Balanced modular parameterizations

Esteban J. Melendez
University of Texas-Pan American

BALANCED MODULAR PARAMETERIZATIONS

A Thesis

by

ESTEBAN J. MELENDEZ

Submitted to the Graduate School of
The University of Texas-Pan American
In partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

August 2014

Major Subject: Mathematics

BALANCED MODULAR

PARAMETERIZATIONS

A Thesis
by
ESTEBAN J. MELENDEZ

COMMITTEE MEMBERS

Dr. Timothy Huber
Chair of Committee

Dr. Sean Lawton
Committee Member

Dr. Virgil Ulysses Pierce
Committee Member

Dr. Andras Balogh
Committee Member

August 2014

ABSTRACT

Melendez, Esteban J., BALANCED MODULAR PARAMETERIZATIONS. Master of Science (MS), August, 2014, 25 pages, 1 table, 2 figures, 17 references, 12 titles.

In this thesis, we show that Classical representations for certain modular forms have symmetric form. These symmetric formulations are interpreted in terms of more general balanced homogeneous polynomial representations resulting from a permutative action of Hecke congruence subgroups on quotients of theta functions. For prime levels between 5 and 19, sets of permuted theta quotients are constructed that generate the corresponding vector spaces of modular forms of weight one.

DEDICATION

I dedicate my dissertation work, first of all to God who has given me the strength, the wisdom, and the inspiration to achieve this new goal in life , to my family and friends. A special feeling of gratitude to my loving father, Elias Melendez Mtz. whose words of encouragement and push for tenacity ring in my ears. My sister and brother, Damaris and Elias who have never left my side and are very special. I also dedicate this dissertation to my many friends and church family who have supported me throughout the process. I dedicate this work and give special thanks to my dear best friend and beloved fiancee, Yamiletz Salinas, I will always appreciate all you and your family have done. I love you sweetheart.

ACKNOWLEDGMENTS

The work contained in this thesis would not been possible without the assistance and encouragement of many stunning individuals. I am in greater debt to my exceptional thesis adviser, Dr. Tim Huber. I continuously have gained profit from Dr. Huber's exceptional and unique competence to teach his perspective. His extraordinary way of working with what we call "Beautiful Mathematics" marks him as a true passionate mathematician. Dr. Huber has guided me out of my comfort zone and clarified my understanding, allowing me to expand my knowledge. The tremendous amount of persistence that Dr. Huber has devoted, led me to never stop proving until everything is proven.

Additionally, I am truly grateful to have had committee members' Sean Lawton, Virgil U. Pierce, and Andras Balogh massive support and exquisite guidance that has contributed greatly to this thesis.

I also thank Dr. Qiao, for making the GAANN Fellowship Award possible for me to continuing my education and earning my master's degree here at the University of Texas Pan American. I express my gratitude to all the math department's staff Mrs. Cisneros, Ms. Chavarria, Ms. Alanis, and Ms. Melgar for their support during my graduate studies at UTPA. Thank you!

Table of Contents

LIST OF TABLES

LIST OF FIGURES

CHAPTER I

INTRODUCTION

Certain polynomial representations for modular forms demonstrate coefficient symmetry. In this thesis, we construct polynomial generators for vector spaces of modular forms of small prime level that parameterize modular forms for congruence subgroups intermediate to $PSL(2,\mathbb{Z})$ in a symmetric way. The search for precise generators and the study of their properties is motivated by the numerous areas where modular forms are implicated.

The symmetric forms studied here are exemplified by the *Klein polynomials*, whose roots encode distinguished points of the stereographically projected circumsphere for a regular icosahedron, are symmetric in absolute value about the middle coefficients

$$K_e(\Lambda) = 1 + 228\Lambda + 494\Lambda^2 - 228\Lambda^3 + \Lambda^4. \tag{I.1}$$

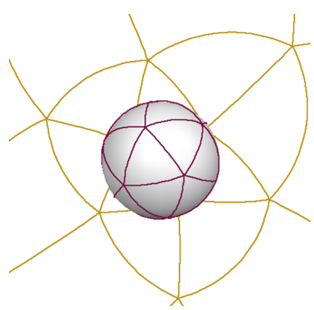


Figure 1.1: stereographic projection of icosahedron

The polynomial $K_e(\Lambda)$ corresponds to representations for Eisenstein series in terms of two modular parameters of level five. The symmetry results from transformation properties for

Eisenstein series and their representations in terms of Klein polynomials are

$$B^{20}K_e(\Lambda) = 1 + 240\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n}, \quad |q|<1, \tag{I.2}$$

$$A^5(q) = q\frac{(q;q)_\infty^2}{(q^2,q^3;q^5)_\infty^5}, \qquad B^5(q) = \frac{(q;q)_\infty^2}{(q,q^4;q^5)_\infty^5}, \qquad \Lambda = A^5/B^5, \tag{I.3}$$

where $(a;q)_n = \prod_{k=0}^{n-1} 1-aq^k$ and $(a_1,a_2,\ldots,a_r;q)_n = \prod_{j-1}^{r}(a_j;q)_n$ for $n \in \mathbb{N}\cup\{\infty\}$.

For small prime levels, these balanced modular parameterizations result from an action of subgroups of Klein's automorphism groups for regular polyhedra and their generalizations on quotients of theta functions. The relevant group actions originate from modular transformation formulas for vector-valued modular forms.

The coefficient symmetry comes from formulations for the modular forms in terms of special bases of theta quotients that generate vector spaces of modular forms of weight one on

$$\Gamma_1(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{Z}) \mid c \equiv 0,\ a \equiv d \equiv 1 \pmod{p} \right\}. \tag{I.4}$$

For each level p, p prime, that we consider, the polynomial ring generators are permuted up to a change of sign, $\{\pm 1\}$, by

$$\Gamma_0(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{Z}) \mid c \equiv 0 \pmod{p} \right\}. \tag{I.5}$$

A key goal of this thesis is to show that balanced polynomial representations for modular forms of weight one on $\Gamma_0(p)$ result from nontrivial permutative action on a set of generators for the vector spaces of modular forms of weight one for $\Gamma_1(p)$ induced by modular transformation formulas.

In this thesis, we primarily analyze the permuted generators of level five. Proceeding similarly we can construct sets of permuted generators for each level p on $\Gamma_0(p), p = 7, 11, 13, 17, 19$, for the vector space of forms on $\Gamma_1(p)$. A set of $\Gamma_0(7)$-permuted generators for the vector space of forms on $\Gamma_1(7)$ is

$$x = q\frac{(q^2,q^5,q^7,q^7;q^7)_\infty}{(q^3,q^4;q^7)_\infty^2}, \quad y = -q\frac{(q,q^6,q^7,q^7;q^7)_\infty}{(q^2,q^5;q^7)_\infty^2}, \quad z = \frac{(q^3,q^4,q^7,q^7;q^7)_\infty}{(q,q^6;q^7)_\infty^2}. \tag{I.6}$$

A set of $\Gamma_0(11)$-permuted generators for the vector space of forms on $\Gamma_1(11)$ is

$$\frac{(q^4,q^7,q^{11},q^{11};q^{11})_\infty}{(q,q^{10},q^2,q^9;q^{11})_\infty},\quad q\frac{(q^5,q^6,q^{11},q^{11};q^{11})_\infty}{(q^3,q^8,q^4,q^7;q^{11})_\infty},\quad q^2\frac{(q,q^{10},q^{11},q^{11};q^{11})_\infty}{(q^3,q^8,q^5,q^6;q^{11})_\infty},\tag{I.7}$$

$$q\frac{(q^3,q^8,q^{11},q^{11};q^{11})_\infty}{(q^2,q^9,q^4,q^7;q^{11})_\infty},\qquad q\frac{(q^2,q^9,q^{11},q^{11};q^{11})_\infty}{(q^5,q^6,q,q^{10};q^{11})_\infty}.\tag{I.8}$$

A set of $\Gamma_0(13)$-permuted generators for the vector space of forms on $\Gamma_1(13)$ is

$$\frac{(q^6,q^7,q^{13},q^{13};q^{13})_\infty}{(q,q^{12},q^3,q^{10};q^{13})_\infty},\quad q\frac{(q^5,q^8,q^{13},q^{13};q^{13})_\infty}{(q^3,q^{10},q^4,q^9;q^{13})_\infty},\quad q\frac{(q,q^2,q^{11},q^{13};q^{13})_\infty}{(q,q^{12},q^4,q^9;q^{13})_\infty},\tag{I.9}$$

$$q\frac{(q^4,q^9,q^{13},q^{13};q^{13})_\infty}{(q^2,q^{11},q^5,q^8;q^{13})_\infty},\quad q^2\frac{(q^3,q^{10},q^{13},q^{13};q^{13})_\infty}{(q^5,q^8,q^6,q^7;q^{13})_\infty},\quad q^2\frac{(q,q^{12},q^{13},q^{13};q^{13})_\infty}{(q^2,q^{11},q^6,q^7;q^{13})_\infty}.\tag{I.10}$$

