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Generalized &thetas;-Parameter Peakon Solutions for a Cubic Camassa-Holm Model

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GENERALIZED θ-PARAMETER PEAKON SOLUTIONS

FOR A CUBIC CAMASSA-HOLM MODEL

A Thesis

by

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ABSTRACT

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In this paper we outline a method for obtaining generalized peakon solutions for a cubic Camassa-Holm model originally introduced by Fokas (1995) and recently shown to have a Lax pair representation and bi-Hamiltonian structure by Qiao et al (2012). By considering an amended signum function – denoted $\text{sgn}_\theta(x)$ – where $\text{sgn}(0) = \theta$ for a constant θ, we explore new generalized peakon solutions for this model. In this context, all previous peakon solutions are of the case $\theta = 0$. Further, we aim to analyze the algebraic quadratic equation resulting from a substitution of the single-peakon ansatz equipped with our amended signum function in order to determine the effects of constants $k_1, k_2, c,$ and θ on the wave height. Moreover, we introduce a new measure R relating the scalars k_1, k_2 of the cubic and quadratic nonlinearity terms which we find has deterministic properties relating to the existence of real vs complex solutions.

DEDICATION

This paper is dedicated to my parents, for instilling the importance of both education and hard work, and for always pushing me to do and be better; my partner, for being my rock and keeping my feet on the ground even on my worst days; my advisor Dr. Zhijun Qiao for guiding me through my master's thesis and introducing me to the world of integrable systems; my community college professors Ted Koukounas and Leslie Buck for showing me that I had the mettle to pursue graduate study; and my high school math teachers Mrs. Mecca, Mrs. Rosler, and Mr. Czartosieski ("Ski") for inculcating a persistent attitude and a love of math, even if it took me a few years to realize it.

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CHAPTER I

BACKGROUND

The purpose of this thesis is to identify and analyze new generalized single-peakon solutions for a generalized cubic Camassa-Holm model first derived by Qiao et al. in [15]. This thesis also aims to investigate the historical development of traveling wave solutions and the Camassa-Holm equation.

1.1 Conventions

In this paper, the following conventions will be used:

1. the variable ξ will always be a traveling wave ansatz, i.e. $\xi = x - ct$, and if this variable appears with no other context it should be assumed to be said ansatz;
2. subscripts on a function will always mean a partial derivative, e.g. $u_x = \partial_x u = \partial u/\partial x$, except in the case (4) below;
3. the prime ($'$) notation refers to the derivative of a single-variable function, e.g. $U' = dU(\xi)/d\xi$;
4. the letter δ refers to the Dirac delta distribution and the subscript notation refers to the distribution with respect to the subscript, e.g. $\delta_x u = \langle \delta(x),\ u \rangle$;
5. we denote the space of test functions (i.e. $C_0^\infty(\Omega)$) by D, and the space of distributions by D'.

1.2 Partial Differential Equations

A partial differential equation is an equation of multiple variables in terms of an unknown function and its (partial) derivatives. In general, a partial differential equation of some equation

$u(\mathbf{x})$, where $\mathbf{x} = (x_1, x_2, ..., x_n)$ is a vector in n-space, is an equation of the form

$$F(\mathbf{x}, u, u_{x_1}, ..., u_{x_n}, u_{x_1 x_1}, ...) = 0.$$

While the study of partial differential equations in an abstract sense is both important and satisfying in its own right, there are a plethora of applications in the physical sciences. Indeed the application of ordinary differential equations to certain problems involving functions of both space *and* time leaves much to be desired. Consider the study of current in electrical circuits where $u = u(t)$ describes the current at a given time t: if one needs to describe the current at a particular time, then an ordinary differential equation may be derived; however, if one needs to describe the current at a particular time *and* space, then the ordinary differential equation will not provide the necessary information. According to Guenther and Lee (1988), if it is "realistic to think of the medium as a continuum, then the attempt to obtain specific information about the system at some point in space at some time leads in general to a partial differential equation, an integral equation, or to systems of such equations" [8]. If we combine any of these equations with some data describing the initial state of the system (e.g. initial current in Amps) then we call it an initial and/or boundary value problem, depending on the context and the goal.

For example, consider the initial value problem $u_{tt} = c^2 u_{xx}$ with initial data $u(x, 0) = \varphi(x)$ and $u_t(x, 0) = \psi(x)$. The equation $u_{tt} = c^2 u_{xx}$ is called the undamped wave equation which models a wide variety of physical phenomena, including vibrating strings, current (voltage) in a wire, and the velocity of gas molecules. A novel solution to this problem was discovered by the French mathematician John le Rond d'Alembert in the 18th century. While it is possible to solve this type of initial value problem via Fourier

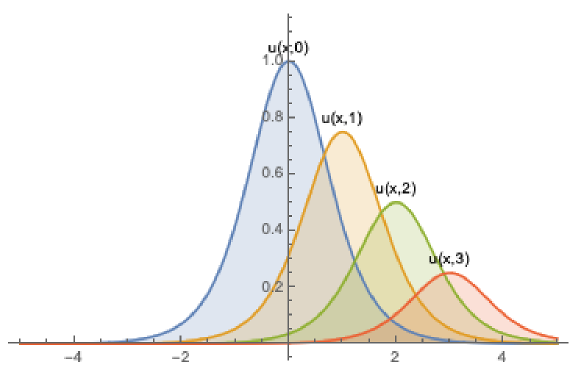


Figure 1.1: A traveling wave.

series, d'Alembert's method introduced the concept of *traveling wave solutions*. A solution is considered a traveling wave solution if x and t appear in the combination $x - ct$, $c \in \mathbb{R}$, i.e. $f(\xi) = f(x - ct)$ by the ansatz $\xi = x - ct$.

1.3 Operators

Operators are mappings from a space into itself acting on an element within that space. Intuitively, one may think of operators as analogous to "verbs," i.e. actions. For example, consider the action of taking the derivative of a function with respect to one of its variables. If $f(x_1,\ x_2,\ ...,\ x_n)$ is a function in n-space, we can express the derivative of f with respect to x_i $(i = 1,\ 2,\ ...,\ n)$ as

$$f_{x_i} = \frac{\partial f}{\partial x_i}.$$

We can take advantage of familiar shorthand to create an operator whose "action" is taking the derivative of f with respect to x_i by defining

$$\partial_{x_i} = \frac{\partial}{\partial x_i}.$$

This is analogous to the more familiar d/dx operator from single-variable calculus. Likewise we can denote higher-order derivatives using the same superscript notation as, e.g., d^2x/dx^2,

$$\partial_x^2 = \frac{\partial^2}{\partial x^2},$$

and further with mixed higher-order derivatives

$$\partial_{xt} = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right)$$

where $u = u(x,t)$. For the rest of this section, we will assume that $u = u(x,t)$ is an infinitely continuously differentiable function in both of its variables, i.e. $u \in C^\infty(\mathbb{R},\mathbb{R})$, and that $f = f(\xi)$ is also infinitely continuously differentiable with respect to ξ, i.e. $f \in C^\infty(\mathbb{R})$.

Example 1.3.1. Consider the expression

$$(\partial_x \, u \, \partial_x) \cdot f.$$

We may begin to simplify this expression by noting an important property: operators only ”act” on the object to their immediate right. Therefore,

$$\partial_x \cdot f = f_x$$

and our expression becomes

$$\partial_x \cdot u \, f_x.$$

Now it may seem as if the remaining operator should act on u alone, but the object to the right of the operator is really the quantity (uf_x), and thus it must act on the entire object:

$$\partial_x \cdot (uf_x) \;=\; (uf_x)_x.$$

We complete our evaluation by applying the chain rule to obtain

$$(uf_x)_x \;=\; uf_{xx} + u_x f_x.$$

We then work ”backwards” by identifying that this is equivalent to

$$u \, \partial_x^2 f + u_x \, \partial_x f \implies (u \, \partial_x^2 \; + \; u_x \partial_x) \cdot f.$$

Therefore, we have the equality

$$\partial_x \, u \, \partial_x \;=\; u \, \partial_x^2 \; + \; u_x \partial_x$$

for all u. □

In this exploratory exercise we have identified a number of properties of operators. For simplicity, let A, B and C be operators.

