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Dynamics for the Compound Burgers-KdV equation

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DYNAMICS FOR THE COMPOUND BURGERS-KDV EQUATION

A Thesis

by

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ABSTRACT

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In this thesis, we study the Two-Dimensional Burgers-Korteweg-de Vries (2D-BKdV) equation and Two-Dimensional Compound Burgers-Korteweg-de Vries (2D-Compound BKdV) by analyzing the first integral equation, which indicates that under some particular conditions, the 2D-BKdV equation and 2D-Compound BKdV have exact travelling wave solutions. By using the elliptic integral and some transformations, travelling wave solution to the 2D-BKdV equation and 2D-Compound BKdV equation are expressed explicitly.

DEDICATION

The completion of my masters studies would not have been possible without the love and support of my family and professor. My father, Fajing Zheng, my mother, Xizhen Huang, my wife, Lu Xu, and my adviser, Dr. Zhaosheng Feng, who inspired, motivated and supported me by all means to accomplish all my dreams until now. Thank you for your love and support.

ACKNOWLEDGMENTS

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CHAPTER I

INTRODUCTION

During the past three decades, the Burgers equation, Korteweg-de Vries (KdV) equation and Burgers-Korteweg-de Vries equation (Burgers-KdV) have attracted a lot of attention from a rather diverse group of scientists such as physicists and mathematicians, because these three equations not only arise from realistic physical phenomena, but can also be widely applied to many physically significant fields such as plasma physics, fluid dynamics, crystal lattice theory, nonlinear circuit theory and astrophysics [1-10].

Consider the Burgers-KdV equation

$$u_t + \alpha u u_x + \beta u_{xx} + s u_{xxx} = 0 \tag{1}$$

where α, β and s are real constants with $\alpha\beta s \neq 0$. Equation (1) is applied as a nonlinear model of the propagation of waves on an elastic tube filled with a viscous fluid [7], the flow of liquids containing gas bubbles [8] and turbulence [9, 10]. It can also be regarded as a combination of the Burgers' equation and KdV equation, since the choices $\alpha \neq 0$, $\beta \neq 0$ and $s = 0$ lead equation (1) to the Burgers' equation

$$u_t + \alpha u u_x + \beta U_{xx} = 0 \tag{2}$$

and the choices $\alpha \neq 0$, $\beta = 0$ and $s \neq 0$ lead equation (1) to the Kdv equation

$$u_t + \alpha u u_x + s u_{xxx} = 0. \tag{3}$$

It is well known that both (2) and (3) are exactly solvable, and have the travelling wave solutions as follows:

$$u(x,t) = \frac{2k}{\alpha} + \frac{2\beta k}{\alpha} tanh\, k(x - 2kt)$$

and

$$u(x,t) = \frac{12sk^2}{\alpha} sech^2 k(x - 4sk^2 t)$$

Respectively. A great number of theoretical issues concerning the Burgers-KdV equation have received considerable attention. In particular, the travelling wave solution to the Burgers-KdV equation has been studied extensively. Johnson examined the travelling wave solutions to the Burgers- KdV equation in the phase plan by means of a perturbation method in the regimes where $\beta \ll s$ and $s \ll \beta$, and developed formal asymptotic expansions for the solution [7]. Grua and Hu used a steady-state version of equation (11) to describe a weak shock profile in plasmas [11]. They studied the same problem using a similar method to that used by Johnson [7], and a related problem was studied by Jeffrey [12]. The numerical investigation of the problem was carried out by Canosa and Gazdag et al [13–15]. Bona and Schonbek studied the existence and uniqueness of bounded travellingwave solutions to (1) which tend to constant states at plus and minus infinity [16]. They also considered the limiting behaviour of the travelling wave solution of (1) as $\to 0$ with s of order 1, and also as $s \to 0$ with of order 1. The case where both β and $s \to 0$ with β/s held fixed was also examined. The asymptotic and global behaviour of the travelling wave solution to (1) as $s > 0$ was undertaken by Guan and Gao, and the applications of the theory to diversified turbulent flow problems were described in detail in [9, 17]. On using variable transformation and the theory of ordinary differential equation, the asymptotic behaviour and the proper analytical solution to (1) were presented by Shu [18]. Gibbon et al showed that equation (1) does not have the Painleve property [19]. Qualitative results concerning the travelling wave solutions to the Burgers-KdV equation were also obtained by the above mentioned authors and others, but they did not find the exact functional form of the travelling wave solution, or any other exact solutions. Since the late 1980s, many mathematicians and physicists have obtained explicit exact solutions to the Burgers-KdV equation by various methods. Among them are Xiong, who obtained an exact solution to (1) when $\alpha = 1, \ = c$ and $s = \beta$ by the analytic method [20], Liu et al who obtained the same solution by the method of undetermined coefficients [21], Jeffrey et al, who obtained an exact solution to (1) by a direct method and a series method [22, 23], Halford and Vlieg-Hulstman, who obtained the same result in [24] by partial use of a Painleve analysis, Wang, who applied the homogeneous balance method to obtain an exact solution [25], which was verified by Parkes by the tanh-function method [26]. However, apart from several minor errors in [25] and [26], the solutions obtained in the previous literature are actually equivalent to one another. That is, the travelling solitary wave solution to (1) can be expressed as a composition of a bell solitary wave and a kink solitary wave. The purpose of this paper is to propose a new approach to the Burgers-KdV equation by using the theory of commutative algebra, which is currently called the first-integral method. The results obtained by this technique coincide with those presented in the previous literature.

The last few decades have seen an enormous growth in the applicability of nonlinear models and in the development of related nonlinear concepts. This has been driven by modern computer power as well as by the discovery of new mathematical techniques, which include two contrasting themes: (i) the theory of dynamical systems, most popularly associated with the study of chaos, and (ii) the theory of integrable systems associated, among other things, with the study of solitons. However, not all systems arising from physical phenomena are integrable, for example, the two-dimensional BurgersKortewegde Vries (2D-BKdV) equation. Therefore, a direct method together with qualitative analysis for treating such nonlinear systems appears to be more powerful and important. Applications of nonlinear models range from atmospheric science to condensed matter physics and to biology, from the smallest scales of theoretical particle physics up to the largest scales of cosmic structure.

Consider the 2D-BKdV equation

$$(U_t + \alpha UU_x + U_{xx} + sU_{xxx})x + U_{yy} = 0 \tag{4}$$

where α, β, s and γ are real constants and $\alpha\beta s\gamma \neq 0$. Equation (1) is a two-dimensional generalization of the Burgers-Korteweg-de Vries equation

$$U_t + \alpha UU_x + \beta U_{xx} + sU_{xxx} = 0 \tag{5}$$

which arises from many different physical contexts as a nonlinear model equation incorporating the effects of dispersion, dissipation and nonlinearity. Johnson derived (2) as the governing equation for waves propagating in a liquid-lled elastic tube [7] and Wijngaarden and Gao used it as a nonlinear model in the ow of liquids containing gas bubbles [8] and turbulence [9]. Grad and Hu used a steady state version of (2) to describe a weak shock prole in plasmas [11]. During the last few decades, many theoretical issues concerning the exact solutions of 2D-BKdV equation have received considerable attention. Barrera and Brugarino applied Lie group analysis to study the similarity solutions of (1) and examined some features of these invariant solutions, but the explicit travelling wave solution to (1) was not shown [35]. Li and Wang used the Hopf-Cole transformation and a computer algebra system to study (1) and found an exact travelling wave solution to (1) [36]. In the mean time, Ma proposed a bounded travelling wave solution to (1) by applying a special solution of square HopfCole type to an ordinary differential equation [37]. These two methods were compared to each other, and the solutions are proved to be equivalent by Parkes [38]. Fan obtained the same result by using an extended tanh-function method for constructing multiple travelling wave solutions of nonlinear partial differential equations in a unied way [39]. Recently, Fan et al [40] claimed that a new complex line soliton for the 2D-BKdV equation was obtained by making use of the same technique as described in [39], and Elwakil et al [41] claimed that a new travelling solitary wave solution was obtained by using a modied extended tanh-function method.

