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# Topological pressure and fractal dimensions for bi-Lipschitz mappings

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TOPOLOGICAL PRESSURE AND FRACTAL DIMENSIONS

FOR BI-LIPSCHITZ MAPPINGS

A Thesis

by

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ABSTRACT

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In this thesis, first we have defined the topological pressure $P(t)$ and then using Banach limit we have determined a unique Borel probability measure $\mu_h$ supported by the invariant set $E$ of a system of bi-Lipschitz mappings where $h$ is the unique zero of the pressure function. Using the topological pressure and the measure $\mu_h$, under certain condition on bi-Lipschitz constants, we have shown that the fractal dimensions such as the Hausdorff dimension, the packing dimension and the box-counting dimension of the set $E$ are all equal to $h$. Moreover, it is shown that the $h$-dimensional Hausdorff measure and the $h$-dimensional packing measure are finite and positive.

## DEDICATION

The accomplishment of my masters studies in Mathematics would not have been possible without the love and support of my family. Thank you to my wife Reyna for supporting me through every step of my college career. Without her, this accomplishment would not be possible. To my children: Montzerrattee, Isaak, and Genessys, thank you for being an inspiration, and making my graduation a dream come true. May God bless them all.

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# CHAPTER I

## INTRODUCTION

A basic task in Fractal Geometry is to determine or estimate the various dimensions of fractal sets. Fractal dimensions are introduced to measure the sizes of fractal sets and are employed in many different disciplines. Many results on fractal dimensions are obtained for fractal sets with a special structure.

Let $E$ be a nonempty bounded subset of $\mathbb{R}^n$, and $s \geq 0$. $\mathscr{H}^s(E)$, $\mathscr{P}^s(E)$, $\dim_{\mathrm{H}} E$, $\dim_{\mathrm{P}} E$, $\underline{\dim}_{\mathrm{B}} E$ and $\overline{\dim}_{\mathrm{B}} E$ denote the $s$-dimensional Hausdorff measure, the $s$-dimensional packing measure, the Hausdorff dimension, the packing dimension, the lower box-counting and the upper box-counting dimension of the set $E$ respectively (for the relevant definitions see the next section).

Assume that $(X,d)$ is a complete metric space and that for each $i \in I$ there is a contractive injection $\varphi_i : X \to X$, where $I := \{1,2,\cdots,N\}$ is a finite index set. By contractivity of a mapping $\varphi$ we mean that there is a constant $0 < s < 1$ so that

$$d(\varphi(x), \varphi(y)) \leq s d(x,y)$$

for every $x, y \in X$. The collection $\{\varphi_i : i \in I\}$ is called an iterated function system (IFS). Then there is a unique nonempty compact set $E \subset X$, called the invariant set or the limit set of the iterated function system such that

$$E = \bigcup_{i \in I} \varphi_i(E).$$

An iterated function system satisfies the open set condition (OSC) if there exists a nonempty open set $U \subset X$ such that $\varphi_i(U) \subset U$ for every $i \in I$ and $\varphi_i(U) \cap \varphi_j(U) = \emptyset$ for every pair $i, j \in I$, $i \neq j$. Furthermore, the system satisfies the strong open set condition (SOSC) if $U$ can be chosen such that

$U \cap E \neq \emptyset$. An IFS is said to satisfy the strong separation condition (SSC) if the images $\varphi_i(E)$, $i \in I$, are pairwise disjoint for the invariant set $E$. Note that SSC implies SOSC, and for a finite iterated function system OSC implies SOSC (see [PRSS, S]). We call the contractive mapping $\varphi_i$ a similitude or a similarity mapping if there is a fixed ratio $0 < c_i < 1$ such that $d(\varphi_i(x), \varphi_i(y)) = c_i d(x,y)$. If all the mappings are similitudes, the invariant set is called self-similar. The following theorem is known (see [H, F2]).

**Theorem A:** Let $\{\varphi_1, \varphi_2, \cdots, \varphi_N\}$ be a system of self-similar mappings with $\varphi_i$ has the similarity ratio $c_i$ and $E$ is the self-similar set. If the open set condition is satisfied then

$$\dim_{\mathrm{H}}(E) = \dim_{\mathrm{P}}(E) = \underline{\dim}_{\mathrm{B}}(E) = \overline{\dim}_{\mathrm{B}}(E) = h, \text{ and } 0 < \mathscr{H}^h(E) \leq \mathscr{P}^h(E) < \infty,$$

where $h$ is the unique positive real number given by $\sum_{j=1}^N c_j^h = 1$.

Write $\varphi_\sigma = \varphi_{\sigma_1} \circ \cdots \circ \varphi_{\sigma_n}$ and $E_\sigma = \varphi_\sigma(E)$ for $\sigma = (\sigma_1, \cdots, \sigma_n) \in I^*$ and $n \in \mathbb{N}$, where $I^* := \cup_{k \geq 1} I^k$ is the set of all words over the symbols in $I$. The IFS is said to be semiconformal if the invariant set $E$ has positive diameter, and there are constants $C \geq 1$ and $0 < \underline{s}_\sigma \leq \overline{s}_\sigma < 1$ for each $\sigma \in I^*$, such that $\overline{s}_\sigma \leq C \underline{s}_\sigma$ and

$$\underline{s}_\sigma d(x,y) \leq d(\varphi_\sigma(x), \varphi_\sigma(y)) \leq \overline{s}_\sigma d(x,y) \tag{1.1}$$

for all $x, y \in X$ and $\sigma \in I^*$. Note that then

$$\frac{C^{-1}}{\operatorname{diam}(X)} \operatorname{diam}(\varphi_\sigma(X)) \leq \underline{s}_\sigma \leq \overline{s}_\sigma \leq \frac{C}{\operatorname{diam}(X)} \operatorname{diam}(\varphi_\sigma(X))$$

for each $\sigma \in I^*$. To know more about semiconformal iterated function systems, one could see [KV, RV]. By (1.1) it follows that the pressure of a semiconformal IFS can be calculated by the formula

$$P(t) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{\sigma \in I^n} s_\sigma^t, \tag{1.2}$$

where each $s_\sigma$, $\sigma \in I^*$, is allowed to be any of the numbers $\operatorname{diam}(E_\sigma)$, $\underline{s}_\sigma$ or $\overline{s}_\sigma$, and $t \in \mathbb{R}$. Let us now state the following proposition (see [RV, Proposition 4.12]).