1. Right-hand distributivity: $(A \pm B) \cdot u \;=\; Au \;\pm\; Bu$
2. Associative law: $(AB)C \;=\; A(BC)$
3. Equality if and only if $Af = Bf$
4. Product law: $ABf = A(Bf)$
5. Commutativity generally does not hold.

We now define an important higher-order object: the **commutator**, given by $[A,\ B]$ where A and B are operators, defined as

$$[A,\ B] = AB - BA.$$

If commutativity is to hold, then we must have $[A,\ B] = 0$. In any other case, we have

$$[A,\ B] = -[B,\ A].$$

We can further scrutinize the behavior of operators by considering a special subset of operators: **linear operators**. An operator is linear if and only if scalar multiplication and "vector" addition are preserved,

$$A(cf(x) + cg(x)) \implies cAf(x) + cAg(x)$$

for all scalars c and "vectors" $f,\ g$.

Example 1.3.2. Consider the differential operator $D \;=\; d/dx$. We will show that D is a linear operator. From calculus we know that

$$D(f(x) + g(x)) = Df(x) + Dg(x)$$

and that

$$D(cf(x)) = cDf(x),$$

from which it follows that $D = d/dx$ is a linear operator. $\square$

Next we explore another important property of operators: for every linear operator[1] L there exists another linear operator L^*, called the **adjoint** of L, such that the inner product $\langle \cdot, \cdot \rangle$ satisfies

$$\langle Ax,\ y\rangle \ = \ \langle x, A^* y\rangle.$$

Moreover, a linear operator L is **self-adjoint** if $\langle Ax,\ y\rangle \ = \ \langle x,\ Ay\rangle$, i.e. if L is its own adjoint. Consider a physical system comprised of a single particle, regarded as a wavefunction Ψ, inside a box. The **Hamiltonian operator** H, defined for any system as $H \ = \ K + P$ where K is the kinetic energy operator and P is the potential energy operator, is given by

$$H \ = \ -\frac{\hbar^2}{2m}\,\partial_{xx} \tag{1.1}$$

where $\hbar$ is the *reduced Planck constant* and m is the mass of the particle. We will now prove that H is self-adjoint.

Proof. Let $f(x)$ and $g(x)$ be smooth functions compatible with the system's boundary data (i.e. $f,\ g \to 0$ as $x \to a$). Multiplying both sides of (1.1) by $f(x)$ on the right and $g(x)$ on the left we have

$$f(x)Hg(x) \ = \ f(x)\left(-\frac{\hbar^2}{2m}\right)\partial_{xx}g(x).$$

Applying the operators and then reorganizing gives

$$f(x)Hg(x) \ = \ -\frac{\hbar^2}{2m}f(x)g''(x).$$

[1]Every linear operator $L:\ H \to H$ where H is a Hilbert space

Taking the integral from 0 to a,

$$\int_0^a f(x)Hg(x)\,dx \;=\; -\frac{\hbar^2}{2m}\int_0^a f(x)g''(x)dx,$$

and then integrating by parts by applying $\int fg' = fg + \int f'g$,

$$-\frac{\hbar^2}{2m}\int_0^a f(x)g''(x)\,dx \;=\; -\frac{\hbar^2}{2m}\left(f(x)g'(x)\big|_0^a - \int_0^a f'(x)g'(x)dx\right).$$

Since f and g vanish at 0 and a by our initial assumption, this cleans up nicely as

$$\int_0^a f(x)g''(x)\,dx \;=\; \int_0^a f'(x)g'(x)\,dx.$$

Integrating by parts and applying the boundary conditions once more, we have

$$\int_0^a f(x)g''(x)\,dx \;=\; \int_0^a f''(x)g(x)\,dx.$$

Generalizing to an indefinite integral to emphasize abstraction of the definition (though one should read the following integrals with the limits of integration $0..a$ to continue the context of the example), his implies that

$$\int fHg\,dx = \int gHf\,dx \;\implies\; \int f^*Hg\,dx = \left(\int g^*Hf\,dx\right)^*,$$

which shows that H is self-adjoint. $\square$

One immediate consequence of H being self-adjoint is that we can now apply what is known as the **spectral theorem** to H. The spectral theorem is an abstraction and extension of the concept of eigenvectors and eigenvalues of matrices in Linear Algebra (recall that if A is a matrix, λ is an eigenvalue of A, and v is the eigenvector corresponding to λ, then $Av = \lambda v$). For the rest of this paragraph, we will continue to refer to H as the Hamiltonian in order to maintain a soft connection to the preceding material, but one may read H as any linear operator on a Hilbert space. Since H is self-adjoint, it is symmetric and can be decomposed into the form $H = UDU^*$ where U is a unitary matrix and D is a diagonal matrix, where the entries D_{ii} $(i = 1, 2, ..., n)$ on the diagonal of D are the n eigenvalues of H, and the columns of U are, individually, the n corresponding eigenvectors of H. The set of eigenvalues of H is called the **spectrum of** H. Moreover, finding the energy contained within this system at some position **x** and time t is equivalent to finding the **eigenfunctions** (and their associated eigenvalues) of $H\Psi$, i.e. applying the Hamiltonian operator H to the wavefunction Ψ.

1.4 Distribution Theory

When one is first introduced to the study of ordinary differential equations it is typical for one to work squarely in the realm of continuous functions, i.e. the various spaces C^k of k-times continuously differentiable functions. In the spectral theory of operators it becomes necessary to work with the space of absolutely continuous functions, where

$$f(x) = \int_{x_0}^{x} g(s)\, ds \,+\, C = f(x_0) \,+\, \int_{x_0}^{x} g'(s)\, ds$$

gives g as the derivative of f, locally integrable on compact subsets [3].

Then, in the study of partial differential equations, we are forced to expand our definition of a ”suitable derivative” to accommodate increasingly complex functions. Let Ω be an open subset of $\mathbb{R}^n$, and let $f(x_1, x_2, ..., x_n)$ and $g(x_1, x_2, ..., x_n)$ be locally integrable on Ω. We say g is an n-th weak derivative of f with respect to x_i if

$$\int_{\Omega} g(x)\varphi(x)\, dx = -\int_{\Omega} f(x) \cdot \partial^{n}_{x_i}\, \varphi(x)\, dx. \tag{1.2}$$

for all test functions $\varphi \in C_0^\infty(\Omega)$, where $C_0^k(S)$ denotes the space of k-times continuously differentiable functions with compact support [7].

Even more generally, sometimes one needs to be able to define a derivative in such a way that it is no longer considered a function. In 1951, the French mathematician Laurent Schwartz introduced the idea of a *distribution*, an object more general than a function, whose definition relies on the idea of *linear functionals*. We introduce the concept of functionals with a simple analogy: if $x_i \mapsto f(x_i)$ is a function of variable x_i, then $f \mapsto f(x_i)$ is a functional of parameter x_i. More rigorously, a functional is a mapping

$$\Lambda \,:\, X \to \mathbb{C}$$

where X is (typically) a function space. Indeed, we can regard the left-hand side of (1.2) as a functional $\Lambda : C_0^\infty(\Omega) \to \mathbb{C}$ such that

$$\Lambda_g : \varphi \mapsto \Lambda_g(\varphi) = \int_\Omega g \cdot \varphi \, dx. \tag{1.3}$$

If we let Λ be a linear functional on $C_0^\infty(\Omega)$ such that

$$-\int_\Omega f \cdot \partial_j \varphi \, dx = \Lambda(\varphi)$$

for all $\varphi \in C_0^\infty(\Omega)$, then we say

$$\partial_j f = \Lambda$$

in the distribution sense even if there is no locally integrable function g such that Λ can be explicitly defined in terms of g as in (1.3) [7].