A set of $\Gamma_0(17)$-permuted generators for the vector space of forms on $\Gamma_1(17)$ is

$$\frac{(q^8,q^9,q^{17},q^{17};q^{17})_\infty}{(q^2,q^{15},q^3,q^{14};q^{17})_\infty},\quad q\frac{(q^5,q^{12},q^{17},q^{17};q^{17})_\infty}{(q^3,q^{14},q^4,q^{13};q^{17})_\infty},\quad q^3\frac{(q,q^{16},q^{17},q^{17};q^{17})_\infty}{(q^4,q^{13},q^6,q^{11};q^{17})_\infty},\tag{I.11}$$

$$q^2\frac{(q^7,q^{10},q^{17},q^{17};q^{17})_\infty}{(q^6,q^{11},q^8,q^9;q^{17})_\infty},\quad q^3\frac{(q^2,q^{15},q^{17},q^{17};q^{17})_\infty}{(q^5,q^{12},q^8,q^9;q^{17})_\infty},\quad q\frac{(q^3,q^{14},q^{17},q^{17};q^{17})_\infty}{(q,q^{16},q^5,q^{12};q^{17})_\infty},\tag{I.12}$$

$$q\frac{(q^4,q^{13},q^{17},q^{17};q^{17})_\infty}{(q,q^{16},q^7,q^{10};q^{17})_\infty},\quad q\frac{(q^6,q^{11},q^{17},q^{17};q^{17})_\infty}{(q^2,q^{15},q^7,q^{10};q^{17})_\infty}.\tag{I.13}$$

and a set of $\Gamma_0(19)$-permuted generators for the vector space of forms on $\Gamma_1(19)$ is

$$\frac{(q^8,q^{11},q^9,q^{10},q^{19},q^{19};q^{19})_\infty}{(q^3,q^{16},q^4,q^{15},q^5,q^{14};q^{19})_\infty},\qquad q\frac{(q^2,q^{17},q^7,q^{12},q^7,q^{12};q^{19})_\infty}{(q,q^{18},q^4,q^{15},q^6,q^{13};q^{19})_\infty},\tag{I.14}$$

$$q\frac{(q^3,q^{16},q^9,q^{10},q^{19},q^{19};q^{19})_\infty}{(q,q^{18},q^5,q^{14},q^8,q^{11};q^{19})_\infty},\qquad q\frac{(q^4,q^{15},q^7,q^{12},q^{19},q^{19};q^{19})_\infty}{(q^2,q^{17},q^5,q^{14},q^6,q^{13};q^{19})_\infty},\tag{I.15}$$

$$q^5\frac{(q,q^{18},q^3,q^{16},q^{19},q^{19};q^{19})_\infty}{(q^6,q^{13},q^8,q^{11},q^9,q^{10};q^{19})_\infty},\qquad q^2\frac{(q^4,q^{15},q^5,q^{14},q^{19},q^{19};q^{19})_\infty}{(q^2,q^{17},q^7,q^{12},q^8,q^{11};q^{19})_\infty},\tag{I.16}$$

$$q^2\frac{(q,q^{18},q^6,q^{13},q^{19},q^{19};q^{19})_\infty}{(q^2,q^{17},q^3,q^{16},q^9,q^{10};q^{19})_\infty},\qquad q^2\frac{(q^5,q^{14},q^8,q^{11},q^{19},q^{19};q^{19})_\infty}{(q^4,q^{15},q^7,q^{12},q^9,q^{10};q^{19})_\infty},\tag{I.17}$$

$$q\frac{(q^2,q^{17},q^6,q^{13},q^{19},q^{19};q^{19})_\infty}{(q,q^{18},q^3,q^{16},q^7,q^{12};q^{19})_\infty}.\tag{I.18}$$

A characteristic aspect of each of sets of permuted generators for levels $p, 5\le p\le 19$, is that any modular form of weight one on a subgroup containing $\Gamma_0(p)$ may be represented as a linear function with a certain coefficient symmetry. The coefficient symmetry is induced by transformation formulas satisfied by the generators. To describe the symmetry displayed in formulations of modular forms in terms of the above parameters of level p, let $\gamma \in PSL(2,\mathbb{Z})$ act on the upper half plane, and define the slash operator on a modular form f of weight k on $\Gamma_1(p)$ [[4], $pg.108$] by

$$|_\gamma (f) = (\gamma_{21}\tau + \gamma_{22})^{-k} f(\gamma\tau), \qquad \gamma = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix} \in PSL(2,\mathbb{Z}). \tag{I.19}$$

Theorem I.1. *The generators for* $\Gamma_1(p)$ *from* (I.3)–(I.18) *are permuted up to change of sign by* $\Gamma_0(p)$ *under action by the slash operator, with permutation representation* $(\mathbb{Z}/p\mathbb{Z})^*/\{\pm 1\}$.

The generators appearing here originate from Eisenstein series, modular on $\Gamma_0(p)$ with Dirichlet character χ modulo p, defined for general weight k by

$$E_{k,\chi}(\tau) = 1 + \frac{2}{L(1-k,\chi)} \sum_{n=1}^{\infty} \chi(n) \frac{n^{k-1} q^n}{1-q^n}, \qquad q = e^{2\pi i \tau}, \tag{I.20}$$

where $L(1-k,\chi)$ is the analytic continuation of the associated Dirichlet L-series and $\chi(-1) = (-1)^k$. Underlying the symmetric parameterizations of this paper is a useful new link between theta functions and Eisenstein series.

Theorem I.2. *For primes* $5 \le p \le 19$*, the action by* $\Gamma_0(p)$ *by* (I.19) *generates a linearly independent set of functions* $x_i(\tau)$*,* $1 \le i \le (p-1)/2$ *over* $\mathbb{C}$ *generating the vector space of modular forms of weight one for* $\Gamma_1(p)$*. These generators are permuted up to change of sign by* $\Gamma_0(p)$ *with permutation representation* $(\mathbb{Z}/p\mathbb{Z})^*/\{\pm 1\}$.

We can now summarize the content of the rest of this thesis. The goal for Chapter II is to elaborate precise forms for parameters appearing in Theorems I.1–I.2. These constructions will be accomplished through elementary elliptic function theory. Sections 2.1- 2.2 draw from the theory of elliptic modular forms to show that the preceding theta quotients generate the vector space of modular forms of weight one for each level. We also give explicit constructions for the vector space of modular forms on $\Gamma_1(p)$ in terms of theta quotients. In Chapter III, we conclude our work by summerizing the main results of this research.

Chapter II

Elliptic Functions

In this Chapter we will introduce requisite knowledge of elliptic modular form theory.

2.1 Elliptic Modular Preliminaries

A lattice [8] is a subgroup which is free over $\mathbb{Z}$ which generates $\mathbb{C}$ over $\mathbb{R}$ with dimension 2. Let L be a lattice in $\mathbb{C}$ by which we mean the set of all integral linear combinations of two given complex numbers w_1 and w_2, where w_1, w_2 do not lie on the same line through the origin. The fundamental parallelogram [7, 8] for w_1, w_2 is defined as

$$\Omega = \{\alpha + aw_1 + bw_2 | \alpha \in \mathbb{C}, \quad 0 \leq a, b \leq 1\}.$$

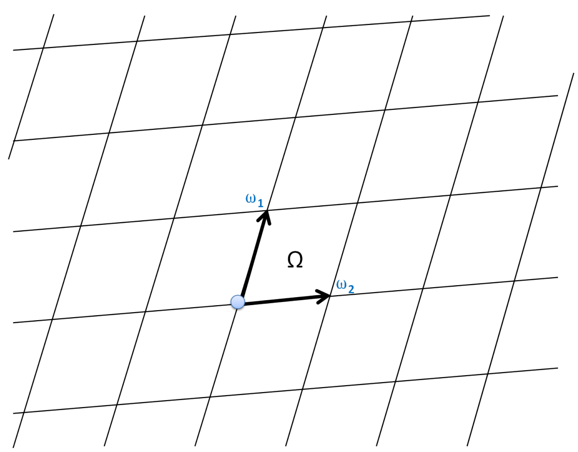


Figure 2.1: Ω-Parallelogram

If w_1, w_2 are basis for the lattice L over $\mathbb{Z}$, then $L = [w_1, w_2]$ for $\mathrm{Im}(w_1/w_2) > 0$, where w_1/w_2 lies on $\mathbb{H}$,defined by $\mathbb{H} = \{x+iy|y>0\}$. An Elliptic function is a meromorphic function $f(z) : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ on $\mathbb{C}$ that satisfies

$$f(z+w) = f(z). \tag{II.1}$$

for all $z \in \mathbb{C}$ and all $w \in L$.

Theorem II.1. *The sum of residues of an elliptic function on its period parallelogram is zero* [5].

Proof. Let f be an elliptic function. Then, f is a doubly periodic function. Since f is an elliptic function, then the poles of f are isolated so we can take an $\alpha \in \mathbb{C}$ so that f is regular (holomorphic) on the boundary $\partial\Omega_\alpha$ of the period parallelogram Ω_α. Then, at each point $x \in \mathbb{C}$

$$\begin{aligned}
\sum_{x\in\Omega_\alpha} res_x f &= \frac{1}{2\pi i}\int_{\partial\Omega_\alpha} f(z)dz \\
&= \frac{1}{2\pi i}\left(\int_\alpha^{\alpha+w_2} f(z)dz + \int_{\alpha+w_2}^{\alpha+w_1+w_2} f(z)dz + \int_{\alpha+w_1+w_2}^{\alpha+w_1} f(z)dz + \int_{\alpha+w_1}^{\alpha} f(z)dz\right) \\
&= \frac{1}{2\pi i}\left(\left[\int_\alpha^{\alpha+w_2} f(z)dz + \int_{\alpha+w_1+w_2}^{\alpha+w_1} f(z)dz\right] + \left[\int_{\alpha+w_2}^{\alpha+w_1+w_2} f(z)dz + \int_{\alpha+w_1}^{\alpha} f(z)dz\right]\right) \\
&= \frac{1}{2\pi i}\left(\left[\int_\alpha^{\alpha+w_2} f(z) - f(z+w_1)dz\right] + \left[\int_\alpha^{\alpha+w_1} f(z+w_2) - f(z)dz\right]\right) \\
&= 0
\end{aligned}$$

by using periodicity of the function f on the period parallelogram and a change of variable. □