**Proposition B:** Let $E$ be the invariant set of a semiconformal IFS $\{\varphi_i : i \in I\}$ defined on a complete metric space. If the SOSC holds, then

$$\dim_{\mathrm{H}}(E) = h$$

where $h$ is unique and is given by $P(h) = 0$.

In the special case, when every $\varphi_i$ is a similitude and $c_i$, $i \in I$, is the corresponding contraction ratio, then for any $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_n) \in I^*$, we get $\underline{s}_\sigma = \overline{s}_\sigma = c_{\sigma_1} c_{\sigma_2} \cdots c_{\sigma_n}$, and so by multinomial theorem, $P(h) = 0$ reduces to

$$\sum_{j=1}^{N} c_j^h = 1,$$

which is the formula used in Theorem A to determine the dimensions of the invariant set $E$. A mapping $\varphi : X \to X$ is said to be a contractive bi-Lipschitz mapping with bi-Lipschitz constants $\underline{s}, \overline{s} \in (0,1)$ if $\underline{s}\, d(x,y) \le d(\varphi(x), \varphi(y)) \le \overline{s}\, d(x,y)$ for all $x, y \in X$. $\{\varphi_1, \varphi_2, \cdots, \varphi_N\}$ is said to be a system of contractive bi-Lipschitz mappings if for each $\sigma \in I^*$, there exist constants $\underline{s}_\sigma, \overline{s}_\sigma \in (0,1)$ such that

$$\underline{s}_\sigma d(x,y) \le d(\varphi_\sigma(x), \varphi_\sigma(y)) \le \overline{s}_\sigma d(x,y),$$

in other words, $\underline{s}_\sigma$ and $\overline{s}_\sigma$ are the bi-Lipschitz constants of $\varphi_\sigma$ for each $\sigma \in I^*$. Note that if we assume $\overline{s}_\sigma \le C \underline{s}_\sigma$ for all $\sigma \in I^*$, then the system of bi-Lipschitz mappings reduces to a semiconformal iterated function system. In this thesis, we have considered a system $\{\varphi_1, \varphi_2, \cdots, \varphi_N\}$ of bi-Lipschitz mappings satisfying the open set condition, and assume the condition that there exists a constant $C \ge 1$ such that $\overline{s}_\sigma \le C \underline{s}_\sigma$ for $\sigma \in I^*$. For such a system, first we have defined the topological pressure function $P(t)$, and show that there exists a unique $h \in (0, +\infty)$ such that $P(h) = 0$. Then using Banach limit we have determined a unique Borel probability measure $\mu_h$ supported by the invariant set $E$. Using the consequence of the topological pressure and the measure $\mu_h$, we have shown that

$$\dim_{\mathrm{H}}(E) = \dim_{\mathrm{P}}(E) = \underline{\dim}_{\mathrm{B}}(E) = \overline{\dim}_{\mathrm{B}}(E) = h, \text{ and } 0 < \mathscr{H}^h(E) \le \mathscr{P}^h(E) < \infty.$$

A self-similar mapping is a special case of a bi-Lipschitz mapping, and so the result in this thesis is a generalization of Theorem A, and is an extension of Proposition B.

## 1.1 Basic Definitions and Proposition

In this section, first we give some basic definitions and proposition that we need to prove the main result in this thesis.

Let $X$ be a nonempty compact metric space and $s \geq 0$. Let $\mathscr{U} = \{U_i\}$ be a countable collection of subsets of $X$. We define

$$\|\mathscr{U}\|^s := \sum_{U_i \in \mathscr{U}} |U_i|^s,$$

where $|A|$ denotes the diameter of a set $A$. Let $E$ be a nonempty subset of $X$ and $\delta > 0$. A countable collection $\mathscr{U} = \{U_i\} \subset X$ is called a $\delta$-covering of the set $E$ if $E \subset \bigcup U_i$ and for each $i$, $0 < |U_i| \leq \delta$. Suppose that $E \subset X$ and $s \geq 0$, define

$$\mathscr{H}^s_\delta(E) = \inf_{\mathscr{U}} \{\|\mathscr{U}\|^s : \mathscr{U} \text{ is a } \delta\text{-covering of } E\},$$

and

$$\mathscr{H}^s(E) = \lim_{\delta \to 0} \mathscr{H}^s_\delta(E).$$

$\mathscr{H}^s(E)$ is called the $s$-dimensional Hausdorff measure of $E$. The Hausdorff dimension of $E$, denoted by $\dim_{\mathrm{H}} E$, is defined by

$$\dim_{\mathrm{H}} E = \sup\{s \geq 0 : \mathscr{H}^s(E) = \infty\} = \inf\{s \geq 0 : \mathscr{H}^s(E) = 0\}.$$

Let $N_\delta(E)$ be the smallest number of sets of diameter at most $\delta$ that can cover $E$. The lower and upper box-counting dimensions of $E$ are defined as

$$\underline{\dim}_{\mathrm{B}} E = \liminf_{\delta \to 0} \frac{\log N_\delta(E)}{-\log \delta}, \text{ and } \overline{\dim}_{\mathrm{B}} E = \limsup_{\delta \to 0} \frac{\log N_\delta(E)}{-\log \delta}.$$