In order to make some future claims regarding distributions rigorously supported, we will now discuss some theoretical preliminaries. In this paper we will frequently make use of the Dirac delta distribution, denoted $\delta(x)$, which is defined as

$$\delta(x) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0 \end{cases} \qquad \text{or} \qquad \int \delta(x)\varphi(x)\, dx = \varphi(0) \;\; \forall\, \varphi \in C_0^\infty$$

and usually used to model a pointwise force (see Fig. 1.2).

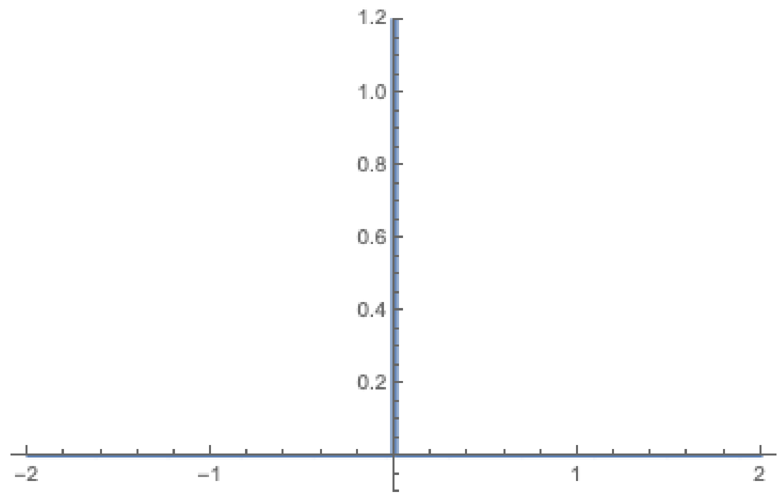


Figure 1.2: The Dirac delta distribution.

We will show that $\delta(x - x_0)$ has compact support. By definition, $\delta = 0$ on every open subset $X \subseteq \mathbb{R} \backslash \{x_0\}$. Consider a subset $X_1 \subset \mathbb{R}$ such that $x_0 \in X_1$, and a test function $\varphi(x) \in D(X_1)$ (i.e. the space of test functions over X_1). Then,

$$\int_{X_1} \delta(x - x_0)\varphi(x)\, dx = \varphi(x_0) \neq 0$$

from which it immediately follows that supp $\delta(x - x_0) = \{x_0\}$.

The δ distribution also has the interesting property that it is the derivative of the Heaviside function $H(x)$, and by extension the sign function sgn(x). We compute the first directly:

$$\int_{\mathbb{R}} H'[x]\varphi(x)\, dx = \int_{\mathbb{R}} -\varphi'(x)H[x]\, dx = \int_0^\infty -\varphi'(x)\, dx = \varphi(0) = \int_{\mathbb{R}} \delta[x]\varphi(x)\, dx$$

and the second likewise:

$$\begin{aligned}
-\int_{\mathbb{R}} \text{sgn}[x]\varphi'(x)\,dx &= \int_{-\infty}^{0} \varphi'(x)\,dx - \int_{0}^{\infty} \varphi'(x)\,dx \\
&= \varphi(0) - \varphi(-\infty) - (\varphi(\infty) - \varphi(0)) \\
&= 2\varphi(0) \\
&= 2\int \delta[x]\varphi(x)\,dx.
\end{aligned}$$

Ergo, $\frac{d}{dx}\text{sgn}(x) = 2\delta[x]$. We will also need to make use of the derivative of the Dirac distribution, denoted in the usual sense by $\delta'[x]$, namely in the context of multiplication by a smooth function:

$$\delta'[x]f(x) = f(0)\delta'[x] - f'(0)\delta[x].$$

1.5 Integrable Systems

Consider a general dynamical system represented by

$$u_t = K(u)$$

where K is some vector field on a manifold M and u is a point on M. This equation is commonly called a **flow** on M whereon a point can be interpreted as literally flowing along a predetermined path called an **orbit**. Recall the definition of the Hamiltonian from Section 1.2, where it is described as the sum of the potential and kinetic energy of a system, i.e. $H = V + T$. We can write V and T in more specific forms, namely

$$V = V(\mathbf{x}), \qquad \mathbf{x} = \{x_1, x_2, ..., x_n\} \in \mathbb{R}^n$$

(i.e. V depends on position) and

$$T = \frac{1}{2}\sum_{i=1}^{n} m_i x_i^2$$

where m_i is the mass of the i-th particle. Since the structure of a system doesn't necessarily depend on the mass of the particles, canonical coordinates are introduced which transforms $\mathbb{R}^n$ into the **phase space** $\mathbb{R}^{2n}$ by

$$q_i = x_i, \quad p_i = m_i x_i'$$

which follows from the fact that momentum $p = mv$ where m denotes mass and v velocity. Since x_i denotes position, x_i' naturally denotes velocity. Then, using the fact that the Hamiltonian is the Legendre transform of the Lagrangian $\mathcal{L}$, where $p_i = \partial\mathcal{L}/\partial q_i'$, we have

$$q_i' = \frac{\partial H}{\partial p_i}, \qquad p_i' = -\frac{\partial H}{\partial q_i},$$

which can also be expressed as

$$u_t = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \nabla H$$

where $u = (q_1, q_2, ...q_n, p_1, p_2, ..., p_n)^T$. [6]

Example 1.5.1. Consider a system with evolution equations $x_t = y$, $y_t = -x$, otherwise known as the harmonic oscillator. This system can be rewritten as

$$\begin{pmatrix} x \\ y \end{pmatrix}_t = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

and clearly $\nabla H = \begin{pmatrix} x & y \end{pmatrix}^T$ where $H = \frac{1}{2}(x^2 + y^2)$ since $\frac{1}{2}\nabla(x^2 + y^2) = \frac{1}{2}\begin{pmatrix} 2x & 2y \end{pmatrix}^T = \begin{pmatrix} x & y \end{pmatrix}^T$. $\square$

Example 1.5.2. Consider the evolution equation for a pendulum given in polar coordinates by

$$\varphi_{tt} + \sin(\varphi) = 0.$$

Converting to canonical coordinates we set $q = \varphi$, $p = \varphi_t$ and $u = \begin{pmatrix} q & p \end{pmatrix}^T$ to obtain

$$u_t = \begin{pmatrix} q \\ p \end{pmatrix}_t = \begin{pmatrix} p \\ -\sin(q) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \sin(q) \\ p \end{pmatrix}$$

since $p_t = (\varphi_t)_t = \varphi_{tt} = -sin(\varphi)$. Then, seeing that $\nabla H = \begin{pmatrix} \sin(q) & p \end{pmatrix}^T$ we compute

$$\int \sin(q)\, dq = -\cos(q) + C_1(p), \qquad \partial_p(-\cos(q) + C_1(p)) = C_1'(p)$$

and therefore $C_1'(p) = p \implies C_1(p) = \frac{1}{2}p^2 + C_2$ which leads to

$$H = -\cos(q) + \frac{1}{2}p^2 + C_2.$$

Finally, by the initial condition $H(0,0) = 0$,

$$H = -\cos(q) + \frac{1}{2}p^2 + 1$$

for all p, q. □

The connection between the Hamiltonian and an integrable system is facilitated by an object called the **Poisson bracket**. Given two differentiable functions $H(x,p)$ and $L(x,p)$ where vectors $x, p \in \mathbb{R}^n$, the Poisson bracket of H with L is given by

$$\{H, L\} = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial p_i} \frac{\partial L}{\partial x_i} - \frac{\partial H}{\partial x_i} \frac{\partial L}{\partial p_i} \right).$$

Moreover, we call L a **first integral** of the system iff

$$\{H, L\} = 0.$$

Finally, we call a system **completely integrable** if it has n first integrals where n is the number of degrees of freedom in the system.