In our recent papers [31–34], we studied equation (1) by utilizing the rst integral method and the Painleve analysis, respectively, and obtained a more general travelling wave solution in terms of elliptic functions

CHAPTER II

FIRST-INTEGRAL METHOD TO STUDY THE BKdV EQUATION

2.1 Preliminaries

Assume that equation (1) has travelling wave solutions in the form $u(x,t) = u(\xi)$, $\xi = x - vt$, $(v \in \mathbb{R})$. Substituting it into equation (1) and integrating once we have

$$u''(\xi) - ru'(\xi) - au^2(\xi) - bu(\xi) - d = 0 \tag{6}$$

where $r = -\frac{\beta}{s}$, $a = -\frac{\alpha}{2s}$, $b = \frac{v}{s}$ and d is an arbitrary integration constant. Equation (6) is a nonlinear ordinary differential equation. It is commonly believed that it is very difficult for us to find exact solutions to equation (6) by usual ways [17]. Let $x = u$, $y = u_\xi$, then equation (6) is equivalent to a planar dynamical system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = ry + ax^2 + bx + d \end{cases} \tag{7}$$

By the qualitative theory of ordinary differential equations [27], if we can find two first-integrals to (7) under the same conditions, then the general solutions to (7) can be expressed explicitly. However, in general, it is really difficult for us to realize this, even for one first-integral, because for a given plane autonomous system, there is no systematic theory that can tell us how to find its first-integrals, nor is there a logical way to tell us what these first-integrals are.

In this section, we are applying the first-integral method to study (1). That is, we will apply the Hilbert-Nullstellensatz theorem to obtain one first-integral to (7) which reduces equation (6) to a first-order integrable ordinary differential equation. An exact solution to (1) is then obtained by solving this equation. At the end of this section, the solutions obtained in the previous literature are compared with ours. For convenience, first let us recall the Hilbert-Nullstellensatz theorem [28].

2.2 Hilbert-Nullstellensatz Theorem

Let k be a field and L an algebraic closure of k.

(i) Every ideal γ of $k[X_1, ..., X_n]$ not containing 1 admits at least one zero in L^n.

(ii) Let $x = (x1, ..., xn)$, $y = (y_1, ..., y_n)$ be two elements of Ln; for the set of polynomials of $k[X1, ..., Xn]$ zero at x to be identical with the set of polynomials of $k[X_1, ..., X_n]$ zero at y, it is necessary and sufficient that there exists a k-automorphisms of L such that $yi = s(x_i)$ for $1 \leq i \leq n$.

(iii) For an ideal α of $k[X_1, ..., X_n]$ to be maximal, it is necessary and sufficient that there exists an x in L^n such that α is the set of polynomials of $k[X_1, ..., X_n]$ zero at x.

(iv) For a polynomial Q of $k[X_1, ..., X_n]$ to be zero on the set of zeros in Ln of an ideal α of $k[X_1, ..., X_n]$, it is necessary and sufficient that there exists an integer $m > 0$ such that $Q^m \in \gamma$.

Following immediately from the Hilbert-Nullstellensatz theorem, we obtain the division theorem for two variables in the complex domain $\mathbb{C}$:

2.3 Division Theorem

Suppose that $P(\omega, z)$ and $Q(\omega, z)$ are polynomials in $\mathbb{C}[\omega, z]$, and $P(\omega, z)$ is irreducible in $\mathbb{C}[\omega, z]$.

If $Q(\omega, z)$ vanishes at all zero points of $P(\omega, z)$, then there exists a polynomial $G(\omega, z)$ in $\mathbb{C}[\omega, z]$ such that

$$Q(\omega, z) = P(\omega, z) \cdot G(\omega, z).$$

2.4 First-Integral to BKdV Equation

Now, we apply the division theorem to seek the first-integral to (7). Suppose that $x = x(\xi)$ and $y = y(\xi)$ are the nontrivial solutions to (7), and $p(x, y) = \Sigma_{i=0}^{m} a_i(x) y^i$ is an irreducible polynomial in $\mathbb{C}[x, y]$ such that

$$p[x(\xi), y(\xi)] = \sum_{i=0}^{m} a_i(x) y^i \tag{8}$$

where $a_i(x)(i = 0, 1, ..., m)$ are polynomials of x and all relatively prime in $\mathbb{C}[x, y]$, and $a_m(x) \neq 0$. Equation (8) is also called the first-integral to (7). We start our study by assuming $m = 2$ in (8). Note that $\frac{dp}{d\xi}$is a polynomial in x and y, and $p[x(\xi), y(\xi)] = 0$ implies $\frac{dp}{d\xi}\Big|_{(5)} = 0$. By the division theorem, there exists a polynomial $H(x, y) = \alpha(x) + \beta(x) y$ in $C[x, y]$ such that

$$\begin{aligned}
\left.\frac{dp}{d\xi}\right|_{(5)} &= \left.\left(\frac{\partial p}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial p}{\partial y}\frac{\partial y}{\partial \xi}\right)\right|_{(5)} \\
&= \sum_{i=0}^{2}[a_i^{'}(x) y^i.y] + \sum_{i=0}^{2}[i a_i(x) y^{i-1}(ry + ax^2 + bx + d)] \\
&= [\alpha(x) + \beta(x) y]\left[\sum_{i=0}^{2} a_i(x) y^i\right]
\end{aligned} \tag{9}$$

On equating the coefficients of $y_i (i = 3, 2, 1, 0)$ on both sides of (9), we have

$$a'(x) = A(x) \cdot a(x) \tag{10}$$

and

$$[0, ax^2 + bx + d, -\alpha(x)] \cdot a(x) = 0 \tag{11}$$

where $a(x) = (a_2(x), a_1(x), a_0(x))^t$, and

$$A(x) = \begin{pmatrix} \beta(x) & 0 & 0 \\ \alpha(x) - 2r & \beta(x) & 0 \\ -2(ax^2 + bx + d) & a(x) - r & \beta(x) \end{pmatrix}$$

Since $a_i(x)(i = 0, 1, 2)$ are polynomials, from (10), we deduce that $a_2(x)$ is a constant and $\beta(x) = 0$. For simplification, taking $a_2(x) = 1$ and solving (10), we have

$$a(x) = \begin{pmatrix} 1 \\ \int[\alpha(x) - 2r]dx \\ \int[a_1(x)\alpha(x) - ra_1(x) - 2(ax^2 + bx + d)]dx \end{pmatrix} \tag{12}$$

By (11) and (12), we conclude that deg $\alpha(x) = 0$, i.e., deg $a_1(x) = 1$. Otherwise, if deg $\alpha(x) = k > 0$, then we deduce deg $a_1(x) = k + 1$ and deg $a_0(x) = 2k + 2$ from (10). This yields a contradiction with (9), since the degree of the polynomial $a_1(x) \cdot (ax^2 + bx + d)$ is $k + 3$, but the degree of the polynomial $a_0(x) \cdot \alpha(x)$ is $3k + 2$.