A $\delta$-packing of the set $E$ is a countable family of disjoint closed balls of radii at most $\delta$ and with centers in $E$. For $s \geq 0$, the $s$-dimensional packing pre-measure is defined as

$$\mathscr{P}^s_0(E) = \lim_{\delta \to 0} \mathscr{P}^s_\delta(E),$$

where $\mathscr{P}^s_\delta(E) = \sup\{\sum_{i=1}^\infty |B_i|^s : \{B_i\}$ is a $\delta$-packing of $E\}$. The $s$-dimensional packing measure is defined as

$$\mathscr{P}^s(E) = \inf\left\{\sum_{i=1}^\infty \mathscr{P}^s_0(E_i) : E \subset \bigcup_{i=1}^\infty E_i\right\}.$$

It is known that $\mathscr{P}^s$ is countably sub-additive, but $\mathscr{P}^s_0$ is only finitely sub-additive. For any $E \subset X$, $\mathscr{P}^s_0(E) \geq \mathscr{P}^s(E)$ and $\mathscr{P}^s_0(E) = \mathscr{P}^s_0(\overline{E})$, where $\overline{E}$ is the closure of $E$. The upper box-counting dimension $\overline{\dim}_B$ and the packing dimension $\dim_P$ can be induced respectively by packing pre-measure and packing measure by

$$\overline{\dim}_B E = \inf\{s \geq 0 : \mathscr{P}^s_0(E) = 0\} = \sup\{s \geq 0 : \mathscr{P}^s_0(E) = \infty\},$$

$$\dim_P E = \inf\{s \geq 0 : \mathscr{P}^s(E) = 0\} = \sup\{s \geq 0 : \mathscr{P}^s(E) = \infty\}.$$

Moreover, for any nonempty set $E$ it is well-known that $\mathscr{H}^s(E) \leq \mathscr{P}^s(E)$, and

$$\dim_H E \leq \dim_P E \leq \overline{\dim}_B E, \text{ and } \dim_H E \leq \underline{\dim}_B E \leq \overline{\dim}_B E. \tag{1.3}$$

For further properties of the above measures and dimensions, one could see [F1, F2, M].

Let us now state the following proposition, which is analogous to Proposition 2.2 in [F2], and can similarly be proved.

**Proposition 1.1.1.** Let $E$ be a Borel subset of a compact metric space $X$, and let $\mu$ be a finite Borel measure on $X$ and $0 < c < +\infty$.

$(a)$ If $\limsup_{r\to 0} \mu(B(x,r))/r^s \leq c$ for all $x \in E$ then $\mathscr{H}^s(E) \geq \mu(E)/c$.

$(a)$ If $\limsup_{r\to 0} \mu(B(x,r))/r^s \geq c$ for all $x \in E$ then $\mathscr{P}^s(E) \leq 2^s \mu(E)/c$.

Let $\ell^\infty$ be the set of all bounded sequences $x = (x_n)_{n\in\mathbb{N}}$ of real or complex numbers, which form a vector space with respect to point-wise addition and multiplication by a scalar. It is equipped with the norm $\|x\| = \sup_n |x_n|$. The normed space $\ell^\infty$ is complete with respect to the metric $\|x - y\|$, and so it forms a Banach space. By the Hahn-Banach theorem (see [Y, p. 102-104]), there exists a linear functional $L : \ell^\infty \to \mathbb{R}$ such that

$(i)$ $L$ is linear;

(*ii*) $L((x_n)_{n\in\mathbb{N}}) = L((x_{n+1})_{n\in\mathbb{N}})$;

(*iii*) $\liminf_{n\to\infty}(x_n) \le L((x_n)_{n\in\mathbb{N}}) \le \limsup_{n\to\infty}(x_n)$.

The functional $L$, defined above, is called a Banach limit. The use of the Banach limit is rather a standard tool in producing an invariant measure from a given measure.

### 1.2 Bi-Lipschitz Iterated Function System and the Topological Pressure

Let $D_0$ be the empty set, and $N \ge 2$. For $n \ge 1$, define

$$D_n = \{1,2,\cdots,N\}^n,\ D_\infty = \lim_{n\to\infty} D_n \text{ and } D = \bigcup_{k=0}^{\infty} D_k.$$

Elements of $D$ are called words. For any $\sigma \in D$ if $\sigma = (\sigma_1,\sigma_2,\cdots,\sigma_n) \in D_n$, we write $\sigma^- = (\sigma_1,\sigma_2,\cdots,\sigma_{n-1})$ to denote the word obtained by deleting the last letter of $\sigma$, $|\sigma| = n$ to denote the length of $\sigma$, and $\sigma|_k := (\sigma_1,\sigma_2,\cdots,\sigma_k)$, $k \le n$, to denote the truncation of $\sigma$ to the length $k$. For any two words $\sigma = (\sigma_1,\sigma_2,\cdots,\sigma_k)$ and $\tau = (\tau_1,\tau_2,\cdots,\tau_m)$, we write $\sigma\tau = (\sigma_1,\cdots,\sigma_k,\tau_1,\cdots,\tau_m)$ to denote the juxtaposition of $\sigma,\tau \in D$. A word of length zero is called the empty word and is denoted by $\emptyset$. For $\sigma \in D$ and $\tau \in D\cup D_\infty$ we say $\tau$ is an extension of $\sigma$, written as $\sigma \prec \tau$, if $\tau|_{|\sigma|} = \sigma$. Two words $\sigma,\tau \in D$ are said to be incomparable if none of them is an extension of the other. For $\sigma \in D_k$, the cylinder set $C(\sigma)$ is defined as $C(\sigma) = \{\tau \in D_\infty : \tau|_k = \sigma\}$. Let $X \subset \mathbb{R}^n$ be a nonempty compact set such that $X = \text{cl}(\text{int}X)$, and $d$ be a metric on $X$ inherited from $\mathbb{R}^n$. Let $\{\varphi_1,\varphi_2,\cdots,\varphi_N\}$ be a set of contractive bi-Lipschitz transformations mapping $X$ into $X$ with the bi-Lipschitz constants $\underline{s}_k$ and $\bar{s}_k$, such that for all $1 \le k \le N$,

$$0 < \underline{s}_k := \inf_{x,y\in X} \frac{d(\varphi_k(x),\varphi_k(y))}{d(x,y)} \le \sup_{x,y\in X} \frac{d(\varphi_k(x),\varphi_k(y))}{d(x,y)} := \bar{s}_k < 1,$$

and they satisfy the open set condition. Let $s = \min\{\underline{s}_j : 1 \le j \le N\}$ and $S = \max\{\bar{s}_j : 1 \le j \le N\}$, and then $0 < s \le S < 1$. For $\sigma = (\sigma_1,\sigma_2,\cdots,\sigma_n) \in D_n$, write $X_\sigma = \varphi_\sigma(X)$ and

$$\varphi_\sigma = \begin{cases} id_{\mathbb{R}^n}, & \sigma = \emptyset, \\ \varphi_{\sigma_1}\circ\cdots\circ\varphi_{\sigma_n}, & |\sigma| = n. \end{cases}$$