CHAPTER II

TRAVELING WAVE SOLUTIONS

2.1 Solitons

In 1834, John Scott Russell noted a very interesting phenomena. While a large boat was being pulled through a narrow canal by a pair of horses, a small wavefront began to travel with it.

> When the boat suddenly stopped, the wave accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind rolled forward with a great velocity assuming the form of a large solitary elevation, a rounded, smooth, and well-defined heap of water which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. [5]

Russell was later able to replicate what he saw which led to him forming four key properties of solitary waves:

1. The waves are stable and can travel over very large distances;
2. the speed depends on the size of the wave, and its width on the depth of water;
3. the waves do not merge: larger waves overtake smaller ones rather than combining; and
4. if a wave is too big for the depth of water, it splits into two waves, one big and one small. [14]

Despite how crucial these initial observations were, Kruskal and Zabusky were technically the first to identify some of the more important properties of solitons, including stability, in the 1960s. Using computer algorithms, they observed that if a system initiates with two solitons, eventually the faster one will overtake the slower one. Moreover, after the two solitons meet, they will both continue on their paths unchanged. Additionally, they noted that any solution of the wave equation with any initial data will decompose as $t \to \infty$ into a finite number of smaller solitons plus a gradually-disappearing tail.

Consider the Korteweg-de Vries (KdV) equation given by

$$u_t + uu_x + u_{xxx} = 0, \qquad (2.1)$$

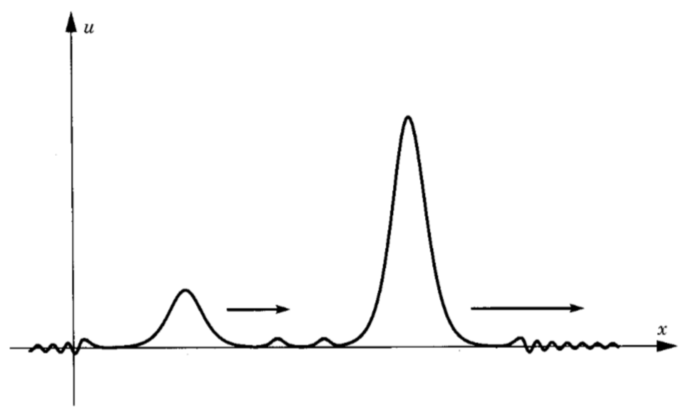


Figure 2.1: A soliton decomposing with tails visible [14].

which governs the motion of water waves in a shallow basin. In the context of traveling wave solutions, $u = u(x, t)$ represents the water's elevation (above its equilibrium elevation) at time t and position x. To better understand the mathematics behind the KdV equation, let us split the equation into two separate equations, a dispersive component and a nonlinear component, given by $u_t + u_{xxx} = 0$ and $u_t + uu_x = 0$, respectively. The dispersive component of the equation endows the waves with the natural physical ability of a liquid to ”spread out” over a given space, and a wave model consisting of only a dispersive component would see the wave packet decompose over time into its individual component wave packets of varying wavelengths. Combined in the combination (2.1), the dispersive and nonlinear effects effectively ”cancel” each other, creating a self-reinforcing wave packet.

It is obvious that this is a third-order nonlinear partial differential equation. The exact solutions to this equation are indeed solitons. In 1967, Gardner, Greener, Kruskal, and Miura discover a new method to solve the KdV initial-value problem, where the initial data is given by $u(x,0) = u_0(x) \; \forall \, x \in \mathbb{R}$. They considered the time-independent Schrodinger equation with $U(x) \equiv -u(x,t)$ to be the potential energy coefficient, and thus identified two contexts for solutions: t-independent solutions (i.e. L^2 energy eigenvalues E), and elementary evolution according to the KdV equation. Then, in 1968, Peter Lax introduced his method of generalizing the methodology of Gardner, Greener, Kruskal and Miura, by identifying **Lax pairs** which allow us to reformulate the KdV equation as a commutator of operators. Finally, in 1974, Segur et al introduce the terminology **inverse scattering** with their inverse scattering theorem, which created a link between the methods identified in 1967 with the Lax method and the Fourier transform. [11]

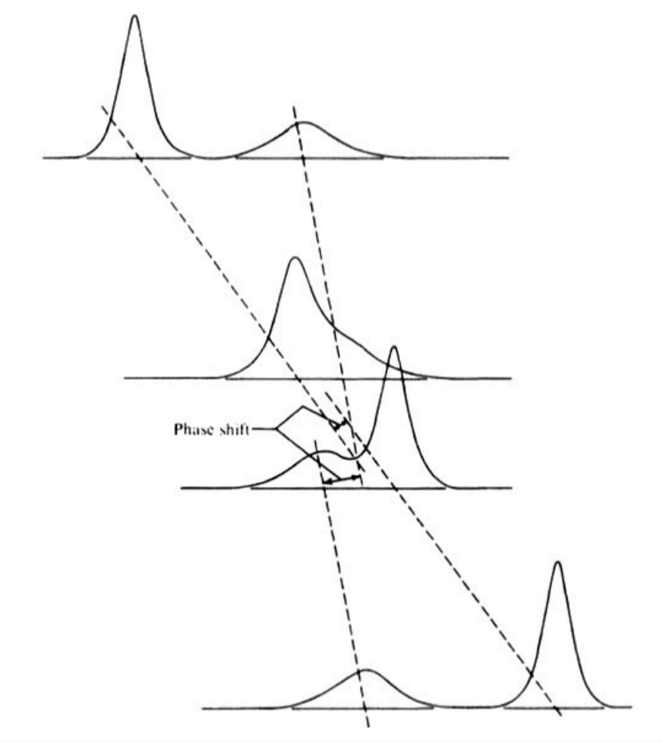


Figure 2.2: Interaction and phase shift of two solitons [1].

The KdV equation can be solved by making a substitution of the ansatz $\xi = x - ct$, yielding the equation $u(x,t) = U(\xi) = U(x-ct)$. The resulting ODE is $-cU' + U''' + UU' = 0$, which then reveals the right-moving soliton solution

$$u(x,t) = -\frac{c}{2}\text{sech}\left(\frac{\sqrt{c}}{2}(x-ct)\right), \quad (2.2)$$

which was well-known to Kruskal and Zabusky. They observed that if, at time $t = 0$, two waves of the form of (2.2) are present with the larger to the left of the smaller, then after a sufficient length of time the larger wave will ”catch up” to and ”merge” with the smaller. After more time, the waves will again separate with the larger wave re-emerging with its original height and speed; the only change is a phase shift, i.e. the ”center” of each wave will be in a different location than it would if there had been no interaction.

It was then in 1968 that Miura discovered an important transformation after researching (and initially identifying four) conservation laws associated with the KdV equation and another form of the equation commonly called the **modified KdV equation**, which is given by

$$u_t + u_{xxx} + \alpha u^2 u_x = 0.$$

Assume that v is a solution to (2.1); then, another solution is given by

$$u = -(v^2 + v_x). \tag{2.3}$$

We can then reformulate (2.1) as

$$-\left(2v + \frac{\partial}{\partial x}\right) M(v) = 0 \tag{2.4}$$

where $M(v)$ is the mKdV equation.