Assume that $a_1(x) = A_1 x + A_0$, $A_1, A_0 \in \mathbb{C}$ with $A_1 \neq 0$. By (10), we deduce that $A_1 = \alpha(x) - 2r$ and $a_0(x) = -\frac{2a}{3}x^3 - bx^2 + \frac{A_1(A_1+r)}{2}x^2 - 2dx + A_0(A_1 + r)x + D$, here D is an arbitrary integration constant. Substituting $a_1(x)$ and $a_0(x)$ into (9) and setting all coefficients of $x_i(i = 3, 2, 1, 0)$ to zero we set

$$\begin{cases} A_1 a & = (-\frac{2a}{3})(A_1 + 2r) \\ A_0 + A_1 b & = [\frac{A_1(A_1+r)}{2} - b] \cdot (A_1 + 2r) \\ A_1 d + A_0 b & = [(A_1 + r)A_0 - 2d] \cdot (A_1 + 2r) \\ A_0 d & = D \cdot (A_1 + 2r). \end{cases} \tag{13}$$

Taking the integrating constant $d = 0$, we have

$$A_1 = -\frac{4r}{5} \quad A_0 = -\frac{12r^3}{125a} - \frac{2br}{5a} \quad D = 0 \tag{14}$$

By the third equation of (13), we obtain

$$b = \frac{6r^2}{25} \quad or \quad b = -\frac{6r^2}{25}. \tag{15}$$

2.5 Exact Travelling Wave Solution to BKdV Equation

In the case $b = \frac{6r^2}{25}$, A_0 in (14) can be simplified as $A_0 = -\frac{4br}{5a}$. Substituting $a_0(x)$ and $a_1(x)$ into (8) we set

$$y^2 - (\frac{4r}{5}x + \frac{4br}{5a})y - \frac{2a}{3}x^3 - bx^2 - \frac{2r^2}{25}x^2 - \frac{4br^2}{25a}x = 0 \tag{16}$$

From (16), y can be expressed in terms of x, i.e.

$$\begin{aligned} y &= \frac{2r}{5}x + \frac{2br}{5a} \pm \sqrt{\frac{2a}{3}x^3 + 2bx^2 + \frac{2b^2}{a}x + \frac{2b^3}{3a^2}} \\ &= \frac{2r}{5}x + \frac{2br}{5a} \pm \sqrt{\frac{2}{3a^2}(ax+b)^3} \end{aligned} \tag{17}$$

Combining (7) and (17), we have

$$\frac{dx}{\frac{2r}{5}x + \frac{2br}{5a} \pm (ax+b)\sqrt{\frac{2}{3a^2}(ax+b)}} = d\xi \tag{18}$$

By a transformation $z = \sqrt{\frac{2}{3a^2}(ax+b)}$, an exact solution to (1) can be obtained as follows by solving (18) directly

$$
\begin{aligned}
u(x,t) &= \frac{3a}{2}\left[\frac{\pm\frac{2r}{5a}e^{\frac{r}{5}\xi}{}^{\frac{r}{5}}}{e}+c\right]^2-\frac{b}{a} \\
&= -\frac{12\beta^2}{25\alpha s}\left[\frac{e^{-\frac{\beta}{3s}\xi}}{e^{-\frac{\beta}{3s}\xi}+c}\right]^2+\frac{2v}{\alpha},
\end{aligned}
\tag{19}
$$

where $\xi = x - vt$ and c is an arbitrary integration constant.

Since $b = \frac{v}{s}$ and $r = -\frac{\beta}{s}$, $b = \frac{6r^2}{25}$ in (15) implies $v = \frac{6\beta^2}{25s}$. By using the equality $4A[\frac{e^{2t}}{1+e^{2t}}] = -A\,sech^2 t + 2A\,tanh\,t + 2A$ and setting $c = 1$ in (17), the explicit travelling solitary wave solution to equation (1) can be written as

$$
u(x,t) = \frac{3\beta^2}{25\alpha s}sech^2\left[\frac{1}{2}\left(-\frac{\beta}{5s}x+\frac{6\beta^3}{125s^2}t\right)\right]-\frac{6\beta^2}{25\alpha s}tanh\left[\frac{1}{2}\left(-\frac{\beta}{5s}x+\frac{6\beta^3}{125s^2}t\right)\right]+\frac{6\beta^2}{25\alpha s} \tag{20}
$$

Similarly, in case $b = -\frac{6r^2}{25}$, an exact solution to (1) is as follows:

$$
u(x,t) = -\frac{12\beta^2}{25\alpha s}\left[\frac{e^{-\frac{\beta}{5s}\xi}}{e^{-\frac{\beta}{5s}\xi}+c}\right]^2 \tag{21}
$$

where $\xi = x - vt$ and c is an arbitrary integration constant.

Note that $b = -\frac{6r^2}{25}$ in (15) implies $v = \frac{6\beta^2}{25s}$. By setting $c = 1$ in (21), explicit travelling solitary wave solutions to equation (1) can be rewritten as

$$
u(x,t) = \frac{3\beta^2}{25\alpha s}sech^2\left[\frac{1}{2}\left(-\frac{\beta}{5s}x-\frac{6\beta^3}{125s^2}t\right)\right]-\frac{6\beta^2}{25\alpha s}tanh\left[\frac{1}{2}\left(-\frac{\beta}{5s}x-\frac{6\beta^3}{125s^2}t\right)\right]-\frac{6\beta^2}{25\alpha s} \tag{22}
$$

Equations (19) and (21) also confirm the qualitative analysis of equation (1) by Guan and Gao in [17].

CHAPTER III

EXACT TRAVELLING WAVE SOLUTION TO 2D-BKdV EQUATION

3.1 From PDE to ODE

Consider the 2D-BKdV equation

$$(U_t + \alpha U U_x + \beta U_{xx} + s U_{xxx})_x + \gamma U_{yy} = 0, \tag{23}$$

where $\alpha, \beta, s,$ and γ are constants and $\alpha\beta s\gamma \neq 0$.

Assume that equation (23) has an exact solution in the form

$$U(x, y, t) \equiv U(\xi), \quad \xi = hx + ly - wt, \tag{24}$$

where h, l, w are real constants to be determined. Substituting equation (24) into equation (23) yields

$$-whU_{\xi\xi} + \alpha h^2 (UU_\xi)_\xi + \beta h^3 U_{\xi\xi\xi} + sh^4 U_{\xi\xi\xi\xi} + \gamma l^2 U_{\xi\xi} = 0.$$

Integrating the above equation twice with respect to ξ, then we have

$$sh^4 U_{\xi\xi} + \beta h^3 U_\xi + \frac{\alpha}{2} h^2 U^2 + \gamma l^2 U - whU = D,$$

where we assume that the first integration constant is zero and the second one is an arbitrary constant D. Rewrite above second-order ordinary differential equation as

$$U^{''}(\xi) - \delta U^{'}(\xi) - aU^2(\xi) - bU(\xi) - d = 0, \tag{25}$$

where $\delta = -\frac{\beta}{sh}$, $a = -\frac{\alpha}{2sh^2}$, $b = -\frac{\gamma l^2 - wh}{sh^4}$ and $d = -\frac{D}{sh^4}$.