Since given $\sigma = (\sigma_1, \sigma_2, \cdots) \in D_\infty$, the diameters of the compact sets $\varphi_{\sigma|_n}(X) = \varphi_{\sigma_1} \circ \cdots \circ \varphi_{\sigma_n}(X)$, $n \geq 1$, converge to zero and since they form a descending family, the set

$$\bigcap_{n=1}^{\infty} \varphi_{\sigma|_n}(X)$$

is a singleton and therefore, if we denote its element by $\pi(\sigma)$, this defines the coding map $\pi : D_\infty \to X$. The main object of our interest is the nonempty compact set

$$E = \pi(D_\infty) = \bigcup_{\sigma \in D_\infty} \bigcap_{n=1}^{\infty} \varphi_{\sigma|_n}(X),$$

called the limit set or the invariant set generated by the bi-Lipschitz mappings $\{\varphi_k : 1 \leq k \leq N\}$. Moreover, $\pi(C(\sigma)) = E \cap J_\sigma$ for $\sigma \in D$. Let $\rho$ be the metric on $D_\infty$ such that for any $\sigma = (\sigma_1, \sigma_2, \cdots), \tau = (\tau_1, \tau_2, \cdots) \in D_\infty$,

$$\rho(\sigma, \tau) = 2^{-\min\{k : \sigma_k = \tau_k\}}$$

with the convention $\rho(\sigma, \sigma) = 0$. Let $E$ be equipped with the Euclidean metric. Then $\pi$ is continuous and onto. Note that $E$ satisfies the following invariance equality:

$$E = \bigcup_{k=1}^{N} \varphi_k(E) \tag{1.4}$$

for each $k \geq 1$. Let $\mathfrak{F}_k = \{J_\sigma : \sigma \in D_k\}$, and $\mathfrak{F} = \cup_{k \geq 0} \mathfrak{F}_k$. The elements of $\mathfrak{F}_k$ are called the rank-$k$ basic elements or basic elements of order $k$, and the elements of $\mathfrak{F}$ are called the basic elements of the limit set $E$. Two basic elements are disjoint by that it is meant that their interiors are disjoint. Let $\underline{s}_\sigma$ and $\overline{s}_\sigma$ be the bi-Lipschitz constants for the bi-Lipschitz mapping $\varphi_\sigma$, where $\sigma \in D$, such that

$$0 < \underline{s}_\sigma := \inf_{x,y \in X} \frac{d(\varphi_\sigma(x), \varphi_\sigma(y))}{d(x,y)} \leq \sup_{x,y \in X} \frac{d(\varphi_\sigma(x), \varphi_\sigma(y))}{d(x,y)} := \overline{s}_\sigma < 1.$$

Then for $\sigma, \tau \in D$, we have

$$\begin{aligned}
\underline{s}_{\sigma\tau} &= \inf_{x,y \in X} \frac{d(\varphi_{\sigma\tau}(x), \varphi_{\sigma\tau}(y))}{d(x,y)} \\
&= \inf_{x,y \in X} \frac{d(\varphi_\sigma(\varphi_\tau(x)), \varphi_\sigma(\varphi_\tau(y)))}{d(\varphi_\tau(x), \varphi_\tau(y))} \cdot \inf_{x,y \in X} \frac{d(\varphi_\tau(x), \varphi_\tau(y))}{d(x,y)} \\
&= \inf_{u,v \in \varphi_\tau(X)} \frac{d(\varphi_\sigma(u), \varphi_\sigma(v))}{d(u,v)} \cdot \inf_{x,y \in X} \frac{d(\varphi_\tau(x), \varphi_\tau(y))}{d(x,y)} \\
&\geq \inf_{u,v \in X} \frac{d(\varphi_\sigma(u), \varphi_\sigma(v))}{d(u,v)} \cdot \inf_{x,y \in X} \frac{d(\varphi_\tau(x), \varphi_\tau(y))}{d(x,y)} \\
&= \underline{s}_\sigma \underline{s}_\tau.
\end{aligned}$$

Similarly, for $\sigma, \tau \in D$ we have $\overline{s}_{\sigma\tau} \leq \overline{s}_\sigma \overline{s}_\tau$. Moreover, for $\sigma, \tau \in D$, we have

$$\operatorname{diam} X_{\sigma\tau} = \operatorname{diam} \varphi_\sigma(\varphi_\tau(X)) = \sup_{u,v \in \varphi_\tau(X)} d(\varphi_\sigma(u), \varphi_\sigma(v)) = \sup_{u,v \in \varphi_\tau(X)} \left[ \frac{d(\varphi_\sigma(u), \varphi_\sigma(v))}{d(u,v)} \cdot d(u,v) \right],$$

and so

$$\operatorname{diam} X_{\sigma\tau} \leq \sup_{u,v \in X} \frac{d(\varphi_\sigma(u), \varphi_\sigma(v))}{d(u,v)} \sup_{u,v \in \varphi_\tau(X)} d(u,v) = \overline{s}_\sigma \operatorname{diam} \varphi_\tau(X). \tag{1.5}$$

