n$
3. $b_1, ..., b_n$

Therefore constructing the following scattering map:

$$[u(x)] \longrightarrow [(r(k), (\kappa_1, ..., \kappa_n, b_1, ..., b_n)] \tag{1.14}$$

Using the KdV equation represented in the form $\dot{L} = LM - ML$ and differentiating $L\phi = \lambda\phi$ in time, we get

$$\begin{aligned} L(\dot{\phi} + M\phi) &= \lambda(\dot{\phi} + M\phi) + \dot{\lambda}\phi & (1.15) \\ &= \lambda(\dot{\phi} + M\phi). & (1.16) \end{aligned}$$

So, $\dot{\phi} + M\phi$ is an eigenfunction for λ and must be a linear combination of $\overline{\phi}$ and ϕ, thus

$$\dot{\phi} + M\psi = \alpha\overline{\phi} + \beta\phi. \tag{1.17}$$

The behavior of the left side at $x \rightarrow +\infty$ is $4ik^3 e^{ikx}$ while the behavior of the right side is $\alpha e^{ikx} + \beta e^{-ikx}$ which implies that α and β must be $4ik^3$ and zero respectively. Thus,

$$\dot{\phi} + M\phi = 4ik^3\phi. \tag{1.18}$$

We consider $\phi = a\overline{\psi} + b\psi$, we proceed by differentiating with respect to time as follows

$$\begin{aligned} \dot{\phi} &= \dot{a}\overline{\psi} + \dot{b}\psi + a\dot{\overline{\psi}} + b\dot{\psi} & (1.19) \\ \Rightarrow \dot{\phi} + M\phi &= \dot{a}\overline{\psi} + \dot{b}\psi + a(\dot{\overline{\psi}} + M\overline{\psi}) + b(\dot{\psi} + M\psi) & (1.20) \\ \Rightarrow 4ik^3(\alpha\overline{\psi} + \beta\psi) &= \dot{a}\overline{\psi} + \dot{b}\psi + 4ik^3 a\overline{\psi} - 4ik^3 b\psi & (1.21) \\ \Rightarrow 8ik^3 b\psi &= \dot{a}\overline{\psi} + \dot{b}\psi & (1.22) \end{aligned}$$

where $\dot{a} = 0$ and $\dot{b} = 8ik^3 b \Rightarrow \dot{r}(k,t) = 8ik^3 r(k,t)$ [2].

Therefore, our scattering data evolves with time in the following way:

$$[r(k), (\kappa_j, b_j)_{j=1}^n] \longrightarrow [r(k)e^{8ik^3 t}, (\kappa_j, b_j e^{8\kappa_j^3 t})_{j=1}^n]. \tag{1.23}$$

The inverse scattering is the problem of reconstructing $u(x,t)$ from the scattering data. We first reconstruct $a(k)$ and $b(k)$ from the scattering data. Then, we define The Fourier transform of

$r(k)$ as follows:

$$\hat{r}(x) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} r(k)e^{ikx}dk, \tag{1.24}$$

$$F(x) = \sum_{j=1}^{n} \frac{b_j e^{-\kappa_j x}}{ia'(k)} + \hat{r}(x), \tag{1.25}$$

$$K(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(y-x)}(\overline{\psi}(x,k)e^{ikx} - 1)dk. \tag{1.26}$$

Then we derive the Gelfand-Levitan-Marchenko (GLM) equation starting from the fact that $\phi(x,k) = a(k)\overline{\psi}(x,k) + b(k)\psi(x,k)$ [2], we multiply both sides by $\frac{e^{iky}}{a(k)}$ and, after subtracting e^{-ikx} (in order to get a well-defined integral), we integrate both sides with respect to k and perform the necessary simplifications and substitutions while making use of the "dressing" operator, and therefore we arrive at

$$\hat{r}(x+y) + \sum_{j=1}^{n} \frac{b_j}{ia'(ik_s)} e^{-k_s(x+y)} + K(x,y) + \int_x^{\infty} K(x,z)(\hat{r}(z+y) + \sum_{j=1}^{n} \frac{b_j e^{-k_j(z+y)}}{ia'(k)})dz = 0. \tag{1.27}$$

Where the sum of the first two terms was called before $F(x)$ [2].

We substitute our previously defined function (1.25) into equation (1.27) and get the Gelfand-Levitan Marchenko (GLM) equation as follows,

$$F(x+y) + K(x,y) + \int_x^{\infty} K(x,z)F(z+y)dz = 0. \tag{1.28}$$

A simplifed version [2].

Substituting ψ with the "dressing operator" in equation (1.6) gives the formula for the potential:

$$u(x,t) = -2K_x(x,x). \tag{1.29}$$

We consider the following theorem:

Theorem 1. *Let* $|H| = det(h_{ij}) = |\delta_{ij} + \int_x^{\infty} \overline{\phi}_i(x)\phi_i(x)dx|$ *(where* h_{ij} *is an n by n matrix). We define* $K(x,z) = \frac{\begin{bmatrix} h_{ij} & \phi_1(x) \\ \overline{\phi}_1(z) \ldots \overline{\phi}_n(z) & 0 \end{bmatrix}}{|H|}$ *and* $F(x,z) = \sum_{i=1}^{n} \overline{\phi}_i(z)\phi_i(x)$, *then these two functions solve the Gelfand-Levitan-Marchenko (GLM) equation.*

We prove it as follows

Proof.

$$
\begin{aligned}
\int_x^\infty K(x,s)F(s,z)ds &= \frac{1}{|H|}\int_x^\infty \begin{bmatrix} h_{ij} & \phi_1(s) \\ \overline{\phi}_1(s)...\overline{\phi}_n(s) & 0 \end{bmatrix} \sum_{i=1}^n \overline{\phi}_i(z)\phi_i(s)ds \\
&= \frac{1}{|H|}\sum_{i=1}^n \overline{\phi}_i(z) \begin{bmatrix} h_{ij} & \phi_1(s) \\ \int_x^\infty \overline{\phi}_1(s)\phi_1(s)ds...\int_x^\infty \overline{\phi}_N(s)\phi_1(s)ds & 0 \end{bmatrix} \\
&= \frac{-1}{|H|}(\sum_{i=1}^n \overline{\phi}_i(z)\phi_i(x)|H| + \begin{bmatrix} h_{ij} & \phi_1(x) \\ \overline{\phi}_1(z)...\overline{\phi}_n(z) & 0 \end{bmatrix}) \\
&= -F(x,z) - K(x,z)
\end{aligned}
$$

□

It can be shown that

$$K(x,x) = \frac{d}{dx}\log(H) = \frac{H_x}{H}, \tag{1.30}$$

therefore

$$u(x,t) = -2K_x(x,x) = -2\frac{d^2}{dx^2}\log(H). \tag{1.31}$$

Now in the case of $n = 1$, we get that

$$H = 1 + \frac{ce^{-2kx+8k^3t}}{2k}. \tag{1.32}$$

Taking second derivatives and performing necessary computations gives the result:

$$u(x,t) = -2\kappa^2\text{sech}^2[\kappa x - 4\kappa^3 t - x_0] \tag{1.33}$$

here the soliton has a phase-shift of $x_0 = \frac{1}{2k}\log\left(\frac{c}{2k}\right)$ and is moving to the right with a constant velocity $4\kappa^2$ [1]. One can perform similar, but much longer calculations and solve for the potential in case of $n = 2$ and get the two-soliton solution.

1.2 Hirota's bilinear form

In 1971 Hirota developed a new method or direct method for constructing multisoliton solutions to integrable nonlinear evolution equations [5]-[6] The basic idea was to formulate a transformation into new variables, in order for their mulisolition solutions to appear in a simple form. This method was successful and therefore ended up being applied to the Korteweg de-Vries (KdV) equation [6], modifed Korteweg de-Vries (mKdV) equation [7], sine-Gordon equation [8], and nonlinear Schrödinger (NLS) equation [10] . Later on it was observed that the new dependent variables (called tau-functions) had very good properties and therefore become a starting point for further developments [5].

Most integrable PDE's that appear in some particular problems tend to not be in the most convenient form to work with. This is where Hirota's bilinear form comes into play.

The basic idea here is that these equations of a non-convenient form can usually be bilinearized by introducing a new dependent variable whose natural degree is 0, i.e. log F or f/g.