3.2 Analyse Planar Dynamical System

When we take $D = 0$, equation (25) can be rewritten into the following planar dynamical system

$$\begin{cases} \frac{dU}{d\xi} = Y, \\ \frac{dY}{d\xi} = \delta Y + aU^2 + bU \end{cases} \tag{26}$$

For any $a \neq 0$, the system (26) has two equilibrium points, defined by $O(0,0)$, $A(-\frac{b}{a}, 0)$.
System (26) is integrable, when we set $b = -\frac{6\delta^2}{25}$, and has a first integral with respect to ξ as follows [29].

$$H_1(U, Y, \xi) = \left(\frac{1}{2}\left(Y - \frac{2}{5}\delta U\right)^2 - \frac{1}{3}aU^3\right)e^{\left(-\frac{6}{5}\delta\xi\right)} = h.$$

Where h is a real arbitrary constant.
Consider the linearization of system (26) when $b = -\frac{6\delta^2}{25}$

$$\begin{bmatrix} P' \\ Q' \end{bmatrix} = J \begin{bmatrix} P \\ Q \end{bmatrix}$$

where J is Jacobian Matrix.Since $f_1(\xi) = Y$ and $f_2(\xi) = \delta Y + aU^2 + bU$,

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial U} & \frac{\partial f_1}{\partial Y} \\ \frac{\partial f_2}{\partial U} & \frac{\partial f_2}{\partial Y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2aU + b & \delta \end{bmatrix}.$$

At the equilibrium point $O(0,0)$, the linearized system has coefficient matrix

$$J_1 = \begin{bmatrix} 0 & 1 \\ -\frac{6\delta^2}{25} & \delta \end{bmatrix}.$$

The characteristic function is

$$\lambda^2 - \delta\lambda + \frac{6\delta^2}{25} = 0.$$

The eigenvalues are

$$\lambda_1 = \frac{\delta + \frac{1}{5}|\delta|}{2}, \quad \lambda_2 = \frac{\delta - \frac{1}{5}|\delta|}{2}.$$

When $\delta < 0$, $Re(\lambda_1) < 0$ and $Re(\lambda_2) < 0$, the equilibrium point $O(0,0)$ is a stable node point.(Fig. 1.1)

When $\delta > 0$, $Re(\lambda_1) > 0$ and $Re(\lambda_2) > 0$, the equilibrium point $O(0,0)$ is an unstable node point.(Fig. 1.2)

Similarly, at the equilibrium point $A(-\frac{b}{a}, 0)$, the linearized system has coefficient matrix

$$J_2 = \begin{bmatrix} 0 & 1 \\ \frac{6\delta^2}{25} & \delta \end{bmatrix}.$$

The characteristic function is

$$\lambda^2 - \delta\lambda - \frac{6\delta^2}{25} = 0.$$

The eigenvalues are

$$\lambda_1 = \frac{\delta + \frac{7}{5}|\delta|}{2}, \quad \lambda_2 = \frac{\delta - \frac{7}{5}|\delta|}{2}.$$

When $\delta < 0$, $Re(\lambda_1) > 0$ and $Re(\lambda_2) < 0$, the equilibrium point $A(-\frac{b}{a}, 0)$ is a saddle point. (Fig. 1.3)

When $\delta > 0$, $Re(\lambda_1) > 0$ and $Re(\lambda_2) < 0$, the equilibrium point $A(-\frac{b}{a}, 0)$ is a saddle point. (Fig. 1.4)

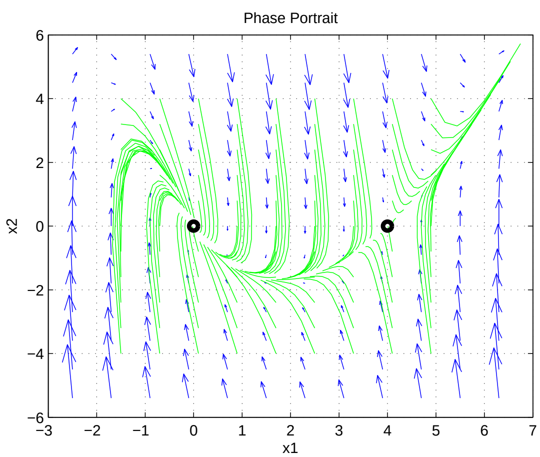

Fig. 1.1 $\delta_1 < 0$, $a > 0$

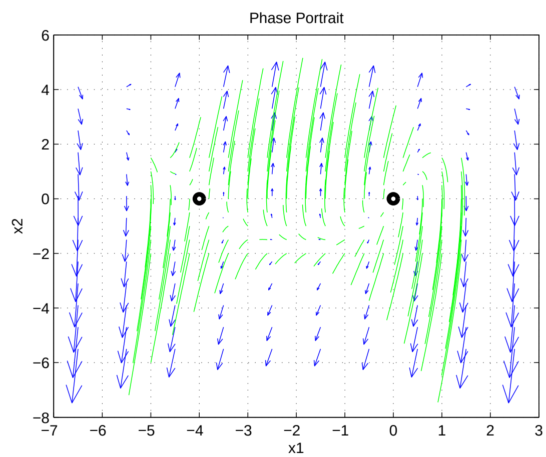

Fig. 1.2 $\delta_1 > 0$, $a < 0$

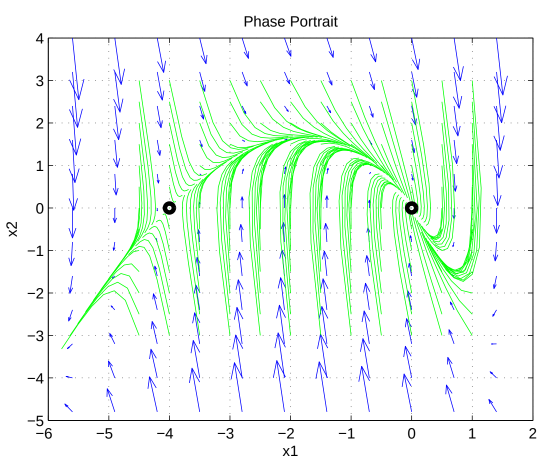

Fig. 1.3 $\delta_1 < 0$, $a < 0$

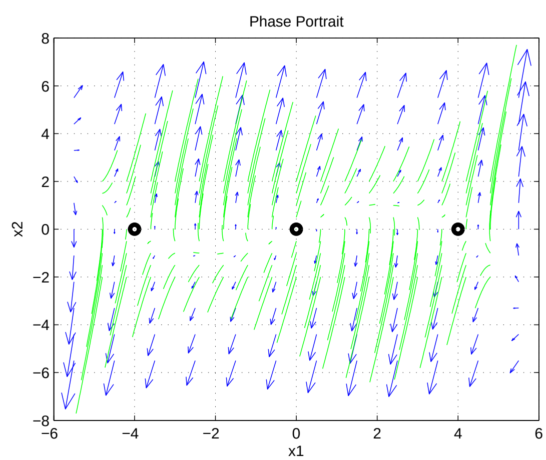

Fig. 1.4 $\delta_1 > 0$, $a > 0$

3.3 Exact Travelling Wave Solution In Two Cases

3.3.1 Case 1.1 $h = 0$

When $h = 0$, we have

$$H_1(U, Y, \xi) = \left(\frac{1}{2}\Big(Y - \frac{2}{5}\delta U\Big)^2 - \frac{1}{3}aU^3\right)e^{\left(-\frac{6}{5}\delta\xi\right)} = 0.$$

Therefore, $Y = \frac{2}{5}\delta U \pm \sqrt{\frac{2}{3}aU^3}$.

Since $\frac{dU}{d\xi} = Y$,

$$\frac{dU}{d\xi} = \frac{2}{5}\delta U \pm \sqrt{\frac{2}{3}aU^3}. \tag{27}$$

Integrating equation (27), we have

$$U_1(\xi) = \frac{6\delta^2}{25a(1 - C_{01}e^{-\frac{1}{5}\delta\xi})}, \qquad U_2(\xi) = \frac{6\delta^2}{25a(1 + C_{02}e^{-\frac{1}{5}\delta\xi})},$$

where C_{01}, C_{02} are constant.