For each prime p, the linearly independent Eisenstein series of weight $k=1$ and primitive character χ generate a subspace of modular forms of weight one for $\Gamma_1(p)$ called the Eisenstein subspace [4],[Theorem 4.8.1]. Let $\mathscr{M}_k(\Gamma)$ denote the vector space of weight k modular forms for $\Gamma \subseteq PSL(2,\mathbb{Z})$. For each prime p with $5 \le p \le 19$, [2, 11]

$$\dim\left(\mathscr{M}_1(\Gamma_1(p))\right) = \frac{p-1}{2}. \tag{II.2}$$

Thus, the Eisenstein subspace forms a basis for $\mathscr{M}_1(\Gamma_1(p))$ over $\mathbb{C}$. To develop the change of bases from Eisenstein series to the permuted bases of products, from (I.3)–(I.18), we will need to construct representations for sums of Eisenstein series in terms of the Dedekind eta function, in Theorem II.3, $\eta(\tau) = q^{1/24}(q;q)_\infty$, a weight $1/2$ modular form for $SL(2,\mathbb{Z})$ with multiplier given explicitly by $[6, p.51]$. The following will be important ingredients for our work. We first define the **Jacobi theta function**

$$\theta_1(z \mid q) = -iq^{1/8} \sum_{n=-\infty}^{\infty} (-1)^n q^{n(n+1)/2} e^{(2n+1)iz}, \tag{II.3}$$

an odd function of z with a simple zero at the origin such that [12, p. 489]

$$\frac{\theta_1'}{\theta_1}(z \mid q) = \cot z + 4\sum_{n=1}^{\infty} \frac{q^n}{1-q^n} \sin 2nz \tag{II.4}$$

$$= i - 2i\sum_{n=1}^{\infty} \frac{q^n e^{2iz}}{1-q^n e^{2iz}} + 2i\sum_{n=0}^{\infty} \frac{q^n e^{-2iz}}{1-q^n e^{-2iz}}. \tag{II.5}$$

The following equations are required for results on II.59–II.63

$$\theta_1(z+n\pi) = (-1)^n \theta_1(z \mid q), \qquad \theta_1(z+n\pi\tau \mid q) = (-1)^n q^{-n^2/2} e^{-2inz} \theta_1(z \mid q). \tag{II.6}$$

The product representations it is being derived from the Jacobi Triple Product expansion given by [12]

$$\theta_1(z \mid q) = -iq^{1/8} e^{iz} (q;q)_\infty (qe^{2iz};q)_\infty (e^{-2iz};q)_\infty. \tag{II.7}$$

The derivative of the Jacobi theta function (II.7) at the origin , is

$$\theta_1'(q) := \lim_{z\to 0} \frac{\theta_1(z \mid q)}{z} = 2q^{1/8}(q;q)_\infty^3. \tag{II.8}$$

In order to characterize the action by $\Gamma_0(p)$ on the generating theta quotients, we will be applying transformations formulas for special values of the Jacobi theta function (II.3) in the form of those for theta constants of odd order k and index ℓ, defined by [5]

$$\varphi_{k,\ell}(\tau) = \theta \begin{bmatrix} \frac{2\ell-1}{k} \\ 1 \end{bmatrix} (0, k\tau), \qquad 1 \le \ell \le \frac{k-1}{2}, \tag{II.9}$$

constructed, in turn, from theta constants of characteristic $[\varepsilon, \varepsilon'] \in \mathbb{R}^2$

$$\theta \begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix} (z,\tau) = \sum_{n\in\mathbb{Z}} \exp 2\pi i \left\{ \frac{1}{2}\left(n+\frac{\varepsilon}{2}\right)^2 \tau + \left(n+\frac{\varepsilon}{2}\right)\left(z+\frac{\varepsilon'}{2}\right) \right\}. \tag{II.10}$$

Theorem II.2. *[5, pp. 215-219] For odd positive integers $k \geq 3$, The $\mathscr{V}_k$ is a vector-valued form of weight $1/2$ on $PSL(2,\mathbb{Z})$ let*

$$\mathscr{V}_k(\tau) = \left[\theta\begin{bmatrix}(k-2)/k\\1\end{bmatrix}(k\tau), \theta\begin{bmatrix}(k-4)/k\\1\end{bmatrix}(k\tau), \ldots, \theta\begin{bmatrix}1/k\\1\end{bmatrix}(k\tau)\right]^T. \tag{II.11}$$

inducing a representation

$$\pi_k : PSL(2,\mathbb{Z}) \to PGL((k-1)/2, \mathbb{C})$$

which is determined by the images of generators for $SL(2,\mathbb{Z})$,

$$S = \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}, \qquad T = \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}$$

$$\mathscr{V}_N(T\tau) = \mathscr{V}_N(\tau+1) = \pi_N(T)\mathscr{V}_N(\tau), \quad \mathscr{V}_N(S\tau) = \mathscr{V}_N(-1/\tau) = \tau^{1/2}\pi_N(S)\mathscr{V}_N(\tau),$$

where the matrices $\pi_N(S)$ and $\pi_N(T)$ have (ℓ, j)th entry, for $1 \leq \ell, j \leq (N-1)/2$,

$$\{\pi_N(T)\}_{(\ell,j)} = \begin{cases}\exp\left(\frac{(N-2\ell)^2\pi i}{4N})\right), & \ell = j,\\ 0, & else,\end{cases} \tag{II.12}$$

$$\{\pi_N(S)\}_{(\ell,j)} = \frac{\left(1+e^{\frac{(2j-N)(N-2\ell)}{k}\pi i}\right)\exp\left(\frac{\left(j(-2N+4\ell+2)+N^2-2(N+1)\ell\right)}{2N}\pi i\right)}{\sqrt{iN}}. \tag{II.13}$$

2.2 Eisenstein Expansions and Permutation Representations

The main objective of this section is to prove claims made in Chapter I for generators of the vector space of modular forms on $\Gamma_1(p)$. We construct the generators and analyze the permutative action by $\Gamma_0(p)$. In Theorem II.3, product expansions are constructed for the normalized sums of weight one Eisenstein series twisted by the odd primitive Dirichlet characters modulo p. By determining the $\Gamma_0(p)$-orbit of the Eisenstein sums under modular transformation, we construct in Theorems II.4 and II.5, bases for the weight one forms on $\Gamma_1(p)$ and precisely characterize the permutative action by $\Gamma_0(p)$. Theorem II.6 shows that the Eisenstein bases from Theorem II.4 can be written as quotients of theta functions. We subsequently show that the image of each of these sums under the $\Gamma_0(p)$-action by the slash operator results in a set of $(p-1)/2$ quotients generating the vector space of weight one modular forms for $\Gamma_1(p)$. The resulting generators are shown to be permuted by $\Gamma_0(p)$.

Theorem II.3. *Define $E_{\chi,k}(\tau)$ as in* (I.20)*. For each prime $5 \le p \le 19$, let*

$$\mathscr{E}_p(\tau) = \frac{2}{p-1} \sum_{\chi(-1)=-1} E_{1,\chi}(\tau), \tag{II.14}$$

where the sum is over the odd primitive Dirichlet characters modulo p. Then

$$\mathscr{E}_5(\tau) = \frac{(q;q)_\infty^2}{(q,q^4;q^5)_\infty^5}, \qquad \mathscr{E}_7(\tau) = \frac{(q^3,q^4,q^7,q^7;q^7)_\infty}{(q,q^6;q^7)_\infty^2}, \tag{II.15}$$

$$\mathscr{E}_{11}(\tau) = \frac{(q^4,q^7,q^{11},q^{11};q^{11})_\infty}{(q,q^{10},q^2,q^9;q^{11})_\infty}, \qquad \mathscr{E}_{13}(\tau) = \frac{(q^6,q^7,q^{13},q^{13};q^{13})_\infty}{(q,q^{12},q^3,q^{10};q^{13})_\infty}, \tag{II.16}$$

$$\mathscr{E}_{17}(\tau) = \frac{(q^8,q^9,q^{17},q^{17};q^{17})_\infty}{(q^2,q^{15},q^3,q^{14};q^{17})_\infty}, \qquad \mathscr{E}_{19}(\tau) = \frac{(q^8,q^{11},q^9,q^{10},q^{19},q^{19};q^{19})_\infty}{(q^3,q^{16},q^4,q^{15},q^5,q^{14};q^{19})_\infty}. \tag{II.17}$$

Proof. To prove each one of them, we use Theorem III.1, that the sum of the residues of an elliptic function on its period parallelogram is zero. We begin by proving the equation involving $\mathscr{E}_5(\tau)$ from (II.15). Let

$$f_5(z) = \frac{e^{-2iz}\theta_1^3(z-\pi\tau \mid q^5)}{\theta_1^2(z \mid q^5)\theta_1(z+2\pi\tau \mid q^5)}. \tag{II.18}$$

Apply (II.6) to verify that $f_5(z)$ is an elliptic function with periods π and $5\pi\tau$. By using properties of the Jacobi theta function, we observe that $f_5(z)$ has a simple pole at $z = -2\pi\tau$ and a double pole at $z=0$. The residue of $f_5(z)$ at $z=-2\pi\tau$ is :

$$\lim_{z\to -2\pi\tau}(z+2\pi\tau)f_5(z) = \lim_{z\to -2\pi\tau}\frac{(z+2\pi\tau)}{\theta_1(z+2\pi\tau \mid q^5)}\lim_{z\to -2\pi\tau}\frac{e^{-2iz}\theta_1^3(z-\pi\tau \mid q^5)}{\theta_1^2(z \mid q^5)}. \tag{II.19}$$

$$= \lim_{z\to -2\pi\tau}\frac{0}{\theta_1(0 \mid q^5)}\cdot\frac{e^{4i\pi\tau}\theta_1^3((-3\pi\tau \mid q^5)}{\theta_1^2(-2\pi\tau \mid q^5)} \tag{II.20}$$