Consider the transform $v = \Psi_x/\Psi$, which linearizes (2.3) as

$$u = -\left(\left(\frac{\Psi_x}{\Psi}\right)^2 + \left(\frac{\Psi_x}{\Psi}\right)_x\right) \implies u = -\frac{\Psi_{xx}}{\Psi} \implies \Psi_{xx} + \Psi u = 0.$$

Accounting for an arbitrary phase shift, Miura, Gardner and Kruskal (1968) considered

$$\Psi_{xx} + (\lambda + u)\Psi = 0 \tag{2.5}$$

which, remarkably, is the time-independent Schrodinger equation. [2]

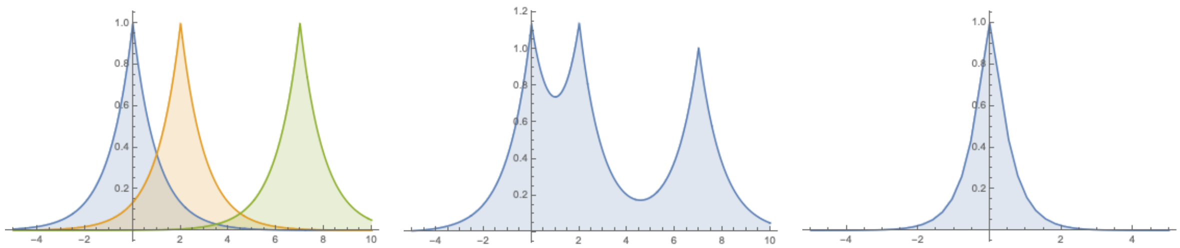
2.2 Peakons

Of course, not all solitons are well-behaved, i.e. have analytic derivatives such that $u \in C^\infty$. Consider the peakon, which is a special type of soliton defined as an n-times continuously differentiable function $u(\xi)$ with peak point ξ_0 such that $u \in C(\mathbb{R})$, $u \in C^n(\mathbb{R} \setminus \{\xi_0\})$, and

$\lim_{\xi \to \pm\infty} u(\xi) = A$. In other words, it is an n-times continuously differentiable function (except at its singular peak position ξ_0 where u attains its global infimum or supremum), which eventually disperses to some constant A and, given the point of n-th order discontinuity ξ_0, we have

$$\lim_{\xi \to \xi_0^-} u^{(n)}(\xi) = -\lim_{\xi \to \xi_0^+} u^{(n)}(\xi),$$

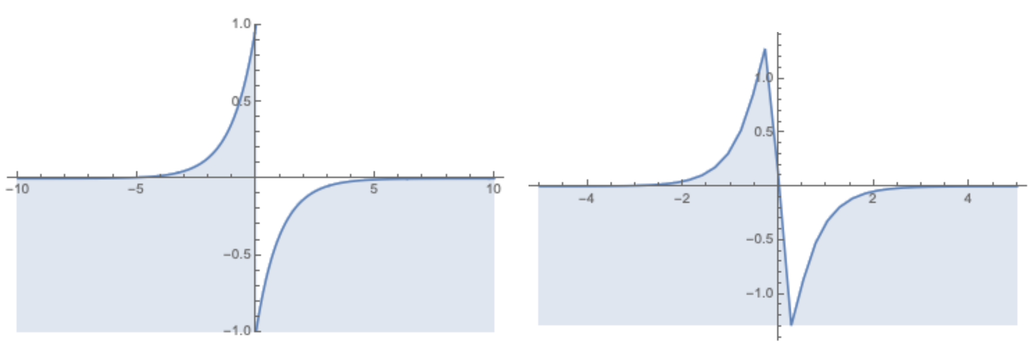
where the limits are finite. Moreover, we can define a yet more specific class of peakons called the pseudo-peakons, which look like peakons but have a continuous first-order derivative.



(a) Three individual peakons. ***(b)*** A three-peakon system. ***(c)*** A pseudo-peakon.

Figure 2.3: A comparison between three individual peakons (a) free, (b) as a system. A pseudo-peakon is shown in (c).

Fig. 2.3(a) shows a regular peakon given by $u(\xi) = Ce^{-|\xi|}$ with $C = 1,\ c = 1$ at time $t = 0,\ 2,\ 7$. Fig. 2.3(c) shows a pseudo-peakon given by $u(\xi) = \frac{\alpha\beta}{\alpha-\beta}\left(\frac{1}{\beta}e^{-\frac{1}{\alpha}|\xi|} - \frac{1}{\alpha}e^{-\frac{1}{\beta}|\xi|}\right)$, with $\alpha = 0.1,\ \beta = 0.5,\ c = 1$ [12]. Illustrated also by Fig. 2.3(c) is the pseudo-peakon's ability to masquerade as a peakon.



(a) The first derivative of a peakon. ***(b)*** The first derivative of a pseudo-peakon.

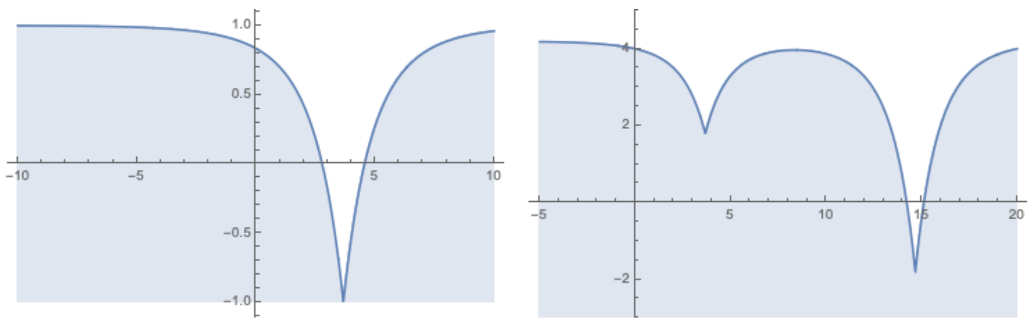
Figure 2.4: The first derivative of a peakon vs a pseudo-peakon, where (a) shows the characteristic jump discontinuity at $x = 0$, and (b) shows continuity at $x = 0$.

2.3 Cuspons

Another unique type of traveling wave is the cuspon, which is very similar to a peakon except we have the property

$$\lim_{\xi \to \xi_0^-} u'(\xi) = \pm\infty = -\lim_{\xi \to \xi_0^+} u'(\xi),$$

i.e. the first left- and right-derivatives are infinite, with one going to positive infinity and the other to negative infinity. Recall that the definition of a regular peakon requires the limits to be finite.



(a) A single cuspon. ***(b)*** A two-cuspon system.

Figure 2.5: A single cuspon vs a two-cuspon system.

CHAPTER III

CAMASSA-HOLM EQUATIONS AS FLUID MODELS

3.1 Hierarchy

Consider the balanced wave equation given by

$$m_t + m_x u + bmu_x = 0, \qquad u = g * m \tag{3.1}$$

otherwise known as the "b-family of fluid transport equations", which was first proposed by Holm and Staley in [9]. Since we have a free parameter b, this equation actually represents a family of equations: when $b = 2$ we obtain the Camassa-Holm equation; when $b = 3$, we have the Degasperis-Procesi (DP) equation. Indeed, it was shown by Novikov in 2002 that this family of equations is only integrable for $b = 2,\ 3$. The $*$ notation denotes the usual convolution

$$u(x) = \int_{-\infty}^{\infty} g(x-y)m(y)\,dy,$$

and therefore one can intuitively regard the um_x term as describing fluid convection, and the bmu_x term as describing fluid stretching.