When $\delta > 0, \xi \to \infty, U_1(\xi) \to 0, U_2(\xi) \to 0.$

When $\delta > 0, \xi \to -\infty, U_1(\xi) \to \frac{6\delta^2}{25a}, U_2(\xi) \to \frac{6\delta^2}{25a}.$

3.3.2 Case 1.2 $h \neq 0$

Using transformation as follows,

$$w = \frac{1}{\sqrt{2}}Ue^{-\frac{2}{5}\delta\xi}, \quad z = \frac{5}{\delta}e^{\frac{1}{5}\delta\xi},$$

we can obtain

$$
\begin{aligned}
&\frac{dw}{d\xi} = \frac{1}{\sqrt{2}}U' e^{-\frac{2}{5}\delta\xi} - \frac{\sqrt{2}}{5} U e^{-\frac{2}{5}\delta\xi},\\
&\frac{dz}{d\xi} = e^{\frac{1}{5}\delta\xi},\\
&\frac{dU}{d\xi} = \sqrt{2} e^{\frac{3}{5}\delta} \frac{dw}{dz} + \frac{2}{5}\delta U.
\end{aligned}
$$

Substituting $\frac{dw}{d\xi}, \frac{dz}{d\xi}, \frac{dU}{d\xi} and\ U = \sqrt{2} w e^{\frac{2}{5}\delta\xi}$ into the first integral equation $H_1(U, Y, \xi) = h$.

This first integral equation can be rewritten in the form

$$
\Big(\frac{dw}{dz}\Big)^2 - \frac{2\sqrt{2}}{3} a w^3 = h. \tag{28}
$$

Simplify equation (28)

$$
\frac{dw}{dz} = \pm\sqrt{h + \frac{2\sqrt{2}}{3} a w^3}.
$$

Therefore, for any $h > 0$, we have an integral of $\frac{dw}{dz}$ in form

$$
\begin{aligned}
z - z_0 &= \pm \int_{-\infty}^{w} \frac{dw}{\sqrt{h}\sqrt{1 + \frac{2\sqrt{2}}{3h} a w^3}}\\
&= \pm \int_{-\infty}^{w} \frac{dw}{\sqrt{h}\sqrt{1 - \Big(-\frac{2\sqrt{2}}{3h} a w^3\Big)}}\\
&= \pm \Big(-\frac{2\sqrt{2}}{3}\Big)^{-\frac{1}{3}} h^{-\frac{1}{6}} \int_{-\infty}^{\tilde{w}} \frac{d\tilde{w}}{\sqrt{1 - \tilde{w}^3}},
\end{aligned} \tag{29}
$$

where $w = \Big(-\frac{3h}{2\sqrt{2}a}\Big)^{\frac{1}{3}} \tilde{w}$ and z_0 is arbitrary integral constant .

Since the elliptic integral has the form

$$\int_{-\infty}^{x} \frac{dx}{\sqrt{1-x^3}} = \frac{1}{\sqrt[4]{3}} cn^{-1}\bigg(\frac{1-\sqrt{3}-x}{1+\sqrt{3}-x}, k_2\bigg),$$

where $k_2 = \frac{\sqrt{2+\sqrt{3}}}{2}$.

Hence equation (29) can be rewritten into the following form

$$z - z_0 = \pm\Big(-\frac{2\sqrt{2}}{3}\Big)^{-\frac{1}{3}} h^{-\frac{1}{6}} \frac{1}{\sqrt[4]{3}} cn^{-1}\bigg(\frac{1-\sqrt{3}-\tilde{w}}{1+\sqrt{3}-\tilde{w}}, k_2\bigg).$$

Solve for $\tilde{w}$, we have

$$\frac{1-\sqrt{3}-\tilde{w}}{1+\sqrt{3}-\tilde{w}} = cn\bigg(\Big(-\frac{2\sqrt{2}}{3}\Big)^{\frac{1}{3}} h^{\frac{1}{6}} \sqrt[4]{3}(z-z_0), k_2\bigg),$$

and then

$$\tilde{w}(z) = 1 + \sqrt{3} + \frac{2\sqrt{3}}{cn\bigg(\Big(-\frac{2\sqrt{2}}{3}\Big)^{\frac{1}{3}} h^{\frac{1}{6}} \sqrt[4]{3}(z-z_0), k_2\bigg) - 1}. \tag{30}$$

Therefore,

$$\begin{aligned} w(z) &= \Big(-\frac{3h}{2\sqrt{2}a}\Big)^{\frac{1}{3}} \tilde{w} \\ &= \Big(-\frac{3h}{2\sqrt{2}a}\Big)^{\frac{1}{3}} \Big(1 + \sqrt{3} + \frac{2\sqrt{3}}{cn(m_1(z-z_0), k_2) - 1}\Big), \end{aligned} \tag{31}$$

where $m_1 = \Big(-\frac{2\sqrt{2}}{3}\Big)^{\frac{1}{3}} h^{\frac{1}{6}} \sqrt[4]{3},\ k_2 = \frac{\sqrt{2+\sqrt{3}}}{2}$, $z = \frac{5}{\delta} e^{\frac{1}{5}\delta\xi}$ and z_0 is an arbitrary constant.

Since $U(\xi) = \sqrt{2} w e^{\frac{2}{5}\delta\xi}$, the equation (23) has exactly solution

$$U(\xi) = \sqrt{2} w e^{\frac{2}{5}\delta\xi} = \Big(-\frac{3h}{a}\Big)^{\frac{1}{3}} \Big(1 + \sqrt{3} + \frac{2\sqrt{3}}{cn(m_1(z-z_0), k_2) - 1}\Big) e^{\frac{2}{5}\delta\xi}. \tag{32}$$

Similarly, for any $h < 0$, and consider another elliptic integral

$$\int_x^{\infty} \frac{dx}{\sqrt{x^3-1}} = \frac{1}{\sqrt[4]{3}} cn^{-1}\left(\frac{x-1-\sqrt{3}}{x-1+\sqrt{3}}, k_1\right),$$

where $k_1 = \frac{\sqrt{2-\sqrt{3}}}{2}$.

We obtain

$$\begin{aligned} z - z_0 &= \pm \int_w^{\infty} \frac{dw}{\sqrt{|h|}\sqrt{\frac{2\sqrt{2}}{3|h|} a w^3 - 1}} \\ &= \pm \Big(\frac{2\sqrt{2}}{3}\Big)^{-\frac{1}{3}} |h|^{-\frac{1}{6}} \int_{\tilde{w}}^{\infty} \frac{d\tilde{w}}{\sqrt{\tilde{w}^3 - 1}} \\ &= \pm \Big(\frac{2\sqrt{2}}{3}\Big)^{-\frac{1}{3}} |h|^{-\frac{1}{6}} \frac{1}{\sqrt[4]{3}} cn^{-1}\left(\frac{\tilde{w}-1-\sqrt{3}}{\tilde{w}-1+\sqrt{3}}, k_1\right). \end{aligned} \tag{33}$$

Therefore,

$$\begin{aligned} w(z) &= \Big(\frac{3|h|}{2\sqrt{2}a}\Big)^{\frac{1}{3}} \tilde{w(z)} \\ &= \Big(\frac{3|h|}{2\sqrt{2}a}\Big)^{\frac{1}{3}} \Big(1 - \sqrt{3} + \frac{2\sqrt{3}}{1 - cn(m_2(z-z_0), k_1)}\Big), \end{aligned} \tag{34}$$

where $m_2 = \pm\Big(\frac{2\sqrt{2}}{3}\Big)^{\frac{1}{3}} |h|^{\frac{1}{6}} \sqrt[4]{3}$.

The equation (23) has the following solution

$$\begin{aligned} U(\xi) &= \sqrt{2} w e^{\frac{2}{5}\delta\xi} \\ &= \Big(\frac{3|h|}{a}\Big)^{\frac{1}{3}} \Big(1 - \sqrt{3} + \frac{2\sqrt{3}}{1 - cn(m_2(z-z_0), k_1)}\Big) e^{\frac{2}{5}\delta\xi}, \end{aligned} \tag{35}$$

where $m_2 = \pm\Big(\frac{2\sqrt{2}}{3}\Big)^{\frac{1}{3}} |h|^{\frac{1}{6}} \sqrt[4]{3},\ k_1 = \frac{\sqrt{2-\sqrt{3}}}{2}$, $z = \frac{5}{\delta} e^{\frac{1}{5}\delta\xi}$ and z_0 is an arbitrary constant.