If $\sigma$ is the empty word $\emptyset$, then we we write $\underline{s}_\sigma = \overline{s}_\sigma = 1$. Let us assume that the bi-Lipschitz constants are such that there exists a constant $C \geq 1$ such that $\overline{s}_\sigma \leq C \underline{s}_\sigma$ for any $\sigma \in D$. Without any loss of generality, we assume $\operatorname{diam}(X) = 1$. Then it follows that

$$\operatorname{diam} X_\sigma = \sup_{x,y \in X} d(\varphi_\sigma(x), \varphi_\sigma(y)) \leq \overline{s}_\sigma \sup_{x,y \in X} d(x,y) = \overline{s}_\sigma,$$

and similarly, $\underline{s}_\sigma \leq \operatorname{diam}(X_\sigma)$. Then for $\sigma, \tau \in D$, by (1.5),

$$\operatorname{diam} \varphi_{\sigma\tau}(X) \leq \overline{s}_\sigma \operatorname{diam} \varphi_\tau(X) \leq \frac{\overline{s}_\sigma}{\operatorname{diam} \varphi_\sigma(X)} \operatorname{diam} \varphi_\sigma(X) \operatorname{diam} \varphi_\tau(X),$$

i.e.,

$$\operatorname{diam} \varphi_{\sigma\tau}(X) \leq \frac{\overline{s}_\sigma}{\underline{s}_\sigma} \operatorname{diam} \varphi_\sigma(X) \operatorname{diam} \varphi_\tau(X) \leq C \operatorname{diam} \varphi_\sigma(X) \operatorname{diam} \varphi_\tau(X),$$

and similarly,

$$C^{-1} \operatorname{diam} \varphi_\sigma(X) \operatorname{diam} \varphi_\tau(X) \leq \operatorname{diam} \varphi_{\sigma\tau}(X).$$

Thus for $\sigma, \tau \in D$,

$$C^{-1}\mathrm{diam}\varphi_\sigma(X)\mathrm{diam}\varphi_\tau(X) \le \mathrm{diam}\varphi_{\sigma\tau}(X) \le C\mathrm{diam}\varphi_\sigma(X)\mathrm{diam}\varphi_\tau(X). \tag{1.6}$$

Hence, by the standard theory of sub-additive sequences, the following limit exists:

$$P(t) := \lim_{k\to\infty} \frac{1}{k} \log \sum_{|\sigma|=k} (\mathrm{diam}X_\sigma)^t, \tag{1.7}$$

where $t \in \mathbb{R}$. The above function $P(t)$ is called the *topological pressure* of the bi-Lipschitz mappings. Since

$$C^{-1}\overline{s}_\sigma \le \underline{s}_\sigma \le \mathrm{diam}(X_\sigma) \le \overline{s}_\sigma \le C\underline{s}_\sigma,$$

the topological pressure $P(t)$ can also be written in any of the following forms:

$$P(t) = \lim_{k\to\infty} \frac{1}{k} \log \sum_{|\sigma|=k} \underline{s}_\sigma^t \;\text{ or }\; P(t) = \lim_{k\to\infty} \frac{1}{k} \log \sum_{|\sigma|=k} \overline{s}_\sigma^t.$$

The following lemma gives some properties of the function $P(t)$.

**Lemma 1.2.1.** *The function $P(t)$ is strictly decreasing, convex and hence continuous on $\mathbb{R}$.*

*Proof.* To prove that $P(t)$ is strictly decreasing let $\delta > 0$. Then,

$$\begin{aligned} P(t+\delta) &= \lim_{k\to\infty} \frac{1}{k} \log \sum_{|\sigma|=k} \overline{s}_\sigma^{t+\delta} \le \lim_{k\to\infty} \frac{1}{k} \log \sum_{|\sigma|=k} \overline{s}_\sigma^t S^{k\delta} \\ &= P(t) + \delta \log S < P(t), \end{aligned}$$

i.e., $P(t)$ is strictly decreasing. For $t_1, t_2 \in \mathbb{R}$ and $a_1, a_2 > 0$ with $a_1 + a_2 = 1$, using Hölder's inequality, we have

$$\begin{aligned} P(a_1t_1 + a_2t_2) &= \lim_{k\to\infty} \frac{1}{k} \log \sum_{|\sigma|=k} \overline{s}_\sigma^{a_1t_1+a_2t_2} = \lim_{k\to\infty} \frac{1}{k} \log \sum_{|\sigma|=k} \overline{s}_\sigma^{a_1t_1}\overline{s}_\sigma^{a_2t_2} \\ &\le \lim_{k\to\infty} \frac{1}{k} \log \Big(\sum_{|\sigma|=k} \overline{s}_\sigma^{t_1}\Big)^{a_1} \Big(\sum_{|\sigma|=k} \overline{s}_\sigma^{t_2}\Big)^{a_2} \\ &= a_1P(t_1) + a_2P(t_2), \end{aligned}$$

i.e., $P(t)$ is convex and hence continuous on $\mathbb{R}$. □

Let us now prove the following lemma.

**Lemma 1.2.2.** *There exists a unique $h \in \mathbb{R}$ such that $P(h) = 0$. In addition, $h \in (0, +\infty)$.*

*Proof.* By Lemma 1.2.1, the function $P(t)$ is strictly decreasing and continuous on $\mathbb{R}$, and so there exists a unique $h \in \mathbb{R}$ such that $P(h) = 0$. Note that

$$P(0) = \lim_{k\to\infty} \frac{1}{k} \log \sum_{|\sigma|=k} 1 = \lim_{k\to\infty} \frac{1}{k} \log N^k = \log N \geq \log 2 > 0.$$

In order to conclude the proof it therefore suffices to show that $\lim_{t\to+\infty} P(t) = -\infty$. Note that $\mathrm{diam} X_\sigma \leq S^k$ for $\sigma \in D_k$, and so for $t > 0$,

$$\begin{aligned} P(t) &= \lim_{k\to\infty} \frac{1}{k} \log \sum_{|\sigma|=k} (\mathrm{diam} X_\sigma)^t \leq \lim_{k\to\infty} \frac{1}{k} \log \sum_{|\sigma|=k} S^{kt} \\ &= \lim_{k\to\infty} \frac{1}{k} \log S^{kt} + \lim_{k\to\infty} \frac{1}{k} \log \sum_{|\sigma|=k} 1 = t \log S + \lim_{k\to\infty} \frac{1}{k} \log N^k = t \log S + \log N. \end{aligned}$$

Since $S < 1$, it follows that $\lim_{t\to+\infty} P(t) = -\infty$, and hence the lemma follows.