So consider that w=α log F, where α is a free parameter. Following from this, we result with

$$F^2 \cdot (..........^2) + 3\alpha(2-\alpha)(2FF'' - F'2)F'2 = 0. \tag{1.34}$$

If we let $\alpha = 2$, then the following quadratic equation results

$$F_{xxxx}F - 4F_{xxx}F_x + 3F_{xx}^2 + F_{xt}F - F_xF_t = 0, \tag{1.35}$$

which in the Hirota bilinear form must be satisfied by two conditions: one, it must be quadratic in the dependent variables, which it is; Two, satisfy the condition of with respect to the derivatives, i.e. they should only appear in combinations that can be expressed using Hirota's D-operator [5]. This is defined as follows

$$D_x^n f \cdot g = (\partial_{x_1} - \partial_{x_2})^n f(x_1)g(x_2) \mid_{x_2=x_1=x}, \tag{1.36}$$

where the operator D operates on the product of two functions like Leibnitz rule [5], with the exception of difference of signs, i.e,

$$D_x f \cdot g = f_x g - f g_x, \tag{1.37}$$

$$D_x D_t f \cdot g = f g_{xt} - f_x g_t - f_t g_x + f g_{xt}. \tag{1.38}$$

Using the D-operator, equation (1.35) can be rewritten as follows

$$(D_x^4 + D_x D_t) F \cdot F = 0. \tag{1.39}$$

This result is made possible by the following dependent variable transformation, u=$2\partial_x^2$log F, which has been integrated once. In the following subsubsection we will show how this procedure is applied to the Korteweg de-Vries (KdV) equation, i.e, show that equation (1.39) is the bilinear form for the KdV equation.

Again, considering the Korteweg de-Vries (KdV) equation

$$u_{xxx} + 6uu_x + u_t = 0. \tag{1.40}$$

Our first step, is to do a bilinearizing transformation to the KdV equation. So, we focus on transforming the equation into a form that is quadratic in the dependent variables. In order to do so we find a transformation that is a leading derivative that should go together with a nonlinear term, and, in particular, have the same number of derivatives. If we count a derivative with respect to x having a degree of 1, then to balance the first two terms for the KdV equation, we should consider that u has a degree of 2. Hence, we introduce the transformation to a dependent variable w, which has a degree of 0, as seen before, i.e., w=α log F.

We let

$$u = \partial_x^2 w, \tag{1.41}$$

then the KdV equation can rewritten as

$$w_{xxxxx} + 6w_{xx}w_{xxx} + w_{xxt} = 0, \tag{1.42}$$

then by integrating with respect to x we result with

$$w_{xxxx} + 3w_{xx}^2 + w_{xt} = 0. \tag{1.43}$$

Therefore, we went from nonlinear to bilinear, using Hirota's bilinear form for the KdV equation [5].

1.3 Sato Theory

The previous sections serve as basis for the concept of Sato Theory. A concept that shows a deep level of algebraic and geometric understanding of integrable systems with infinitely many degrees of freedom and their solutions [14]. It's main idea is that these systems are not just solely isolated and should be seen as belonging to infinite families, i.e., hierarchies of mutually compatible systems. Systems that are governed by an infinite set of evolution parameters in terms of which their (common) solutions can be expressed [14].

Sato's theory can be described as follows:

"Start from an ordinary differential equation and suppose that its solutions satisfy certain dispersion relations, for a set of supplementary parameters. Then, as conditions on the coefficients of this ordinary differential equation, we obtain a set of integrable nonlinear partial differential equations [14]."

Thus, we will use Sato Theory to formulate the KP hierarchy in the following chapter.

CHAPTER II

SATO FORMULATION

2.1 The KP Equation

The KP (Kadomtsev-Petviashvili) equation is given by the following partial differential equation:

$$(-4u_t + 6uu_x + u_{xxx})_x + 3u_{yy} = 0. \tag{2.1}$$

If we let $u = 2w_x$, then equation (2.1) becomes

$$(-4w_{xt} + 12w_x w_{xx} + w_{xxxx}) + 3w_{xyy} = 0. \tag{2.2}$$

We then integrate with respect to x once and taking zero to be the integration constant and thus result with the potential form of the KP equation as follows

$$(-4w_t + 6w_x^2 + w_{xxx})_x + 3w_{yy} = 0. \tag{2.3}$$

Which is sometimes referred as the KP equation [12]. This version of the KP equation will play a role in a later chapter.

2.2 Lax Equation

To begin formulating the KP hierarchy, we consider the following operators as follows:
The pseudo-differential operator

$$L = \partial + u_1\partial^{-1} + u_2\partial^{-2} + u_3\partial^{-3} + ... = \partial + \sum_{n=1}^{\infty} u_n\partial^{-n}, \tag{2.4}$$

where,

$$u_i = u_i(t_1, t_2, .., t_n, ..). \tag{2.5}$$

The differential operator

$$B_n = (L^n)_+. \tag{2.6}$$

Following these two operators we consider the lax equations

$$\frac{\partial L}{\partial t_n} = [B_n, L], \tag{2.7}$$

$$\frac{\partial L^m}{\partial t_n} = [B_n, L^m]. \tag{2.8}$$

With $B_n = (L^n)_{\geq 0}$, where $n = 1, 2, ...$

2.3 The Zero Curvature Equation (Zakharov-Shabat Equation)

Now, we consider the zero curvature equation (Zakharov-Shabat Equation) lemma

$$\frac{\partial B_n}{\partial t_m} - \frac{\partial B_m}{\partial t_n} + [B_n, B_m] = 0, \tag{2.9}$$

which is satisfied by $B_n = (L^n)_{\geq 0}$. We prove it as follows

Proof. First we note,

$$\begin{aligned}
\frac{\partial L^n}{\partial t_m} &= \sum_{k=0}^{m-1} L^k \frac{\partial L}{\partial t_m} L^{n-k-1} \\
&= \sum_{k=0}^{n-1} L^k [B_m, L] L^{n-k-1} \\
&= \sum_{k=0}^{n-1} L^k B_m L^{n-k} - \sum_{k=0}^{n-1} L^{k+1} B_m L^{n-k-1} \\
&= B_m L^n - L^n B_m \\
&= [B_m, L^n].
\end{aligned}$$

Then using the decomposition $L = B_n + (L^n)_{<0}$, and (2.8) we have

$$\begin{aligned}
\frac{\partial L^n}{\partial t_m} - \frac{\partial L^m}{\partial t_n} &= [B_m, L^n] - [B_n, L^m] \\
&= [B_m, B_n] + [B_m, (L^n)_{<0}] - [B_n, L^m] \\
&= [B_m, B_n] + [B_m, (L^n)_{<0}] + [(L^n)_{<0}, L^m] \\
&= [B_m, B_n] - [(L^m)_{<0}, (L^n)_{<0}].
\end{aligned}$$

Then taking the differential part of the above equation, will result in

$$\frac{\partial B_n}{\partial t_m} - \frac{\partial B_m}{\partial t_n} + [B_m, B_n] = 0.$$

Hence proved [12]. □

With this result, we prove the following theorem:

Theorem 2. *Equations (2.7) and (2.8) are compatible, i.e.*

$$\frac{\partial^2 L}{\partial t_n \partial t_m} = \frac{\partial^2 L}{\partial t_m \partial t_n}, \tag{2.10}$$

for every $m, n \in N$.