3.2 Camassa-Holm

In 1993, Camassa and Holm identified a new type of completely integrable dispersive shallow-water equation now known as the **Camassa-Holm equation**, which is given by

$$u_t + 2\kappa u_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx} \tag{3.2}$$

where $m = u - u_{xx}$ is the momentum and $u(x,t)$ represents the fluid's horizontal velocity (i.e. wave height relative to its equilibrium position). Derived from Euler's equations for an inviscid incompressible fluid of uniform density, this equation was a novel discovery due to the existence of not only soliton solutions, but peakon solutions. It was shown through this work that the Camassa-Holm equation describes two-dimensional propagating waves in shallow water with a flat basin.

Moreover, the Camassa-Holm model of water waves (i.e. the Camassa-Holm equation applied to the theory of water waves) was a significant achievement in that it ”dethroned” the Korteveg-de Wries equation from its position as the ”go-to” model. This is largely due to the fact that the Camassa-Holm model accurately describes wave-breaking, while the KdV model does not [10]. Wave-breaking is the phenomenon that occurs when a wave's height reaches a critical level at which point the wave packet can no longer physically support itself and, as a result, energy is dissipated.

The Camassa-Holm equation is bi-Hamiltonian, meaning that it admits two Hamiltonian structures, namely

$$m_t = -J_n \frac{\delta H_n[m]}{\delta m}$$

where H_n and J_n, $n = 1, 2$, are given by

$$H_1 = \frac{1}{2}\int mu\, dx, \qquad H_2 = \frac{1}{2}\int u^3 + uu_x^2 + 2\kappa u^2\, dx,$$

where δ denotes the variational derivative, and $J_1 = 2\kappa\partial + m\partial + \partial m$, $J_2 = \partial - \partial^3$ [10]. Moreover, the Camassa-Holm equation also admits a Lax pair given by

$$\begin{aligned} \Psi_{xx} &= \left(\frac{1}{4} + \lambda(m+\kappa)\right)\Psi \\ \Psi_t &= \left(\frac{1}{2\lambda} - u\right)\Psi_x + \frac{1}{2}u_x\Psi + \gamma\Psi \end{aligned}$$

with arbitrary constant γ.

In 2006, Qiao et al. investigated a weak traveling wave solution to the Camassa-Holm equation, given by

$$u(x,t) = -a\sinh(|x-ct-\xi_0|) + ce^{-|x-ct-\xi_0|}$$

where $a \in \mathbb{R}$ is an arbitrary constant, c is wave speed, and $\xi_0 = x_0 - ct_0$ is an arbitrary real constant [13]. Setting $a = 0$ we have

$$u(x,t) = ce^{-|x-ct-\xi_0|}$$

which is a single peakon solution, first described by Camassa and Holm in [4]. Qiao investigated further, aiming to find *all* soliton solutions, which led to the following discovery: assuming that $u(x,t)$ is a single soliton solution of the Camassa-Holm equation satisfying the boundary condition $\lim_{\xi\to\pm\infty} U(\xi) = A$,

1. if $A = 0$, the only single soliton solution is the peakon $u = ce^{-|x-ct|}$, where $U(0) = c$, $U(\pm\infty) = 0$, $U'(0)^+ = -c$, $U'(0)^- = c$;

2. if $0 < A \leq c \leq 3A$, or $3A \leq c \leq A < 0$, there is no solitary wave solution;

3. if $A > 0$ with $c < A$, or if $A < 0$ with $c > A$, the solitary wave is a cuspon given by

$$u(x,t) = (c-2A) + \frac{2A}{1-(F_1^{-1}(|x-ct|))^2},$$

   where $F_1(x) = a\,\text{arctanh}(\frac{1}{a}x) - \text{arctanh}(x)$ (see [13]) and $U(0) = c$, $U(\pm\infty) = A$, $U'(0)^+ = \text{sgn}(A)\infty$, $U'(0)^- = -\text{sgn}(A)\infty$;

4. and finally, if $0 < 3A < c$ or $c < 3A < 0$, the solitary wave is smooth and given by

$$u(x,t) = (c-2A) + \frac{2A}{1-(F_2^{-1}(|x-ct|))^2}$$

   where $F_2(x) = |\xi|$ (see [13]) and $U(0) = c-2A$, $U(\pm\infty) = A$, $U'(0) = 0$.

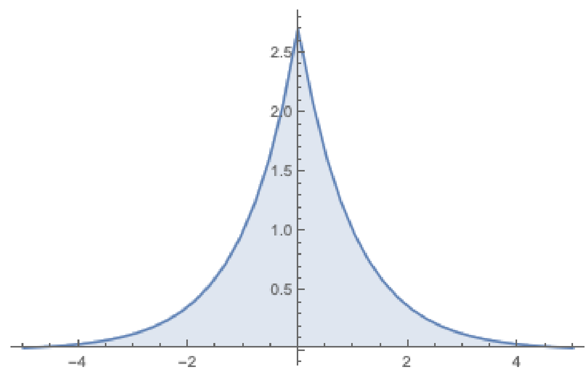


Figure 3.1: The peakon $u = Ce^{|x-ct|}$, with $C = 1,\ c = 1$ at $t = 0$.

Qiao's work not only revealed new explicit solutions to the Camassa-Holm equation, but also neatly packaged them into distinct classes based on the choices of constants.

In addition to single soliton solutions, it was shown by Camassa and Holm in [4] that the equation allows for a generalized N-peakon solution where we have a superposition of the usual peakons of type $u(\xi) = \exp(-|\xi|)$, given by

$$u(x,t) = \sum_{i=1}^{N} p_i(t) e^{-|x-q_i(t)|}$$

where $p_i(t)$ and $q_i(t)$ emerge from converting to a $2N$-dimensional dynamical system with canonical coordinates.

CHAPTER IV

θ-PARAMETER SOLUTIONS TO THE GCCH MODEL

Consider the Generalized Cubic Camassa-Holm (GCCH) Model, given by

$$m_t = bu_x + \frac{1}{2}k_1[m(u^2 - u_x^2)]_x + \frac{1}{2}k_2(2mu_x + m_x u), \tag{4.1}$$

where $m = u - u_{xx}$ as usual, which was first described by Qiao et al in [15]. Setting $k_1 = 0,\ k_2 = -2$, we obtain the Camassa-Holm equation,

$$\begin{aligned} m_t &= bu_x + \frac{1}{2} \cdot 0 \cdot [m(u^2 - u_x^2)]_x + \frac{1}{2} \cdot -2 \cdot (2mu_x + m_x u) \\ &= bu_x - 2mu_x - m_x u \end{aligned}$$

and if we set $k_1 = 2,\ k_2 = 0$, we obtain the cubic nonlinear equation

$$m_t = bu_x + [m(u^2 - u_x^2)]_x \tag{4.2}$$

which shows that the GCCH model is a combination of the typical Camassa-Holm equation (3.2) and the cubic nonlinear equation (4.2), which leads us to describe it as a generalized Camassa-Holm equation. In the context of using the equation as a model for surface water waves, we refer to it as the GCCH model.

In 2015, Qiao et al identified single-soliton, single- and multi-peakon, single weak-kink, and kink-peakon interaction solutions to the GCCH model, including explicit equations for a 2-peakon dynamical system [15].