□

**Definition 1.2.3.** *Let $E$ be the limit set of the bi-Lipschitz mappings satisfying the open set condition. For some $0 < r < 1$, the family of the basic elements $\mathscr{U}_r = \{X_\sigma : |X_\sigma| \leq r < |X_{\sigma^-}|\} \subset \mathfrak{F}$ is called the $r$-Moran covering of $E$ provided it is a covering of $E$, i.e., $E \subseteq \bigcup_{X_\sigma \in \mathscr{U}_r} X_\sigma$.*

From the definition it follows that elements of a Moran covering are disjoint, have almost equal sizes, and are often of different ranks. Let us now prove the following proposition.

**Proposition 1.2.4.** Let $0 < r < 1$, and let $\mathscr{U}_r$ be the $r$-Moran covering of $E$. Then there exists a positive integer $M$ such that the ball $B(x, r)$ of radius $r$, where $x \in X$, intersects at most $M$ elements of $\mathscr{U}_r$.

*Proof.* For $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_k) \in D_k$, $k \geq 1$,

$$|X_\sigma| = \mathrm{diam}\varphi_\sigma(X) \geq C^{-1} \mathrm{diam}\varphi_{\sigma^-}(X) \mathrm{diam}\varphi_{\sigma_k}(X) \geq C^{-1} \mathrm{diam}\varphi_{\sigma^-}(X) \underline{s}_{\sigma_k} \geq C^{-1} s |X_{\sigma^-}|,$$

and so $|X_\sigma| \geq c|X_{\sigma^-}|$ where $c = C^{-1}s$. Let $\mathscr{U}_r$ be the $r$-Moran covering of $E$. Fix any $x \in X$, and write $V = B(x,r)$. Define,

$$Q_V = \{X_\sigma \in \mathscr{U}_r : X_\sigma \cap V \neq \emptyset, \sigma \in D\}.$$

Since the set $X$ contains a ball of radius $a$, where $a > 0$ is a constant, each set $X_\sigma$ in $Q_V$ contains a ball of radius $a|X_\sigma| \geq ac|X_{\sigma^-}| > acr$, and all such balls are disjoint. Again any element $X_\sigma$ of $Q_V$ is contained in a ball of radius $2r$ concentric with $V$, and so comparing the volumes, we have

$$V_{2r} > (\# Q_V) V_{acr},$$

where $V_\ell$ represents the volume of a ball of radius $\ell$, and thus $\# Q_V < V_{2r}/V_{acr}$ (in this regard one could also see [F1, Lemma 9.2]). Hence $M := \lfloor V_{2r}/V_{acr} \rfloor$ fulfills the statement of the proposition, where $\lfloor x \rfloor$ of a number $x$ represents the greatest integer not exceeding $x$.

□

In the next section, we state and prove the main result of the thesis.

### 1.3 Main Result

The relationship between the unique zero $h$ of the pressure function $P(t)$ and the fractal dimensions of the limit set $E$ is given by the following theorem. Moreover, it shows that the $h$-dimensional Hausdorff measure and the $h$-dimensional packing measure are finite and positive.

**Theorem 1.3.1.** *Let $E$ be the limit set of a set of bi-Lipschitz mappings $\{\varphi_1, \cdots, \varphi_N\}$ such that the bi-Lipschitz constants satisfies the condition $\overline{s}_\sigma \leq C\underline{s}_\sigma$ as defined before, and $h \in (0, +\infty)$ be unique such that $P(h) = 0$. Then*

$$dim_H(E) = dim_P(E) = \underline{dim}_B(E) = \overline{dim}_B(E) = h, \text{ and } 0 < \mathscr{H}^h(E) \leq \mathscr{P}^h(E) < \infty.$$

Let us now prove the following proposition, which plays a vital role in the thesis.

**Proposition 1.3.2.** Let $h \in (0, +\infty)$ be unique such that $P(h) = 0$, and let $s_*$ and $s^*$ be any two arbitrary real numbers with $0 < s_* < h < s^*$. Then for all $n \geq 1$,

$$C^{-s_*} < \sum_{\sigma \in D_n} |X_\sigma|^{s_*} \text{ and } \sum_{\sigma \in D_n} |X_\sigma|^{s^*} < C^{s^*}.$$

*Proof.* Let $s_* < h$. As the pressure function $P(t)$ is strictly decreasing, $P(s_*) > P(h) = 0$. Then for any positive integer $n$, by (1.6), we have

$$0 < P(s_*) = \lim_{p\to\infty} \frac{1}{np} \log \sum_{\omega\in D_{np}} |X_\omega|^{s_*} \leq \lim_{p\to\infty} \frac{1}{np} \log C^{(p-1)s_*} \left( \sum_{\sigma\in D_n} |X_\sigma|^{s_*} \right)^p,$$

which implies

$$0 < \frac{1}{n} \log \left( C^{s_*} \sum_{\sigma\in D_n} |X_\sigma|^{s_*} \right) \text{ and so } \sum_{\sigma\in D_n} |X_\sigma|^{s_*} > C^{-s_*}.$$