CHAPTER IV

EXACT TRAVELLING WAVE SOLUTION TO 2D-CBKdV EQUATION

4.1 From PDE to ODE

Consider the 2D-CBKdV equation

$$(U_t + \alpha_1 UU_x + \beta_1 UU_x + \mu_1 U_{xx} + s_1 U_{xxx})_x + \gamma_1 U_{yy} = 0, \tag{36}$$

where $\alpha_1, \beta_1, \mu_1, s_1$, and γ_1 are constants and $\alpha_1\beta_1\mu_1 s_1\gamma_1 \neq 0$.
Assume that equation (36) has an exact solution in the form

$$U(x, y, t) \equiv U(\xi), \quad \xi = hx + ly - wt, \tag{37}$$

where h, l, w are real constants to be determined. Substituting equation (37) into equation (36) yields

$$-whU_{\xi\xi} + \alpha_1 h^2 (UU_\xi)_\xi + 2\beta_1 h^2 UU_\xi^2 + \beta h_1^2 UU_{\xi\xi} + \mu_1 h^3 U_{\xi\xi\xi} + s_1 h^4 U_{\xi\xi\xi\xi} + \gamma_1 l^2 U_{\xi\xi} = 0.$$

Integrating the above equation twice with respect to ξ, then we have

$$s_1 h^4 U_{\xi\xi} + \frac{\beta_1 h^2}{3} U^3 + \mu_1 h^3 U_\xi + \frac{\alpha_1}{2} h^2 U^2 + \gamma_1 l^2 U - whU = D,$$

where we assume that the first integration constant is zero and the second one is arbitrary constant D. Rewrite above second-order ordinary differential equation as

$$U^{''}(\xi) - \delta_1 U^{'}(\xi) - a_1 U^3(\xi) - b_1 U^2(\xi) - c_1 U(\xi) - d_1 = 0, \tag{38}$$

where $\delta_1 = -\frac{\beta_1}{\mu_1 h}$, $a_1 = -\frac{\beta_1}{3s_1 h^2}$, $b_1 = \frac{\alpha_1}{2s_1 h^2}$, $c_1 = -\frac{\gamma_1 l^2 - wh}{s_1 h^4}$ and $d = -\frac{D}{s_1 h^4}$.

4.2 Analyse Planar Dynamical System

When we take $D = 0$, equation (38) can be rewritten into the following planar dynamical system

$$\begin{cases} \frac{dU}{d\xi} = Y, \\ \frac{dY}{d\xi} = \delta_1 Y + a_1 U^3 + b_1 U^2 + c_1 U + d_1 \end{cases} \tag{39}$$

Let $U = \tilde{U} - \frac{b_1}{3a_1}$ and $d_1 = \frac{c_1 b_1}{3a_1} - \frac{2b_1^3}{27a^2}$, the system will be rewritten into form

$$\begin{cases} \frac{d\tilde{U}}{d\xi} = Y, \\ \frac{dY}{d\xi} = \delta_1 Y + a_1 \tilde{U}^3 + n\tilde{U} \end{cases} \tag{40}$$

where $n = c_1 - \frac{b_1^2}{3a_1}$.

When $b_1 = -\frac{6\delta_1^2}{25}$, $na_1 < 0$, for any $a_1 \neq 0$, the system (40) has three equilibrium points, defined by $O(0,0)$, $A(-\sqrt{-\frac{n}{a_1}}, 0)$, $B(\sqrt{-\frac{n}{a_1}}, 0)$.

System (40) is integrable, when we set $b_1 = -\frac{6\delta_1^2}{25}$, and has a first integral of system (40) with respect to ξ as follows [29].

$$H_2(\tilde{U}, Y, \xi) = \left(\frac{1}{2}\Big(Y - \frac{1}{3}\delta_1 \tilde{U}\Big)^2 - \frac{1}{4}a_1 \tilde{U}^4\right) e^{\left(-\frac{4}{3}\delta_1 \xi\right)} = h,$$

where h is a real arbitrary constant.

Consider the linearization of system (40) when $b = -\frac{6\delta^2}{25}$.

$$\begin{bmatrix} P' \\ Q' \end{bmatrix} = J \begin{bmatrix} P \\ Q \end{bmatrix}$$

Where J is Jacobian Matrix. Since $g_1(\xi) = Y$ and $g_2(\xi) = \delta_1 Y + a_1 \tilde{U}^3 + n\tilde{U}$.

$$J = \begin{bmatrix} \frac{\partial g_1}{\partial U} & \frac{\partial g_1}{\partial Y} \\ \frac{\partial g_2}{\partial U} & \frac{\partial g_2}{\partial Y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 3a_1\tilde{U}^2 + n & \delta_1 \end{bmatrix}.$$

At the equilibrium point $O(0,0)$, the linearized system has coefficient matrix

$$J_1 = \begin{bmatrix} 0 & 1 \\ n & \delta_1 \end{bmatrix}.$$

The characteristic function is

$$\lambda^2 - \delta_1\lambda - n = 0$$

The eigenvalues are

$$\lambda_1 = \frac{\delta_1 + \sqrt{\delta_1^2 + 4n}}{2}, \ \lambda_2 = \frac{\delta_1 - \sqrt{\delta_1^2 + 4n}}{2}$$

When $n > 0$,

1. $\delta_1 < 0$, $Re(\lambda_1) > 0$ and $Re(\lambda_2) < 0$, the equilibrium point $O(0,0)$ is a saddle point. (Fig. 2.1)

2. $\delta_1 > 0$, $Re(\lambda_1) > 0$ and $Re(\lambda_2) < 0$, the equilibrium point $O(0,0)$ is a saddle point. (Fig. 2.2)

When $n < 0$,

3. $\delta_1 < 0$, $Re(\lambda_1) < 0$ and $Re(\lambda_2) < 0$, the equilibrium point $O(0,0)$ is a stable node point. (Fig. 2.3)

4. $\delta_1 > 0$, $Re(\lambda_1) > 0$ and $Re(\lambda_2) > 0$, the equilibrium point $O(0,0)$ is an unstable node point. (Fig. 2.4)

Similarly, at the equilibrium point $A(-\sqrt{-\frac{n}{a_1}}, 0)$, $B(\sqrt{-\frac{n}{a_1}}, 0)$, the linearized system has coefficient matrix

$$J_2 = \begin{bmatrix} 0 & 1 \\ 2n & \delta_1 \end{bmatrix}.$$

The characteristic function is

$$\lambda^2 - \delta_1 \lambda + 2n = 0$$

The eigenvalues are

$$\lambda_1 = \frac{\delta_1 + \sqrt{\delta_1^2 - 8n}}{2}, \quad \lambda_2 = \frac{\delta_1 - \sqrt{\delta_1^2 - 8n}}{2}.$$

When $0 < n < \frac{\delta_1^2}{8}$,

1. $\delta_1 < 0$, $Re(\lambda_1) < 0$ and $Re(\lambda_2) < 0$, equilibrium points $A(-\sqrt{-\frac{n}{a_1}}, 0)$ and $B(\sqrt{-\frac{n}{a_1}}, 0)$ are stable node points.(Fig. 2.1)

2. $\delta_1 > 0$, $Re(\lambda_1) > 0$ and $Re(\lambda_2) > 0$, equilibrium points $A(-\sqrt{-\frac{n}{a_1}}, 0)$ and $B(\sqrt{-\frac{n}{a_1}}, 0)$ are unstable node points.(Fig. 2.2)