4.1 Generalized Peakon Solutions

In this section we outline exactly what we mean by ”generalized” peakon solutions and how they are obtained. Consider the single-peakon ansatz given by

$$u = Ce^{-|x-ct|}.$$

Due to the presence of the absolute value function in the exponential argument, we cannot take classical derivatives of this solution due to the lack of differentiability along the line $x = ct$, and thus it cannot satisfy (4.1) in the classical sense. However, we can make use of the notion of weak derivatives via distribution theory as discussed in Chapter 1, Section 3. Moreover, in the course of our calculations we will frequently obtain terms of the form

$$\text{sgn}^2(x)\delta(x)$$

which, when the sign function is classically interpreted, equals zero since $\text{sgn}(0) = 0$. However, we will consider an alternative sign function given by

$$\text{sgn}_\theta(x) = \begin{cases} 1, & \text{if } x > 0 \\ \theta, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases}$$

which is equal almost everywhere to the traditional signum function. In a weak sense, the derivative of the absolute value function $\text{abs}(x)$ is any function f such that $f(x) = -1$ a.e. for $x < 0$, and $f(x) = 1$ a.e. for $x > 0$; this implies that we may choose any value for $f(0)$, which we denote by θ in our alternative sign function sgn_θ. Moreover, we have $\text{sgn}'_\theta(x) = 2\delta(x)$ where δ is the Dirac delta distribution since it is equal to 0 a.e. except at its point of discontinuity, $x = 0$. Since our choice of arbitrary θ does not affect the height of the peakon, we are free to reinterpret the sign function as such.

Lemma 4.1.1. For every $x \in \mathbb{R}$ and constant θ, we have

$$\text{sgn}_\theta(x)\delta(x) = 0, \tag{4.3}$$

$$\text{sgn}^2_\theta(x)\delta(x) = \theta^2\delta(x). \tag{4.4}$$

Proof. We prove (3.3) first. Consider the integral

$$\int_{-\infty}^{\infty} \delta_x \,\text{sgn}_\theta(x)\varphi(x)\; dx.$$

Since clearly the point $x = 0$ poses a problem in terms of continuity and differentiability, we split the integral over two intervals

$$\int_{-\infty}^{0} \delta_x \,\text{sgn}_\theta(x)\varphi(x)\; dx \;+\; \int_{0}^{\infty} \delta_x \,\text{sgn}_\theta(x)\varphi(x)\; dx.$$

Since $\text{sgn}_\theta(x) = -1$ for all $x < 0$, and $\text{sgn}_\theta(x) = 1$ for all $x > 0$, we can simplify these integrals as

$$-\int_{-\infty}^{0} \delta_x\varphi(x)\; dx \;+\; \int_{0}^{\infty} \delta_x\varphi(x)\; dx \;\implies\; \int_{0}^{-\infty} \delta_x\varphi(x)\; dx \;+\; \int_{0}^{\infty} \delta_x\varphi(x)\; dx.$$

Finally, applying the change of variable $x = -t$, $dx = -dt$, we have

$$\begin{aligned}
-\int_{0}^{\infty} \delta_{-t}\varphi(-t)\; dt \;+\; \int_{0}^{\infty} \delta_x\varphi(x)\; dx \;&\implies\; \int_{0}^{\infty} -\delta_{-t}\varphi(-t) \;+\; \delta_t\varphi(t)\; dt \\
&\implies\; -\varphi(0) \;+\; \varphi(0) \\
&\implies\; 0
\end{aligned}$$

as desired. The proof of (3.4) follows from the definition of $\text{sgn}_\theta(x)$, since

$$\delta_x \,\text{sgn}^2_\theta(x) = \delta_x \,\text{sgn}^2_\theta(0) = \theta^2\delta_x,$$

and thus the proof is complete. □

4.2 Single Peakon Solutions

Consider the single-peakon solution to (4.1), given by

$$u = Ce^{-|x-ct|}.$$

Taking the derivative with respect to x, we have

$$\partial_x u = Ce^{-|x-ct|} \cdot - \operatorname{sgn}_\theta(x - ct) = -Ce^{-|x-ct|} \operatorname{sgn}_\theta(x - ct)$$

where $\partial_x(|x|) = \operatorname{sgn}_\theta(x)$. Similarly, we have

$$\partial_t u = Ce^{-|x-ct|} \cdot - \operatorname{sgn}(x - ct) \cdot -c = cCe^{-|x-ct|} \operatorname{sgn}_\theta(x)$$

by the chain rule.

In order to determine an appropriate expression for m, we make use of the Dirac distribution δ introduced in Chapter 1. First, we compute u_{xx},

$$\begin{aligned} u_{xx} &= \partial_x(u_x) \\ &= -C \operatorname{sgn}_\theta(x - ct) \cdot e^{-|x-ct|} \cdot - \operatorname{sgn}_\theta(x - ct) + e^{-|x-ct|} \cdot -C \cdot 2\delta(x - ct) \\ &= C \operatorname{sgn}_\theta^2(x - ct)e^{-|x-ct|} - 2C\delta(x - ct)e^{-|x-ct|} \end{aligned}$$

where the derivative of $\operatorname{sgn}_\theta(x - ct)$ is given in the distribution sense as $2\delta(x - ct)$. We then compute m as

$$
\begin{aligned}
m &= u - u_{xx} \\
&= Ce^{-|x-ct|} - C\operatorname{sgn}_\theta^2(x-ct)e^{-|x-ct|} + 2C\delta(x-ct)e^{-|x-ct|} \\
&= 2C\delta(x-ct)
\end{aligned}
$$

where $\operatorname{sgn}_\theta^2(x-ct) = 1$ a.e., and the product $\delta(\xi)e^{-|\xi|} = \delta(\xi)e^0 = \delta(\xi)$ by the property of multiplying δ by a smooth function.

Substituting everything we have just calculated into (4.1) and setting $\xi = x - ct$ for brevity, we have

$$
\begin{aligned}
-2cC\delta'(\xi) = -bCe^{-|\xi|}\operatorname{sgn}_\theta(\xi) &+ \frac{1}{2}k_1\left[2C\delta(\xi)(C^2e^{-2|\xi|} - C^2\operatorname{sgn}_\theta^2(\xi)e^{-2|\xi|})\right]_x \\
&+ \frac{1}{2}k_2\left[-4Ce^{-|\xi|}\operatorname{sgn}_\theta(\xi)\delta(\xi) + 2C^2e^{-|\xi|}\delta'(\xi)\right]
\end{aligned}
$$

Since C is the wave height, and a wave with zero height is not very interesting, we allow ourselves to divide both sides of the equation by C and restrict $C \neq 0$. We can simplify further by applying properties of the product of the δ distribution with a smooth function, as well as our results from (4.3) and (4.4) to obtain

$$
\begin{aligned}
-2c\delta'(\xi) &= -be^{-|\xi|}\operatorname{sgn}_\theta(\xi) + \frac{1}{2}k_1\left[2C^2\delta(\xi) - 2C^2\theta^2\delta(\xi)\right]_x + \frac{1}{2}k_2\left[-2Ce^{-|\xi|}\delta'(\xi)\right] \\
&= -be^{-|\xi|}\operatorname{sgn}_\theta(\xi) + k_1C^2(1-\theta^2)\left[\delta(\xi)\right]_x - k_2C\left[e^{-|\xi|}\delta'(\xi)\right] \\
&= -be^{-|\xi|}\operatorname{sgn}_\theta(\xi) + k_1C^2(1-\theta^2)\delta'(\xi) - k_2C\left[\delta'(\xi) + e^{-|\xi|}\operatorname{sgn}_\theta(\xi)\delta(\xi)\right] \\
&= -be^{-|\xi|}\operatorname{sgn}_\theta(\xi) + k_1C^2(1-\theta^2)\delta'(\xi) - k_2C\delta'(\xi)
\end{aligned}
$$