For the first limit we must use L'Hopital rule and (III.8). Then, we obtain (II.21)

$$= \lim_{z\to -2\pi\tau}\frac{(z+2\pi\tau)'}{\theta_1(z+2\pi\tau \mid q^5)'}\cdot\frac{e^{4i\pi\tau}\theta_1^3((-3\pi\tau \mid q^5)}{\theta_1^2(-2\pi\tau \mid q^5)} \tag{II.22}$$

$$= \frac{q^2\theta_1^3(-3\pi\tau \mid q^5)}{2q^{5/8}(q^5;q^5)_\infty^3\theta_1^2(-2\pi\tau \mid q^5)}. \tag{II.23}$$

Now, we use the Jacobi Theta Product expansion and simplify

$$\lim_{z\to -2\pi\tau}(z+2\pi\tau)f_5(z) = \frac{q^2\theta_1^3(-3\pi\tau \mid q^5)}{2q^{5/8}(q^5;q^5)_\infty^3\theta_1^2(-2\pi\tau \mid q^5)} \tag{II.24}$$

$$= \frac{q^2(-iq^{5/8}e^{i(-3\pi\tau)}(q^5;q^5)_\infty}{2q^{5/8}(q^5;q^5)_\infty^3(-iq^{5/8}e^{i(-2\pi\tau)}(q^5;q^5)_\infty} \times \frac{(q^5e^{2i(-3\pi\tau)};q^5)_\infty(e^{-2i(-3\pi\tau)};q^5)_\infty)^3}{(q^5e^{2i(-2\pi\tau)};q^5)_\infty(e^{-2i(-2\pi\tau)};q^5)_\infty)^2} \tag{II.25}$$

$$= \frac{q^2iq^{15/8}q^{-9/2}(q^5;q^5)_\infty^3(q^5q^{-3};q^5)_\infty^3(q^3;q^5)_\infty^3}{2q^{5/8}(q^5;q^5)_\infty^3 i^2q^{10/8}q^{-2}(q^5;q^5)_\infty^2(q^5q^{-2};q^5)_\infty^2(q^2;q^5)_\infty^2} \tag{II.26}$$

$$= \frac{q^{-5/8}(q^5;q^5)_\infty^3(q^2;q^5)_\infty^3(q^3;q^5)_\infty^3}{2iq^{-1/8}(q^5;q^5)_\infty^3(q^5;q^5)_\infty^2(q^3;q^5)_\infty^2(q^2;q^5)_\infty^2} \tag{II.27}$$

$$= \frac{(q^2;q^5)_\infty(q^3;q^5)_\infty}{2iq^{1/2}(q^5;q^5)_\infty^2} \tag{II.28}$$

The residue of $f_5(z)$ at $z=0$ is

$$\lim_{z\to 0}(z^2f_5(z))' = 2zf_5(z)+z^2f_5'(z) \tag{II.29}$$

$$= \lim_{z\to 0}z^2\left(\frac{2z}{z^2}f_5(z)+f_5'(z)\right) \tag{II.30}$$

$$= \lim_{z\to 0}\left(z^2f(z)\right)\left(\frac{2}{z}+\frac{f_5'(z)}{f_5(z)}\right) \tag{II.31}$$

$$= \left(\lim_{z\to 0}\frac{z^2}{\theta_1^2(z \mid q^5)}\right)\left(\lim_{z\to 0}\frac{e^{-2iz}\theta_1^3(z-\pi\tau \mid q^5)}{(z+2\pi\tau \mid q^5)}\right)\left(\lim_{z\to 0}\frac{2}{z}+\frac{f_5'(z)}{f_5(z)}\right) \tag{II.32}$$

$$= \frac{-1}{(2q^{5/8}(q^5;q^5)_\infty^3)^2}\cdot\frac{\theta_1^3(\pi\tau \mid q^5)}{(2\pi\tau \mid q^5)}\left(\lim_{z\to 0}\frac{2}{z}+\frac{f_5'(z)}{f_5(z)}\right) \tag{II.33}$$

$$= \frac{-(-iq^{5/8}e^{i(\pi\tau)}(q^5;q^5)_\infty(q^5e^{2i(\pi\tau)};q^5)_\infty(e^{-2i(\pi\tau)};q^5)_\infty)^3}{4q^{10/8}(q^5;q^5)_\infty^6(-iq^{5/8}e^{i(2\pi\tau)}(q^5;q^5)_\infty(q^5e^{2i(2\pi\tau)};q^5)_\infty(e^{-2i(2\pi\tau)};q^5)_\infty)} \times \left(\lim_{z\to 0}\frac{2}{z}+\frac{f_5'(z)}{f_5(z)}\right) \tag{II.34}$$

$$= \frac{-iq^{27/8}(q^5;q^5)_\infty^3(q^6;q^5)_\infty^3(q^{-1};q^5)_\infty^3}{-4iq^{23/8}(q^5;q^5)_\infty^7(q^7;q^5)_\infty(q^{-2};q^5)_\infty}\left(\lim_{z\to 0}\frac{2}{z}+\frac{f_5'(z)}{f_5(z)}\right) \tag{II.35}$$

$$= \frac{-iq^{27/8}(q^5;q^5)_\infty^3(1-q)^3(q;q^5)_\infty^3(1-q^{-1})^3(q^4;q^5)_\infty^3}{-4iq^{23/8}(q^5;q^5)_\infty^7(1-q^2)(q^2;q^5)_\infty(1-q^{-2})(q^3;q^5)_\infty}\left(\lim_{z\to 0}\frac{2}{z}+\frac{f_5'(z)}{f_5(z)}\right) \tag{II.36}$$

$$= \frac{-(q;q^5)_\infty^3(q^4;q^5)_\infty^3}{4q^{1/2}(q^5;q^5)_\infty^4(q^2;q^5)_\infty(q^3;q^5)_\infty}\left(\lim_{z\to 0}\frac{2}{z}+\frac{f_5'(z)}{f_5(z)}\right) \tag{II.37}$$

(II.38)

Since the sum of the residues of $f_5(z)$ by Theorem II.1 is zero, we get the following:

$$2i\frac{(q^2,q^3,q^5;q^5)_\infty^2}{(q,q^4,q^5;q^5)_\infty^3}(q^5;q^5)_\infty^3 = \lim_{z\to 0}\frac{2}{z}+\frac{f_5'(z)}{f_5(z)}. \tag{II.39}$$

By applying identities (II.4)-(II.5), and the Laurent expansion for $\cot z$, we derive

$$\lim_{z\to 0}\frac{2}{z}+\frac{f_5'(z)}{f_5(z)} = \lim_{z\to 0}\left(\frac{2}{z}-2\frac{\theta_1'}{\theta_1}(z\mid q^5)\right)-2i-3\frac{\theta_1'}{\theta_1}(\pi\tau\mid q^5)-\frac{\theta_1'}{\theta_1}(2\pi\tau\mid q^5) \tag{II.40}$$

$$= -2i-3\frac{\theta_1'}{\theta_1}(\pi\tau\mid q^5)-\frac{\theta_1'}{\theta_1}(2\pi\tau\mid q^5) \tag{II.41}$$

$$= -2i+6i\sum_{n=1}^{\infty}\frac{q^{5n+1}}{1-q^{5n+1}}-6i\sum_{n=1}^{\infty}\frac{q^{5n-1}}{1-q^{5n-1}}+2i\sum_{n=1}^{\infty}\frac{q^{5n+2}}{1-q^{5n+2}}-2i\sum_{n=1}^{\infty}\frac{q^{5n-2}}{1-q^{5n-2}} -4i-6i\left(\frac{q^{-1}}{1-q^{-1}}\right)-2i\left(\frac{q^{-2}}{1-q^{-2}}\right) \tag{II.42}$$

Notice that:

$$-4i-6i\left(\frac{q^{-1}}{1-q^{-1}}\right)-2i\left(\frac{q^{-2}}{1-q^{-2}}\right) = -4i-6i\left(\frac{\frac{1}{q}}{\frac{q-1}{q}}\right)-2i\left(\frac{\frac{1}{q^2}}{\frac{q^2-1}{q^2}}\right) \tag{II.43}$$

$$= -4i+6i\left(\frac{1}{1-q}\right)+2i\left(\frac{1}{1-q^2}\right) \tag{II.44}$$

$$= -4i+6i\left(\frac{q}{1-q}\right)+2i\left(\frac{q^2}{1-q^2}\right) \tag{II.45}$$

(II.46)

Therefore,

$$\lim_{z\to 0}\frac{2}{z}+\frac{f_5'(z)}{f_5(z)}=2i+\sum_{n=1}^{\infty}\frac{c_nq^n}{1-q^n}, \tag{II.47}$$

where, from (II.5), $\{c_n\}_{n=1}^{\infty}$ is a periodic sequence modulo five such that

$$c_1=6i,\qquad c_2=2i,\qquad c_3=-2i,\qquad c_4=-6i,\qquad c_5=0. \tag{II.48}$$

If we denote the two odd primitive Dirichlet characters modulo five by

$$\langle\chi_{2,5}(n)\rangle_{n=0}^{4}=\langle 0,1,i,-i,-1\rangle,\qquad \langle\chi_{4,5}(n)\rangle_{n=0}^{4}=\langle 0,1,-i,i,-1\rangle, \tag{II.49}$$

then, since for χ non-principal modulo p, we may write $[4,pp.136-137]$,

$$L(\chi,0)=\sum_{n=0}^{p-1}\chi(n)\left(\frac{1}{2}-\frac{n}{p}\right), \tag{II.50}$$

it follows from (II.49) and (II.50) that

$$c_n=\frac{2i\chi_{2,5}(n)}{L(\chi_{2,5},0)}+\frac{2i\chi_{4,5}(n)}{L(\chi_{4,5},0)} \tag{II.51}$$