Proof. First, we compute the right hand side

$$\begin{aligned}
\frac{\partial^2 L}{\partial t_m \partial t_n} &= [\frac{\partial B_n}{\partial t_m}, L] + [B_n, \frac{\partial L}{\partial t_m}] \\
&= [\frac{\partial B_n}{\partial t_m}, L] + [B_n, [B_m, L]].
\end{aligned}$$

Next, the left hand side

$$\frac{\partial^2 L}{\partial t_n \partial t_m} = [\frac{\partial B_m}{\partial t_n}, L] + [B_m, [B_n, L]].$$

Using both results, we obtain

$$\begin{aligned}\frac{\partial^2 L}{\partial t_m \partial t_n} - \frac{\partial^2 L}{\partial t_n \partial t_m} &= [\frac{\partial B_n}{\partial t_m} - \frac{\partial B_m}{\partial t_n}, L] + [B_n, [B_m, L]] - [B_m, [B_n, L]] \\ &= [\frac{\partial B_n}{\partial t_m} - \frac{\partial B_m}{\partial t_n}, L] + [[B_n, B_m], L] \\ &= [\frac{\partial B_n}{\partial t_m} - \frac{\partial B_m}{\partial t_n} + [B_n, B_m], L].\end{aligned}$$

Then by using the Jacobi identity for the commutator we have that

$$[\frac{\partial B_n}{\partial t_m} - \frac{\partial B_m}{\partial t_n} + [B_n, B_m], L] = 0.$$

Now from the zero curvature equation (Zakharov-Shabat Equation) Lemma, we deduce that

$$\frac{\partial^2 L}{\partial t_m \partial t_n} - \frac{\partial^2 L}{\partial t_n \partial t_m} = 0.$$

Which implies that

$$\frac{\partial^2 L}{\partial t_n \partial t_m} = \frac{\partial^2 L}{\partial t_m \partial t_n}.$$

Hence, proved [12]. □

2.4 The Dressing Operator

Now, one can eliminate all the variables u_i from lax operator L by a gauge operator W. We call this gauge operator the dressing operator. It is written in the following form:

$$W = 1 + w_1 \partial^{-1} + w_2 \partial^{-2} + ... = 1 + \sum_{k=1}^{\infty} w_k \partial^{-k} \quad (2.11)$$

such that

$$L = W \partial W^{-1}. \quad (2.12)$$

We have that the zero curvature equation (Zakharov-Shabat Equation) is the compatibility condition for the following linear problems

$$\partial t_m \psi = B_m \psi \tag{2.13}$$

$$\partial t_n \psi = B_n \psi \tag{2.14}$$

$$L\psi = z\psi \tag{2.15}$$

We denote as follows

$$\xi(t,z) = xz + t_2 z^2 + t_3 z^3 + ... \tag{2.16}$$

So we will find the solution to (2.13) and (2.14) in the form

$$\psi = (1 + \frac{\xi_1}{z_1} + \frac{\xi_2}{z_2} + ...)e^{\xi(t,z)}, \tag{2.17}$$

where the cofficients ξ_i depend only on x and on t_j. Using the dressing operator, W, in (2.17) then we can find the common solutions to (2.13) , (2.14), and (2.15). So we have that

$$\psi = We^{\xi(t,z)} = (1 + \xi_1 \partial^{-1} + \xi \partial^{-2} + ...)e^{\xi(t,z)}, \tag{2.18}$$

where ∂^{-1} acts to the exponential function according to the rule $\partial^{-1} e^{xz} = z^{-1} e^{xz}$.
Equation (2.18) is also called the $Baker - Akhiezer$ function [15].

We prove (2.13) as follows

Proof.

$$\begin{aligned}\partial t_m \psi &= z^m W e^{\xi(t,z)} + \partial t_m W e^{\xi(t,z)} \\ &= (z^m W + \partial t_m W) e^{\xi(t,z)} \\ &= (W z^m - (W \partial^m W^{-1})_- W) e^{\xi(t,z)} \\ &= (W \partial^m - (W \partial^m W^{-1})_- W) e^{\xi(t,z)} \\ &= (W \partial^m W^{-1} - (W \partial^m W^{-1})_-) W e^{\xi(t,z)} \\ &= (L^m - (L^m)_-) \psi \\ &= B_m \psi.\end{aligned}$$

Thus, it holds true [15]. □

The same follows for (2.14)

Proof.

$$\begin{aligned}\partial t_n \psi &= z^n W e^{\xi(t,z)} + \partial t_n W e^{\xi(t,z)} \\ &= (z^n W + \partial t_n W) e^{\xi(t,z)} \\ &= (W z^n - (W \partial^n W^{-1})_- W) e^{\xi(t,z)} \\ &= (W \partial^n - (W \partial^n W^{-1})_- W) e^{\xi(t,z)} \\ &= (W \partial^n W^{-1} - (W \partial^n W^{-1})_-) W e^{\xi(t,z)} \\ &= (L^n - (L^n)_-) \psi \\ &= B_n \psi.\end{aligned}$$

Thus, it holds true [15]. □

Now the proof for (2.15)

Proof.

$$\begin{aligned} L\psi &= L[We^{\xi(t,z)}] \\ &= W\partial e^{\xi(t,z)} \\ &= Wze^{\xi(t,z)} \\ &= z\psi. \end{aligned}$$

Thus, it holds true [15]. □

2.5 The Sato Equation

So, we impose that the dressing operator W satisfies the Sato equation,

$$\frac{\partial W}{\partial t_n} = B_n W - W\partial^n, \tag{2.19}$$

when $n = 1, 2, ...$ and where B_n is now given by $B_n = (W\partial W^{-1})_{\geq 0}$ [12]. Thus, we have the following theorem, in which the Sato equation is used:

Theorem 3. *If the dressing operator, W, satisfies the Sato equation, then the operator $L = W\partial W^{-1}$ satisfies the lax equation for the KP hierarchy,*

$$\frac{\partial L}{\partial t_n} = [B_n, L], \tag{2.20}$$

where $B_n = (L^n)_{\geq 0}$.

Here is the proof as follows

Proof. By direct calculation we have

$$
\begin{aligned}
\frac{\partial L}{\partial t_n} &= \frac{\partial}{\partial t_n} L \\
&= \frac{\partial}{\partial t_n}(W\partial W^{-1}) \\
&= \frac{\partial W}{\partial t_n}\partial W^{-1} + W\partial \frac{\partial}{\partial t_n}(W^{-1}) \\
&= \frac{\partial W}{\partial t_n}\partial W^{-1} - W\partial W^{-1}\frac{\partial W}{\partial t_n}W^{-1} \\
&= (B_n W - W\partial^n)\partial W^{-1} - W\partial W^{-1}(B_n W - W\partial^n)W^{-1} \\
&= B_n W\partial W^{-1} - W\partial^{n+1}W^{-1} - (W\partial B_n + W\partial^{n+1})K^{-1} \\
&= B_n W\partial W^{-1} - W\partial^{n+1}W^{-1} - W\partial W^{-1}B_n + W\partial^{n+1}W^{-1} \\
&= B_n W\partial W^{-1} - W\partial W^{-1}B_n \\
&= [B_n, L].
\end{aligned}
$$

Hence, proved [12]. □

Also $L = W\partial W^{-1}$, i.e., $LW = W\partial$ gives the following correspondences between the coefficients of L and those of W,

$$LW = (\partial + \sum_{k=1}^{+\infty} u_{k+1}\partial^{-k})(1 - \sum_{k=1}^{+\infty} w_k\partial^{-k}) \tag{2.21}$$

$$= \partial - \sum_{k=1}^{+\infty}(w_k\partial^{1-k} + w_{k,x}\partial^{-k}) + \sum_{k=1}^{+\infty}(u_{k+1}\partial^{-k} - u_{k+1}\sum_{j=1}^{+\infty}\sum_{l=1}^{+\infty}\binom{-k}{l}(\partial^l w_j)\partial^{-k-j-l}) \tag{2.22}$$

$$= \partial - \sum_{k=1}^{+\infty}(w_k\partial^{1-k} + w_{k,x}\partial^{-k}) + \sum_{k=1}^{+\infty}(u_{k+1}\partial^{-k} - u_{k+1}\sum_{m=2}^{+\infty}\sum_{l=0}^{m-k-l\geq 1}\binom{-k}{l}(\partial^l w_{m-k-l)}\partial^{-m}), \tag{2.23}$$

then

$$LW = \partial - w_1 - (w_2 + w_{1,x} - u_2)\partial^{-1}$$
$$- \sum_{m=2}^{+\infty} (w_{m+1} + w_{m,x} - u_{m+1} + \sum_{k=1} \sum_{l=0}^{m-k-l\geq 1} u_{k+1} \binom{-k}{l} (\partial^l w_{m-k-l}))\partial^{-m}. \quad (2.24)$$

And

$$W\partial = \partial + \sum_{m=0}^{+\infty} w_{m+1}\partial^{-m}. \quad (2.25)$$

So, comparing these cofficients of ∂^{-m}, we result with

$$\begin{cases} u_2 = w_{1,x} \\ u_3 = w_{2,x} + w_1 w_{1,x} \\ u_4 = w_{3,x} + w_1 w_{2,x} + w_2 w_{1,x} - w_{1,x}^2 + w_1^2 w_{1,x} \\ \ldots \end{cases} \quad (2.26)$$

This means that finding the solution for the W- equation is enough for the KP equation.