Now if $h < s^*$, then $P(s^*) < 0$ as $P(t)$ is strictly decreasing. Then for any positive integer $n$, by (1.6), we have

$$0 > P(s^*) = \lim_{p\to\infty} \frac{1}{np} \log \sum_{\omega\in D_{np}} |X_\omega|^{s^*} \geq \lim_{p\to\infty} \frac{1}{np} \log C^{-(p-1)s^*} \left( \sum_{\sigma\in D_n} |X_\sigma|^{s^*} \right)^p,$$

which implies

$$0 > \frac{1}{n} \log \left( C^{-s^*} \sum_{\sigma\in D_n} |X_\sigma|^{s^*} \right) \text{ and so } \sum_{\sigma\in D_n} |X_\sigma|^{s^*} < C^{s^*}.$$

Thus the proposition is obtained. □

**Corollary 1.3.3.** Let $s_*$ and $s^*$ be any two arbitrary real numbers with $0 < s_* < h < s^*$ and $h$ is fixed. Then from the above proposition it follows that for all $n \geq 1$,

$$C^{-h} \leq \sum_{\sigma\in D_n} |X_\sigma|^h \leq C^h.$$

*Proof.* Let us first prove $\sum_{\sigma\in D_n} |X_\sigma|^h \leq C^h$. If not let $\sum_{\sigma\in D_n} |X_\sigma|^h > C^h$. Define $f_n(t) = C^t - \sum_{\sigma\in D_n} |X_\sigma|^t$. Then $f_n(t)$ is a real valued continuous function. Moreover, $f_n(s^*) > 0$, and $f_n(h) < 0$. Then by intermediate value theorem, there exists a real number $s$, where $h < s < s^*$, such that $f_n(s) = 0$, which implies $\sum_{\sigma\in D_n} |X_\sigma|^s = C^s$. But $h < s^*$ is arbitrary for which $\sum_{\sigma\in D_n} |X_\sigma|^{s^*} < C^{s^*}$, and so a contradiction arises. Hence $\sum_{\sigma\in D_n} |X_\sigma|^h \leq C^h$. Similarly, $\sum_{\sigma\in D_n} |X_\sigma|^h \geq C^{-h}$. Thus the corollary follows.

□

The following proposition plays an important role in the rest of the thesis.

**Proposition 1.3.4.** Let $h \in (0, +\infty)$ be such that $P(h) = 0$. Then there exists a unique Borel probability measure $\mu_h$ supported by $E$ such that for any $\sigma \in D$,

$$C^{-3h}|X_\sigma|^h \le \mu_h(X_\sigma) \le C^{3h}|X_\sigma|^h.$$

*Proof.* For $\sigma \in D$, $n \ge 1$, define

$$\nu_n(C(\sigma)) = \frac{\sum_{\tau \in D_n} (\mathrm{diam} X_{\sigma\tau})^h}{\sum_{\tau \in D_{|\sigma|+n}} (\mathrm{diam} X_\tau)^h}.$$

Then using (1.6) and Corollary 1.3.3, we have

$$\nu_n(C(\sigma)) \le \frac{C^h (\mathrm{diam} X_\sigma)^h \sum_{\tau \in D_n} (\mathrm{diam} X_\tau)^h}{\sum_{\tau \in D_{|\sigma|+n}} (\mathrm{diam} X_\tau)^h} \le C^{3h} (\mathrm{diam} X_\sigma)^h,$$

and similarly, $\nu_n(C(\sigma)) \ge C^{-3h}(\mathrm{diam} X_\sigma)^h$. Thus for a given $\sigma \in D$, $\{\nu_n(C(\sigma))\}_{n=1}^\infty$ is a bounded sequence of real numbers, and so Banach limit, denoted by Lim, is defined. For $\sigma \in D$, let

$$\nu(C(\sigma)) = \mathrm{Lim}_{n\to\infty} \nu_n(C(\sigma)).$$

Then

$$\sum_{j=1}^N \nu(C(\sigma j)) = \mathrm{Lim}_{n\to\infty} \sum_{j=1}^N \frac{\sum_{\tau \in D_n} (\mathrm{diam} X_{\sigma j \tau})^h}{\sum_{\tau \in D_{|\sigma|+1+n}} (\mathrm{diam} X_\tau)^h} = \mathrm{Lim}_{n\to\infty} \frac{\sum_{\tau \in D_{n+1}} (\mathrm{diam} X_{\sigma\tau})^h}{\sum_{\tau \in D_{|\sigma|+n+1}} (\mathrm{diam} X_\tau)^h},$$

and so

$$\sum_{j=1}^N \nu(C(\sigma j)) = \mathrm{Lim}_{n\to\infty} \nu_{n+1}(C(\sigma)) = \mathrm{Lim}_{n\to\infty} \nu_n(C(\sigma)) = \nu(C(\sigma)).$$

Thus by Kolmogorov's extension theorem, $\nu$ can be extended to a unique Borel probability measure $\gamma$ on $D_\infty$. Let $\mu_h$ be the image measure of $\gamma$ under the coding map $\pi$, i.e., $\mu_h = \gamma \circ \pi^{-1}$. Then $\mu_h$ is a unique Borel probability measure supported by $E$. Moreover, for any $\sigma \in D$,

$$\mu_h(X_\sigma) = \gamma(C(\sigma)) = \mathrm{Lim}_{n\to\infty} \nu_n(C(\sigma)) \le \mathrm{Lim}_{n\to\infty} C^{3h} (\mathrm{diam} X_\sigma)^h = C^{3h} (\mathrm{diam} X_\sigma)^h,$$

and similarly,

$$\mu_h(X_\sigma) \ge C^{-3h} (\mathrm{diam} X_\sigma)^h.$$

Thus the proof of the proposition is complete. □

**Proposition 1.3.5.** Let $h \in (0, +\infty)$ be the unique root of the pressure function as defined before. Then

$$0 < \mathscr{H}^h(E) < \infty \text{ and } \dim_{\mathrm{H}}(E) = h.$$

*Proof.* For any $n \geq 1$, the set $\{X_\sigma : \sigma \in D_n\}$ is a covering of the $E$ and so by Corollary 1.3.3,

$$\mathscr{H}^h(E) \leq \liminf_{n\to\infty} \sum_{\sigma \in D_n} |X_\sigma|^h \leq C^h < \infty,$$

which yields $\dim_H(E) \leq h$.