Therefore, from (II.75) and identities (II.47), (II.51), and (I.20),

$$\lim_{z\to 0}\frac{2}{z}+\frac{f_5'(z)}{f_5(z)}=2i+\sum_{n=1}^{\infty}\frac{c_nq^n}{1-q^n} \tag{II.52}$$

$$=2i+\sum_{n=1}^{\infty}\frac{(\frac{2i\chi_{2,5}(n)}{L(\chi_{2,5},0)}+\frac{2i\chi_{4,5}(n)}{L(\chi_{4,5},0)})q^n}{1-q^n} \tag{II.53}$$

$$=2i+2i\sum_{n=1}^{\infty}\left(\frac{\chi_{2,5}(n)}{L(\chi_{2,5},0)}+\frac{\chi_{4,5}(n)}{L(\chi_{4,5},0)}\right)\cdot\frac{q^n}{1-q^n} \tag{II.54}$$

$$=2i\left(1+\sum_{n=1}^{\infty}(\frac{\chi_{2,5}(n)}{L(\chi_{2,5},0)}+\frac{\chi_{4,5}(n)}{L(\chi_{4,5},0)})\cdot\frac{q^n}{1-q^n}\right) \tag{II.55}$$

$$=2i\left(1+\left(\sum_{n=1}^{\infty}\frac{\chi_{2,5}(n)}{L(\chi_{2,5},0)}\cdot\frac{q^n}{1-q^n}+\sum_{n=1}^{\infty}\frac{\chi_{4,5}(n)}{L(\chi_{4,5},0)}\cdot\frac{q^n}{1-q^n}\right)\right) \tag{II.56}$$

$$=2i\left(\left(\frac{1}{2}+\sum_{n=1}^{\infty}\frac{\chi_{2,5}(n)}{L(\chi_{2,5},0)}\cdot\frac{q^n}{1-q^n}\right)+\left(\frac{1}{2}+\sum_{n=1}^{\infty}\frac{\chi_{4,5}(n)}{L(\chi_{4,5},0)}\cdot\frac{q^n}{1-q^n}\right)\right) \tag{II.57}$$

$$=2i\mathscr{E}_5(q). \tag{II.58}$$

This completes the proof for $\mathscr{E}_5(q)$ equation of (II.15). A construction of $\mathscr{E}_7(q)$ equation of (II.15) from (II.59) may be derived from $[9, Eq.(3.23)]$. For the next levels $7 \leq p \leq 19$, the product expansions for the Eisenstein sums $\mathscr{E}_p(\tau)$ mentioned before may be obtained by applying the residue theorem with the elliptic functions $f_p(z)$ of period $\pi, p\pi\tau$, defined by

$$f_7(z) = e^{2iz}\frac{\theta_1^2(z+\pi\tau \mid q^7)\theta_1(z+2\pi\tau \mid q^7)}{\theta_1^2(z \mid q^7)\theta_1(z-3\pi\tau \mid q^7)}, \tag{II.59}$$

$$f_{11}(z) = e^{-2iz}\frac{\theta_1(z-2\pi\tau \mid q^{11})\theta_1(z-3\pi\tau \mid q^{11})\theta_1(z-5\pi\tau \mid q^{11})}{\theta_1^2(z \mid q^{11})\theta_1(z+\pi\tau \mid q^{11})}, \tag{II.60}$$

$$f_{13}(z) = e^{-2iz}\frac{\theta_1(z-3\pi\tau \mid q^{13})\theta_1(z-4\pi\tau \mid q^{13})\theta_1(z-5\pi\tau \mid q^{13})}{\theta_1^2(z \mid q^{13})\theta_1(z+\pi\tau \mid q^{13})}, \tag{II.61}$$

$$f_{17}(z) = e^{-2iz}\frac{\theta_1(z-3\pi\tau \mid q^{17})\theta_1(z-5\pi\tau \mid q^{17})\theta_1(z-7\pi\tau \mid q^{17})}{\theta_1^2(z \mid q^{17})\theta_1(z+2\pi\tau \mid q^{17})}, \tag{II.62}$$

$$f_{19}(z) = e^{-2iz}\frac{\theta_1(z-4\pi\tau \mid q^{19})\theta_1(z-5\pi\tau \mid q^{19})\theta_1(z-7\pi\tau \mid q^{19})}{\theta_1^2(z \mid q^{19})\theta_1(z+3\pi\tau \mid q^{19})}, \tag{II.63}$$

These can be shown to be periodic, with periods π and $p\pi\tau$, by applying (II.6) Now we let $\Gamma_0(p)$ act on the series $\mathscr{E}_p(\tau)$ to obtain a change of basis for each level p from the Eisenstein basis for $\mathscr{M}_1(\Gamma_1(p))$ to a corresponding basis of products. $\square$

For the bases constructed below, we require that the first nonzero coefficient in the q-expansion of the image of $\mathscr{E}_p(\tau)$ under the slash operator to be 1. Since this action depends only on the lower right entry of $\gamma \in \Gamma_0(p)$, we list only this element in the following results, and denote the operator of

$$|_\gamma = \langle \gamma_{22} \rangle \tag{II.64}$$

Theorem II.4. *Define* $\langle\cdot\rangle$ *by* (II.64) *and define* $\mathscr{E}_p$ *by* (II.75)*. For prime* $5 \leq p \leq 19$*, and a set of distinct elements* $\{a_{k,p}\}_{k=1}^{(p-1)/2} \subset (\mathbb{Z}/p\mathbb{Z})^*/\{\pm 1\}$*, there exists a basis decomposition*

$$\mathscr{M}_1(\Gamma_1(p)) = \bigoplus_{k=1}^{(p-1)/2} \mathbb{C}\langle a_{k,p}\rangle(\mathscr{E}_p). \tag{II.65}$$

Moreover, if the constants $a_{k,p}$ *are as follows, the basis elements* $\langle a_{k,p}\rangle(\mathscr{E}_p)$ *of* (II.65) *are normalized so that the first nonzero coefficient in their q-expansion is* 1*:*

$$(a_{k,5})_{k=1}^{2} = (1,2), \qquad (a_{k,7})_{k=1}^{3} = (1,2,3), \qquad (a_{k,11})_{k=1}^{5} = (1,2,3,5,7),$$

$$(a_{k,13})_{k=1}^{6} = (1,2,3,4,5,7), \qquad (a_{k,17})_{k=1}^{8} = (1,2,3,5,7,8,11,13),$$

$$(a_{k,19})_{k=1}^{9} = (1,2,3,4,5,7,9,11,13).$$

Proof. The orthogonality of the Dirichlet characters modulo p may be used to derive

$$\sum_{\chi(-1)=-1} \chi(a)\overline{\chi}(b) = \begin{cases} \pm\varphi(p)/2, & a \equiv \pm b \pmod{p}, \\ 0, & a \not\equiv \pm b \pmod{p}, \end{cases} \tag{II.66}$$

Therefore, if $\{a_{k,p}\}_{k=1}^{(p-1)/2} = (\mathbb{Z}/p\mathbb{Z})^*/\{\pm 1\}$ and $\{\chi_{2s}\}_{s=1}^{(p-1)/2}$ are odd, the rows of

$$(B)_{k,s} = \chi_{2s}(a_{k,p}), \qquad 1 \le k,s \le (p-1)/2, \tag{II.67}$$

are orthogonal under the standard Hermitian inner product. Since an orthogonal set of vectors is linearly independent, the matrix B is an invertible linear transformation corresponding to the change of basis for $\mathscr{M}_1(\Gamma_1(p))$

$$B\Big(E_{1,\chi_2}(\tau),\ldots,E_{1,\chi_{2p}}(\tau)\Big)^T = \Big(\langle a_{1,p}\rangle(\mathscr{E}_p),\ldots,\langle a_{(p-1)/2,p}\rangle(\mathscr{E}_p)\Big)^T. \tag{II.68}$$

The normalization claims of Theorem II.4 may be verified from q-expansions for the linear combination of Eisenstein series defining each basis element in the image. □

Theorem II.5. *The basis elements from* (II.65) *are permuted up to a change of sign by* $\Gamma_0(p)$ *under* $\langle\cdot\rangle$*, with permutation representation isomorphic to* $(\mathbb{Z}/p\mathbb{Z})^*/\{\pm 1\}$.