We know that u_2 satisfies the KP equation as follows

$$\frac{\partial}{\partial x}(4\frac{\partial u_2}{\partial t_3} - 12u_2 u_{2,x} - u_{2,xxx}) - 3\frac{\partial^2 u_2}{\partial t_2^2} = 0. \quad (2.27)$$

Also, the relation $u_2 = w_{1,x}$ implies that w_1 satisfies the potential KP equation, mentioned in section 2.1, as follows

$$\frac{\partial}{\partial x}(-4\frac{\partial w_1}{\partial t_3} + 6w_{1,x}^2 + w_{1,xxx}) + 3\frac{\partial^2 w_1}{\partial t_2^2} = 0. \quad (2.28)$$

Now, if we truncate the series W, by considering a finite reduction, we get

$$W = 1 + w_1\partial^{-1} + w_2\partial^{-2} + \ldots + w_N\partial^{-N}. \quad (2.29)$$

Then we define the differential operator as follows

$$W_N = W\partial^N = \partial^N + w_1\partial^{N-1} + w_2\partial^{N-2} + .. + w_N. \tag{2.30}$$

As proved in Theorem 3, the dressing operator W satisfies the Sato equation, we can also show that W_N satisfies

$$\frac{\partial W_N}{\partial t_n} = B_n W_N - W_N \partial^n, \tag{2.31}$$

where $B_n = (W\partial W^{-1})_{\geq 0}$ and $n = 1, 2, ...$

We consider the following theorem:

Theorem 4. *Suppose that f be a solution of the linear hierarchy, $\frac{\partial f}{\partial t_n} = \frac{\partial^n f}{\partial x^n}$ for $n = 1, 2, ...$ Then the N-th order differential equation $W_N f = 0$ is invariant with respect to any flow parameter t_n.*

Proof. This can be proved by simple computation,

$$\begin{aligned} \frac{\partial}{\partial t_n}(W_N f) &= \frac{\partial W_N}{\partial t_n} f + W_N \frac{\partial f}{\partial t_n} \\ &= (B_n W_N - W_N \partial^n) f + W_N \frac{\partial f}{\partial t_n} \\ &= B_n(W_N f) + W_N\left(\frac{\partial f}{\partial t_n} - \frac{\partial^n f}{\partial x^n}\right) \\ &= 0. \end{aligned}$$

Hence, proved [12]. □

From Theorem 4 and it's proof we have the following

$$\begin{cases} W_N f = 0 \\ \frac{\partial f}{\partial t_n} = \frac{\partial^n f}{\partial x^n} \end{cases} \tag{2.32}$$

where $n = 1, 2, ...$

Recall that $W_N f = 0$ [12] gives us that

$$f^{(N)} = w_1 f^{(n-1)} + w_2 f^{(N-2)} + ... + w_{N-1} f^{(1)} + w_N f, \tag{2.33}$$

then, with the linearly independent solution $f_i : i = 1, 2, ..., N$(a fundamental set of solutions), w_1 is given by

$$w_1 = \frac{\partial}{\partial x} \ln Wr(f_1, f_2, ..., f_N), \tag{2.34}$$

where $Wr(f_1, .., f_N)$ is the Wronskian determinant. Therefore the τ- function is given by

$$\tau = Wr(f_1, f_2, .., f_N). \tag{2.35}$$

2.6 Tau-function of the KP hierarchy and the Hirota bilinear equation

The Tau-function is used in the formulation of the KP hierarchy. It is regarded as a dependent variable, which allows to formulate the KP hierarchy as an infinite set of compatible equations for just one functions, rather then an infinite number of compatible equations.

Now, in terms of the tau-functions, all of the equations of the KP hierarchy become bilinear and have a single "generating equation", known as the Hirota's bilinear equation.

We consider the following theorem:

Theorem 5. *For the ψ -function of any solution to the KP-Hierarchy the following bilinear identity holds*

$$res_z(\psi(t,z)\psi^*(t',z)) = 0, \tag{2.36}$$

where $t = t_j$ and $t' = t'_j$ are two arbitrary sets of times.

We previously mentioned the following

$$\psi = We^{\xi(t,z)} = (1 + \xi_1 \partial^{-1} + \xi \partial^{-2} + ...)e^{\xi(t,z)},$$

the *Baker − Akhiezer* function, equation (2.18) from section 2.4. Now consider the following theorem where the tau-function is used:

Theorem 6. *Let ψ and ψ^* be the Baker-Ahkiezer functions of the KP hierarchy, then there exists a function $\tau(t_1,t_2,t_3,...)$ such that*

$$\psi(t,z) = e^{\xi(t,z)} \frac{\tau(t_1 - \frac{1}{z}, t_2 - \frac{1}{2z^2}, ...)}{\tau(t, t_2, t_3, ...)}, \tag{2.37}$$

and

$$\psi^*(t,z) = e^{-\xi(t,z)} \frac{\tau(t_1 + \frac{1}{z}, t_2 + \frac{1}{2z^2}, ...)}{\tau(t, t_2, t_3, ...)}. \tag{2.38}$$

I will describe the process of proving equation (2.37).

So we write $\psi(t,z)$ as follows:

$$\psi(t,z) = e^{\xi(t,z)} w(t,z), \tag{2.39}$$

and the same for $\psi^*(t,z)$:

$$\psi^*(t,z) = e^{-\xi(t,z)} w^*(t,z) \tag{2.40}$$

we take the logarithmic derivative with respect to z on both sides and result with

$$\partial_z \log w(t,z) = \sum_{m\geq 1} \frac{\partial \log w(t,z)}{\partial t_m} z^{-m-1} + \sum_{m\geq 1} \frac{\partial \log \tau}{\partial_{t_m}} z^{-m-1} \tag{2.41}$$

which is also equal to

$$\frac{\partial \log \tau}{\partial t_n} = res_z(z^n(\partial_z - \partial(z)) \log w(t,z)), \tag{2.42}$$

where

$$\partial(z) := \sum_{j\geq 1} z^{-j-1} \frac{\partial}{\partial t_j}. \tag{2.43}$$

So, to show that the tau-function exists is enough to prove the following expression

$$res_z(z^n(\partial_z - \partial(z))\partial_{t_m} \log w(t,z)) \tag{2.44}$$

is symmetric under the permutation of m and n.

The overall idea of the proof is based on the bilinear identity. We let $t'_j = t_j - \zeta^{-j}/j$ in the bilinear idenity and write as follows

$$res_z(\psi(t,z)\psi^*(t-[\zeta^{-1}],z)) = 0, \tag{2.45}$$

where we conveniently introduce the follwing

$$F(t \pm [z]) \equiv F(t_1 \pm z, t_2 \pm \frac{z^2}{2}, t_3 \pm \frac{z^3}{3}, ..). \tag{2.46}$$

We substitute equation (2.39) and equation (2.40) into equation (2.45) and result with

$$res_z(\frac{w(t,z)w^*(t-[\zeta^{-1}],z)}{1-\frac{z}{\zeta}}) = 0. \tag{2.47}$$

One can see that for any series $f(z) = 1 + \sum_{j\geq 1} f_j z^{-j}$ the identity

$$res_z(\frac{f(z)}{1-\frac{z}{\zeta}}) = \sum_{j\geq 1} f_j \zeta^{1-j} = \zeta(f(\zeta)-1), \tag{2.48}$$

holds.

Applying to the previous equality we get the relation between w and w^* as follows

$$w(t,z)w^*(t,z)(t-[z^{-1}],z) = 1. \tag{2.49}$$

In a similar fashion, from the bilinear identity

$$res_z(\psi(t,z)\psi^*(t-[\zeta^{-1}]-[\zeta^{-2}],z)) = 0 \tag{2.50}$$

can be rewritten as follows

$$res_z(\frac{w(t,z)w^*(t-[\zeta^{-1}]-[\zeta^{-2}],z)}{(1-\frac{z}{\zeta_1})(1-\frac{z}{\zeta_2})})=0 \tag{2.51}$$

it follows that

$$w(t,z)w^*(t-[\zeta^{-1}]-[\zeta^{-2}],\zeta_1)=w(t,\zeta_2)w^*(t-[\zeta^{-1}]-[\zeta^{-2}],\zeta_2), \tag{2.52}$$

where the following identity is used

$$\frac{1/\zeta_1-1/\zeta_2}{(1-\frac{z}{\zeta_1})(1-\frac{z}{\zeta_2})}=\frac{1}{\zeta_1(1-\frac{z}{\zeta_1})}-\frac{1}{\zeta_2(1-\frac{z}{\zeta_2})}, \tag{2.53}$$

and equation (2.48).