When $n < 0$,

3. $\delta_1 < 0$, $Re(\lambda_1) > 0$ and $Re(\lambda_2) < 0$, equilibrium points $A(-\sqrt{-\frac{n}{a_1}}, 0)$ and $B(\sqrt{-\frac{n}{a_1}}, 0)$ are stable node points.(Fig. 2.3)

4. $\delta_1 > 0$, $Re(\lambda_1) > 0$ and $Re(\lambda_2) < 0$, equilibrium points $A(-\sqrt{-\frac{n}{a_1}}, 0)$ and $B(\sqrt{-\frac{n}{a_1}}, 0)$ are saddle points. (Fig. 2.4)

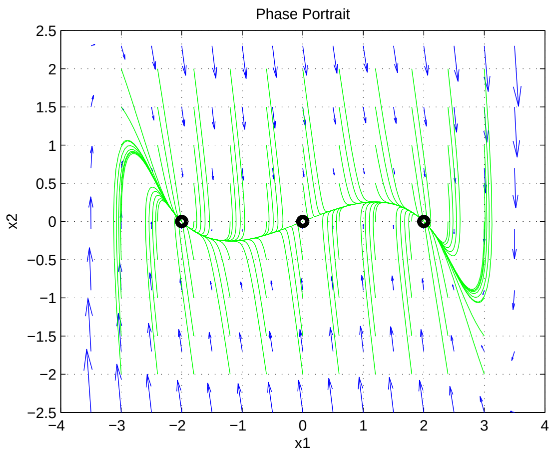

Fig. 2.1 $\delta_1 < 0$, $0 < n < \frac{\delta_1^2}{8}$

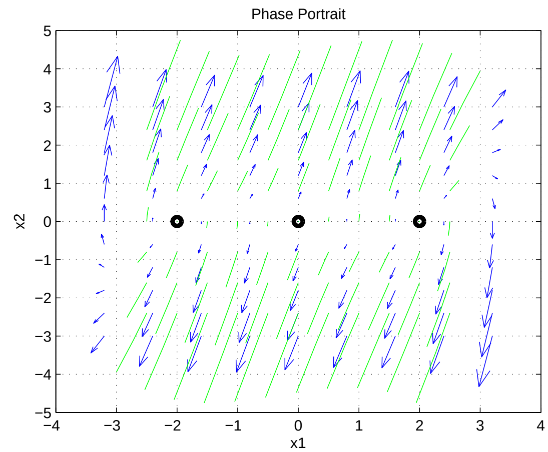

Fig. 2.2 $\delta_1 > 0$, $0 < n < \frac{\delta_1^2}{8}$

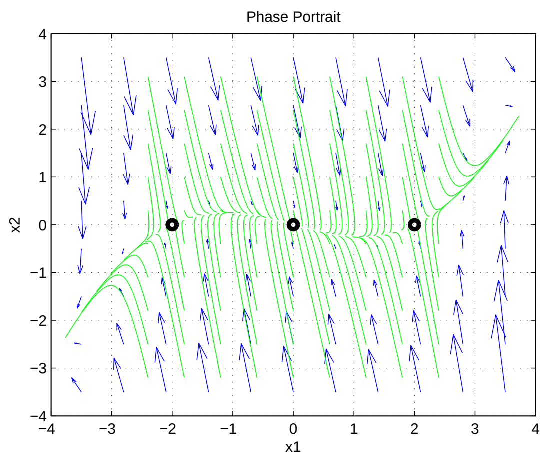

Fig. 2.3 $\delta_1 > 0$, $n < 0$

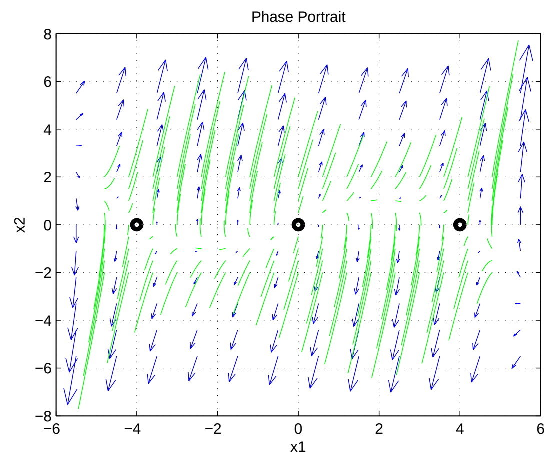

Fig. 2.4 $\delta_1 < 0$, $n < 0$

4.3 Exact Travelling Wave Solution In Two Cases

4.3.1 Case 2.1 $h = 0$

When $h = 0$, we have

$$H_2(\tilde{U}, Y, \xi) = \left(\frac{1}{2}\left(Y - \frac{1}{3}\delta_1\tilde{U}\right)^2 - \frac{1}{4}a_1\tilde{U}^4\right)e^{\left(-\frac{4}{3}\delta_1\xi\right)} = 0$$

Therefore, $Y = \frac{1}{3}\delta_1 \tilde{U} \pm \sqrt{\frac{1}{2}a}\tilde{U}^2$

Since $\frac{d\tilde{U}}{d\xi} = Y$,

$$\frac{d\tilde{U}}{d\xi} = \frac{1}{3}\delta_1 \tilde{U} \pm \sqrt{\frac{1}{2}a}\tilde{U}^2 \tag{41}$$

Integrating equation (41), we have

$$\tilde{U}_1(\xi) = \frac{\delta_1}{3(\sqrt{\frac{a_1}{2}} + C_{11}e^{-\frac{1}{3}\delta_1\xi})}, \quad \tilde{U}_2(\xi) = -\frac{\delta_1}{3(\sqrt{\frac{a_1}{2}} + C_{12}e^{-\frac{1}{3}\delta_1\xi})},$$

where C_{11}, C_{12} are constants.

When $\xi \to \infty, \tilde{U}_1(\xi) \to 0, \tilde{U}_2(\xi) \to 0.$

When $\xi \to -\infty, \tilde{U}_1(\xi) \to -\frac{1}{3}\sqrt{\frac{2}{a}}|\delta_1|, \tilde{U}_2(\xi) \to \frac{1}{3}\sqrt{\frac{2}{a}}|\delta_1|.$

4.3.2 Case 2.2 $h \neq 0$

Using transformation as follows,

$$w = \frac{1}{\sqrt{2}}\tilde{U}e^{-\frac{1}{3}\delta_1\xi}, \quad z = \frac{3}{\delta_1}e^{\frac{1}{3}\delta\xi}.$$

We can obtain

$$\begin{aligned} \frac{dw}{d\xi} &= \frac{1}{\sqrt{2}}\tilde{U}'e^{-\frac{1}{3}\delta_1\xi} - \frac{1}{3\sqrt{2}}\tilde{U}\delta_1 e^{-\frac{1}{3}\delta_1\xi}, \\ \frac{dz}{d\xi} &= e^{-\frac{1}{3}\delta_1\xi}, \\ \frac{d\tilde{U}}{d\xi} &= \sqrt{2}e^{\frac{2}{3}\delta_1}\frac{dw}{dz} + \frac{1}{3}\delta_1\tilde{U}. \end{aligned}$$

Substituting $\frac{dw}{d\xi}, \frac{dz}{d\xi}, \frac{d\tilde{U}}{d\xi} and \ \tilde{U} = \sqrt{2}we^{\frac{1}{3}\delta_1\xi}$ into the first integral equation $H_1(U, Y, \xi) = h$.