Proof. Let $\delta : \Gamma_0(p) \to (\mathbb{Z}/p\mathbb{Z})^*$ be defined by

$$\delta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = d(mod p) \tag{II.69}$$

and $\kappa : (\mathbb{Z}/p\mathbb{Z})^* \to PGL\left(\frac{p-1}{2},\mathbb{C}\right)$ be defined by

$$\delta(d) = diag(\chi_2(d),\chi_4(d),\cdots,\chi_{2s}(d)). \tag{II.70}$$

Since δ is an onto homomorphism from $\Gamma_0(p)$ to $(\mathbb{Z}/p\mathbb{Z})^*$,

$$\delta(\Gamma_0(p)) = (\mathbb{Z}/p\mathbb{Z})^*. \tag{II.71}$$

Since the kernel of $\kappa \in (\mathbb{Z}/p\mathbb{Z})^*$ is $\{\pm 1\}$, we get, using the first isomorphism theorem,

$$\kappa \circ \delta(\Gamma_0(p)) = \delta(\Gamma_0(p))/Ker\kappa = (\mathbb{Z}/p\mathbb{Z})^*/\{\pm 1\}. \tag{II.72}$$

If we identify scalar multiples of modular forms, then the slash operator defines a group action $\Gamma_0(p).X$ on a set of generators for the vector space of weight one modular forms on $\Gamma_1(p)$

$$X = \mathscr{M}_1(\Gamma_1(p))$$

To determine the permutation representations for this group action, we note that each generator given in theorem II.4 for $\mathscr{M}_1(\Gamma_1(p))$ is uniquely representable as a linear combination of Eisenstein series $E_{1,\chi}(\tau)$ of weight one for $\Gamma_0(p)$ twisted by an odd Dirichlet character χ. The Eisenstein series transform under $\Gamma_0(p)$ according to

$$E_{1,\chi}(\gamma\tau) = \chi(d)(c\tau + d)E_{1,\chi}(\tau), \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{II.73}$$

Therefore, a linear combination of the Eisenstein series transforms according to

$$\langle\gamma\rangle \left(\sum_{s=1}^{\frac{p-1}{2}} c_s E_{1,\chi_{2s}}(\gamma,\tau) \right) = \sum_{s=1}^{\frac{p-1}{2}} c_s \chi_{2s}(d) E_{1,\chi_{2s}}(\gamma,\tau). \tag{II.74}$$

Since one of the generators for $\mathscr{M}_1(\Gamma_1(p))$ is

$$\mathscr{E}_p(\tau) = \frac{2}{p-1} \sum_{\chi(-1)=-1} E_{1,\chi}(\tau), \tag{II.75}$$

we see that the orbit of $\mathscr{E}_p(\tau)$ under the action of $\Gamma_0(p)$ on $X = \mathscr{M}_1(\Gamma_1(p))$ is

$$\left\{ \frac{2}{p-1} \sum_{s=1}^{\frac{p-1}{2}} \chi_{2s}(d) E_{1,\chi_{2s}}(\gamma,\tau) | d \in (\mathbb{Z}/p\mathbb{Z})^* \right\}.$$

Hence, a group representation for the permutation of X by $\Gamma_0(p)$ is given by the image of

$$\kappa \circ \delta : \Gamma_0(p) \to PGL\left(\frac{p-1}{2}, \mathbb{C}\right). \tag{II.76}$$

Since we have proven above that

$$\kappa \circ \delta(\Gamma_0(p)) = (\mathbb{Z}/p\mathbb{Z})^*/\{\pm 1\}, \tag{II.77}$$

the permutation representation for the action of $\Gamma_0(p)$ on X is

$$(\mathbb{Z}/p\mathbb{Z})^*/\{\pm 1\}. \tag{II.78}$$

In other words, the action of $\Gamma_0(p)$ under the slash operator permutes the given generators for $\mathscr{M}_1(\Gamma_1(p))$ with permutation representation

$$(\mathbb{Z}/p\mathbb{Z})^*/\{\pm 1\}. \tag{II.79}$$

□

The proof of Theorem II.5 implies that the action of $d \in (\mathbb{Z}/p\mathbb{Z})^*$ on $\mathscr{E}_p$ satisfies

$$\langle d\rangle\langle a_{k,p}\rangle(\mathscr{E}_p) = \langle d\cdot a_{k,p}\rangle(\mathscr{E}_p) = \pm\langle a'_{k,p}\rangle(\mathscr{E}_p), \quad \pm d\cdot a_{k,p} \equiv a'_{k,p} \in \{a_{k,p}\}_{k=1}^{(p-1)/2}. \tag{II.80}$$

A basis for $\mathscr{M}_1(\Gamma_1(7))$ is $\langle 1\rangle\mathscr{E}_7(\tau), \langle 2\rangle\mathscr{E}_7(\tau), \langle 3\rangle\mathscr{E}_7(\tau)$. This basis is permuted up to a change of sign by $\Gamma_0(7)$. An example is given below.

Example (II.6), Take $a_{2,7}$ from Theorem II.4 and let $d = 2$. Then,

$$\langle d\rangle\langle 2\rangle\mathscr{E}_7(\tau) = \langle d\rangle\left(\langle 2\rangle\mathscr{E}_7(\tau)\right) \tag{II.81}$$

$$= \langle d\rangle\left(\frac{2}{7-1}\chi_{2,7}(2)E_{\chi_{2,7}}(\tau) + \frac{2}{7-1}\chi_{4,7}(2)E_{\chi_{4,7}}(\tau) + \frac{2}{7-1}\chi_{6,7}(2)E_{\chi_{6,7}}(\tau)\right) \tag{II.82}$$

$$= \langle d\rangle\left(\frac{1}{3}\chi_{2,7}(2)E_{\chi_{2,7}}(\tau) + \frac{1}{3}\chi_{4,7}(2)E_{\chi_{4,7}}(\tau) + \frac{1}{3}\chi_{6,7}(2)E_{\chi_{6,7}}(\tau)\right) \tag{II.83}$$

$$= \left(\frac{1}{3}\chi_{2,7}(2)\chi_{2,7}(d)E_{\chi_{2,7}}(\tau) + \frac{1}{3}\chi_{4,7}(2)\chi_{4,7}(d)E_{\chi_{4,7}}(\tau) + \frac{1}{3}\chi_{6,7}(2)\chi_{6,7}(d)E_{\chi_{6,7}}(\tau)\right) \tag{II.84}$$

$$= \left(\frac{1}{3}\chi_{2,7}(4)E_{\chi_{2,7}}(\tau) + \frac{1}{3}\chi_{4,7}(4)E_{\chi_{4,7}}(\tau) + \frac{1}{3}\chi_{6,7}(4)E_{\chi_{6,7}}(\tau)\right) \tag{II.85}$$

$$= \left(\frac{1}{3}\chi_{2,7}(-3)E_{\chi_{2,7}}(\tau) + \frac{1}{3}\chi_{4,7}(-3)E_{\chi_{4,7}}(\tau) + \frac{1}{3}\chi_{6,7}(-3)E_{\chi_{6,7}}(\tau)\right) \tag{II.86}$$

$$= \Big(\frac{1}{3}\chi_{2,7}(-1)\chi_{2,7}(3)E_{\chi_{2,7}}(\tau) + \frac{1}{3}\chi_{4,7}(-1)\chi_{4,7}(3)E_{\chi_{4,7}}(\tau) + \frac{1}{3}\chi_{6,7}(-1)\chi_{6,7}(3)E_{\chi_{6,7}}(\tau)\Big) \tag{II.87}$$

$$= \left(-\frac{1}{3}\chi_{2,7}(3)E_{\chi_{2,7}}(\tau) - \frac{1}{3}\chi_{4,7}(3)E_{\chi_{4,7}}(\tau) - \frac{1}{3}\chi_{6,7}(3)E_{\chi_{6,7}}(\tau)\right) \tag{II.88}$$

$$= -\left(\frac{2}{7-1}\chi_{2,7}(3)E_{\chi_{2,7}}(\tau) + \frac{2}{7-1}\chi_{4,7}(3)E_{\chi_{4,7}}(\tau) + \frac{2}{7-1}\chi_{6,7}(3)E_{\chi_{6,7}}(\tau)\right) \tag{II.89}$$

$$= -\langle 3\rangle\mathscr{E}_7(\tau). \tag{II.90}$$

Thus, $\langle d\rangle\langle 2\rangle\mathscr{E}_7(\tau)=-\langle 3\rangle\mathscr{E}_7(\tau)$.

We now show that the normalized Eisenstein sums from Theorem II.4 are synonymous with the products (I.3)–(I.18) from the Introduction. To prove this, we show that each basis element of level p for $\mathscr{M}_1(\Gamma_1(p))$ from Theorem II.4 is representable as a quotient of modified theta constants with Jacobi triple product representation [5, p. 141]

$$\theta\begin{bmatrix} m/n \\ 1 \end{bmatrix}(n\tau)=\exp\left(\frac{\pi i m}{2n}\right)q^{m^2/(8n)}(q^{(n-m)/2};q^n)_\infty(q^{(n+m)/2};q^n)_\infty(q^n;q^n)_\infty. \tag{II.91}$$

We derive each theta quotient by writing the product formulations of Theorem II.3 in terms of modified theta constants and applying transformations for the theta constants.