Using equation (2.53), we can express w^* through w and let $\zeta_1=z$, and $\zeta_2=\zeta$.

So, the result is

$$\frac{w(t,z)}{w(t-[\zeta^{-1}],z)}=\frac{w(t,\zeta)}{w(t-[z^{-1}],\zeta)}. \tag{2.54}$$

Now, we take the log of the equality (2.54) and use the operator $\partial_z-\partial(z)$ and get

$$(\partial_z-\partial(z))\log w(t,z)-(\partial_z-\partial(z))\log w(t-[\zeta^{-1}],z)=-\partial\log w(t,\zeta). \tag{2.55}$$

To make it simple we let $Y_n(t):=res_z(z^n(\partial_z-\partial(z))\log w(t,z))$ [15]. Thus, multiplying both sides of equation (2.55) by z^n and taking the residue, we result with

$$Y_n(t)-Y_n(t-[\zeta^{-1}])=-\partial_{t_n}\log w(t,\zeta). \tag{2.56}$$

So we differentiate with respect to t_m and subtracting a similar equality with interchanged m, n, thus yielding

$$\partial_{t_m}Y_n(t)-\partial_{t_n}Y_m(t)=\partial_{t_m}Y_n(t-[\zeta^{-1}])-\partial_{t_n}Y_m(t-[\zeta^{-1}]). \tag{2.57}$$

We denote the left hand side by $F_{mn}(t)$. Then expand $F_{mn}(t-[\zeta^{-1}])$ in powers of ζ as follows

$$F_{mn}(t-[\zeta^{-1}]) = F_{mn}(t) - \zeta^{-1}\partial_{t_1}F_{mn}(t) + \frac{1}{2}\zeta^{-2}(\partial_{t_1}^2 F_{mn}(t) - \partial_{t_2}F_{mn}(t)) + ... \tag{2.58}$$

Since $F_{mn}(t-[\zeta^{-1}]) = F_{mn}(t)$ for all t_k and all solutions, the comparison of coefficients at ζ^{-1} implies that $\partial_{t_1}F_{mn}(t) = 0$ for all t, meaning that F_{mn} does not depend on t_1. The same goes for ζ^{-2}. Therefore this goes on beyond ζ^{-3} and conclude that F_{mn} does not depend on all times, i.e. it is constant. Of course the trivial solution, $u_i = 0$ the constant is 0. Since F_{mn} is a differential polynomial of u_i, then this constant is equal to 0 for any solution. Hence, $\partial_{t_m}Y_n(t) = \partial_{t_n}Y_m(t)$, implying the existence of the tau-function and that both equation (2.37) and equation (2.42) hold true. A similar proof for equation (2.38) can be shown as well [15].

Through the tau-function, ψ and ψ^* can be expressed using the "Japanese" formulas. Doing so, results the following bilinear relation of the tau-function:

$$res_z(\tau(t-[z^{-1}])\tau(t'+[z^{-1}])e^{\xi(t-t',z)}) = 0, \tag{2.59}$$

or

$$\oint e^{\xi(t-t',z)}\tau(t-[z^{-1}])\tau(t'+[z^{-1}])dz = 0. \tag{2.60}$$

Equation (2.60) is equal to an infinite system of bilinear differential equations of the tau-function. To acquire these, we equal the expansion coefficients in the Taylor series for the left hand side in $t'-t$ to zero.

Now if we substitute $t_i - T_i$, and $t_i + T_i$ for t_i and t_i', then

$$res_z[\tau(t-T-[z^{-1}])\tau(t+T+[z^{-1}])e^{-2\xi(T,z)}] = res_z[e^{\xi(\overline{\partial}_T,z^{-1})}](\tau(t-T)\tau(t+T))e^{-2\xi(T,z)}] \tag{2.61}$$

$$= res_z[\sum_{j\geq 0} z^{-j}h_j(\overline{\partial}_T)(\tau(t-T)\tau(t+T))\sum_{l\geq 0} z^l h_l(-2T)] \tag{2.62}$$

$$= \sum_{j\geq 0} h_j(-2T)h_{j+1}(\overline{\partial}_T)\tau(t-T)\tau(t+T) \tag{2.63}$$

$$= 0. \tag{2.64}$$

So

$$\sum_{j\geq 0} h_j(-2T)h_{j+1}(\overline{\partial}_T)\tau(t-T)\tau(t+T) = 0. \tag{2.65}$$

Equation (2.65) can also be expressed as follows

$$\sum_{j\geq 0} h_j(-2T)h_{j+1}(\overline{\partial}_X)e^{\sum_{l\geq 1} T_l D_l}\tau(t-X)\tau(t+X) = 0, \tag{2.66}$$

where $X_m = 0$.

In this defined rule

$$P(D)f(t)\cdot g(t) := P(\partial_X)(f(t-X)(g(t+X)) = 0 \tag{2.67}$$

where $X = 0$ [15]. We use the symbol D_j of the "Hirota derivative" [15] for any polynomial $P(D)$ of D_i , we can write equation (2.66) in the following form

$$\sum_{j\geq 0} h_j(-2T)h_{j+1}(\overline{D})e^{\sum_{l\geq 1} T_l D_l}\tau(t)\cdot\tau(t) = 0. \tag{2.68}$$

Where, the relation contains all of the bilinear Hirota equations for the KP hierarchy.

In equation (2.59) we let $t' = t - [\lambda_1{}^{-1}] - [\lambda_2{}^{-1}] - [\lambda_3{}^{-1}]$, where $\lambda_{1,2,3}$ are arbitrary complex parameters, i.e.

$$t_k' = t_k - \frac{\lambda_1{}^{-k}}{k} - \frac{\lambda_2{}^{-k}}{k} - \frac{\lambda_3{}^{-k}}{k}. \tag{2.69}$$

With these in mind, the bilinear relation takes on the following form

$$res_z\left(\frac{\tau(t-[z^{-1}])\tau(t-[\lambda_1{}^{-1}]-[\lambda_2{}^{-1}]-[\lambda_3{}^{-1}]+[z^{-1}])}{(1-\frac{z}{\lambda_1})(1-\frac{z}{\lambda_2})(1-\frac{z}{\lambda_3})}\right)=0. \tag{2.70}$$

Now, we use the identity (2.48) to represent the product of the pole factors as a sum of poles:

$$\frac{(\frac{1}{\lambda_1}-\frac{1}{\lambda_2})(\frac{1}{\lambda_1}-\frac{1}{\lambda_3})(\frac{1}{\lambda_2}-\frac{1}{\lambda_3})}{(1-\frac{z}{\lambda_1})(1-\frac{z}{\lambda_2})(1-\frac{z}{\lambda_3})}=\frac{\frac{(\frac{1}{\lambda_2}-\frac{1}{\lambda_3})}{\lambda_1{}^2}}{1-\frac{z}{\lambda_1}}+(231)+(312), \tag{2.71}$$

where the last two terms are obtained from the first one by cyclic permutations of indices $(1\to 2, 2\to 3, 3\to 1)$ and $(1\to 3, 2\to 1, 3\to 2)$ [15].

Again using identity (2.48), we result in

$$\begin{aligned}(\lambda_2-\lambda_3)\tau(t-[\lambda_1{}^{-1}])\tau(t-[\lambda_2{}^{-1}]-[\lambda_3{}^{-1}])\\ +(\lambda_3-\lambda_1)\tau(t-[\lambda_2{}^{-1}])\tau(t-[\lambda_3{}^{-1}]-[\lambda_1{}^{-1}])\\ +(\lambda_1-\lambda_2)\tau(t-[\lambda_3{}^{-1}])\tau(t-[\lambda_1{}^{-1}]-[\lambda_2{}^{-1}])=0.\end{aligned} \tag{2.72}$$

This result is satisfied by the tau-function of the KP hierarchy, and is called the Hirota-Miwa equation.