This first integral equation can be rewritten in the form

$$\left(\frac{dw}{dz}\right)^2 - a_1w^4 = h. \tag{42}$$

Simplify equation (42)

$$\frac{dw}{dz} = \pm\sqrt{h + a_1w^4}.$$

Therefore, for any $h > 0$, we have an integral of $\frac{dw}{dz}$ in form

$$\begin{aligned} z - z_0 &= \pm\int_{-\infty}^{w} \frac{dw}{\sqrt{h}\sqrt{1 + \frac{a_1}{h}w^4}} \\ &= \pm\left(a_1h\right)^{-\frac{1}{4}} \int_{\tilde{w}}^{\infty} \frac{d\tilde{w}}{\sqrt{1 + \tilde{w}^4}}, \end{aligned} \tag{43}$$

where $w = \left(\frac{h}{a_1}\right)^{\frac{1}{4}}\tilde{w}$ and z_0 is an arbitrary integral constant .

Since the elliptic integral has the form

$$\int_{x}^{\infty} \frac{dt}{\sqrt{1 + t^4}} = \frac{1}{2}cn^{-1}\left(\frac{x^2 - 1}{x^2 + 1}, k_1\right),$$

where $k_1 = \frac{\sqrt{2}}{2}$.

Hence equation (43) can be rewritten into the following form

$$\begin{aligned} z - z_0 &= \pm\left(a_1h\right)^{-\frac{1}{4}} \int_{\tilde{w}}^{\infty} \frac{d\tilde{w}}{\sqrt{1 + \tilde{w}^4}} \\ &= \pm\frac{1}{2}\left(a_1h\right)^{-\frac{1}{4}}cn^{-1}\left(\frac{\tilde{w}^2 - 1}{\tilde{w}^2 + 1}, k_1\right). \end{aligned} \tag{44}$$

Solve for $\tilde{w}$, we have

$$\frac{\tilde{w}^2 - 1}{\tilde{w}^2 + 1} = cn\Bigg(\pm 2\Big(a_1 h\Big)^{\frac{1}{4}}(z - z_0), k_1\Bigg),$$

and then

$$\tilde{w}(z) = \pm\sqrt{\frac{cn\Big(\pm 2\big(a_1 h\big)^{\frac{1}{4}}(z - z_0), k_1\Big) + 1}{cn\Big(\pm 2\big(a_1 h\big)^{\frac{1}{4}}(z - z_0), k_1\Big) - 1}}\ . \tag{45}$$

Therefore,

$$\begin{aligned} w(z) &= \Big(\frac{h}{a_1}\Big)^{\frac{1}{4}} \tilde{w}(z) \\ &= \Big(\frac{h}{a_1}\Big)^{\frac{1}{4}} \sqrt{\frac{cn\Big(\pm 2\big(a_1 h\big)^{\frac{1}{4}}(z - z_0), k_1\Big) + 1}{cn\Big(\pm 2\big(a_1 h\big)^{\frac{1}{4}}(z - z_0), k_1\Big) - 1}}, \end{aligned} \tag{46}$$

where $k_1 = \frac{\sqrt{2}}{2}$, $z = \frac{3}{\delta} e^{\frac{1}{3}\delta\xi}$ and z_0 is an arbitrary constant.

Since $U(\xi) = \sqrt{2} w e^{\frac{1}{3}\delta\xi}$, the equation (24) has exactly solution

$$\begin{aligned} U(\xi) &= \sqrt{2} w e^{\frac{1}{3}\delta\xi} \\ &= \sqrt{2}\Big(\frac{h}{a_1}\Big)^{\frac{1}{4}} \tilde{w}(z) \\ &= \Big(\frac{h}{a_1}\Big)^{\frac{1}{4}} e^{\frac{1}{3}\delta\xi} \sqrt{\frac{cn\Big(\pm 2\big(a_1 h\big)^{\frac{1}{4}}(z - z_0), k_1\Big) + 1}{cn\Big(\pm 2\big(a_1 h\big)^{\frac{1}{4}}(z - z_0), k_1\Big) - 1}}\ . \end{aligned} \tag{47}$$

Similarly, for any $h < 0$, and consider another elliptic integral in form

$$
\begin{aligned}
z - z_0 &= \pm \int_{-\infty}^{w} \frac{dw}{\sqrt{|h|}\sqrt{\frac{a_1}{|h|}w^4 - 1}} \\
&= \pm\Big(a_1|h|\Big)^{-\frac{1}{4}} \int_{\tilde{w}}^{\infty} \frac{d\tilde{w}}{\sqrt{\tilde{w}^4 - 1}} \ ,
\end{aligned}
\tag{48}
$$

where $w = \left(\frac{|h|}{a_1}\right)^{\frac{1}{4}}\tilde{w}$ and z_0 is arbitrary integral constant .

Since the elliptic integral has the form

$$
\int_{x}^{\infty} \frac{dt}{\sqrt{t^4 - 1}} = \frac{1}{\sqrt{2}} cn^{-1}\left(\frac{1}{x}, k_1\right),
$$

where $k_1 = \frac{\sqrt{2}}{2}$.

Hence equation (48) can be rewritten into the following form

$$
\begin{aligned}
z - z_0 &= \pm\Big(a_1|h|\Big)^{-\frac{1}{4}} \int_{\tilde{w}}^{\infty} \frac{d\tilde{w}}{\sqrt{\tilde{w}^4} - 1} \\
&= \pm\frac{1}{\sqrt{2}}\Big(a_1 h\Big)^{-\frac{1}{4}} cn^{-1}\left(\frac{1}{\tilde{w}}, k_1\right)
\end{aligned}
\tag{49}
$$

Solve for $\tilde{w}$, we have

$$
\frac{1}{\tilde{w}} = cn\left(\pm\sqrt{2}\Big(a_1|h|\Big)^{\frac{1}{4}}(z - z_0), k_1\right),
$$

and then

$$\tilde{w}(z) = \frac{1}{cn\Bigg(\pm\sqrt{2}\Big(a_1|h|\Big)^{\frac{1}{4}}(z-z_0), k_1\Bigg)}$$
$$= nc\Bigg(\pm\sqrt{2}\Big(a_1|h|\Big)^{\frac{1}{4}}(z-z_0), k_1\Bigg). \tag{50}$$

Therefore,

$$w = \Big(\frac{|h|}{a_1}\Big)^{\frac{1}{4}}\tilde{w}$$
$$= \Big(\frac{|h|}{a_1}\Big)^{\frac{1}{4}} nc\Bigg(\pm\sqrt{2}\Big(a_1|h|\Big)^{\frac{1}{4}}(z-z_0), k_1\Bigg), \tag{51}$$

Since $U(\xi) = \sqrt{2}we^{\frac{1}{3}\delta\xi}$, the equation (24) has exact solution

$$U(\xi) = \sqrt{2}we^{\frac{1}{3}\delta\xi}$$
$$= \sqrt{2}\Big(\frac{|h|}{a_1}\Big)^{\frac{1}{4}}\tilde{w}(z)e^{\frac{1}{3}\delta\xi}$$
$$= \sqrt{2}\Big(\frac{|h|}{a_1}\Big)^{\frac{1}{4}} e^{\frac{1}{3}\delta\xi} nc\Bigg(\pm\sqrt{2}\Big(a_1|h|\Big)^{\frac{1}{4}}(z-z_0), k_1\Bigg). \tag{52}$$

where $k_1 = \frac{\sqrt{2}}{2}$, $z = \frac{3}{\delta}e^{\frac{1}{3}\delta\xi}$ and z_0 is an arbitrary constant.

CHAPTER V

CONCLUSION

In this thesis, we studied the Two-Dimensional Burgers-Korteweg-de Vries (2D-BKdV) equation and Two-Dimensional Compound Burgers-Korteweg-de Vries (2D-Compound BKdV) by analyzing the first integral equation, which indicates that under some particular conditions, the 2D-BKdV equation and 2D-Compound BKdV have exact travelling wave solutions. By using the elliptic integral and some transformations, travelling wave solution to the 2D-BKdV equation and 2D-Compound BKdV equation are expressed explicitly.

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BIOGRAPHICAL SKETCH

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