Theorem II.6. *Define* $\varphi_{\ell,k}$ *by* (II.9)*, and, for* $[b_1,\ldots,b_{(p-1)/2}]\in\mathbb{Z}^{(p-1)/2}$*, denote*

$$\mathfrak{T}_p[b_1,\ldots,b_{(p-1)/2}](\tau)=\eta^3(p\tau)\prod_{\ell=1}^{(p-1)/2}\exp\left(-\frac{\pi i b_\ell(2-1)}{2p}\right)\varphi_{p,\ell}^{b_\ell}(\tau). \tag{II.92}$$

The bases for $\mathscr{M}_1(\Gamma_1(p))$ *from Theorem II.4 have the theta quotient representations:*

Level, p	*Basis for* $\mathscr{M}_1(\Gamma_1(p))$
5	$\langle 1\rangle(\mathscr{E}_5)=\mathfrak{T}_5[2,-3]$, $\langle 2\rangle(\mathscr{E}_5)=\mathfrak{T}_5[-3,2]$
7	$\langle 1\rangle(\mathscr{E}_7)=\mathfrak{T}_7[1,0,-2]$, $\langle 2\rangle(\mathscr{E}_7)=\mathfrak{T}_7[-2,1,0]$, $\langle 3\rangle(\mathscr{E}_7)=\mathfrak{T}_7[0,-2,1]$
11	$\langle 1\rangle(\mathscr{E}_{11})=\mathfrak{T}_{11}[0,1,0,-1,-1]$, $\langle 2\rangle(\mathscr{E}_{11})=\mathfrak{T}_{11}[-1,0,0,1,-1]$ $\langle 3\rangle(\mathscr{E}_{11})=\mathfrak{T}_{11}[1,-1,-1,0,0]$, $\langle 5\rangle(\mathscr{E}_{11})=\mathfrak{T}_{11}[0,-1,1,-1,0]$ $\langle 7\rangle(\mathscr{E}_{11})=\mathfrak{T}_{11}[-1,0,-1,0,1]$
13	$\langle 1\rangle(\mathscr{E}_{13})=\mathfrak{T}_{13}[1,0,0,-1,0,-1]$, $\langle 2\rangle(\mathscr{E}_{13})=\mathfrak{T}_{13}[-1,-1,0,1,0,0]$, $\langle 3\rangle(\mathscr{E}_{13})=\mathfrak{T}_{13}[0,0,-1,0,1,-1]$, $\langle 4\rangle(\mathscr{E}_{13})=\mathfrak{T}_{11}[0,1,-1,-1,0,0]$, $\langle 5\rangle(\mathscr{E}_{13})=\mathfrak{T}_{13}[0,-1,1,0,-1,0]$, $\langle 7\rangle(\mathscr{E}_{13})=\mathfrak{T}_{13}[-1,0,0,0,-1,1]$
17	$\langle 1\rangle(\mathscr{E}_{17})=\mathfrak{T}_{17}[1,0,0,0,0,-1,-1,0]$, $\langle 2\rangle(\mathscr{E}_{17})=\mathfrak{T}_{17}[0,-1,0,0,1,0,0,-1]$, $\langle 3\rangle(\mathscr{E}_{17})=\mathfrak{T}_{17}[0,0,0,-1,0,1,0,-1]$, $\langle 5\rangle(\mathscr{E}_{17})=\mathfrak{T}_{17}[0,0,0,1,-1,-1,0,0]$,

	$\langle 7\rangle(\mathscr{E}_{17}) = \mathfrak{T}_{17}[0,-1,1,0,0,0,-1,0]$,
	$\langle 8\rangle(\mathscr{E}_{17}) = \mathfrak{T}_{17}[0,0,-1,0,-1,0,0,1]$
	$\langle 11\rangle(\mathscr{E}_{17}) = \mathfrak{T}_{17}[-1,1,-1,0,0,0,0,0]$,
	$\langle 13\rangle(\mathscr{E}_{17}) = \mathfrak{T}_{17}[-1,0,0,-1,0,0,1,0]$,
19	$\langle 1\rangle(\mathscr{E}_{19}) = \mathfrak{T}_{19}[1,1,0,0,-1,-1,-1,0,0]$,
	$\langle 2\rangle(\mathscr{E}_{19}) = \mathfrak{T}_{19}[0,-1,-1,0,1,1,0,-1,0]$,
	$\langle 3\rangle(\mathscr{E}_{19}) = \mathfrak{T}_{19}[1,-1,0,0,-1,0,1,0,-1]$,
	$\langle 4\rangle(\mathscr{E}_{19}) = \mathfrak{T}_{19}[0,0,1,-1,0,-1,0,1,-1]$,
	$\langle 5\rangle(\mathscr{E}_{19}) = \mathfrak{T}_{19}[0,0,-1,1,0,0,-1,1,-1]$,
	$\langle 7\rangle(\mathscr{E}_{19}) = \mathfrak{T}_{19}[0,0,1,-1,-1,1,0,-1,0]$
	$\langle 9\rangle(\mathscr{E}_{19}) = \mathfrak{T}_{19}[-1,-1,0,-1,0,0,1,0,1]$,
	$\langle 11\rangle(\mathscr{E}_{19}) = \mathfrak{T}_{19}[-1,0,0,1,0,0,-1,-1,1]$
	$\langle 13\rangle(\mathscr{E}_{19}) = \mathfrak{T}_{19}[-1,1,-1,0,1,-1,0,0,0]$

Table 2.1: bases for $\mathscr{M}_1(\Gamma_1(p))$

Proof. The theta quotient representations for $\mathscr{E}_p(\tau) = \langle 1\rangle(\mathscr{E}_p)(\tau)$ may be deduced from the product representations proved in Theorem II.4. Transformation formula for these theta quotients under $\Gamma_0(p)$, in turn, may be deduced from corresponding modular transformation formulas for $\eta(\tau)$ from $[6, p.51]$ and those for vectors of modified theta constants $\mathscr{V}_N(\tau)$, defined by (II.11), under generators for the full modular group from Theorem II.2. For each prime p, we may deduce the product representations for each normalized Eisenstein sum $\langle a_{k,p}\rangle(\mathscr{E}_p)$ from the modular transformation formulas for these building blocks. We illustrate the general procedure with $p = 5$. From Theorem II.3 and (II.91),

$$\langle 1\rangle(\mathscr{E}_5) = \frac{(q;q)_\infty^2}{(q,q^4;q^5)_\infty^5} = \eta^3(5\tau)\frac{e^{-2\pi i/10}\varphi_{5,1}^2(\tau)}{e^{-9\pi i/10}\varphi_{5,2}^3(\tau)} = \mathfrak{T}_5[2,-3](\tau). \tag{II.93}$$

A set of generators for $\Gamma_0(5)$ is given by [10]

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \alpha = \begin{pmatrix} 2 & -1 \\ 5 & -2 \end{pmatrix}, \quad \beta = \begin{pmatrix} 3 & -2 \\ 5 & -3 \end{pmatrix}. \tag{II.94}$$

We now employ transformation formulas up to a constant multiple for the weight $1/2$ vector of modified theta constants $[\varphi_{5,2}, \varphi_{5,1}]^{tr}$. We begin with parameterizations for the generators of $\Gamma_0(5)$ in terms of those for the full modular group

$$\alpha = TST^2ST^3S, \qquad \beta = TST^3ST^2S. \tag{II.95}$$

Up to a constant multiple, transformation matrices for the vectors of modified theta constants may be computed from their images in $PGL((p-1)/2, \mathbb{C})$ via the representation π_p given in (II.12)–(II.13)

$$\pi_5(T) = \begin{pmatrix} e^{\frac{9\pi i}{20}} & 0 \\ 0 & e^{\frac{\pi i}{20}} \end{pmatrix}, \ \pi_5(\alpha) = \begin{pmatrix} 0 & e^{\frac{\pi i}{20}} \\ -e^{\frac{9\pi i}{20}} & 0 \end{pmatrix}, \ \pi_5(\beta) = \begin{pmatrix} 0 & e^{\frac{\pi i}{4}} \\ -e^{\frac{\pi i}{4}} & 0 \end{pmatrix}. \tag{II.96}$$

Hence, by (II.93), and the modular transformation formula for $\eta(\tau)$, we deduce that up to a constant multiple, C,

$$\langle 2\rangle(\mathscr{E}_5) = (5\tau-3)^{-1}\mathfrak{T}_5[2,-3](\beta\tau) \tag{II.97}$$

$$= (5\tau-3)^{-1}\eta^3(5\beta\tau)\prod_{\ell=1}^{2}\exp\left(-\frac{\pi i b_\ell(2-1)}{2(5)}\right)\varphi_{5,\ell}^{b_\ell}(\beta\tau) \tag{II.98}$$

$$= (5\tau-3)^{-1}\eta^3(5\beta\tau)\left[\exp\left(-\frac{\pi i b_1(2-1)}{2(5)}\right)\varphi_{5,1}^{b_1}(\beta\tau)\cdot\exp\left(-\frac{\pi i b_2(2-1)}{2(5)}\right)\varphi_{5,2}^{b_2}(\beta\tau)\right] \tag{II.99}$$

$$= (5\tau-3)^{-1}\eta^3(5\beta\tau)\left[e^{\left(-\frac{\pi i(2)}{10}\right)}\varphi_{5,1}^{2}(\beta\tau)\cdot e^{\left(-\frac{\pi i(-3)}{10}\right)}\varphi_{5,2}^{-3}(\beta\tau)\right] \tag{II.100}$$

$$= (5\tau-3)^{-1}\eta^3(5\beta\tau)\left[\frac{e^{\left(\frac{3\pi i}{10}\right)}}{e^{\left(\frac{2\pi i}{10}\right)}}\cdot\frac{\varphi_{5,1}^{2}(\beta\tau)}{\varphi_{5,2}^{3}(\beta\tau)}\right] \tag{II.101}$$

From (II.96) we note that, up to a constant multiple, the transformation corresponding to β is

$$\pi_5(\beta)\cdot\begin{pmatrix}\theta\begin{bmatrix}\frac{3}{5}\\1\end{bmatrix}(5\tau)\\ \theta\begin{bmatrix}\frac{1}{5}\\1\end{bmatrix}(5\tau)\end{pmatrix}=\begin{pmatrix}0 & e^{\frac{\pi i}{4}}\\ -e^{\frac{\pi i}{4}} & 0\end{pmatrix}\cdot\begin{pmatrix}\theta\begin{bmatrix}\frac{3}{5}\\1\end{bmatrix}(5\tau)\\ \theta\begin{bmatrix}\frac{1}{5}\\1\end{bmatrix}(5\tau)\end{pmatrix} \tag{II.102}$$

$$=\begin{pmatrix}e^{\frac{\pi i}{4}}\theta\begin{bmatrix}\frac{1}{5}\\1\end{bmatrix}(5\tau)\\ -e^{\frac{\pi i}{4}}\theta\begin{bmatrix}\frac{3}{5}\\1\end{bmatrix}(5\tau).\end{pmatrix}. \tag{II.103}$$