Collecting terms, we have

$$
be^{-|\xi|}\operatorname{sgn}_\theta(\xi) = \delta'(\xi)\left(k_1(1-\theta^2)C^2 - k_2C + 2c\right)
$$

which, for $b \neq 0$, is a non-homogeneous problem. Setting $b = 0$ we have the homogeneous variant

$$0 = \delta'(\xi)\left(k_1(1-\theta^2)C^2 - k_2 C + 2c\right)$$

which can be solved with the zero-product law, namely we have a solution if

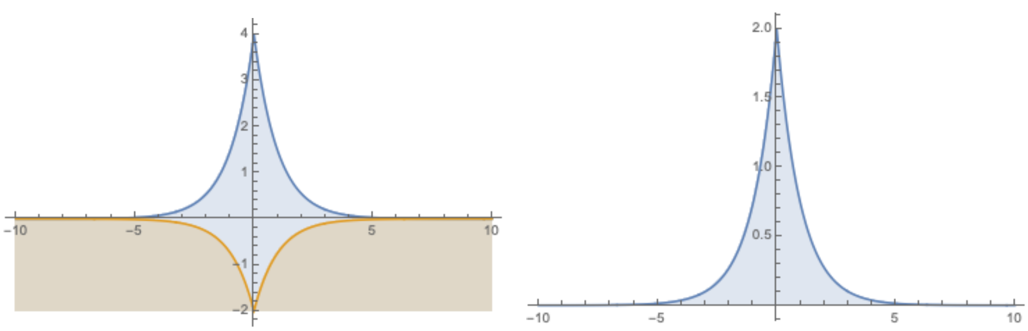
$$k_1(1-\theta^2)C^2 - k_2 C + 2c = 0.$$

An application of the quadratic formula yields

$$C = \frac{-k_2 \pm \sqrt{k_2^2 - 8k_1 c(1-\theta^2)}}{2k_1(1-\theta^2)}. \tag{4.5}$$

4.2.1 Wave Height Analysis

The constant C controls the height of the peakon. In Figure 3.2.1, two peakons u_1, u_2 are graphed on the same coordinate plane: u_1 has a negative $C_1 = -2$, and u_2 has a positive $C_2 = 4$ that is twice as large as $|C_1|$. The sum $u_3 = u_1 + u_2$ graphed in Figure 3.2.2 looks exactly as expected since it is a linear combination of two solutions.



(a) Two peakons with $C_1 = -2$, $C_2 = 4$.   ***(b)*** The sum of the peakons in Figure 3.2.1.

Figure 4.1: Two individual peakons vs their sum

Clearly, the equation for C is not exactly a trivial one, and it may behoove one to understand the sorts of combinations of the constants k_1, k_2, c, and θ lead to positive vs negative or real vs complex wave height.

Theorem 4.2.1. Let $u = Ce^{-|x-ct|}$, with C as in (4.5), be a solution to (4.1), and define $R = k_2^2/k_1$. If $k_1 > 0$, then the restriction $R \geq 8c(1-\theta^2)$ generates real-valued peakons and $R < 8c(1-\theta^2)$ generates complex-valued peakons. Likewise, if $k_1 < 0$, then the restriction $R \leq 8c(1-\theta^2)$ generates real-valued peakons, and $R > 8c(1-\theta^2)$ generates complex-valued peakons.

Proof: Consider the discriminant of (4.5). It is clear from elementary algebra that if the value of the discriminant is less than zero, then the quadratic formula reveals complex-valued solutions. Let us consider the case where the discriminant is positive, namely that

$$k_2^2 - 8k_1c(1-\theta^2) \geq 0 \implies k_2^2 \geq 8k_1c(1-\theta^2).$$

If we wish to divide this inequality by k_1 we must consider the possible repercussions of the sign of k_1: specifically, first assume that k_1 is positive (and thus the inequality does not need to flip after division), so we obtain

$$R \geq 8c(1-\theta^2).$$

Next assume that $k_1 < 0$, so the inequality must flip to become

$$R < 8c(1-\theta^2).$$

It follows that complex solutions may be obtained by taking the complement of each inequality.

□

Interpreting R as a function of θ with parameter c, we really have a family of parabolas (which we will call the R-family) denoted by the set

$$R(\theta) = \{\ 8c(1-\theta^2)\ :\ c \in [0,\infty)\ \}$$

which illustrate in the real plane the boundary between real and complex solutions for a specific wave speed.

It can clearly be seen in Fig. 4.1 that as the wave speed increases, the wave is able to ac-

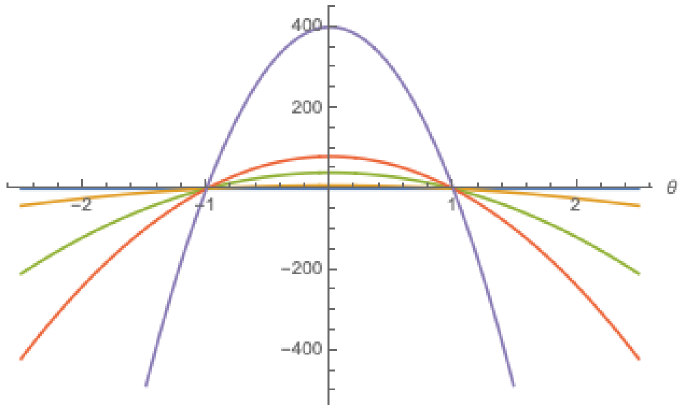


Figure 4.2: Members of the R-family for $c = 0, 1, 5, 10,$ and 50.

commodate more values of θ which allow it to remain a real solution. For example, if $c = 20$ then we have the R-family member $R = 160(1 - \theta^2)$. Consider the situation where $c \to \infty$. The individual successive parabolas $R_{n+1}(\theta)$ begin to form what is essentially a rectangular bounded region between the lines $\theta = \pm 1$, as shown in Fig. 4.2, which implies that as wave speed increases to infinity, only values of θ which are outside the open ball $B_1(0)$ will produce real-valued peakons for small enough values of R. Of course the parabolas corresponding to large values of c aren't necessarily forming rectangular regions but rather masquerading as rectangular regions for small enough subsets of the range.

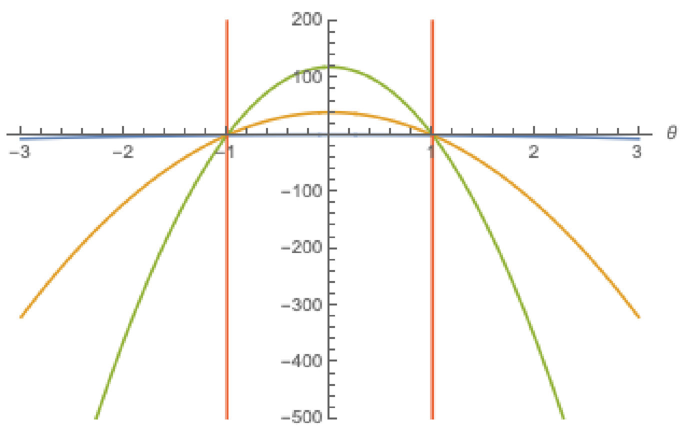


Figure 4.3: Comparison between low-speed members and a single high-speed member of the R-family.

4.3 Further Study

We are interested to see what effect the amended sign function has on the N-peakon solution as well as the resulting dynamics. We are also interested to see what other equations have solutions affected by the amended sign function.

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BIOGRAPHICAL SKETCH

Born and raised on Long Island, N.Y., Michael earned his Master of Science degree in Applied Mathematics from The University of Texas at Rio Grande Valley, his Bachelor of Arts degree in Mathematics from Indiana University, and his Associate of Arts degree in Liberal Arts from Suffolk County Community College. During his time at UTRGV, Michael worked with Dr. Zhijun (George) Qiao researching various types of traveling wave solutions to assorted expressions of the Holm-Staley b-family, including the Camassa-Holm equation in typical form as well as in a generalized cubic form and a fifth-order form. With a passion for teaching, Michael has directed mathematics instruction at a private learning center in Wading River, N.Y., for middle- and high-school students struggling with all levels of mathematics and science. In his free time, Michael enjoys programming and listening to hard rock music.