CHAPTER III

GENERAL SOLUTION TO THE KP HIERARCHY AND IT'S REDUCTION

3.1 The General Solution

We first show that the KP hierarchy indeed contains equation (2.1) (the KP equation). We substitute $n = 2$ and $m = 3$ into $B_n = (L^n)_{\geq 0}$ and get that

$$\begin{cases} B_2 = (L^2)_+ = \partial^2 + 2u_1 \\ B_3 = (L^3)_+ = \partial^3 + 3u_1\partial + 3(u_{1,x} + u_2). \end{cases} \tag{3.1}$$

Substituting (3.1) into the Zakharov-Shabat equation gives us:

$$0 = \frac{\partial B_2}{\partial t_3} - \frac{\partial B_3}{\partial t_2} + [B_2, B_3], \tag{3.2}$$

or

$$0 = (-3\frac{\partial u_1}{\partial t_2} + 3u_{1,xx} + 6u_{2,x})\partial + 2\frac{\partial u_1}{\partial t_3} - 3\frac{\partial}{\partial t_2}(u_{1,x} + u_2) + u_{1,xxx} + 3u_{2,xx} - 6u_1u_{1,x}. \tag{3.3}$$

From here

$$\begin{cases} \frac{\partial u_1}{\partial t_2} = u_{1,xx} + 2u_{2,x} \\ 2\frac{\partial u_1}{\partial t_3} = 3\frac{\partial}{\partial t_2}(u_{1,x} + u_2) - u_{1,xxx} - 3u_{2,xx} + 6u_1u_{1,x} \end{cases} \tag{3.4}$$

Eliminating u_2 and letting $(u_1 = \frac{u}{2}, t_2 = y, t_3 = t)$ gives us the KP equation. Also, we can let $(u_1 = -w_{1_x}, t_2 = y, t_3 = t)$ and get the potential form of the KP equation as seen in section 2.1,

$$(-4w_t + 6w_{1,x}^2 + w_{1,xxx})_x + 3w_{1,yy} = 0 \tag{3.5}$$

Now, we would like to find a general solution to equation (3.5).

Truncating W and multiplying it by ∂^N defines the following differential operator:

$$W_N = W\partial^N = \partial^N + w_1\partial^{N-1} + w_2\partial^{N-2} + ... + w_N,$$

as seen in section 2.5 as equation (2.30).

Also, from section 2.5 we have that the dressing operator W satisfies the Sato equation, then W_N satisfies

$$\frac{\partial W_N}{\partial t_n} = B_n W_N - W_N \partial^n,$$

which is equation (2.31).

Another result was from the proof of Theorem 4 in section 2.5, which was

$$\begin{cases} W_N f = 0 \\ \frac{\partial f}{\partial t_n} = \frac{\partial^n f}{\partial x^n} \end{cases}$$

that is equation (2.32).

So, we have the following functions that satisfy Theorem 4 from section 2.5:

$$f = \sum_{i=1}^{N} a_i E_i \tag{3.6}$$

such that

$$E_i = e^{k_i x + k_i^2 t_2 + k_i^3 t_3 + ...} \tag{3.7}$$

Now, by Cramer's rule, we get

$$-f^{(N)} = w_1 f^{(N-1)} + w_2 f^{(N-2)} + ... + w_{N-1} f^{(1)} + w_N f \tag{3.8}$$

then, for the set of linearly independent solutions, $(f_i : i = 1, 2, ..., N)$, we denote $\tau = Wr(f_1, f_2, ..., f_N)$. Therefore we have that

$$w_1 = -(\ln \tau)_x \tag{3.9}$$

Since w_1 satisfies equation (3.5), we have that

$$u = -2w_{1,x} = 2(\ln\tau)_{xx} \tag{3.10}$$

satisfies equation (2.1). Thus, finding the general solution to the KP hierarchy.

3.2 One-Soliton Solution for the KP Equation

If we let $N = 2$ (we get $N-1 = 1$ solitons), $a_1 = a_2 = 1$, and consider a regular three-dimensional KP equation with $t_2 = y$, $t_3 = t$. then our solutions to the KP hierarchy, equation (3.5), take the following form:

$$f_1(x,y,t) = e^{k_1x+k_1^2y+k_1^3t} + e^{k_2x+k_2^2y+k_2^3t} \tag{3.11}$$

From equation (3.9), we have that

$$w_1 = -(\ln f_1)_x = \frac{1}{2}(k_1+k_2) + \frac{1}{2}(k_1-k_2)\tanh(\frac{\xi_1-\xi_2}{2}),$$

where $\xi_i = k_ix + k_i^2y + k_i^3t$. Thus, equation (3.10) becomes:

$$u = \frac{1}{2}(k_1-k_2)^2\operatorname{sech}^2[\frac{1}{2}(k_1-k_2)x + \frac{1}{2}(k_1^2-k_2^2)y + \frac{1}{2}(k_1^3-k_2^3)t],$$

which is a one-soliton solution to the KP equation.

3.3 2-Reduction

We use the Gelf'and - Dikii reduction:

$$L^N = B_N = (L^N)_{\geq 0} \tag{3.12}$$

Then equation (2.7), the KP hierarchy, becomes

$$\frac{\partial B_N}{\partial t_n} = [B_n, B_N]. \tag{3.13}$$

When $N = 2$, we get that

$$L^2 = \partial^2 + 2u_1 + (u_{1,x} + 2u_2)\partial^{-1} + (u_{2,x} + u_1^2 + 2u_3)\partial^{-2} + ... \tag{3.14}$$

Now, we know that

$$B_2 = (L^2)_+ = \partial^2 + 2u_1, \tag{3.15}$$

iterating, we get

$$\begin{cases} u_2 = -\frac{u_{1,x}}{2} \\ u_3 = -\frac{u_{1,xx}}{4} - \frac{u_1^2}{2} \\ ... \end{cases} \tag{3.16}$$

Where all u_j's are determined by a single variable u_1. Now, we have that

$$B^3 = (\partial + u_1\partial^{-1} + u_2\partial^{-2})(\partial^2 + 2u_1) = \partial^3 + 3u_1\partial + \frac{3u_{1,x}}{2}. \tag{3.17}$$

Therefore, calculating $[B_3, B_2]$ with

$$\begin{cases} B_3B_2 = \partial^5 + 5u_1\partial^3 + \frac{15}{2}u_{1,x}\partial^2 + 6(u_{1,xx} + u_1^2)\partial + 2u_{1,xxx} + 9u_1u_{1,x} \\ B_2B_3 = \partial^5 + 5u_1\partial^3 + \frac{15}{2}u_{1,x}\partial^2 + 6(u_{1,xx} + u_1^2)\partial + \frac{3}{2}u_{1,xxx} + 3u_1u_{1,x} \end{cases} \tag{3.18}$$

gives us that

$$\frac{\partial u_1}{\partial t_3} = \frac{1}{4}u_{1,xxx} + 3u_1u_{1,x}. \tag{3.19}$$

Substituting $u_1 = \frac{u}{2}$ and $t_3 = t$ yields

$$-4u_t + 6uu_x + u_{xxx} = 0, \tag{3.20}$$

which is a KdV equation. Now, we can easily get a one-soliton solution to the KdV equation from the one-soliton solution to the KP equation by letting $k_1 = -k_2$ which will cancel the y component from equation (3.11) and give us the following:

$$u = 2k_2^2\text{sech}^2[-\frac{1}{2}k_2x - k_2^3t], \tag{3.21}$$

which is a one-soliton solution to equation (3.20) - the KdV equation.

3.4 3-Reduction

Now, we let N=3 and get that

$$L^3 = \partial^3 + 3u_1 + 3(u_2 + u_{1,x}) + (3u_3 + 3u_{2,x} + 3u_1^2 + u_{1,xx})\partial^{-1} + \dots \tag{3.22}$$

We have that

$$B_3 = \partial^3 + 3u_1\partial + \frac{3u_{1,x}}{2}. \tag{3.23}$$

Now, letting $u = 3u_1$ and $v = 3u_2 + 3u_{1,x}$, we get that

$$\begin{cases} u_1 = \frac{u}{3} \\ u_2 = \frac{1}{3}(v - u_x) \\ u_3 = \frac{1}{9}(2u_{xx} - 3v_x - u^2) \\ \dots \end{cases} \tag{3.24}$$

We plug in $n = 2$ and $N = 3$ into equation (3.6) which yields

$$\frac{\partial B_3}{\partial t_2} = [B_2, B_3]. \tag{3.25}$$

Now, we calculate $[B_2, B_3]$ and get the following system of equations:

$$\begin{cases} \frac{\partial u}{\partial t_2} = -u_{xx} + 2v_x \\ \frac{\partial v}{\partial t_2} = v_{xx} - \frac{2}{3}(u_{xxx} + uu_x) \end{cases} \tag{3.26}$$

Eliminating v from the system and substituting $t_2 = y$ along with $u = \frac{3u}{2}$, we obtain

$$3u_{yy} + (u_x xx + 6uu_x)_x = 0, \tag{3.27}$$

which is the Boussinesq equation. Similarly to the KdV case, we can get a one-soliton solution to the

Boussinesq equation from the one-soliton solution to the KP equation by letting $k_1 = k_2 e^{-\frac{2\pi i}{3}}$ which will cancel the t component from equation (3.11) and give us the following:

$$u = (\frac{3}{4} + \frac{3\sqrt{3}i}{4})k_2^2 \text{sech}^2[(-\frac{3}{4} - -\frac{\sqrt{3}i}{4})k_2 x + (-\frac{3}{4} + \frac{\sqrt{3}i}{4})k_2^2 y], \tag{3.28}$$

which is a one-soliton solution to equation (3.27) - the Boussinesq equation.