Therefore, up to a constant multiple, $\varphi_{5,1}$ is sent to $\varphi_{5,2}$ under β and vice versa. By applying the modular transformation, including that for $\eta(\tau)$, we derive

$$\langle 2\rangle(\mathscr{E}_5)=(5\tau-3)^{-1}\eta^3(5\tau)\left[\frac{e^{\left(\frac{3\pi i}{10}\right)}}{e^{\left(\frac{2\pi i}{10}\right)}}\cdot\frac{\left(-e^{\frac{\pi i}{4}}\varphi_{5,2}(\tau)\right)^2}{\left(e^{\frac{\pi i}{4}}\varphi_{5,1}(\tau)\right)^3}\right] \tag{II.104}$$

$$=C\eta^3(5\tau)\frac{\varphi_{5,2}^2(\tau)}{\varphi_{5,1}^3(\tau)} \tag{II.105}$$

$$=C\frac{e^{6\pi i/10}}{e^{3\pi i/10}}q+O(q^2). \tag{II.106}$$

On the other hand, from transformation formulas satisfied by $E_{\chi_{2,5},1}(\tau)$ and $E_{\chi_{4,5},1}(\tau)$ from (II.49) and (II.73),

$$\langle 2\rangle(\mathscr{E}_5)=\mathscr{E}_5(\beta\tau)=\frac{\chi_{2,5}(2)}{2}E_{\chi_{2,5},1}(\tau)+\frac{\chi_{4,5}(2)}{2}E_{\chi_{4,5},1}(\tau) \tag{II.107}$$

$$=\frac{\chi_{2,5}(2)}{2}\left(1+\frac{2}{L(0,\chi_{2,5})}\sum_{n=1}^{\infty}\frac{\chi_{2,5}(n)q^n}{1-q^n}\right)+\frac{\chi_{4,5}(2)}{2}\left(1+\frac{2}{L(0,\chi_{4,5})}\sum_{n=1}^{\infty}\frac{\chi_{4,5}(n)q^n}{1-q^n}\right) \tag{II.108}$$

$$=\left(\frac{\chi_{2,5}(2)}{2}+\frac{2\chi_{2,5}(2)}{2L(0,\chi_{2,5})}\sum_{n=1}^{\infty}\frac{\chi_{2,5}(n)q^n}{1-q^n}\right)+\left(\frac{\chi_{4,5}(2)}{2}+\frac{2\chi_{4,5}(2)}{2L(0,\chi_{4,5})}\sum_{n=1}^{\infty}\frac{\chi_{4,5}(n)q^n}{1-q^n}\right) \tag{II.109}$$

$$=\left(\frac{i}{2}+\frac{i}{L(0,\chi_{2,5})}\sum_{n=1}^{\infty}\frac{\chi_{2,5}(n)q^n}{1-q^n}\right)+\left(\frac{(-i)}{2}+\frac{(-i)}{L(0,\chi_{4,5})}\sum_{n=1}^{\infty}\frac{\chi_{4,5}(n)q^n}{1-q^n}\right) \tag{II.110}$$

$$=\frac{i}{L(0,\chi_{2,5})}\sum_{n=1}^{\infty}\left(\sum_{d|n}\chi_{2,5}(d)\right)q^n+\frac{(-i)}{L(0,\chi_{4,5})}\sum_{n=1}^{\infty}\left(\sum_{d|n}\chi_{4,5}(d)\right)q^n \tag{II.111}$$

$$=q+O(q^2). \tag{II.112}$$

Therefore, $C = e^{-3\pi i/10}$, and so

$$\langle 2\rangle(\mathscr{E}_5) = e^{-3\pi i/10}\eta^3(5\tau)\frac{\varphi_{5,2}^2(\tau)}{\varphi_{5,1}^3(\tau)} = \mathfrak{T}_5[-3,2](\tau) = q\frac{(q;q)_\infty^2}{(q^2,q^3;q^5)_\infty^5}. \tag{II.113}$$

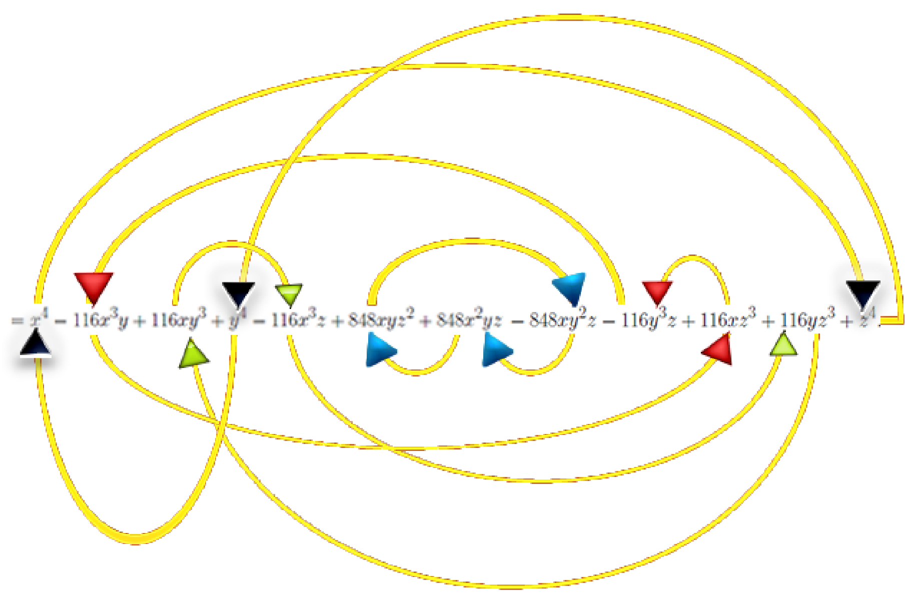
For $5 \leq p \leq 19$ we obtain the theta quotient representations of the bases for $\mathscr{M}_1(\Gamma_1(p))$ from those for $\mathscr{E}_p(\tau)$. In each case, we permute the theta quotients according to the image of π_N and apply the transformation formulas for Eisenstein series on $\Gamma_0(p)$, $(\gamma_{21}\tau+\gamma_{22})^{-k}E_{k,\chi}(\gamma\tau) = \chi(\gamma_{22})E_{k,\chi}(\tau)$ for $\gamma \in \Gamma_0(p)$, to each componenent of $\mathscr{E}_p(\tau)$. We then compare the first nonzero entry in the resulting q-expansions. By repeating this process with each independent set of generators for $\mathscr{M}_1(\Gamma_0(p))$ from Theorem II.4, we ultimately obtain the linearly independent sets of theta quotient representations claimed in Theorem II.6. For higher levels $7 \leq p \leq 19$, we similarly use the fact that the image of $\Gamma_0(p)$ under the presentation π_p defined by Theorem II.2 is a matrix with a single nonzero entry in each row and column. □

Chapter III

Conclusion

The reason why the results in this thesis are important is that they continue my colleagues' work [3]. Richard Charles et al. showed that Eisenstein series for $PSL(2,\mathbb{Z})$ are representable symmetrically in terms of certain infinite products that are modular forms of level $N = 5$. His work uses recursion formulas satisfied by the Eisenstein series. In this thesis, we have given an algebraic explanation for the symmetries of these polynomials and have described the symmetry precisely. This work also extends the phenomenon beyond the level $N = 5$ case. The following is an example of the phenomenon for level $N = 7$ exhibiting the symmetry in a formulation for the Eisenstein series of weight four (I.3) from chapter I:

$1 + 240\sum_{n=1}^{\infty} \frac{n^3q^n}{1-q^n}$



BIBLIOGRAPHY

[1] L. A. BORISOV AND P. E. GUNNELLS, *Toric modular forms of higher weight*, arXiv preprint math/0203242, (2002).

[2] K. BUZZARD, *Computing weight one modular forms over c and fp*, in Submitted to Proceedings of the Summer School and Conference,âĂIJComputations with Modular Forms, 2011.

[3] R. CHARLES, T. HUBER, AND A. MENDOZA, *Parameterizations for quintic eisenstein series*, Journal of Number Theory, 133 (2013), pp. 195–214.

[4] F. DIAMOND AND J. M. SHURMAN, *A first course in modular forms*, vol. 228, Springer, 2005.

[5] H. M. FARKAS AND I. KRA, *Theta constants, Riemann surfaces and the modular group*, American Mathematical Soc., 2001.

[6] M. I. KNOPP, *Modular functions in analytic number theory*, vol. 337, American Mathematical Soc., 2008.

[7] N. KOBLITZ, *Introduction to elliptic curves and modular forms*, vol. 97, Springer, 1993.

[8] S. LANG, *Elliptic functions*, Springer, 1987.

[9] Z.-G. LIU, *On certain identities of ramanujan*, Journal of Number Theory, 83 (2000), pp. 59–75.

[10] W. STEIN ET AL., *Sage Mathematics Software (Version 6.2.0)*, The Sage Development Team, 2014. `http://www.sagemath.org`.

[11] W. A. STEIN, *Modular forms, a computational approach*, vol. 79, American Mathematical Soc., 2007.

[12] E. T. WHITTAKER AND G. N. WATSON, *A course of modern analysis*, Cambridge university press, 1927.

BIOGRAPHICAL SKETCH

Esteban J. Melendez was born in Brownsville, Texas on September 30th, 1988 and raised in Monterrey, Nuevo Leon, Mexico. He is the oldest of three children born to his parents Mr. Elias Melendez Mtz. and Mrs. Cruz Santillan A.(†). He received his B.S. in Mathematics with Pure Mathematical Concentration on December 2012 from The University of Texas-Pan American, and his M.S. in Mathematical Sciences on August 2014. For correspondence he may be reached at estebanjms@gmail.com.