Let $\mu_h$ be the probability measure defined in Proposition 1.3.4. Let $r > 0$ and let $\mathscr{U}_r = \{X_\omega : |X_\omega| \leq r < |X_{\omega^-}|\}$ be the $r$-Moran covering of $E$. Then by Proposition 1.2.4, we get

$$\mu_h(B(x,r)) \leq \sum_{X_\omega \cap B(x,r) \neq \emptyset} \mu_h(X_\omega) \leq C^{3h} \sum_{X_\omega \cap B(x,r) \neq \emptyset} |X_\omega|^h \leq C^{3h} M r^h.$$

Thus

$$\limsup_{r\to 0} \frac{\mu_h(B(x,r))}{r^h} \leq C^{3h} M,$$

and so by Proposition 1.1.1, $\mathscr{H}^h(E) \geq C^{-3h} M^{-1} > 0$, which implies $\dim_H(E) \geq h$. Thus the proposition is yielded. □

Let us now prove the following lemma.

**Lemma 1.3.6.** *Let $h \in (0, +\infty)$ be such that $P(h) = 0$. Then $\overline{\dim}_B(E) \leq h$.*

*Proof.* Let $\mu_h$ be the probability measure defined in Proposition 1.3.4, and for $r > 0$ let $\mathscr{U}_r = \{X_\omega : |X_\omega| \leq r < |X_{\omega^-}|\}$ be the $r$-Moran covering of $E$. Then for any $X_\sigma \in \mathscr{U}_r$, we get $\mu_h(X_\sigma) \geq C^{-3h} |X_\sigma|^h$. Thus it follows that

$$\|\mathscr{U}_r\|^h = \sum_{X_\sigma \in \mathscr{U}_r} |X_\sigma|^h \leq C^{3h} \sum_{X_\sigma \in \mathscr{U}_r} \mu_h(X_\sigma) = C^{3h}.$$

Again for any $X_\sigma \in \mathscr{U}_r$, it follows that $|X_\sigma| \geq C^{-1} s |X_{\sigma^-}| > C^{-1} s r$. Hence, $\left(C^{-1} s r\right)^h N_r(E) \leq \|\mathscr{U}_r\|^h \leq C^{3h}$, where $N_r(E)$ is the smallest number of sets of diameter at most $r$ that can cover $E$, which implies $N_r(E) \leq C^{3h} \left(C^{-1} s r\right)^{-h} = C^{4h} s^{-h} r^{-h}$ and so

$$\log N_r(E) \leq \log\left[C^{4h} s^{-h}\right] - h \log r,$$

which yields

$$\overline{\dim}_{\mathrm{B}} E = \limsup_{r\to 0} \frac{\log N_r(E)}{-\log r} \le h,$$

and thus the lemma is obtained. □

Let us now prove the following proposition.

**Proposition 1.3.7.** Let $h \in (0,+\infty)$ be such that $P(h) = 0$, and then $\mathscr{P}^h(E) < \infty$.

*Proof.* Let $\mu_h$ be the probability measure defined in Proposition 1.3.4, then $\mu_h(X_\sigma) \ge C^{-3h}|X_\sigma|^h$. Let $\mathscr{U}_r = \{X_\omega : |X_\omega| \le r < |X_{\omega^-}|\}$ be the $r$-Moran covering of $E$ for some $r > 0$. Let $x \in X_\sigma$ for some $X_\sigma \in \mathscr{U}_r$, and then $X_\sigma \subset B(x,r)$. Again

$$|X_\sigma| \ge C^{-1} s |X_{\sigma^-}|.$$

Therefore,

$$\mu_h(B(x,r)) \ge \mu_h(X_\sigma) \ge C^{-3h}|X_\sigma|^h > C^{-4h} s^h r^h,$$

which implies

$$\liminf_{r\to 0} \frac{\mu_h(B(x,r))}{r^h} \ge C^{-4h} s^h,$$

and so by Proposition 1.1.1,

$$\mathscr{P}^h(E) \le 2^h C^{4h} s^{-h} < \infty,$$

and thus the proposition is obtained. □

**Proof of Theorem 1.3.1**

Proposition 1.3.5 tells us that $\dim_{\mathrm{H}}(E) = h$, and Lemma 1.3.6 gives that $\overline{\dim}_{\mathrm{B}} E \le h$. Combining these with the inequalities in (1.3), we have

$$\dim_{\mathrm{H}}(E) = \dim_{\mathrm{P}}(E) = \underline{\dim}_{\mathrm{B}}(E) = \overline{\dim}_{\mathrm{B}}(E) = h.$$

Again from Proposition 1.3.5 and Proposition 1.3.7 it follows that

$$0 < \mathscr{H}^h(E) \le \mathscr{P}^h(E) < \infty.$$

Thus the proof of the theorem is complete.

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## BIOGRAPHICAL SKETCH

Hugo E. Olvera, is an ARMY veteran who served in Operation Iraq Freedom and in the Republic of South Korea from 2004-2008. He was born in Toluca, Edo. de Mexico. In 2008, after fulfilling his military service, he started his college career at South Texas College, where he earned an Associate degree in Mathematics. In Spring of 2011, he transferred to the University of Texas Pan-American, he earned a Bachelors degree in Pure Mathematics. In Fall of 2012, he entered the Master's program in Mathematics at the University of Texas Pan-American. His main research includes fractal dimensions such as the Hausdorff dimension, the packing dimension and the box-counting dimension of the limit set generated by a set of contractive bi-Lipschitz mappings. He is dedicated to his work, and proud of his educational accomplishments. In his spare time, he loves to spend time with his family, and read to his children. He likes listening to the music of The Beatles, and play outdoor sports. Currently, he lives in Weslaco, Texas, with his wife Mrs. Reyna Olvera, and their children: Montzerrattee, Isaak, and Genessys.