CHAPTER IV

MULTI-SOLITON SOLUTION

4.1 Hirota's direct method

In section 3.2 we gave the one-soliton solution for the KP equation. Now we will work towards a multi-soliton solution for it. To do so we will use the direct method developed by Hirota as stated in section 1.2, that is used for deriving multiple-soliton solutions for completely integrable equations [3]-[4]. The following bilinear differential operators were introduced in this method:

$$D_t^m D_x^n (a * b) = (\frac{\partial}{\partial t} - \frac{\partial}{\partial t'})^m (\frac{\partial}{\partial x} - \frac{\partial}{\partial x'})^n a(x,t) b(x',t'), \tag{4.1}$$

where $x = x'$ and $t = t'$.

The solution for the KP equation can be expressed as follows

$$u = 2\frac{\partial^2}{\partial x^2} \ln f. \tag{4.2}$$

Where f is represented by following expansion

$$f = 1 + \sum_{n=1}^{\infty} \varepsilon^n f_n(x,y,t), \tag{4.3}$$

and ε is a formal expansion parameter. In our case, multiple-soliton solutions, we set

$$f = 1 + \varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 f_3 + ... \tag{4.4}$$

where the functions $f_1, f_2, f_3, ...$ can be found via Hirota bilinear formalism or direct substitution of (4.3) into the correct equation, as will be seen later.

4.2 A simplified version of Hirota's direct method

Hirota's direct method always leads to a bilinear form, if such form exists. In this method it was shown that soliton solutions are just polynomials of exponentials. This method uses (127) to show the dependent variable transformation

$$u = 2(\ln f)_{xx}, \tag{4.5}$$

that transforms (2.1) into the bilinear form

$$B(f,f) = (D^4 + D_x D_t \pm D_y^2) f \cdot f = 0, \tag{4.6}$$

or more so

$$[f(f_{xt} + f_{4x} \pm f_{2y})] - [f_x f_t + 4 f_x f_{3x} - 3 f_{2x}^2 \pm f_y^2] = 0. \tag{4.7}$$

A simplified way of Hirota method was introduced by Hereman et,al. [4,5].

The method is as follows:

Equation (4.7) can be transformed into the linear operator L and the nonlinear operator N as follows

$$L = \frac{\partial^2}{\partial x \partial t} + \frac{\partial^4}{\partial x^4} \pm \frac{\partial^2}{\partial y^2}, \tag{4.8}$$

$$N(f,f) = -f_x f_t - 4 f_x f_{3x} + 3 f_{2x} f_{2x} \pm f_y^2. \tag{4.9}$$

Now, let f have an expansion of the form as in equation (4.3) but this time ε is a non-small formal expansion parameter.

With both the direct and simplified version of Hirota's method we substitute (4.3) into (4.7) and by equating the powers of ε to zero, we result with

$$\begin{cases} O(\varepsilon^1): Lf_1 = 0, \\ O(\varepsilon^2): Lf_2 = -N(f_1, f_1), \\ O(\varepsilon^3): Lf_3 = -f_1Lf_2 - N(f_1, f_2) - N(f_2, f_1), \\ O(\varepsilon^4): Lf_4 = -f_1Lf_3 - f_2Lf_2 - f_3Lf_1 - N(f_1, f_3) - N(f_2, f_2) - N(f_2, f_1), \\ \ldots \\ O(\varepsilon^n): Lf_n = -\sum_{j=1}^{n-1}[f_jLf_{n-1} + N(f_j, f_{n-j})] = 0. \end{cases} \tag{4.10}$$

Thus, the N-soliton solution is obtained from

$$f_1 = \sum_{i=1}^{N} e^{\theta_i}, \tag{4.11}$$

where $\theta_i = k_ix + m_iy - c_it$ and where k_i, m_i, and c_i are arbitrary constants.

Plugging (156) into $O(\varepsilon^1): Lf_1 = 0$, we get the dispersion relation

$$c_i = \frac{k_i^4 \pm m_i^2}{k_i}. \tag{4.12}$$

Thus,

$$\theta_i = k_ix + m_iy - \frac{k_i^4 \pm m_i^2}{k_i}. \tag{4.13}$$

Therefore,

$$f_1 = e^{\theta_1} = e^{k_1x + m_1y - \frac{k_1^4 \pm m_1^2}{k_1}}, \tag{4.14}$$

which is acquired by setting $N = 1$ in (4.11).

4.3 The Application

We will show two, three, and four soliton solutions.

First, the two-soliton solution:

Set $N = 2$ in (4.11). We get

$$f_1 = e^{\theta_1} + e^{\theta_2}, \tag{4.15}$$

and we have that

$$f = 1 + e^{\theta_1} + e^{\theta_2} + f_2(x, y, t). \tag{4.16}$$

Now we substitute equation (4.16) into $O(\varepsilon^2) : Lf_2 = -N(f_1, f_1)$ and then evaluate the right hand side and equating it to the left hand side we obtain

$$f_2 = \sum\nolimits_{1 \le i < j \le 2} a_{ij} e^{\theta_1 + \theta_2}, \tag{4.17}$$

in which

$$a_{12} = \frac{3k_1^2 k_2^2 (k_1 - k_2)^2 - (k_1 m_2 - k_2 m_1)^2}{3k_1^2 k_2^2 (k_1 + k_2)^2 - (k_1 m_2 - k_2 m_1)^2}, \tag{4.18}$$

and θ_1, θ_2 are given by $\theta_i = k_i x + m_i y - c_i t$.

Since we are using $1 \le i < j \le 2$, we obtain the following

$$f = 1 + e^{\theta_1} + e^{\theta_2} + a_{12} e^{\theta_1 + \theta_2} \tag{4.19}$$

$$= 1 + e^{\theta_1} + e^{\theta_2} + \frac{3k_1^2 k_2^2 (k_1 - k_2)^2 - (k_1 m_2 - k_2 m_1)^2}{3k_1^2 k_2^2 (k_1 + k_2)^2 - (k_1 m_2 - k_2 m_1)^2} e^{\theta_1 + \theta_2}, \tag{4.20}$$

the two-soliton solution.

To find the two-soliton solution explicitly, use equation (4.5) for the function f into equation (4.20).

Next, the three-soliton solution:

Set $N = 3$ in (4.11). We get

$$f_1 = e^{\theta_1} + e^{\theta_2} + e^{\theta_3}. \tag{4.21}$$

Also

$$f_2 = a_{12} e^{\theta_1 + \theta_2} + a_{23} e^{\theta_2 + \theta_3} + a_{13} e^{\theta_1 + \theta_3}. \tag{4.22}$$

Overall, we have that

$$f = 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + a_{12} e^{\theta_1 + \theta_2} + a_{23} e^{\theta_2 + \theta_3} + a_{13} e^{\theta_1 + \theta_3} + f_3(x, y, t). \tag{4.23}$$

Now we substitute equation (4.23) into $O(\varepsilon^3): Lf_3 = -f_1Lf_2 - N(f_1, f_2) - N(f_2, f_1)$, we result with

$$f_3 = b_{123}e^{\theta_1+\theta_2+\theta_3}, \tag{4.24}$$

where

$$a_{ij} = \frac{3k_i^2k_j^2(k_i-k_j)^2-(k_im_j-k_jm_i)^2}{3k_i^2k_j^2(k_i+k_j)^2-(k_im_j-k_jm_i)^2}, \tag{4.25}$$

$1 \le i < j \le 3$, since it is a three-soliton solution case, and also that $b_{123} = a_{12}a_{13}a_{23}$, where $\theta_1, \theta_2, \theta_3$ are given by $\theta_i = k_ix + m_iy - c_it$.

Therefore, we obtain that

$$f = 1+e^{\theta_1}+e^{\theta_2}+e^{\theta_3}+a_{12}e^{\theta_1+\theta_2}+a_{13}e^{\theta_1+\theta_3}+a_{23}e^{\theta_2+\theta_3}+b_{123}e^{\theta_1+\theta_2+\theta_3}. \tag{4.26}$$

or equivalently

$$\begin{aligned} f = 1+e^{\theta_1}+e^{\theta_2}+e^{\theta_3}+\frac{3k_1^2k_2^2(k_1-k_2)^2-(k_1m_2-k_2m_1)^2}{3k_1^2k_2^2(k_1+k_2)^2-(k_1m_2-k_2m_1)^2}e^{\theta_1+\theta_2} \\ +\frac{3k_2^2k_3^2(k_2-k_3)^2-(k_2m_3-k_3m_2)^2}{3k_2^2k_3^2(k_2+k_3)^2-(k_2m_3-k_3m_2)^2}e^{\theta_2+\theta_3} \\ +\frac{3k_1^2k_3^2(k_1-k_3)^2-(k_1m_3-k_3m_1)^2}{3k_1^2k_3^2(k_1+k_3)^2-(k_1m_3-k_3m_1)^2}e^{\theta_1+\theta_3} \\ +(\frac{3k_1^2k_2^2(k_1-k_2)^2-(k_1m_2-k_2m_1)^2}{3k_1^2k_2^2(k_1+k_2)^2-(k_1m_2-k_2m_1)^2})(\frac{3k_1^2k_3^2(k_1-k_3)^2-(k_1m_3-k_3m_1)^2}{3k_1^2k_3^2(k_1+k_3)^2-(k_1m_3-k_3m_1)^2}) \\ ((\frac{3k_2^2k_3^2(k_2-k_3)^2-(k_2m_3-k_3m_2)^2}{3k_2^2k_3^2(k_2+k_3)^2-(k_2m_3-k_3m_2)^2})e^{\theta_1+\theta_2+\theta_3}). \end{aligned} \tag{4.27}$$

the three-soliton solution.

To find the three-soliton solution explicitly, use equation (4.5) for the function f into equation (4.27).

Then the four-soliton solution:

Set $N = 4$ in (4.11). We get

$$f_1 = e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + e^{\theta_4}, \tag{4.28}$$

$$f_2 = a_{12}e^{\theta_1+\theta_2} + a_{13}e^{\theta_1+\theta_3} + a_{14}e^{\theta_1+\theta_4} + a_{23}e^{\theta_2+\theta_3} + a_{24}e^{\theta_2+\theta_4} + a_{34}e^{\theta_3+\theta_4}, \tag{4.29}$$

$$f_3 = b_{123}e^{\theta_1+\theta_2+\theta_3} + b_{124}e^{\theta_1+\theta_2+\theta_3} + b_{134}e^{\theta_1+\theta_3+\theta_4} + b_{234}e^{\theta_2+\theta_3+\theta_4}. \tag{4.30}$$

Overall, we have that

$$\begin{aligned} f = 1 &+ e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + e^{\theta_4} + a_{12}e^{\theta_1+\theta_2} + a_{13}e^{\theta_1+\theta_3} + a_{14}e^{\theta_1+\theta_4} + a_{23}e^{\theta_2+\theta_3} + a_{24}e^{\theta_2+\theta_4} + a_{34}e^{\theta_3+\theta_4} \\ &+ b_{123}e^{\theta_1+\theta_2+\theta_3} + b_{124}e^{\theta_1+\theta_2+\theta_3} + b_{134}e^{\theta_1+\theta_3+\theta_4} + b_{234}e^{\theta_2+\theta_3+\theta_4} + f_4(x,y,t). \end{aligned} \tag{4.31}$$

Now we substitute equation (4.31) into

$O(\varepsilon^4) : Lf_4 = -f_1Lf_3 - f_2Lf_2 - f_3Lf_1 - N(f_1, f_3) - N(f_2, f_2) - N(f_2, f_1),$

we result with

$$f_4 = c_{1234}(\theta_1 + \theta_2 + \theta_3 + \theta_4), \tag{4.32}$$

where $c_{1234} = a_{12}a_{13}a_{14}a_{23}a_{24}a_{34}$ and $\theta_1, \theta_2, \theta_3, \theta_4$ are given by $\theta_i = k_ix + m_iy - c_it$.

Since we are doing four-soliton solution, we use $1 \le i < j \le 4$ and therefore obtain that

$$\begin{aligned} f = 1 &+ e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + e^{\theta_4} + a_{12}e^{\theta_1+\theta_2} + a_{13}e^{\theta_1+\theta_3} + a_{14}e^{\theta_1+\theta_4} + a_{23}e^{\theta_2+\theta_3} + a_{24}e^{\theta_2+\theta_4} + a_{34}e^{\theta_3+\theta_4} \\ &+ b_{123}e^{\theta_1+\theta_2+\theta_3} + b_{124}e^{\theta_1+\theta_2+\theta_3} + b_{134}e^{\theta_1+\theta_3+\theta_4} + b_{234}e^{\theta_2+\theta_3+\theta_4} + c_{1234}(\theta_1 + \theta_2 + \theta_3 + \theta_4). \end{aligned} \tag{4.33}$$

which is the four-soliton solution.

Where

$$a_{ij} = \frac{3k_i^2k_j^2(k_i - k_j)^2 - (k_im_j - k_jm_i)^2}{3k_i^2k_j^2(k_i + k_j)^2 - (k_im_j - k_jm_i)^2}, 1 \le i < j \le 4, \tag{4.34}$$

$b_{ijr} = a_{ij}a_{ir}a_{jr}$, $1 \le i < j < r \le 4$, and $c_{1234} = a_{12}a_{13}a_{14}a_{23}a_{24}a_{34}$.

To find the four-soliton solution explicitly, use equation (4.5) for the function f into (4.33).

So the multi-soliton solutions for the KP equation are as follows

Two-soliton solution:

$$f = 1 + e^{\theta_1} + e^{\theta_2} + a_{12}e^{\theta_1+\theta_2}.$$

equation (4.19).

Three-soliton solution:

$$f = 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + a_{12}e^{\theta_1+\theta_2} + a_{13}e^{\theta_1+\theta_3} + a_{23}e^{\theta_2+\theta_3} + b_{123}e^{\theta_1+\theta_2+\theta_3}.$$

equation (4.26).

Four-soliton solution:

$$\begin{aligned} f = 1 &+ e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + e^{\theta_4} + a_{12}e^{\theta_1+\theta_2} + a_{13}e^{\theta_1+\theta_3} + a_{14}e^{\theta_1+\theta_4} + a_{23}e^{\theta_2+\theta_3} + a_{24}e^{\theta_2+\theta_4} + a_{34}e^{\theta_3+\theta_4} \\ &+ b_{123}e^{\theta_1+\theta_2+\theta_3} + b_{124}e^{\theta_1+\theta_2+\theta_3} + b_{134}e^{\theta_1+\theta_3+\theta_4} + b_{234}e^{\theta_2+\theta_3+\theta_4} + c_{1234}(\theta_1+\theta_2+\theta_3+\theta_4). \end{aligned}$$

equation (4.33).

Similarly, this method follows for a five-soliton solution and beyond, i.e, $N \geq 6$, as stated by Wazwaz [13]. Hirota [6]-[7]-[8]-[9]-[11], implied that the soliton solutions are just polynomials of exponentials and also that higher-soliton solutions do not have new free parameters other than a_{ij} derived from the two-soliton solution.

CHAPTER V

SUMMARY AND CONCLUSION

By looking into the soltion theory of the Kortewegde-Vries equation or KdV equation via inverse scattering transform, and how Hirota's bilinear method applied to the KdV equation. We were able to, with the help of Sato Theory, reduce the KP hierarchy into the KdV equation (2-reduction) and into the Boussinesq equation (3-reduction), find it's general solution and one-soliton solution, and applying Hirota's bilinear form (direct method) to find it's multiple soliton solutions.

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BIOGRAPHICAL SKETCH

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